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Decentralized Systems with Design Constraints

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In the Name of

the All-Compassionate, the All-Merciful.

*"Glory be to **ALLAH**. We have no knowledge except what you have taught us." Verily, it is **ALLAH**, the All-Knower, the All-Wise.*

*Dedicated to the **M**-family*
Medhat, Monda, Mohamed, Menna,
Malak M., Mostafa M.
*and the big **S**'s Sakina and Salwa*

Preface

The past decades have witnessed various applications of systems engineering methodologies to urban planning, economic models, power systems, industrial processes, transportation networks, and others. Due to economic factors and socio-political constraints, a fundamental constituent of these applications is frequently described by the following attributes: *multidimensional*, *highly interacting*, *and complex models*. Several approaches have been developed [2–4, 6, 11, 14] to deal with these models with the intention of reducing some measure of complexity in the course of analysis and design. Concepts and key ideas from economics, management science, and operation research have been exploited successfully and generalized in a dynamic framework. These continuous efforts systematically establish a body of theories pertaining to interconnected systems (ICS). The voluminous literature on theories and applications of large-scale systems (LSS), interconnected systems (ICS) or complex dynamical systems (CDS) includes survey articles and textbooks and monographs [1, 3, 5] and [7–13].

Throughout this book and in view of our technical experience, we will adopt *decentralized systems* (DS) as the most convenient designation for LSS, ICS or CDS since the common denominator in these systems is to deploy decentralization in the analysis, control, filtering and processing tasks. Equivalently stated, the effort of any task is essentially *distributed* among various units who are cooperating to achieve the desired objective.

It is often true that a book is developed through a long tour that consists of many tiny steps and interactions with many people. While the major idea of writing a book on decentralized systems has been in the back of my mind for quite long time, the thrust behind this volume started in July 2009 when I met with Oliver Jackson during the Systems and Control Conference in Saint Petersburg, Russia. It has been a good opportunity to start a fruitful communication channel that ended with writing the present book.

Over the past decades it was highly interesting and extensive activity to watch and interact with the global scientific/engineering development of decentralized systems leading to thousands of papers published and/or talks presented in journals and conferences about various related aspects. This book is basically an outgrowth of my academic research work and postgraduate teaching activities. It provides an in-depth treatment to problems of interconnected systems which some requirements are imposed in the course of analysis and/or design.

In engineering and economic organizations, one can easily recognize the presence of several decision makers (DMs) that

1. generate decisions and control variables by acting on the same system,
2. have access to different information coming from the controlled system and
3. pursue different goals.

Such organizations are addressed in the wide research area called "game theory."

For the purpose of uniformity, we will adopt the following definition of an interconnected system throughout this book: *a dynamical system which contains a number of interdependent constituents which serve particular functions, share resources, and are governed by a set of interrelated goals and constraints.*

It is manifested that "complexity" is an essential and dominating problem in systems theory and practice. It leads to severe difficulties that are encountered in the tasks of analyzing, designing, and implementing appropriate control strategies and algorithms. With focus on the control design goal, these difficulties arise mainly from the underlying multi-modes of operation and gain perturbations, which from now onwards we term them as *design constraints*. Given the advanced development in robust control and time-delay theories, we treat uncertain time-delay systems as basic module in our subsequent analysis.

From this perspective, the notion of DS introduced in the context of control engineering problems arose when it became clear that there are real world control problems that cannot be solved by using conventional approaches. Such typical problems arise in the control of interconnected power systems with strong interactions, water systems which are widely distributed in space, traffic systems with many external signal, or large-space flexible structures. These problems recall for new ideas for dividing the analysis and synthesis of the overall system into independent or almost independent subproblems, for dealing with the incomplete information about the system, for coping with the uncertainties and for dealing with time-delays.

This book is written about recent advances in decentralized systems theories and methods with design constraints. It aims at providing a rigorous framework for studying analysis, stability and control problems of DS while addressing the dominating sources of design constraints. The primary objective is to focus on robust decentralized methods based on linear matrix inequalities framework while tacking into consideration possible design considerations and/or constraints. Such constraints include the presence of quantizers, nonlinear/overflow elements, encoder/decoder and networks.

The main features of the book are:

(I) It provides key concepts of decentralized systems with their proofs followed by efficient computational method;
(II) It establishes decentralized control techniques under design constraints; and
(III) It gives some representative applications.

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Acknowledgements

In my academic career, I have benefited from listening, discussing, collaborating with several colleagues across the globe. Therefore, the various topics discussed in this book have constituted an integral part of my academic research investigations over the past several years. In writing this volume, I took the approach of referring within the text to papers and/or books from which I learned some concepts, ideas and methods. I further complemented this by adding some remarks and notes within and at the end of each chapter to shed some light on other related results. I apologize in advance in case I committed injustice and assure all of the colleagues that any mistake was definitely unintentional.

This book is intended for graduate students and researchers in control systems design including robust, reliable and resilient methods with the hope to provide a guided tour through decentralized systems (DS) results and a source of new research problems. I had the privilege of teaching graduate courses on decentralized control, large-scale systems and hierarchical systems at different institutions including Cairo University (Egypt), University of Manchester Institute of Science and Technology (UK), Kuwait University (Kuwait), KFUPM (Saudi Arabia). The course notes, updated and expanded over years, were instrumental in generating different chapters of this book and valuable comments and suggestions as well as several detailed critiques made by graduate students were greatly helpful. For the development of the book, I am immensely pleased for many stimulating discussions and interactions with colleagues, students and friends world-wide which have definitely enriched my knowledge and experience. Most of all however, I would like to express my deepest gratitude to all members of my family and especially my wife *Salwa* for her elegant style and compassion. Without their constant devotion, incredible amount of patience and (mostly) generous support this volume would not have been finished.

I would appreciate any comments, questions, criticisms, or corrections that readers may take the trouble of communicating to me at msmahmoud@kfupm.edu.sa or magdim@yahoo.com.

Dhahran, Saudi Arabia Magdi S. Mahmoud

Contents

Abbreviations[1]

Notation and Symbols

I^+	the set of positive integers
$\Re$	the set of real numbers
$\Re_+$	the set of non-negative real numbers
$\Re^n$	the set of all n-dimensional real vectors
$\Re^{n \times m}$	the set of $n \times m$-dimensional real matrices
$\mathcal{C}^-$	the open right-half complex plane
$\mathcal{C}^+$	the closed right-half complex plane
$\in$	belong to or element of
$\subset$	subset of
$\bigcup$	union
$\bigcap$	intersection
$\gg$	much greater than
$\ll$	much less than
I	an identity matrix of arbitrary order
I_s	the identity matrix of dimension $s \times s$
e_j	the jth column of matrix I
x^t or A^t	the transpose of vector x or matrix A
$\lambda(A)$	an eigenvalue of matrix A
$\varrho(A)$	the spectral radius of matrix A
$\lambda_j(A)$	the jth eigenvalue of matrix A
$\lambda_m(A)$	the minimum eigenvalue of matrix A where $\lambda(A)$ are real
$\lambda_M(A)$	the maximum eigenvalue of matrix A where $\lambda(A)$ are real
A^{-1}	the inverse of matrix A
$A^\dagger$	the Moore-Penrose-inverse of matrix A

[1]Throughout this book, the following terminologies, conventions and notations have been adopted. All of them are quite standard in the scientific media and only vary in form or character. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. In symmetric block matrices or complex matrix expressions, we use the symbol $\bullet$ to represent a term that is induced by symmetry.

$P > 0$	matrix P is real symmetric and positive-definite
$P \geq 0$	matrix P is real symmetric and positive semi-definite
$P < 0$	matrix P is real symmetric and negative-definite
$P \leq 0$	matrix P is real symmetric and negative semi-definite
$A(i, j),\ A_{ij}$	the ij-th element of matrix A
$\det(A)$	the determinant of matrix A
$\text{trace}(A)$	the trace of matrix A
$\text{rank}(A)$	the rank of matrix A
$\mathcal{L}_2(-\infty, \infty)$	space of time domain square integrable functions
$\mathcal{L}_2[0, \infty)$	subspace of $\mathcal{L}_2(-\infty, \infty)$ with functions zero for $t < 0$
$\mathcal{L}_2(-\infty, 0]$	subspace of $\mathcal{L}_2(-\infty, \infty)$ with functions zero for $t > 0$
$\mathcal{L}_2(j\Re)$	square integrable functions on $\mathcal{C}_0$ including at ∞
$\mathcal{H}_2$	subspace of $\mathcal{L}_2(j\Re)$ with functions analytic in $\text{Re}(s) > 0$
$\mathcal{L}_\infty(j\Re)$	subspace of functions bounded on $\text{Re}(s) = 0$ including at ∞
$\mathcal{H}_\infty$	the set of $\mathcal{L}_\infty(j\Re)$ functions analytic in $\text{Re}(s) > 0$
$\lvert a \rvert$	the absolute value of scalar a
$\lVert x \rVert$	the Euclidean norm of vector x
$\lVert A \rVert$	the induced Euclidean norm of matrix A
$\lVert x \rVert_p$	the ℓ_p norm of vector x
$\lVert A \rVert_p$	the induced ℓ_p norm of matrix A
$\text{Im}(A)$	the image of operator/matrix A
$\text{Ker}(A)$	the kernel of operator/matrix A
$\max \mathbf{D}$	the maximum element of set $\mathbf{D}$
$\min \mathbf{D}$	the minimum element of set $\mathbf{D}$
$\sup \mathbf{D}$	the smallest number that is larger than or equal to each element of set $\mathbf{D}$
$\inf \mathbf{D}$	the largest number that is smaller than or equal to each element of set $\mathbf{D}$
$\arg\max \mathbf{D}$	the index of maximum element of ordered set $\mathbf{S}$
$\arg\min \mathbf{D}$	the index of minimum element of ordered set $\mathbf{S}$
$\mathbf{B}_r$	the ball centered at the origin with radius r
$\mathbf{R}_r$	the sphere centered at the origin with radius r
$\mathcal{N}$	the fixed index set $\{1, 2, \ldots, N\}$
$[a, b)$	the real number set $\{t \in \Re : a \leq t < b\}$
$[a, b]$	the real number set $\{t \in \Re : a \leq t \leq b\}$
$\mathbf{S}$	the set of modes $\{1, 2, \ldots, s\}$
iff	if and only if
$\otimes$	the Kronecker product
$\mathbf{O}(.)$	order of (.)
$\text{diag}(\ldots)A$	diagonal matrix with given diagonal elements
$\text{spec}(A)$	the set of eigenvalues of matrix A (spectrum)
$\text{min-poly}(A)(s)$	the minimal polynomial of matrix A

Acronyms

ARE	algebraic Riccati equation
DC	decentralized control

HC	hierarchical control
LMI	linear matrix inequality
SISO	single-input single-output
MIMO	multi-input multi-output
BIBS	bounded-input bounded-state
iISS	integral-input-to-state stable
UGAS	uniformly globally asymptotically stable
OLD	overlapping decomposition
OLC	overlapping control
TDS	time-delay system
TDUS	time-delay uncertain system
LKF	Lyapunov-Krasovskii functional
DFC	decentralized feedback control
DHC	decentralized $\mathcal{H}_\infty$ control
SVD	singular value decomposition
DNS	decentralized nonlinear systems
LBD	Lyapunov-based design
DTS	discrete-time systems
LQC	linear quadratic control
LMCR	liquid-metal cooled reactor
DSMP	decentralized servomechanism problem
DIP	distributed information processing
CIP	centralized information processing
SMC	sliding mode control

Chapter 1
Introduction

The book covers some of the past and present results pertaining to the area of large-scale, interconnected or complex systems. An emphasis is laid on decentralization, decomposition, and robustness. These methodologies serve as effective tools to overcome specific difficulties arising in large-scale complex systems such as high dimensionality, information structure constraints, uncertainty, and delays. Several prospective topics for future research are introduced in this contents. The subsequent chapters are focused on recent decomposition approaches in interconnected dynamic systems due to their potential in providing the extension of decentralized control into networked control systems.

1.1 Introduction

The past several decades have witnessed an increasing amount of attention paid to the general subject area of large-scale systems. This comes quite naturally from the relatively rapid growth of our societal needs which often result in multidimensional, highly interacting, complex systems which are frequently stochastic in nature. Though the existence of large-scale systems as objects for understanding and management is repeatedly affirmed, there has yet been proposed no precise definition for largeness nor generally acceptable quantitative measures of scale. From the viewpoint of developing analytical models, a system is large when its input-output behavior cannot be understood without curtailing it, partitioning it into modules, and/or aggregating its modularized subsystems. On the other hand, from a systems viewpoint, a system is large if it exceeds the capacity of a single control structure. Thus one can enumerate several viewpoints regarding scale. The definition of a large-scale system we will adopt here is a system which contains a number of interdependent constituents which serve particular functions, share resources, and are governed by a set of interrelated goals and constraints [9].

Motivated by the prominent structural aspects of an organization and some facets from the area of automation and control of complex industrial systems and general

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man made communication problems, theoretical investigations were conducted at the Systems Research Center of Case Western Reserve University starting about 1961. The overall goal was to develop a conceptual framework to the mathematical theory of complex multi goal decision making systems. Basically, the main idea behind this approach is the recognition of the hierarchical order in living systems as well as many existing physical systems. In fact, this approach takes the position that for a mathematical theory to claim to be dealing with large-scale complex systems, the complexity of the real systems must be reflected in the structure of the model [7].

Although there is no universal definition of a large-scale system, it is commonly accepted that such systems possess the following characteristics [5]:

1. Large-scale systems are often controlled by more than one controller or decision maker involving "decentralized" computations,
2. The controllers have different but correlated "information" available to them, possibly at different times,
3. Large-scale systems can also be controlled by local controllers at one level whose control actions are being coordinated at another level in a "hierarchical" (multi-level) structure,
4. Large-scale systems are usually represented by imprecise "aggregate" models,
5. Controllers may operate in a group as a "team" or in a "conflicting" manner with single- or multiple-objective or even conflicting-objective functions, and
6. Large-scale systems may be satisfactorily optimized by means of suboptimal or near-optimum controls, sometimes termed a "satisfying" strategy.

1.2 Feedback Control

At first sight, feedback control of large-scale systems poses the 'classical' control problem: for a given process with control input $u(t)$ and observed output $y(t)$ find a controller that ensures closed-loop stability and asymptotic regulation and assigns the loop a suitable input-output (I/O) behavior. This problem is usually solved in two steps [6]:

1. *The design phase*: for a given model of the plant and expected classes of disturbances $d(t)$ and command signals $v(t)$ a control law

$$u_c(t) = K(y(t) - v(t))$$

   is chosen which satisfies the specifications given for the closed-loop system.
2. *The execution phase*: a controller with the control law $u_c(t)$ is applied to the process, that is at every instant of time t the observed signal $y(t)$ and the command $v(t)$ are combined according to the control law in order to determine the control input $u_c(t)$.

This well-known control problem has been treated by classical and modern control theory under the crucial assumptions that there is a unique plant with a unique controller and that all computations are based on the whole information about the plant. This means that the design problem is solved for a model that describes the process as a whole. In this case, the controller receives all sensor data available and determines all input signals of the plant. In other words, there is unit thought of as a *centralized decision maker* in charge of all information available for a single unit that designs and applies the controller to the plant. Hence, multivariable control theory deals with the centralized design of centralized controllers. Obviously, such an assumption can hardly be satisfied if modern technological or societal systems have to be controlled.

Practical control technologies rely on the cooperation of many different operational units or transportation systems and all their parts are linked by common resources, by material flows or through information networks. Consequently, neither a complete model (*a priori information*) nor a complete set of measurement data (*a posteriori information*) can be made available for a centralized decision maker.

For reliability considerations, the overall design problem has to be broken down into different, albeit coupled, subproblems. As a result, the overall plant is no longer controlled by a single controller but by several independent controllers, which are called *control stations* and which all together represent a *decentralized controller*. These control stations are no longer designed simultaneously on the basis of a complete knowledge of the plant, but in different design steps by means of models that describe only the relevant parts of the plant.

This fundamental difference between feedback control of 'small' and 'large' systems is usually described by the idea of information structure. The information structure describes the way in which a priori and a posteriori information is transferred among decision-making units.

1.2.1 Information Structure

One of the major issues that manifests large-scale systems is the role governed by the idea of information structure. Initially, in case of centralized systems, refer to Fig. 1.2, the basic feedback problem is to find control input vector $u(\cdot)$ on the basis of the a priori knowledge of the plant S described by its *design model* in the presence of a class of disturbances $v(\cdot)$ and the control goal given in the form of the design requirements $\{\mathbf{C}\}$ and the a posteriori information about the outputs $y(\cdot)$ and the command signals $r(\cdot)$. Classical information structure corresponds to centralized control as illustrated by Fig. 1.1. It is important to note that the controller receives all sensor data available and determines all input signals of the plant. In other words, all information is assumed to be available for a single unit that designs and applies the controller to the plant. In present-day technologies where several different units are coexist side by side, neither a complete model (a priori information) nor a complete set of measurement data (a posteriori information) can be made available for a centralized decision maker. Instead, the overall design problem has to be broken down into different, albeit coupled, subproblems. As a result, the overall

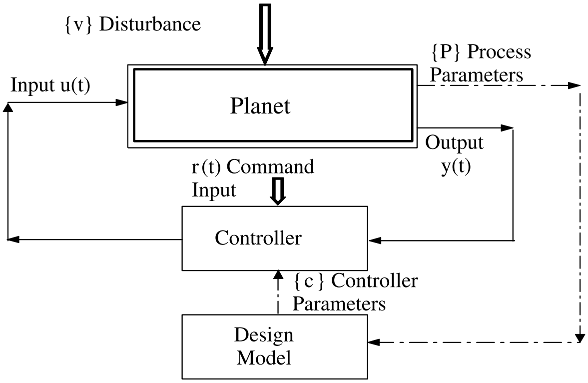

Fig. 1.1 Information structure: centralized control

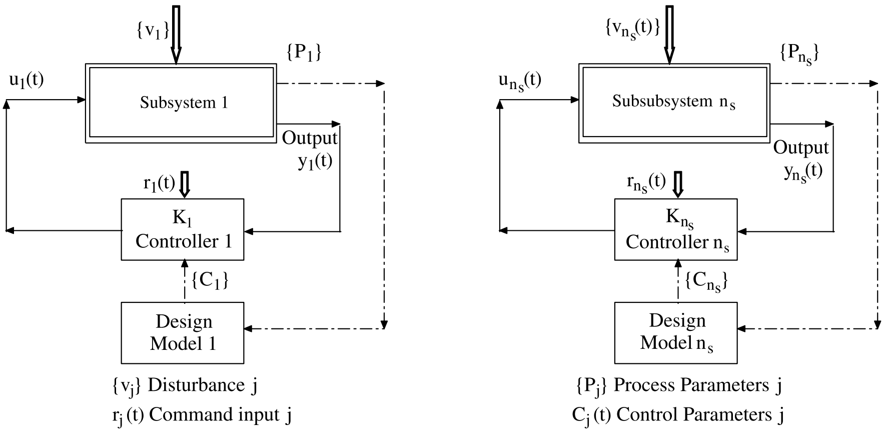

Fig. 1.2 Information structure: decentralized control

plant is no longer controlled by a single controller but by several independent controllers constituting a decentralized controller structure. Moreover, these controllers are no longer designed simultaneously on the basis of a complete knowledge of the plant, but in different design steps by means of models that describe only the relevant parts of the plant. This amounts to non-classical information structure which arises in decentralized design schemes as shown in Fig. 1.2.

1.2.2 System Representation

There are available two main structures of the models of large-scale systems distinguished by the degree to which they reflect the internal structure of the overall dynamic system. These structures are called multi-channel systems entailing the presence of multi-controllers and interconnected systems incorporating coordinated

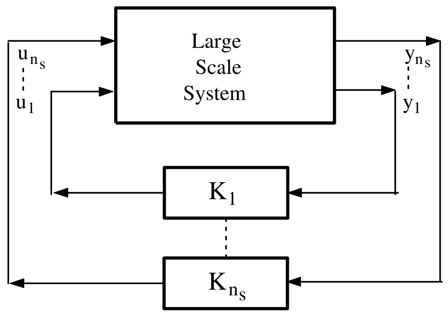

Fig. 1.3 Multi-controller structure

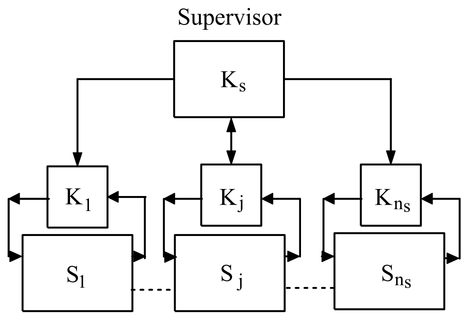

Fig. 1.4 Coordinated control structure

controllers as illustrated in Figs. 1.3 and 1.4. In multi-channel systems, the associated input and output vectors are decomposed into subvectors constituting n_s channels, while the system is considered as one whole. More on this type of systems will be mentioned in later chapters.

1.2.3 Team Problems

In engineering and economic organizations, there may be several decision makers (DMs) that

(A) generate decisions and control variables by acting on the same system;
(B) have access to different information coming from the controlled system; and
(C) pursue different goals.

Such organizations are addressed in the wide research area called "game theory".

As to point (C), if all the decision makers cooperate on the accomplishment of a common goal, the organization becomes a team and the related optimization problems are named team optimal control problems [4].

1.2.4 General Methodologies

Interconnected systems operate with interactions among subsystems. They are represented by signals through which subsystems interact among themselves. These signal are internal signals of the overall system.

To cope with the aforementioned appearance of the complexity issues, several general methodologies have been and are being elaborated. Most of them belong to one of the following three groups [8]:

1. *Model simplification*,
2. *Decomposition*,
3. *Decentralization*

The idea of *model simplification* is to come up with a reasonable model that preserves or inherits most of the main trends (features or dominant modes) of the original large-scale/complex system, see [1–3, 5, 8] for further elaboration. The *decomposition* (tearing) process amounts to generating a group of subsystems (smaller in size) from the original large-scale/complex system. This could be achieved for numerical purposes or along the boundaries of coupled units. In turns out that decomposition is only a part of two-step procedure, the second of which is *coordination* (recomposing) which amounts to synthesizing the overall solution from the generated solutions of the subsystems (subsolutions). There are two aspects of *decentralization*: the first issue is concerned with the information structure inherent in the solution of the given control problem and refers to the subdivision of the process in terms of the model and the design goals. The other issue is associated with on-line information about the state and the command to generate the decentralized control law. The net result is that a completely independent implementation of the controllers is made viable. There is a variety of different motivating reasons for the decentralization of the design process such as weak coupling of subsystems, subsystems have contradictory goals, subsystems are assigned to different authorities, or the high dimensionality of the overall system. Following [6], the principal ways of decentralizing the design tasks belong to two groups: *decentralized design for strongly coupled subsystems* and *decentralized design for weakly coupled subsystems*.

The *decentralized design for strongly coupled subsystems* means that at least an approximate model of all other subsystems must be considered for the design of any subsystem under the current design, while the coupling can be neglected during the design of individual control stations when considering the decentralized design for weakly coupled subsystems.

1.2.5 Hierarchical Systems

One of the fundamental approaches in dealing with large-scale static systems was the idea of *decomposition* treated theoretically in mathematical programming by treating large linear programming problems possessing special structures. The objective was to gain computational efficiency and design simplification. There are two basic approaches for dealing with such problems:

1. The *coupled* approach where the problem's structure kept intact while taking advantage of the structure to perform efficient computations [4], and
2. The *decoupled* approach which divides the original system into a number of subsystems involving certain values of parameters. Each subsystem is solved independently for a fixed value of the so-called "decoupling" parameter, whose value is subsequently adjusted by a coordinator in an appropriate fashion so that the subsystems resolve their problems and the solution to the original system is obtained.

1.3 Outline of the Book

During the past several decades, there have been real world system applications for which the associated control design problems cannot be solved by using one-shot approaches. Typical applications arise in the areas of interconnected power systems with strong coupling ties among network elements, water systems which are widely distributed in space, traffic systems with many external signal, or large-space flexible structures with interacting modes. Models of such systems are frequently complex in nature, multidimensional and/or composed of highly interacting subsystems. Several approaches to deal with these systems have been developed based on key ideas from economics, management sciences and operations research. Over the years, such approaches have been dynamically evolved into a body of "large-scale systems (LSS) theories".

This book is written about large-scale systems theories. It aims at providing a rigorous framework for studying analysis, stability and control problems of LSS while addressing the dominating sources of difficulties due to: dimensionality; information structure constraints; parametric uncertainty and time-delays. The primary objective is three-fold: to review past methods and results from a contemporary perspective, to examine presents trends and approaches and to provide future possibilities, focusing on robust, reliable and/or resilient decentralized design methods based on linear matrix inequalities framework.

The main features of the book are:

1. It provide an overall assessment of the large-scale systems theories over the past several decades,
2. It addresses several issues like model-order reduction, parametric uncertainties, time-delays, control/estimator gain perturbations,
3. It presents key concepts with their proofs followed by efficient computational method,
4. It establishes decentralized control techniques for time-delay and delay-free systems, and
5. It gives some representative applications.

1.3.1 Methodology

Throughout the book, our methodology in each chapter/section is composed of five-steps:

- *Mathematical Modeling* in which we discuss the main ingredients of the state-space model under consideration.
- *Definitions and/or Assumptions*—here we state the definitions and/or constraints on the model variables to pave the way for subsequent analysis.
- *Analysis and Examples*—this signifies the core of the respective sections and subsections which contains some solved examples for illustration.
- *Results* which are provided most of the time in the form of theorems, lemmas and corollaries.
- *Remarks* which are given to shed some light of the relevance of the developed results vis-a-vis published work.

In the sequel, theorems (lemmas, corollaries) are keyed to chapters and stated in *italic* font with bold titles, for example, **Theorem 3.4** means Theorem 4 in Chap. 3 and so on. For convenience, we have provided an appropriate list of references cited at the end of each chapter. Relevant notes and research issues are offered at the end of each chapter for the purpose of stimulating the reader.

We hope that this way of articulating the information will attract the attention of a wide-spectrum of readership.

1.3.2 Book Organization

The book is primarily intended for researchers and engineers in the system and control community. It can also serve as complementary reading for large-scale system theory at the post-graduate level. The book is divided into nine chapters.

Chapter 1 provides an overview of the concepts and techniques of large-scale dynamic systems and introduces the system description and motivation of the study. Then it sets forth formal definitions pertaining to the scope and objectives of the book.

Chapter 2 treats the first part of decentralized control methods for some classes of nonlinear interconnected dynamical systems.

Chapter 3 deals with the second part of decentralized control methods for some classes of nonlinear interconnected dynamical systems.

Chapter 4 examines stabilization and feedback control of decentralized systems using multi-controller structures.

Chapter 5 focuses on decentralized control in the presence of quantizers within continuous-time and discrete-time systems switched. Once again, the analytical development starts with time-delay systems then generates the ordinary systems as important special cases.

Chapter 6 examines large-scale traffic systems and identifies their features. Appropriate models are derived using continuous and discrete formalisms. Flexible routing policies are derived under different operating conditions.

Chapter 7 considers large-scale systems with Markovian jumping parameters. The analytical development deals with ordinary systems as well as time-delay systems.

Chapter 8 deals with decentralized adaptive control strategies of interconnected systems.

Chapter 9 contains some relevant mathematical lemmas, basic algebraic inequalities and standard stability theorems.

Throughout the book and seeking computational convenience, all the developed results are cast in the format of family of LMIs. In writing up the different topics, emphasis is primarily placed on major developments attained thus far and then reference is made to other related work.

In summary, this book covers decentralized control for interconnected systems under alternative design considerations which is supplemented with rigorous proofs of closed-loop stability properties and simulation studies. The material contained in this book not only organized to focus on the new developments in the analysis and control methods for LSS, but it also integrates the impact of the design constraints like delay-factor, information structures, interaction pattern, quantization and overflow, switching among multi-controllers. After an introductory chapter, it is intended to divide the book into self-contained chapters with each chapter being equipped with illustrative examples, problems and questions. Each chapter of the book will be supplemented by an extended bibliography, appropriate appendices and indexes. It is planned while organizing the material that this book would be appropriate for use either as graduate-level textbook in applied mathematics as well as different engineering disciplines (electrical, mechanical, civil, chemical, systems), a good volume for independent study or a suitable reference for graduate-students, practicing engineers, interested readers and researchers from wide-spectrum of engineering disciplines, science and mathematics.

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Chapter 2
Decentralized Control of Nonlinear Systems I

In this chapter, we examine decentralized control techniques for classes of nonlinear interconnected systems. We identify classes for the system structure along with the underlying assumptions and emphasize the information and design constraints. The subsequent sections focus on a class of large-scale interconnected minimum-phase nonlinear systems with parameter uncertainty and nonlinear interconnections. The uncertain parameters are allowed to be time-varying and enter the systems nonlinearly. The interconnections are bounded by nonlinear functions of states. The problem we address is to design a decentralized robust controller such that the closed-loop large-scale interconnected nonlinear system is globally asymptotically stable for all admissible uncertain parameters and interconnections. It is shown that decentralized global robust stabilization of the system can be achieved using a control law obtained by a recursive design method together with an appropriate Lyapunov function.

The problem of decentralized output-feedback tracking with disturbance attenuation is addressed for a new class of large-scale and minimum-phase nonlinear systems. Common assumptions like matching and growth conditions are not required for the underlying decentralized system with a diagonal structure. An observer-based decentralized controller design is presented. The proposed decentralized output-feedback laws achieve asymptotic tracking and internal Lagrange stability when the disturbance inputs disappear, and, guarantee external stability in the presence of disturbance inputs. These external stability properties include Sontag's ISS and iISS conditions and standard $\mathcal{L}_2$-gain property.

2.1 Classes of Nonlinear Interconnected Systems

In what follows, we summarize the classes of nonlinear interconnected systems (NIS) that will be treated in the subsequent sections. We focus on the features of each class before addressing the topics of stability analysis and decentralized output-feedback control design.

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2.1.1 Class I

In recent years, modern control methods have found their way into decentralized design of interconnected systems leading to a wide variety of new concepts and results. This includes, but not limited to, the framework of $\mathcal{H}_\infty/\mathcal{H}_2$ design and linear matrix inequalities (LMIs) [1] which has been shown [6, 44] to be very attractive particularly when coping with high dimensional systems. Applications having sophisticated theoretical generalizations of the underlying concepts have been in control of multi-agent systems, such as platoons of vehicles on highways and in the air, interconnected spatially-invariant systems, and large-scale power systems [5–7]. It turns out that, the decentralized control designs imply, either explicitly or implicitly, that the system, with local feedback loops closed around the subsystems, remains stable despite changes in its interconnection topology [4, 60, 66, 67].

2.1.1.1 System Description

According to this class, a nonlinear interconnected system **S** is considered to be composed of a finite number N of subsystems represented by

$$\begin{aligned} \mathbf{S}_j\colon \quad \dot{x}_j &= A_j x_j + B_j u_j + h_j(t,x), \\ y_j &= C_j x_j, \end{aligned} \tag{2.1}$$

where $x_j \in \Re^{n_j}$, $u_j \in \Re^{m_j}$ and $y_j \in \Re^{p_j}$ are the subsystem state, input and output vectors, respectively, $x = [x_1^t, \dots, x_N^t]^t$ is the global state vector with $\sum_{i=1}^N n_j = n$ and $h_j(t,x)\colon \Re^{n+1} \to \Re^{n_j}$ are piecewise continuous vector functions in both arguments, satisfying in their domains of continuity the following quadratic inequalities

$$h_j^t(t,x) h_j(t,x) \le \widetilde{\sigma}_j^{-2} x^t \widetilde{H}_j^t \widetilde{H}_j x, \tag{2.2}$$

where $\widetilde{\sigma}_j > 0$ are bounding parameters and $\widetilde{H}_j$ are constant $\alpha_j \times n$ matrices, $j = 1, \dots, N$.

The interconnected system can be represented as

$$\begin{aligned} \mathbf{S}\colon \quad \dot{x} &= Ax + Bu + h(t,x), \\ y &= Cx, \end{aligned} \tag{2.3}$$

where

$$u = [u_1^t, \dots, u_N^t]^t, \qquad y = [y_1^t, \dots, y_N^t]^t,$$
$$h(t,x) = [h_1(t,x)^t, \dots, h_N(t,x)^t]^t$$

are the global input, output and interconnection vectors, respectively, with

$$\sum_{i=1}^{N} m_j = m, \qquad \sum_{i=1}^{N} p_j = p,$$

$$A = \mathrm{diag}[A_1, \ldots, A_N], \qquad B = \mathrm{diag}[B_1, \ldots, B_N], \qquad C = \mathrm{diag}[C_1, \ldots, C_N]$$

and $h(t, x)$ is the global interconnection function. Proceeding further, define $\bar{H}^t = [H_1^t \vdots \ldots \vdots H_N^t]$, where $\widetilde{H}_j$, $j = 1, \ldots, N$, are defined in (2.2), and

$$\widetilde{\Gamma} = \mathrm{diag}[\tilde{\gamma}_1 I_{\alpha 1}, \ldots, \tilde{\gamma}_N I_{\alpha N}], \quad \tilde{\gamma}_j = \bar{\alpha}_j^{-2}, \ I_{\alpha j} \in \Re^{\alpha j \times \alpha_j}$$

then, it is always possible to find matrices H, Γ such that

$$h(t, x)^t h(t, x) \leq x^t \bar{H}^t \bar{\Gamma}^{-1} \bar{H}_x \leq x^t H^t \Gamma^{-1} H x, \tag{2.4}$$

where

$$H = \mathrm{diag}[H_1, \ldots, H_N], \quad H_j \in \Re^{\alpha_j \times n_j}, \qquad \Gamma = \mathrm{diag}[\gamma_1 I_{\alpha 1}, \ldots, \gamma_N I_{\alpha N}],$$
$$j = 1, \ldots, N.$$

It is not difficult to show that matrices H and Γ satisfy

$$\lambda_M(\bar{H}^t \bar{H}) \min_i \bar{\gamma}_j \leq \max_i \gamma_j \min_i \lambda_{\min}(H_j H_j^t)$$

represent a possible choice; different structures can be chosen in accordance with the problem under consideration, see [55, 58] for further elaboration.

Remark 2.1 The main feature of this class is its suitability to develop an LMI-based method for designing dynamic output feedback for robust decentralized stabilization of interconnected systems. This scheme is selected as a methodological basis for several reasons [55]. First, the method applies to systems composed of linear subsystems coupled by nonlinear interconnections. This type of model is attractive since, in most practical situations, local subsystem models are known with sufficient precision to make the linearization successful, while the interconnections are largely unknown: only their bounds are available for control design. Second, the scheme allows for maximization of interconnection bounds, and third, the resulting closed-loop system is connectively stable. Elaborations of the basic scheme in [55] presented in the literature have been related either to the state feedback [55], or to output feedback schemes containing an observer of Luenberger type [9, 46–54, 56–65, 67–89].

As will be shown later on that by assuming decentralized dynamic linear output feedback with a general structure, we apply the classical methodology of $\mathcal{H}_\infty$ controller design [6, 11, 24] to the basic scheme from [55]. As a result, a new two-step LMI-based design procedure is obtained, providing at the first step the block-diagonal Lyapunov matrix, together with the robustness degree vector, and at the second step the decentralized controller parameters.

2.1.2 Class II

Large-scale systems, frequently consisting of a set of small-interconnected subsystems, can be found in many applications such as electric power systems, industrial manipulators, computer networks, to name a few. On one hand, the centralized control of these systems is usually infeasible mainly due to the requirement of a formidable amount of information exchange. In this regard, decentralized control is often preferable [60] whereby a control law based only on local information is designed and implemented. In view of the interconnections among subsystems, the design of a decentralized control is in general more difficult than that of a centralized control. On the other hand, due to their complexity, exact modeling of large-scale systems is usually impossible. Therefore, it is of practical significance that decentralized control must reflect such design constraints by taking into account possible modeling uncertainties. Usually, the uncertainties for interconnected systems appear not only in local subsystems but also in interconnections.

From the literature, decentralized robust control for interconnected linear systems with uncertainties satisfying the so-called strict matching conditions was investigated in [3, 17, 56] and references cited therein. The interconnections among subsystems treated in these works are mostly bounded by first-order polynomials. It was pointed out in [13, 18, 38, 56] that interconnected systems with a decentralized control based on the first-order bounded interconnections may become unstable when the interconnections are of higher order. Decentralized robust stabilization was considered in [20] for systems with interconnections bounded by some nonlinear functions and uncertainties satisfying the so-called matching conditions. Decentralized adaptive control for a class of interconnected nonlinear systems was proposed in [22, 25] based on exact linearization by following the development of centralized control of nonlinear systems [23, 32, 39] and where the strict matching condition was relaxed and higher-order interconnections among subsystems were introduced.

2.1.2.1 System Description

The second class of systems considered in this chapter looks at a large-scale nonlinear system as comprised of N interconnected subsystems with time-varying unknown parameters and/or disturbances entering nonlinearly into the state equation. The jth subsystem is given as

$$\begin{aligned}
\dot{z}_j &= f_{i0}(z_j, x_{j1}) + g_{j0}(z_j, \bar{x}_{j0}, Z_j, Y_j; \theta)x_{j1}, \\
\dot{x}_{j1} &= x_{j2} + g_{j1}(z_j, \bar{x}_{j1}, Z_j, Y_j; \theta), \\
\dot{x}_{j2} &= x_{j3} + g_{j2}(z_j, \bar{x}_{j2}, Z_j, Y_j; \theta), \\
&\ \vdots
\end{aligned} \tag{2.5}$$

$$\dot{x}_{j,r_j-1} = x_{j,r_j} + \phi_{j,r_j-1}(z_j, \bar{x}_{j,r_j-1}, Z_j, Y_j; \theta),$$
$$\dot{x}_{j,r_j} = v_j + \phi_{j,r_j}(z_j, \bar{x}_{j,r_j}, Z_j, Y_j; \theta),$$
$$y_j = x_{j1},$$

where $\bar{x}_{j,k} = [x_{j1}\, x_{j2}\, \ldots\, x_{jk}]^t$ with $\bar{x}_{j0} = x_{j1}$, $x_j = \bar{x}_{jr_j}$, (z_j, x_j) is the state vector of the jth subsystem with

$$z_j \in \Re^{n_j - r_j}, \quad Z_j = [z_1^t\, z_2^t\, \ldots\, z_{j-1}^t\, z_{j+1}^t\, \ldots\, z_N^t]^t,$$
$$Y_j = [y_1\; y_2\; \ldots\; y_{j-1}\; y_{j+1}\; \ldots\; y_N]^t$$

and $v_j \in \Re$ is the control input, $y_j \in \Re$ is the output, θ is a vector of unknown, time-varying piecewise continuous parameters and/or disturbances which belong to a known compact set Ω, the vector fields f_{j0} and ϕ_{jk} are smooth with $f_{j0}(0,0) = 0$ and $g_{jk}(0,0,0,0;\theta) = 0$, $\forall \theta \in \Omega$, $1 \le j \le N$, $0 \le j \le r_j$. Observe that the vector (g_{jk}), $k = 0, 1, 2, \ldots, r_j$, represents the interconnections of the ith subsystem with the other subsystems.

Remark 2.2 In what follows, we consider the decentralized robust control problem for a wider class of interconnected systems with partially feedback linearizable subsystems and nonlinear parameterization of time-varying parametric uncertainty. Observe from (2.5) that the interconnections involve the zero-dynamics and outputs of other subsystems. This is in contrast to [25] where an adaptive stabilization was considered for a class of interconnected nonlinear systems whose subsystems are exactly feedback linearizable and with linear parameterization of parameter uncertainty. Geometrical conditions on the isolated subsystems and interconnections such that the interconnected nonlinear systems are transformable into the so-called decentralized strict feedback form has been characterized in [25].

Remark 2.3 Similar to the centralized case discussed in [35, 40], the zero dynamics of each subsystem in (2.5) are independent of the uncertain parameter vector θ.

In the sequel, we assume that $n_j = n$, $r_j = r$, $1 \le j \le N$. Then, by considering $y_j = x_{j1}$, system (2.5) becomes

$$\begin{aligned}
\dot{z}_j &= f_{j0}(z_j, x_{j1}) + g_{j0}(z_j, \bar{x}_{i0}, Z_j, X_{j1}; \theta)x_{j1},\\
\dot{x}_{j1} &= x_{j2} + g_{j1}(z_j, \bar{x}_{j1}, Z_j, X_{j1}; \theta),\\
\dot{x}_{j2} &= x_{j3} + g_{j2}(z_j, \bar{x}_{j2}, Z_j, X_{j1}; \theta),\\
&\vdots\\
\dot{x}_{j,r-1} &= x_{j,r} + g_{j,r-1}(z_j, \bar{x}_{j,r-1}, Z_j, X_{j1}; \theta),\\
\dot{x}_{j,r} &= v_j + g_{j,r}(z_j, \bar{x}_{j,r}, Z_j, X_{j1}; \theta),
\end{aligned} \tag{2.6}$$

where $X_{j1} = Y_j = [x_{11}\; x_{21}\; \ldots x_{j-1,1}\; x_{j+1,1}\; \ldots\; x_{N1}]^t$.

The following assumptions are made for system (2.6).

Assumption 2.1 There exist some smooth real-valued functions

$$V_{j0}(z_j), \quad j = 1, 2, \ldots, N,$$

which are positive definite and proper (radially unbounded), such that

$$\frac{\partial V_{j0}}{\partial z_j} f_{j0}(z_j, 0) \leq -\nu_j \|z_j\|^2, \quad 1 \leq j \leq N, \tag{2.7}$$

for some positive real numbers $\nu_j > 0$.

Assumption 2.2 The nonlinear interconnections g_{jk} in (2.6) satisfy

$$\begin{aligned} &|g_{jk}(z_j, \bar{x}_{jk}, Z_j, X_{j1}; \theta) - \phi_{jk}(z_j, \bar{x}_{jk}, 0, 0, \theta)| \\ &\quad \leq \sum_{\ell=1}^{N} \eta_{jk\ell}(z_j, \bar{x}_{jk})[\zeta_{jk\ell}^0(\|z_l\|)\|z_l\| + \zeta_{jk\ell}^1(z_\ell, x_{\ell 1})|x_{\ell 1}|] \\ &\quad \leq \sum_{\ell=1}^{N} \eta_{jk\ell}(z_j, \bar{x}_{jk})\zeta_{jk\ell}(\|(z_\ell, x_{\ell 1})\|), \end{aligned} \tag{2.8}$$

for any $\theta \in \Omega$, $\eta_{jk\ell}(\cdot)$, $\zeta_{jk\ell}^0(\cdot)$ and $\zeta_{jk\ell}^1(\cdot)$, $\ell = 1, 2, \ldots, N$, $0 \leq k \leq r$, $1 \leq j \leq N$ are nonnegative smooth functions with $\zeta_{jki}^0(\cdot) = \zeta_{jkj}^1(\cdot) \equiv 0$.

Remark 2.4 By the well-known converse Lyapunov theorem [29, 31], the zero dynamics of each subsystem are globally asymptotically stable if and only if there exists a positive definite and proper Lyapunov function V_{j0} such that $(\partial V_{j0}/\partial z_j) f_{j0}(z_j, 0) < 0$, $\forall z_j \neq 0$. Indeed, the requirement (2.7) is more restrictive than this. However, a globally exponentially minimum-phase nonlinear system (that is, the zero-dynamics of the system are globally exponentially stable) always satisfies condition (2.7).

Remark 2.5 The interconnections in Assumption 2.2 are very general, including many types of interconnections considered in existing literature as special cases, for example, interconnections bounded by linear (first-order) polynomials [3], and higher-order polynomials [56]. By contrast to the work in [3, 20, 27, 56], no matching conditions are imposed for system (2.6).

Later on, we will deal with the decentralized global robust stabilization problem for system (2.6) satisfying Assumptions 2.1 and 2.2. More precisely, we are concerned with the design of decentralized robust control laws $v_j = v_j(z_j, x_j)$, $j = 1, \ldots, N$, such that the overall closed-loop interconnected system (2.6) with the control laws is globally asymptotically stable for all admissible uncertainties and interconnections.

2.1.3 Class III

Recent years have seen steady progress in the field of decentralized control of both linear and nonlinear systems. Decentralized control issues naturally arise from controlling large complex systems found in the power industry, aerospace and chemical engineering applications, and telecommunication networks, to name only a few. Among the main characteristics of decentralized control are the dramatic reduction of computational complexity and the enhancement of robustness and reliability against interacting operation failures. Many researchers have made significant contributions to the development of decentralized control theory for large-scale, or interconnected, dynamic systems ([60] and a rather complete list of earlier references cited therein).

In Class III of this chapter, we study a broad class of large-scale nonlinear systems with output measurements. This problem, usually referred to as decentralized output-feedback control, is technically challenging because of the lack of a general theory for nonlinear observer design and the nonlinear version of the well-known "Separation Principle".

2.1.3.1 System Description

According to this Class III, a large-scale nonlinear system is viewed as comprised of N interconnected subsystems with time-varying unknown parameters and/or disturbances entering nonlinearly into the state equation. The jth subsystem is given as

$$\dot{x}_j = F_j(x_j) + G_j(x_j)u_j + \Pi_{j1}(y_1, \ldots, y_N)x_j + \Pi_{j2}(y_1, \ldots, y_N)w_j, \tag{2.9}$$

$$y_j = h_j(x_j), \tag{2.10}$$

where $1 \le j \le N$, $x_j \in \Re^{n_j}$, $u_j \in \Re$ and $y_j \in \Re$ represent the state, the single control input and the single output of the local jth subsystem, respectively, and $w_j \in \Re^{n_{w_j}}$ is the disturbance input. Also, $F_j, G_j, h_j, \Lambda_{j1}$ and Π_{j2} are sufficiently smooth functions. In the absence of the interacting terms Π_{j1} and Π_{j2}, the system (2.9)–(2.10) collapses to an isolated single-input single-output SISO system and has been extensively studied in the recent literature. Various constructive control algorithms have been developed for large classes of centralized nonlinear systems in special normal forms. Similar questions in the decentralized context should be addressed, that is, in the presence of strong interactions among local systems of the form (2.9)–(2.10). In the sequel, attention is focused on large-scale dynamic systems of type (2.9)–(2.10) transformable to

$$\begin{aligned}
\dot{z}_j &= Q_j z_j + f_{j0}(y_1, \ldots, y_N) + p_{j0}(y_1, \ldots, y_N)w_j, \\
\dot{x}_{j1} &= x_{i2} + f_{j1}(y_1, \ldots, y_N) + g_{j1}(y_1, \ldots, y_N)z_j + p_{j1}(y_1, \ldots, y_N)w_j, \\
&\;\;\vdots
\end{aligned} \tag{2.11}$$

$$\dot{x}_{jn_j} = u_j + f_{jn_j}(y_1, \dots, y_N) + g_{jn_j}(y_1, \dots, y_N) z_j + p_{jn_j}(y_1, \dots, y_N) w_j,$$
$$y_j = x_{j1},$$

where for each $1 \le j \le N$, $z_j \in \Re^{n_{z_j}}$ and $x_j = (x_{i1}, \dots, x_{in_j}) \in \Re^{n_j}$ are the states of the ith transformed subsystem. For every j, Q_j is a constant matrix with appropriate dimension, f_{jk}, g_{jk} and p_{ij} are known and smooth functions.

In the sequel, the following minimum-phase condition is assumed.

Condition A For every $1 \le j \le N$, Q_j is a Hurwitz matrix.

The structure involved in (2.11) is commonly utilized in the past literature in both centralized and decentralized control, the reader is referred to [20, 23, 26, 32, 40, 48, 56, 80]. In view of the existing results on geometric nonlinear control [23, 29, 32, 40], necessary and sufficient geometric conditions can be easily derived under which a nonlinear system (2.9), (2.10) is transformed into (2.11), yielding the so-called "disturbed decentralized output-feedback form".

Remark 2.6 It is worth noting that the nonlinearities in (2.9) depend only on the output $y = (y_1, \dots, y_N)$ and that the unmeasured states $X_j[z_j, x_{j2}, \dots, x_{jn_j}]$ in (2.11) appear linearly. This feature is found appealing in recent studies in global output-feedback control for both centralized and decentralized nonlinear systems, in the framework of robust and/or adaptive control. As a matter of fact, simple counterexamples found in [43] reveal the fundamental limitation of global output-feedback control for systems with strong nonlinearities due to unmeasured states. For example, it has been shown in [43] that there is no continuous (static or dynamic) output-feedback control law that can globally asymptotically stabilize a nonlinear system $\dot{x}_1 = x_2, \dot{x}_2 = x_2^n + u$ with output $y = x_1$ whenever $n \ge 3$.

2.2 Dynamic Output Feedback: Class I

The objective of this section is to propose an approach to robust stabilization of systems which are composed of linear subsystems coupled by nonlinear time-varying interconnections satisfying quadratic constraints. The proposed algorithms, which are formulated within the convex optimization framework, employ linear dynamic feedback structure involving local Luenberger-type observers. It is also shown how the new methodology can produce improved results if interconnections have linear parts that are known a priori. Examples of output stabilization of inverted pendulums and decentralized control of a platoon of vehicles are used to illustrate the applicability of the design method.

With the emergence of the powerful convex optimization toolboxes involving linear matrix inequalities (LMIs), solving problems of controller design within the convex optimization framework became very attractive, see [1, 6, 10, 11, 14, 21, 24, 61]. Of wide-spread interest have been the control problems of formulating sufficient conditions for computing output feedback control laws using convex optimization methods due to the fact that the necessary and sufficient conditions are known to be non convex, in general. These problems become increasingly more difficult to solve when decentralized information structure constraints are imposed in the control design [2, 15, 16, 49, 59, 83, 85]. These information structures can be found in important applications, such as power systems [86], control of formations of unmanned vehicles [65] and control of large structures [34], to name few.

2.2.1 Observer-Based Control Design

In what follows, we propose novel sufficient conditions for the design of decentralized dynamic output controllers in the convex optimization context for stabilization of interconnected systems with linear subsystems and nonlinear time-varying interconnections. Controllers are designed to guarantee robust stability of the overall system and, in addition, maximize the bounds of unknown interconnection terms, starting from the methodology proposed in [55]. In what follows, we adopt here the controller structure containing local observers of Luenberger type. Several algorithms are proposed in the general case of full order observers, differing by complexity and the degree of interdependence between the observer and the feedback gains, where no additional constraints on the parameters of the system model are imposed [46, 58]. It is also shown how the proposed scheme can be used to build reduced-order observers. Particular attention is paid to the case when linear parts of interconnections are known a priori, and an algorithm is proposed which takes advantage of this knowledge to come up with improved results. To illustrate the application of the proposed schemes we include two examples, the first dealing with interconnected pendulums, and the second with the problem of platoons of vehicles in the case when the velocity and acceleration of the neighboring vehicles are not accessible.

Reference is made to model of Class I as described by (2.1)–(2.4). To proceed further, we consider that

1. *The dynamic controller* **F** *for* **S** *is linear*,
2. *It obeys the decentralized information structure constraint requiring that each subsystem is controlled using its own local output* and
3. *It is composed of an observer of Luenberger-type and a static observer state feedback.*

This motivates us to express controller **F** into the

$$\mathbf{F}: \quad \dot{w} = Aw + Bu + L(y - Cw), \qquad u = Kw, \tag{2.12}$$

where $w \in \Re^n$ is the observer state, with $w = [w_1^t, \dots, w_N^t]^t$, $w_j \in \Re^{n_j}$ and

$$K = \mathrm{diag}\{K_1, \dots, K_N\}, \qquad L = \mathrm{diag}\{L_1, \dots, L_N\}$$

represent the global controller parameter matrices while pairs (K_j, L_j) determine the local dynamic controllers.

The resulting closed-loop system $\mathbf{S}_c = (\mathbf{S}, \mathbf{F})$ can be expressed as

$$\mathbf{S}_c: \quad \dot{z} = A_c z + h_c(t, z), \qquad y = C_c z, \tag{2.13}$$

where z is the state vector. Defining

$$z = [z_1^t, z_2^t]^t, \quad z_1 = w, \; z_2 = w - x$$

we obtain

$$\begin{aligned} A_c &= \begin{bmatrix} A + Bk & -LC \\ 0 & A - LC \end{bmatrix}, \qquad C_c = \left[C \,\vdots\, -C \right], \\ h_c(t, z) &= \left[0 \,\vdots\, -h(z_1 - z_2)^t \right]^t. \end{aligned} \tag{2.14}$$

In view of (2.4), we have now

$$h_c(t, z)^t h_c(t, z) \le z^t H_c^t \Gamma^{-1} H_c z, \tag{2.15}$$

where $H_c = [H \,\vdots\, -H]$.

We now address the key feature of dynamic controller $\mathbf{F}$, that is, it must robustly stabilizes $\mathbf{S}$. According to the results of [55, 58], it is shown that $\boldsymbol{S}$ is robustly stabilized with vector degree $\alpha = [\alpha_1, \ldots, \alpha_N]^t$ if the equilibrium $z = 0$ of the closed-loop system $\mathbf{S}_c = (\mathbf{S}, \mathbf{F})$ is globally asymptotically stable for all $h_c(t, z)$ satisfying (2.15) for some H_c and Γ.

It turns out that the controller stabilizes the linear part of $\mathbf{S}$ and, at the same time, maximizes its tolerance to uncertain nonlinear interconnections and perturbations. This is nicely expressed by the following LMI-based formulation:

System $\mathbf{S}_c = (\mathbf{S}, \mathbf{F})$ i*s robustly stable with vector degree* α *if the following problem is feasible*:

$$\begin{aligned} &\min \ \mathrm{Tr}\,\Gamma \\ &\text{subject to} \quad X_c > 0, \quad \begin{bmatrix} X_c A_c + A_c^t X_c & X_c & H_c^t \\ \bullet & -I & 0 \\ \bullet & \bullet & -\Gamma \end{bmatrix} < 0. \end{aligned} \tag{2.16}$$

It must be observed that, by and large, observer-based feedback design cannot be completed directly using (2.16). The main reason for this is that the second matrix inequality is not an LMI in both X_c and the feedback parameter matrix.

Remark 2.7 At this stage we should recall some basic results from [1, 16]. In the case of state-feedback the problem can be readily transformed into an LMI problem by a simple change of variables (convexification procedure). However, in the case of dynamic output feedback the problem becomes far more complex. A decoupled quadratic Lyapunov function with block-diagonal weighting matrix has been used in

[58] to determine the dynamic controller parameters. However, the proposed design procedure imposes additional constraints on the system model characteristics.

In what follows we will provide some modifications of problem (2.16) obtained by convexifying the constraints. Solutions to these problems will provide guaranteed feasible solutions to (2.16) and the upper bound of the objective function $\mathrm{Tr}\,\Gamma$.

2.2.1.1 Full Order Observer

Introducing the following matrices

$$\begin{aligned} Q &= \mathrm{diag}\{Q_1, \ldots, Q_N\}, \quad Q_j \in \Re^{n_j \times n_j}, \\ P &= \mathrm{diag}\{P_1, \ldots, P_N\}, \quad P_j \in \Re^{n_j \times n_j}, \\ W &= \mathrm{diag}\{W_1, \ldots, W_N\}, \quad W_j \in \Re^{m_j \times n_j}, \\ V &= \mathrm{diag}\{V_1, \ldots, V_N\}, \quad V_j \in \Re^{n_j \times p_j}. \end{aligned}$$

For the purpose of simplifying the subsequent analysis, we define the matrix function

$$\Psi(S, L, M, \Gamma) = \begin{bmatrix} S & L & M \\ \bullet & -I & 0 \\ \bullet & \bullet & -\Gamma \end{bmatrix}, \tag{2.17}$$

for some S, L, M, Γ matrices with appropriate dimensions.

Problem 2.1

$$\begin{aligned} &\min\ \mathrm{Tr}\,\Gamma \\ &\text{subject to} \quad Q > 0, \quad P > 0, \\ &\qquad \Psi(S_1, I, QH^t, \Gamma) < 0, \quad \Psi(S_2, P, -H^t, \Gamma) < 0, \end{aligned} \tag{2.18}$$

where $S_1 = AQ + QA^t + BW + W^tB^t$ and $S_2 = PA + A^tP - VC - C^tV^t$.

We have the following result:

Theorem 2.1 *System* $\boldsymbol{S}$ *is robustly stabilized by the controller* $\boldsymbol{F}$ *if Problem 2.1 is feasible. The controller parameters are given by*

$$K = WQ^{-1}, \qquad L = P^{-1}V. \tag{2.19}$$

Proof In what follows it will be shown that there exists a real number $\lambda > 0$ such that the matrix $X_c = \mathrm{diag}\{\lambda^{-1}Q^{-1}, P\}$ satisfies LMIs (2.16) for some $\Gamma > 0$, where

P and Q are solutions of Problem 2.1. Substituting (2.14) and X_c into (2.16), we obtain

$$\begin{bmatrix} \lambda S_1 & -LC & I & 0 & \lambda QH^t \\ \bullet & S_2 & 0 & P & -H^t \\ \bullet & \bullet & -I & 0 & 0 \\ \bullet & \bullet & \bullet & -I & 0 \\ \bullet & \bullet & \bullet & \bullet & -\Gamma \end{bmatrix} < 0. \tag{2.20}$$

By Schur complements, we obtain the following conditions equivalent to (2.20):

$$\begin{aligned} &\Xi_1 < 0, \qquad \lambda \Xi_3(\Gamma_\lambda) - \Xi_2 \Xi_1^{-1} \Xi_2^t + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} < 0, \\ &\Xi_1 = \begin{bmatrix} -I & P \\ P & S_2 \end{bmatrix}, \qquad \Xi_2 = \begin{bmatrix} 0 & -LC \\ 0 & -H \end{bmatrix}, \qquad \Xi_3(X) = \begin{bmatrix} S_1 & QH^t \\ HQ & -X \end{bmatrix}, \\ &\Gamma_\lambda = \lambda^{-1}\Gamma, \qquad X \in \Re^{n\times n}. \end{aligned} \tag{2.21}$$

Now let $\Gamma_0 = \mathrm{diag}\{\gamma_1^0 I_{l_1}, \ldots, \gamma_N^0 I_{l_N}\}$ is the optimal Γ obtained by solving Problem 2.1 and define

$$\nu = \lambda_{\min}(\Xi_1^{-1}), \qquad a = \lambda_M(\Xi_2 \Xi_2^t), \qquad \mu = \lambda_M(\Xi_3(\Gamma_0)).$$

It is easy to see that Ξ_1 and $\Xi_3(\Gamma_0)$ represent principal minors of the matrices $\Psi(S_1, I, QH^t, \Gamma_0) < 0$ and $\Psi(S_2, P, -H^t, \Gamma_0) < 0$ and hence both eigenvalues μ and ν are negative.

Selecting $\Gamma = \lambda^* \Gamma_0, \lambda^* > |\theta|/|\mu|, \theta = -1 + a\nu$ and assuming that $0 < \lambda < \lambda^*$, it follows that

$$\lambda_M\{\Xi_3(\Gamma_\lambda)\} = \lambda_M\{\Xi_3((\lambda^*/\lambda)\Gamma_0)\} \le \lambda_M\{\Xi_3(\Gamma_0)\} = \mu$$

bearing in mind that $\lambda^*/\lambda > 1$. For this selection of Γ and λ, (2.21) is implied by

$$\mu\lambda - \theta < 0, \tag{2.22}$$

which holds true for $|\theta|/|\mu| < \lambda < \lambda^*$. Therefore, the desired λ exists and the proof is completed. □

Remark 2.8 The local robustness degrees defined by

$$\alpha_j = 1\Big/\sqrt{\gamma_j^0 |\theta|/|\mu|}, \quad j = 1, \ldots, N$$

guaranteed from Theorem 2.1 are generally conservative. More realistic values can be obtained by plugging the controller parameters obtained by (2.19) into (2.16) and by solving the corresponding minimization problem with variables X_c and Γ. This will be demonstrated in the numerical examples presented later on.

Remark 2.9 It is interesting to note that Problem 2.1 implements the separation principle. The constituent problems

$$Q > 0, \qquad \Psi(S_1, I, QH^t, \Gamma^1) < 0, \qquad P > 0, \qquad \Psi(S_2, P, -H^t, \Gamma^2) < 0$$

can be readily solved independently, the first providing K as in the state feedback design and the second L, robustly stabilizing the observer, so that

$$\Gamma = \text{diag}\{\max(\gamma_1^1, \gamma_1^2)I_{\ell_1}, \dots, \max(\gamma_N^1, \gamma_N^2)I_{\ell_N}\}.$$

Remark 2.10 An alternative procedure to simplify LMIs in (2.18) is as follows:

Problem 2.2

$$\begin{aligned} &\min\ \text{Tr}\,\Gamma \\ &\text{subject to} \quad Q > 0, \quad P > 0, \quad \Xi_3(\Gamma) < 0, \quad \Xi_1 < 0 \end{aligned} \tag{2.23}$$

while the controller parameters are obtained by using (2.19).

Generally speaking, the achievable robustness degree is lower than the one obtained by solving Problem 2.1. Specifically, it is possible to show using the methodology of Theorem 2.1 that if Q_0, W_0 and Γ_0 are obtained by solving Problem 2.2, then there exist $\rho > 0$ and $\beta > 1$ such that $\Psi(\rho(AQ_0 + Q_0A^t + BW_0 + W_0^tB^t), I, \rho Q_0H^t, \beta\Gamma_0) < 0$.

By taking into consideration the interdependence between K and L in the LMIs (2.16), we will attempt to exploit the structure of (2.20) to construct improved algorithms with higher robustness degree.

Problem 2.3

$$\begin{aligned} &\min\ \text{Tr}\,\Gamma \\ &\text{subject to} \quad P > 0, \quad \Psi(S_2, P, -H^t, \Gamma) < 0. \end{aligned} \tag{2.24}$$

1. Use the solutions $P, S_2, \Gamma, L = P^{-1}V$.
2.

$$\begin{aligned} &\min\ \text{Tr}\,\Delta \\ &\text{subject to} \quad Q > 0, \\ &\qquad\qquad \begin{bmatrix} S_1 & I & -LC & 0 & QH^t \\ \bullet & -I & 0 & 0 & 0 \\ \bullet & \bullet & S_2 & P & -H^t \\ \bullet & \bullet & \bullet & -I & 0 \\ \bullet & \bullet & \bullet & \bullet & -\Gamma\Delta \end{bmatrix} < 0, \end{aligned} \tag{2.25}$$

where $\Delta = \text{diag}\{\delta_1 I_{l_1}, \dots, \delta_N I_{l_N}\}$, $\delta_j > 0, \forall j$.

The following result stands-out:

Theorem 2.2 *System* ***S*** *is robustly stabilized by the controller* ***F*** *if Problem 2.3 is feasible. Controller parameters are given by (2.19). The robustness degree bounds are given by* $\alpha_j = 1/\sqrt{\gamma_j \delta_j}$.

Proof It is readily seen that the second inequality in (2.25) is identical to inequality (2.16) for $X_c = \mathrm{diag}\{Q^{-1}, P\}$, with Γ replaced by $\Gamma\Delta$ and hence the desired result. □

Remark 2.11 It should be noted that Steps 1 and 2 have to be performed consecutively and not simultaneously, like in Problems 2.1 and 2.2. Alternative algorithms could be derived if one takes, for example,

$$z = [z_1^t, z_2^t]^t, \quad z_1 = x, \; z_2 = x - w,$$
$$A_c = \begin{bmatrix} A + BK & -BK \\ 0 & A - LC \end{bmatrix}, \qquad C_c = \begin{bmatrix} C \vdots 0 \end{bmatrix},$$
$$h_c(t, z) = \begin{bmatrix} h^t(z_1) \vdots h^t(z_1) \end{bmatrix}^t$$

and arrives at a problem similar to Problem 2.3, in which K is determined in the first step, and L in the second step.

2.2.1.2 Reduced Order Observer

The results of the foregoing section can be directly extended to the design of controllers with decentralized reduced order observers. For this purpose, we assume that $C_j = [0_{(n_j - p_j)\times n_j} \vdots I_{Pi}]$, $p_j \le n_j$ if x_j is divided into

$$x_j = [(x_j^a)^t, (x_j^c)^t]^t, \quad x_j^a \in \Re^{n_j - p_j}, \; x_j^c \in \Re^{p_j}$$

then $y_j = x_j^c$ and the output $w_j \in \Re^{n_j - p_j}$ of the local reduced order observer is an estimate of x_j^a. Similar to [33], we assume that the local dynamic controllers F_j have the form:

$$\dot{w}_j = A_j^{11} w_j + A_j^{12} y_j + B_j^1 u_j + L_j[\dot{y}_j - A_j^{21} w_j - A_j^{22} y_j - B_j^2 u_j], \tag{2.26}$$
$$u_j = G_j w_j + J_j y_j = K_j \xi_j, \tag{2.27}$$

where

$$A_j = \begin{bmatrix} A_j^{11} & A_j^{12} \\ A_j^{21} & A_j^{22} \end{bmatrix}, \qquad B_j = \begin{bmatrix} B_j^1 \\ B_j^2 \end{bmatrix},$$
$$\xi_j = [w_j^t, y_j^t]^t = [w_j^t, (x_j^c)^t]^t.$$

Note that differentiation of y_j in (2.26) can be avoided by standard transformation of variables. Defining

$$\eta_j = w_j - x_j^a, \qquad \xi = [\xi_1^t, \dots, \xi_N^t]^t, \qquad \eta = [\eta_1^t, \dots, \eta_N^t]^t$$

we take $z = [\xi^t, \eta^t]^t$ as a new state vector for $S_c = (S, F)$, and obtain

$$\mathbf{S}_f\colon \quad \dot{z} = \begin{bmatrix} A + BK & \bar{L}A^{12} \\ 0 & A^{11} - LA^{21} \end{bmatrix} z + h_c(t, z), \tag{2.28}$$

where

$$\begin{aligned} A^{11} &= \operatorname{diag}\{A_1^{11}, \dots, A_N^{11}\}, \qquad A^{12} = \operatorname{diag}\{A_1^{12}, \dots, A_N^{12}\}, \\ A^{21} &= \operatorname{diag}\{A_1^{21}, \dots, A_N^{21}\}, \qquad K = \operatorname{diag}\{K_1, \dots, K_N\}, \\ L &= \operatorname{diag}\{L_1, \dots, L_N\}, \qquad \bar{L} = \operatorname{diag}\{\bar{L}_1, \dots, \bar{L}_N\}, \\ \bar{L}_j &= \begin{bmatrix} -L_j^t & -I_{pi} \end{bmatrix}^t, \\ h_c(t, z) &= \Big[\big[0_{n_1 - p_1}^t \,\vdots\, h_1^c(x)^t \big], \dots, \big[0_{n_N - p_N}^t \,\vdots\, h_N^c(x)^t \big], -h_1^a(x)^t, \dots, -h_N^a(x)^t \Big]^t \end{aligned}$$

where the decomposition $h_j(x) = (h_j^a(x)^t, h_j^c(x)^t)^t$ is induced by the decomposition of x_j into x_j^a and x_j^c. This leads to

$$h_c(t, z)^t h_c(t, z) \le \alpha^2 z^t \bar{H}_c^t \bar{H}_c z, \tag{2.29}$$

where $\bar{H}_c = [H \,\vdots\, -\bar{H}]$, $\bar{H}^t = [\bar{H}_1^t \,\vdots\, \dots \,\vdots\, \bar{H}_N^t]$, while $\bar{H}_j$ is an $l_j \times (n_j - p_j)$ matrix containing the first $n_j - p_j$ columns of H_j, having in mind that $H_j x = H_j \xi - \bar{H}_j \eta$.

The structure of the closed-loop model (2.28) shows that controller design can be entirely based on the methodology developed earlier. Hence, Problem 2.1 and Theorem 2.1 yield

Corollary 2.1 *System* $\mathbf{S}$ *in which*

$$\begin{aligned} C &= \operatorname{diag}\big\{ \big[0_{(n_1 - p_1) \times p_1} \,\vdots\, I_{p1} \big], \dots, \big[0_{(n_N - p_N) \times p_N} \,\vdots\, I_{p_N} \big] \big\}, \\ & p_j \le j = 1, \dots, N \end{aligned}$$

is robustly stabilized by the dynamic controller $\boldsymbol{F}$ *defined by* (2.26), (2.27) *if the following problem is feasible*:

$$\begin{aligned} &\min \ \operatorname{Tr} \Gamma \\ &\text{subject to} \quad Q > 0, \quad \bar{P} > 0, \\ &\qquad\qquad\quad \Psi(S_1, I, QH^t, \Gamma) < 0, \\ &\qquad\qquad\quad \Psi(\bar{S}_2, \bar{P}, -H^t, \Gamma) < 0, \end{aligned} \tag{2.30}$$

where

$$\bar{S}_2 = \bar{P}A^{11} + (A^{11})^t\bar{P} - \bar{V}A^{21} - (A^{21})^t\bar{V}^t \in \Re^{n_j-p_j \times n_j-p_j},$$
$$\bar{P} = \text{diag}\{\bar{P}_1, \ldots, \bar{P}_N\} \in \Re^{n_j-p_j\times p_j}, \qquad \bar{V} = \text{diag}\{\bar{V}_1, \ldots, \bar{V}_N\}.$$

The controller parameters are obtained by using (2.19).

2.2.1.3 Important Special Case

We now look at the special case where the interconnections between the subsystems $\mathbf{S}_j$ in $\mathbf{S}$ is known, linear and can be represented by a full matrix $A_s \in \Re^{n\times n}$ containing off diagonal interconnection blocks, so that $A + A_s$ becomes the new state matrix in the linear part of $\mathbf{S}$ in (2.2). The function $h(t, x)$ still represents the unknown part of interconnections.

The foregoing design methodology can be extended to this case while aiming to exploit the additional *a priori* information constraint. A point of caution must be entertained here. By replacing $\mathbf{A}$ by $A + A_s$ in the observer equation for $\mathbf{F}$ in (2.12) one violates the adopted information structure constraint, i.e. the dynamic controller ceases to be decentralized. Inserting $A + A_s$ only in the state model (2.3), we obtain

$$A_c = \begin{bmatrix} A + BK & \vdots & -LC \\ -A_\delta & \vdots & A + A_s - LC \end{bmatrix}.$$

This fact indicates that the design scheme could now be based on modifying the problems described in Sects. 2.2.1.1 and 2.2.1.2 by inserting the new information in the form of A_s at the corresponding places in the related LMIs. Robust stabilization is achievable however, when the interconnections are sufficiently weak. For example, Problem 2.1 turns to be:

Problem 2.4

$$\min \ \text{Tr}\,\Gamma \tag{2.31}$$
$$\text{subject to} \quad P > 0, \quad Q > 0, \tag{2.32}$$
$$\Psi(S_1, I, QH^t, \Gamma) < 0, \tag{2.33}$$
$$\Psi(S_{2s}, P, -H^t, \Gamma) < 0, \tag{2.34}$$

where $S_{2s} = P(A + A_s) + (A + A_s)^t P - VC - C^t V^t$.

Theorem 2.3 *The system* $\mathbf{S}$ *with known linear interconnections* (*modeled by adding* A_δ *to* A *in* (2.3)) *is robustly stabilized by the decentralized dynamic controller* $\boldsymbol{F}$ *in* (2.12) *if Problem* 2.4 *is feasible and*

$$\delta < \frac{\mu^2}{8\theta_\delta \nu_\delta \lambda_p \lambda_Q}, \tag{2.35}$$

where $\delta = \lambda_M(A_s^t A_s)$, $\lambda_P = \lambda_M(P^2)$, $\lambda_Q = \lambda_M(Q^2)$, $\nu_\delta = \lambda_m(\Xi_{1\delta}^{-1})$, *matrix* Ξ_{1s} *is obtained from* Ξ_1 *in* (2.21) *by replacing* S_2 *with* S_{2s}, *and* $\theta_s = -1 + 2a\nu_s$.

Proof The proof is based on a line of thought similar to that applied in Theorem 2.1. Inserting

$$X_c = \text{diag}\{\lambda^{-1}Q^{-1}, P\}, \qquad A_c = \begin{bmatrix} A + BK & -LC \\ -A_s & A + A_s - LC \end{bmatrix}$$

into (2.16) we obtain

$$\begin{bmatrix} \lambda S_1 & -L_\delta & I & 0 & \lambda Q H^t \\ \bullet & S_{2s} & 0 & P & -H^t \\ \bullet & \bullet & -I & 0 & 0 \\ \bullet & \bullet & \bullet & -I & 0 \\ \bullet & \bullet & \bullet & \bullet & -\Gamma \end{bmatrix} < 0, \tag{2.36}$$

where $L_s = LC + \lambda Q A_s^t P$. The last inequality is equivalent to $\Xi_{1s} < 0$ and

$$\lambda \Xi_3(\Gamma_s) - (\Xi_2 + \lambda \Xi_{2s})\Xi_{1\delta}^{-1}(\Xi_2 + \lambda \Xi_{2\delta})^t + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} < 0, \tag{2.37}$$

where

$$\Xi_{2s} = \begin{bmatrix} 0 & -QA_s^t P \\ 0 & 0 \end{bmatrix}.$$

By similarity to Theorem 2.1, we let $P = \lambda^* \Gamma_0$ for some $\lambda^* > 0$, where Γ_0 is the optimal value obtained by solving Problem 2.4. Assume that $0 < \lambda < \lambda^*$. Then, (2.37) is implied by

$$-2\delta\nu_s\lambda_P\lambda_Q\lambda^2 + \mu\lambda - \theta_s < 0, \tag{2.38}$$

bearing in mind that $\lambda_M\{\Xi_3(\Gamma_\lambda)\}\lambda_M\{\Xi_3(\Gamma_0)\} = \mu$. Observe that $\nu_s < 0$ by assumption, as a consequence of the feasibility of Problem 2.4. The existence of $\lambda > 0$ satisfying (2.38) is guaranteed if (2.35) holds, since then we have $D = \mu^2 - 8\delta\theta_s\nu_s\lambda_P\lambda_Q > 0$. Consequently, we choose

$$\frac{-\mu - \sqrt{D}}{-4\delta\nu_s\lambda_P\lambda_Q} = \lambda_1 < \lambda^* \le \lambda_2 = \frac{-\mu + \sqrt{D}}{-4\delta\nu_s\lambda_P\lambda_Q},$$

where $0 < \lambda_1 < \lambda_2$ since $\mu < 0$ and $\sqrt{D} \le |\mu|$, so that λ can take any value in the interval $[\lambda_1, \lambda^*]$. The local guaranteed robustness degree bounds are now $\alpha_j = 1/\sqrt{\gamma_j^0 \lambda_1}$, $j = 1, \dots, N$, which concludes the proof. □

2.2.2 Simulation Example 2.1

Consider the motion of two inverted pendulums connected by a spring which can slide up and down the rods of the pendulums in jumps of unpredictable size and direction between the support and the height equal to 1 [55]. An appropriate linearized and normalized model is given by

$$\begin{aligned}
\mathbf{S}:\quad \dot{x} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u + h(t,x), \\
y &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x, \\
h(t,x) &= e(t,x)Gx, \quad G = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix},
\end{aligned} \tag{2.39}$$

where $e(t,x) : \Re^5 \to [0, 1]$ represents a normalized interconnection parameter.

It is required to compute a decentralized control law which would connectively stabilize the system for all values of $e(t,x) \in [0, 1]$.

A decentralized state-feedback is designed to provide $\alpha = 4.4950$ with the local controller gain matrix $K = [-725.9085\ -40.4346]$ and the corresponding closed-loop poles $\{-20 \pm j17.8093\}$.

Computer simulation shows that the system is not stabilizable by static output feedback, since two coefficients of the characteristic equation remain fixed to zero irrespective of the controller parameters.

Turning to dynamic output feedback obtained by the proposed algorithms, Table 2.1 provides results on robustness degree α. In this table, Case A corresponds to the situation in which $H = G$ in the three algorithms from Sect. 2.2.1.1. Case B refers to $H = G$ with

$$A_s = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5 & 0 & -0.5 & 0 \end{bmatrix}$$

when the algorithms derived from Problems 2.1–2.3 in accordance with the methodology of Problem 2.4 and Theorem 2.3. Case C represents the situation with no a

Table 2.1 Robustness degree α for different algorithms

	Problem 2.1	Problem 2.2	Problem 2.3
Case A	5.6450	0.3813	21.7304
Case B	4.3840	0.5787	15.2214
Case C	0.6564	0.3191	0.7003

priori knowledge, when $H = I$ and $A_s = 0$, and the algorithms from Sect. 2.2.1.1 are applied.

The ensuing results lead to the conclusion that the best results are obtained by solving Problem 2.3; the worst case corresponds to Problem 2.2. This is quite expected. In view of the results of [55], we note that in Case C none of the algorithms ensures connective stability. For Problem 2.2, connective stability is achieved only in Case B, when the information about the interconnections is included. This corresponds in Case B to have, in fact, $e(t,x) = 0.5 + e^a(t,x)$, where $e^a(t,x) \in [-0.5, 0.5]$, so that any value of $\alpha > 0.5$ is sufficient for connective stability. All values of K and L and the corresponding modes are not presented because of the lack of space. For example, for Problem 2.1 and Case A we have $K_j = [-79.1666 \;\; -11.2883]$, $L_j^t = [27.7711 \;\; 15.7991]$, with local closed-loop poles $\{-27.2275, -0.5435, -0.5441 \pm j6.8052\}$.

2.2.3 Simulation Example 2.2

This example is concerned with the decentralized control of a platoon of vehicles. A feedback-linearized state space model of a platoon of N automotive vehicles is based, according to [65], on the following feedback linearized individual vehicle model:

$$\dot{d}_j = v_{j-1} - v_j, \qquad \dot{v}_j = a_j, \qquad \dot{a}_j = -\tau_j^{-1} a_j + \tau_j^{-1} u_j, \tag{2.40}$$

where $d_j = x_{j-1} - x_j$ is the distance between two consecutive vehicles, x_{j-1} and x_j being their positions, v_j and a_j are the velocity and acceleration of ith vehicle, respectively, u_j the input signal chosen to make the closed-loop system satisfy certain performance criteria, and τ_j the time constant of the engine. After obtaining the overall platoon state space model with the state

$$X = (d_1 - d_r, v_1 - v_r, a_1 - a_r, \ldots, d_N - d_r, v_N - v_r, a_N - a_r)^t$$

and input

$$u = (u_1, u_2, \ldots, u_N)^t,$$

where d_r, v_r, a_r are the reference values for inter-vehicle distance, velocity and acceleration, respectively, and applying the state and input expansion by using convenient full-rank linear transformations, the following model in the expanded space is obtained [65]:

$$\tilde{\mathbf{S}}: \quad \dot{\xi} = \tilde{A}\xi + \tilde{B}\zeta, \tag{2.41}$$

where

$$\xi = [\xi_1^t, \ldots, \xi_N^t]^t, \quad \zeta = [\zeta_1^t, \ldots, \zeta_N^t]^t,$$
$$\tilde{A} = \mathrm{diag}\{A_1, \ldots, A_N\}, \quad \tilde{B} = \mathrm{diag}\{B_1, \ldots, B_N\}$$

with vectors ξ_j and ζ_j and matrices A_j and B_j are defined within the formally defined subsystem models connected to each vehicle:

$$\begin{aligned}\mathbf{S}_j: \quad \dot{\xi}_j &= A_j\xi_j + B_j\zeta_j \\ &= \begin{bmatrix} A_j^l & 0 \\ A_d & A_j^v \end{bmatrix}\xi_j + \begin{bmatrix} B_j^l & 0 \\ 0 & B_j^v \end{bmatrix}\zeta_j,\end{aligned} \tag{2.42}$$

with $\xi_j = [v_{j-1} - v_r, a_{i-1} - a_r, d_j - d_r, v_j - v_r, a_j - a_r]^t$ being the state vector of jth subsystem, $\zeta_j = (u_{j-1}, u_j)^t$ represents its control vector, while

$$A_j^\ell = \begin{bmatrix} 0 & 1 \\ 0 & \tau_j^{-1} \end{bmatrix}, \qquad \bar{A}_d^t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B_j^\ell = \begin{bmatrix} 0 \\ \tau_j^{-1} \end{bmatrix},$$

$$A_j^v = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\tau_j^{-1} \end{bmatrix}, \qquad B_j^v = \begin{bmatrix} 0 \\ 0 \\ \tau_j^{-1} \end{bmatrix}.$$

This model is treated in [65] where it is shown that a decentralized dynamic control law can be designed for the expanded system using the methodology from Sect. 2.2.1.2, supposing that only the subsystem states $d_j - d_r$, $v_j - v_r$ and $a_j - a_r$ are exactly known in jth vehicle (subsystem), that is, v_{j-1} and a_{j-1} are not accessible in ith vehicle. Applying the results of Sect. 2.2.1.2, the reduced-order Luenberger observer for $\xi_j^1 = (v_{j-1} - v_r, a_{j-1} - a_r)^t$ is given by

$$\dot{w}_j = A_j^l w_j + B_j^l u_{i-1} + L_j[\dot{\xi}_j^2 - \bar{A}_d w_j - A_j^v \xi_j^2], \tag{2.43}$$

where $\xi_j^2 = (d_j - d_r, v_j - v_r, a_j - a_r)^t$. The local control law has the following specific structure:

$$u_{j-1} = G_j^1 w_j, \qquad u_j = G_j^2 w_j + J_j^2 \xi_j^2, \tag{2.44}$$

having in mind that $(j-1)$th vehicle does not have any information about ith vehicle. Matrices

$$K_j = \begin{bmatrix} G_j^1 & 0 \\ G_j^2 & J_j^2 \end{bmatrix}, \quad L_j, \quad j = 1, \ldots, N$$

can now be obtained by using the algorithm from Corollary 2.1, exploiting the specific lower-block-triangular structure of K_j.

For $\tau_j = \tau = 0.1$, one obtains:

$$G_j = G = [-38.6940\ -2.1224], \qquad G_j^1 = G^1 = [-38.6940\ -2.1224],$$
$$G_j^2 = G^2 = [0.0095\ 0.0005],$$
$$J_j^2 = J^2 = [351.4028\ -319.3970\ -13.2356],$$
$$L_j = L = 10^4 \begin{bmatrix} 0.0001 & 0 & 0 \\ 3.2068 & 0 & 0 \end{bmatrix}, \qquad \alpha_j = \alpha = 1/4.080$$

generating the closed loop poles

$$10^2\{-1.1480, -0.0116, -0.1561 \pm j0.1197, -0.2640, -320.68, -0.00004\}.$$

Obviously, it is also possible to apply the alternative design schemes from Sect. 2.2.1.1. By using the expansion/contraction matrices as in [60] and [65], the obtained controller has to be finally contracted to the original space for implementation.

2.2.4 Simulation Example 2.3

The third example considered here is a linearized two-tank system modeled in the form (2.1) with data

$$A_1 = \begin{bmatrix} 0.703 & 0 & 0.395 & -0.320 \\ -0.052 & 0 & 0 & -0.137 \\ 0 & 0 & 0 & 0.619 \\ 0 & 1.028 & 1.752 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -0.402 & 0.978 \\ 0 & 0 \\ -0.263 & 0.159 \\ 0 & 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.423 & 0 & 0 & 0.317 \\ 0 & 0.137 & 0.576 & 0.340 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.695 & 0.013 & 0.315 & -0.414 \\ -0.193 & 0 & 0 & 0.258 \\ 0 & 0 & 0 & -0.834 \\ 0 & 0.879 & 0.978 & 0.015 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -0.375 & 0.888 \\ 0 & 0 \\ -0.249 & 0.147 \\ 0 & 0 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0.462 & 0 & 0 & 0.351 \\ 0 & 0.098 & 0.685 & 0.742 \end{bmatrix},$$

$$h(t,x) = f(t,x)Mx, \quad M = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix},$$

$$M_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and $f(t,x) : \Re^4 \rightarrow [0, 1]$ represents a normalized coupling parameter. Exploring decentralized control design, we get state-feedback results with local gains as

$$K_1 = \begin{bmatrix} -6.222 & 18.345 & 28.367 & 17.793 \\ -1.893 & -8.148 & -13.479 & -8.542 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} -13.028 & 34.718 & 52.797 & 33.092 \\ -5.766 & 12.867 & 18.465 & 11.302 \end{bmatrix},$$

which do not stabilize the two-tank system. On the other hand, the output feedback gains are given by

$$K_1 = \begin{bmatrix} -36.856 & 10.094 \\ 15.441 & 9.313 \end{bmatrix}, \qquad K_2 = \begin{bmatrix} -12.188 & 16.158 \\ 35.738 & 2.568 \end{bmatrix},$$

which stabilize the system with robustness degree $\alpha = 3.8436$.

2.2.5 Dynamic Control Design

Extending on the foregoing section, we now consider a general linear time-invariant dynamic controller **F** for **S** which obeys the decentralized information structure constraint. This entails that each subsystem is controlled using only its own local output. Therefore,

$$\mathbf{F}: \quad \dot{w} = F_c w + L_c y, \qquad u = K_c w + G_c y, \tag{2.45}$$

where $w \in \Re^s$ is the global observer state, and

$$w = \begin{bmatrix} w_1 \dots w_N \end{bmatrix}^t, \quad w_j \in \Re^{s_j}, \; s = \sum_{j=1}^{N} s_j,$$

$$F_c = \text{diag}\{F_{c1}, \dots, F_{cN}\}, \qquad L_c = \text{diag}\{L_{c1}, \dots, L_{cN}\},$$
$$K_c = \text{diag}\{K_{c1}, \dots, K_{cN}\}, \qquad G_c = \text{diag}\{G_{c1}, \dots, G_{cN}\}.$$

For simplicity in exposition, we denote

$$J_j = \begin{bmatrix} F_{cj} & L_{cj} \\ K_{cj} & G_{cj} \end{bmatrix} \in \Re^{s_j + m_j \times s_j + p_j}$$

the local controller parameter matrices, and by $J = \text{diag}\{J_1, \dots, J_N\}$ the global controller parameter matrix.

By standard algebraic manipulations, the resulting closed-loop system $\mathbf{S}_c = (\mathbf{S}, \mathbf{F})$ can be represented by

$$\mathbf{S}_c: \quad \dot{z} = A_c z + h_c(t, z), \qquad y = C_c z, \tag{2.46}$$

where

$$z = \begin{bmatrix} x_1^t \; w_1^t \dots x_N^t \; w_N^t \end{bmatrix}^t, \qquad A_c = \text{diag}\{A_{c1}, \dots, A_{cN}\},$$

$$C_c = \text{diag}\{C_{c1}, \dots, C_{cN}\}, \qquad h_c(t, z)^t = \begin{bmatrix} h_{c_1}^t(t, z) \dots h_{c_N}^t(t, z) \end{bmatrix}^t,$$

$$A_{cj} = \begin{bmatrix} A_j + B_j G_j C_j & B_j K_j \\ L_j C_j & F_j \end{bmatrix}, \qquad C_{cj} = \begin{bmatrix} C_j \vdots 0 \end{bmatrix},$$

$$h_{cj} = \begin{bmatrix} h_j^t(t,x) \vdots 0 \end{bmatrix}^t.$$

In view of the structural constraint (2.4), we have

$$h_{c_j}^t(t,z) h_{c_j}(t,z) \le z^t \bar{H}_c^t \bar{\Gamma}^{-1} \bar{H}_c z \le z^t \tilde{H}^t \Gamma^{-1} \tilde{H} z, \tag{2.47}$$

where

$$\bar{H}^f = \begin{bmatrix} \bar{H}_1^{f^t} \vdots \ldots \vdots \bar{H}_j^{f^t} \end{bmatrix}^t,$$

$$\bar{H}_j^f = \begin{bmatrix} \bar{H}_j^1 \vdots 0 \vdots \bar{H}_j^2 \vdots 0 \vdots \ldots \vdots \bar{H}_j^N \vdots 0 \end{bmatrix}$$

in which $\nu_j \times n_j$ matrices $\bar{H}_j^t$ $(j = 1, \ldots, N)$ follow from the decomposition $\bar{H}_j = [\bar{H}_j^1 \vdots \ldots \vdots \bar{H}_j^N]$ while $\tilde{H} = \text{diag}\{\tilde{H}_1, \ldots, \tilde{H}_N\}$ with $\tilde{H}_j = [H_j \vdots 0]$.

Our immediate objective is to design the dynamic controller $\mathbf{F}$ which robustly stabilizes $\mathbf{S}$. Following the results of [9, 46–54, 56–58], it follows that

System $\mathbf{S}$ *is robustly stabilized with vector degree*

$$\bar{\alpha} = \begin{bmatrix} \bar{\alpha}_1 \ldots \bar{\alpha}_N \end{bmatrix}^t = \begin{bmatrix} 1/\sqrt{\bar{\gamma}_1} \ldots 1/\sqrt{\bar{\gamma}_N} \end{bmatrix}^t$$

if the equilibrium $x = 0$ of the closed-loop system $\mathbf{S}_f = (\mathbf{S}, \mathbf{F})$ is globally asymptotically stable for all $h(t,z)$ satisfying (2.4) for some given $\bar{H}$ and $\bar{\alpha}$, according to the first inequalities in (2.4) and (2.47).

It turns out that maximizing $\bar{\alpha}$, the controller stabilizes the linear part of $\mathbf{S}$ and, at the same time, maximizes its tolerance to uncertain nonlinear interconnections and perturbations. In this regard, the nonlinear interconnections bound is represented by a full matrix $\bar{H}$. Bearing in mind that the system model sparsity implied by (2.1) and (2.3) and the developed controller structure in (2.45) designates the perfectly decentralized control [52, 53], the corresponding controller subspace is not quadratically invariant. This entails that the related optimization problem is not convex.

In order to convexify the problem under consideration, we invoke further decompositions by applying the second (right hand side) inequalities in (2.4) and (2.47), and formulate the following modified robust stabilization problem:

System $\mathbf{S}_f = (\mathbf{S}, \mathbf{F})$ *is robustly stable with vector degree* $\alpha = (\alpha_1, \ldots, \alpha_N)^t = (1/\sqrt{\gamma_1}, \ldots, 1/\sqrt{\gamma_N})^t$ *if the following problem is feasible*:

$$\text{Minimize} \quad \sum_{i=1}^{N} \gamma_j$$

$$\text{subject to} \quad \tilde{X} > 0, \quad \begin{bmatrix} \tilde{X} A^f + A^{f^t} \tilde{X} & \tilde{X} & \tilde{H}^t \\ \tilde{X} & -I & 0 \\ \tilde{H} & 0 & -\Gamma \end{bmatrix} < 0, \tag{2.48}$$

where $\tilde{X}$ *is the global Lyapunov matrix.*

It must be noted that the matrix $\tilde{H}$ is block-diagonal in accordance with the assumed system sparsity, that is, with the subsystem dimensions. The second matrix inequality in (2.48) however is still not an LMI in both $\tilde{X}$ and the controller parameter matrix.

In the next section, we show that the above general robust stabilization problem can also be formulated as an LMI problem.

2.2.6 Robust Decentralized Design

Having in mind the availability of the system structure, together with the *a priori* knowledge about the interconnection bounds, it is quite natural to consider global Lyapunov matrices $\tilde{X}$ structurally adapted to $\mathbf{S}$ and $\mathbf{F}$:

Assumption 2.3 Matrix $\tilde{X}$ in (2.48) possesses the block-diagonal structure, that is, $\tilde{X} = \mathrm{diag}\{\tilde{X}_1, \ldots, \tilde{H}_N\}$ where $\tilde{X}_j \in \Re^{n_j+s_j \times n_j+s_j}$, $j = 1, \ldots, N$ are the local Lyapunov matrices.

It must be emphasized that this choice does not represent a significant restriction, giving the fact that the original problem has been already decomposed in (2.48) into N independent robust dynamic output feedback design problems.

Proceeding further, we let

$$\bar{A} = \mathrm{diag}\{\bar{A}_1, \ldots, \bar{A}_N\}, \qquad \bar{B} = \mathrm{diag}\{\bar{B}_1, \ldots, \bar{B}_N\}, \qquad \bar{C} = \mathrm{diag}\{\bar{C}_1, \ldots, \bar{C}_N\},$$
$$\bar{A}_j = \begin{bmatrix} A_j & 0 \\ 0 & 0 \end{bmatrix}, \qquad \bar{B}_j = \begin{bmatrix} 0 & B_j \\ I & 0 \end{bmatrix}, \qquad \bar{C}_j = \begin{bmatrix} 0 & I \\ C_j & 0 \end{bmatrix}$$

and then write $\widetilde{A} = \bar{A} + \bar{B} J \bar{C}$, where J is the global controller parameter matrix. Consequently, the second inequality in (2.48) can be written as

$$\widetilde{R} + \widehat{B} J \widetilde{C} + \widetilde{C}^t J^t \widehat{B}^t < 0, \tag{2.49}$$

where $\widetilde{R} = \mathrm{diag}\{\tilde{R}_1, \ldots, \tilde{R}_N\}$, $\widetilde{B} = \mathrm{diag}\{\widetilde{B}_1, \ldots, \widetilde{B}_N\}$, $\tilde{C} = \mathrm{diag}\{\widetilde{C}_1, \ldots, \widetilde{C}_N\}$

$$\widetilde{R}_j = \begin{bmatrix} \tilde{X}_j \bar{A}_j + \bar{A}_j^t \tilde{X}_j & \tilde{X}_j & \tilde{H}_j^t \\ \bullet & -I & 0 \\ \bullet & \bullet & -\gamma_j I \end{bmatrix}, \qquad \widehat{B}_j = \begin{bmatrix} \tilde{X}_j \bar{B}_j \\ 0 \\ 0 \end{bmatrix}, \qquad \widetilde{C}_j^t = \begin{bmatrix} \bar{C}_j^t \\ 0 \\ 0 \end{bmatrix}.$$

It is interesting to note that the problem (2.49) resembles a compact formulation of a set of N local classical $\mathcal{H}_\infty$ problems for virtual subsystems defined as

$$\dot{x}_j = A_j x_j + B_j u_j + w_j, \qquad z_j = H_j x_j,$$

where the immediate objective is to compute local controllers that render the $\mathcal{H}_\infty$-norms of the transfer functions between w_j and z_j are less than γ_j.

An important observation arises here. The block matrix (2.48) contains the entire Lyapunov matrix $\tilde{X}$, and not of $\tilde{X}\,\mathrm{col}[I, ., 0]$ as it should be in the case of the classical $\mathcal{H}_\infty$ problems [6, 10, 11].

The following lemma provides a pertinent result:

Lemma 2.1 *Let Assumption 2.3 hold and let $\tilde{X} > 0$. Then, (2.49) holds if and only if*

$$\begin{aligned}
&\widetilde{B}^{\perp}\widetilde{T}\widetilde{B}^{\perp t} < 0, \qquad \tilde{C}^{t\perp}\tilde{R}\tilde{C}^{t\perp t} < 0,\\
&\widetilde{T} = \mathrm{diag}\{\widetilde{T}_1, \dots, \widetilde{T}_N\},\\
&\widetilde{T}_j = \begin{bmatrix} \tilde{X}_j^{-1}\bar{A}_j^t + \bar{A}_j\tilde{X}_j^{-1} & I_j & \tilde{X}_j^{-1}\tilde{H}_j^t \\ \bullet & -I_j & 0 \\ \bullet & \bullet & -\gamma_j I_j \end{bmatrix},\\
&\widetilde{B}^{\perp} = \mathrm{diag}\{\widetilde{B}_1^{\perp}, \dots, \widetilde{B}_N^{\perp}\},
\end{aligned} \tag{2.50}$$

where $\widetilde{B}_j^t = [\,\bar{B}_j^t\ 0\ 0\,]$.[1]

Proof The structure of $\tilde{X}$ and J implies that (2.49) decouples into N independent inequalities

$$\widetilde{R}_j + \widehat{B}_j J_j \widetilde{C}_j + \widetilde{C}_j^t J_j^t \widehat{B}_j^t < 0$$

with general $(s_j + m_j) \times (s_j + p_j)$ matrices J_j. According to the elimination lemma [1], the necessary and sufficient conditions for these inequalities are

$$\widehat{B}_j^{\perp}\tilde{R}_j\widehat{B}_j^{\perp t} < 0, \quad \tilde{C}_j^{t\perp}\tilde{R}_j\tilde{C}_j^{t\perp t} < 0, \quad j = 1, \dots, N. \tag{2.51}$$

Note that $\widehat{B}_j^{\perp}\tilde{R}_j\widehat{B}_j^{\perp t} < 0$ holds if and only if $\widetilde{B}_j^{\perp}\widetilde{T}_j\widetilde{B}_j^{\perp t} < 0$.

Since $\widehat{B}_j = S_j[\bar{B}_j^t\ 0\ 0]^t$, $S_j = \mathrm{diag}\{\tilde{X}_j, I, I\}$, we have $\widehat{B}_j^{\perp} = \tilde{B}_j^{\perp}S_j^{-1}$, taking into consideration that $S_j^{-1}\tilde{R}_jS_j^{-1} = \tilde{T}_j$ and $\tilde{X} > 0$. This concludes the proof. □

Proceeding further, we follow the approach of [6] and introduce the decompositions:

$$\tilde{X}_j = \begin{bmatrix} X_j & X_{2j} \\ X_{2j}^t & X_{3j} \end{bmatrix}, \qquad \tilde{Y}_j = \tilde{X}_j^{-1} = \begin{bmatrix} Y_j & Y_{2j} \\ Y_{2j}^t & Y_{3j} \end{bmatrix}, \tag{2.52}$$

where $0 < X_j = X_j^t$ and $0 < Y_j = Y_j^t$ are $n_j \times n_j$ real matrices for $j = 1, \dots, N$. The following result is established:

---

[1] $A^{\perp}$ denotes a matrix with the properties $\mathcal{N}(A^{\perp}) = \mathcal{R}(A)$ and $A^{\perp}A^{\perp t} > 0$, where $\mathcal{N}(.)$, $\mathcal{R}(.)$ denote the null space and the range space of an indicated matrix.

Lemma 2.2 *Let Assumption 2.3 hold, let* $\tilde{X} > 0$, *and let* X_j, Y_j *and* X_{2j} *be given by* (2.52), $j = 1, \dots, N$. *Then inequalities* (2.50) *hold if and only if*

$$E^c V E^{ct} < 0, \qquad E^b W E^{bt} < 0, \tag{2.53}$$

where

$$\begin{aligned} V &= \mathrm{diag}\{V_1, \dots, V_N\}, & W &= \mathrm{diag}\{W_1, \dots, W_N\}, \\ E^c &= \mathrm{diag}\{E_1^c, \dots, E_N^c\}, & E^b &= \mathrm{diag}\{E_1^b, \dots, E_N^b\}, \\ E_j^c &= \begin{bmatrix} C_j^{t\perp} & 0 \\ 0 & I \end{bmatrix}, & E_j^b &= \begin{bmatrix} B_j^{\perp} & 0 \\ 0 & I \end{bmatrix} \end{aligned}$$

$$V_j = \begin{bmatrix} X_j A_j + A_j^t X_j & X_j & X_{2j} & H_j^t \\ \bullet & -I & 0 & 0 \\ \bullet & \bullet & -I & 0 \\ \bullet & \bullet & \bullet & -\gamma_j I \end{bmatrix},$$

$$W_j = \begin{bmatrix} Y_j A_j^t + A_j Y_j & I & Y_j H_j^t \\ \bullet & -I & 0 \\ \bullet & \bullet & -\gamma_j I \end{bmatrix}.$$

Proof By definition, we have

$$\tilde{R}_j = \begin{bmatrix} X_j A_j + A_j^t X_j & A_j^t X_{2j} & X_j & X_{2j} & H_j^t \\ \bullet & 0 & X_{2j}^t & X_{3j} & 0 \\ \bullet & \bullet & -I_j & 0 & 0 \\ \bullet & \bullet & \bullet & -I_j & 0 \\ \bullet & \bullet & \bullet & \bullet & -\gamma_j I_j \end{bmatrix}.$$

On the other hand, taking into consideration the structure of $\tilde{C}_j$ and $\bar{C}_j$, we have

$$\tilde{C}_j^{t\perp} = \begin{bmatrix} C_j^{t\perp} & 0 & 0 \\ 0 & 0 & I_j \end{bmatrix}.$$

As the second block-column in $\tilde{C}_j^{t\perp}$ contains only zero matrices, the second inequality in (2.50) gives the first inequality in (2.53).

Turning to the second inequality in (2.53), it is not difficult to show that it can be obtained analogously. From

$$\tilde{T}_j = \begin{bmatrix} A_j Y_j + Y_j A_j^t & A_j Y_{2j} & I & 0 & Y_j H_j^t \\ \bullet & 0 & 0 & I_j & Y_{2j}^t H_j^t \\ \bullet & \bullet & -I_j & 0 & 0 \\ \bullet & \bullet & \bullet & -I_j & 0 \\ \bullet & \bullet & \bullet & \bullet & -\gamma_j I_j \end{bmatrix}$$

and deleting the unnecessary block-rows and block-columns, we arrive at the desired result. □

It is readily seen that the matrices X_j, Y_j and X_{2j} are constrained by (2.53). This is in contrast to the standard $\mathcal{H}_\infty$ design [6, 10, 24]) where only the first diagonal blocks of the global Lyapunov matrix and its inverse are constrained by the corresponding LMIs.

Once X_j, Y_j and X_{2j} are determined, the next problem is to find $\tilde{X}_j > 0$ satisfying (2.52), $j = 1, \dots, N$.

Lemma 2.3 *Assume that*:

(1) $s_j = n_j$,
(2) X_{2j} *in* (2.52) *is nonsingular, and*
(3) $Q_j = \left[\begin{smallmatrix} X_j & I \\ I & Y_j \end{smallmatrix}\right] > 0$. *Then*,

$$X_{3j} = X_{2j}^t (X_j - Y_1^{-1})^{-1} X_{2j} \quad \Longrightarrow \quad \tilde{X}_j > 0, \quad j = 1, \dots, N. \tag{2.54}$$

Proof From (2.52), we obtain $Y_{2j}^t = X_{2j}^{-1}(I - X_j Y_j)$, yielding directly (2.54). Obviously, $\tilde{X}_j > 0$, since $X_j > 0$ and $X_j - X_{2j} X_{2j}^{-1}(X_j - Y_j^{-1})(X_{2j}^t)^{-1} X_{2j}^t = Y_j^{-1} > 0$, which completes the proof. □

By combining the foregoing results, we have the following theorem:

Theorem 2.4 *Under Assumption* 2.3, *system* $\boldsymbol{S}$ *in* (2.3) *is robustly stabilized by the dynamic controller* $\boldsymbol{F}$ *in* (2.45) *with* $s_j = n_j$ *if the following problem is feasible*:

$$\begin{aligned} &\textit{minimize} \quad \sum_{j=1}^{N} \gamma_j \\ &\textit{subject to} \quad X > 0, \quad Y > 0, \quad Q > 0, \quad Z > 0, \quad \bar{E}^c \bar{V} \bar{E}^{ct} < 0, \\ &\qquad\qquad\quad E^b W E^{bt} < 0, \end{aligned} \tag{2.55}$$

where $X = \text{diag}\{X_1, \dots, X_N\}$, $Y = \text{diag}\{Y_1, \dots, Y_N\}$, $Q = \text{diag}\{Q_1, \dots, Q_N\}$, $Z = \text{diag}\{Z_1, \dots, Z_N\}$, $\bar{V} = \text{diag}\{\bar{V}_1, \dots, \bar{V}_N\}$,

$$\bar{V}_j = \begin{bmatrix} X_j A_j + A_j^t X_j + Z_j & X_j & H_j^t \\ \bullet & -I_j & 0 \\ \bullet & \bullet & -\gamma_j I_j \end{bmatrix}$$

while matrix $\bar{E}^c$ *is a matrix having the same structure as* E^c *in* (2.53), *but with the elements* $\bar{E}_j^c$ *obtained from* $E_j^c = \begin{bmatrix} C_j^{t\perp} & 0 \\ 0 & I \end{bmatrix}$ *in such a way that the dimension of the identity matrix ensures compatibility of the product with* $\bar{V}_j$ *(instead of* V_j*)*, $j = 1, \dots, N$.

Proof Notice that the inequality $\bar{E}^c \bar{V} \bar{E}^{ct} < 0$ from the problem (2.55) follows immediately from the first inequality in (2.53) in Lemma 2.2 after applying the Schur's complement formula and replacing $X_{2j} X_{2j}^t$ by Z_j in view of the expression for $\bar{V}_j$. Condition $Z > 0$ results from the requirement that the matrices X_{2j} are nonsingular, $j = 1, \dots, N$. The inequality $E^b W E^{bt} < 0$ is identical to the second inequality in (2.55). This completes the proof. □

Remark 2.12 Solving (2.55), one gets $X > 0$, $Y > 0$ and $Z > 0$. Nonsingular matrices X_{2j} can always be constructed from any given $Z_j > 0$; one gets X_{3j} from (2.54), and, consequently, $\tilde{X}_j > 0$ from (2.52), $j = 1, \dots, N$. Then, we come back to the original inequality (2.49), which represents then a system of N independent LMIs with unconstrained matrix variables J_j, $j = 1, \dots, N$. Any solution to these LMIs gives the required block-diagonal parameter matrix $J = \text{diag}\{J_1, \dots, J_N\}$, that is, a robustly stabilizing decentralized dynamic controller **F** for **S**.

The underlying assumptions in Lemma 2.3 are important for the formulation of Theorem 2.1 in terms of LMIs. In general, in the case of reduced order observers (when $s_j < n_j$), one is faced with the problem of the existence of solutions for Y_{2j}, Y_{3j} and X_{3j} satisfying (2.52); notice that in the case of $\mathcal{H}_\infty$ design we have the rank condition in addition to the condition of the type $Q_j > 0$ [6]. The obtained estimates of the robustness degree α may appear to be too conservative. A better insight into the real robustness can be obtained by calculating A^f with the obtained parameter matrix J, replacing it in (2.48), and solving (2.48) for $\tilde{X}$ and Γ. An even more realistic and less conservative estimate can be obtained by using (2.48) with $\tilde{H}$ being replaced by $\bar{H}$ and Γ by $\bar{\Gamma}$, and by solving the corresponding LMI problem for $\tilde{X}$ and $\bar{\Gamma}$. By limiting the norm of the gain matrices J_j via the procedure of [55, 58] some benefits are anticipated.

Remark 2.13 In the case that the interconnection function in **S** is in the form $h(t,x) = h_L(t,x) + h_N(t,x)$, where $h_L(t,x) = A^h x$ is a known linear part in which A^h is a constant $N \times N$ block-matrix with blocks A^h_{jk}, $j,k = 1, \dots, N$, and $h_N(t,x)$ is an unknown nonlinear part satisfying inequality (2.4). Taking $A^* = A + A^h$ as a new state matrix in (2.3), instead of (2.49) we have

$$\tilde{R}^* + \Delta\tilde{R} + \tilde{B}^x J \tilde{C}^t + \tilde{C}^t J^t \tilde{B}^{xt} < 0, \tag{2.56}$$

where $\Delta\tilde{R}$ is an $N \times N$ block-matrix with blocks

$$\Delta\tilde{R}_{ij} = \begin{bmatrix} \tilde{X}_j \bar{A}^h_{ij} + \bar{A}^{ht}_{ji} \tilde{X}_j & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\bar{A}^h_{ij} = \begin{bmatrix} A^h_{jk} & 0 \\ 0 & 0 \end{bmatrix}, \quad j,k = 1, \dots, N, \; j \neq k, \qquad \Delta\tilde{R}_{mm} = 0, \quad m = 1, \dots, N$$

and $\tilde{R}^*$ is obtained from $\tilde{R}$ in (2.49) by replacing A_j by $A^*_{ii} = A_j + A^h_{ii}$. Bearing in mind that $\tilde{R}^* + \Delta\tilde{R}$ is not block-diagonal, Theorem 2.1 cannot be directly applied to (2.56). However, (2.56) can have a solution satisfying Assumption 2.3; it is reasonable to expect that the resulting controller provides better performance than the one obtained in the absence of the assumed a priori knowledge about linear interconnections.

2.2.7 Simulation Example 2.4

This examples uses the model of two inverted pendulums connected by a spring treated in the simulation Example 2.1.

From [55], the decentralized robust linear static state feedback provides $\alpha^* = \alpha_1 = \alpha_2 = 4.4950$, with the local gain matrix $K = [-725.909\ -40.435]$ and the local closed-loop poles $\{-20 \pm j17.8093\}$. It easy to see that the system is not stabilizable by any linear static output feedback.

The local dynamic output feedback controller parameters obtained on the basis of Theorem 2.1, with $H_j = I$ are

$$F_j = 10^4 \begin{bmatrix} -0.4670 & -1.4182 \\ -1.0131 & -3.1931 \end{bmatrix}, \quad L_j = 10^4 \begin{bmatrix} -3.3926 \\ 1.5118 \end{bmatrix},$$
$$K_j = \begin{bmatrix} 243.5166\ 767.0817 \end{bmatrix}, \quad G_j = -333.7029, \quad j = 1, 2$$

with the local closed-loop poles

$$\{-3.6543 \times 10^4, -0.0390 \times 10^4, -0.7455 \pm j0.5605\},$$

with $\alpha^* = 0.5670 < 1$—that is, the desired property is not achieved.

Assuming now that $e(t, x) = 0.5 + \acute{e}(t, x)$, where $\acute{e}(t, x) \in [-0.5, 0.5]$ one obtains the structure with known linear interconnections with

$$A^h = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5 & 0 & -0.5 & 0 \end{bmatrix}, \qquad \bar{H} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}.$$

In this case the LMI (2.56) is feasible and one gets a decentralized stabilizing controller with $\alpha^* = 1.3526$, ensuring stability for all spring positions. The local controller parameter matrices are in this case

$$F_j = 10^6 \begin{bmatrix} -1.4090 & -1.5774 \\ -1.0938 & -1.2571 \end{bmatrix}, \qquad L_j = 10^6 \begin{bmatrix} -5.0635 \\ 5.3116 \end{bmatrix},$$
$$K_j = 10^6 [0.5753\ 0.6573], \qquad G_j = 10^6 \times -1.69250,$$

and the local closed loop poles

$$\{-2.6487 \times 10^6, -1.7217 \times 10^4, -84.866, -1.8834\}.$$

A direct comparison with the results presented in relation with the same example in [67] shows that a better performance is obtained by using an observer of Luenberger type, incorporating the state matrix of the system model and leaving a smaller number of free parameters in the controller design procedure.

2.3 Robust Control Design: Class II

In this section, we investigate the problem of robust decentralized control for a wider class of large-scale nonlinear systems with parametric uncertainty and nonlinear interconnections. This class of systems was labeled in Sect. 2.1.2 as Class II. In this class, each subsystem of the interconnected system is assumed to be partially feedback linearizable and minimum phase. The uncertain parameters and/or disturbances are allowed to be time-varying and enter the system nonlinearly. The nonlinear interconnections are bounded by general nonlinear functions of the zero-dynamics and outputs of other subsystems. Inspired by the centralized nonlinear control results [9, 23, 35, 39, 51], we show in the sequel that decentralized global robust stabilization can be achieved for the uncertain interconnected systems by employing a Lyapunov-based recursive controller design method. Our result relies on a proper construction of Lyapunov function for the interconnected systems.

2.3.1 Construction Procedure

In what follows, we first present the following lemma which provides the first step of the induction in the construction of robust decentralized state feedback control laws of system (2.6).

Lemma 2.4 *Consider the first two state equations of system* (2.6):

$$\begin{aligned} \dot{z}_j &= f_{j0}(z_j, x_{j1}) + \phi_{j0}(z_j, x_{j1}, Z_j, X_{j1}; \theta)x_{j1}, \\ \dot{x}_{j1} &= x_{j2} + \phi_{j1}(z_j, x_{j1}, Z_j, X_{j1}, \theta), \\ y_j &= x_{j1}, \end{aligned} \tag{2.57}$$

satisfying Assumptions 2.1 *and* 2.2. *Then, there exists a smooth function* $x_{i2}^*(z_j, x_{j1})$ *with* $x_{j2}^*(0,0) = 0$ *such that system* (2.12) *with the control* $x_{j2} = x_{j2}^*(z_j, x_{j1})$ *in the coordinates*

$$z_j = z_j, \qquad \tilde{x}_{j1} = x_{j1}$$

satisfies

$$\dot{V}_{j1} \le \frac{dW_j(V_{j0})}{dV_{j0}} \frac{\partial V_{j0}}{\partial z_j} f_{h00} - b_j(z_j, x_{j1}) x_{j1}^2$$
$$- r\tilde{x}_{j1}^2 + \|z_j\|^2 + \frac{1}{2}\sum_{l=1}^{N} \delta_{j1l}(\|(z_l, x_{l1})\|), \tag{2.58}$$

where

$$V_{j1} = W_j(V_{j0}) + \frac{1}{2}\tilde{x}_{j1}^2, \tag{2.59}$$

with V_{j0} *given in Assumption 2.1*, $W_j(\cdot)$ *and* $b_j(\cdot,\cdot)$ *are*, *respectively*, *a smooth* $\mathcal{K}_\infty$*-function and a smooth function to be chosen*; *and*

$$f_{j00}(z_j) = f_{j0}(z_j, 0), \tag{2.60}$$
$$\delta_{j1l}(\|(z_l, x_{l1})\|) = \beta_{j0l}^{-1}(\zeta_{j0l}(\|(z_l, x_{l1})\|))^2 + \beta_{j1l}^{-1}(\zeta_{j1l}(\|(z_l, x_{l1})\|))^2, \tag{2.61}$$

with β_{j0l} *and* β_{j1l} *being positive scaling constants*.

Proof First, since $f_{j0}(z_j, x_{j1})$ of (2.12) is a smooth vector with $f_{j0}(0,0) = 0$, there exists a smooth vector $f_{j1}(z_j, x_{j1})$ such that

$$f_{j0}(z_j, x_{j1}) = f_{j00}(z_j) + f_{j1}(z_j, x_{j1}) x_{j1},$$

where $f_{j00}(z_j)$ is as in (2.60). By virtue of Assumption 2.2 and along the state trajectory of system (2.57), we have

$$\dot{V}_{j1} = \frac{dW_j}{dV_{j0}} \frac{\partial V_{j0}}{\partial z_j}(f_{j0} + \phi_{j0} x_{j1}) + x_{j1}[x_{j2} + \phi_{j1}(z_j, x_{j1}, Z_j, X_{j1}; \theta)]$$
$$= \frac{dW_j}{dV_{j0}} \frac{\partial V_{j0}}{\partial z_j}(f_{j00} + f_{j1} x_{j1}) + x_{j1} x_{j2} + x_{j1} \sum_{j=0}^{1} \psi_{j1}^{1}(z_j) \phi_{il}(z_j, x_{j1}, 0, 0; \theta)$$
$$+ x_{j1} \sum_{j=0}^{1} \psi_{j1}^{1}(z_j) \phi_{jl}(z_j, x_{j1}, Z_j, X_{j1}; \theta) - \phi_{jl}(z_j, x_{j1}, 0, 0, \theta)), \tag{2.62}$$

where

$$\psi_{j1}^{0}(z_j) = \frac{dW_j}{dV_{j0}} \frac{\partial V_{j0}}{\partial z_j}, \qquad \psi_{j1}^{1}(z_j) = 1.$$

Since $\phi_{j0}(0,0,0,0;\theta) = \phi_{j1}(0,0,0,0;\theta) = 0, \forall \theta$, there exists some function $\alpha_{j1}(z_j, x_{j1})$ such that

$$\left| x_{j1} \sum_{\iota=0}^{1} \psi_{i1}^{\iota}(z_j) \phi_{i\iota}(z_j, x_{j1}, 0, 0; \theta) \right| \le |x_{j1}| \alpha_{j1}(z_j, x_{j1})(\|z_j\| + \|x_{j1}\|). \tag{2.63}$$

In view of Assumption 2.2, it follows from (2.62) with some algebraic manipulations that

$$
\begin{aligned}
\dot{V}_{j1} \leq{}& \frac{dW_j}{dV_{j0}}\frac{\partial V_{j0}}{\partial z_j}(f_{j00}+f_{j1}x_{j1})+x_{j1}x_{j2}\\
&+|x_{j1}|\left|\frac{dW_j}{dV_{j0}}\right|\left\|\frac{\partial V_{j0}}{\partial z_j}\right\|\sum_{\ell=1}^{N}\eta_{j0\ell}(z_j,x_{j1})\zeta_{j0\ell}(\|(z_l,x_{\ell 1})\|)\\
&+|x_{j1}|\sum_{\ell=1}^{N}\eta_{j1\ell}(z_j,x_{j\ell})\zeta_{j1\ell}(\|(z_j,x_{\ell 1})\|)\\
&+|x_{j1}|\alpha_{j1}(z_j,x_{j1})(\|z_j\|+\|x_{j1}\|)\\
\leq{}& \frac{dW_j}{dV_{j0}}\frac{\partial V_{j0}}{\partial z_j}(f_{j00}+f_{j1}x_{j1})+x_{j1}(x_{j2}+x_{j1}\alpha_{j1}(z_j,x_{j1}))\\
&+\frac{1}{2}x_{j1}^2\left|\frac{dW_j}{dV_{j0}}\right|^2\left\|\frac{\partial V_{j0}}{\partial z_j}\right\|^2\sum_{\ell=1}^{N}\beta_{j0\ell}\eta_{j0\ell}^2(z_j,x_{j1})\\
&+\frac{1}{2}\sum_{\ell=1}^{N}\beta_{j0\ell}^{-1}(\zeta_{j0\ell}(\|(z_\ell,x_{\ell 1})\|))^2\\
&+\frac{1}{2}x_{i1}^2\sum_{l=1}^{N}\beta_{i1l}\eta_{i1l}^2(z_j,x_{i1})+\frac{1}{2}\sum_{l=1}^{N}\beta_{i1l}^{-1}(\zeta_{j1\ell}(\|(z_\ell,x_{\ell 1})\|))^2\\
&+\frac{1}{4}x_{j1}^2\alpha_{j1}^2(z_j,x_{j1})+\|z_j\|^2\\
={}& \frac{dW_j}{dV_{j0}}\frac{\partial V_{j0}}{\partial z_j}f_{j00}+x_{j1}(x_{j2}+M_{j1}(z_j,x_{j1}))\\
&+\|z_j\|^2+\frac{1}{2}\sum_{\ell=1}^{N}\delta_{j1\ell}(\|(z_\ell,x_{\ell 1})\|),
\end{aligned}
\tag{2.64}
$$

where $\delta_{j1\ell}$ is given in (2.61) and

$$
\begin{aligned}
M_{j1}(z_j,x_{j1}) ={}& \frac{dW_j}{dV_{j0}}\frac{\partial V_{j0}}{\partial z_j}f_{j1}+\frac{1}{2}x_{j1}\left|\frac{dW_j}{dV_{j0}}\right|^2\left\|\frac{\partial V_{j0}}{\partial z_j}\right\|^2\\
&\times\sum_{\ell=1}^{N}\beta_{j0\ell}\eta_{j0\ell}^2(z_j,x_{j1})+\frac{1}{2}\sum_{\ell=1}^{N}\beta_{j1\ell}\eta_{j1\ell}^2(z_j,x_{j1})\\
&+x_{j1}\alpha_{j1}(z_j,x_{j1})+\frac{1}{4}x_{j1}\alpha_{j1}^2(z_j,x_{j1}).
\end{aligned}
\tag{2.65}
$$

Now, select

$$x_{j2} = x^*_{j2} = -M_{j1} - b_j(z_j, x_{j1})x_{j1} - rx_{j1}, \tag{2.66}$$

where $b_j(\cdot,\cdot)$ is a smooth function to counteract the effect of the interconnections and is to be determined. Then, (2.58) is obtained and the proof of Lemma 2.4 is now completed. □

Remark 2.14 For the case when $r = 1$, that is, $x_{j2} = v_j$ in (2.57) is the actual control input, it can be shown, refer to the proof of Theorem 2.5, that the design functions $b_j(\cdot,\cdot)$ and $W_j(\cdot)$, $j = 1, 2, \dots, N$ can be chosen such that the decentralized state feedback control $v_j = x^*_{j2}(z_j, x_{j1})$ solves the robust decentralized stabilization problem.

2.3.2 Recursive Design

Next, we proceed toward the systematic recursive design methodology for constructing robust decentralized control laws for the system (2.6) when $r \geq 2$. A preliminary result is provided.

Lemma 2.5 *Consider the first $\rho + 1$ state equations of system* (2.6):

$$\begin{aligned}
\dot{z}_j &= f_{j0}(z_j, x_{j1}) + \phi_{j0}(z_j, x_{j1}, Z_j, X_{j1}; \theta)x_{j1}, \\
\dot{x}_{j1} &= x_{j2} + \phi_{j1}(z_j, x_{j1}, Z_j, X_{j1}; \theta), \\
\dot{x}_{j2} &= x_{j3} + \phi_{j2}(z_j, \bar{x}_{j2}, Z_j, X_{j1}; \theta), \\
&\vdots \\
\dot{x}_{j,\rho-1} &= x_{j,\rho} + \phi_{j,\rho-1}(z_j, \bar{x}_{j,\rho-1}, Z_j, X_{j1}; \theta), \\
\dot{x}_{j,\rho} &= x_{j,\rho+1} + \phi_{j,\rho}(z_j, \bar{x}_{j\rho}, Z_j, X_{j1}; \theta),
\end{aligned} \tag{2.67}$$

satisfying Assumptions 2.1 *and* 2.2. *Suppose that for any given index $\rho = m$ ($1 \leq m \leq r - 1$), there exist smooth functions*

$$\begin{aligned}
&x^*_{j2}(z_j, x_{j1}), \quad x^*_{j3}(z_j, \bar{x}_{j2}), \quad \dots, \quad x^*_{j,m+1}(z_j, \bar{x}_{jm}); \\
&x^*_{jk}(0, 0) = 0, \quad 2 \leq k \leq m + 1
\end{aligned}$$

such that system (2.67) *with the control $x_{j,m+1} = x^*_{j,m+1}(z_j, \bar{x}_{j,m})$ in the new coordinates*

$$\begin{aligned}
&z_j = z_j, \qquad \tilde{x}_{j1} = x_{j1}, \\
&\tilde{x}_{j2} = x_{j2} - x^*_{j2}(z_j, x_{j1}), \quad \dots, \quad \tilde{x}_{jm} = x_{jm} - x^*_{j,m}(z_j, \bar{x}_{j,m-1}),
\end{aligned}$$

satisfies

$$\dot{V}_{jm} \le \frac{dW_j}{dV_{j0}} \frac{\partial V_{j0}}{\partial z_j} f_{j00} - b_j(z_j, x_{j1}) x_{j1}^2 - (r - m + 1) \sum_{k=1}^{m} \tilde{x}_{jk}^2 + m \|z_j\|^2$$
$$+ \frac{1}{2} \sum_{\ell=1}^{N} \delta_{jm\ ell}(\|(z_\ell, x_{\ell 1})\|), \tag{2.68}$$

where

$$V_{im} = W_j(V_{j0}) + \frac{1}{2} \sum_{k=1}^{m} \tilde{x}_{jk}^2,$$

with V_{j0} as given in Assumption 2.1 and

$$\begin{aligned} &\delta_{j0\ell}(\|(z_\ell, x_{\ell 1})\|) \equiv 0, \\ &\delta_{jk\ell}(\|(z_\ell, x_{\ell 1})\|) = \delta_{j,k-1,\ell}(\|(z_\ell, x_{\ell 1})\|) \\ &\qquad + \sum_{\iota=0}^{k} \beta_{\ell\iota\ell}^{-1}(\zeta_{\ell\iota\ell}(\|(z_\ell, x_{\ell 1})\|))^2, \quad 1 \le k \le r. \end{aligned} \tag{2.69}$$

Then for system (2.67) with $\rho = m + 1$, there exists a smooth decentralized state feedback control law

$$x_{j,m+2} = x_{j,m+2}^*(z_j, \bar{x}_{j,m+1}); \qquad x_{j,m+2}^*(0, 0) = 0 \tag{2.70}$$

such that system (2.67) with (2.70) in the new coordinates

$$\begin{aligned} &z_j = z_j, \quad \tilde{x}_{jk}, \quad 1 \le k \le m, \\ &\tilde{x}_{j,m+1} = x_{j,m+1} - x_{j,m+1}^*(z_j, \bar{x}_{j,m}), \end{aligned}$$

satisfies

$$\dot{V}_{j,m+1} \le \frac{dW_j}{dV_{j0}} \frac{\partial V_{j0}}{\partial z_j} f_{j00} - b_j(z_j, x_{j1}) x_{j1}^2 - (r - m) \sum_{k=1}^{m+1} \tilde{x}_{jk}^2$$
$$+ (m + 1)\|z_j\|^2 + \frac{1}{2} \sum_{\ell=1}^{N} \delta_{\ell,m+1,\ell}(\|(z_\ell, x_{\ell 1})\|), \tag{2.71}$$

where

$$V_{j,m+1} = V_{jm} + \frac{1}{2} \tilde{x}_{j,m+1}^2.$$

Proof Initially, the derivative of $\tilde{x}_{j,m+1} = x_{j,m+1} - x^*_{j,m+1}$ is given by

$$\begin{aligned}\dot{\tilde{x}}_{j,m+1} &= x_{j,m+2} + a_{j,m+1}(z_j, \bar{x}_{j,m+1}) \\ &\quad + \sum_{\iota=0}^{m+1} \psi^{\iota}_{j,m+1}(z_j, \bar{x}_{j,m})\phi_{j\iota}(z_j, \bar{x}_{j\iota}, Z_j, X_{j1}; \theta),\end{aligned}$$

where

$$\begin{aligned}a_{j,m+1}(z_j, \bar{x}_{j,m+1}) &= -\frac{\partial x^*_{j,m+1}}{\partial z_j} f_{j0}(z_j, x_{j1}) - \sum_{\iota=1}^{m} \frac{\partial x^*_{j,m+1}}{\partial x_{j,\iota}} x_{j,\iota+1}, \\ \psi^0_{j,m+1}(z_j, \bar{x}_{j,m}) &= -\frac{\partial x^*_{j,m+1}}{\partial z_j} x_{j1}, \\ \psi^{\iota}_{j,m+1}(z_j, \bar{x}_{j,m}) &= -\frac{\partial x^*_{j,m+1}}{\partial x_{j,\iota}}, \quad 1 \le \iota \le m, \\ \psi^{m+1}_{j,m+1}(z_j, \bar{x}_{j,m}) &= 1.\end{aligned}$$

The time derivative of $V_{j,m+1}$ is given by

$$\begin{aligned}\dot{V}_{j,m+1} &= \dot{V}_{j,m} + \tilde{x}_{j,m+1}\Bigg[x_{j,m+2} + a_{j,m+1} \\ &\quad + \sum_{\iota=0}^{m+1} \psi^{\iota}_{j,m+1}(z_j, \bar{x}_{i,m})\phi_{j\iota}(z_j, \bar{x}_{j\iota}, Z_j, X_{j1}; \theta)\Bigg] \\ &= \dot{V}_{jm} + \tilde{x}_{j,m+1}(x_{j,m+2} + a_{j,m+1}) + \tilde{x}_{j,m+1} \sum_{\iota=0}^{m+1} \psi^{\iota}_{j,m+1}\phi_{i\iota}(z_j, \bar{x}_{i\iota}, 0, 0; \theta) \\ &\quad + \tilde{x}_{j,m+1} \sum_{\iota=0}^{m+1} \psi^{\iota}_{j,m+1}[\phi_{j\iota}(z_j, \bar{x}_{j\iota}, Z_j, X_{j1}; \theta) \\ &\quad - \phi_{j\iota}(z_j, \bar{x}_{j\iota}, 0, 0; \theta)]. \qquad (2.72)\end{aligned}$$

Define

$$\begin{aligned}\tilde{\phi}_{j\iota}(z_j, \bar{\tilde{x}}_{j\iota}; \theta) &= \phi_{j\iota}(z_j, \bar{x}_{j\iota}, 0, 0; \theta) \\ &= \phi_{j\iota}(z_j, \bar{\tilde{x}}_{j\iota} + \bar{x}^*_{j\iota}, 0, 0; \theta), \quad 2 \le \iota \le m+1 \qquad (2.73)\end{aligned}$$

where $\bar{\tilde{x}}_{j\iota} = (\tilde{x}_{j1}, \ldots, \tilde{x}_{j\iota})$ and $\bar{x}^*_{j\iota} = (x^*_{j1}, x^*_{j2}, \ldots, x^*_{j\iota})$ with $\bar{\tilde{x}}_{j0} = \tilde{x}_{j1}$ and $\bar{x}^*_{j0} = x^*_{j1}$.

Now since $\phi_{j\iota}(0, 0, 0, 0; \theta) = 0, \forall\theta \in \Omega, 0 \le \iota \le m+1$, it is easy to verify that $\tilde{\phi}_{j\iota}(0, 0; \theta) = 0, \forall\theta \in \Omega$. Thus, there exist smooth bounding functions $\alpha_{j\iota}(z_j, \bar{\tilde{x}}_{j,\iota})$,

$\iota = 0, 1, \ldots, m+1$ such that

$$
\begin{aligned}
&|\phi_{j0}(z_j, x_{j1}, 0, 0; \theta)| = |\tilde{\phi}_{j0}(z_j, \tilde{x}_{j1}; \theta) \le \alpha_{j0}(z_j, \tilde{x}_{j1})(\|z_j\| + \|\tilde{x}_{j1}\|), \\
&|\phi_{j\iota}(z_j, \bar{x}_{j\iota}, 0, 0; \theta)| = |\tilde{\phi}_{i\iota}(z_j, \bar{\tilde{x}}_{j\iota}; \theta) \le \alpha_{j\iota}(z_j, \bar{\tilde{x}}_{j,\ell})\left[\|z_j\| + \sum_{k=1}^{\ell} |\tilde{x}_{jk}|\right], \\
&\quad 1 \le \ell \le m+1.
\end{aligned}
\tag{2.74}
$$

Hence, the second last term of (2.72) satisfies

$$
\begin{aligned}
&\tilde{x}_{j,m+1} \sum_{\ell=0}^{m+1} \psi_{j,m+1}^{j\iota} \phi_{j\iota}(z_j, \bar{x}_{j\iota}, 0, 0; \theta) \\
&\quad \le |\tilde{x}_{j,m+1}| \left[\psi_{j,m+1}^{0} |\alpha_{j0}(\|z_j\| + |\tilde{x}_{j1}|) + \sum_{\ell=1}^{m+1} |\psi_{j,m+1}^{\ell}| \alpha_{j\iota} \left(\|z_j\| + \sum_{k=1}^{\ell} |\tilde{x}_{jk}| \right) \right] \\
&\quad = |\tilde{x}_{j,m+1}| \left[|\psi_{j,m+1}^{0}| \alpha_{j0}(\|z_j\| + |\tilde{x}_{j1}|) + \sum_{\ell=1}^{m} |\psi_{j,m+1}^{\ell}| \alpha_{i\iota} \left(\|z_j\| + \sum_{k=1}^{\ell} |\tilde{x}_{jk}| \right) \right] \\
&\qquad + |\tilde{x}_{j,m+1}| \alpha_{j,m+1} \left(\|z_j\| + \sum_{k=1}^{m} |\tilde{x}_{jk}| \right) + \alpha_{j,m+1} \tilde{x}_{j,m+1}^2 \\
&\quad \le \tilde{x}_{ij,m+1}^2 \sum_{\ell=0}^{m} (\psi_{j,m+1}^{\ell})^2 \alpha_{j\iota}^2 (m+1)(\ell+1) \\
&\qquad + \frac{1}{4(m+1)} \left[(\|z_j\| + |\tilde{x}_{j1}|)^2 + \sum_{\ell=1}^{m} \frac{1}{(\ell+1)} \left(\|z_j\| + \sum_{k=1}^{\ell} |\tilde{x}_{jk}| \right)^2 \right] \\
&\qquad + \frac{1}{2}(m+1) \tilde{x}_{j,m+1}^2 \alpha_{j,m+1}^2 + \frac{1}{2(m+1)} \left(\|z_j\| + \sum_{k=1}^{m} |\tilde{x}_{jk}| \right)^2 + \alpha_{j,m+1} \tilde{x}_{j,m+1}^2 \\
&\quad \le \tilde{x}_{j,m+1}^2 \sum_{\ell=0}^{m} (\psi_{j,m+1}^{\ell})^2 \alpha_{i\iota}^2 (m+1)(\ell+1) + \frac{1}{2}\|z_j\|^2 + \frac{1}{2} \sum_{k=1}^{m} |\tilde{x}_{jk}|^2 \\
&\qquad + \frac{1}{2}(m+1) \tilde{x}_{j,m+1}^2 \alpha_{j,m+1}^2 + \frac{1}{2} \left(\|z_j\|^2 + \sum_{k=1}^{m} |\tilde{x}_{jk}|^2 \right) + \alpha_{j,m+1} \tilde{x}_{j,m+1}^{*} \\
&\quad = \left[\sum_{\ell=0}^{m} (\psi_{j,m+1}^{\ell})^2 \alpha_{i\iota}^2 (m+1)(\ell+1) + \frac{1}{2}(m+1) \alpha_{j,m+1}^2 + \alpha_{j,m+1} \right] \tilde{x}_{j,m+1}^2 \\
&\qquad + \|z_{jk}\|^2 + \sum_{k=1}^{m} |\tilde{x}_{jk}|^2 \\
&\quad \le \tilde{x}_{j,m+1}^2 E_{j,m+1}(z_j, \bar{\tilde{x}}_{j,m+1}) + \|z_j\|^2 + \sum_{k=1}^{m} |\tilde{x}_{jk}|^2.
\end{aligned}
\tag{2.75}
$$

Invoking Assumption 2.2 and (2.75), it follows that (2.72) can be written as

$$
\begin{aligned}
\dot{V}_{j,m+1} \le{}& \dot{V}_{jm} + \tilde{x}_{j,m+1}(x_{j,m+2} + a_{j,m+1}) \\
&+ |\tilde{x}_{j,m+1}| \sum_{\ell=0}^{m+1} |\psi_{j,m+1}^{\ell}| \sum_{\ell=1}^{N} \eta_{j\iota\ell}(z_j, \bar{x}_{j\iota})\zeta_{j\ell\ell}(\|(z_\ell, x_{\ell 1})\|) \\
&+ \tilde{x}_{j,m+1}^2 E_{j,m+1} + \|z_j\|^2 + \sum_{k=1}^{m} |\tilde{x}_{jk}|^2 \\
\le{}& \dot{V}_{jm} + \tilde{x}_{j,m+1}(x_{j,m+2} + a_{j,m+1}) + \tilde{x}_{j,m+1}^2 E_{j,m+1} \\
&+ \|z_j\|^2 + \sum_{k=1}^{m} |\tilde{x}_{jk}|^2 \\
&+ \frac{1}{2}\tilde{x}_{j,m+1}^2 \sum_{\ell=0}^{m+1} \sum_{\ell=1}^{N} (\psi_{j,m+1}^{\ell})^2 (\eta_{j\iota\ell}(z_j, \bar{x}_{j\iota}))^2 \beta_{j\iota\ell} \\
&+ \frac{1}{2} \sum_{\ell=0}^{m+1} \sum_{\ell=1}^{N} (\zeta_{j\ell\ell}(\|(z_\ell, x_{\ell 1})\|))^2 \beta_{j\iota\ell}^{-1} \\
\le{}& \frac{dW_j}{dV_{j0}} \frac{\partial V_{j0}}{\partial z_j} f_{j00} - b_j(z_j, x_{j1}) x_{j1}^2 - (r - m + 1) \sum_{k=1}^{m} \tilde{x}_{jk}^2 \\
&+ m\|z_j\|^2 + \frac{1}{2} \sum_{\ell=1}^{N} \delta_{jm\ell}(\|(z_\ell, x_{\ell 1})\|) + \tilde{x}_{jm}\tilde{x}_{j,m+1} \\
&+ \tilde{x}_{j,m+1}(x_{j,m+2} + M_{j,m+1}) + \|z_j\|^2 + \sum_{k=1}^{m} \tilde{x}_{jk}^2 \\
&+ \frac{1}{2} \sum_{\iota=0}^{m+1} \sum_{\ell=1}^{N} (\zeta_{j\iota\ell}(\|(z_\ell, x_{\ell 1})\|))^2 \beta_{\ell\ell l}^{-1},
\end{aligned}
\tag{2.76}
$$

where

$$
\begin{aligned}
M_{i,m+1}(z_j, \bar{\bar{x}}_{j,m+1}) ={}& a_{j,m+1} + \tilde{x}_{j,m+1} E_{j,m+1} \\
&+ \frac{1}{2}\tilde{x}_{j,m+1} \sum_{\iota=0}^{m+1} \sum_{\ell=1}^{N} (\psi_{j,m+1}^{\ell})^2 (\eta_{j\iota\ell}(z_j, \bar{x}_{j\iota}))^2 \beta_{j\iota\ell}.
\end{aligned}
\tag{2.77}
$$

Select

$$
x_{j,m+2} = x_{j,m+2}^*(z_j, x_{j1}, \ldots, x_{j,m+1}) = -M_{j,m+1} - \tilde{x}_{jm} - (r - m)\tilde{x}_{j,m+1}. \tag{2.78}
$$

This makes (2.71) in Lemma 2.5 is valid, which completes the proof. □

By combining Lemmas 2.4 and 2.5 the construction of robust decentralized control law stabilizing the uncertain interconnected nonlinear systems (2.6) can be completed. This is demonstrated below.

Theorem 2.5 *Consider the uncertain interconnected system* (2.6) *satisfying Assumptions* 2.1 *and* 2.2. *Then there exists a decentralized control law*, $v_j = v_j(z_j, x_j)$, $j = 1, 2, \dots, N$, *such that the overall system with the decentralized controller is globally asymptotically stable for all admissible uncertainties and interconnections. Indeed*; *a suitable decentralized controller is given by*

$$v_j := x^*_{j,r+1}(z_j, \bar{x}_{j,r}) = -M_{jr} - \tilde{x}_{j,r-1} - \tilde{x}_{jr}, \tag{2.79}$$

where M_{jr} *is given in* (2.35) *with* $m + 1 = r$.

Proof By Lemma 2.4, it is not difficult to show that the induction hypotheses of Lemma 2.5 is satisfied. This motivates us to build a Lyapunov-based recursive decentralized control law by applying Lemma 2.5 repeatedly until the rth step. Therefore, we can construct $x^*_{j2}(z_j, x_{j1}), \dots, x^*_{j,r+1}(z_j, \bar{x}_{jr})$ such that under the new coordinates

$$z_j, \quad \tilde{x}_{j1} = x_{j1}, \quad \tilde{x}_{j2} = x_{j2} - x^*_{j2}(z_j, x_{j1}), \quad \dots, \quad \tilde{x}_{jr} = x_{jr} - x^*_{j,r}(z_j, \bar{x}_{j,r-1})$$

system (2.2) with control law (2.79) satisfies

$$\begin{aligned}\dot{V}_{jr} \le{}& \frac{dW_j}{dV_{j0}} \frac{\partial V_{j0}}{\partial z_j} f_{ij00} - b_j(z_j, x_{j1}) x_{j1}^2 - \sum_{k=1}^{r} \tilde{x}_{jk}^2 + r\|z_j\|^2 \\ &+ \frac{1}{2} \sum_{\ell=1}^{N} \delta_{jr\ell}(\|(z_\ell, x_{\ell 1})\|),\end{aligned} \tag{2.80}$$

where $V_{jr} = W_j(V_{j0}) + \frac{1}{2}\sum_{k=1}^{r} \tilde{x}_{jk}^2$ and

$$\begin{aligned}\delta_{jr\ell}(\|(z_\ell, x_{\ell 1})\|) ={}& r\beta_{j0\ell}^{-1}(\zeta_{j0\ell}(\|(z_\ell, x_{\ell 1})\|))^2 \\ &+ \sum_{\iota=1}^{r} (r - \iota + 1)\beta_{j\iota\ell}^{-1}(\zeta_{j\iota\ell}(\|(z_\ell, x_{\ell 1})\|))^2.\end{aligned} \tag{2.81}$$

By Assumption 2.2, we have

$$\begin{aligned}\delta_{jr\ell}(\|(z_\ell, x_{\ell 1})\|) ={}& r\beta_{j0\ell}^{-1}(\zeta_{j0\ell}^0(\|z_\ell\|)\|z_\ell\| + \zeta_{j0\ell}^1(z_\ell, x_{\ell 1})|x_{\ell 1}|)^2 \\ &+ \sum_{\ell=1}^{r} (r - \ell + 1)\beta_{j\iota\ell}^{-1}(\zeta_{j\iota\ell}^0(\|z_\ell\|)\|z_\ell\| + \zeta_{j\ell\ell}^1(z_\ell, x_{\ell 1})|x_{\ell 1}|)^2 \\ \le{}& 2r\beta_{j0\ell}^{-1}((\zeta_{j0\ell}^0(\|z_\ell\|))^2\|z_\ell\|^2 + (\zeta_{j0\ell}^1(z_\ell, x_{\ell 1}))^2 x_{\ell 1}^2)\end{aligned}$$

$$
\begin{aligned}
&+ 2\sum_{\ell=1}^{r}(r-\ell+1)\beta_{j\iota\ell}^{-1} \\
&\times ((\zeta_{j\ell\ell}^{0}(\|z_\ell\|))^2\|z_\ell\|^2 + (\zeta_{j\iota\ell}^{1}(z_\ell, x_{\ell 1}))^2 x_{\ell 1}^2) \\
\le{}& 2\Delta_{j\ell}(\|z_\ell\|)\|z_\ell\|^2 + 2D_{j\ell}(z_\ell, x_{\ell 1})x_{\ell 1}^2, \qquad (2.82)
\end{aligned}
$$

where

$$
\Delta_{j\ell}(\|z_\ell\|) = r\beta_{j0\ell}^{-1}(\zeta_{j0\ell}^{0}(\|z_\ell\|))^2 + \sum_{\iota=1}^{r}(r-\iota+1)\beta_{j\iota\ell}^{-1}(\zeta_{j\iota\ell}^{0}(\|z_\ell\|))^2, \qquad (2.83)
$$

$$
D_{j\ell}(z_\ell, x_{\ell 1}) = r\beta_{j0\ell}^{-1}(\zeta_{j0\ell}^{1}(z_\ell, x_{\ell 1}))^2 + \sum_{\iota=1}^{r}(r-\iota+1)\beta_{j\iota\ell}^{-1}(\zeta_{j\iota\ell}^{1}(z_\ell, x_{\ell 1}))^2. \quad (2.84)
$$

Define

$$
V = \sum_{i=1}^{N} V_j r.
$$

Observing the interconnection structural constraint

$$
\begin{aligned}
&\sum_{j=1}^{N}\sum_{\ell=1}^{N}[\Delta_{j\ell}(\|z_\ell\|)\|z_\ell\|^2 + D_{j\ell}(z_\ell, x_\ell)x_{\ell 1}^2] \\
&= \sum_{j=1}^{N}\sum_{\ell=1}^{N}[\Delta_{\ell j}(\|z_j\|)\|z_j\|^2 + D_{\ell j}(z_j, x_j)x_{j1}^2]
\end{aligned}
$$

and Assumption 2.1 and by noting that $W_j(V_{j0})$ is a $\mathcal{K}_\infty$ function of V_{j0}, we have

$$
\begin{aligned}
\dot{V}_{jr} \le{}& \sum_{j=1}^{N}\left\{\frac{dW_j}{dV_{j0}}\frac{\partial V_{j0}}{\partial z_j}f_{j00} - b_j(z_j, x_{j1})x_{j1}^2 - \sum_{k=1}^{r}\tilde{x}_{jk}^2 + r\|z_j\|^2\right. \\
&\left. + \sum_{\ell=1}^{N}[\Delta_{\ell j}(\|z_j\|)\|z_j\|^2 + D_{\ell j}(z_j, x_j)x_{j1}^2]\right\} \\
\le{}& \sum_{j=1}^{N}\left\{-\frac{dW_j}{dV_{j0}}\nu_j\|z_j\|^2 + \left[r + \sum_{\ell=1}^{N}\Delta_{\ell j}(\|z_j\|)\right]\|z_j\|^2\right. \\
&\left. - \sum_{k=1}^{r}\tilde{x}_{jk}^2 - \left[b_j(z_j, x_{j\ell})x_{j\ell}^2 - \sum_{\ell=1}^{N}D_{\ell j}(z_j, x_{j\ell})\right]x_{j\ell}^2\right\}. \qquad (2.85)
\end{aligned}
$$

Since $V_{j0}(z_j)$ in Assumption 2.1 is radially unbounded and positive definite, there exists a $\mathcal{K}_\infty$ function $\kappa_{\ell j}$ such that

$$\Delta_{\ell j}(\|z_j\|) \le \Delta_{\ell j}(0) + \kappa_{\ell j}(V_{j0}). \tag{2.86}$$

Now select

$$b_j(z_j, x_{j1}) = \sum_{\ell=1}^{N} D_{\ell j}(z_j, x_{j1}) \tag{2.87}$$

and

$$\frac{dW_j}{dV_{j0}} = k_j + \frac{1}{\nu_j}\left[r + \sum_{\ell=1}^{N}(\Delta_{\ell j}(0) + \kappa_{\ell j}(V_{j0}))\right], \qquad W_j(0) = 0, \tag{2.88}$$

where $k_j > 0$ is a constant. It is obvious that $W_j(\cdot)$ is a smooth $\mathcal{K}_\infty$-function. Then it follows that

$$\dot{V} \le \sum_{j=1}^{N}\left\{\left(-k_j \nu_j \|z_j\|^2 - \sum_{k=1}^{r} \tilde{x}_{jk}^2\right)\right\}. \tag{2.89}$$

Therefore, due to the onto-relation between (z_j, x_j) and $(z_j, \tilde{x}_j)$, where $\tilde{x}_j = (\tilde{x}_{j1}, \ldots, \tilde{x}_{ir})$, the closed-loop interconnected system of (2.2) with the decentralized controller (2.79) is globally asymptotically stable for all admissible uncertainties and interconnections. □

Remark 2.15 Observe from Theorem 2.5 that the functions $b_j(z_j, x_{i1})$ and $W_j(V_{i0})$, $i = 1, 2, \ldots, N$, can be chosen before we start the recursive design of the robust decentralized stabilization controller.

Remark 2.16 Theorem 2.5 presents a decentralized global stabilization result for uncertain interconnected minimum-phase nonlinear systems with parametric uncertainty and interconnections bounded by general nonlinear functions. This result extends centralized results in [35, 39] to decentralized control of large-scale interconnected systems.

2.3.3 Simulation Example 2.5

Consider the following large-scale system which is composed of two subsystems:

$$\begin{aligned} \textit{Subsystem } 1: \quad \dot{z}_1 &= -2z_1 + z_1 x_{11}, \\ \dot{x}_{11} &= x_{12} + x_{11} z_1 \sin\theta_1 + x_{21}^2 z_2 \cos\theta_1^2, \\ \dot{x}_{12} &= u_1 + x_{12}^2(x_{11} z_1 + z_1^2)\sin\theta_1 + x_{21} z_2 \cos(\theta_1 z_1); \end{aligned} \tag{2.90}$$

$$\begin{aligned} \textit{Subsystem } 2: \quad & \dot{z}_2 = -z_2 + x_{21}^2, \\ & \dot{x}_{21} = x_{22} + (x_{11}^2 z_1 + x_{21}^2 z_2)\sin(z_2\theta_2), \\ & \dot{x}_{22} = u_2 + x_{22}^2(x_{11}z_1^2 + x_{21}z_2^2)\sin\theta_2 + x_{22}^2 z_2^3 \cos(\theta_2^2 z_2^2), \end{aligned} \tag{2.91}$$

where $\theta_1, \theta_2 \in [-2, 2]$.

It is easy to verify that the interconnections in the above interconnected system satisfy Assumption 2.2. Choose $\beta_{jkm} = 1$, $j, k, m = 1, 2$. It follows from (2.83) and (2.84) that

$$\begin{aligned} & \Delta_{11} = \Delta_{12} = \Delta_{21} = \Delta_{22} = 0 \\ & D_{11} = 0, \qquad D_{12} = 2x_{21}^2 z_2^2 + z_2^2, \qquad D_{21} = 2x_{11}^2 z_1^2 + z_1^4, \qquad D_{22} = 0. \end{aligned}$$

1. Let $V_{10} = \frac{1}{2}z_1^2$ and $V_{20} = \frac{1}{2}z_2^2$. Then,

$$\frac{\partial V_{10}}{\partial z_1} f_{10}(z_1, 0) = -2z_1^2; \qquad \frac{\partial V_{20}}{\partial z_2} f_{20}(z_2, 0) = z_2^2.$$

Obviously, Assumption 2.1 is satisfied with $\nu_1 = 2$ and $\nu_2 = 1$.

It also follows from (2.86) that

$$\kappa_{11}(V_{10}) = \kappa_{21}(V_{10}) = \kappa_{12}(V_{20}) = \kappa_{22}(V_{20}) = 0.$$

By choosing $k_1 = k_2 = 3$, according to (2.87) and (2.88), we have

$$\frac{dW_1}{dV_{10}} = 4, \qquad \frac{dW_2}{dV_{20}} = 5$$

and

$$b_1 = D_{11} + D_{21}, \qquad b_2 = D_{12} + D_{22}.$$

It follows from (2.18) and (2.20) that

$$\alpha_{11} = x_{11}^2 + 0:25, \qquad \alpha_{21} = x_{21}^2$$

and

$$\begin{aligned} M_{11} &= \frac{dW_1}{dV_{10}} z_1^2 + 0.5x_{11} + x_{11}\alpha_{11} + 0.25\alpha_{11}^2, \\ M_{21} &= \frac{dW_2}{dV_{20}} z_2 x_{21} + 0.5x_{21} + x_{21}\alpha_{21} + 0.25\alpha_{21}^2. \end{aligned}$$

Hence, we can compute the virtual control

$$\begin{aligned} x_{12}^* &= -M_{11} - b_1 x_{11} - 2x_{11}, \\ x_{22}^* &= -M_{21} - b_2 x_{21} - 2x_{21}. \end{aligned}$$

2. Letting $\tilde{x}_{i2} = x_{i2} - x^*_{i2}$, $i = 1, 2$, we have

$$\psi^0_{12} = -\frac{\partial x^*_{12}}{\partial z_1} x_{11}, \qquad \psi^0_{22} = -\frac{\partial x^*_{22}}{\partial z_2} x_{21},$$

$$\psi^1_{12} = -\frac{\partial x^*_{12}}{\partial x_{11}}, \qquad \psi^1_{22} = -\frac{\partial x^*_{22}}{\partial x_{21}}, \qquad \psi^2_{21} = \psi^2_{22} = 1,$$

$$a_{12} = -\frac{\partial x^*_{12}}{\partial z_1}(-2z_1 + x_{11}z_1) - \frac{\partial x^*_{12}}{\partial x_{11}} x_{12},$$

$$a_{22} = -\frac{\partial x^*_{22}}{\partial z_2}(-z_2 + x^2_{21}) - \frac{\partial x^*_{22}}{\partial x_{21}} x_{22}.$$

According to (2.74), we can choose

$$\alpha_{12} = x^2_{12}(z^2_1 + 0.25), \qquad \alpha_{22} = x^2_{22} z^2_2.$$

Hence, it follows from (2.77) that

$$M_{12} = a_{12} + \tilde{x}_{12}(4(\psi^1_{12})^2\alpha^2_{11} + \alpha^2_{12} + \alpha_{12}) + 0.5\tilde{x}_{12}((\psi^1_{12})^2 + (\psi^2_{12})^2),$$
$$M_{22} = a_{22} + \tilde{x}_{22}(4(\psi^1_{22})^2\alpha^2_{21} + \alpha^2_{22} + \alpha_{22}) + 0.5\tilde{x}_{22}((\psi^1_{22})^2 + (\psi^2_{22})^2 x^4_{22}).$$

The control law can be obtained from (2.78) as follows:

$$u_1 = -x_{11} - M_{12} - \tilde{x}_{12}, \tag{2.92}$$

$$u_2 = -x_{21} - M_{22} - \tilde{x}_{22}. \tag{2.93}$$

Systems (2.90)–(2.91) were simulated with the controller (2.92) and (2.93) to demonstrate the effectiveness of the decentralized robust control design procedure. The initial conditions are set to be

$$z_1 = 1.0, \qquad x_{11} = -1.0, \qquad x_{12} = 1.5,$$
$$z_2 = 1.0, \qquad x_{21} = -1.0, \qquad x_{22} = 1.5$$

and the uncertainties θ_1 and θ_2 are given by $\theta_1 = 2\sin t$ and $\theta_2 = 2\cos t^2$. Obviously, the uncertainties are time-varying ones and belong to the set $[-2, 2]$. The closed-loop responses for the two subsystems are plotted in Figs. 2.1 and 2.2 from which the stability is clearly seen.

2.4 Decentralized Tracking: Class III

In this section, we attend to the problem of class III that was presented in Sect. 2.1.3. In the problem description there, attention was given to a class of large-scale nonlinear systems which is comprised of N interconnected subsystems with time-varying

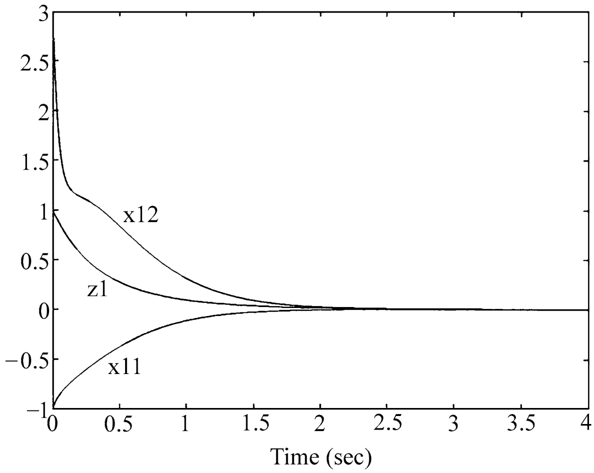

Fig. 2.1 Closed-loop responses of subsystem 1

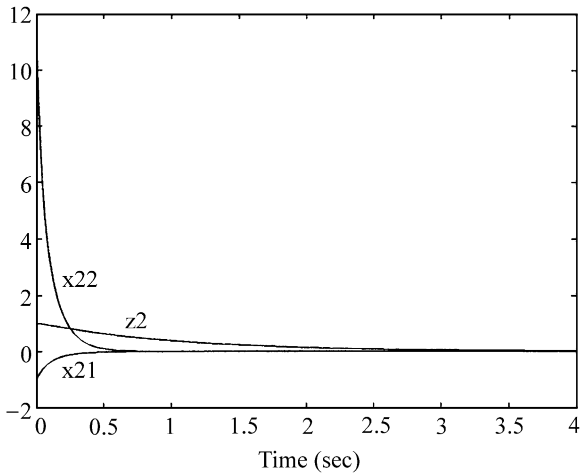

Fig. 2.2 Closed-loop responses of subsystem 2

unknown parameters and/or disturbances entering nonlinearly into the state equation as modeled by (2.9) and (2.10).

In what follows, we focus on studying the problem of decentralized output-feedback tracking with disturbance attenuation. Thus, with reference to the model (2.9) and (2.10), for every $1 \le j \le N$ and a given time-varying signal $y_{ir}(t)$ whose derivatives up to order n_j are bounded over $[0, \infty)$, our objective hereafter is to design a smooth, decentralized, dynamic, output-feedback controller of the form

$$\dot{x}_j = \nu_j(x_j, y_j, t), \quad u_j = \mu_j(x_j, y_j, t), \quad x_j \in \Re^{\bar{n}_j} \tag{2.94}$$

such that the following properties hold for the resulting closed-loop large-scale nonlinear system (2.11), (2.94):

1. *When the signal* $w_j \equiv 0$ *for all* $1 \le j \le N$, *the tracking error signal* $y_j - y_{ir}$ *goes to zero asymptotically and all other closed-loop signals remain bounded over* $[0, \infty)$.
2. *When* $w_j \neq 0$ *for all* $1 \le j \le N$, *the closed-loop system is bounded-input bounded-state* **BIBS** *stable and, in appropriate coordinates, is integral-input-to-state stable* **iISS** *with respect to the disturbance input* w [63]. *In particular, there exists a class-*$\mathcal{K}$ *function* γ_d (*that is,* γ_d *is continuous, strictly increasing and vanishes at the origin) such that, for any* $\rho > 0$, *the controller* (2.94) *can be tuned to satisfy the inequality*

$$\int_{t_0}^{t} |y(\tau) - y_r(t)|^2 d\tau \le \rho \int_0^t \gamma_d(|w(\tau)|) d\tau + \eta_0(z(0), x(0), x(0)) \quad \forall t \ge 0, \tag{2.95}$$

where η_0 is a nonnegative C^0 function, and

$$z(0) = [z_1^t(0), \ldots, z_N^t(0)]^t, \quad x(0) = [x_1^t(0), \ldots, x_N^t(0)]^t,$$
$$x(0) = [x_1^t(0), \ldots, x_N^t(0)]^t.$$

Remark 2.17 Property (1) above means that decentralized asymptotic tracking is achieved for each local jth subsystem (2.11) in the absence of disturbance inputs. Property (2) with (2.95) in implies that, in the presence of disturbances, the decentralized output-feedback controller (2.94) has the ability to attenuate the effect of the disturbances on the tracking error arbitrarily for a fixed class-K gain-function γ_d. As we shall see later, $\gamma_d(s) = s^2 + s^4 + s^8$ in our case.

In the sequel, sufficient conditions are provided to yield the standard $\mathcal{L}_2$-gain disturbance rejection property—that is, $\gamma_d(s) = s^2$ in (2.95). It is interesting to note that a similar problem has been studied in [41] in the framework of centralized output-feedback tracking with almost disturbance decoupling.

The control problem formulated above will be solved in two steps demonstrated in the following sections. We first introduce a (partially) decentralized observer in order to obtain an augmented decentralized system with partial-state information. Then, we base the decentralized controller design on this enlarged dynamic system.

2.4.1 Partially Decentralized Observer

Owing to the structure in every local system of (2.11), for each $1 \le j \le N$, we introduce the following state estimator for the (z_j, x_j)-subsystem:

$$\begin{aligned}
\dot{\hat{z}}_j &= Q_j \hat{z}_j + f_{j0}(y_{1r}, \ldots, y_{Nr}), \\
\dot{\hat{x}}_{j1} &= \hat{x}_{j2} + L_{j1}(y_j - x_{j1}) + f_{j1}(y_{1r}, \ldots, y_{Nr}) \\
&\quad + g_{j1}(y_{1r}, \ldots, y_{Nr})\hat{z}_j, \\
&\vdots \\
\hat{x}_{jn_j} &= u_j + L_{jn_j}(y_j - \hat{x}_{j1}) + f_{jn_j}(y_{1r}, \ldots, y_{Nr}) \\
&\quad + g_{jn_j}(y_{1r}, \ldots, y_{Nr})\hat{z}_j,
\end{aligned} \tag{2.96}$$

$$A_j = \begin{bmatrix} -L_{j1} & \\ -L_{j2} & I_{n_j-1} \\ \vdots & \\ -L_{jn_j} & 0\ldots0 \end{bmatrix}. \tag{2.97}$$

Notice that the eigenvalues of A_j can be assigned to any desired location in the open left-half plane via the choice of appropriate constants $\{L_{jm}\}_{m=1}^{n_m}$, provided complex conjugate eigenvalues appear in pair. In (2.97), I_{n_j-1} is the unit matrix of order $n_j - 1$.

Introducing the new variables

$$\tilde{z}_j = z_j - \hat{z}_j, \quad \tilde{x}_{jk} = x_{jk} - \hat{x}_{jk}, \quad 1 \le k \le n_j, \ 1 \le j \le N. \tag{2.98}$$

Then from (2.11) and (2.96), it follows that:

$$\begin{aligned}
\dot{\tilde{z}}_j &= Q_j \tilde{z}_j + f_{j0}(y_1, \ldots, y_N) - f_{j0}(y_{1r}, \ldots, y_{Nr}) \\
&\quad + p_{j0}(y_1, \ldots, y_N) w_j,
\end{aligned} \tag{2.99}$$

$$\begin{aligned}
\dot{\tilde{x}}_j &= A_j \tilde{x}_j + f_j(y_1, \ldots, y_N) - f_j(y_{1r}, \ldots, y_{Nr}) \\
&\quad + g_j(y_1, \ldots, y_N) z_j - g_j(y_{1r}, \ldots, y_{Nr})\hat{z}_j \\
&\quad + p_j(y_1, \ldots, y_N) w_j,
\end{aligned} \tag{2.100}$$

where

$$\begin{aligned}
\tilde{x}_j &= (\tilde{x}_{j1}, \ldots, \tilde{x}_{jn_j})^t, & f_j &= (f_{j1}, \ldots, f_{jn_j})^t, \\
g_j &= (g_{i1}, \ldots, g_{jn_j})^t, & p_j &= (p_{j1}, \ldots, p_{jn_j})^t.
\end{aligned}$$

Since every f_{jk} is a smooth function and every y_{jr} is a bounded signal, there exist a finite number of nonnegative smooth functions $\{\varphi_{j0k}\}_{k=1}^{N}$, $\{\varphi_{jk}\}_{k=1}^{N}$ such that

$$|f_{j0}(y_1, \ldots, y_N) - f_{j0}(y_{1r}, \ldots, y_{Nr})| \le \sum_{k=1}^{N} |\tilde{x}_{k1}| \varphi_{j0k}(\tilde{x}_{k1}), \tag{2.101}$$

$$|f_j(y_1, \ldots, y_N) - f_j(y_{1r}, \ldots, y_{Nr})| \le \sum_{k=1}^{N} |\tilde{x}_{k1}| \varphi_{jk}(\tilde{x}_{k1}). \tag{2.102}$$

In a similar way, we can obtain a functional bound for $g_j(y_1, \dots, y_N)z_j - g_j(y_{1r}, \dots, y_{Nr})\hat{z}_j$. Indeed, we have

$$
\begin{aligned}
& g_j(y_1, \dots, y_N)z_j - g_j(y_{1r}, \dots, y_{Nr})\hat{z}_j \\
& \quad = g_j(y_1, \dots, y_N)\tilde{z}_j + (g_j(y_1, \dots, y_N) - g_j(y_{1r}, \dots, y_{Nr}))\hat{z}_j. \qquad (2.103)
\end{aligned}
$$

Using the Mean-Value Theorem [29], there exist nonnegative smooth functions ϕ_{ik} $(1 \le k \le N)$ such that

$$
\begin{aligned}
& |g_j(y_1, \dots, y_N)z_j - g_j(y_{1r}, \dots, y_{Nr})\hat{z}_j| \\
& \quad \le |g_j(y_1, \dots, y_N)||\tilde{z}_j| + \sum_{k=1}^{N} |\tilde{x}_{k1}|\phi_{ik}(\tilde{x}_{k1})|\hat{z}_j|. \qquad (2.104)
\end{aligned}
$$

It must be noted that, by means of these inequalities (2.101)–(2.104), it is easy to show that, in the absence of disturbance inputs, the solutions $(\tilde{z}_j(t), \tilde{x}_j(t))$ of the cascade system (2.99)–(2.100) go to zero, if $y_j(t) - y_{ir}(t) \to 0$ for all $1 \le j \le N$. The latter property will be guaranteed with the help of the decentralized controller to be designed next.

Remark 2.18 It should be mentioned that the observer (2.96) is not asymptotic and is totally decentralized only if the reference signals $y_{jr} = 0$ for all $1 \le j \le N$. Proceeding further, we select a partially decentralized observer so that; in appropriate coordinates; the system (2.105) has an equilibrium point and therefore there is a solution to decentralized asymptotic tracking. In general, when $y_{jr}(t)$ are general time-varying signals, the system augmented with a totally decentralized observer does not have a fixed equilibrium. Thus, only practical tracking can be achieved by means of high-gain feedback [60].

2.4.2 Design Procedure

From the forgoing development of partially decentralized observers, we derive the following controller-observer combined system for the purpose of feedback design:

$$
\begin{aligned}
\dot{\tilde{z}}_j &= Q_j\tilde{z}_j + f_{j0}(y_1, \dots, y_N) - f_{i0}(y_{1r}, \dots, y_{Nr}) \\
&\quad + p_{j0}(y_1, \dots, y_N)w_j, \\
\dot{\tilde{x}}_j &= A_j\tilde{x}_j + f_j(y_1, \dots, y_N) - f_j(y_{1r}, \dots, y_{Nr}) \\
&\quad + g_j(y_1, \dots, y_N)z_j - g_j(y_{1r}, \dots, y_{Nr})\hat{z}_j \\
&\quad + p_j(y_1, \dots, y_N)w_j, \\
\dot{y}_j &= \hat{x}_{j2} + \tilde{x}_{i2} + f_{j1}(y_1, \dots, y_N) + g_{j1}(y_1, \dots, y_N)z_j
\end{aligned} \qquad (2.105)
$$

$$
\begin{aligned}
&\quad + p_{i1}(y_1, \ldots, y_N) w_j, \\
\dot{\hat{x}}_{j2} &= \hat{x}_{j3} + L_{i2}(y_j - \hat{x}_{i1}) + f_{j2}(y_{1r}, \ldots, y_{Nr}) \\
&\quad + g_{j2}(y_{1r}, \ldots, y_{Nr}) \hat{z}_j, \\
&\ \ \vdots \\
\dot{\hat{x}}_{in_j} &= u_j + L_{jn_j}(y_j - \hat{x}_{i1}) + f_{jn_j}(y_{1r}, \ldots, y_{Nr}) \\
&\quad + g_{jn_j}(y_{1r}, \ldots, y_{Nr}) \hat{z}_j.
\end{aligned}
$$

Notice that the state variables $(y_j, \hat{x}_{j1}, \hat{x}_{j2}, \ldots, \hat{x}_{jn_j})$, and then $\tilde{x}_{j1}$, are available for feedback design. Also note that the states $(\tilde{z}_j, \tilde{x}_j)$ are unmeasured and that the outputs y_j, with $j \neq i$, of other subsystems are unavailable for the design of the regional input u_j.

We now direct attention to the jth local system (2.105) with u_j as the control input. For the sake of clarity, the arguments of a function are often omitted in case no possible confusion arises. For notational simplicity, denote

$$\tilde{f}_{j0} = f_{j0}(y_1, \ldots, y_N) - f_{j0}(y_{1r}, \ldots, y_{Nr}), \tag{2.106}$$

$$\tilde{f}_j = f_j(y_1, \ldots, y_N) - f_j(y_{1r}, \ldots, y_{Nr}), \tag{2.107}$$

$$\tilde{g}_j = g_j(y_1, \ldots, y_N) z_j - g_j(y_{1r}, \ldots, y_{Nr}) \hat{z}_j. \tag{2.108}$$

A step-by-step constructive controller design procedure is now developed, leading to an improved solution to the decentralized problem under consideration with desired tracking controllers.

Step J.1: Starting with the first $(\tilde{z}_j, \tilde{x}_j, y_j)$-subsystem of (2.105). Introduce the new variable $\xi_{j1} = y_j - y_{jr}$ $(= \tilde{x}_{j1})$ and consider the Lyapunov function

$$V_{j1} = \lambda_{j1} \tilde{z}_j^t P_{j1} \tilde{z}_j + \lambda_{j2} (\tilde{z}_j^t P_{j1} \tilde{z}_{j1})^2 + \tilde{x}_j^t P_{j2} \tilde{x}_j + \frac{1}{2} \xi_{i1}^2, \tag{2.109}$$

where $\lambda_{j1}, \lambda_{j2} > 0$ are design parameters, $P_{j1} = P_{j1}^t > 0$ and $P_{i2} = P_{i2}^t > 0$ satisfy

$$P_{i1} Q_j + Q_j^t P_{i1} = -2 I_{n_{z_j}}, \tag{2.110}$$

$$P_{i2} A_j + A_j^t P_{i2} = -2 I_{n_j}. \tag{2.111}$$

This guarantees that $V_{j1} > 0$. Then by evaluating the time derivative of V_{i1} along the solutions of (2.105), we obtain

$$
\begin{aligned}
\dot{V}_{j1} &= (\lambda_{j1} + 2\lambda_{j2} \tilde{z}_j^t P_{j1} \tilde{z}_j)(-2|\tilde{z}_j|^2 + 2\tilde{z}_j^t P_{j1}(\tilde{f}_{j0} + p_{j0} w_j)) \\
&\quad - 2|\tilde{x}_j|^2 + 2\tilde{x}_j^t P_{j2}(\tilde{f}_j + \tilde{g}_j + p_j w_j) + \xi_{j1}(\hat{x}_{j2} + \tilde{x}_{j2} \\
&\quad + f_{j1}(y_1, \ldots, y_N) + g_{j1}(y_1, \ldots, y_N) z_j \\
&\quad + p_{j1}(y_1, \ldots, y_N) w_j - \dot{y}_{ir}).
\end{aligned} \tag{2.112}
$$

We first examine the term $2\tilde{z}_j^t P_{j1}(\tilde{f}_{j0} + p_{i0} w_j)$. Using (2.106) and (2.101), with the help of Young's inequality (see Chap. 9) and some algebraic manipulations, it follows that:

$$
\begin{aligned}
&2(\lambda_{j1} + 2\lambda_{j2}\tilde{z}_j^t P_{j1}\tilde{z}_j)\tilde{z}_j^t P_{j1}(\tilde{f}_{j0} + p_{j0} w_j) \\
&\quad \le \lambda_{j1}|\tilde{z}_j|^2 + \frac{3\lambda_{i2}}{\lambda_{\max}(P_{i1})}(\tilde{z}_j^t P_{j1}\tilde{z}_j)^2 + \sum_{k=1}^{N} \xi_{k1}^2 \psi_{ik1}(\xi_{k1}) \\
&\qquad + c_{j2}|w_j|^2 + c_{j3}|w_j|^4 + |w_j|^8,
\end{aligned}
\tag{2.113}
$$

where $c_{j1}, c_{j2}, c_{j3} > 0$ and ψ_{jk1} is a nonnegative smooth function.

In a similar way, there exist positive constants κ_{j1}, c_{j4} and a nonnegative smooth function ψ_{jk2} such that

$$
\begin{aligned}
&2\tilde{x}_j^t P_{j2}(\tilde{f}_j + \tilde{g}_j + p_j w_j) \\
&\quad \le |\tilde{x}_j|^2 + \kappa_{j1}|\tilde{z}_j|^2 + |\tilde{z}_j|^4 + \sum_{k=1}^{N} \xi_{k1}^2 \psi_{jk2}(\xi_{k1}) + c_{j4}|w_j|^2 + |w_j|^4,
\end{aligned}
\tag{2.114}
$$

where we have used the fact that $\hat{z}_j$ is bounded.

By substituting (2.113) and (2.114) into (2.112), we readily obtain

$$
\begin{aligned}
\dot{V}_{i1} \le{}& -(\lambda_{j1} + \lambda_{j2}\tilde{z}_j^t P_{j1}\tilde{z}_j)|\tilde{z}_j|^2 - |\tilde{x}_j|^2 + \sum_{k=1}^{N} \xi_{k1}^2(\psi_{ik1} + \psi_{jk2}) \\
&+ \kappa_{j1}|\tilde{z}_j|^2 + |\tilde{z}_j|^4 + (c_{j2} + c_{j4})|w_j|^2 + (c_{j3} + 1)|w_j|^4 \\
&+ |w_j|^8 + \xi_{j1}(\hat{x}_{i2} + \tilde{x}_{j2} + f_{j1}(y_1, \ldots, y_N) \\
&+ g_{j1}(y_1, \ldots, y_N)z_j + p_{j1}(y_1, \ldots, y_N)w_j - \dot{y}_{jr}).
\end{aligned}
\tag{2.115}
$$

It is significant to note that κ_{j1} does not depend on λ_{j1} and λ_{j2} while c_{jk}'s may depend on λ_{j1} and λ_{j2}.

Proceeding further, using (2.102) and (2.104), we have

$$
\begin{aligned}
&\xi_{j1}(\tilde{x}_{j2} + \tilde{f}_{j1} + \tilde{g}_{j1} + p_{j1} w_j) \\
&\quad \le \frac{1}{2}|\tilde{x}_j|^2 + \sum_{k=1}^{N} \xi_{k1}^2 \psi_{jk3}(\xi_{k1}) + |\tilde{z}_j|^2 + |w_j|^2,
\end{aligned}
\tag{2.116}
$$

where ψ_{jk3} is a nonnegative smooth function.

Taking into consideration the decomposition in (2.107) and (2.108) and letting $\hat{\psi}_{jk1} = \psi_{jk1} + \psi_{jk2} + \psi_{jk3}$, the following holds true:

$$\begin{aligned}\dot{V}_{j1} \le &-(\lambda_{j1} + \lambda_{j2}\tilde{z}_j^t P_{j1}\tilde{z}_j - \kappa_{j1} - 1 - |\tilde{z}_j|^2)|\tilde{z}_j|^2 \\ &- \frac{1}{2}|\tilde{x}_j|^2 + (c_{j2} + c_{j4} + 1)|w_j|^2 + (c_{j3} + 1)|w_j|^4 \\ &+ |w_j|^8 + \xi_{j1}(\hat{x}_{j2} + f_{j1}(y_{1r}, \dots, y_{Nr}) \\ &+ g_{j1}(y_{1r}, \dots, y_{Nr})\hat{z}_j - \dot{y}_{jr}) + \sum_{k=1}^{N} \xi_{k1}^2 \hat{\psi}_{ik1}. \end{aligned} \tag{2.117}$$

This motivates us to choose a control function $\xi_{j1}^\star$ and a new variable ξ_{j2} in the form

$$\begin{aligned}\xi_{j1}^\star = &-k_{j1}\xi_{j1} - \xi_{j1}K_j(\xi_{j1}) - f_{j1}(y_{1r}, \dots, y_{Nr}) \\ &- g_{j1}(y_{1r}, \dots, y_N r)\hat{z}_j + \dot{y}_{jr},\end{aligned} \tag{2.118}$$

$$\xi_{j2} = \hat{x}_{j2} - \xi_{j1}^\star(y_j, y_{1r}, \dots, y_{Nr}, \dot{y}_{jr}, \hat{z}_j), \tag{2.119}$$

where $k_{j1} > 0$ is a design parameter and K_j is a nonnegative, smooth function such that

$$K_{j1}(\xi_{j1}) \ge \sum_{k=1}^{N} \hat{\psi}_{kj1}(\xi_{j1}). \tag{2.120}$$

This leads us to

$$\begin{aligned}\dot{V}_{j1} \le &-(\lambda_{j1} + \lambda_{j2}\tilde{z}_j^t P_{j1}\tilde{z}_j - \kappa_{j1} - 1 - |\tilde{z}_j|^2)|\tilde{z}_j|^2 \\ &- \frac{1}{2}|\tilde{x}_j|^2 + (c_{j2} + c_{j4} + 1)|w_j|^2 \\ &+ (c_{j3} + 1)|w_j|^4 + |w_j|^8 - k_{i1}\xi_{j1}^2 - \xi_{j1}^2 K_j(\xi_{i1}) \\ &+ \sum_{k=1}^{N} \xi_{k1}^2 \hat{\psi}_{jk1}(\xi_{k1}) + \xi_{j1}\xi_{i2}. \end{aligned} \tag{2.121}$$

Step J.*k* $(2 \le k \le n_k)$: Consider the $(\tilde{z}_j, \tilde{x}_j, y_j, \hat{x}_{i2}, \dots, \hat{x}_{jk})$-subsystem of (2.105) with $\hat{x}_{j,jk+1}$ as the virtual control. For notational simplicity, we define $\hat{x}_{j,n_k+1} := u_k$.

Rolling over from Step J.1 to Step J.$k-1$, we assume that we have designed intermediate control functions $\{\xi_{j\ell}^\star\}_{\ell=1}^{k-1}$, and that we have introduced new variables

$$\begin{aligned}&\xi_{j,\ell+1} = \hat{x}_{j,\ell+1} - \xi_{j\ell}^\star(y_j, \hat{x}_{j2}, \dots, \hat{x}_{j\ell}, y_{\ell r}, \dots, y_{Nr}, \dot{y}_{jr}, \dots, y_{jr}^{(\ell)}, \hat{z}_j) \\ &\forall 1 \le \ell \le k-1\end{aligned} \tag{2.122}$$

and a positive-definite and proper function

$$V_{j,k-1}(\tilde{z}_j, \tilde{x}_j, \xi_{j\ell}, \dots, \xi_{j,k-1}) = V_{j\ell}(\tilde{z}_j, \tilde{x}_j, \xi_{j\ell} + \sum_{\ell=2}^{k-1} \frac{1}{2}\xi_{j\ell}^2. \tag{2.123}$$

It is further assumed that the time derivative of $V_{j,k-1}$ along the solutions of (2.105) satisfies

$$\begin{aligned}\dot{V}_{j,k-1} \leq & -(\lambda_{j1} + \lambda_{j2}\tilde{z}_j^t P_{j1}\tilde{z}_j - \kappa_{j1} - k + 1 - |\tilde{z}_j|^2)|\tilde{z}_j|^2 \\ & - \frac{1}{2^{k-1}}|\tilde{x}_j|^2 + (k - 1 + c_{k2} + c_{k4})|w_j|^2 \\ & + (c_{j3} + 1)|w_j|^4 + |w_j|^8 - \sum_{\ell=1}^{k-1} k_{j\ell}\xi_{j\ell}^2 - \xi_{j\ell}^2 K_j(\xi_{j\ell}) \\ & + \sum_{m=1}^{N} \xi_{m1}^2 \hat{\psi}_{jm(k-1)}(\xi_{m1}) + \xi_{j,k-1}\xi_{jk} \end{aligned} \tag{2.124}$$

with $k_{j\ell}$ $(1 \leq \ell \leq k-1)$ positive design parameters and $\hat{\psi}_{jm(k-1)}$ a nonnegative smooth function being independent of K_j.

The objective is to prove that a similar property to the above also holds for the subsystem

$$(\tilde{z}_j, \tilde{x}_j, y_j, \hat{x}_{j2}, \ldots, \hat{x}_{jk})$$

of (2.105) when $\hat{x}_{j,k+1}$ is considered as the (virtual) input.

Toward this end, consider the positive-definite and proper function

$$V_{jk} = V_{j,k-1}(\tilde{z}_j, \tilde{x}_j, \xi_{j1}, \ldots, \xi_{j,k-1}) + \frac{1}{2}\xi_{jk}^2. \tag{2.125}$$

Evaluating the time-derivative of V_{jk} along the solutions of (2.105) yields

$$\begin{aligned}\dot{V}_{jk} = \dot{V}_{j,k-1}\xi_{jk} \Bigg[& \hat{x}_{j,k+1} + L_{jk}(y_j - \hat{x}_{j1}) \\ & + f_{jk}(y_{1r}, \ldots, y_{Nr}) + g_{jk}(y_{1r}, \ldots, y_N r)\hat{z}_j \\ & - \sum_{m=2}^{k-1} \frac{\partial \xi_{j,k-1}^\star}{\partial \hat{x}_{jm}} (\hat{x}_{j,m+1} + L_{jm}(y_j - \hat{x}_{j1}) \\ & + f_{jm}(y_{1r}, \ldots, y_{Nr}) + g_{jm}(y_{1r}, \ldots, y_{Nr})\hat{z}_j) \\ & - \sum_{m=1}^{N} \frac{\partial \xi_{j,k-1}^\star}{\partial y_{mr}} \dot{y}_{mr} - \sum_{m=1}^{k-1} \frac{\partial \xi_{j,k-1}^\star}{\partial y_{jr}^{(m+1)}} y_{jr}^{(m+1)} \\ & - \frac{\partial \xi_{j,k-1}^\star}{\partial \hat{z}_j} (Q_j \hat{z}_j + f_{k0}(y_{1r}, \ldots, y_{Nr})) \\ & - \frac{\partial \xi_{j,k-1}^\star}{\partial y_j} (\hat{x}_{i2} + \hat{x}_{k2} + f_{k1} + g_{k1} z_j + p_{k1} w_j) \Bigg]. \end{aligned} \tag{2.126}$$

Adopting similar arguments to Step J.1, after algebraic routine manipulations, it follows the existence of nonnegative smooth functions $\{\psi_{jmk}\}_{m=1}^{N}$ and κ_{jk} such that:

$$-\xi_{jk}\frac{\partial\xi^{\star}_{j,k-1}}{\partial y_j}(\tilde{x}_{j2}+\tilde{f}_{j1}+\tilde{g}_{j1}+p_{j1}w_j)$$
$$\le\frac{1}{2^j}\tilde{x}_j^2+\xi_{jk}^2\kappa_{jk}+\sum_{m=1}^{N}\xi_{m1}^2\psi_{jmk}(\xi_{m1})+|\tilde{z}_j|^2+|w_j|^2. \qquad (2.127)$$

Observe that κ_{jk} is a function of

$$(y_j,\hat{x}_{j2},\dots,\hat{x}_{jk},y_{1r},\dots,y_{Nr},\dot{y}_{jr},\dots,y_{jr}^{(\ell)},\hat{z}_j)$$

and that every ψ_{jmk} does not depend on K_j.

This motivates us to select the following control function:

$$\begin{aligned}\xi^{\star}_{jk}=&-k_{jk}\xi_{jk}-\xi_{j,k-1}-\xi_{jk}\kappa_{jk}-L_{jk}(y_j-\hat{x}_{j1})\\&-f_{jk}(y_{1r},\dots,y_{Nr})-g_{jk}(y_{1r},\dots,y_{Nr})\hat{z}_j\\&+\frac{\partial\xi^{\star}_{j,k-1}}{\partial y_j}(\hat{x}_{j2}+f_{j1}(y_{1r},\dots,y_{Nr})+g_{j1}(y_{1r},\dots,y_{Nr})\hat{z}_j)\\&+\sum_{m=2}^{k-1}\frac{\partial\xi^{\star}_{j,k-1}}{\partial\hat{x}_{jm}}(\hat{x}_{j,m+1}+L_{jm}(y_j-\hat{x}_{j1})\\&+f_{jm}(y_{1r},\dots,y_{Nr})+g_{jm}(y_{1r},\dots,y_{Nr})\hat{z}_j)\\&+\sum_{m=1}^{N}\frac{\partial\xi^{\star}_{j,k-1}}{\partial y_{mr}}\dot{y}_{mr}+\sum_{m=1}^{j-1}\frac{\partial\xi^{\star}_{j,k-1}}{\partial y_{jr}^{(m)}}y_{jr}^{(m+1)}\\&+\frac{\partial\xi^{\star}_{j,k-1}}{\partial\hat{z}_j}(Q_j\hat{z}_j+f_{j0}(y_{1r},\dots,y_{Nr})),\end{aligned} \qquad (2.128)$$

where $k_{jk}>0$ is a design parameter.

Denoting $\xi_{j,k+1}=\hat{x}_{j,k+1}-\xi^{\star}_{jk}$ and combining (2.124) with (2.126)–(2.128), we obtain

$$\begin{aligned}\dot{V}_{jk}\le&-(\lambda_{j1}+\lambda_{j2}\tilde{z}_j^tP_{j1}\tilde{z}_j-\kappa_{j1}-j-|\tilde{z}_j|^2)|\tilde{z}_j|^2\\&-\frac{1}{2^j}|\tilde{x}_j|^2+(j+c_{j2}+c_{j4})|w_j|^2+(c_{j3}+1)|w_j|^4\\&+|w_j|^8-\sum_{\ell=1}^{j}k_{jl}\xi_{jl}^2-\xi_{j1}^2K_j(\xi_{j1})\\&+\sum_{m=1}^{N}\xi_{m1}^2(\hat{\psi}_{jm(k-1)}(\xi_{m1})+\psi_{jmk}(\xi_{m1}))+\xi_{jk}\xi_{j,k+1}.\end{aligned} \qquad (2.129)$$

That is, property (2.124) holds for the $(\tilde{z}_j, \tilde{x}_j, y_j, \hat{x}_{j2}, \ldots, \hat{x}_{jk})$-subsystem with

$$\hat{\psi}_{jmk} = \hat{\psi}_{ik(j-1)} + \psi_{ikj}.$$

By induction, at Step n_j, setting the control law

$$u_j = \xi^{\star}_{jn_j}(y_j, \hat{x}_{j2}, \ldots, \hat{x}_{jn_j}, y_{1r}, \ldots, y_{Nr}, \dot{y}_{ir}, \ldots, y_{jr}^{(n_j)}, \hat{z}_j) \tag{2.130}$$

leads us to

$$\begin{aligned}\dot{V}_{jn_j} \le & -(\lambda_{j1} + \lambda_{j2}\tilde{z}_j^t P_{j1}\tilde{z}_j - \kappa_{j1} - n_j - |\tilde{z}_j|^2)|\tilde{z}_j|^2 \\ & - \frac{1}{2^{n_j}}|\tilde{x}_j|^2 + (n_j + c_{i2} + c_{i4})|w_j|^2 \\ & + (c_{j3}+1)|w_j|^4 + |w_j|^8 - \sum_{\ell=1}^{n_j} k_{j\ell}\xi_{j\ell}^2 - \xi_{j1}^2 K_j(\xi_{j1}) \\ & + \sum_{m=1}^{N} \xi_{m1}^2 \hat{\psi}_{jmn_j}(\xi_{m1}),\end{aligned} \tag{2.131}$$

where by construction, $\hat{\psi}_{jmn_j}$ is independent of the design function K_j.

Consider now the positive-definite and proper Lyapunov function for the entire closed-loop interconnected system

$$V(\tilde{z}, \tilde{x}, \xi) = \sum_{j=1}^{N} V_{jn_j}(\tilde{z}_j, \tilde{x}_j, \xi_{j1}, \ldots, \xi_{jn_j}), \tag{2.132}$$

where

$$\tilde{z} = (\tilde{z}_1^t, \ldots, \tilde{z}_N^t)^t, \qquad \tilde{x} = (\tilde{x}_1^t, \ldots, \tilde{x}_N^t)^t, \qquad \xi = \xi_1^t, \ldots, \xi_N^t)^t.$$

Notice that the positive definiteness and properness of V in (2.132) follows from the foregoing recursive construction.

To eliminate the positive sum of the last term of (2.131), which also appears in the time derivative of V, we pick a set of appropriate smooth functions $\{K_j\}_{j=1}^N$ to check on the inequalities $(1 \le j \le N)$

$$K_j(\xi_{j1}) \ge \sum_{m=1}^{N} \hat{\psi}_{mjn_m}\xi_{j1}. \tag{2.133}$$

Obviously, such a design function K_j always exists.

2.4.3 Design Results

When applying the above-described control design to the uncertain large-scale system (2.11), we establish the following result.

Theorem 2.6 *The problem of decentralized output-feedback tracking with disturbance attenuation is solvable for the minimum-phase large-scale system* (2.11) *subject to Condition A.*

Proof By differentiating V defined by (2.132), along the solutions of the closed-loop system (2.11) and (2.130), it yields

$$\begin{aligned}\dot{V} \le & -\sum_{j=1}^{N}(\lambda_{j1}+\lambda_{j2}\tilde{z}_j^t P_{j1}\tilde{z}_j-\kappa_{j1}-n_j-|\tilde{z}_j|^2)|\tilde{z}_j|^2\\ & -\sum_{j=1}^{N}\left(\frac{1}{2^{n_j}}|\tilde{x}_j|^2+\sum_{\ell=1}^{n_j}k_{j\ell}\xi_{j\ell}^2\right)\\ & +\sum_{j=1}^{N}[(n_j+c_{j2}+c_{j4})|w_j|^2+(c_{j3}+1)|w_j|^4+|w_j|^8]. \end{aligned}\tag{2.134}$$

By selecting sufficiently large design parameters λ_1 and λ_2 such that

$$\begin{aligned}&(\lambda_{j1}+\lambda_{j2}\tilde{z}_j^t P_{j1}\tilde{z}_j-\kappa_{j1}-n_j-|\tilde{z}_j|^2)|\tilde{z}_j|^2\\ &\quad\ge\frac{\lambda_{j1}}{2}\tilde{z}_j P_{j1}\tilde{z}_j+\frac{\lambda_{j2}}{2}(\tilde{z}_j P_{j1}\tilde{z}_j)^2\end{aligned}\tag{2.135}$$

it follows from (2.134) and (2.132) that

$$\begin{aligned}\dot{V} \le -\lambda V &+\sum_{j=1}^{N}[(n_j+c_{j2}+c_{j4})|w_j|^2\\ &+(c_{j3}+1)|w_j|^4+|w_j|^8],\end{aligned}\tag{2.136}$$

where

$$\lambda=\min\left\{\frac{1}{2},\,1/2^{n_j}\lambda_{\max}(P_{j2}),\ k_{j\ell}\mid 1\le j\le N,\ 1\le\ell\le n_j\right\}.$$

The BIBS and iISS property (2) follows readily for the (transformed) closed-loop system (2.11), (2.130) by either applying the technique in [64] or the Gronwall-Bellman lemma [32] to (2.136). When $w_j=0$ for all $1\le j\le N$, the null solution is uniformly globally asymptotically stable (UGAS), leading to the asymptotic convergence of the tracking error $y-y_r$ because $\xi_1=y-y_r$.

Now from (2.134), for any pair of instants $0\le t_0\le t$, we obtain

$$\begin{aligned}\int_{t_0}^{t}|\xi_1(\tau)|^2 d\tau \le V(z(t_0),x(t_0),\xi(t_0))&+\rho\int_{t_0}^{t}(|w(\tau)|^2\\ &+|w(\tau)|^4+|w(\tau)|^8)d\tau\end{aligned}\tag{2.137}$$

where $\rho > 0$ is defined by

$$\rho = \max\left\{\frac{\max\{n_j + c_{j2} + c_{j3} | 1 \le j \le N\}}{\min\{k_{j1} | 1 \le j \le N\}}, \frac{\max\{c_{j3} + 1 | 1 \le j \le N\}}{\min\{k_{j1} | 1 \le i \le N\}}, \frac{1}{\min\{k_{j1} | 1 \le j \le N\}}\right\}.$$

It must be noted that ρ can be made as small as possible by selecting sufficiently large values of the constants k_{j1}. In the present case, (2.95) is met with $\gamma_d(s) = s^2 + s^4 + s^8$. The proof of Theorem 2.6 is now completed. □

Remark 2.19 It is of interest to observe that, in the absence of disturbance inputs w, (2.136) yields that V converges to zero at an exponential rate and; therefore; the tracking error $y(t) - y_r(t)$ goes to zero exponentially.

Remark 2.20 By similarity to the centralized output-feedback tracking with almost disturbance decoupling [41], Condition A can be weakened and the z_j-system in (2.11) can be broadened as follows:

$$\dot{z}_j = \Gamma_j(y_1, \dots, y_N) z_j + f_{i0}(y_1, \dots, y_N) + p_{i0}(y_1, \dots, y_N) w_j. \quad (2.138)$$

Assume that, for each $1 \le j \le N$, there are a pair of constant matrices $(0 < P_j = P_j^t, 0 < M_j = M_j^t)$ such that

$$\Gamma_j^t(y_1, \dots, y_N) P_j + P_j \Gamma_j(y_1, \dots, y_N) \le -M_j. \quad (2.139)$$

Under this hypothesis, the $\hat{z}_j$-system in the decentralized observer (2.96) is replaced by

$$\dot{\hat{z}}_j = \Gamma_j(y_{1r}, \dots, y_{Nr}) \hat{z}_j + f_{j0}(y_{1r}, \dots, y_{Nr}). \quad (2.140)$$

Using the same techniques as in Sect. 2.4.2, Theorem 2.6 can be extended to this situation.

To proceed further, we examine the situation when the developed controller design procedure yields a decentralized output-feedback law guaranteeing the standard $\mathcal{L}_2$-gain disturbance attenuation property (2.95) holds with $\gamma_d(s) = s^2$. The following additional sufficient condition is recalled.

Condition B For all $1 \le j \le N$ and $1 \le k \le n_k$, the function p_{jk} is bounded by a constant. Furthermore, $p_{j0} = 0$ for each $1 \le j \le N$.

The following lemma provides the desired result:

Lemma 2.6 *Under Condition A and Condition B, the problem of decentralized output-feedback tracking with $\mathcal{L}_2$-gain disturbance attenuation is solvable for the class of minimum-phase large-scale systems (2.11).*

Proof We initially note that the only place where $|w_j|^4$ and $|w_j|^8$ occur is Step J.1 during the controller development in Sect. 2.4.2. More precisely, they are brought up in the inequalities (2.113) and (2.114). Under Condition B, the function V_{j1} satisfies the following inequality, instead of (2.121):

$$\begin{aligned}\dot{V}_{j1} \le& -(\lambda_{j1} + \lambda_{j2}\tilde{z}_j^t P_{j1}\tilde{z}_j - \kappa_{j1} - 1 - |\tilde{z}_j|^2)|\tilde{z}_j|^2 \\ &- \frac{1}{2}|\tilde{x}_j|^2 + (c_{j2} + c_{j4} + 1)|w_j|^2 - k_{j1})\xi_{j1}^2 \\ &- \xi_{i1}^2 K_j(\xi_{i1}) + \sum_{m=1}^{N} \xi_{m1}^2 \hat{\psi}_{jm1}(\xi_{m1}) + \xi_{j1}\xi_{j2}. \end{aligned} \tag{2.141}$$

The above Lyapunov function V satisfies

$$\dot{V} \le -\lambda V + \sum_{j=1}^{N} [(n_j + c_{j2} + c_{j4})|w_j|^2]. \tag{2.142}$$

From (2.142), the standard $\mathcal{L}_2$-gain property from w to $\xi_1 = y - y_r$ follows readily. The proof of Lemma 2.6 is thus completed. □

Remark 2.21 As an immediate corollary of Theorem 2.6, the standard $\mathcal{L}_2$-gain property from w to $\xi_1 = y - y_r$ can also be established when all functions f_{jk}, g_{jk} in the decentralized system (2.11) are bounded by linear functions and the functions p_{jk} $(1 \le j \le N, 0 \le k \le n_k)$ are bounded by some constants (in this case, $p_{j0} \ne 0$). The derived decentralized output-feedback controllers are linear.

Remark 2.22 The main features are four-fold:

(i) identifying a wide class of large-scale nonlinear systems in disturbed decentralized output-feedback form;
(ii) proposing an effective systematic output-feedback controller design procedure for decentralized systems in the presence of strong nonlinearities appearing in the subsystems and interactions and
(iii) guaranteeing decentralized asymptotic tracking when the disturbance inputs disappear and achieving desirable external stability properties when the disturbance inputs are present;
(iv) extending further the earlier results of [23, 29, 32, 40] to uncertain large complex systems.

2.5 Decentralized Guaranteed Cost Control

In recent years, the problem of the decentralized robust control of large-scale systems with parameter uncertainties has been widely studied. Although there have been numerous studies on the decentralized robust control of large-scale uncertain systems, much effort has been made toward finding a controller that guarantees robust stability. However, when controlling such systems, it is also desirable to design control systems that guarantee not only robust stability but also an adequate level of performance. One approach to this problem is the so-called guaranteed cost control approach [47]. This approach has the advantage of providing an upper bound on a given performance index.

Recent advances in the LMI theory have allowed a revisiting of the guaranteed cost control approach [82]. In [82], the guaranteed cost control technique for interconnected systems by means of the LMI approach has been discussed. In the literature, the guaranteed cost control for nonlinear uncertain large-scale systems under gain perturbations has been considered. However, the time delays have not been considered in those reports. If the system does not have delays, the theoretical behavior would usually be more tractable. However, if delays are present, they may result in instability or serious deterioration in the performance of the resulting control systems. Therefore, the study of the control, considering these time delays on the guaranteed cost stability, is very important.

In what follows, the guaranteed cost control problem of the decentralized robust control for uncertain nonlinear large-scale systems that have delay in both state and control input is considered. It should be noted that although the robust control design method for parameter uncertain ordinary dynamic systems that have delay in both state and control input has been considered, the guaranteed cost control for nonlinear uncertain large-scale systems that have delay in both state and control input has never been discussed. A sufficient condition for the existence of the decentralized robust feedback controllers is derived in terms of the LMI. The main result shows that the guaranteed cost controllers can be constructed by solving the LMI. The crucial difference between the existing results [82] and that of the present study is that the controller that guarantees the stability and the adequate level of performance of the large-scale delay systems is given. Thus, the applicability of the resulting controllers can be extended to more practical large-scale systems. Moreover, since the construction of the guaranteed cost controller consists of an LMI-based control design, the proposed method is computationally attractive and useful.

2.5.1 Analysis of Robust Performance

To demonstrate ideas, we consider in the sequel a class of continuous-time autonomous uncertain nonlinear large-scale interconnected delay systems, which consist of N subsystems of the form:

$$\begin{aligned} \dot{x}_j(t) &= [\bar{A}_j + \Delta\bar{A}_j(t)]x_j(t) + [A_j^d + \Delta A_j^d(t)]xi(t-\tau_j) \\ &\quad + [H_j^d + \Delta H_j^d(t)]x_j(t-h_j) \\ &\quad + \sum_{j=1, j\neq k}^{N} [G_{kj} + \Delta G_{kj}(t)]g_{kj}(x_j, x_k), \end{aligned} \tag{2.143}$$

$$\begin{aligned} x_j(t) &= \phi_j(t), \quad t \in [-d_j, 0], \\ d_j &= \max\{\tau_j, h_j\}, \quad j = 1, \ldots, N, \end{aligned} \tag{2.144}$$

where $x_j(t) \in \Re^{n_j}$ are the states. $\tau_j > 0$ and $h_j > 0$ are the delay constants, and $\phi_j(t)$ are the given continuous vector valued initial functions. $\bar{A}_j$, A_j^d, and H_j^d are the constant matrices of appropriate dimensions. $G_{ij} \in \Re^{n_j \times l_j}$ are the interconnection matrices between the ith subsystems and other subsystems. $g_{kj}(x_j, x_k) \in \Re^{\ell_j}$ are unknown nonlinear vector functions that represent nonlinearity. The parameter uncertainties considered here are assumed to be of the following form:

$$[\Delta\bar{A}_j(t)\Delta A_j^d(t)\Delta H_j^d(t)] = D_j F_j(t)[\bar{E}_j^1 E_j^{1d} \bar{E}_j^{d_h}], \tag{2.145}$$

$$\Delta G_{jk}(t) = D_{jk} F_{jk}(t) E_{jk}, \tag{2.146}$$

where D_j, $\bar{E}_j^1$, E_j^{1d}, $\bar{E}_j^{d_h}$, D_{ij}, and E_{ij} are known constant real matrices of appropriate dimensions. $F_j(t) \in \Re^{p_j \times q_j}$ and $F_{ij}(t) \in \Re^{r_{ij} \times s_{jk}}$ are unknown matrix functions with Lebesgue measurable elements and satisfy

$$F_j^t(t)F_j(t) \le I_{qi}, \qquad F_{ij}^t(t)F_{ij}(t) \le I_{sij}. \tag{2.147}$$

We make the following assumptions concerning the unknown nonlinear vector functions.

(A1) *There exist known constant matrices* V_j *and* W_{jk} *such that for all* $j, k, t \ge 0, x_j \in \Re^{n_j}$ and $x_j \in \Re^{n_j}$

$$\|g_{jk}(x_j, x_k)\| \le \|V_j x_j\| + \|W_{jk} x_j\|.$$

(A2) *For all* j, k

$$U_j := 2 \sum_{j=1, j\neq k}^{N} (V_j^t V_j + W_{jk}^t W_{jk}) > 0.$$

The cost function of the associated system (2.143) is given as

$$J = \sum_{j=1}^{N} \int_0^\infty x_j^t(t)\bar{Q}_j x_j(t)dt, \quad 0 < \bar{Q}_j = \bar{Q}_j^t. \tag{2.148}$$

The following definition of the cost matrix for the uncertain large-scale interconnected delay systems is given in [47]:

Definition 2.1 The set of matrices $0 < P_j = P_j^t$ is said to be the quadratic cost matrix for the uncertain nonlinear large-scale interconnected delay systems (2.143) if the following inequality holds

$$\sum_{i=1}^{N}\left(\frac{d}{dt}x_j^t(t)P_jx_j(t) + x_j^t(t)\bar{Q}_jx_j(t)\right) < 0, \tag{2.149}$$

for all nonzero $x_j \in \Re^{n_j}$ and all uncertainties (2.145).

Theorem 2.7 *Under assumptions* (A1) *and* (A2), *suppose there exist matrices* $0 < P_j = P_j^t \in \Re^{n_j \times n_j}, 0 < S_j = S_j^t \in \Re^{n_j \times n_j}, 0 < T_j = T_j^t \in \Re^{n_j \times n_j}$ *such that for all admissible uncertainties satisfying* (2.145) *the following matrix inequality holds*:

$$\Lambda_j = \begin{bmatrix} \Xi_j & P_j\tilde{A}_j^d & P_j\tilde{H}_j^d & P_j\tilde{G}_{j1} & \dots & P_j\tilde{G}_{jN} \\ \bullet & -S_j & 0 & 0 & \dots & 0 \\ \bullet & \bullet & -T_j & 0 & \dots & 0 \\ \bullet & \bullet & \bullet & -I_{l_1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \bullet & \bullet & \bullet & \bullet & \dots & -I_{l_N} \end{bmatrix} < 0, \tag{2.150}$$

where

$$\Lambda_j \in \Re^{\bar{N} \times \bar{N}}, \quad \bar{N} = 3n_j + \sum_{m=1, j \neq m}^{N} \ell_m,$$

$$\Xi_j := \tilde{A}_j^t P_j + P_j \tilde{A}_j + U_j + \bar{Q}_j + S_j + T_j, \quad \tilde{A}_j := \bar{A}_j + \Delta A_j(t),$$

$$\tilde{A}_j^d := A_j^d + \Delta A_j^d(t), \qquad \tilde{H}_j^d := H_j^d + \Delta H_j^d(t),$$

$$\tilde{G}_{jk} := G_{jk} + \Delta G_{jk}(t).$$

Then the free uncertain nonlinear large-scale interconnected systems (2.143) *are quadratically stable, and the corresponding value of the cost function* (2.148) *satis-*

fies the following inequality:

$$J < \sum_{i=1}^{N}\Big[\phi_j^t(0)P_j\phi_j(0) + \int_{-\tau_j}^{0}\phi_j^t(s)S_j\phi_j(s)ds + \int_{-h_j}^{0}\phi_j^t(s)T_j\phi_j(s)ds\Big]. \qquad (2.151)$$

Proof Based on the definitions $\tilde{A}_j, \tilde{A}_j^d, \tilde{H}_j^d$ and $\tilde{G}_{jk}$, we can change the form (2.143) to

$$\dot{x}_j(t) = \tilde{A}_j x_j(t) + \tilde{A}_j^d x_j(t-\tau_j) + \tilde{H}_j^d x_j(t-h_j) + \sum_{k=1, j\neq k}^{N}\tilde{G}_{jk}g_{jk}(x_j, x_k). \qquad (2.152)$$

There exist matrices $0 < P_j = P_j^t \in \Re^{n_j\times n_j}$, $0 < S_j = S_j^t \in \Re^{n_j\times n_j}$, $0 < T_j = T_j^t \in \Re^{n_j\times n_j}$, $j = 1, \ldots, N$ such that the matrix inequality (2.150) holds for all admissible uncertainties (2.145). To prove the asymptotic stability of the interconnected delay systems (2.152), we introduce the following Lyapunov function candidate

$$V\big(x(t)\big) = \sum_{i=1}^{N}\Big[x_j^t(t)P_jx_j(t) + \int_{t-\tau_j}^{t}x_j^t(s)S_jx_j(s)ds + \int_{t-h_j}^{t}x_j^t(s)T_jx_j(s)ds\Big], \qquad (2.153)$$

where $x(t) = [x_1^t(t) \ \ldots \ x_N^t(t)]^t$. Note by default that $V(x(t)) > 0$ whenever $x(t) \neq 0$. The time derivative of $V(x(t))$ along any trajectory of the interconnected delay systems (2.152) is given by

$$\frac{d}{dt}V(x(t)) = \sum_{i=1}^{N}z_j^t(t)\Lambda_j z_j(t) - \sum_{i=1}^{N}x_j^t(t)\bar{Q}_j x_j(t) - \sum_{i=1}^{N}\sum_{k=1, j\neq k}^{N}(2x_j^tV_j^tV_jx_j + 2x_j^tW_{jk}^tW_{jk}x_j - g_{jk}^tg_{jk}),$$

where

$$z_j = [x_j^t(t) \ x_j^t(t-\tau_j) \ x_j^t(t-h_j) \ g_{j1}^t \ \ldots \ g_{jN}^t]^t \in \Re^{\bar{N}}$$

and Ξ_j and Λ_j are given in (2.151).

Under assumption (A1), it is easy to verify that the following inequality holds

$$2x_j^tV_j^tV_jx_j + 2x_j^tW_{jk}^tW_{jk}x_j \geq g_{jk}^tg_{jk}. \qquad (2.154)$$

With inequalities (2.150) and (2.154) hold, it immediately follows that

$$\frac{d}{dt}V(x(t)) < -\sum_{j=1}^{N} x_j^t(t)\bar{Q}_j x_j(t) < 0, \tag{2.155}$$

which assures that $V(x(t))$ is a Lyapunov function for the interconnected delay system (2.152). Therefore, system (2.152) is asymptotically stable. Furthermore, by integrating both sides of the inequality (2.155) from 0 to T and using the initial conditions, we obtain

$$V(x(T)) - V(x(0)) < -\sum_{j=1}^{N} \int_0^t x_j^t(t)\bar{Q}_j x_j(t)dt. \tag{2.156}$$

Since system (2.152) is asymptotically stable, that is, $x(T) \to 0$ when $T \to \infty$, we obtain $V(x(T)) \to 0$. Thus we obtain

$$\begin{aligned} J &= \sum_{j=1}^{N} \int_0^t x_m^t(t)\bar{Q}_j x_j(t)dt < V(x(0)) \\ &= \sum_{j=1}^{N} \left[\phi_j^t(0)P_j\phi_j(0) + \int_{-\tau_j}^{0} \phi_j^t(s)S_j\phi_j(s)ds + \int_{-h_j}^{0} \phi_j^t(s)T_j\phi_j(s)ds \right]. \end{aligned}$$

This completes the proof of Theorem 2.7. □

2.5.2 *Including Input Delays*

In what follows, we consider the problem of decentralized guaranteed cost control via the state feedback to the class of nonlinear uncertain interconnected systems with input delays. The class of system under consideration is described by

$$\begin{aligned} \dot{x}_j(t) &= [A_j + \Delta A_j(t)]x_j(t) + [B_j + \Delta B_j(t)]u_j(t) \\ &\quad + [A_{dj} + \Delta A_{dj}(t)]x_j(t-\tau_j) + [B_{dj} + \Delta B_{dj}(t)]u_j(t-h_j) \\ &\quad + \sum_{k=1, j\neq k}^{N} [G_{jk} + \Delta G_{jk}]g_{jk}(x_j, x_k), \end{aligned} \tag{2.157}$$

$$x_j(t) = \phi_j(t), \quad t \in [-d_j, 0], \quad d_j = \max\{\tau_j, h_j\}, \quad j = 1, \ldots, N, \tag{2.158}$$

where $u_j(t) \in \Re^{m_j}$ are the control inputs of the jth subsystems. The parameter uncertainties satisfy

$$[\Delta A_j(t)\ \Delta B_j(t)\ \Delta A_{dj}(t)\ \Delta B_{dj}(t)] = D_j F_j(t)[E_{1j}\ E_{2j}\ E_{1dj}\ E_{2dj}]. \tag{2.159}$$

$A_j, B_j, E_{1j}, E_{2j}, E_{d1j}, E_{d2j}$ are constant matrices of appropriate dimensions. The remaining constant real matrices and parameter uncertainties are the same as those in system (2.143). Moreover, it is assumed that Assumptions (A1) and (A2) hold for the unknown nonlinear vector functions $g_{jk}(x_j, x_k) \in \Re^{\ell_j}$.

Associated with system (2.157) is the cost function

$$J = \sum_{j=1}^{N} \int_0^\infty [x_j^t(t) Q_j x_j(t) + u_j^t(t) R_j u_j(t)] dt,$$
$$0 < Q_j = Q_j^t, \quad 0 < R_j = R_j^t. \tag{2.160}$$

In view of the results of [47], the definition of the guaranteed cost control for the class of uncertain interconnected systems (2.157) is now provided:

Definition 2.2 A decentralized control law $u_j(t) = K_j x_j(t)$ is said to be a quadratic guaranteed cost control related to the set of matrices $0 < P_j = P_j^t$ for the uncertain interconnected system (2.157) and cost function (2.160) if the closed-loop system is quadratically stable and the closed-loop value of the cost function (2.160) satisfies the bound $J \leq J^*$ for all admissible uncertainties, that is,

$$\sum_{j=1}^{N} \left(\frac{d}{dt} x_j^t(t) P_j x_j(t) + x_j^t(t)[Q_j + K_j^t R_j K_j] x_j(t) \right) < 0, \tag{2.161}$$

for all nonzero $x_j \in \Re^{n_j}$.

The objective now is to design a decentralized guaranteed cost controller

$$u_j(t) = K_j x_j(t), \quad j = 1, \ldots, N,$$

for the uncertain large-scale interconnected delay system (2.157).

2.5.3 *Decentralized Design Results*

We now present the LMI design approach to the construction of a guaranteed cost controller.

Theorem 2.8 *Under assumptions* (A1) *and* (A2), *suppose there exist scalar parameters* $\mu_j > 0, \varepsilon_j > 0$ *and matrices* $0 < X_j = X_j^t \in \Re^{n_j \times n_j}, 0 < \bar{S}_j = \bar{S}_j^t \in \Re^{n_j \times n_j}, 0 < X_j = X_j^t \in \Re^{n_j \times n_j}, Y_j = \in \Re^{m_j \times n_j}$, *such that for all* $j = 1, \ldots, N$ *the* *following LMI*

$$
\left[\begin{array}{ccccccc}
\Phi_j & A_{dj}\bar{S}_j & B_{dj}Y_j & (E_{1j}X_j + E_{2j}Y_j)^t & G_{j1} & 0 & \dots \\
\bullet & -\bar{S}_j & 0 & \bar{S}_j E^t_{1dj} & 0 & 0 & \dots \\
\bullet & \bullet & -Z_j & Y^t_j E^t_{2dj} & 0 & 0 & \dots \\
\bullet & \bullet & \bullet & -\mu_j I_{qj} & 0 & 0 & \dots \\
\bullet & \bullet & \bullet & \bullet & -I_{\ell_1} & E^t_{1j} & \dots \\
\bullet & \bullet & \bullet & \bullet & \bullet & -\varepsilon_j I_{s_{j\ell}} & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots
\end{array}\right.
$$

$$
\left.\begin{array}{cccccc}
G_{jN} & 0 & X_j & Y^t_j & X_j & X_j \\
0 & 0 & 0 & 0 & 0 & 0 \\
Y^t_j B^t_{dj} & 0 & -Z_j & Y^t_j E^t_{2dj} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-I_{\ell_N} & E^t_{jN} & 0 & 0 & 0 & 0 \\
E_{jN} & -\varepsilon_j I_{s_{jN}} & 0 & 0 & 0 & 0 \\
\bullet & \bullet & -Q^{-1}_j & 0 & 0 & 0 \\
\bullet & \bullet & \bullet & -R^{-1}_j & 0 & 0 \\
\bullet & \bullet & \bullet & \bullet & -\bar{S}_j & 0 \\
\bullet & \bullet & \bullet & \bullet & \bullet & -U^{-1}_j
\end{array}\right] < 0, \qquad (2.162)
$$

has a feasible solution, *where*

$$
\Phi_j := A_j X_j + B_j Y_j + (A_j X_j + B_j Y_j)^t + Z_j + \mu_j D_j D^t_j + H_j,
$$

$$
H_j := \sum_{j=1, j \neq k}^{N} D_{jk} D^t_{jk}.
$$

Moreover, *the decentralized linear state feedback control laws*

$$
u_j(t) = K_j x_j(t) = Y_j X^{-1}_j x_j(t), \quad j = 1, \dots, N \qquad (2.163)
$$

are the guaranteed cost controllers and

$$J < \sum_{i=1}^{N} \Bigg[\phi_j^t(0) X_j^{-1} \phi_j(0) + \int_{-\tau_j}^{0} \phi_j^t(s) \bar{S}_j^{-1}(s) ds \\ + \int_{-h_j}^{0} \phi_j^t(s) X_j^{-1} Z_j X_j^{-1} \phi_j(s) ds \Bigg] \tag{2.164}$$

is the associated guaranteed cost.

Proof Introducing the matrices $X_j := P_j^{-1}, Y_j := K_j P_j^{-1}, \bar{S}_j := S_j^{-1}$ and $Z_j := P_j^{-1} T_j P_j^{-1}$. Pre-and post-multiplying both sides of the inequality (2.162) by

$$\text{blockdiag}[P_j \; S_j \; P_j \; I_{qj} \; I_{l1} \; I_{si1} \; \ldots \; I_{l_N} \; I_{siN} \; I_{n_j} \; I_{m_j} \; I_{n_j} \; I_{n_j}]$$

yields

$$\begin{bmatrix} \Psi_j & P_j A_{dj} & P_j B_{dj} K_j & \bar{E}_j^t & P_j G_{j1} & 0 & P_j G_{jN} & 0 & I_{n_j} & K_j^t & I_{n_j} & I_{n_j} \\ \bullet & -S_j & 0 & E_{1dj}^t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & -T_j & K_j^t E_{2dj}^t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\mu_j I_{qi} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -I_{\ell_1} & E_{j1}^t & 0 & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\varepsilon_j I_{s_{i1}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \ldots & -I_{\ell_N} & E_{jN}^t & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \ldots & E_{jN} & -\varepsilon_j I_{s_{jN}} & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \ldots & 0 & 0 & -Q_j^{-1} & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \ldots & 0 & 0 & 0 & -R_j^{-1} & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \ldots & 0 & 0 & 0 & 0 & -S_j^{-1} & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \ldots & 0 & 0 & 0 & 0 & 0 & -U_j^{-1} \end{bmatrix} \\ < 0, \tag{2.165}$$

where

$$\Psi_j := \bar{A}_j^t P_j + P_j \bar{A}_j + T_j + \mu_j P_j D_j D_j^t P_j + P_j H_j P_j, \\ \bar{A}_j := A_j + B_j K_j, \bar{E}_j := E_j^1 + E_{2j} K_j.$$

Using Schur complement, the matrix inequality (2.165) holds if and only if, the following inequality holds:

$$F_j := \begin{bmatrix} \Gamma_j & P_j A_{dj} + \mu_j^{-1} \bar{E}_j^t E_{1dj} & P_j B_{dj} K_j + \mu_j^{-1} \bar{E}_j^t E_{2dj} K_j & P_j G_{j1} & \ldots & P_j G_{jN} \\ \bullet & \mu_j^{-1} E_{1dj}^t E_{1dj} - S_j & \mu_j^{-1} E_{1dj}^t E^{2dj} K_j & 0 & \ldots & 0 \\ \bullet & \bullet & \mu_j^{-1} K_j^t E_{2dj}^t E_{2dj} K_j - T_j & 0 & \ldots & 0 \\ \bullet & \bullet & 0 & \Theta_1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \bullet & \bullet & 0 & 0 & \ldots & \Theta_N \end{bmatrix} \\ < 0, \tag{2.166}$$

where

$$\begin{aligned}\Gamma_j &:= \bar{A}_j^t P_j + P_j \bar{A}_j + U_j + \bar{R}_j + S_j + T_j + \mu_j P_j D_j D_j^t P_j + P_j H_j P_j \\ &\quad + \mu_j^{-1} \bar{E}_j^t \bar{E}_j, \\ \bar{R}_j &:= Q_j + K_j^t R_j K_j, \qquad \Theta_j := \varepsilon_j^{-1} E_{jk}^t E_{jk} - I_{\ell_j}.\end{aligned}$$

Using a standard matrix inequality [30] for all admissible uncertainties (2.145) and (2.159), the following matrix inequality holds:

$$\begin{aligned} 0 > {} & F_j \\ \geq {} & \begin{bmatrix} \bar{A}_j^t P_j + P_j \bar{A}_j + U_j + \bar{R}_j + S_j + T_j & P_j A_{dj} & P_j B_{dj} K_j & P_j G_{j1} & \dots & P_j G_{jN} \\ \bullet & -S_j & 0 & 0 & \dots & 0 \\ \bullet & \bullet & -T_j & 0 & \dots & 0 \\ \bullet & \bullet & \bullet & -I_{\ell_1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \bullet & \bullet & \bullet & \bullet & \dots & -I_{\ell_N} \end{bmatrix} \\ & + \begin{bmatrix} P_j D_j \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} F_j(t) \begin{bmatrix} \bar{E}_j^t \\ E_{1dj}^t \\ K_j^t E_{2dj}^t \\ 0 \\ \vdots \\ 0 \end{bmatrix}^t + \begin{bmatrix} \bar{E}_j^t \\ E_{1dt}^t \\ K_j^t E_{2dj}^t \\ 0 \\ \vdots \\ 0 \end{bmatrix} F_j^t(t) \begin{bmatrix} P_j D_j \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^t \\ & + \begin{bmatrix} 0 & 0 & 0 & P_j D_{j1} & \dots & P_j D_{jN} \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & F_{j1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & F_{jN} \end{bmatrix} \\ & \times \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & E_{j1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & E_{jN} \end{bmatrix} \end{aligned}$$

$$+\begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & E_{j1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & E_{jN} \end{bmatrix}^t \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & F_{j1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & F_{jN} \end{bmatrix}^t$$

$$\times \begin{bmatrix} 0 & 0 & 0 & P_j D_{j1} & \dots & P_j D_{jN} \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}^t = L_j. \tag{2.167}$$

Taking into consideration

$$\begin{aligned} \bar{A}_{dj} &= A_{dj} + D_j F_j(t) E_{1dj}, \qquad \tilde{G}_{jk} = G_{jk} + D_{jk} F_{jk}(t) E_{jk}, \\ \tilde{A}_j &= \bar{A}_j + D_j F_j(t) \bar{E}_j = \bar{A}_j + \Delta \bar{A}_j(t), \\ B_{dj} K_j &= H_{dj}, \qquad \Delta B_{dj}(t) K_j = \Delta H_{dj}(t), \\ Q_j + K_j^t R_j K_j &= \bar{R}_j = \bar{Q}_j \end{aligned}$$

we readily obtain $L_j = \Lambda_j$. Hence, the individual closed-loop systems are asymptotically stable under Theorem 2.8. The results of the cost bound (2.164) can be proved by using similar arguments for the proof of Theorem 2.7. □

Remark 2.23 Since LMI (2.162) consists of a solution set of $(\mu_j, \varepsilon_j X_j, Y_j, \bar{S}_j, Z_j)$, various efficient convex optimization algorithms can be applied. Moreover, its solutions represent the set of guaranteed cost controllers. This parameterized representation can be exploited to design the guaranteed cost controllers, which minimizes the value of the guaranteed cost for the closed-loop uncertain interconnected delay systems.

Consequently, to determine the optimal cost bound we solve the following optimization problem:

$$\begin{aligned} D_0: \quad & \min_{X_j} \sum_{i=1}^{N} \bar{J}_j = J^*, \\ & \bar{J}_j := \alpha_j + \operatorname{Tr} M_j + c_j^2 \|N_j N_j^t\|_2 \operatorname{Tr} Z_j, \\ & X_j \in (\mu_j, \varepsilon_j X_j, Y_j, \bar{S}_j, Z_j, \alpha_j, M_j), \end{aligned} \tag{2.168}$$

such that (2.162) is satisfied and

$$\begin{bmatrix} -\alpha_j & \phi_j^t(0) \\ \bullet & -X_j \end{bmatrix} < 0, \tag{2.169}$$

$$\begin{bmatrix} -M_j & M_j^t \\ \bullet & -\bar{S}_j \end{bmatrix} < 0, \tag{2.170}$$

$$\begin{bmatrix} -c_j I_{n_j} & I_{n_j} \\ \bullet & -X_j \end{bmatrix} < 0, \tag{2.171}$$

where $c_j > 0$ are prescribed constants and

$$M_j M_j^t := \int_{-\tau_j}^{0} \phi_j(s)\phi_j^t(s)ds, \qquad N_j N_j^t := \int_{-h_j}^{0} \phi_j(s)\phi_j^t(s)ds.$$

The main design result is summarized by the following theorem:

Theorem 2.9 *If the foregoing optimization problem has the solution*

$$\mu_j, \varepsilon_j, X_j, Y_j, \bar{S}_j, Z_j, , \alpha_j, M_j,$$

then the control laws of the form (2.163) *are the decentralized linear state feedback control laws, which ensure the minimization of the guaranteed cost* (2.164) *for the uncertain interconnected delay systems.*

Proof By Theorem 2.8, the control laws (2.163) constructed from the feasible solutions

$$\mu_j, \varepsilon_j, X_j, Y_j, \bar{S}_j, Z_j, \alpha_j, M_j$$

are the guaranteed cost controllers of the uncertain interconnected delay systems (2.157). Applying the Schur complement to the LMI (2.169) and using the following inequality [12]:

$$\mathrm{Tr}\, XY \le \|X\|_2 \,\mathrm{Tr}\, Y, \quad Y = Y^t \ge 0, \quad X = X^t,$$

we have the following

1.

$$\phi_j^t(0) X_j^{-1} \phi_j(0) < \alpha_j,$$

2.

$$\begin{aligned}\int_{-\tau_j}^{0} \phi_j^t(s)\bar{S}_j^{-1}\phi_j(s)ds &= \int_{\tau_j}^{0} \mathrm{Tr}[\phi_j^t(s)\bar{S}_j^{-1}\phi_j(s)]ds \\ &= \mathrm{Tr}[M_j^t \bar{S}_j^{-1} M_j] < \mathrm{Tr}[M_j],\end{aligned}$$

3.

$$\begin{aligned}
&\int_{-h_j}^{0} \phi_j^t(s) X_j^{-1} Z_j X_j^{-1} \phi_j(s) ds \\
&= \int_{-h_j}^{0} \mathrm{Tr}[\phi_j^t(s) X_j^{-1} Z_j X_j^{-1} \phi_j(s)] ds \\
&= \mathrm{Tr}[N_j^t X_j^{-1} Z_j X_j^{-1} N_j] \le \|N_j N_j^t\|_2 \|X_j^{-1}\|_2^2 \,\mathrm{Tr}\, Z_j \\
&< c_j^2 \|N_j N_j^t\|_2 \,\mathrm{Tr}\, Z_j.
\end{aligned}$$

It follows that

$$\begin{aligned}
J &< \sum_{j=1}^{N} \Big[\phi_j^t(0) X_j^{-1} \phi_j(0) + \int_{-\tau_j}^{0} \phi_j^t(s) \bar{S}_j^{-1} \phi_j(s) ds \\
&\quad + \int_{-h_j}^{0} \phi_j^t(s) X_j^{-1} Z_j X_j^{-1} \phi_j(s) ds \Big] \\
&< \sum_{i=1}^{N} (\alpha_j + \mathrm{Tr}[M_j] + c_j^2 \|N_j N_j^t\|_2 \cdot \mathrm{Tr}[Z_j]) \\
&\le \min_{X_j} \sum_{j=1}^{N} \bar{J}_j = J^*.
\end{aligned} \tag{2.172}$$

Thus, the minimization of $\sum_{i=1}^{N} \bar{J}_j$ implies the minimum value J^* of the guaranteed cost for the interconnected uncertain delay systems (2.157). The optimality of the solution of the optimization problem follows from the convexity of the objective function under the LMI constraints. This is the required result. □

Remark 2.24 It must be noted that the original optimization problem for the guaranteed cost (2.168) can be appropriately decomposed into the following reduced optimization problems (2.173) since each optimization problem (2.173) is independent of each other. Hence, we only have to solve the optimization problems (2.173) for each independent subsystem:

$$\begin{aligned}
&\min_{X_j} \sum_{j=1}^{N} \bar{J}_j = \sum_{j=1}^{N} \min_{X_j} \bar{J}_j, \\
&X_j \in (\mu_j, \varepsilon_j X_j, Y_j, \bar{S}_j, Z_j, \alpha_j, M_j), \quad D_j\!: \quad \min_{X_j} \bar{J}_j, \quad j = 1, \ldots, N, \\
&\bar{J}_j := \alpha_j + \mathrm{Tr}[M_j] + c_j^2 \|N_j N_j^t\|_2 \cdot \mathrm{Tr}[Z_j].
\end{aligned} \tag{2.173}$$

Remark 2.25 The constant parameter c_j, which is included in the inequality (2.171), needs to be optimized as the LMI constraints. In this case, it is hard to obtain the optimum guaranteed cost, because the resulting problem is nonconvex optimization problem. As an alternative, the above suboptimal guaranteed cost control is solved instead of solving the non convex optimization problem. Consequently, the decentralized robust suboptimal guaranteed cost controller, which minimizes the value of the guaranteed cost for the closed-loop uncertain delay systems, can be easily solved by using the LMI. The selected constant parameter c_j needs to be as small as since the matrix X_j is constrained by the inequality (2.169).

2.6 Global Robust Stabilization

2.6.1 Introduction

The decentralized control schemes, different from the classical centralized information structures, have been considered with significant interests for the control of interconnected systems in recent years. The main objectives of decentralized control are to find some feedback laws for adapting the interactions from the other subsystems where no state information is transferred. The advantage of decentralized control design is to reduce complexity and this therefore allows the control implementation to be more feasible.

Unlike centralized control design, decentralized control cannot have access to the entire state information. Therefore, interconnections between subsystems need to be analyzed, so that their influence on the system performance can be properly addressed by the control. As far as asymptotic stability of interconnected systems is concerned, there are two main approaches for the treatment of the interconnections in the literature. The first is to assume that the interconnections satisfy the matching conditions bounded by first-order polynomials of states [3] or higher-order polynomials [38, 56]. The second is to require that the interconnections meet a triangular structure bounded by first-order polynomials of states [79] or higher-order polynomials [25]. The matching condition guarantees that Lyapunov redesign is applicable, which begins with Lyapunov functions for nominal subsystems and then attempts to use these Lyapunov functions to design decentralized feedback laws. Most of the work in the literature falls into this category. On the other hand, the triangular structure makes it possible to apply backstepping technique to design the decentralized controllers. The backstepping design idea, which was initially introduced in [28] for nonlinear adaptive control and in [8] for nonlinear robust control, was applied to construct decentralized robust controllers in [79] and used in decentralized adaptive control by [25]. In the latter, we note that decentralized adaptive control design is addressed for a class of large-scale interconnected nonlinear systems with decentralized strict feedback form and single input subsystems. In the literature, the interconnections are assumed to be bounded by higher order polynomials of the states in the first integrator of every subsystem, whose coefficients admit a lower triangular structure.

One of the important problems in decentralized control is to relax restrictions on the interconnections and uncertainties. There exist two kinds of restrictions, such as matching conditions and strict feedback conditions in the literature. Many physical systems, such as power systems in [62], do not satisfy these conditions, so the study of relaxing these restrictions is of theoretical and practical importance.

Hereafter, the main objective is to investigate the problem of decentralized robust stabilization for a class of large-scale nonlinear systems with parameter uncertainties and nonlinear interconnections. Each system of the interconnected system is assumed to be controlled by multiple inputs and to be in a nested structure, which was first introduced by [37]. The uncertain parameters and/or disturbances are allowed to be time-varying and enter the system nonlinearly. The nonlinear interconnections are bounded by higher-order polynomials in the decentralized strict feedback form. Inspired by the recent work of centralized nonlinear control [36], it is proved that the global decentralized robust asymptotic stabilization problem can be solved for the uncertain interconnected nonlinear systems by applying a recursive design procedure.

2.6.2 Problem Formulation and Assumptions

Consider a large-scale nonlinear system composed of N interconnected subsystems with m inputs. The ith subsystem is given as

$$\begin{aligned}
\dot{x}^i &= f^i(x^i, \xi_{11}^i) + \sum_{n=1}^{m} \Phi_{n0}^{in}(\bar{x}^N, \bar{\xi}_1^N, \dots, \bar{\xi}_n^N, \theta)\xi_{n1}^i, \\
\xi_{j1}^i &= \xi_{j2}^i + \Psi_{j1}^i(\bar{x}^N, \bar{\xi}_1^N, \dots, \bar{\xi}_j^N, \theta) \\
&\quad + \sum_{n=j+1}^{m} \Phi_{j1}^{in}(\bar{x}^N, \bar{\xi}_1^N, \dots, \bar{\xi}_n^N, \theta)\xi_{n1}^i, \\
&\ \vdots \\
\xi_{j,r_{j-1}}^i &= \xi_{jr_j}^i + \Psi_{j,r_{j-1}}^i(\bar{x}^N, \bar{\xi}_1^N, \dots, \bar{\xi}_j^N, \theta), \\
&\quad + \sum_{n=j+1}^{m} \Phi_{j,r_{j-1}}^{in}(\bar{x}^N, \bar{\xi}_1^N, \dots, \bar{\xi}_n^N, \theta)\xi_{n1}^i, \\
\xi_{jr_j}^i &= u_j^i + \Psi_{jr_j}^i(\bar{x}^N, \bar{\xi}_1^N, \dots, \bar{\xi}_j^N, \theta) \\
&\quad + \sum_{n=j+1}^{m} \Phi_{jr_j}^{in}(\bar{x}^N, \bar{\xi}_1^N, \dots, \bar{\xi}_n^N, \theta)\xi_{n1}^i,
\end{aligned} \tag{2.174}$$

where

$$x^j \in \Re^{n_j}, \quad \bar{x}^N = [(x^1)^t, \dots, (x^N)^t]^t, \quad \bar{\xi}^j_{jd} = [\xi^j_{j1}, \dots, \xi^j_{jd}]^t,$$
$$\bar{\xi}^N_j = [(\bar{\xi}^1_{jr_j})', \dots, (\bar{\xi}^N_{jr_j})^t]^t, \quad i = 1, \dots, N, \ j = 1, \dots, m, \ d = 1, \dots, r_j.$$

The vector $\theta \in \Re^q$ is a time-varying uncertain parameters. All functions are smooth and vanishing at the origin for any θ.

Remark 2.26 Every subsystem in (2.174) possesses a nested structure, that is, the $(x^j, \bar{\xi}^j_{1r_1}, \dots, \bar{\xi}^j_{jr_j})$-blocks are nested in the $\bar{\xi}^j_{j+1,r_{j+1}}$-block through feedback connections between these blocks. Moreover, each block has a strict feedback structure with unmatched interconnections. Such a structure can be easily seen from (5).

Our objective is to design decentralized robust controllers

$$u^j_1 = u^j_1(x^j, \bar{\xi}^j_{1r_1}), \quad \dots, \quad u^j_m = u^j_m(x^j, \bar{\xi}^j_{1r_1}, \dots, \xi^j_{mr_m}), \quad j = 1, \dots, N$$

such that the origin of the corresponding closed-loop system is globally asymptotically stable for any θ. The recursive design technique, that is, back stepping with the aid of augmentation, developed in [36], will be applied to construct decentralized robust controllers for the system (2.174).

To this end, we impose the following assumptions:

Assumption 2.4 There exist positive definite and proper smooth functions

$$V^j(x^j), \quad j = 1, \dots, N, \quad p^{jt}_0 > 0$$

such that

$$\sum_{j=1}^{N} \frac{\partial V^j}{\partial x^j} f^j(x^j, 0) \le -\sum_{j=1}^{N}\sum_{t=1}^{\rho} p^{jt}_0 \|x^j\|^{2t}. \tag{2.175}$$

Assumption 2.5 There exist a series of non-negative smooth functions

$$\Psi^{ikt}_{jd0}(x^j, \bar{\xi}^j_{1r_1}, \dots, \bar{\xi}^j_{j-1,r_{j-1}}, \bar{\xi}^j_{jd}), \quad \Psi^{iit}_{jdls}(x^j, \bar{\xi}^j_{1r_1}, \dots, \bar{\xi}^j_{j-1,r_{j-1}}, \bar{\xi}^j_{jd}),$$
$$\Psi^{iit}_{jdjs}(x^j, \bar{\xi}^j_{1r_1}, \dots, \bar{\xi}^j_{j-1,r_{j-1}}, \bar{\xi}^j_{jd}), \quad \Psi^{ikt}_{jdl1}(x^j, \bar{\xi}^j_{1r_1}, \dots, \bar{\xi}^j_{j-1,r_{j-1}}, \bar{\xi}^j_{jd})$$

such that

$$\begin{aligned} &\|\Psi^j_{jd}(\bar{x}^N, \bar{\xi}^N_1, \dots, \bar{\xi}^N_j, \theta)\| \\ &\quad \le \sum_{k=1}^{N}\sum_{t=1}^{\rho} \Psi^{ikt}_{jd0}\|x^k\|^t + \sum_{l=1}^{j-1}\sum_{s=2}^{r_1}\sum_{t=1}^{\rho} \Psi^{iit}_{jdls}|\xi^j_{ls}|^t \\ &\qquad + \sum_{s=2}^{d}\sum_{t=1}^{\rho} \Psi^{iit}_{jdjs}|\xi^j_{js}|^t + \sum_{k=1}^{N}\sum_{l=1}^{j}\sum_{t=1}^{\rho} \Psi^{ikt}_{jdl1}|\xi^k_{l1}|^t \end{aligned} \tag{2.176}$$

for $j = 1, \dots, N$, $k = 1, \dots, m$ and $d = 1, \dots, r_j$.

Assumption 2.6 There exist a series of non-negative smooth functions

$$\Phi_{jd0}^{inkt}(x^j, \bar{\xi}_{1r_1}^j, \ldots, \bar{\xi}_{n-1,r_{n-1}}^j), \quad \Phi_{jdls}^{init}(x^j, \bar{\xi}_{1r_1}^j, \ldots, \bar{\xi}_{n-1,r_{n-1}}^j),$$
$$\Phi_{jdl1}^{inkt}(x^j, \bar{\xi}_{1r_1}^j, \ldots, \bar{\xi}_{n-1,r_{n-1}}^j),$$

such that

$$\begin{aligned}
&\|\Phi_{jd}^{in}(\bar{x}^N, \bar{\xi}_1^N, \ldots, \bar{\xi}_n^N, \theta)\| \\
&\quad \le \sum_{k=1}^{N}\sum_{t=1}^{\rho} \Phi_{jd0}^{inkt}\|x^k\|^t + \sum_{l=1}^{n-1}\sum_{s=2}^{r_1}\sum_{t=1}^{\rho} \Phi_{jdls}^{init}|\xi_{ls}^j|^t \\
&\qquad + \sum_{k=1}^{N}\sum_{l=1}^{n}\sum_{t=1}^{\rho} \Phi_{jdl1}^{inkt}|\xi_{l1}^k|^t
\end{aligned} \tag{2.177}$$

for $j = 1, \ldots, N$, $k = 1, \ldots, m$, $n = j+1, \ldots, m$ and $d = 0, \ldots, r_j$.

Remark 2.27 It must be noted that Assumptions 2.5 and 2.6 imply that the interconnections are bounded by polynomial-type nonlinearities with the decentralized strict feedback form. In particular, the interconnections in the ith subsystem are bounded by polynomial-type nonlinearities which are composed of two parts: higher-order polynomials of its own states, i.e. the second and the third terms on the right-hand side of (2.176) and the second terms on the right-hand side of (2.177); higher-order polynomials of the states from other subsystems, that is. the first terms on the right-hand side of (2.176) and (2.177) which are comprised of all the zero-dynamic considered in [25], the last terms in (2.176) and (2.177) which are comprised of the first states of each subsystem.

Remark 2.28 Note also that the restrictions on the interconnections imposed in Assumptions 2.5 and 2.6 are very general which include many types of interconnections considered in the existing literature as special cases, for example, the interconnections bounded by first-order polynomials [3], higher-order polynomials [25, 38]. Compared with the work in [3, 56], no matching conditions are imposed in Assumptions 2.5 and 2.6. Furthermore, the kth subsystem's state variables x^k are allowed to appear in the higher-order polynomials in Assumptions 2.5 and 2.6.

Remark 2.29 In the literature, the decentralized robust stabilization problem has been addressed for a class of large-scale nonlinear systems of the form (2.178). In what follows, we consider the same problem for a wider class of large-scale systems with more than one input and less restrictions on interconnections.

2.6.3 Robust Control Design

We now look for designing decentralized robust controllers for the large-scale system (2.174). The design will be carried out step by step.

1. Consider system (2.174) with $m=1$, that is

$$\begin{aligned}
\dot{x}^j &= f^j(x^j, \xi_{11}^j) + \Psi_{10}^{i1}(\bar{x}^N, \bar{\xi}_1^N, \theta)\xi_{11}^j, \\
\xi_{11}^j &= \xi_{12}^j + \Psi_{11}^j(\bar{x}^N, \bar{\xi}_1^N, \theta), \\
&\vdots \\
\xi_{1,r_1-1}^j &= \xi_{1r_1}^j + \Psi_{1,r_1-1}^j(\bar{x}^N, \bar{\xi}_1^N, \theta), \\
\xi_{1r_1}^j &= u_1^j + \Psi_{1r_1}^j(\bar{x}^N, \bar{\xi}_1^N, \theta),
\end{aligned} \tag{2.178}$$

where Φ_{10}^{i1} and Ψ_{1d}^j satisfy the following conditions:

$$\begin{aligned}
\|\Phi_{10}^{i1}(\bar{x}^N, \bar{\xi}_1^N, \theta)\| \le{}& \sum_{k=1}^{N}\sum_{t=1}^{\rho} \Phi_{100}^{i1kt}(x^j, \xi_{11}^j)\|x^k\|^t \\
&+ \sum_{k=1}^{N}\sum_{t=1}^{\rho} \Phi_{1011}^{i1kt}(x^j, \xi_{11}^j)|\xi_{11}^k|^t,
\end{aligned} \tag{2.179}$$

$$\begin{aligned}
\|\Psi_{1d}^j(\bar{x}^N, \bar{\xi}_1^N, \theta)\| \le{}& \sum_{k=1}^{N}\sum_{t=1}^{\rho} \Psi_{1d0}^{ikt}(x^j, \bar{\xi}_{1d}^j)\|x^k\|^t \\
&+ \sum_{s=2}^{d}\sum_{t=1}^{\rho} \Psi_{1d1s}^{iit}(x^j, \bar{\xi}_{1d}^j)|\xi_{1s}^j|^t \\
&+ \sum_{k=1}^{N}\sum_{t=1}^{\rho} \Psi_{1d11}^{ikt}(x^j, \bar{\xi}_{1d}^j)|\xi_{11}^k|^t,
\end{aligned} \tag{2.180}$$

which follows from Assumptions 2.5 and 2.6. It is readily seen that system (2.178) is quite general. Furthermore, conditions (2.179) and (2.180) are less restrictive due to the presence of the higher polynomial terms $|\xi_{1s}^j|^t$ in (2.180) and the interconnection terms $\|x^k\|^t$ in (2.179) and (2.180). With Assumption 2.4, (2.179) and (2.180), an appropriate design procedure can be applied to system (2.178), the result can be summarized by the following lemma:

Lemma 2.7 *Consider system* (2.178) *with Assumption* 2.4 *and* (2.179) *and* (2.180). *There exist a change of coordinates* $z_{1d}^j = \xi_{1d}^j - \alpha_{1,d-1}^j(x^j, \bar{\xi}_{1,d-1}^j)$ *with* $\alpha_{10}^j = 0$

and decentralized feedback laws $u_1^j = u_1^j(x^j, \bar{\xi}_{1r_1}^j)$ *such that the Lyapunov function*

$$W_1 = \sum_{i=1}^{N} V^j + \sum_{i=1}^{N}\sum_{d=1}^{r_1} \frac{1}{2}(z_{1d}^j)^2 \tag{2.181}$$

satisfies

$$\begin{aligned} \dot{W}_1 &\le -\sum_{i=1}^{N}\sum_{t=1}^{\rho} p_1^{it}\|x^j\|^{2t} - \sum_{i=1}^{N}\sum_{d=1}^{r_1}\sum_{t=1}^{\rho} c_{1d1}^{it}(z_{1d}^j)^{2t}, \\ p_1^{it} &> 0, \quad c_{1d2}^{it} > 0 \end{aligned} \tag{2.182}$$

along the solutions to system (2.178) *with* $u_1^j = u_1^j(x^j, \bar{\xi}_{1r_1}^j)$.

Remark 2.30 Note that Lemma 2.7 is an extension of the results given in the literature. The proof presented there can be modified to verify Lemma 2.7. However, a major modification should be made, that is, the terms like $|\xi_{1s}^j|$ should be expressed in terms of x^j and z_{1d}^j for $d = 1, \dots, s$. Observe that $\alpha_{1,s-1}^j$ can be put into the form

$$\alpha_{1,s-1}^j = \bar{\alpha}_{1,s-1,0}^j(x^j)x^j + \sum_{d=1}^{s-1} \bar{\alpha}_{1,s-1,d}^j(x^j, \xi_{11}^j, \dots, \xi_{1d})z_{1d}^j$$

due to the smoothness of $\alpha_{1,s-1}^j$ and $\alpha_{1,s-1}^j(0) = 0$. It follows from

$$\xi_{1s}^j = z_{1s}^j + \alpha_{1,s-1}^j(x^j, \xi_{11}^j, \dots, \xi_{1,s-1}^j)$$

that

$$\xi_{1s}^j = z_{1s}^j + \bar{\alpha}_{1,s-1,0}^j(x^j)x^j + \sum_{d=1}^{s-1} \bar{\alpha}_{1,s-1,d}^j(x^j, \xi_{11}^j, \dots, \xi_{1d})z_{1d}^j$$

which implies, according to Lemma 2.8 in Sect. 2.6.5, that

$$\begin{aligned} |\xi_{1s}^j|^t &\le (s+1)^{t-1}[|z_{1s}^j|^t + \|\bar{\alpha}_{1,s-1,0}^j(x^j)\|^t\|x^j\|^t] \\ &\quad + (s+1)^{t-1}\sum_{d=1}^{s-1} |\bar{\alpha}_{1,s-1,d}^j(x^j, \xi_{11}^j, \dots, \xi_{1d})|^t |z_{1d}^j|^t. \end{aligned}$$

Step *T*: Consider system (2.174) with $m = T$, $T \ge 2$, that is,

$$\begin{aligned} \dot{x}^j &= f^j(x^j, \xi_{11}^j) + \sum_{n=1}^{t} \Phi_{n0}^{in}(\bar{x}^N, \bar{\xi}_1^N, \dots, \bar{\xi}_n^N, \theta)\xi_{n1}^j, \\ \xi_{j1}^j &= \xi_{j2}^j + \Psi_{j1}^j(\bar{x}^N, \bar{\xi}_1^N, \dots, \bar{\xi}_j^N, \theta) \end{aligned}$$

$$
\begin{aligned}
&\qquad + \sum_{n=j+1}^{t} \Phi_{j1}^{in}(\bar{x}^N, \bar{\xi}_1^N, \ldots, \bar{\xi}_n^N, \theta)\xi_{n1}^j, \\
&\vdots \\
\xi_{j,r_j-1}^j &= \xi_{jr_j}^j + \Psi_{j,r_j-1}^j(\bar{x}^N, \bar{\xi}_1^N, \ldots, \bar{\xi}_j^N, \theta) \\
&\qquad + \sum_{n=j+1}^{t} \Phi_{j,r_j-1}^{in}(\bar{x}^N, \bar{\xi}_1^N, \ldots, \bar{\xi}_n^N, \theta)\xi_{n1}^j, \\
\xi_{jr_j}^j &= u_j^j + \Psi_{jr_j}^j(\bar{x}^N, \bar{\xi}_1^N, \ldots, \bar{\xi}_j^N, \theta) \\
&\qquad + \sum_{n=j+1}^{t} \Phi_{jr_j}^{in}(\bar{x}^N, \bar{\xi}_1^N, \ldots, \bar{\xi}_n^N, \theta)\xi_{n1}^j,
\end{aligned}
\tag{2.183}
$$

where $u_1^j = u_1^j(x^j, \bar{\xi}_{1r_1}^j)$, $u_2^j = u_2^j(X_1^j, \bar{\xi}_{2r_2}^j)$, $\ldots$, $u_{T-1}^j = u_{T-1}^j(X_{T-2}^j, \bar{\xi}_{T-1,r_{T-1}}^j)$ are determined in the first $T-1$ steps with

$$
\begin{aligned}
X_1^j &= [(x^j)', (\bar{\xi}_{1r_1}^j)']', \\
X_1^N &= [(X_1^1)', \ldots, (X_1^N)']', \\
F_1^j &= [(f_0^j + \Phi_{10}^j \xi_{11}^j)', \xi_{12}^j + \Psi_{11}^j, \ldots, \xi_{1r_1}^j \\
&\qquad + \Psi_{1,r_1-1}^j, u_1^j(X_1^j) + \Psi_{1r_1}^j]', \\
\bar{\Phi}_1^{i2} &= [(\Phi_{20}^{i2})', \Phi_{11}^{i2}, \ldots, \Phi_{1,r_1-1}^{i2}, \Phi_{1r_1}^{i2}]', \\
&\vdots \\
X_{T-2}^j &= [(X_{T-3}^j)', \xi_{T-2,1}^j, \ldots, \xi_{T-2,r_{T-2}}^j]', \\
X_{T-2}^N &= [(X_{T-2}^1)', \ldots, (X_{T-2}^N)']', \\
F_{T-2}^j &= [(F_{T-3}^j + \bar{\Phi}_{T-3}^{i,T-2} \xi_{T-2,1}^j)', \xi_{T-2,2}^j + \Psi_{T-2,1}^j, \ldots, \\
&\qquad \xi_{T-2,r_{T-2}}^j + \Psi_{T-2,r_{T-2}-1}^j, u_{T-1}^j(X_{T-2}^j) + \Psi_{T-2,r_{T-2}}^j]', \\
\bar{\Phi}_{T-2}^{i,T-1} &= [(\Phi_{T-1,0}^{i,T-1})', \Phi_{11}^{i,T-1}, \ldots, \Phi_{1r_1}^{i,T-1}, \ldots, \\
&\qquad \Phi_{T-1,1}^{i,T-1}, \ldots, \Phi_{T-2,r_{T-2}-1}^{i,T-1}, \Phi_{T-2,r_{T-2}}^{i,T-1}]'.
\end{aligned}
$$

Such a system can be alternatively put into the following form:

$$
\begin{aligned}
\dot{X}_{T-1}^j &= F_{T-1}^j(\bar{X}_{T-1}^N, \theta) + \bar{\Phi}_{T-1}^{iT}(\bar{X}_{T-1}^N, \bar{\xi}_T^N, \theta)\xi_{T1}^j, \\
\xi_{T1}^j &= \xi_{T2}^j + \Psi_{T1}^j(\bar{X}_{T-1}^N, \bar{\xi}_T^N, \theta),
\end{aligned}
$$

$$
\begin{aligned}
&\vdots \\
\xi^j_{T2} &= \xi^j_{T3} + \Psi^j_{T2}(\bar{X}^N_{T-1}, \bar{\xi}^N_T, \theta), \\
\xi^j_{Tr_T} &= u^j_T + \Psi^j_{Tr_T}(\bar{X}^N_{T-1}, \bar{\xi}^N_T, \theta),
\end{aligned}
\tag{2.184}
$$

where

$$
\begin{aligned}
X^j_{T-1} &= [(X^j_{T-2})', \xi^j_{T-1,1}, \ldots, \xi^j_{T-1,r_{T-1}}]', \\
X^N_{T-1} &= [(X^1_{T-1})', \ldots, (X^N_{T-1})']', \\
F^j_{T-1} &= [(F^j_{T-2} + \bar{\Phi}^{i,T-1}_{T-2}\xi^j_{T-1,1})', \xi^j_{T-1,2} + \Psi^j_{T-1,1}, \ldots, \\
&\qquad \xi^j_{T-1,r_{T-1}} + \Psi^j_{T-1,r_{T-1}}, u^j_{T-1}(X^j_{T-1}) + \Psi^j_{T-1,r_{T-1}}]', \\
\bar{\Phi}^{iT}_{T-1} &= [(\Phi^{iT}_{T0})', \Phi^{iT}_{11}, \ldots, \Phi^{iT}_{1r_1}, \ldots, \Phi^{iT}_{T-1,1}, \ldots, \\
&\qquad \Phi^{iT}_{T-1,r_{T-1}-1}, \Phi^{iT}_{T-1,r_{T-1}}]'.
\end{aligned}
$$

According to Step $T-1$, F^j_{T-1} satisfies the following inequality:

$$
\begin{aligned}
&\sum_{i=1}^{N} \frac{\partial W_{T-1}}{\partial X^j_{T-1}} F^j_{T-1}(X^N_{T-1}, \theta) \\
&\quad \le -\sum_{i=1}^{N}\sum_{t=1}^{\rho} p^{it}_{T-1}\|x^j\|^{2t} - \sum_{i=1}^{N}\sum_{j=1}^{T-1}\sum_{d=1}^{r_j}\sum_{t=1}^{\rho} c^{it}_{jd,T-1}(z^j_{jd})^2 t\cdot
\end{aligned}
\tag{2.185}
$$

It follows from Assumptions 2.5 and 2.6 that Φ^{iT}_{T-1} and Ψ^j_{Td} satisfy the following inequalities:

$$
\begin{aligned}
\|\Phi^{iT}_{T-1}(\bar{X}^N_{T-1}, \bar{\xi}^N_T, \theta)\| &\le \|\Phi^{iT}_{T0}\| + \sum_{j=1}^{T-1}\sum_{d=1}^{r_j}\|\Phi^{iT}_{jd}\| \\
&\le \sum_{k=1}^{N}\sum_{t=1}^{\rho} \bar{\Phi}^{iTkt}_{T-1,0}(X^j_{T-1}, \xi^j_{T1})\|x^k\|^t \\
&\quad + \sum_{k=1}^{N}\sum_{l=1}^{T-1}\sum_{s=2}^{r_j}\sum_{t=1}^{\rho} \bar{\Phi}^{iTit}_{T-1,ls}(X^j_{T-1}, \xi^j_{T1})|\xi^j_{ls}|^t \\
&\quad + \sum_{k=1}^{N}\sum_{l=1}^{T-1}\sum_{t=1}^{\rho} \bar{\Phi}^{iTkt}_{T-1,l1}(X^j_{T-1}, \xi^j_{T1})|\xi^j_{l1}|^t,
\end{aligned}
\tag{2.186}
$$

$$\begin{aligned}\|\Psi_{Td}^{j}(\bar{X}_{k-1}^{N},\bar{\xi}_{T}^{N},\theta)\| \le &\sum_{k=1}^{N}\sum_{t=1}^{\rho}\Psi_{Td0}^{ikt}(X_{T-1}^{j},\bar{\xi}_{Td}^{j})\|x^{k}\|^{t}\\ &+\sum_{l=1}^{T-1}\sum_{s=2}^{r_1}\sum_{t=1}^{\rho}\Psi_{Tdls}^{iit}(X_{T-1}^{j},\bar{\xi}_{Td}^{j})|\xi_{ls}^{j}|^{t}\\ &+\sum_{s=2}^{d}\sum_{t=1}^{\rho}\Psi_{TdTs}^{iit}(X_{T-1}^{j},\bar{\xi}_{Td}^{j})|\xi_{Ts}^{j}|^{t}\\ &+\sum_{k=1}^{N}\sum_{l=1}^{T-1}\sum_{t=1}^{\rho}\Psi_{Tdl1}^{ikt}(X_{T-1}^{j},\bar{\xi}_{Td}^{j})|\xi_{l1}^{k}|^{t}. \end{aligned}\tag{2.187}$$

With (2.185)–(2.187), it follows from Lemma 2.7 that there exits a change of coordinates $z_{Td}^{j}=\xi_{Td}^{j}-\alpha_{T,d-1}^{j}(X_{T-1}^{j},\bar{\xi}_{T,d-1}^{j})$ with $\alpha_{T0}^{j}=0$ and decentralized feedback laws $u_{T}^{j}=u_{k}^{j}(X_{T-1}^{j},\bar{\xi}_{Tr_T}^{j})$ so that the Lyapunov function

$$W_T=W_{T-1}+\sum_{i=1}^{N}\sum_{d=1}^{r_T}\frac{1}{2}(z_{Td}^{j})^{2t}\tag{2.188}$$

satisfies

$$\dot{W}_T\le-\sum_{i=1}^{N}\sum_{t=1}^{\rho}p_T^{it}\|x^j\|-\sum_{i=1}^{N}\sum_{j=1}^{t}\sum_{d=1}^{r_j}\sum_{t=1}^{\rho}c_{jdT}^{it}(z_{jd}^{j})^{2t}\tag{2.189}$$

along the solution of (2.183) with

$$u_1^j=u_1^j(x^j,\bar{\xi}_{1r_1}^{j}),\quad u_2^j=u_2^j(X_1^j,\bar{\xi}_{2r_2}^{j}),\quad \ldots,\quad u_T^j=u_2^j(X_{T-1}^j,\bar{\xi}_{Tr_T}^{j}).$$

From the foregoing analysis, we have the following result for system (2.174):

Theorem 2.10 *Suppose that Assumptions 2.4–2.6 are satisfied. Then, system (2.174) can be globally asymptotically stabilized by decentralized robust control laws* $u_1^j=u_1(x^j,\bar{\xi}_{1r_1}^{j}),\ldots,u_m^j=u_m(x^j,\bar{\xi}_{1r_1}^{j},\ldots,\bar{\xi}_{mr_m}^{j})$.

2.6.4 Simulation Example 2.7

To illustrate the theoretical developments, we consider the large-scale nonlinear system

$$\begin{aligned}\dot{x}^j &= -x^1-(x^1)^3+\xi_{11}^1(x^1)^2\theta\sin t\\ &\quad+\frac{1}{\Delta}\xi_{21}^1[(\xi_{11}^2)^2+(\xi_{21}^2)^2],\end{aligned}$$

$$
\begin{aligned}
\xi_{11}^1 &= \xi_{12}^1 + \xi_{21}^1(\xi_{11}^2)^2\theta\cos t,\\
\xi_{12}^1 &= u_1^1,\\
\xi_{21}^1 &= \xi_{22}^1 + \frac{1}{\Delta}[x^1\sin t + (\xi_{21}^2)^2\theta\cos t],\\
\xi_{22}^1 &= u_2^1,\\
\xi_{11}^2 &= \xi_{12}^2 + \xi_{21}^2\xi_{11}^1\xi_{11}^2\theta\cos t,\\
\xi_{12}^2 &= u_1^2,\\
\xi_{21}^2 &= \xi_{22}^2 + \frac{1}{\Delta}[x^1\sin t + (\xi_{21}^1)^2\theta\cos t],\\
\xi_{22}^2 &= u_2^2,
\end{aligned}
\tag{2.190}
$$

where $\Delta = 1 + (x^1)^2 + \sum_{i=1}^2\sum_{j=1}^2(\xi_{1j}^j)^2 + \sum_{i=1}^2\sum_{j=1}^2(\xi_{2j}^j)^2$ and $|\theta| < 1$.

For this purpose we choose $V = \frac{1}{2}p(x^1)^2$, $p > 0$. Then, a simple calculation shows that

$$
\frac{\partial V}{\partial x^1}[-x^1 - (x^1)^3] \le -p(x^1)^2 - \frac{p}{2}(x^1)^4
$$

which implies that Assumption 2.4 is satisfied. In addition, it is not difficult to prove that Assumptions 2.5 and 2.6 are satisfied as well. Therefore, the design procedure developed in Sect. 2.6.3 is applicable. Note that the approach in [81] cannot be used to solve the problem for (2.190) because there exist interconnected terms, that is, the last terms in the first equation, the second equation, the fourth equation, the sixth equation, and the eighth equation of system (2.190).

First, consider the following system:

$$
\begin{aligned}
\dot{x}^1 &= -x^1 - (x^1)^3 + \xi_{11}^1(x^1)^2\theta\sin t,\\
\xi_{11}^1 &= \xi_{12}^1,\\
\xi_{12}^1 &= u_1^1,\\
\xi_{11}^2 &= \xi_{12}^2,\\
\xi_{12}^2 &= u_1^2.
\end{aligned}
\tag{2.191}
$$

It follows from Step 1 in Sect. 2.6.3 that the following controllers can be constructed:

$$
\begin{aligned}
u_1^1 = {} & -c_{121}^{11}z_{12}^1 - \xi_{11}^1 - \frac{\partial\alpha_{11}^1}{\partial x^1}[x^1 + (x^1)^3]\\
& - \frac{1}{2}z_{12}^1\left(\frac{\partial\alpha_{11}^1}{\partial x^1}z_{11}^1\right)^2 + \frac{\partial\alpha_{11}^1}{\partial z_{11}^1}(z_{12}^1 + \alpha_{11}^1),\\
u_1^2 = {} & -c_{121}^{21}z_{12}^2 - z_{11}^2 + \frac{\partial\alpha_{11}^2}{\partial z_{11}^2}(z_{12}^2 + \alpha_{11}^2),
\end{aligned}
$$

so that the Lyapunov function

$$W_1 = V + \sum_{i=1}^{2}\sum_{d=1}^{2}(z_{1d}^{j})^2$$

satisfies

$$\dot{W}_1 \le -p(x^1)^2\left(\frac{p}{2}-\frac{1}{2}\right)(x^1)^4 - c_{111}^{11}(z_{11}^1)^2 \\ - c_{111}^{21}(z_{11}^2)^2 - c_{111}^{22}(z_{11}^2)^4 - c_{121}^{11}(z_{12}^1)^2 - c_{121}^{21}(z_{12}^2)^2,$$

where $z_{11}^1 = \xi_{11}^1$, $z_{11}^2 = \xi_{11}^2$, $z_{12}^1 = \xi_{12}^1 - \alpha_{11}^1$, $z_{12}^2 = \xi_{12}^2 - \alpha_{11}^2$, $\alpha_{11}^1 = -c_{111}^{11}\xi_{11}^1 - \frac{1}{2}p\xi_{11}^1(x^1)^2$ and $\alpha_{11}^2 = -c_{111}^{21}\xi_{11}^2 - c_{111}^{22}(\xi_{11}^2)^3$.

Second, consider (2.190) and carry out Step 2 in Sect. 2.6.3. We obtain the following controllers:

$$u_2^1 = -z_{22}^1 - z_{21}^1 - \psi_{22}^1 - \delta_{22}^1 z_{22}^1,$$
$$u_2^2 = -z_{22}^2 - z_{21}^2 - \psi_{22}^2 - \delta_{22}^2 z_{22}^2,$$

where $z_{21}^1 = \xi_{21}^1$, $z_{21}^2 = \xi_{21}^2$, $z_{22}^1 = \xi_{22}^1 - \alpha_{21}^1$, $z_{22}^2 = \xi_{22}^2 - \alpha_{21}^2$, and

$$\alpha_{21}^1 = -2z_{21}^1 - (z_{21}^1)^3 - z_{21}^1\left[\left(px^1 - z_{12}^1\frac{\partial\alpha_{11}^1}{\partial x^1}\right)^2 + \frac{1}{2}\left(z_{11}^1 - z_{12}^1\frac{\partial\alpha_{11}^1}{\partial z_{11}^1}\right)^2\right],$$

$$\alpha_{21}^2 = -2z_{21}^2 - 2(z_{21}^2)^3 - z_{21}^2\left[\frac{1}{2}\left(z_{11}^2 - \frac{\partial\alpha_{11}^2}{\partial z_{11}^2}\right)^2(z_{11}^2)^2 + \frac{1}{2}(z_{21}^2)^2\right],$$

$$\psi_{22}^1 = \frac{\partial\alpha_{21}^1}{\partial x^1}[x^1 + (x^1)^3] - \frac{\partial\alpha_{21}^1}{\partial z_{11}^1}(z_{12}^1 + \alpha_{11}^1) \\ - \frac{\partial\alpha_{21}^1}{\partial z_{12}^1}u_1^1 - \frac{\partial\alpha_{21}^1}{\partial z_{21}^1}(z_{22}^1 + \alpha_{21}^1),$$

$$\delta_{22}^1 = \frac{1}{2}\left(\frac{\partial\alpha_{21}^1}{\partial x^1}z_{11}^1\right)^2 + \left(\frac{\partial\alpha_{21}^1}{\partial x^1}z_{21}^1\right)^2 + \frac{1}{2}\left(\frac{\partial\alpha_{21}^1}{\partial z_{11}^1}z_{21}^1\right)^2 + \left(\frac{\partial\alpha_{21}^1}{\partial z_{21}^1}\right)^2,$$

$$\psi_{22}^2 = \frac{\partial\alpha_{21}^2}{\partial z_{11}^2}(z_{12}^2 + \alpha_{11}^2) - \frac{\partial\alpha_{21}^2}{\partial z_{12}^2}u_1^2 - \frac{\partial\alpha_{21}^2}{\partial z_{21}^2}(z_{22}^2 + \alpha_{21}^2),$$

$$\delta_{22}^2 = \frac{1}{2}\left(\frac{\partial\alpha_{21}^2}{\partial z_{21}^2}z_{21}^2 z_{11}^2\right)^2 + \frac{1}{2}\left(\frac{\partial\alpha_{12}^2}{\partial\xi_{21}^2}\right)^2 + \left(\frac{\partial\alpha_{21}^2}{\partial z_{21}^2}\right)^2.$$

The derived controllers stabilize the system (2.190) because they render the Lyapunov function

$$W_2 = W_1 + \sum_{i=1}^{2}\sum_{d=1}^{2}(z_{2d}^{j})^2$$

satisfy

$$\dot{W}_2 \le -(p-2)(x^1)^2 - \left(\frac{p}{2}-1\right)(x^1)^4 - (c_{111}^{11}-1)(z_{11}^1)^2 - c_{111}^{21}(z_{11}^2)^2$$
$$-(c_{111}^{22}-2)(z_{11}^2)^4 - c_{121}^{11}(z_{12}^1)^2 - c_{121}^{21}(z_{12}^2)^2 - \sum_{i=1}^{2}\sum_{j=1}^{2} c_{2j2}^{i1}(z_{2j}^{j})^2.$$

For the purpose of demonstration, simulation is carried out for the initial conditions $x^1 = 0.9$, $\xi_{11}^1 = -0.9$, $\xi_{11}^2 = 0.5$, $\xi_{12}^1 = 0.5$, $\xi_{12}^2 = -0.7$, $\xi_{21}^1 = 0.7$, $\xi_{21}^2 = 0.8$, $\xi_{22}^1 = -0.8$, $\xi_{22}^2 = 0.9$ and the parameters $p = 3$, $c_{111}^{11} = 2$, $c_{111}^{21} = 1$, $c_{111}^{22} = 2$, $c_{121}^{11} = 1$, $c_{121}^{21} = 1$, $c_{2j2}^{i1} = 1$ for $i, j = 1, 2$. The responses for the closed-loop system are plotted in Fig. 2.3.

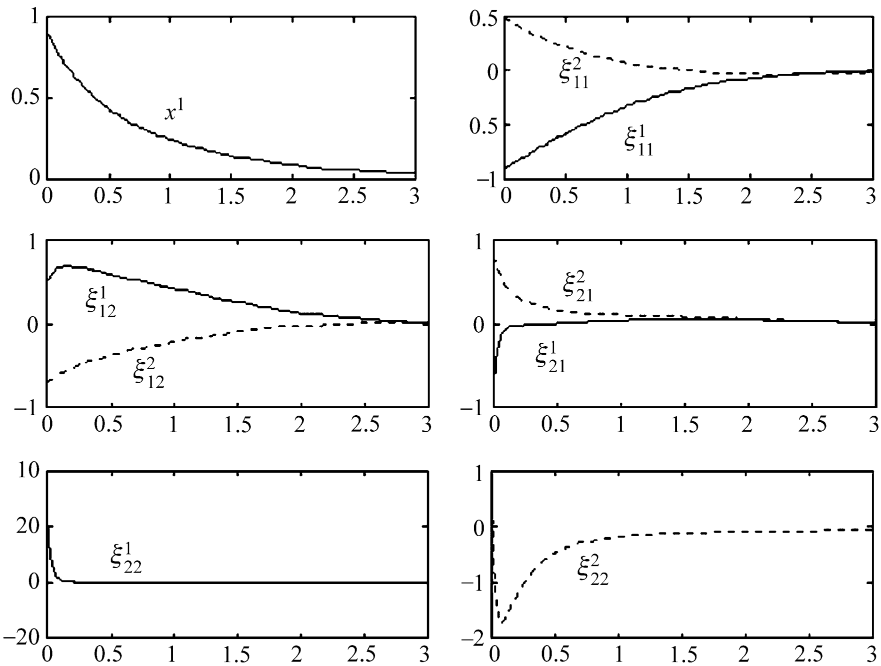

Fig. 2.3 Trajectories of the closed-loop system

2.6.5 Proof of Lemma 2.8

Lemma 2.8

$$(a_1 + \cdots + a_n)^t \le n^{t-1}(|a_1|^t + \cdots + |a_n|^t).$$

Proof It is obvious that

$$(a_1 + \cdots + a_n)^t \le (|a_1| + \cdots + |a_n|)^t.$$

Set $\bar{a} = (|a_1| + \cdots + |a_n|)/n$ and $f(x) = x^t$ for $t \ge 1$ and $x \ge 0$. Because $f(x)$ is c^∞ function, by Taylor expansion, there exists a real value ξ between x and $\bar{a}$, satisfying

$$f(x) = f(\bar{a}) + \dot{f}(\bar{a})(x - \bar{a}) + \frac{1}{2}\ddot{f}(\xi)(x - \bar{a})^2$$

which implies that

$$f(x) \ge f(\bar{a}) + \dot{f}(\bar{a})(x - \bar{a})$$

because $\ddot{f}(\xi)(x - \bar{a})^2 \ge 0$. Therefore

$$\begin{aligned} f(|a_n|) &\ge f(\bar{a}) + \dot{f}(\bar{a})(|a_1| - \bar{a}), \\ &\vdots \\ f(|a_n|) &\ge f(\bar{a}) + \dot{f}(\bar{a})(|a_n| - \bar{a}). \end{aligned}$$

Adding all these equations together gives

$$|a_1|^t + \cdots + |a_n|^t \ge nf(\bar{a}) = n(\bar{a})^t = \frac{(|a_1| + \cdots + |a_n|)^t}{n^{t-1}}$$

which implies that

$$\begin{aligned} (a_1 + \cdots + a_n)^t &\le (|a_1|^t + \cdots + |a_n|)^t \\ &\le n^{t-1}(|a_1| + \cdots + |a_n|^t). \end{aligned}$$

□

2.7 Notes and References

This chapter provided a critical overview of decentralized control techniques for classes of nonlinear interconnected continuous-time systems. The area of nonlinear control is so wide to accommodate new and research directions along the productive ideas [9, 19, 22, 23, 29, 40, 42, 43, 45, 46]. In particular, the topic of nonlinear interconnected discrete-time systems has not been fully investigated in the literature.

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Chapter 3
Decentralized Control of Nonlinear Systems II

In this chapter, we start our examination of the development of decentralized control techniques for interconnected systems where we focus on the classes of nonlinear continuous-time systems. We focus on interconnected minimum-phase nonlinear systems with parameter uncertainty and bounded and/or strong nonlinear interconnections. The objective is to design a robust decentralized controller such that the closed-loop large-scale interconnected nonlinear system is globally asymptotically stable for all admissible uncertain parameters and interconnections. The design is recursive in nature. By employing $\mathcal{H}_\infty$ performance, the solution of the decentralized control problem is attained via the Hamilton-Jacobi-Isaacs (HJI) inequalities. Finally, a decentralized output-feedback tracking problem with disturbance attenuation is addressed for a new class of large-scale and minimum-phase nonlinear systems. Application of decentralized stabilization and excitation controls of multimachine power systems are demonstrated.

3.1 Introduction

Large-scale systems consisting of a set of small-interconnected subsystems can be found in many applications such as electric power systems, industrial manipulators, computer networks, etc. The centralized control of large-scale systems is usually infeasible due to the requirement of a formidable amount of information exchange. Hence, decentralized control, a control law based only on local information, is often preferable [46]. Certainly, because of the interconnections among subsystems the design of a decentralized control is in general, more difficult than that of a centralized control.

On the other hand, exact modeling is usually impossible for physical systems, not to mention large-scale systems due to their complexity. Therefore, a decentralized control design which takes into account possible modeling uncertainties is of practical significance. Usually, the uncertainties for large-scale interconnected systems appear not only in local subsystems but also in interconnections. Decentralized robust control for interconnected linear systems with uncertainties satisfying

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the so-called strict matching conditions has been investigated in [6, 11, 12, 44, 45] and references cited therein. The interconnections among subsystems treated in these papers are mostly bounded by first-order polynomials. It was pointed out in [44, 45] that interconnected systems with a decentralized control based on the first-order bounded interconnections may become unstable when the interconnections are of higher order. In [15], decentralized robust stabilization was considered for large-scale systems with interconnections bounded by some nonlinear functions and uncertainties satisfying the so-called matching conditions. Very recently, following the development of centralized control of nonlinear systems [17, 27, 32], a decentralized adaptive control for a class of large-scale interconnected nonlinear systems was proposed in [21] where the strict matching condition was relaxed and higher-order interconnections among subsystems were introduced. Notice that the system under consideration in [21] is assumed to be exactly linearizable and have a linear parameterization of uncertain parameters and/or disturbances.

In this section, we investigate the problem of decentralized robust control for a wider class of large-scale nonlinear systems with parametric uncertainty and nonlinear interconnections. Each subsystem of the interconnected system is assumed to be partially feedback linearizable and minimum phase. The uncertain parameters and/or disturbances are allowed to be time-varying and enter the system nonlinearly. The nonlinear interconnections are bounded by general nonlinear functions of the zero-dynamics and outputs of other subsystems. Inspired by the recent work of centralized nonlinear control [7, 17, 29, 32, 42], we show that decentralized global robust stabilization can be achieved for the uncertain interconnected large-scale systems by employing a recursive controller design method. Our result relies on a proper construction of Lyapunov function for the interconnected systems. A numerical example is presented to demonstrate the effectiveness of the proposed robust decentralized control technique.

3.1.1 System Description

Consider a large-scale nonlinear system comprised of N interconnected subsystems with time-varying unknown parameters and/or disturbances entering nonlinearly into the state equation. The ith subsystem is given as

$$\begin{aligned}
\dot{z}_j &= f_{j0}(z_j, x_{j1}) + \phi_{j0}(z_j, \bar{x}_{j0}, Z_j, Y_j; \theta) x_{j1}, \\
\dot{x}_{j1} &= x_{j2} + \phi_{j1}(z_j, \bar{x}_{j1}, Z_j, Y_j; \theta), \\
\dot{x}_{j2} &= x_{j3} + \phi_{j2}(z_j, \bar{x}_{j2}, Z_j, Y_j; \theta), \\
&\vdots \\
\dot{x}_{j,r_j-1} &= x_{j,r_j} + \phi_{j,r_j-1}(z_j, \bar{x}_{j,r_j-1}, Z_j, Y_j; \theta), \\
\dot{x}_{j,r_j} &= v_j + \phi_{j,r_j}(z_j, \bar{x}_{j,r_j}, Z_j, Y_j; \theta), \\
y_j &= x_{j1},
\end{aligned} \tag{3.1}$$

where

$$\bar{x}_{j,k} = [x_{j1}\ x_{j2}\ \dots\ x_{jk}]^t, \quad \bar{x}_{j0} = x_{j1}, \quad x_j = \bar{x}_{jr_j},$$

(z_j, x_j) is the state vector of the jth subsystem with

$$z_j \in \Re^{n_j - r_j}, \quad Z_j = [z_1^t\ z_2^t\ \dots\ z_{j-1}^t\ z_{j+1}^t\ \dots\ z_N^t]^t,$$
$$Y_j = [y_1\ y_2\ \dots\ y_{j-1}\ y_{j+1}\ \dots\ y_N]^t,$$

and $v_j \in \Re$ is the control input, $y_j \in \Re$ is the output, θ is a vector of unknown, time-varying piecewise continuous parameters and/or disturbances which belong to a known compact set Ω. The vector fields f_{j0} and ϕ_{jk} are smooth with $f_{j0}(0,0) = 0$ and $\phi_{jk}(0,0,0,0;\theta) = 0$, $\forall\theta \in \Omega$, $1 \le j \le N$, $0 \le k \le r_j$. Observe that the vector (ϕ_{jk}), $k = 0, 1, 2, \dots, r_j$, represents the interconnections of the jth subsystem with the other subsystems.

In what follows, we consider the decentralized robust control problem for a wider class of interconnected systems with partially feedback linearizable subsystems and nonlinear parameterization of time-varying parametric uncertainty. Observe from (3.1) that the interconnections involve the zero-dynamics and outputs of other subsystems.

Remark 3.1 Similar to the centralized case discussed in [29, 34], the zero dynamics of each subsystem in (3.1) are independent of the uncertain parameter vector θ. For notional simplicity, in the sequel, we assume that $n_j = n$, $r_j = r$, $1 \le j \le N$. Then, by considering $y_j = x_{j1}$, system (3.1) becomes

$$\begin{aligned}
\dot{z}_j &= f_{j0}(z_j, x_{j1}) + \phi_{j0}(z_j, \bar{x}_{i0}, Z_j, X_{j1};\theta)x_{j1},\\
\dot{x}_{j1} &= x_{j2} + \phi_{j1}(z_j, \bar{x}_{j1}, Z_j, X_{j1};\theta),\\
\dot{x}_{j2} &= x_{j3} + \phi_{j2}(z_j, \bar{x}_{j2}, Z_j, X_{j1};\theta),\\
&\ \ \vdots\\
\dot{x}_{j,r-1} &= x_{j,r} + \phi_{j,r-1}(z_j, \bar{x}_{j,r-1}, Z_j, X_{j1};\theta),\\
\dot{x}_{j,r} &= v_j + \phi_{j,r}(z_j, \bar{x}_{j,r}, Z_j, X_{j1};\theta),
\end{aligned} \tag{3.2}$$

where $X_{j1} = Y_j = [x_{11}\ x_{21}\ \dots\ x_{j-1,1}\ x_{j+1,1}\ \dots\ x_{N1}]^t$.

We make the following assumptions for system (3.2).

Assumption 3.1 There exist some smooth real-valued functions

$$V_{j0}(z_j), \quad j = 1, 2, \dots, N$$

which are positive definite and proper (radially unbounded), such that

$$\frac{\partial V_{j0}}{\partial z_j} f_{j0}(z_j, 0) \le -\nu_j \|z_j\|^2, \quad 1 \le j \le N \tag{3.3}$$

for some positive real numbers $\nu_j > 0$.

Assumption 3.2 The nonlinear interconnections ϕ_{jk} in (3.2) satisfy

$$\begin{aligned}
&|\phi_{jk}(z_j, \bar{x}_{jk}, Z_j, X_{j1}; \theta) - \phi_{jk}(z_j, \bar{x}_{jk}, 0, 0, \theta)| \\
&\quad \le \sum_{\ell=1}^{N} \eta_{jk\ell}(z_j, \bar{x}_{jk})[\zeta_{jk\ell}^{0}(\|z_l\|)\|z_l\| + \zeta_{jk\ell}^{1}(z_\ell, x_{\ell 1})|x_{\ell 1}|] \\
&\quad \le \sum_{\ell=1}^{N} \eta_{jk\ell}(z_j, \bar{x}_{jk})\zeta_{jk\ell}(\|(z_\ell, x_{\ell 1})\|), \qquad (3.4)
\end{aligned}$$

for any $\theta \in \Omega$, $\eta_{jk\ell}(\cdot)$, $\zeta_{jk\ell}^{0}(\cdot)$ and $\zeta_{jk\ell}^{1}(\cdot)$, $\ell = 1, 2, \ldots, N$, $0 \le k \le r$, $1 \le j \le N$ are nonnegative smooth functions with $\zeta_{jk\ell}^{0}(\cdot) = \zeta_{jk\ell}^{1}(\cdot) \equiv 0$.

Remark 3.2 Note that by the well-known converse Lyapunov theorem, the zero dynamics of each subsystem are globally asymptotically stable if and only if there exists a positive definite and proper Lyapunov function V_{j0} such that

$$(\partial V_{j0}/\partial z_j) f_{j0}(z_j, 0) < 0, \quad \forall z_j \neq 0.$$

Indeed, the requirement (3.3) is more restrictive than this. However, a globally exponentially minimum-phase nonlinear system (that is, the zero-dynamics of the system are globally exponentially stable) always satisfies condition (3.3).

Remark 3.3 The interconnections in Assumption 3.2 are very general, including many types of interconnections considered in existing literature as special cases, for example, interconnections bounded by linear (first-order) polynomials [6, 11] and higher-order polynomials [45]. Furthermore, unlike the work in [6, 11, 15, 45], no matching conditions are imposed for system (3.2).

In the sequel, we deal with the decentralized global robust stabilization problem for system (3.2) satisfying Assumptions 3.1 and 3.2. More precisely, we are concerned with the design of decentralized robust control laws $v_j = v_j(z_j, x_j)$, $j = 1, \ldots, N$, such that the overall closed-loop interconnected system (3.2) with the control laws is globally asymptotically stable for all admissible uncertainties and interconnections.

3.1.2 Robust Control Design

In this section, we shall show that the interconnected system of (3.2) is globally asymptotically stabilizable by decentralized state feedback controllers. It is demonstrated that the decentralized robust controllers can be constructed effectively by employing a Lyapunov-based recursive design procedure.

To establish the main result, we shall first present the following lemma which provides the first step of the induction in the construction of robust decentralized state feedback control laws of system (3.2).

Lemma 3.1 *Consider the first two state equations of system* (3.2):

$$\begin{aligned} \dot{z}_j &= f_{j0}(z_j, x_{j1}) + \phi_{j0}(z_j, x_{j1}, Z_j, X_{j1}; \theta)x_{j1}, \\ \dot{x}_{j1} &= x_{j2} + \phi_{j1}(z_j, x_{j1}, Z_j, X_{j1}, \theta), \\ y_j &= x_{j1}, \end{aligned} \tag{3.5}$$

satisfying Assumptions 3.1 *and* 3.2. *Then, there exists a smooth function* $x^*_{j2}(z_j, x_{j1})$ *with* $x^*_{j2}(0, 0) = 0$ *such that system* (3.5) *with the control* $x_{j2} = x^*_{j2}(z_j, x_{j1})$ *in the coordinates*

$$z_j = z_j, \qquad \tilde{x}_{j1} = x_{j1}$$

having

$$V_{j1} = W_j(V_{j0}) + \frac{1}{2}\tilde{x}^2_{j1}, \tag{3.6}$$

$$\begin{aligned} \dot{V}_{j1} \le{}& \frac{dW_j(V_{j0})}{dV_{j0}} \frac{\partial V_{j0}}{\partial z_j} f_{j00} - b_j(z_j, x_{j1})x^2_{j1} - r\tilde{x}^2_{j1} + \|z_j\|^2 \\ &+ \frac{1}{2}\sum_{\ell=1}^{N} \delta_{j1\ell}(\|(z_\ell, x_{\ell 1})\|) \end{aligned} \tag{3.7}$$

with V_{j0} *being given in Assumption* 3.1 *and* $W_j(\cdot)$ *and* $b_j(., .)$ *are, respectively, a smooth* $\mathcal{K}_\infty$*-function and a smooth function to be chosen. Moreover,*

$$f_{j00}(z_j) = f_{j0}(z_j, 0), \tag{3.8}$$

$$\delta_{j1\ell}(\|(z_\ell, x_{\ell 1})\|) = \beta^{-1}_{j0\ell}(\zeta_{j0\ell}(\|(z_\ell, x_{\ell 1})\|))^2 + \beta^{-1}_{j1\ell}(\zeta_{j1\ell}(\|(z_\ell, x_{\ell 1})\|))^2 \tag{3.9}$$

with $\beta_{j0\ell}$ *and* $\beta_{j1\ell}$ *being positive scaling constants.*

Proof Since $f_{j0}(z_j, x_{j1})$ of (3.5) is a smooth vector with $f_{j0}(0, 0) = 0$, there exists a smooth vector $f_{j1}(z_j, x_{j1})$ such that

$$f_{j0}(z_j, x_{j1}) = f_{j00}(z_j) + f_{j1}(z_j, x_{j1})x_{j1},$$

where $f_{j00}(z_j)$ is given by (3.8). In view of Assumption 3.2 and along the state trajectory of system (3.5), we have

$$\begin{aligned} \dot{V}_{j1} &= \frac{dW_j}{dV_{j0}} \frac{\partial V_{j0}}{\partial z_j}(f_{j0} + \phi_{j0}x_{j1}) + x_{j1}[x_{j2} + \phi_{j1}(z_j, x_{j1}, Z_j, X_{j1}; \theta)] \\ &= \frac{dW_j}{dV_{j0}} \frac{\partial V_{j0}}{\partial z_j}(f_{j00} + f_{j1}x_{j1}) + x_{j1}x_{j2} + x_{j1}\sum_{j=0}^{1} \psi^1_{j1}(z_j)\phi_{jl}(z_j, x_{j1}, 0, 0; \theta) \\ &\quad + x_{j1}\sum_{j=0}^{1} \psi^1_{j1}(z_j)(\phi_{jl}(z_j, x_{j1}, Z_j, X_{i1}; \theta) - \phi_{jl}(z_j, x_{j1}, 0, 0, \theta)), \end{aligned} \tag{3.10}$$

where

$$\psi_{j1}^{0}(z_j) = \frac{dW_j}{dV_{j0}} \frac{\partial V_{j0}}{\partial z_j}, \qquad \psi_{j1}^{1}(z_j) = 1.$$

Since $\phi_{j0}(0,0,0,0;\theta) = \phi_{j1}(0,0,0,0;\theta) = 0$, $\forall\theta$, there exists some function $\alpha_{j1}(z_j, x_{j1})$ such that

$$\left| x_{j1} \sum_{\ell=0}^{1} \psi_{j1}^{\ell}(z_j)\phi_{j\ell}(z_j, x_{j1}, 0, 0; \theta) \right| \le |x_{j1}|\alpha_{j1}(z_j, x_{j1})(\|z_j\| + \|x_{j1}\|). \quad (3.11)$$

Recalling Assumption 3.2, it follows from (2.17) that

$$\begin{aligned}
\dot{V}_{j1} \le\ & \frac{dW_j}{dV_{j0}} \frac{\partial V_{j0}}{\partial z_j}(f_{j00} + f_{j1}x_{j1}) + x_{j1}x_{j2} \\
& + |x_{j1}| \left| \frac{dW_j}{dV_{j0}} \right| \left\| \frac{\partial V_{j0}}{\partial z_j} \right\| \sum_{\ell=1}^{N} \eta_{j0\ell}(z_j, x_{j1})\zeta_{j0\ell}(\|(z_\ell, x_{\ell 1})\|) \\
& + |x_{j1}| \sum_{\ell=1}^{N} \eta_{j1\ell}(z_j, x_{j1})\zeta_{j1\ell}(\|(z_j, x_{\ell 1})\|) \\
& + |x_{j1}|\alpha_{j1}(z_j, x_{j1})(\|z_j\| + \|x_{j1}\|) \\
\le\ & \frac{dW_j}{dV_{j0}} \frac{\partial V_{j0}}{\partial z_j}(f_{j00} + f_{j1}x_{j1}) + x_{j1}(x_{j2} + x_{j1}\alpha_{j1}(z_j, x_{j1})) \\
& + \frac{1}{2}x_{j1}^2 \left| \frac{dW_j}{dV_{j0}} \right|^2 \left\| \frac{\partial V_{j0}}{\partial z_j} \right\|^2 \sum_{\ell=1}^{N} \beta_{j0\ell}\eta_{j0\ell}^2(z_j, x_{j1}) \\
& + \frac{1}{2} \sum_{\ell=1}^{N} \beta_{j0\ell}^{-1}(\zeta_{j0\ell}(\|(z_\ell, x_{\ell 1})\|))^2 \\
& + \frac{1}{2}x_{j1}^2 \sum_{\ell=1}^{N} \beta_{j1\ell}\eta_{j1\ell}^2(z_j, x_{j1}) + \frac{1}{2} \sum_{\ell=1}^{N} \beta_{j1\ell}^{-1}(\zeta_{j1\ell}(\|(z_\ell, x_{\ell 1})\|))^2 \\
& + \frac{1}{4}x_{j1}^2\alpha_{j1}^2(z_j, x_{j1}) + \|z_j\|^2 \\
=\ & \frac{dW_j}{dV_{j0}} \frac{\partial V_{j0}}{\partial z_j} f_{j00} + x_{j1}(x_{j2} + M_{j1}(z_j, x_{j1})) \\
& + \|z_j\|^2 + \frac{1}{2} \sum_{\ell=1}^{N} \delta_{j1\ell}(\|(z_\ell, x_{\ell 1})\|), \qquad (3.12)
\end{aligned}$$

where $\delta_{j1\ell}$ is given in (3.9) and

$$M_{j1}(z_j, x_{j1}) = \frac{dW_j}{dV_{j0}}\frac{\partial V_{j0}}{\partial z_j} f_{j1} + \frac{1}{2}x_{j1}\left|\frac{dW_j}{dV_{j0}}\right|^2 \left\|\frac{\partial V_{j0}}{\partial z_j}\right\|^2 \sum_{\ell=1}^{N} \beta_{j0\ell}\eta_{j0\ell}^2(z_j, x_{j1})$$
$$+ \frac{1}{2}\sum_{\ell=1}^{N} \beta_{j1\ell}\eta_{j1\ell}^2(z_j, x_{j1}) + x_{j1}\alpha_{j1}(z_j, x_{j1})$$
$$+ \frac{1}{4}x_{j1}\alpha_{j1}^2(z_j, x_{j1}). \tag{3.13}$$

Choosing

$$x_{j2} = x_{j2}^* = -M_{j1} - b_j(z_j, x_{j1})x_{j1} - r x_{j1}, \tag{3.14}$$

where $b_j(.,.)$ is a smooth function to counteract the effect of the interconnections and yet to be determined. Then, (3.7) is obtained and the proof of Lemma 3.1 is completed. □

Remark 3.4 Considering the case when $r = 1$, that is, $x_{j2} = v_j$ in (3.5) is the actual control input. Then it can be easily shown that the design functions $b_j(.,.)$ and $W_j(\cdot)$, $j = 1, 2, \ldots, N$ can be chosen such that the decentralized state feedback control $v_j = x_{j2}^*(z_j, x_{j1})$ solves the robust decentralized stabilization problem.

3.1.3 Recursive Method

Next, we proceed to establish the systematic recursive design methodology for constructing robust decentralized control laws for the system (3.2) when $r \geq 2$. We need this technical result.

Lemma 3.2 *Consider the first $\rho + 1$ state equations of system* (3.2):

$$\begin{aligned}
\dot{z}_j &= f_{j0}(z_j, x_{j1}) + \phi_{j0}(z_j, x_{j1}, Z_j, X_{j1}; \theta)x_{i1},\\
\dot{x}_{j1} &= x_{j2} + \phi_{j1}(z_j, x_{i1}, Z_j, X_{j1}; \theta),\\
\dot{x}_{j2} &= x_{j3} + \phi_{j2}(z_j, \bar{x}_{i2}, Z_j, X_{j1}; \theta),\\
&\vdots\\
\dot{x}_{j,\rho-1} &= x_{j,\rho} + \phi_{j,\rho-1}(z_j, \bar{x}_{i,\rho-1}, Z_j, X_{j1}; \theta),\\
\dot{x}_{j,\rho} &= x_{j,\rho+1} + \phi_{j,\rho}(z_j, \bar{x}_{i\rho}, Z_j, X_{j1}; \theta),
\end{aligned} \tag{3.15}$$

satisfying Assumptions 3.1 and 3.2. Suppose that for any given index $\rho = m$ ($1 \leq m \leq r - 1$), there exist smooth functions

$$x_{j2}^*(z_j, x_{j1}), \quad x_{i3}^*(z_j, \bar{x}_{j2}), \quad \ldots, \quad x_{j,m+1}^*(z_j, \bar{x}_{im}),$$
$$x_{jk}^*(0,0) = 0, \quad 2 \le k \le m+1$$

such that system (3.15) *with the control* $x_{j,m+1} = x_{j,m+1}^*(z_j, \bar{x}_{j,m})$ *in the new coordinates*

$$z_j = z_j, \quad \tilde{x}_{j1} = x_{j1},$$
$$\tilde{x}_{j2} = x_{j2} - x_{j2}^*(z_j, x_{j1}), \quad \ldots, \quad \tilde{x}_{jm} = x_{jm} - x_{j,m}^*(z_j, \bar{x}_{j,m-1}),$$

having

$$V_{jm} = W_j(V_{j0}) + \frac{1}{2}\sum_{k=1}^{m} \tilde{x}_{jk}^2,$$
$$\dot{V}_{jm} \le \frac{dW_j}{dV_{j0}} \frac{\partial V_{i0}}{\partial z_j} f_{j00} - b_j(z_j, x_{j1}) x_{j1}^2 - (r-m+1)\sum_{k=1}^{m} \tilde{x}_{jk}^2 + m\|z_j\|^2 \tag{3.16}$$

with V_{j0} *as given in Assumption* 3.1 *and*

$$\delta_{j0\ell}(\|(z_\ell, x_{\ell 1})\|) \equiv 0,$$
$$\delta_{jk\ell}(\|(z_\ell, x_{\ell 1})\|) = \delta_{j,k-1,\ell}(\|(z_\ell, x_{\ell 1})\|) + \sum_{k=0}^{k} \beta_{jk\ell}^{-1}(\zeta_{jk\ell}(\|(z_\ell, x_{\ell 1})\|))^2, \quad 1 \le k \le r. \tag{3.17}$$

Then for system (3.15) *with* $\rho = m+1$, *there exists a smooth decentralized state feedback control law*

$$x_{j,m+2} = x_{i,m+2}^*(z_j, \bar{x}_{j,m+1}); \quad x_{j,m+2}^*(0,0) = 0 \tag{3.18}$$

such that system (3.15) *with* (3.18) *in the new coordinates*

$$z_j = z_j, \quad \tilde{x}_{jk}, \quad 1 \le k \le m,$$
$$\tilde{x}_{j,m+1} = x_{j,m+1} - x_{j,m+1}^*(z_j, \bar{x}_{j,m})$$

satisfies

$$\dot{V}_{j,m+1} \le \frac{dW_j}{dV_{j0}} \frac{\partial V_{j0}}{\partial z_j} f_{j00} - b_j(z_j, x_{j1}) x_{j1}^2 - (r-m)\sum_{k=1}^{m+1} \tilde{x}_{jk}^2 + (m+1)\|z_j\|^2 + \frac{1}{2}\sum_{\ell=1}^{N} \delta_{j,m+1,\ell}(\|(z_\ell, x_{\ell 1})\|), \tag{3.19}$$

where

$$V_{j,m+1} = V_{jm} + \frac{1}{2}\tilde{x}_{j,m+1}^2.$$

Proof By evaluating the derivative of $\tilde{x}_{j,m+1} = x_{j,m+1} - x^*_{j,m+1}$, we obtain

$$\begin{aligned}\dot{\tilde{x}}_{j,m+1} &= x_{j,m+2} + a_{j,m+1}(z_j, \bar{x}_{j,m+1}) \\ &\quad + \sum_{\ell=0}^{m+1} \psi^{\ell}_{j,m+1}(z_j, \bar{x}_{j,m})\phi_{j\ell}(z_j, \bar{x}_{j\ell}, Z_j, X_{j1};\theta),\end{aligned}$$

where

$$\begin{aligned}a_{j,m+1}(z_j, \bar{x}_{j,m+1}) &= -\frac{\partial x^*_{j,m+1}}{\partial z_j} f_{j0}(z_j, x_{j1}) - \sum_{\ell=1}^{m} \frac{\partial x^*_{j,m+1}}{\partial x_j, \ell} x_{i,\ell+1}, \\ \psi^0_{j,m+1}(z_j, \bar{x}_{j,m}) &= -\frac{\partial x^*_{j,m+1}}{\partial z_j} x_{j1}, \\ \psi^{\ell}_{j,m+1}(z_j, \bar{x}_{j,m}) &= -\frac{\partial x^*_{j,m+1}}{\partial x_{j,\ell}}, \quad 1 \le \ell \le m, \\ \psi^{m+1}_{j,m+1}(z_j, \bar{x}_{j,m}) &= 1.\end{aligned}$$

Then, the time derivative of $V_{j,m+1}$ is given by

$$\begin{aligned}\dot{V}_{j,m+1} = \dot{V}_{j,m} + \tilde{x}_{j,m+1}\Bigg[& x_{j,m+2} + a_{j,m+1} \\ & + \sum_{\ell=0}^{m+1} \psi^{\ell}_{j,m+1}(z_j, \bar{x}_{j,m})\phi_{j\ell}(z_j, \bar{x}_{j\ell}, Z_j, X_{j1};\theta)\Bigg] \\ = \dot{V}_{jm} &+ \tilde{x}_{j,m+1}(x_{j,m+2} + a_{j,m+1}) \\ &+ \tilde{x}_{j,m+1} \sum_{\ell=0}^{m+1} \psi^{\ell}_{j,m+1}\phi_{j\ell}(z_j, \bar{x}_{j\ell}, 0, 0;\theta) \\ &+ \tilde{x}_{j,m+1} \sum_{\ell=0}^{m+1} \psi^{\ell}_{j,m+1}[\phi_{j\ell}(z_j, \bar{x}_{j\ell}, Z_j, X_{j1};\theta) - \phi_{j\ell}(z_j, \bar{x}_{j\ell}, 0, 0;\theta)].\end{aligned} \tag{3.20}$$

Define

$$\begin{aligned}\tilde{\phi}_{i\iota}(z_j, \bar{\tilde{x}}_{j\ell};\theta) &= \phi_{j\ell}(z_j, \bar{x}_{j\ell}, 0, 0;\theta) \\ &= \phi_{j\ell}(z_j, \bar{\tilde{x}}_{j\ell} + \bar{x}^*_{j\ell}, 0, 0;\theta), \quad 2 \le \ell \le m+1,\end{aligned} \tag{3.21}$$

where

$$\bar{\tilde{x}}_{j\ell} = (\tilde{x}_{j1}, \ldots, \tilde{x}_{j\iota}), \quad \bar{x}^*_{j\ell} = (x^*_{j1}, x^*_{j2}, \ldots, x^*_{j\ell}), \quad \bar{\tilde{x}}_{j0} = \tilde{x}_{j1}, \quad \bar{x}^*_{j0} = x^*_{i1}.$$

Since $\phi_{j\ell}(0,0,0,0;\theta) = 0$, $\forall\theta \in \Omega$, $0 \le \ell \le m+1$, it is easy to verify that $\tilde{\phi}_{j\ell}(0,0;\theta) = 0$, $\forall\theta \in \Omega$. Thus, there exist smooth bounding functions $\alpha_{i\iota}(z_j, \bar{\tilde{x}}_{j,\ell})$, $\ell = 0, 1, \ldots, m+1$ such that

$$\begin{aligned}
&|\phi_{i0}(z_j, x_{j1}, 0, 0; \theta)| = |\tilde{\phi}_{j0}(z_j, \tilde{x}_{j1}; \theta) \le \alpha_{j0}(z_j, \tilde{x}_{j1})(\|z_j\| + \|\tilde{x}_{j1}\|), \\
&|\phi_{j\ell}(z_j, \bar{x}_{j\ell}, 0, 0; \theta)| = |\tilde{\phi}_{j\ell}(z_j, \bar{\tilde{x}}_{j\ell}; \theta) \le \alpha_{i\iota}(z_j, \bar{\tilde{x}}_{j,\ell}) \left[\|z_j\| + \sum_{k=1}^{\iota} |\tilde{x}_{jk}| \right], \quad (3.22) \\
&\quad 1 \le \ell \le m+1.
\end{aligned}$$

In view of this, the second last term of (3.20) satisfies

$$\begin{aligned}
&\tilde{x}_{j,m+1} \sum_{\iota=0}^{m+1} \psi^{\iota}_{j,m+1} \phi_{i\iota}(z_j, \bar{x}_{j\ell}, 0, 0; \theta) \\
&\quad \le |\tilde{x}_{j,m+1}| \left[\psi^0_{j,m+1} |\alpha_{j0}(\|z_j\| + |\tilde{x}_{j1}|) + \sum_{\ell=1}^{m+1} |\psi^{\iota}_{j,m+1}| \alpha_{j\ell} \left(\|z_j\| + \sum_{k=1}^{\iota} |\tilde{x}_{jk}| \right) \right] \\
&\quad = |\tilde{x}_{j,m+1}| \Bigg[|\psi^0_{j,m+1}| \alpha_{j0}(\|z_j\| + |\tilde{x}_{j1}|) \\
&\qquad + \sum_{\ell=1}^{m} |\psi^{\iota}_{j,m+1}| \alpha_{i\iota} \left(\|z_j\| + \sum_{k=1}^{\iota} |\tilde{x}_{jk}| \right) \Bigg] \\
&\qquad + |\tilde{x}_{j,m+1}| \alpha_{j,m+1} \left(\|z_j\| + \sum_{k=1}^{m} |\tilde{x}_{jk}| \right) + \alpha_{j,m+1} \tilde{x}^2_{j,m+1} \\
&\quad \le \tilde{x}^2_{j,m+1} \sum_{\ell=0}^{m} (\psi^{\iota}_{j,m+1})^2 \alpha^2_{j\ell} (m+1)(\ell+1) \\
&\qquad + \frac{1}{4(m+1)} \left[(\|z_j\| + |\tilde{x}_{j1}|)^2 + \sum_{\ell=1}^{m} \frac{1}{(\ell+1)} \left(\|z_j\| + \sum_{k=1}^{\ell} |\tilde{x}_{jk}| \right)^2 \right] \\
&\qquad + \frac{1}{2}(m+1) \tilde{x}^2_{i,m+1} \alpha^2_{j,m+1} + \frac{1}{2(m+1)} \left(\|z_j\| + \sum_{k=1}^{m} |\tilde{x}_{jk}| \right)^2 + \alpha_{j,m+1} \tilde{x}^2_{j,m+1} \\
&\quad \le \tilde{x}^2_{j,m+1} \sum_{\ell=0}^{m} (\psi^{\iota}_{j,m+1})^2 \alpha^2_{j\ell} (m+1)(\ell+1) + \frac{1}{2} \|z_j\|^2 + \frac{1}{2} \sum_{k=1}^{m} |\tilde{x}_{jk}|^2
\end{aligned}$$

$$+\frac{1}{2}(m+1)\tilde{x}_{j,m+1}^2\alpha_{j,m+1}^2+\frac{1}{2}\left(\|z_j\|^2+\sum_{k=1}^{m}|\tilde{x}_{jk}|^2\right)+\alpha_{j,m+1}\tilde{x}_{j,m+1}^*$$

$$=\left[\sum_{\ell=0}^{m}(\psi_{j,m+1}^{\ell})^2\alpha_{j\ell}^2(m+1)(\ell+1)+\frac{1}{2}(m+1)\alpha_{j,m+1}^2+\alpha_{j,m+1}\right]\tilde{x}_{j,m+1}^2$$

$$+\|z_{jk}\|^2+\sum_{k=1}^{m}|\tilde{x}_{jk}|^2$$

$$\leq \tilde{x}_{j,m+1}^2E_{j,m+1}(z_j,\bar{\bar{x}}_{j,m+1})+\|z_j\|^2+\sum_{k=1}^{m}|\tilde{x}_{jk}|^2. \tag{3.23}$$

In view of Assumption 3.2 and (3.23), it follows that (3.20) can be written as

$$\begin{aligned}
\dot{V}_{j,m+1} \leq\ & \dot{V}_{jm}+\tilde{x}_{j,m+1}(x_{j,m+2}+a_{j,m+1}) \\
&+|\tilde{x}_{j,m+1}|\sum_{\ell=0}^{m+1}|\psi_{j,m+1}^{\ell}|\sum_{s=1}^{N}\eta_{j\ell s}(z_j,\bar{x}_{j\ell})\zeta_{j\ell s}(\|(z_s,x_{s1})\|) \\
&+\tilde{x}_{j,m+1}^2E_{j,m+1}+\|z_j\|^2+\sum_{k=1}^{m}|\tilde{x}_{jk}|^2 \\
\leq\ & \dot{V}_{jm}+\tilde{x}_{j,m+1}(x_{j,m+2}+a_{j,m+1})+\tilde{x}_{j,m+1}^2E_{j,m+1} \\
&+\|z_j\|^2+\sum_{k=1}^{m}|\tilde{x}_{ik}|^2 \\
&+\frac{1}{2}\tilde{x}_{j,m+1}^2\sum_{\ell=0}^{m+1}\sum_{s=1}^{N}(\psi_{j,m+1}^{\ell})^2(\eta_{j\ell s}(z_j,\bar{x}_{j\ell}))^2\beta_{j\ell s} \\
&+\frac{1}{2}\sum_{\ell=0}^{m+1}\sum_{s=1}^{N}(\zeta_{j\ell s}(\|(z_s,x_{s1})\|))^2\beta_{j\ell s}^{-1} \\
\leq\ & \frac{dW_j}{dV_{j0}}\frac{\partial V_{j0}}{\partial z_j}f_{j00}-b_j(z_j,x_{j1})x_{j1}^2-(r-m+1)\sum_{k=1}^{m}\tilde{x}_{jk}^2 \\
&+m\|z_j\|^2+\frac{1}{2}\sum_{\ell=1}^{N}\delta_{jm\ell}(\|(z_\ell,x_{\ell 1})\|)+\tilde{x}_{jm}\tilde{x}_{j,m+1} \\
&+\tilde{x}_{j,m+1}(x_{j,m+2}+M_{j,m+1})+\|z_j\|^2+\sum_{k=1}^{m}\tilde{x}_{jk}^2 \\
&+\frac{1}{2}\sum_{\ell=0}^{m+1}\sum_{s=1}^{N}(\zeta_{j\ell s}(\|(z_s,x_{s1})\|))^2\beta_{j\ell s}^{-1},
\end{aligned} \tag{3.24}$$

where

$$M_{j,m+1}(z_j,\bar{\bar{x}}_{j,m+1}) = a_{j,m+1} + \tilde{x}_{j,m+1}E_{j,m+1} \\ + \frac{1}{2}\tilde{x}_{j,m+1}\sum_{\ell=0}^{m+1}\sum_{s=1}^{N}(\psi^{\ell}_{j,m+1})^2(\eta_{j\ell s}(z_j,\bar{x}_{j\ell}))^2\beta_{j\ell s}. \tag{3.25}$$

Choosing

$$x_{j,m+2} = x^*_{j,m+2}(z_j,x_{j1},\dots,x_{j,m+1}) = -M_{j,m+1} - \tilde{x}_{jm} - (r-m)\tilde{x}_{j,m+1} \tag{3.26}$$

assures that (3.19) in Lemma 3.2 holds and hence the proof is completed. □

By applying Lemmas 3.1 and 3.2, the construction of robust decentralized control law which stabilizes the uncertain interconnected nonlinear systems (3.2) can be readily completed. This is seen by the following theorem:

Theorem 3.1 *Consider the uncertain interconnected system (3.2) satisfying Assumptions 3.1 and 3.2. There exists a decentralized control law, $v_j = v_j(z_j, x_j)$, $j = 1, 2, \dots, N$, such that the overall system under the decentralized controller is globally asymptotically stable for all admissible uncertainties and interconnections. A suitable decentralized controller is given by*

$$v_j = x^*_{j,r+1}(z_j,\bar{x}_{j,r}) = -M_{jr} - \tilde{x}_{j,r-1} - \tilde{x}_{jr}, \tag{3.27}$$

where M_{jr} is given in (3.25) with $m+1=r$.

Proof Based on Lemma 3.1, it follows that Lemma 3.2 is satisfied. Extending on this and applying Lemma 3.2 repeatedly until the rth step, we readily obtain a Lyapunov-based recursive decentralized control law. Therefore, we can construct $x^*_{j2}(z_j,x_{j1}),\dots,x^*_{j,r+1}(z_j,\bar{x}_{jr})$ such that under the new coordinates

$$z_j,\quad \tilde{x}_{j1} = x_{j1},\quad \tilde{x}_{j2} = x_{j2} - x^*_{j2}(z_j,x_{j1}),\quad \dots,\quad \tilde{x}_{jr} = x_{jr} - x^*_{j,r}(z_j,\bar{x}_{j,r-1}),$$

system (3.2) with control law (3.27) and

$$V_{jr} = W_j(V_{j0}) + \frac{1}{2}\sum_{k=1}^{r}\tilde{x}^2_{jk} \tag{3.28}$$

satisfies

$$\dot{V}_{jr} \le \frac{dW_j}{dV_{j0}}\frac{\partial V_{j0}}{\partial z_j}f_{j00} - b_j(z_j,x_{j1})x^2_{i1} - \sum_{k=1}^{r}\tilde{x}^2_{jk} + r\|z_j\|^2 \\ + \frac{1}{2}\sum_{\ell=1}^{N}\delta_{jr\ell}(\|(z_\ell,x_{\ell 1})\|), \tag{3.29}$$

where

$$\delta_{jr\ell}(\|(z_\ell, x_{\ell 1})\|) = r\beta_{j0\ell}^{-1}(\zeta_{j0\ell}(\|(z_\ell, x_{\ell 1})\|))^2 + \sum_{s=1}^{r}(r-s+1)\beta_{js\ell}^{-1}(\zeta_{js\ell l}(\|(z_s, x_{s1})\|))^2. \quad (3.30)$$

By virtue of Assumption 3.2, we get

$$\begin{aligned}
&\delta_{jr\ell}(\|(z_\ell, x_{\ell 1})\|) \\
&= r\beta_{j0\ell}^{-1}(\zeta_{j0\ell}^0(\|z_\ell\|)\|z_\ell\| + \zeta_{j0\ell}^1(z_\ell, x_{\ell 1})|x_{\ell 1}|)^2 \\
&\quad + \sum_{s=1}^{r}(r-s+1)\beta_{js\ell}^{-1}(\zeta_{js\ell}^0(\|z_\ell\|)\|z_\ell\| + \zeta_{js\ell}^1(z_\ell, x_{\ell 1})|x_{\ell 1}|)^2 \\
&\le 2r\beta_{j0\ell}^{-1}((\zeta_{j0\ell}^0(\|z_\ell\|))^2\|z_\ell\|^2 + (\zeta_{j0\ell}^1(z_\ell, x_{\ell 1}))^2 x_{\ell 1}^2) \\
&\quad + 2\sum_{s=1}^{r}(r-s+1)\beta_{js\ell}^{-1}((\zeta_{js\ell}^0(\|z_\ell\|))^2\|z_\ell\|^2 + (\zeta_{js\ell}^1(z_\ell, x_{\ell 1}))^2 x_{\ell 1}^2) \\
&\le 2\Delta_{j\ell}(\|z_\ell\|)\|z_\ell\|^2 + 2D_{j\ell}(z_\ell, x_{\ell 1})x_{\ell 1}^2, \quad (3.31)
\end{aligned}$$

where

$$\Delta_{jl}(\|z_\ell\|) = r\beta_{j0\ell}^{-1}(\zeta_{j0\ell}^0(\|z_\ell\|))^2 + \sum_{s=1}^{r}(r-s+1)\beta_{js\ell}^{-1}(\zeta_{js\ell}^0(\|z_\ell\|))^2, \quad (3.32)$$

$$D_{j\ell}(z_\ell, x_{\ell 1}) = r\beta_{j0\ell}^{-1}(\zeta_{j0\ell}^1(z_\ell, x_{\ell 1}))^2 + \sum_{s=1}^{r}(r-s+1)\beta_{js\ell}^{-1}(\zeta_{js\ell}^1(z_\ell, x_{\ell 1}))^2. \quad (3.33)$$

Proceeding further, we define

$$V = \sum_{j=1}^{N} V_{jr}$$

and invoking the structural identity

$$\begin{aligned}
&\sum_{j=1}^{N}\sum_{\ell=1}^{N}[\Delta_{jl}(\|z_\ell\|)\|z_\ell\|^2 + D_{jl}(z_\ell, x_\ell)x_{\ell 1}^2] \\
&\quad = \sum_{j=1}^{N}\sum_{\ell=1}^{N}[\Delta_{\ell j}(\|z_j\|)\|z_j\|^2 + D_{\ell j}(z_j, x_j)x_{j1}^2]
\end{aligned}$$

in view of Assumption 3.1 and by noting that $W_j(V_{j0})$ is a K_∞ function of V_{j0}, we arrive at

$$\begin{aligned}\dot{V}_{jr} &\leq \sum_{j=1}^{N}\left\{\frac{dW_j}{dV_{ij0}}\frac{\partial V_{j0}}{\partial z_j}f_{j00} - b_j(z_j, x_{j1})x_{j1}^2 - \sum_{k=1}^{r}\tilde{x}_{jk}^2 + r\|z_j\|^2\right.\\ &\quad \left. + \sum_{\ell=1}^{N}[\Delta_{\ell j}(\|z_j\|)\|z_j\|^2 + D_{\ell j}(z_j, x_j)x_{j1}^2]\right\}\\ &\leq \sum_{j=1}^{N}\left\{-\frac{dW_j}{dV_{j0}}\nu_j\|z_j\|^2 + \left[r + \sum_{\ell=1}^{N}\Delta_{\ell j}(\|z_j\|)\right]\|z_j\|^2\right.\\ &\quad \left. - \sum_{k=1}^{r}\tilde{x}_{jk}^2 - \left[b_j(z_j, x_{j1})x_{j1}^2 - \sum_{\ell=1}^{N}D_{\ell j}(z_j, x_{j1})\right]x_{j1}^2\right\}. \end{aligned} \tag{3.34}$$

According to Assumption 3.1 $V_{j0}(z_j)$ is radially unbounded and positive definite and therefore there exists a $\mathcal{K}_\infty$ function $\kappa_{\ell j}$ such that

$$\Delta_{\ell j}(\|z_j\|) \leq \Delta_{\ell j}(0) + \kappa_{\ell j}(V_{j0}). \tag{3.35}$$

On selecting

$$b_j(z_j, x_{j1}) = \sum_{\ell=1}^{N} D_{\ell j}(z_j, x_{j1}), \tag{3.36}$$

$$\frac{dW_j}{dV_{j0}} = k_j + \frac{1}{\nu_j}\left[r + \sum_{\ell=1}^{N}(\Delta_{\ell j}(0) + \kappa_{\ell j}(V_{j0}))\right], \quad W_j(0) = 0, \tag{3.37}$$

where $k_j > 0$ is a constant, it is readily evident that $W_j(\cdot)$ is a smooth $\mathcal{K}_\infty$-function. Then it follows that

$$\dot{V} \leq \sum_{j=1}^{N}\left\{\left(-k_j\nu_j\|z_j\|^2 - \sum_{k=1}^{r}\tilde{x}_{jk}^2\right)\right\}. \tag{3.38}$$

Due to the onto-relation between (z_j, x_j) and $(z_j, \tilde{x}_j)$, where $\tilde{x}_j = (\tilde{x}_{j1}, \ldots, \tilde{x}_{jr})$, the closed-loop interconnected system of (3.2) under the decentralized controller (3.27) is globally asymptotically stable for all admissible uncertainties and interconnections. □

Remark 3.5 It is interesting to observe from Theorem 3.1 that the functions

$$b_j(z_j, x_{j1}), \quad W_j(V_{j0}), \quad j = 1, 2, \ldots, N$$

can be selected *a priori* before the recursive design of the robust decentralized stabilization controller. Moreover, Theorem 3.1 provides a decentralized global stabilization result for uncertain interconnected minimum-phase nonlinear systems with parametric uncertainty and interconnections bounded by general nonlinear functions. This result essentially extends existing centralized results in [29, 32] to decentralized control of interconnected systems.

3.1.4 Simulation Example 3.1

Consider an interconnected system composed of two subsystems:

- subsystem 1:

$$\begin{aligned}\dot{z}_1 &= -2z_1 + z_1 x_{11},\\ \dot{x}_{11} &= x_{12} + x_{11} z_1 \sin\theta_1 + x_{21}^2 z_2 \cos\theta_1^2,\\ \dot{x}_{12} &= u_1 + x_{12}^2(x_{11}z_1 + z_1^2)\sin\theta_1 + x_{21}z_2\cos(\theta_1 z_1),\end{aligned}$$

- subsystem 2:

$$\begin{aligned}\dot{z}_2 &= -z_2 + x_{21}^2,\\ \dot{x}_{21} &= x_{22} + (x_{11}^2 z_1 + x_{21}^2 z_2)\sin(z_2\theta_2),\\ \dot{x}_{22} &= u_2 + x_{22}^2(x_{11}z_1^2 + x_{21}z_2^2)\sin\theta_2 + x_{22}^2 z_2^3\cos(\theta_2^2 z_2^2),\end{aligned}$$

where $\theta_1, \theta_2 \in [-2, 2]$.

It is easy to verify that the interconnections in the system under consideration satisfy Assumption 3.2. Initially, set $\beta_{jkm} = 1$, $j, k, m = 1, 2$. It follows from (3.32) and (3.33) that

$$\begin{aligned}&\Delta_{11} = \Delta_{12} = \Delta_{21} = \Delta_{22} = 0,\\ &D_{11} = 0, \quad D_{12} = 2x_{21}^2 z_2^2 + z_2^2, \quad D_{21} = 2x_{11}^2 z_1^2 + z_1^4, \quad D_{22} = 0.\end{aligned}$$

Letting $V_{10} = \frac{1}{2}z_1^2$ and $V_{20} = \frac{1}{2}z_2^2$. Then,

$$\frac{\partial V_{10}}{\partial z_1} f_{10}(z_1, 0) = -2z_1^2; \qquad \frac{\partial V_{20}}{\partial z_2} f_{20}(z_2, 0) = z_2^2.$$

It is readily evident that Assumption 3.1 is satisfied with $\nu_1 = 2$ and $\nu_2 = 1$.

From (3.35), it follows that

$$\kappa_{11}(V_{10}) = \kappa_{21}(V_{10}) = \kappa_{12}(V_{20}) = \kappa_{22}(V_{20}) = 0.$$

By selecting $k_1 = k_2 = 3$, then we have from (3.36) and (3.37):

$$\frac{dW_1}{dV_{10}} = 4, \qquad \frac{dW_2}{dV_{20}} = 5,$$
$$b_1 = D_{11} + D_{21}, \qquad b_2 = D_{12} + D_{22}.$$

It follows from (3.11) and (3.13) that

$$\alpha_{11} = x_{11}^2 + 0.25, \qquad \alpha_{21} = x_{21}^2$$

and

$$M_{11} = \frac{dW_1}{dV_{10}} z_1^2 + 0.5x_{11} + x_{11}\alpha_{11} + 0.25\alpha_{11}^2,$$
$$M_{21} = \frac{dW_2}{dV_{20}} z_2 x_{21} + 0.5x_{21} + x_{21}\alpha_{21} + 0.25\alpha_{21}^2.$$

Using (3.10), the virtual control is now computed as

$$x_{12}^* = -M_{11} - b_1 x_{11} - 2x_{11},$$
$$x_{22}^* = -M_{21} - b_2 x_{21} - 2x_{21}.$$

Next, letting $\tilde{x}_{j2} = x_{j2} - x_{j2}^*$, $j = 1, 2$, we obtain

$$\psi_{12}^0 = -\frac{\partial x_{12}^*}{\partial z_1} x_{11}, \qquad \psi_{22}^0 = -\frac{\partial x_{22}^*}{\partial z_2} x_{21},$$
$$\psi_{12}^1 = -\frac{\partial x_{12}^*}{\partial x_{11}}, \qquad \psi_{22}^1 = -\frac{\partial x_{22}^*}{\partial x_{21}}, \qquad \psi_{21}^2 = \psi_{22}^2 = 1,$$
$$a_{12} = -\frac{\partial x_{12}^*}{\partial z_1}(-2z_1 + x_{11}z_1) - \frac{\partial x_{12}^*}{\partial x_{11}} x_{12},$$
$$a_{22} = -\frac{\partial x_{22}^*}{\partial z_2}(-z_2 + x_{21}^2) - \frac{\partial x_{22}^*}{\partial x_{21}} x_{22}.$$

According to (3.22), we can select

$$\alpha_{12} = x_{12}^2(z_1^2 + 0.25), \qquad \alpha_{22} = x_{22}^2 z_2^2.$$

It then follows from (3.25) that

$$M_{12} = a_{12} + \tilde{x}_{12}(4(\psi_{12}^1)^2\alpha_{11}^2 + \alpha_{12}^2 + \alpha_{12}) + 0.5\tilde{x}_{12}((\psi_{12}^1)^2 + (\psi_{12}^2)^2),$$
$$M_{22} = a_{22} + \tilde{x}_{22}(4(\psi_{22}^1)^2\alpha_{21}^2 + \alpha_{22}^2 + \alpha_{22}) + 0.5\tilde{x}_{22}((\psi_{22}^1)^2 + (\psi_{22}^2)^2 x_{22}^4).$$

The decentralized control law can be obtained from (3.26) as follows:

$$u_1 = -x_{11} - M_{12} - \tilde{x}_{12},$$
$$u_2 = -x_{21} - M_{22} - \tilde{x}_{22}.$$

The interconnected system under consideration was simulated with the developed decentralized controller in order to demonstrate the effectiveness of the decentralized robust control design procedure. The initial conditions were set to be

$$z_1 = 1.0, \qquad x_{11} = -1.0, \qquad x_{12} = 1.5,$$
$$z_2 = 1.0, \qquad x_{21} = -1.0, \qquad x_{22} = 1.5$$

and the uncertainties θ_1 and θ_2 are given by $\theta_1 = 2\sin t$ and $\theta_2 = 2\cos t^2$. It is quite evident that the uncertainties are time-varying ones and belong to the set $[-2, 2]$. The closed-loop responses for the two subsystems are plotted in Figs. 3.1 and 3.2 from which the stability is clearly assured.

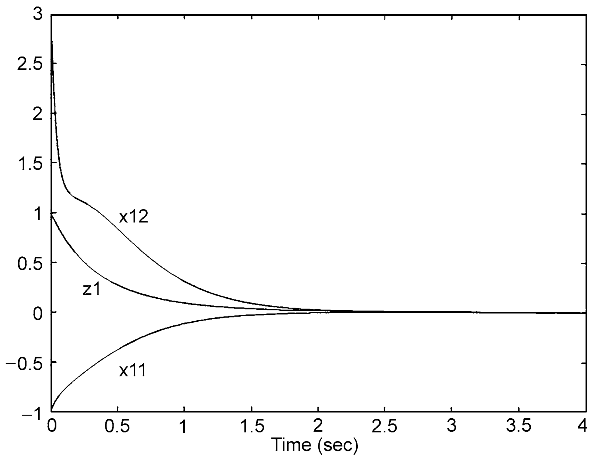


Fig. 3.1 Closed-loop responses of subsystem 1

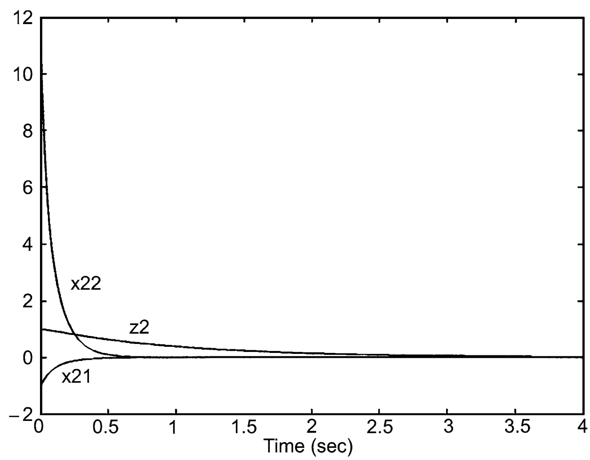


Fig. 3.2 Closed-loop responses of subsystem 2

3.2 Global Almost Disturbance Decoupling

We have learned before that the decentralized stabilization problem for interconnected linear systems with uncertainties satisfying the so-called strict matching conditions has been investigated in [6, 11, 45, 46], and references therein. It has been customary to treat the interconnections among subsystems to be bounded by first-order polynomials of state. In [15], decentralized robust stabilization was considered for interconnected systems bounded by some nonlinear functions with matching uncertainties.

3.2.1 Introduction

The decentralized $\mathcal{H}_\infty$ control problem for linear systems has been considered in [37] where it was shown that the design of each local $\mathcal{H}_\infty$ control law depends on the solution of a higher-order algebraic Riccati equation associated with the overall interconnected system. In [55] a design approach was provided for composite linear systems. In spite of significant advance in centralized $\mathcal{H}_\infty$ control for nonlinear systems [2, 19, 52] and references therein—few results on decentralized $\mathcal{H}_\infty$ control of interconnected nonlinear systems are available in the literature. Note that all these results on $\mathcal{H}_\infty$ control of nonlinear systems require solution of the *Hamilton-Jacobi-Isaacs* (HJI) partial differential equations, which imposes an intricate difficulty and especially in practical applications. On the other hand, the problems of global disturbance attenuation and almost disturbance decoupling for a class of nonlinear systems with lower triangular structure [19, 33]. An interesting feature of these results is that a solution of the HJI equations or inequalities is not required. However, their basic limitation is that there was no penalty on control efforts which in turn represents a serious drawback as it would result in a poor dynamic performance and large control effort. This issue has been addressed in [20], where a global $\mathcal{L}_2$-gain design methodology was developed for minimum-phase nonlinear systems in the lower triangular form. In the light of the results in [30, 52], the relationship between an $\mathcal{L}_2$-gain of a nonlinear system and that of its linearized system has become quite transparent. Accordingly, if the $\mathcal{H}_\infty$ control problem for the linearized system is solvable, one can find a local solution to the $\mathcal{H}_\infty$ control problem of the original nonlinear system. The pioneering results of [20] suggest that *for linearizable systems or minimum-phase nonlinear systems with triangular structure, the solution to the problem of disturbance attenuation for the linearized system suffices to determine a feedback law that solves the global disturbance attenuation problem with internal stability for the corresponding nonlinear system*. Note that in [20], a weighting function is fixed *a posteriori* and only the problem of inverse global $\mathcal{L}_2$-gain analysis is addressed, that is, determine a weighting function $r(x)$ and a globally stabilizing feedback law $\alpha(x)$ (constructed by starting from the solution of a strict Riccati inequality) that solve the problem of global $\mathcal{H}_\infty$ disturbance attenuation.

In what follows, a global decentralized $\mathcal{H}_\infty$ control problem via state feedback control for a class of interconnected nonlinear systems is investigated. First, we consider a rather general interconnected nonlinear system with strong nonlinear interconnections. The global decentralized $\mathcal{H}_\infty$ control problem of the system is shown to be converted into the centralized $\mathcal{H}_\infty$ control problems for a set of auxiliary nonlinear systems without interconnections. It is well known that solutions to the latter problems are related to the HJI equations. This result extends the decentralized $\mathcal{H}_\infty$ control problem for linear interconnected systems to the nonlinear case.

Bearing in mind the difficulty of solving HJI equations globally, a global decentralized almost disturbance decoupling problem (DADDP) is considered for a class of interconnected systems which are transformable to interconnected systems with lower triangular structure. It is then shown that a solution to the DADDP can be obtained via recursive design technique. We focus next on the $\mathcal{H}_\infty$ control problem. Specifically, a set of decentralized state feedback control laws as well as state-dependent weights of the control inputs are sought such that the associated global $\mathcal{H}_\infty$ control problem is solvable.

3.3 Decentralized $\mathcal{H}_\infty$ Control

Consider a large-scale nonlinear system composed of N interconnected subsystems of the form

$$\begin{aligned} \dot{x}_j &= A_j(x_j) + B_j(x_j)u_j + p_j(x_j)\omega_j + h_j(x_j), \\ y_j &= C_j(x_j), \\ z_j &= (y_j^t y_j + u_j^t R_j(x_j)u_j)^{\frac{1}{2}},\ j = 1, 2, \ldots, N, \end{aligned} \tag{3.39}$$

where $x_j \in \Re^{n_j}$ is the state of jth subsystem, $j = 1, 2, \ldots, N$, $x = [x_1^t \ \ldots \ x_N^t]^t$ is the state of the overall interconnected system, $u_j \in \Re^{m_j}$, $\omega_j \in \Re^{q_j}$ and $z_j \in \Re$ are the control input, the disturbance input and the penalty output, respectively. The functions

$$A_j(x_j), \quad B_j(x_j), \quad C_j(x_j), \quad p_j(x_j), \quad R_j(x_j)$$

are smooth with appropriate dimensions and satisfy

$$A_j(0) = 0, \qquad C_j(0) = 0, \qquad h_j(0) = 0, \qquad \Re_j(x_j) \geq 0, \quad \forall x_j \in \Re^{n_j}.$$

Assumption 3.3 The nonlinear interconnections

$$h_j(x) = [h_{j1}(x)\ h_{j2}(x)\ \ldots\ h_{jn_j}(x)]^t$$

are bounded by

$$|h_{jk}(x)| \leq \eta_{jk}(x_j) \sum_{\ell=1}^{N} \zeta_{jk\ell}(x_\ell), \tag{3.40}$$

where $\eta_{jk}(x_j)$, $\zeta_{jk\ell}(x_\ell)$, $1 \leq k \leq n_j$, $1 \leq j, \ell \leq N$, are nonnegative continuous functions with $\zeta_{jk\ell}(0) = 0$.

Remark 3.6 It is interesting to note that the interconnections $h_j(x_j)$, $j = 1, 2, \ldots, N$ in Assumption 3.3 are quite general and include, as special cases the interconnections bounded by linear first-order polynomials [6, 11, 15] and higher-order polynomials [15]. In addition, no matching conditions are imposed.

We direct attention to the global decentralized $\mathcal{H}_\infty$ control problem for the system (3.39) satisfying Assumption 3.3. Formally, given scalars $\gamma_j > 0$, $j = 1, 2, \ldots, N$, we are interested in the design of local decentralized control laws, $u_j = u_j(x_j)$, $j = 1, 2, \ldots, N$, at the subsystem level such that the overall closed-loop interconnected system (3.39) is globally asymptotically stable and the $\mathcal{L}_2$-gain from the disturbance $\omega = [\omega_1^t \ \ldots \ \omega_N^t]^t$ to the controlled output $z = [z_1^t \ \ldots \ z_N^t]^t$ is less than $\gamma = [\gamma_1 \ \ldots \ \gamma_N]^t$ in the following sense

$$\sum_{j=1}^{N} \int_0^\infty z_j^t z_j \, dt < \sum_{j=1}^{N} \gamma_j^2 \int_0^\infty \omega_j^t \omega_j \, dt + \delta(x_0) \tag{3.41}$$

for all $\omega_j \in \mathcal{L}_2[0, \infty)$, where $\delta(x_0)$ is a real-valued function of the initial state $x_0 = [x_1^t(0) \ \ldots \ x_N^t(0)]^t$ satisfying $\delta(0) = 0$.

Remark 3.7 In stating the foregoing problem, γ_j, $j = 1, 2, \ldots, N$, can be regarded as the prespecified level of $\mathcal{H}_\infty$disturbance attenuation for each sub-system. When $\gamma_j = \gamma_0, \forall j$, (3.41) becomes

$$\|z\|_2^2 \leq \gamma_0^2 \|\omega\|_2^2 + \delta(x_0) \tag{3.42}$$

for all $\omega_j \in \mathcal{L}_2[0, \infty)$, where $z = [z_1 \ z_2 \ \ldots \ z_N]^t$ and $\omega = [\omega_1^t \ \ldots \ \omega_N^t]^t$.

In this case, a standard decentralized $\mathcal{H}_\infty$ control problem is recovered.

To pave the way toward a result on decentralized nonlinear $\mathcal{H}_\infty$ control, it is crucial to recall the definition of global disturbance attenuation for nonlinear systems. For this purpose, consider a nonlinear system of the form

$$\begin{aligned} \dot{x} &= A(x) + B(x)u + p(x)\omega, \\ y &= C(x), \\ z &= (y^t y + u^t R(x) u)^{\frac{1}{2}}, \end{aligned} \tag{3.43}$$

where $\omega \in \Re^q$, $u \in \Re^m$ and $z \in \Re$ are the disturbance input, the control input and the penalty output, respectively, with $A(0) = 0$, $C(0) = 0$ and $R(x) \geq 0$ for all $x \in \Re^n$ and pose the following

Definition 3.1 Given a real number $\gamma > 0$, system (3.43) is said to have global $\mathcal{H}_\infty$ disturbance attenuation γ if there exists a feedback control laws $u = \alpha(x)$ with $\alpha(0) = 0$ such that, for some proper function $V(x) > 0$, the HJI matrix inequality

$$\begin{bmatrix} \frac{\partial V(x)}{\partial x}(A(x) + B(x)\alpha(x)) + \frac{1}{4\gamma^2}\big(\frac{\partial V(x)}{\partial x}p(x)\big)^2 & C^t(x) & \alpha^t(x)R(x) \\ \bullet & -I & 0 \\ \bullet & \bullet & -R(x) \end{bmatrix} < 0 \tag{3.44}$$

is satisfied for all nonzero x.

Remark 3.8 It is well known that if (3.44) holds, then the feedback law $u = \alpha(x)$ globally asymptotically stabilizes the equilibrium $x = 0$ of the system (3.43) when $\omega = 0$, and render the $\mathcal{L}_2$-gain from ω to z less than or equal to γ [52].

Associated with the interconnected system (3.39), we introduce the following auxiliary systems:

$$\begin{aligned} \dot{x}_j &= A_j(x_j) + B_j(x_j)u_j + \big[p_j(x_j)\ \beta^{\frac{1}{2}}\gamma_j\eta_j(x_j)\big]\tilde{\omega}_j, \\ \tilde{y}_j &= \begin{bmatrix} C_j(x_j) \\ \beta_j^{-\frac{1}{2}}(d_j(x_j))^{\frac{1}{2}} \end{bmatrix}, \\ \tilde{z}_j &= (\tilde{y}_j^t\tilde{y}_j + u_j^t R_j(x_j)u_j)^{\frac{1}{2}}, \quad j = 1, 2, \ldots, N, \end{aligned} \tag{3.45}$$

where x_j is the state, $\tilde{\omega}_j$ is the disturbance input, u_j is the control input, $\tilde{z}_j$ is the penalty output, $A_j(x_j)$, $B_j(x_j)$, $C_j(x_j)$, $p_j(x_j)$ and $R_j(x_j)$ are the same as in the system (3.39), β_j, $j = 1, 2, \ldots, N$, are some positive scalars and $\beta = \sum_{l=1}^N \beta_l$. Moreover,

$$\eta_j(x_j) = \mathrm{diag}\{\eta_{i1}(x_j), \ldots, \eta_{in_j}(x_j)\},$$

$$\zeta_{il}(x_l) = [\zeta_{il1}(x_l), \ldots, \zeta_{in_jl}(x_l)]^t, \qquad d_j(x_j) = \sum_{l=1}^N \zeta_{li}^t\zeta_{li}.$$

The following theorem establishes that to solve the global decentralized $\mathcal{H}_\infty$ control problem for the system (3.39), it suffices to solve the $\mathcal{H}_\infty$ control problem for the auxiliary system (3.45).

Theorem 3.2 *Consider the interconnected system* (3.39) *satisfying Assumption* 3.3. *Given* $\gamma_j > 0$, $j = 1, 2, \ldots, N$, *suppose that there exist state feedback control laws* $u_j = \alpha_j(x_j)$ *with* $\alpha_j(0) = 0$ *such that the system* (3.45) *has global* $\mathcal{H}_\infty$ *disturbance* *attenuation γ_j from the disturbance $\tilde{\omega}_j$ to the penalty output $\tilde{z}_j$. Then the decentralized control laws $u_j = \alpha_j(x_j)$ render the system (3.39) globally asymptotically stable with the L_2-gain from the disturbance $\omega = [\omega_1^t \ \dots \ \omega_N^t]^t$ to the controlled output $z = [z_1^t \ \dots \ z_N^t]^t$ less than $\gamma = [\gamma_1 \ \dots \ \gamma_N]^t$ for all $\omega_j \in \mathcal{L}_2[0,\infty)$, $j = 1, 2, \dots, N$.*

Proof Suppose that system (3.45) has global $\mathcal{H}_\infty$ disturbance attenuation γ_j. By Definition 3.1, there exist feedback control laws $u_j = \alpha_j(x_j)$ with $\alpha_j(0) = 0$ such that, for some proper function $V_j(x_j) > 0$, the HJI inequalities

$$\begin{aligned}
&\frac{\partial V_j(x_j)}{\partial x_j}(A_j(x_j) + B_j(x_j)\alpha(x_j)) + \frac{1}{4\gamma_j^2}\left(\frac{\partial V_j(x_j)}{\partial x_j}p_j(x_j)\right)^2 \\
&\quad + \frac{1}{4}\beta \frac{\partial V_j(x_j)}{\partial x_j}\eta_j(x_j)\eta_j^t(x_j)\left(\frac{\partial V_j(x_j)}{\partial x_j}\right)^t \\
&\quad + C_j^t(x_j)C_j(x_j) + \beta_j^{-1}d_j(x_j) + \alpha_j^t(x_j)R_j(x_j)\alpha_j(x_j) < 0, \\
&\quad j = 1, 2, \dots, N
\end{aligned} \tag{3.46}$$

are satisfied $\forall x_j \neq 0$.

By defining $V = \sum_{i=1}^N V_j$ and evaluating the derivative along the state trajectory of the interconnected system (3.39), we obtain

$$\begin{aligned}
\dot{V} &= \sum_{j=1}^N \dot{V}_j \\
&= \sum_{j=1}^N \frac{\partial V_j(x_j)}{\partial x_j}[A_j(x_j) + B_j(x_j)u_j + p_j(x_j)\omega_j + h_j(x)] \\
&= \sum_{j=1}^N \frac{\partial V_j(x_j)}{\partial x_j}[A_j(x_j) + B_j(x_j)u_j + p_j(x_j)\omega_j] + \sum_{j=1}^N \sum_{k=1}^{n_j} \frac{\partial V_j(x_j)}{\partial x_{jk}} h_{jk}(x) \\
&\le \sum_{j=1}^N \frac{\partial V_j(x_j)}{\partial x_j}[A_j(x_j) + B_j(x_j)u_j + p_j(x_j)\omega_j] \\
&\quad + \sum_{j=1}^N \sum_{k=1}^{n_j} \left|\frac{\partial V_j(x_j)}{\partial x_{jk}}\right| \eta_{jk}(x_j) \sum_{\ell=1}^N \zeta_{jk\ell}(x_\ell) \\
&\le \sum_{j=1}^N \frac{\partial V_j(x_j)}{\partial x_j}[A_j(x_j) + B_j(x_j)u_j + p_j(x_j)\omega_j] \\
&\quad + \sum_{j=1}^N \sum_{k=1}^{n_j} \sum_{\ell=1}^N \frac{1}{4}\beta_l \left(\frac{\partial V_j(x_j)}{\partial x_{jk}}\eta_{jk}(x_j)\right)^2 + \sum_{j=1}^N \sum_{k=1}^{n_j} \sum_{\ell=1}^N \beta_l^{-1}(\zeta_{jk\ell}(x_\ell))^2
\end{aligned}$$

$$= \sum_{j=1}^{N} \frac{\partial V_j(x_j)}{\partial x_j}[A_j(x_j) + B_j(x_j)u_j + p_j(x_j)\omega_j]$$

$$+ \sum_{j=1}^{N} \frac{1}{4}\beta_\ell \frac{\partial V_j(x_j)}{\partial x_j}\eta_j(x_j)\eta_j^t(x_j)\left(\frac{\partial V_j(x_j)}{\partial x_j}\right)^t + \sum_{j=1}^{N} \beta_j^{-1} d_j(x_j). \tag{3.47}$$

In view of (3.46), and letting $u_j = \alpha_j(x_j)$ in (3.47) while invoking Schur complements, we have

$$\begin{aligned}
\mathsf{H} :=& \sum_{i=1}^{N}(\dot{V}_j - \gamma_j^2 \omega_j^t \omega_j + z_j^t z_j) \\
\leq& \sum_{i=1}^{N}\left\{ \frac{\partial V_j(x_j)}{\partial x_j}[A_j(x_j) + B_j(x_j)\alpha_j(x_j)] \right. \\
&+ \frac{1}{4\gamma_j^2}\left(\frac{\partial V_j(x_j)}{\partial x_j} p_j(x_j)\right)^2 C_j^t(x_j)C_j(x_j) + \alpha_j^t(x_j)R_j(x_j)\alpha_j(x_j) \\
&\left. + \frac{1}{4}\beta \frac{\partial V_j(x_j)}{\partial x_j}\eta_j(x_j)\eta_j^t(x_j)\left(\frac{\partial V_j(x_j)}{\partial x_j} p_j(x_j)\right)^t + \beta_j^{-1} d_j(x_j)\right\} \\
<& \ 0
\end{aligned} \tag{3.48}$$

for all nonzero $x = [x_1^t \dots x_N^t]^t$.

On setting $\omega = 0$, it follows from (3.48) that the overall closed-loop interconnected system is globally asymptotically stable. Alternatively by integrating (3.48) over $[0, \infty)$, we have

$$\sum_{j=1}^{N} \int_0^\infty z_j^t z_j \, dt < \sum_{j=1}^{N} \gamma_j^2 \int_0^\infty \omega_j^t \omega_j t + \delta(x_0),$$

where $\delta(x_0) = \sum_{j=1}^{N} V_j(x_j(0))$. This completes the proof. □

It must be noted that Theorem 3.2 established that the decentralized $\mathcal{H}_\infty$ control problem for interconnected nonlinear systems can be cast into the associated centralized $\mathcal{H}_\infty$ control problems whose solutions are related to the HJI equations. This result naturally extends the decentralized $\mathcal{H}_\infty$ control of interconnected linear systems [55] to the nonlinear case.

3.3.1 The Local Disturbance Problem

Next we will look at the local disturbance problem. In particular, we examine the possibility that the local solution of the decentralized $\mathcal{H}_\infty$ control problem of the system (3.39) can be obtained by solving the $\mathcal{H}_\infty$ control problem for the linearized system of (3.39).

Toward our goal, consider the linear interconnected system given by

$$\begin{aligned}\dot{x}_j &= A_j x_j + P_j \omega_j + B_j u_j + \Gamma_j \sum_{\ell=1}^{N} \zeta_{j\ell} x_\ell, \\ y_j &= C_j x_j, \\ z_j &= (y_j^t y_j + u_j^t R_j u_j)^{\frac{1}{2}},\end{aligned} \tag{3.49}$$

where $x_j \in \Re^{n_j}$ and $u_j \in \Re^{m_j}$ are the state and the control input, respectively, $\omega_j \in \Re^{q_j}$ is the disturbance input, z_j is the controlled output, with $R_j > 0$ and the matrices

$$A_j, \quad B_j, \quad C_j, \quad P_j, \quad \Gamma_j, \quad \zeta_{j\ell}$$

are constants with appropriate dimensions.

Following the earlier development, we associate with the system (3.49) an auxiliary linear systems of the form:

$$\begin{aligned}\dot{x}_j &= A_j x_j + [P_j \beta^{\frac{1}{2}} \; \gamma_j \Gamma_j] \tilde{\omega}_j + B_j u_j, \\ \tilde{y}_j &= \begin{bmatrix} C_j x_j \\ \beta_j^{-\frac{1}{2}} (d_j^1)^{\frac{1}{2}} x_j \end{bmatrix}, \\ \tilde{z}_j &= (\tilde{y}_j^t \tilde{y}_j + u_j^t R_j u_j)^{\frac{1}{2}},\end{aligned} \tag{3.50}$$

where $\beta_j > 0$, $\beta = \sum_{\ell=1}^{N} \beta_\ell$ and $d_j^\ell = \sum_{\ell=1}^{N} \zeta_{\ell j}^t \zeta_{\ell j}$.

The following theorem provides a solution to the decentralized $\mathcal{H}_\infty$ control problem of the linear interconnected system (3.49).

Theorem 3.3 *Given some real numbers $\gamma_j > 0$ and matrices $R_j > 0$, $i = 1, 2, \ldots, N$, consider the interconnected linear system (3.49). Suppose that, for each i, there exist some constants β_ℓ, $\ell = 1, 2, \ldots, N$, and a feedback control law $u_j = K_j x_j$ with $K_j \in \Re^{p_j \times n_j}$, such that the resulting closed-loop system of (3.50) is asymptotically stable and the $\mathcal{L}_2$-gain from $\tilde{\omega}_j$ to $\tilde{z}_j$ is less than γ_j. Then, the decentralized control laws $u_j = K_j x_j$, $1 \le j \le N$, asymptotically stabilize the interconnected linear system (3.49) and render its $\mathcal{L}_2$-gain from the disturbance input $\omega = [\omega_1^t \; \ldots \; \omega_N^t]^t$ to the controlled output $z = [z_1 \; \ldots \; z_N]^t$ less than $\gamma = [\gamma_1 \; \ldots \; \gamma_N]^t$ in the sense that*

$$\sum_{j=1}^{N} \int_0^\infty z_j^t z_j \, dt < \sum_{j=1}^{N} \gamma_j^2 \int_0^\infty \omega_j^t \omega_j \, dt + \delta(x_0)$$

for all $\omega_j \in \mathcal{L}_2[0, \infty)$, where $\delta(x_0)$ is a function of the initial state

$$x_0 = [x_1^t(0) \; \ldots \; x_N^t(0)]^t$$

satisfying $\delta(0) = 0$.

Proof In light of the assumption of the theorem, there exist matrices $0 < Y_j = Y_j^t \in \Re^{n_j \times n_j}$ such that

$$\begin{bmatrix} \begin{matrix} Y_j(A_j + B_j K_j) \\ +(A_j + B_j K_j)^t Y_j + \beta_j^{-1} d_j^1 \end{matrix} & C_j^t & Y_j P_j & \beta Y_j \Gamma_j & K_j^t R_j \\ \bullet & -I & 0 & 0 & 0 \\ \bullet & \bullet & -\gamma_j^2 I & 0 & 0 \\ \bullet & \bullet & \bullet & -\beta I & 0 \\ \bullet & \bullet & \bullet & \bullet & -R_j \end{bmatrix} < 0. \quad (3.51)$$

Let $V_j = x_j^t Y_j x_j$ and $V = \sum_{j=1}^N V_j$ and following the same line of reasoning as in the proof of Theorem 3.2, we reach the desired result. □

Remark 3.9 It is a simple task to prove, based on the first assumptions of Theorem 3.3, that the solution to the decentralized $\mathcal{H}_\infty$ control problem of the linear interconnected system (3.49), namely $V_j = x_j^t Y_j x_j$ and $u_j = K_j x_j$, actually satisfies the HJI inequality (3.46) for all x in a neighborhood of $x = 0$, see [30, 52]. In turn, the solution of the $\mathcal{H}_\infty$ control problem for the linearized system of (3.39) also yields a local solution of the $\mathcal{H}_\infty$ control problem for the non-linear system (3.39).

3.3.2 Results for Non-minimum Phase Systems

Consider a class of interconnected nonlinear systems which are transformable to interconnected nonlinear systems extended form [21]:

$$\begin{aligned} \dot{\chi}_j &= f_{j0}(\chi_j, \xi_{j1}) + P_{j0}(\chi_j, \xi_{j1})\omega_j + \phi_{j0}(\chi_j, \xi_{j1}; X_{j1}), \\ \dot{\xi}_{j1} &= \xi_{j2} + p_{j1}(\chi_j, \xi_{j1})\omega_j + \phi_{j1}(\chi_j, \xi_{j1}; X_{j1}), \\ \dot{\xi}_{ij2} &= \xi_{j3} + p_{j2}(\chi_j, \bar{\xi}_{j2})\omega_j + \phi_{j2}(\chi_j, \bar{\xi}_{j2}; X_{j1}), \\ &\vdots \\ \dot{\xi}_{j,\tau_j-1} &= \xi_{j,\tau_j} + p_{j,\tau_j-1}(\chi_j, \bar{\xi}_{j,\tau_j-1})\omega_j + \phi_{j,\tau_j-1}(\chi_j, \bar{\xi}_{j,\tau_j-1}; X_{j1}), \\ \xi_{j,\tau_j} &= u_j + p_{j,\tau_j}(\chi_j, \bar{\xi}_{j,\tau_j})\omega_j + \phi_{j,\tau_j}(\chi_j, \bar{\xi}_{j,\tau_j}; X_{j1}), \\ y_j &= C_j(\chi_j, \xi_{j1}), \end{aligned} \quad (3.52)$$

where

$$\chi_j \in \Re^{n_j - \tau_j}, \quad \bar{\xi}_{jk} = [\xi_{j1}\ \xi_{j2}\ \dots\ \xi_{jk}]^t, \quad i = 1, 2, \dots, N,\ j = 1, 2, \dots, \tau_j,$$
$$X_{j1} = [\xi_{11}\ \xi_{21}\ \dots\ \xi_{j-1,1}\ \xi_{j+1,1}\ \dots\ \xi_{N1}]^t,$$

$u_j \in \Re$ is the local control input, $p_{i0}(\chi_j, \xi_{j1}), \dots, p_{j\tau_j}(\chi_j, \bar{\xi}_{j\tau_j})$,

$$\phi_{j0}(\chi_j, \xi_{j1}, X_{j1}), \quad \dots, \quad \phi_{j\tau_j}(\chi_j, \bar{\xi}_{j,\tau_j}, X_{j1})$$

and $C_j(\chi_j, \xi_{j1})$ are smooth with

$$\phi_{jk}(0;0) = 0, \quad C_j(0,0) = 0, \quad k = 0, \dots, \tau_j, \ j = 1, \dots, N.$$

The following assumptions about (3.52) are made [18, 49]:

Assumption 3.4 The χ_j-subsystem of the ith subsystem in (3.52) can be decomposed into two cascade-subsystems as follows:

$$\begin{aligned}\dot{\chi}_{j1} &= f_{j01}(\chi_j, \xi_{j1}) + p_{j01}(\chi_j, \xi_{j1})\omega_j + \phi_{j01}(\chi_j, \xi_{j1}; X_{j1}),\\ \dot{\chi}_{j2} &= f_{j02}(\chi_{j2}, \xi_{j1}),\end{aligned} \tag{3.53}$$

where $\chi_j = [\chi_{j1}^t \ \chi_{j2}^t]^t$ with $\chi_{i1} \in \Re^{n_{j1}}$, $\chi_{j2} \in \Re^{n_{j2}}$ and $n_{j1} + n_{j2} = n_j - \tau_j$.

Assumption 3.5 There exists a smooth real-valued positive definite and proper function $V_{i01}(\chi_{i1})$ such that

$$\begin{aligned}&\frac{\partial V_{j01}}{\partial \chi_{j1}}\{f_{j01}(\chi_j, \xi_{j1}) + \phi_{j01}(\chi_j, \xi_{j1}; 0) + [p_{j01}(\chi_j, \xi_{j1})\ \beta^{\frac{1}{2}}\gamma_j \eta_{j0} I_{n_{j1}}]\tilde{\omega}_j\}\\ &\quad \le -\alpha_{j01}\|\chi_{j1}\|^2 + \gamma_{i0}^2\|\tilde{\omega}_j\|^2 + k_{j1}(\chi_{j2}, \xi_{j1})\end{aligned} \tag{3.54}$$

for some definite function $k_{j1}(\chi_{j2}, \xi_{j1})$, some positive real numbers α_{j01}, β and γ_{j0} and $\tilde{\omega}_j \in \mathcal{L}_2[0, \infty)$, where $I_{n_{j1}}$ is the identity matrix of dimensions $n_{j1} \times n_{j1}$, $j = 1, 2, \dots, N$.

Assumption 3.6 There exist a smooth real-valued function $\nu_{j02}(\chi_{j2})$ with $\nu_{j02}(0) = 0$, and a smooth real-valued proper function $\nu_{j02}(\chi_{j2}) > 0$, such that

$$\begin{aligned}\frac{\partial V_{j02}}{\partial \chi_{j2}} f_{j02}(\chi_{j2}, \nu_{j02}(\chi_{j2})) &\le -\alpha_{j02}(\chi_{j2}),\\ \alpha_{j02}\|\chi_{j2}\|^2 &\le V_{j02}(\chi_{j2})\end{aligned} \tag{3.55}$$

for some real numbers $\alpha_{j02} > 0$ and $\alpha_{j03} > 0$.

Assumption 3.7 The control output y_j of the system (3.52) can be expressed in the form

$$y_j = C_{j0}(\chi_{j2}, \xi_{j1}), \tag{3.56}$$

where $C_{j0}(\chi_{j2}, \xi_{j1})$ is a smooth real function with $C_{j0}(0,0) = 0$.

Assumption 3.8 The nonlinear interconnections in (3.52) are bounded by strong nonlinearities in X_{j1}

$$\|\phi_{j01}(\chi_j, \bar{\xi}_{jk}; x_{j1}) - \phi_{j01}(\chi_j, \bar{\xi}_{jk}; 0)\| \le \sum_{\ell=1, \ell \ne j}^{N} |\eta_{j0}(\chi_j, \bar{\xi}_{jk})||\zeta_{j0l}(\xi_{\ell 1})|, \tag{3.57}$$

$$|\phi_{jk}(\chi_j, \bar{\xi}_{jk}; X_{j1})| - \phi_{jk}(\chi_j, \bar{\xi}_{ij}; 0)| \le \sum_{\ell=1, \ell\neq j}^{N} |\eta_{jk}(\chi_j, \bar{\xi}_{jk})||\zeta_{jk\ell}(\xi_{\ell 1})|, \quad (3.58)$$

where $\eta_{jk}(\chi, \bar{\xi}_{jk})$ and $\zeta_{jk\ell}(\xi_{\ell 1})$, $0 \le k \le \tau_j$, $1 \le j, \ell \le N$, are smooth functions with $\zeta_{jk\ell}(0) = 0$.

Remark 3.10 It is noted that Assumption 3.5 amounts to the input-state stability with respect to the disturbance input ω_j and bounded-input bounded-state stability with respect to χ_{j2} and ξ_{j1}, whereas Assumption 3.6 implies that the subsystem χ_{j2} is asymptotically stabilized by the feedback $\xi_{j1} = \nu_{j02}(\chi_{j1})$. Interestingly enough, these assumptions are similar to those in centralized $\mathcal{H}_\infty$ control [18].

In what follows, we proceed to deal with the global DADDP for the interconnected nonlinear system (3.52) phrased as follows:

Given any real numbers $\gamma_j > 0$, $j = 1, 2, \dots, N$, it is desired to find decentralized feedback laws $u_j = \alpha_j(x_j)$, $\alpha_j(0) = 0$, such that the overall closed-loop system is internally asymptotically stable with the $\mathcal{L}_2$-gain between the disturbance input $\omega = [\omega_1^t \ \omega_2^t \ \dots \ \omega_N^t]^t$ and the output $y = [y_1 \ y_2 \ \dots \ y_N]^t$ less than $\gamma = [\gamma_1 \ \dots \ \gamma_N]^t$ in the following sense

$$\sum_{j=1}^{N} \int_0^\infty y_j^t y_j \, dt < \sum_{j=1}^{N} \gamma_j^2 \int_0^\infty \omega_j^t \omega_j \, dt + \delta(x_0)$$

for all $\omega_j \in \mathcal{L}_2[0, \infty)$ and all admissible nonlinear interconnections, where $\delta(x_0)$ is a function of the initial state x_0 satisfying $\delta(0) = 0$.

Taking into account Assumption 3.8 and noting that $\zeta_{jk\ell}(\xi_{\ell 1})$, $j = 1, 2, \dots, N$, $k = 0, 1, 2, \dots, \tau_j$, $\ell = 1, 2, \dots, N$, are smooth with $\zeta_{jk\ell}(0) = 0$, there exist smooth functions $\tilde{\zeta}_{jk\ell}(\xi_{\ell 1})$ such that

$$\zeta_{jk\ell}(\xi_{\ell 1}) = \tilde{\zeta}_{jk\ell}(\xi_{\ell 1})\xi_{\ell 1}.$$

Now, turning to Theorem 3.2 and introducing an auxiliary systems associated with (3.52) satisfying Assumptions 3.4–3.8 of the form:

$$\begin{aligned} \dot{x}_j &= f_j(x_j) + B_j u_j + [P_j(x_j) \, \beta^{\frac{1}{2}} \, \gamma_j \eta_j(x_j)]\tilde{\omega}_j, \\ \tilde{y}_j &= \begin{bmatrix} C_{j0}(\chi_{j2}, \xi_{j1}) \\ \beta_j^{-\frac{1}{2}} (\tilde{d}_j(\xi_{j1}))^{\frac{1}{2}} \xi_{j1} \end{bmatrix}, \end{aligned} \quad (3.59)$$

where $x_j = [\chi_j^t \ \xi_{j1} \ \dots \ \xi_{j\tau_j}]^t$, $j = 1, 2, \dots, N$, is the state, $\tilde{\omega}_j$ is the disturbance input, u_j is the control input, and β_j, $j = 1, 2, \dots, N$ are some positive scalars,

$\beta = \sum_{\ell=1}^{N} \beta_\ell$ and

$$f_j(x_j) = \begin{bmatrix} f_{j01}(\chi_j, \xi_{j1}) + \phi_{j01}(\chi_j, \xi_{j1}; 0) \\ f_{j02}(\chi_{j2}, \xi_{j1}) \\ \xi_{j2} + \phi_{j1}(\chi_j, \underline{\xi}_{j1}; 0) \\ \xi_{j3} + \phi_{j2}(\chi_j, \bar{\xi}_{j2}; 0) \\ \vdots \\ \xi_{j\tau_j} + \phi_{j,\tau_j-1}(\chi_j, \bar{\xi}_{j,\tau_j-1}; 0) \\ \phi_{j,\tau_j}(\chi_j, \bar{\xi}_{j,\tau_j}; 0) \end{bmatrix}; \qquad B_j = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$p_j(x_j) = \begin{bmatrix} p_{j01}(\chi_j, \xi_{j1}) \\ 0 \\ p_{j1}(\chi_j, \underline{\xi}_{j1}) \\ p_{j2}(\chi_j, \bar{\xi}_{j2}) \\ \vdots \\ p_{j,\tau_j-1}(\chi_j, \bar{\xi}_{j,\tau_j-1}) \\ p_{j,\tau_j}(\chi_j, \bar{\xi}_{j,\tau_j}) \end{bmatrix},$$

$$\eta_j(x_j) = \mathrm{diag}\{\eta_{j0}(\chi_j, \xi_{j1}) I_{n_{j1}}, 0_{n_{j2}\times n_{j2}}, \eta_{j1}(\chi_j, \xi_{j1}), \ldots, \eta_{j\tau_j}(\chi_j, \bar{\xi}_{j\tau_j})\},$$

$$\tilde{\zeta}_{j\ell}(\xi_{\ell 1}) = \begin{cases} \left[[\tilde{\zeta}_{j0l}\xi_{\ell 1}\ \tilde{\zeta}_{j\ell l}(\xi_{\ell 1})\ \tilde{\zeta}_{j2\ell}(\xi_{\ell 1}) \ldots \tilde{\zeta}_{j\tau_j\ell}(\xi_{\ell 1}) \right]^t & \text{if } \ell \neq j, \\ \left[0\, 0 \ldots 0 \right]^t & \text{if } \ell = j, \end{cases}$$

$$\tilde{d}_j(\xi_{j1}) = \sum_{\ell=1,\ell\neq j}^{N} \tilde{\zeta}_{\ell j}^t(\xi_{j\ell})\tilde{\zeta}_{\ell j}(\xi_{j1}) = \sum_{\ell=1}^{N} \tilde{\zeta}_{\ell j}^t(\xi_{j1})\tilde{\zeta}_{\ell j}(\xi_{j1}).$$

In view of Theorem 3.2, the following theorem is easily established:

Theorem 3.4 *Consider the interconnected system* (3.52) *satisfying Assumptions* 3.4–3.8. *Given any* $\gamma_j > 0$, $j = 1, 2, \ldots, N$, *suppose that, for some* $\beta_j > 0$, *the control law* $u_j = u_j(x_j)$ *with* $u_j(0) = 0$ *solves the almost disturbance decoupling problem for the system* (3.59), *that is,* u_j *globally asymptotically stabilizes the system* (3.59) *and render the* L_2*-gain from the disturbance* $\tilde{\omega}_j$ *to the penalty output* $\tilde{y}_j$ *less than* γ_j *in the sense that*

$$\int_0^\infty \tilde{y}_j^t \tilde{y}_j\, dt < \gamma_j^t \int_0^\infty \omega_j^t \omega_j dt + \delta(x_{i0})$$

for all $\omega_j \in \mathcal{L}_2[0, \infty)$, $j = 1, 2, \ldots, N$, *where* $\delta(x_j 0)$ *is a function of the initial state* x_{j0} *satisfying* $\delta(0) = 0$. *Then given* $\gamma = [\gamma_1 \ldots \gamma_N]$ *with* $\gamma_j > 0$, *the same control laws* $u_j = u_j(x_j)$ *will solve the global DADDP for the interconnected system* (3.52).

Remark 3.11 It is quite evident from Theorem 3.4 that the global decentralized almost disturbance decoupling for interconnected non-minimum phase nonlinear system (3.52) is converted into the global almost disturbance decoupling problem for the system (3.59) without interconnections. The latter problem can be solved by a recursive Lyapunov-based design approach [18, 49], which does not involve solving HJI equations (inequalities). In this way, the developed design procedure is systematic and applicable to wide class of interconnected systems.

3.4 Global Inverse Control of Nonlinear Systems

In Sect. 3.3, it has been shown in principle that a global solution to the problem of decentralized $\mathcal{H}_\infty$ nonlinear control can be obtained from the global solutions of the HJI inequalities (3.46). The HJI inequalities are generally difficult to solve and for technical reasons it is usually impossible to solve the HJI inequality of the form (3.46) globally.

An alternative way is to study the problem of global inverse $\mathcal{H}_\infty$ control rather than the regular global $\mathcal{H}_\infty$ control problem. In this way, the aim is to seek not only a set of decentralized feedback control laws but also weighting functions for control inputs such that the associated $\mathcal{H}_\infty$ problem is solvable globally for a class of interconnected nonlinear systems.

To put the main issues in proper perspectives, we consider a class of interconnected non-linear systems which are transformable to interconnected nonlinear systems of the form:

$$\begin{aligned}
\dot{x}_{j1} &= x_{j2} + p_{j1}(x_{j1})\omega_j + \phi_{j1}(x_{j1}; X_{j1}),\\
\dot{x}_{j2} &= x_{j3} + p_{j2}(\bar{x}_{j2})\omega_j + \phi_{j2}(\bar{x}_{j2}; X_{j1}),\\
&\vdots\\
\dot{x}_{j,n_j-1} &= x_{j,n_j} + p_{j,n_j-1}(\bar{x}_{j,n_j-1})\omega_j + \phi_{j,n_j-1}(\bar{x}_{j,n_j-1}; X_{j1}),\\
\dot{x}_{j,n_j} &= u_j + p_{j,n_j}(\bar{x}_{j,n_j})\omega_j + \phi_{j,n_j}(\bar{x}_{j,n_j}; X_{j1}),\\
y_j &= x_{j1},
\end{aligned} \tag{3.60}$$

where $u_j \in \Re$ is the local control input,

$$\begin{aligned}
\bar{x}_{jk} &= [x_{j1}\ x_{j2}\ \dots\ x_{jk}]^t, \quad j = 1, 2, \dots, N,\ j = 1, 2, \dots, n_j,\\
X_{j1} &= [x_{11}\ x_{21}\ \dots\ x_{j-1,1}\ x_{j+1,1}\ \dots\ x_{N1}]^t,
\end{aligned}$$

$P_{i1}(x_{i1}), \dots, P_{in_j}(\bar{x}_{in_j})$ and $\phi_{j1}(x_{j1}; X_{j1}), \dots, \phi_{jn_j}(\bar{x}_{j,n_1}; X_{j1})$ are smooth with $\phi_{jk}(0; 0) = 0$, $j = 1, \dots, N$, $k = 1, \dots, n_j$.

For simplicity in exposition, we consider $n_j = n$, $1 \le j \le N$, and express the system in the following form:

$$\begin{aligned}
\dot{x}_j &= A_j x_j + f_j(x_j) + p_j(x_j)\omega_j + B_j u_j + h_j(x_j; X_{j1}),\\
y_j &= C_j x_j,
\end{aligned} \tag{3.61}$$

where $x_j = [x_{j1}\ x_{j2}\ \dots\ x_{jn}]^t = \bar{x}_{jn}$ and

$$A_j = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \in \Re^{n \times n}; \qquad B_j = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \Re^n,$$

$$C_j = \begin{bmatrix} 1\ 0 \dots 0 \end{bmatrix} \in \Re^{1 \times n},$$

$$f_j(x_j) = \begin{bmatrix} \phi_{j1}(x_{j1};0) \\ \phi_{j2}(\bar{x}_{j2};0) \\ \vdots \\ \phi_{jn}(\bar{x}_{jn};0) \end{bmatrix} \in \Re^n; \qquad p_j(x_j) = \begin{bmatrix} p_{j1}(x_{j1}) \\ p_{j2}(\bar{x}_{j2}) \\ \vdots \\ p_{jn}(\bar{x}_{jn}) \end{bmatrix} \in \Re^n,$$

$$h_j(x_j, X_{j1}) = \begin{bmatrix} \phi_{j1}(x_{j1}; X_{j1}) - \phi_{j1}(x_{j1};0) \\ \phi_{j2}(\bar{x}_{j2}; X_{j1}) - \phi_{j2}(\bar{x}_{j2};0) \\ \vdots \\ \phi_{jn}(\bar{x}_{jn}; X_{j1}) - \phi_{jn}(\bar{x}_{jn};0) \end{bmatrix} \in \Re^n.$$

The following assumption is made:

Assumption 3.9 The nonlinear interconnections $h_j(x_j; X_{j1})$, $j = 1, 2, \dots, N$, in (3.61) are bounded by nonlinearities in X_{j1}:

$$|\phi_{jk}(\bar{x}_{jk}; X_{j1}) - \phi_{jk}(\bar{x}_{jk};0)| \le \sum_{\ell=1;\ell \ne i}^{N} |\eta_{jk}(\bar{x}_{jk})||\zeta_{jk\ell}(x_{\ell 1})|, \tag{3.62}$$

where $\eta_{jk}(\bar{x}_{jk})$ and $\zeta_{jk\ell}(x_{\ell 1})$, $1 \le k \le n$, $1 \le j, \ell \le N$, are smooth functions with $\zeta_{jk\ell}(0) = 0$.

It must be emphasized that Assumption 3.9 represents a fairly general form of interconnections which includes those in [6, 11, 45, 46] as special cases.

In the sequel, we shall focus on the global decentralized inverse control problem of nonlinear systems, phrased as follows:

Given some real numbers $\gamma_j > 0$, $j = 1, 2, \dots, N$, it is desired to find decentralized feedback laws $u_j = \alpha_j(x_j), \alpha_j(0) = 0$, and some continuous functions $0 \le r_j(x_j)$ such that the overall closed-loop system is internally asymptotically stable with the $\mathcal{L}_2$-gain between the disturbance input $\omega = [\omega_1^t\ \omega_2^t\ \dots\ \omega_N^t]^t$ and the controlled output $z = [z_1\ z_2\ \dots\ z_N]^t$, where $z_j = (y_j^2 + r_j(x_j)u_j^2)^{\frac{1}{2}}$, less than $\gamma = [\gamma_1\ \dots\ \gamma_N]^t$ in the following sense

$$\sum_{j=1}^{N} \int_0^\infty z_j^t z_j dt < \sum_{j=1}^{N} \gamma_j^2 \int_0^\infty \omega_j^t \omega_j dt + \delta(x_0)$$

$\forall \omega_j \in \mathcal{L}_2[0,\infty)$ and all admissible nonlinear interconnections, where $\delta(x_0)$ is a function of the initial state x_0 satisfying $\delta(0)=0$.

In view of Assumption 3.9 and observing that

$$\zeta_{jk\ell}(x_{\ell 1}), \quad j=1,2,\ldots,N,\ k=1,2,\ldots,n,\ \ell=1,2,\ldots,N,\ \zeta_{jk\ell}(0)=0,$$

are smooth, there exist smooth functions $\tilde{\zeta}_{jk\ell}(x_{\ell 1})$ such that $\zeta_{jk\ell}(x_{\ell 1}) = \tilde{\zeta}_{jk\ell}(x_{\ell 1})x_{\ell 1}$.

Extending on Theorem 3.2, we introduce the following auxiliary systems associated with (3.61) satisfying Assumption 3.9:

$$\begin{aligned} \dot{x}_j &= A_j(x_j) + f_j(x_j) + B_j u_j + [P_j(x_j)\ \beta^{\frac{1}{2}}\gamma_j\eta_j(x_j)]\tilde{\omega}_j, \\ \tilde{y}_j &= \begin{bmatrix} C_j x_j \\ \beta_j^{-\frac{1}{2}}(\tilde{d}_j(x_{i1}))^{\frac{1}{2}} C_j x_j \end{bmatrix}, \end{aligned} \tag{3.63}$$

where x_j, $j=1,2,\ldots,N$, is the state $\tilde{\omega}_j$ is the disturbance input, u_j is the control input, $A_j(x_j)$, $B_j(x_j)$, $C_j(x_j)$, $f_j(x_j)$ and $p_j(x_j)$ are the same as in system (3.61) and $0<\beta_j$, $j=1,2,\ldots,N$, are some scalars, $\beta=\sum_{\ell=1}^{N}\beta_\ell$, $j=1,2,\ldots,N$ and

$$\begin{aligned} \eta_j &= \operatorname{diag}\{\eta_{j1}(x_{j1}), \eta_{j2}(\bar{x}_{j2}), \ldots, \eta_{jn}(\bar{x}_{jn})\}, \\ \tilde{\zeta}_{jl}(x_{\ell 1}) &= \begin{cases} [\tilde{\zeta}_{j1\ell}(x_{j1})\ \tilde{\zeta}_{j2\ell}(x_{\ell 1})\ \ldots\ \tilde{\zeta}_{jn\ell}(x_{\ell 1})]^t & \text{if } \ell \neq j, \\ [0\,0\ldots 0]^t & \text{if } \ell = j, \end{cases} \\ \tilde{d}_j(x_{j1}) &= \sum_{\ell=1,\ell\neq j}^{N} \tilde{\zeta}_{\ell j}^t(x_{j1})\tilde{\zeta}_{\ell j}(x_{j1}) = \sum_{\ell=1}^{N} \tilde{\zeta}_{\ell j}^t(x_{j1})\tilde{\zeta}_{\ell j}(x_{j1}). \end{aligned}$$

Let P_j denote the value of $p_j(x_j=0)$. Since $f_j(x_j)$, $f_j(0)=0$ is smooth, it can be rewritten as

$$\begin{aligned} f_j(x_j) &= \begin{bmatrix} f_{j1}(x_{j1}) \\ f_{j2}(\bar{x}_{j2}) \\ \vdots \\ f_{jn}(\bar{x}_{jn}) \end{bmatrix} \\ &= \begin{bmatrix} \phi_{j11}(x_{j1}) & 0 & \ldots & 0 \\ \phi_{j21}(\bar{x}_{j2}) & \phi_{j22}(\bar{x}_{i2}) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{jn1}(\bar{x}_{jn}) & \phi_{jn2}(\bar{x}_{in}) & \ldots & \phi_{jnn}(\bar{x}_{jn}) \end{bmatrix} \begin{bmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{in} \end{bmatrix} \\ &= \Phi_{jf}(x_j)x_j, \end{aligned}$$

where all the involved functions are smooth. It is readily seen that

$$F_j = \Phi_{if}(0) = \left.\frac{\partial f_j}{\partial x_j}\right|_{x_j=0}$$

and is a lower triangular matrix. The linearized system of (3.63) at $x_j = 0$ is

$$\begin{aligned}\dot{x}_j &= \tilde{A}_j(x_j) + B_j u_j + [P_j\ \beta^{\frac{1}{2}} \gamma_j \Gamma_j]\tilde{\omega}_j,\\ \tilde{y}_j &= \begin{bmatrix} C_j x_j \\ \beta_j^{-\frac{1}{2}} (d_{j0})^{\frac{1}{2}} C_j x_j \end{bmatrix},\end{aligned} \tag{3.64}$$

where $\bar{A}_j = A_j + F_j$, $\Gamma_j = \eta_j(0)$ and $d_{j0} = \tilde{d}_j(0)$.

A solution to the global decentralized inverse control problem for the interconnected nonlinear system (3.61) is summarized in the following theorem:

Theorem 3.5 *Consider the interconnected system* (3.61) *satisfying Assumption* 3.9 *and given* $\bar{r}_j > 0$, $j = 1, 2, \dots, N$ *be constant weighting factors. Suppose that, for each* j, *there exists a linear feedback law* $u_j = K_j x_j$ *for* (3.64) *which internally stabilizes the system and render its* $\mathcal{L}_2$*-gain, between the disturbance input* $\tilde{\omega}_j$ *and the penalty output* $\tilde{z}_j = (\tilde{y}_j^t \tilde{y}_j + \bar{r}_j u_j^2)^{\frac{1}{2}}$, *less than a prescribed number* $\gamma_j > 0$. *There exist weighting factors* $r_j(x_j)$, *continuously depending on* x_j *and satisfying* $r_j(0) = \bar{r}_j$ *and* $0 \le r_j(x_j) \le \bar{r}_j$, *and smooth decentralized feedback control laws* $\alpha_j(x_j)$, $i = 1, 2, \dots, N$, *for* (3.61) *which globally stabilize the interconnected system* (3.61) *and render its* $\mathcal{L}_2$*-gain, between the disturbance input* $\omega = [\omega_1^t\ \omega_2^t\ \dots\ \omega_N^t]^t$ *and the penalty output* $z = [z_1\ z_2\ \dots\ z_N]^t$, *where* $z_j = (y_j^2 + r_j(x_j)u_j^2)^{\frac{1}{2}}$, *less than* $\gamma = [\gamma_1\ \dots\ \gamma_N]^t$ *for all* $\omega_j \in \mathcal{L}_2[0, \infty)$ *and all admissible non-linear interconnections.*

Proof Using Theorem 3.2 and extending the result of [20, 60] backward and forward to system (3.64) and (3.63) with lower triangular structures, Theorem 3.5 can be readily established. Details are given in [20]. □

Remark 3.12 Theorem 3.5 presents a constructive solution to the global decentralized inverse control problem based on an explicit use of the weighting factors, the proper, positive definite Lyapunov functions and the decentralized control laws satisfying the HJI inequalities (3.46). The key point lies in overcoming the strong nonlinear interconnections by casting the decentralized $\mathcal{H}_\infty$ control problem into an associated centralized $\mathcal{H}_\infty$ control. The latter is then solved by extending the result in [20].

Remark 3.13 It is significant to assure that at the equilibrium point $x_j = 0$, $j = 1, 2, \dots, N$, the constructed Lyapunov functions, the decentralized nonlinear control laws and the control weighting factors reduce to those associated with the decentralized $\mathcal{H}_\infty$ control of the linearized interconnected system. Looked at in this light, Theorem 3.5 provides an important link between the linear (local) decentralized $\mathcal{H}_\infty$ control of linearized interconnected system (the interconnected nonlinear system) and the global $\mathcal{H}_\infty$ control of the interconnected nonlinear system.

3.4.1 Disturbance Attenuating Trackers

In what follows, we continue our study to interconnected nonlinear systems with output measurements. This problem, usually referred to as decentralized output-feedback control, is technically challenging because of the lack of a general theory for nonlinear observer design and the nonlinear version of the well-known *Separation Principle*. The central focus is three-fold:

(i) identifying a wide class of large-scale nonlinear systems in disturbed decentralized output-feedback form;
(ii) proposing an improved systematic output-feedback controller design procedure for decentralized systems in the presence of strong nonlinearities appearing in the subsystems and interactions;
(iii) guaranteeing decentralized asymptotic tracking when the disturbance inputs disappear and achieving desirable external stability properties when the disturbance inputs are present.

In this regard, we record that constructive control design methods for classes of highly nonlinear systems were developed in [17, 24, 27, 34]). In a related work on decentralized adaptive control, the work of [23] presents a systematic method for a class of interconnected systems under matching conditions and weakly nonlinear disturbances. The results of [23] have been generalized in various ways in [11, 15, 45, 59]). In most of the available results, the trend has been to restrict the location of uncertainties [11, 15, 45] and impose growth conditions on the subsystem and interacting nonlinearities [11, 13, 22, 43, 45, 59].

In this section, we proceed to extend recent developments in nonlinear $\mathcal{L}_2$-gain feedback control [17, 23, 33, 35, 51] to the important problem of asymptotic tracking with disturbance attenuation property within the context of interconnected nonlinear systems with output measurements. In the sequel, we assume that the unmeasured states appear linearly. To reconstruct the unmeasured states, an effective full-order decentralized observer is introduced. On the basis of an enlarged decentralized system comprising the observer, an output-feedback decentralized controller is designed via the recursive backstepping technique. In order to achieve the desired control objective of asymptotic tracking with disturbance attenuation for the decentralized system in question, a non quadratic Lyapunov function is used and turns out to be necessary.

3.4.2 System Description

Consider a large-scale nonlinear system comprised of N interconnected subsystems with time-varying unknown parameters and/or disturbances entering nonlinearly into the state equation. The ith subsystem is given as

$$\dot{X}_j = F_j(X_j) + G_j(X_j)u_j + \varLambda_{j1}(y_1, \dots, y_N)X_j + \varLambda_{j2}(y_1, \dots, y_N)w_j, \tag{3.65}$$

$$y_j = h_j(X_j), \tag{3.66}$$

where $1 \le j \le N$, $X_j \in \Re^{N_j}$, $u_j \in \Re$ and $y_j \in \Re$ represent the state, the control input and the output of the local ith subsystem, respectively, and $w_j \in \Re^{n_{w_j}}$ is the disturbance input. The functions F_j, G_j, h_j, $\varLambda_{j1}$, $\varLambda_{j2}$ are sufficiently smooth. In the absence of the interacting terms $\varLambda_{j1}$ and $\varLambda_{i2}$, the system (3.65)–(3.66) reduces to an isolated SISO system. From the literature, we found various constructive control algorithms developed for wide classes of centralized nonlinear systems in special normal forms. It is quite naturally to seek similar results in the decentralized context, that is, in the presence of strong interactions among local systems of the form (3.65)–(3.66). For the simplicity in exposition, we will examine the following class of interconnected dynamic systems of the type (3.65)–(3.66) which is transformable to

$$\begin{aligned}
\dot{z}_j &= Q_j z_j + f_{j0}(y_1, \dots, y_N) + p_{j0}(y_1, \dots, y_N)w_j,\\
\dot{x}_{j1} &= x_{j2} + f_{j1}(y_1, \dots, y_N) + g_{j1}(y_1, \dots, y_N)z_j + p_{j1}(y_1, \dots, y_N)w_j,\\
&\ \vdots\\
\dot{x}_{jn_j} &= u_j + f_{jn_j}(y_1, \dots, y_N) + g_{in_j}(y_1, \dots, y_N)z_j + p_{jn_j}(y_1, \dots, y_N)w_j,\\
y_j &= x_{j1},
\end{aligned} \tag{3.67}$$

where for each $1 \le j \le N$, $z_j \in \Re^{n_{z_j}}$ and $x_j = (x_{j1}, \dots, x_{jn_j}) \in \Re^{n_j}$ are the states of the jth transformed subsystem. For every j, Q_j is a constant matrix with appropriate dimension, f_{jk}, g_{jk} and p_{jk} are known and smooth functions. The following minimum-phase condition is recalled.

Assumption 3.10 For every $1 \le j \le N$, Q_j is a Hurwitz matrix.

Remark 3.14 We assert that the structure of (3.67) is commonly seen in the literature in both centralized and decentralized control [11, 15, 17, 22, 27, 34, 40, 45, 59]. Employing elements of geometric nonlinear control [17, 24, 27, 34], necessary and sufficient conditions were derived under which the nonlinear system (3.65)–(3.66) can be transformed into (3.67), the so-called "disturbed decentralized output-feedback form". The nonlinearities in (3.65) depend only on the output $y = (y_1, \dots, y_N)$ and that the unmeasured states X_j or $(z_j, x_{j2}, \dots, x_{jn_j})$ in (3.67) appear linearly. This feature is quite standard in recent studies on global output-feedback control for both centralized and decentralized nonlinear systems. Simple counterexamples in [36] revealed the fundamental limitation of global output-feedback control for systems with strong nonlinearities due to unmeasured states.

We now address the following control problem:

For every $1 \le j \le N$ *and a given time-varying signal* $y_{jr}(t)$ *whose derivatives up to order* n_j *are bounded over* $[0, \infty)$, *it is desired to design a smooth, decentralized, dynamic, output-feedback controller of the form*

$$\dot{x}_j = \nu_j(x_j, y_j, t), \quad u_j = \mu_j(x_j, y_j, t), \quad x_j \in \Re^{\bar{n}_j} \tag{3.68}$$

such that the following properties hold for the closed-loop large-scale nonlinear system (3.67)–(3.68):

1. *When* $w_j = 0$ *for all* $1 \le j \le N$, *the tracking error* $y_j - y_{ir}$ *goes to zero asymptotically and all other closed-loop signals remain bounded over* $[0, \infty)$.
2. *When* $w_j \neq 0$ for all $1 \le j \le N$, *the closed-loop system is bounded-input bounded-state* (*BIBS*) *stable and, in appropriate coordinates, is integral-input-to-state stable* (*iISS*) *with respect to the disturbance input* w [47]. *In particular, there exists a class-*$\mathcal{K}$ *function* γ_d (*that is,* γ_d *is continuous, strictly increasing and vanishes at the origin*) *such that, for any* $\rho > 0$, *the controller* (3.68) *can be tuned to satisfy the inequality*

$$\int_{t_0}^{t} |y(\tau) - y_r(t)|^2 \, d\tau \le \rho \int_0^t \gamma_d(|w(\tau)|) \, d\tau + \eta_0(z(0), x(0), x(0)) \quad \forall t \ge 0, \tag{3.69}$$

where η_0 *is a nonnegative* C^0 *function, and* $z(0) = (z_1^t(0), \ldots, z_N^t(0))^t, x(0) = (x_1^t(0), \ldots, x_N^t(0))^t$ *and* $x(0) = (x_1^t(0), \ldots, x_N^t(0))^t$.

Property 1 means that decentralized asymptotic tracking is achieved for each local jth subsystem (3.67) in the absence of disturbance inputs. Note in Property 2 that (3.69) implies, in the presence of disturbances, that the decentralized output-feedback controller (3.68) has the ability to attenuate the effect of the disturbances on the tracking error arbitrarily for a fixed class-$\mathcal{K}$ gain-function γ_d, later on we have $\gamma_d(s) = s^2 + s^4 + s^8$.

3.4.3 Output Feedback Tracking

The control problem addressed before will be solved in the sequel in two steps. We first introduce a “partially” decentralized observer to produce an augmented decentralized system with partial-state information. Then, we base the decentralized controller design on this enlarged dynamic system.

3.4.4 Partially Decentralized Observer

Owing to the structure in every local system of (3.67), for each $1 \le j \le N$, we introduce the following state estimator for the (z_j, x_j)-subsystem:

$$\begin{aligned}\dot{\hat{z}}_j &= Q_j \hat{z}_j + f_{j0}(y_{1r}, \ldots, y_{Nr}),\\ \dot{\hat{x}}_{j1} &= \hat{x}_{j2} + L_{j1}(y_j - x_{j1}) + f_{j1}(y_{1r}, \ldots, y_{Nr}) + g_{j1}(y_{1r}, \ldots, y_{Nr})\hat{z}_j,\\ &\vdots\\ \hat{x}_{jn_j} &= u_j + L_{jn_j}(y_j - \hat{x}_{j1}) + f_{jn_j}(y_{1r}, \ldots, y_{Nr}) + g_{jn_j}(y_{1r}, \ldots, y_{Nr})\hat{z}_j,\end{aligned} \tag{3.70}$$

$$A_j = \begin{bmatrix} -L_{j1} & \\ -L_{j2} & I_{n_j-1} \\ \vdots & \\ -L_{jn_j} & 0 \ldots 0 \end{bmatrix}. \tag{3.71}$$

Notice that the eigenvalues of A_j can be assigned to any desired location in the open left-half plane via the choice of appropriate constants $\{L_{jk}\}_{k=1}^{n_j}$, provided complex conjugate eigenvalues appear in pair. In (3.71), I_{n_j-1} is the unit matrix of order $n_j - 1$.

Introducing the new variables

$$\tilde{z}_j = z_j - \hat{z}_j, \quad \tilde{x}_{ijk} = x_{jk} - \hat{x}_{jk}, \quad 1 \le k \le n_j,\ 1 \le i \le N. \tag{3.72}$$

From (3.67) and (3.70), it follows that:

$$\dot{\tilde{z}}_j = Q_j \tilde{z}_j + f_{j0}(y_1, \ldots, y_N) - f_{j0}(y_{1r}, \ldots, y_{Nr}) + p_{j0}(y_1, \ldots, y_N)w_j, \tag{3.73}$$

$$\begin{aligned}\dot{\tilde{x}}_j &= A_j \tilde{x}_j + f_j(y_1, \ldots, y_N) - f_j(y_{1r}, \ldots, y_{Nr})\\ &\quad + g_j(y_1, \ldots, y_N)z_j - g_j(y_{1r}, \ldots, y_{Nr})\hat{z}_j + p_j(y_1, \ldots, y_N)w_j,\end{aligned} \tag{3.74}$$

where

$$\tilde{x}_j = (\tilde{x}_{j1}, \ldots, \tilde{x}_{jn_j})^t, \quad f_j = (f_{j1}, \ldots, f_{jn_j})^t,$$
$$g_j = (g_{j1}, \ldots, g_{jn_j})^t, \quad p_j = (p_{i1}, \ldots, p_{jn_j})^t.$$

Since every f_{jk} is a smooth function and every y_{jr} is a bounded signal, there exist a finite number of nonnegative smooth functions $\{\varphi_{j0k}\}_{k=1}^N$, $\{\varphi_{jk}\}_{k=1}^N$ such that

$$|f_{j0}(y_1, \ldots, y_N) - f_{j0}(y_{1r}, \ldots, y_{Nr})| \le \sum_{k=1}^N |\tilde{x}_{k1}|\varphi_{j0k}(\tilde{x}_{k1}), \tag{3.75}$$

$$|f_j(y_1, \ldots, y_N) - f_j(y_{1r}, \ldots, y_{Nr})| \le \sum_{k=1}^N |\tilde{x}_{k1}|\varphi_{jk}(\tilde{x}_{k1}). \tag{3.76}$$

In a similar way, we can obtain a functional bound for

$$g_j(y_1,\dots,y_N)z_j - g_j(y_{1r},\dots,y_{Nr})\hat{z}_j.$$

Indeed, we have

$$\begin{aligned} &g_j(y_1,\dots,y_N)z_j - g_j(y_{1r},\dots,y_{Nr})\hat{z}_j \\ &\quad = g_j(y_1,\dots,y_N)\tilde{z}_j + (g_j(y_1,\dots,y_N) - g_j(y_{1r},\dots,y_{Nr}))\hat{z}_j. \end{aligned} \tag{3.77}$$

By the Mean Value Theorem, there exist nonnegative smooth functions ϕ_{jk} $(1 \le k \le N)$ such that

$$\begin{aligned} &|g_j(y_1,\dots,y_N)z_j - g_j(y_{1r},\dots,y_{Nr})\hat{z}_j| \\ &\quad \le |g_j(y_1,\dots,y_N)||\tilde{z}_j| + \sum_{k=1}^{N} |\tilde{x}_{k1}|\phi_{jk}(\tilde{x}_{k1})|\hat{z}_j|. \end{aligned} \tag{3.78}$$

Combining these inequalities (3.75), (3.76) and (3.78), it is easy to show, in the absence of disturbance inputs, that the solutions $(\tilde{z}_j(t), \tilde{x}_j(t))$ of the cascade system (3.73)–(3.74) go to zero, if $y_j(t) - y_{jr}(t) \to 0$ for all $1 \le j \le N$. The latter property will be shown to be guaranteed with the help of the decentralized controller to be designed shortly.

Remark 3.15 It must be emphasized that the observer (3.70) is not asymptotic and is totally decentralized only if the reference signals $y_{jr} = 0$ for all $1 \le j \le N$. Therefore, we select a partially decentralized observer so that; in appropriate coordinates; system (3.79) has an equilibrium point and consequently, there is a solution to decentralized asymptotic tracking. When $y_{jr}(t)$ are general time-varying signals, the augmented system with a totally decentralized observer does not have a fixed equilibrium. In effect, only practical tracking can be achieved by means of high-gain feedback [46].

3.4.5 Controller Design Procedure

From the development of partially decentralized observers, we derive the following controller-observer combined system for feedback design:

$$\begin{aligned} \dot{\tilde{z}}_j &= Q_j\tilde{z}_j + f_{j0}(y_1,\dots,y_N) - f_{j0}(y_{1r},\dots,y_{Nr}) + p_{j0}(y_1,\dots,y_N)w_j, \\ \dot{\tilde{x}}_j &= A_j\tilde{x}_j + f_j(y_1,\dots,y_N) - f_j(y_{1r},\dots,y_{Nr}) \\ &\qquad + g_j(y_1,\dots,y_N)z_j - g_j(y_{1r},\dots,y_{Nr})\hat{z}_j + p_j(y_1,\dots,y_N)w_j, \\ \dot{y}_j &= \hat{x}_{j2} + \tilde{x}_{j2} + f_{j1}(y_1,\dots,y_N) + g_{i1}(y_1,\dots,y_N)z_j \\ &\qquad + p_{j1}(y_1,\dots,y_N)w_j, \end{aligned} \tag{3.79}$$

$$\dot{\hat{x}}_{j2} = \hat{x}_{j3} + L_{j2}(y_j - \hat{x}_{j1}) + f_{j2}(y_{1r}, \ldots, y_{Nr}) + g_{j2}(y_{1r}, \ldots, y_{Nr})\hat{z}_j,$$

$$\vdots$$

$$\dot{\hat{x}}_{jn_j} = u_j + L_{jn_j}(y_j - \hat{x}_{j1}) + f_{jn_j}(y_{1r}, \ldots, y_{Nr}) + g_{jn_j}(y_{1r}, \ldots, y_{Nr})\hat{z}_j.$$

Notice that the state variables $(y_j, \hat{x}_{j1}, \hat{x}_{i2}, \ldots, \hat{x}_{jn_j})$, and then $\tilde{x}_{j1}$, are available for feedback design. The states $(\tilde{z}_j, \tilde{x}_j)$ are unmeasured and the outputs y_j, with $k \neq j$, of other subsystems are unavailable for the design of the regional input u_j.

We now direct attention to the jth local system (3.79) with u_j being the control input. For the simplicity in exposition, denote

$$\tilde{f}_{j0} = f_{j0}(y_1, \ldots, y_N) - f_{j0}(y_{1r}, \ldots, y_{Nr}), \tag{3.80}$$

$$\tilde{f}_j = f_j(y_1, \ldots, y_N) - f_j(y_{1r}, \ldots, y_{Nr}), \tag{3.81}$$

$$\tilde{g}_j = g_j(y_1, \ldots, y_N)z_j - g_j(y_{1r}, \ldots, y_{Nr})\hat{z}_j. \tag{3.82}$$

In the sequel, we develop a step-by-step constructive controller design procedure, leading to an effective solution to the desired decentralized problem and tracking controllers.

Step $j.1$**.** Start with the first $(\tilde{z}_j, \tilde{x}_j, y_j)$-subsystem of (3.79). Introduce the new variable $\xi_{j1} = y_j - y_{jr}$ $(= \tilde{x}_{j1})$ and consider the proper function

$$V_{i1} = \lambda_{j1}\tilde{z}_j^t P_{j1}\tilde{z}_j + \lambda_{j2}(\tilde{z}_j^t P_{j1}\tilde{z}_{j1})^2 + \tilde{x}_j^t P_{j2}\tilde{x}_j + \frac{1}{2}\xi_{j1}^2 > 0, \tag{3.83}$$

where $\lambda_{j1}, \lambda_{j2} > 0$ are design parameters and $P_{j1} = P_{j1}^t > 0$ and $P_{j2} = P_{j2}^t > 0$ satisfy the local Lyapunov equations

$$P_{j1}Q_j + Q_j^t P_{j1} = -2I_{n_{z_j}}, \tag{3.84}$$

$$P_{j2}A_j + A_j^t P_{j2} = -2I_{n_j}. \tag{3.85}$$

Evaluating the time derivative of V_{j1} along the solutions of (3.79) it yields

$$\begin{aligned}
\dot{V}_{j1} = {} & (\lambda_{j1} + 2\lambda_{j2}\tilde{z}_j^t P_{j1}\tilde{z}_j)(-2|\tilde{z}_j|^2 + 2\tilde{z}_j^t P_{j1}(\tilde{f}_{j0} + p_{j0}w_j)) \\
& - 2|\tilde{x}_j|^2 + 2\tilde{x}_j^t P_{j2}(\tilde{f}_j + \tilde{g}_j + p_j w_j) + \xi_{j1}(\hat{x}_{j2} + \tilde{x}_{j2}) \\
& + f_{j1}(y_1, \ldots, y_N) + g_{j1}(y_1, \ldots, y_N)z_j \\
& + p_{j1}(y_1, \ldots, y_N)w_j - \dot{y}_{jr}.
\end{aligned} \tag{3.86}$$

Focusing on the term $2\tilde{z}_j^t P_{j1}(\tilde{f}_{j0} + p_{j0}w_j)$ and using (3.80) and (3.75), with the help of Young's inequality (see Chap. 10) and after some tedious calculations, it follows that:

$$
\begin{aligned}
&2(\lambda_{j1} + 2\lambda_{j2}\tilde{z}_j^t P_{j1}\tilde{z}_j)\tilde{z}_j^t P_{j1}(\tilde{f}_{j0} + p_{j0}w_j) \\
&\quad \le \lambda_{j1}|\tilde{z}_j|^2 + \frac{3\lambda_{j2}}{\lambda_{\max}(P_{j1})}(\tilde{z}_j^t P_{j1}\tilde{z}_j)^2 \\
&\qquad + \sum_{k=1}^{N} \xi_{k1}^2 \psi_{ik1}(\xi_{k1}) + c_{j2}|w_j|^2 + c_{j3}|w_j|^4 + |w_j|^8,
\end{aligned} \tag{3.87}
$$

where $c_{j1}, c_{j2}, c_{j3} > 0$ and ψ_{jk1} is a nonnegative smooth function.

In a similar way, there exist positive constants κ_{j1}, c_{j4} and a smooth function $0 \le \psi_{jk2}$ such that

$$
\begin{aligned}
2\tilde{x}_j^t P_{j2}(\tilde{f}_j + \tilde{g}_j + p_j w_j) \le |\tilde{x}_j|^2 + \kappa_{j1}|\tilde{z}_j|^2 + |\tilde{z}_j|^4 + \sum_{k=1}^{N} \xi_{k1}^2 \psi_{jk2}(\xi_{k1}) \\
+ c_{j4}|w_j|^2 + |w_j|^4,
\end{aligned} \tag{3.88}
$$

where we have used the fact that $\hat{z}_j$ is bounded because of Assumption 3.10.

By substituting (3.87) and (3.88) into (3.86), we obtain

$$
\begin{aligned}
\dot{V}_{j1} \le &-(\lambda_{j1} + \lambda_{j2}\tilde{z}_j^t P_{j1}\tilde{z}_j)|\tilde{z}_j|^2 - |\tilde{x}_j|^2 + \sum_{k=1}^{N} \xi_{k1}^2 (\psi_{jk1} + \psi_{jk2}) \\
&+ \kappa_{j1}|\tilde{z}_j|^2 + |\tilde{z}_j|^4 + (c_{j2} + c_{j4})|w_j|^2 + (c_{j3} + 1)|w_j|^4 \\
&+ |w_j|^8 + \xi_{j1}(\hat{x}_{j2} + \tilde{x}_{j2} + f_{j1}(y_1, \ldots, y_N) \\
&+ g_{j1}(y_1, \ldots, y_N) z_j + p_{j1}(y_1, \ldots, y_N) w_j - \dot{y}_{jr}).
\end{aligned} \tag{3.89}
$$

It must be noted that κ_{j1} does not depend on λ_{j1} and λ_{j2} while c_{jk}'s may depend on λ_{j1} and λ_{j2}. Proceeding further, using (3.76) and (3.78), we have

$$
\begin{aligned}
&\xi_{j1}(\tilde{x}_{j2} + \tilde{f}_{j1} + \tilde{g}_{j1} + p_{j1}w_j) \\
&\quad \le \frac{1}{2}|\tilde{x}_j|^2 + \sum_{k=1}^{N} \xi_{k1}^2 \psi_{ik3}(\xi_{k1}) + |\tilde{z}_j|^2 + |w_j|^2,
\end{aligned} \tag{3.90}
$$

where $\psi_{jk3} \le 0$ is a smooth function.

Keeping in mind the decomposition in (3.81) and (3.82) and letting

$$
\hat{\psi}_{jk1} = \psi_{jk1} + \psi_{jk2} + \psi_{jk3}
$$

it follows that

$$\begin{aligned}\dot{V}_{j1} \leq &- (\lambda_{j1} + \lambda_{j2}\tilde{z}_j^t P_{j1}\tilde{z}_j - \kappa_{j1} - 1 - |\tilde{z}_j|^2)|\tilde{z}_j|^2 \\ &- \frac{1}{2}|\tilde{x}_j|^2 + (c_{j2} + c_{j4} + 1)|w_j|^2 + (c_{j3} + 1)|w_j|^4 \\ &+ |w_j|^8 + \xi_{j1}(\hat{x}_{j2} + f_{j1}(y_{1r}, \dots, y_{Nr}) \\ &+ g_{j1}(y_{1r}, \dots, y_{Nr})\hat{z}_j - \dot{y}_{jr}) + \sum_{k=1}^{N} \xi_{k1}^2 \hat{\psi}_{jk1}. \end{aligned} \tag{3.91}$$

This motivates choosing a control function ξ_{j1}^* and a new variable ξ_{j2} as

$$\begin{aligned}\xi_{j1}^* = &-k_{j1}\xi_{j1} - \xi_{j1}K_j(\xi_{j1}) - f_{j1}(y_{1r}, \dots, y_{Nr}) - g_{j1}(y_{1r}, \dots, y_N r)\hat{z}_j \\ &+ \dot{y}_{jr}, \end{aligned} \tag{3.92}$$

$$\xi_{i2} = \hat{x}_{i2} - \xi_{i1}^*(y_j, y_{1r}, \dots, y_{Nr}, \dot{y}_{ir}, \hat{z}_j), \tag{3.93}$$

where $k_{j1} > 0$ is a design parameter and $K_j \leq 0$ is a smooth function such that

$$K_{j1}(\xi_{j1}) \geq \sum_{k=1}^{N} \hat{\psi}_{kj1}(\xi_{j1}). \tag{3.94}$$

Consequently, we get

$$\begin{aligned}\dot{V}_{j1} \leq &-(\lambda_{j1} + \lambda_{j2}\tilde{z}_j^t P_{j1}\tilde{z}_j - \kappa_{j1} - 1 - |\tilde{z}_j|^2)|\tilde{z}_j|^2 \\ &- \frac{1}{2}|\tilde{x}_j|^2 + (c_{j2} + c_{j4} + 1)|w_j|^2 \\ &+ (c_{j3} + 1)|w_j|^4 + |w_j|^8 - k_{j1}\xi_{j1}^2 - \xi_{j1}^2 K_j(\xi_{j1}) \\ &+ \sum_{k=1}^{N} \xi_{k1}^2 \hat{\psi}_{jk1}(\xi_{k1}) + \xi_{j1}\xi_{j2}. \end{aligned} \tag{3.95}$$

Step $j.k$ $(2 \leq k \leq n_j)$**.** Consider the $(\tilde{z}_j, \tilde{x}_j, y_j, \hat{x}_{i2}, \dots, \hat{x}_{jk})$-subsystem of (3.79) with $\hat{x}_{j,k+1}$ as the virtual control. For notational simplicity, we define $\hat{x}_{j,n_j+1} := u_j$.

Assume that, from Step $j.1$ to Step $j.(k-1)$, we have designed intermediate control functions $\{\xi_{j\ell}^*\}_{\ell=1}^{k-1}$, and that we have introduced new variables

$$\begin{aligned}&\xi_{j,\ell+1} = \hat{x}_{j,\ell+1} - \xi_{jl}^*(y_j, \hat{x}_{j2}, \dots, \hat{x}_{ijl}, y_{1r}, \dots, y_{Nr}, \dot{y}_{jr}, \dots, y_{ir}^{(l)}, \hat{z}_j) \\ &\forall 1 \leq \ell \leq k-1 \end{aligned} \tag{3.96}$$

and a proper function

$$V_{j,k-1}(\tilde{z}_j, \tilde{x}_j, \xi_{j1}, \dots, \xi_{j,k-1}) = V_{j1}(\tilde{z}_j, \tilde{x}_j, \xi_{j1}) + \sum_{\ell=2}^{k-1} \frac{1}{2}\xi_{j\ell}^2 > 0. \tag{3.97}$$

It is further assumed that the time derivative of $V_{j,k-1}$ along the solutions of (3.79) satisfies

$$\begin{aligned}\dot{V}_{j,k-1} \le & -(\lambda_{j1} + \lambda_{j2}\tilde{z}_j^t P_{j1}\tilde{z}_j - \kappa_{j1} - k + 1 - |\tilde{z}_j|^2)|\tilde{z}_j|^2 \\ & - \frac{1}{2^{k-1}}|\tilde{x}_j|^2 + (k - 1 + c_{j2} + c_{j4})|w_j|^2 \\ & + (c_{j3} + 1)|w_j|^4 + |w_j|^8 - \sum_{\ell=1}^{k-1} k_{j\ell}\xi_{j\ell}^2 - \xi_{j1}^2 K_j(\xi_{j1}) \\ & + \sum_{m=1}^{N} \xi_{m1}^2 \hat{\psi}_{jm(k-1)}(\xi_{m1}) + \xi_{j,k-1}\xi_{jk}\end{aligned} \tag{3.98}$$

with $k_{j\ell}$ $(1 \le \ell \le k-1)$ being positive design parameters and $\hat{\psi}_{jm(k-1)}$ a nonnegative smooth function being independent of K_j.

To proceed further, it is desired to establish that a similar property also holds for the

$$(\tilde{z}_j, \tilde{x}_j, y_j, \hat{x}_{i2}, \ldots, \hat{x}_{ij})\text{-}subsystem$$

of (3.79) when $\hat{x}_{j,k+1}$ is considered as the (virtual) input. For this purpose, consider the proper function

$$V_{jk} = V_{j,k-1}(\tilde{z}_j, \tilde{x}_j, \xi_{j1}, \ldots, \xi_{j,k-1}) + \frac{1}{2}\xi_{jk}^2 > 0. \tag{3.99}$$

Differentiating V_{jk} along the solutions of (3.79) gives

$$\begin{aligned}\dot{V}_{jk} = \dot{V}_{j,k-1}\xi_{jk}\Bigg[& \hat{x}_{j,k+1} + L_{jk}(y_j - \hat{x}_{j1}) + f_{jk}(y_{1r}, \ldots, y_{Nr}) \\ & + g_{jk}(y_{1r}, \ldots, y_N r)\hat{z}_j - \sum_{m=2}^{k-1} \frac{\partial \xi_{j,k-1}^*}{\partial \hat{x}_{jm}}(\hat{x}_{j,m+1} + L_{jm}(y_j - \hat{x}_{j1}) \\ & + f_{jm}(y_{1r}, \ldots, y_{Nr}) + g_{jm}(y_{1r}, \ldots, y_{Nr})\hat{z}_j) \\ & - \sum_{m=1}^{N} \frac{\partial \xi_{j,k-1}^*}{\partial y_{mr}}\dot{y}_{mr} - \sum_{m=1}^{k-1} \frac{\partial \xi_{j,k-1}^*}{\partial y_{jr}^{(m+1)}} y_{jr}^{(m+1)} \\ & - \frac{\partial \xi_{j,k-1}^*}{\partial \hat{z}_j}(Q_j\hat{z}_j + f_{j0}(y_{1r}, \ldots, y_{Nr})) \\ & - \frac{\partial \xi_{j,k-1}^*}{\partial y_j}(\hat{x}_{j2} + \hat{x}_{j2} + f_{j1} + g_{j1}z_j + p_{j1}w_j)\Bigg].\end{aligned} \tag{3.100}$$

With the help of similar arguments as in Step $j.1$, after lengthy but routine manipulation, it follows the existence of nonnegative smooth functions $\{\psi_{jmk}\}_{m=1}^{N}$ and κ_{jk}

such that:

$$
\begin{aligned}
&-\xi_{jk}\frac{\partial \xi^*_{j,k-1}}{\partial y_j}(\tilde{x}_{j2}+\tilde{f}_{j1}+\tilde{g}_{j1}+p_{j1}w_j)\\
&\quad\le \frac{1}{2^j}\tilde{x}_j^2+\xi_{jk}^2\kappa_{jk}+\sum_{m=1}^{N}\xi_{m1}^2\psi_{jmk}(\xi_{m1})+|\tilde{z}_j|^2+|w_j|^2. \qquad (3.101)
\end{aligned}
$$

It must be noted that κ_{jk} is a function of $(y_j, \hat{x}_{j2}, \dots, \hat{x}_{jk}, y_{1r}, \dots, y_{Nr}, \dot{y}_{jr}, \dots, y_{jr}^{(l)}, \hat{z}_j)$ and that every ψ_{jmk} does not depend on K_j.

We are now motivated to choose the following control function:

$$
\begin{aligned}
\xi^*_{jk} &= -k_{jk}\xi_{jk}-\xi_{j,k-1}-\xi_{jk}\kappa_{jk}-L_{jk}(y_j-\hat{x}_{j1})\\
&\quad -f_{jk}(y_{1r},\dots,y_{Nr})-g_{jk}(y_{1r},\dots,y_{Nr})\hat{z}_j\\
&\quad +\frac{\partial \xi^*_{j,k-1}}{\partial y_j}(\hat{x}_{j2}+f_{j1}(y_{1r},\dots,y_{Nr})+g_{j1}(y_{1r},\dots,y_{Nr})\hat{z}_j)\\
&\quad +\sum_{m=2}^{k-1}\frac{\partial \xi^*_{j,k-1}}{\partial \hat{x}_{jmk}}(\hat{x}_{j,m+1}+L_{jm}(y_j-\hat{x}_{j1})\\
&\quad + f_{jm}(y_{1r},\dots,y_{Nr})+g_{jm}(y_{1r},\dots,y_{Nr})\hat{z}_j)\\
&\quad +\sum_{m=1}^{N}\frac{\partial \xi^*_{j,k-1}}{\partial y_{mr}}\dot{y}_{mr}+\sum_{m=1}^{k-1}\frac{\partial \xi^*_{j,k-1}}{\partial y_{jr}^{(m)}}y_{jr}^{(m+1)}\\
&\quad +\frac{\partial \xi^*_{j,k-1}}{\partial \hat{z}_j}(Q_j\hat{z}_j+f_{j0}(y_{1r},\dots,y_{Nr})), \qquad (3.102)
\end{aligned}
$$

where $k_{jk}>0$ is a design parameter.

In terms of the deviation vector $\xi_{j,k+1}=\hat{x}_{j,k+1}-\xi^*_{jk}$ and combining (3.98), (3.100), (3.101) and (3.102) together, we obtain

$$
\begin{aligned}
\dot{V}_{jk} &\le -(\lambda_{j1}+\lambda_{j2}\tilde{z}_j^t P_{j1}\tilde{z}_j-\kappa_{j1}-j-|\tilde{z}_j|^2)|\tilde{z}_j|^2\\
&\quad -\frac{1}{2^j}|\tilde{x}_j|^2+(j+c_{j2}+c_{j4})|w_j|^2+(c_{j3}+1)|w_j|^4\\
&\quad +|w_j|^8-\sum_{\ell=1}^{k}k_{j\ell}\xi_{j\ell}^2-\xi_{j1}^2K_j(\xi_{j1})\\
&\quad +\sum_{m=1}^{N}\xi_{m1}^2(\hat{\psi}_{jm(k-1)}(\xi_{m1})+\psi_{jmk}(\xi_{m1}))+\xi_{jk}\xi_{j,k+1}. \qquad (3.103)
\end{aligned}
$$

This implies that inequality (3.98) holds for the $(\tilde{z}_j, \tilde{x}_j, y_j, \hat{x}_{j2}, \dots, \hat{x}_{jk})$-subsystem with $\hat{\psi}_{jmk}=\hat{\psi}_{jm(k-1)}+\psi_{jmk}$.

Now by induction, at Step n_j and setting the control law

$$u_j = \xi^*_{jn_j}(y_j, \hat{x}_{j2}, \ldots, \hat{x}_{jn_j}, y_{1r}, \ldots, y_{Nr}, \dot{y}_{jr}, \ldots, y_{jr}^{(n_j)}, \hat{z}_j). \tag{3.104}$$

It turn, it leads to

$$\begin{aligned}\dot{V}_{jn_j} \leq & -(\lambda_{j1} + \lambda_{j2}\tilde{z}_j^t P_{j1}\tilde{z}_j - \kappa_{j1} - n_j - |\tilde{z}_j|^2)|\tilde{z}_j|^2 \\ & - \frac{1}{2^{n_j}}|\tilde{x}_j|^2 + (n_j + c_{j2} + c_{j4})|w_j|^2 \\ & + (c_{j3} + 1)|w_j|^4 + |w_j|^8 - \sum_{\ell=1}^{n_j} k_{j\ell}\xi_{j\ell}^2 - \xi_{j1}^2 K_j(\xi_{j1}) \\ & + \sum_{m=1}^{N} \xi_{m1}^2 \hat{\psi}_{jmn_j}(\xi_{m1}),\end{aligned} \tag{3.105}$$

where we recall by construction that $\hat{\psi}_{ijmn_j}$ is independent of the design function K_j.

By considering the overall proper Lyapunov function for the entire closed-loop interconnected system

$$V(\tilde{z}, \tilde{x}, \xi) = \sum_{j=1}^{N} V_{jn_j}(\tilde{z}_j, \tilde{x}_j, \xi_{j1}, \ldots, \xi_{jn_j}) > 0, \tag{3.106}$$

where

$$\tilde{z} = (\tilde{z}_1^j, \ldots, \tilde{z}_N^j)^j, \quad \tilde{x} = (\tilde{x}_1^j, \ldots, \tilde{x}_N^j)^j, \quad \xi = \xi_1^j, \ldots, \xi_N^j)^j$$

and the positive definiteness and properness of V in (3.106) follows from the foregoing recursive construction.

Finally, to eliminate the last positive term of (3.105), which also appears in the time derivative of V, we select an appropriate set of smooth functions $\{K_j\}_{j=1}^N$ satisfying the inequalities ($1 \leq j \leq N$)

$$K_j(\xi_{j1}) \geq \sum_{m=1}^{N} \hat{\psi}_{mjn_m}(\xi_{j1}). \tag{3.107}$$

It is evident that such a design function K_j always exists.

3.4.6 Control Design Results

By applying the foregoing design procedure to the uncertain interconnected system (3.67), we establish the following result.

Theorem 3.6 *The problem of decentralized output-feedback tracking with disturbance attenuation is solvable for the minimum-phase interconnected system* (3.67) *subject to Assumption* 3.10.

Proof By differentiating V of (3.106) along the solutions of the closed-loop system (3.67) with (3.104), it yields

$$\begin{aligned}\dot{V} \le & -\sum_{j=1}^{N}(\lambda_{j1} + \lambda_{j2}\tilde{z}_j^t P_{j1}\tilde{z}_j - \kappa_{j1} - n_j - |\tilde{z}_j|^2)|\tilde{z}_j|^2 \\ & - \sum_{j=1}^{N}\left(\frac{1}{2^{n_j}}|\tilde{x}_j|^2 + \sum_{\ell=1}^{n_j} k_{j\ell}\xi_{j\ell}^2\right) \\ & + \sum_{j=1}^{N}[(n_j + c_{j2} + c_{j4})|w_j|^2 + (c_{j3}+1)|w_j|^4 + |w_j|^8]. \end{aligned} \tag{3.108}$$

Selecting sufficiently large design parameters λ_1 and λ_2 such that

$$\begin{aligned}&(\lambda_{j1} + \lambda_{j2}\tilde{z}_j^t P_{j1}\tilde{z}_j - \kappa_{j1} - n_j - |\tilde{z}_j|^2)|\tilde{z}_j|^2 \\ &\quad \ge \frac{\lambda_{j1}}{2}\tilde{z}_j P_{j1}\tilde{z}_j + \frac{\lambda_{j2}}{2}(\tilde{z}_j P_{j1}\tilde{z}_j)^2. \end{aligned} \tag{3.109}$$

It follows from (3.106) and (3.108) that

$$\dot{V} \le -\lambda V + \sum_{j=1}^{N}[(n_j + c_{j2} + c_{j4})|w_j|^2 + (c_{j3}+1)|w_j|^4 + |w_j|^8], \tag{3.110}$$

where

$$\lambda_a = \frac{1}{2}, \quad \lambda_c = \frac{\lambda_M P_{j2}}{2^{n_j}}, \quad \lambda = \min\{\lambda_a, \lambda_c, kj\ell\}, \quad 1 \le j \le N,\ 1 \le \ell \le n_j.$$

Applying the Gronwall Lemma [27] to (3.110), the BIBS condition and iISS property 2 follow immediately for the transformed closed-loop system (3.67) with (3.104). Moreover, when $w_j \equiv 0$, $\forall 1 \le j \le N$, the null solution is uniformly globally asymptotically stable, leading to the asymptotic convergence of the tracking error $y - y_r$ because $\xi_1 = y - y_r$. It must be emphasized that same result could have been attained by following parallel procedure to [48].

Finally from (3.108), for any pair of instants $0 \le t_0 \le t$, we obtain

$$\begin{aligned}\int_{t_0}^{t} |\xi_1(\tau)|^2 \, d\tau \le V(z(t_0), x(t_0), \xi(t_0)) + \rho \int_{t_0}^{t} (|w(\tau)|^2 \\ + |w(\tau)|^4 + |w(\tau)|^8) \, d\tau, \end{aligned} \tag{3.111}$$

where $\rho > 0$ defined by

$$\rho = \max\left\{\frac{\max\{n_j + c_{j2} + c_{j3} | 1 \le j \le N\}}{\min\{k_{j1} | 1 \le j \le N\}}, \frac{\max\{c_{j3} + 1 | 1 \le j \le N\}}{\min\{k_{j1} | 1 \le j \le N\}}, \frac{1}{\min\{k_{j1} | 1 \le j \le N\}}\right\}$$

and observe that ρ can be made as small as possible by selecting sufficiently large values of the constants k_{j1}. In the present case, (3.69) is met with $\gamma_d(s) = s^2 + s^4 + s^8$ which completes the proof of Theorem 3.6. □

The following remarks stand out:

Remark 3.16 It is of interest to note that, in the absence of disturbance inputs w, (3.110) eventually yields that V converges to zero at an exponential rate and; therefore; the tracking error $y(t) - y_r(t)$ goes to zero exponentially.

Remark 3.17 In centralized output-feedback tracking with almost disturbance decoupling [35], Assumption 3.10 can be weakened and the z_j-system in (3.67) can be broadened as follows:

$$\dot{z}_j = \Gamma_j(y_1, \dots, y_N) z_j + f_{j0}(y_1, \dots, y_N) + p_{j0}(y_1, \dots, y_N) w_j. \tag{3.112}$$

Considering that, for each $1 \le j \le N$, there are a pair of constant, matrices ($0 < P_j^t = P_j,\ 0 < M_j^t = M_j$) such that

$$\Gamma_j^t(y_1, \dots, y_N) P_j + P_j \Gamma_j(y_1, \dots, y_N) \le -M_j. \tag{3.113}$$

Under this condition, the $\hat{z}_j$-system in the decentralized observer (3.70) is replaced by

$$\dot{\hat{z}}_j = \Gamma_j(y_{1r}, \dots, y_{Nr}) \hat{z}_j + f_{j0}(y_{1r}, \dots, y_{Nr}). \tag{3.114}$$

By using the same techniques as in Sect. 3.4.5, Theorem 3.6 can be extended to this situation.

3.4.7 $\mathcal{L}_2$-Gain Disturbance Attenuation

In what follows, we examine whether the controller design procedure yields a decentralized output-feedback law guaranteeing the standard $\mathcal{L}_2$-gain disturbance attenuation property, that is, (3.69) holds with $\gamma_d(s) = s^2$. In this case, the following additional sufficient condition is needed.

Assumption 3.11 *For all $1 \le j \le N$ and $1 \le k \le n_j$, the function p_{jk} is bounded by a constant. Furthermore, $p_{j0} = 0$ for each $1 \le j \le N$.*

Proposition 3.1 *Under Assumptions 3.10 and 3.11, the problem of decentralized output-feedback tracking with standard $\mathcal{L}_2$-gain disturbance attenuation is solvable for the class of minimum-phase interconnected systems* (3.67).

Proof It suffices to note that the only place where $|w_j|^4$ and $|w_j|^8$ occur are Step $j.1$ during the controller development in Sect. 3.4.5. These terms are entered into the inequalities (3.87) and (3.88). Under Assumption 3.11, the function V_{j1} satisfies the following inequality, in replace of (3.95):

$$\begin{aligned}\dot{V}_{j1} \le & -(\lambda_{j1} + \lambda_{j2}\tilde{z}_j^t P_{j1}\tilde{z}_j - \kappa_{j1} - 1 - |\tilde{z}_j|^2)|\tilde{z}_j|^2 \\ & - \frac{1}{2}|\tilde{x}_j|^2 + (c_{j2} + c_{j4} + 1)|w_j|^2 - k_{j1}\xi_{j1}^2 \\ & - \xi_{j1}^2 K_j(\xi_{j1}) + \sum_{m=1}^{N} \xi_{m1}^2 \hat{\psi}_{jm1}(\xi_{m1}) + \xi_{j1}\xi_{j2}. \end{aligned} \tag{3.115}$$

Consequently, in replace of (3.110), this Lyapunov function V satisfies

$$\dot{V} \le -\lambda V + \sum_{j=1}^{N}[(n_j + c_{j2} + c_{j4})|w_j|^2]. \tag{3.116}$$

Finally, from (3.116), the standard $\mathcal{L}_2$-gain property from w to $\xi_1 = y - y_r$ follows readily. This concludes the proof of Proposition 3.1. □

Remark 3.18 As a corollary of Theorem 3.6, the standard $\mathcal{L}_2$-gain property from w to $\xi_1 = y - y_r$ can similarly be proven when all functions f_{jk}, g_{jk} in decentralized system (3.67) are bounded by linear functions and the functions p_{jk} ($1 \le j \le N$, $0 \le k \le n_j$, $p_{i0} \ne 0$) are bounded by some constants. The resulting decentralized output-feedback controllers would be linear.

3.5 Application to Power Systems

Power systems are increasingly called upon to operate transmission lines at high transmission level due to economic considerations. In a lot of cases, transient stability transfer limits are more constraining than steady-state limits under contingency. On the other hand, operating conditions of modern large scale power systems are always varying to satisfy different load demands. The control systems are therefore required to have the ability to damp the system oscillations that might threaten the system stability as load demands increase or after a major fault occurs, and maintain the system stability under a diversity of operating conditions and different system configurations.

In the design of conventional control systems, approximately linearized power system models are employed. Normally, the system is simplified as single-machine to infinite bus model and approximately linearized at one operating point. Then conventional controllers are designed based on the simplified linear model. It is obvious that when a major fault occurs, the behavior of the power system may change significantly. Conventional linear controllers do not guarantee the system stability under such circumstances.

In recent years, a great deal of attention has been given to the control of power systems using the recent developed nonlinear control theory, particularly to improve system transient stability [1, 40, 47, 48]. Rather than using an approximately linearized model as in the design of the usual power system stabilizer, nonlinear models are used and nonlinear feedback linearization techniques are employed to linearize the power system models, thereby alleviating the operating point dependent nature of the linear designs. Using nonlinear controllers, power system transient stability can be improved significantly. However, nonlinear controllers are of more complicated structure and harder to be implemented in practice compared with linear controllers. In addition, feedback linearization schemes need exact plant parameters to cancel the inherent system nonlinearities and make the stability analysis a formidable task. The design of decentralized linear controllers to enhance the stability of interconnected nonlinear power systems within the whole operating region is still a challenging task [41].

In this section, we will consider the linear controller design problem of an N-machine nonlinear power system. Unlike the approximately linearized model normally used, a nonlinear fourth order classical model, including the governor/turbine dynamics of multi-machine power systems, will be considered. Robust control technique [53, 55, 56, 62, 63], will be employed to develop a linear control scheme for power system transient stability enhancement. Nonlinear interconnections are treated similar to parametric uncertainties [57] and the control of each generator is derived separately by solving an algebraic Riccati equation. Although the proposed scheme is a decentralized linear controller, it can guarantee the stability of the nonlinear power system model in the whole operating region. The design of the controller only requires local measurements and can be easily implemented.

3.5.1 Power System Model

An N-machine power system with steam valve control can be described by the interconnection of N subsystems as follows [3, 28]:

$$\dot{x}_j(t) = A_j x_j(t) + B_j u_j(t) + \sum_{j=1, j\neq 1}^{N} p_{ij} G_{ij} g_{ij}(x_j, x_j), \tag{3.117}$$

where $i \neq N$; we define the Nth machine as the slack machine,

$$x_j^t(t) = [\Delta\delta_j(t)\ \omega_j(t)\ \Delta P_{M_j}(t)\ \Delta X_{E_j}(t)],$$
$$\Delta\delta_j(t) = \delta_j(t) - \delta_{i0}, \quad \Delta P_{M_j}(t) = P_{M_j}(t) - P_{M_{j0}},$$
$$\Delta X_{E_j}(t) = X_{E_j}(t) - X_{E_j 0}$$

with

$$A_j = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{D_j}{2H_j} & \frac{\omega_0}{2H_j}(1 - F_{IP_j}) & \frac{\omega_0}{2H_j} F_{IP_j} \\ 0 & 0 & -\frac{1}{T_{M_j}} & \frac{k_{M_j}}{T_{M_j}} \\ 0 & -\frac{k_{E_j}}{T_{E_j} R_j \omega_0} & 0 & -\frac{1}{T_{E_j}} \end{bmatrix},$$

$$B_j = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{T_{E_j}} \end{bmatrix}; \qquad G_{ij} = \begin{bmatrix} 0 \\ -\frac{\omega_0 E_{qi} E_{qj} B_{ij}}{2H_j} \\ 0 \\ 0 \end{bmatrix},$$

$$g_{ij}(x_j, x_j) = \sin(\omega_j - \omega_j) - \sin(\omega_{i0} - \omega_{j0}),$$

where

- p_{ij} constant of either 1 or 0 ($p_{ij} = 0$ means that jth machine has no connection with ith machine);
- H_j inertia constant for jth machine, in seconds;
- D_j damping coefficient for jth machine, in p.u.;
- F_{IP_j} fraction of the turbine power generated by the intermediate pressure (IP) section;
- T_{M_j} time constant of jth machine's turbine with typical numerical valve of 0.2 to 2.0 s;
- K_{M_j} gain of jth machine's turbine;
- T_{E_j} time constant of jth machine's speed governor, typically around 0.2 s;
- K_{E_j} gain of jth machine's speed governor; $K_{M_j} K_{E_j} = 1$;
- R_j regulation constant of jth machine in p.u., typically 0.05;
- B_{ij} ith row and jth column element of nodal susceptance matrix at the internal nodes after eliminated all physical buses, in p.u.;
- P_{M_j} mechanical power for jth machine, in p.u.;
- X_{E_j} steam valve opening for jth machine, in p.u.;
- P_{C_j} power control input of jth machine;
- u_j $P_{C_j} - P_{M_{j0}}$;
- ω_j relative speed for jth machine, in radian/s;
- ω_0 the synchronous machine speed; $\omega_0 = 2\Pi f_0$;
- δ_j rotor angle for jth machine, in radian;
- E_{qi} internal transient voltage for ith machine, in p.u., which is assumed to be constant;
- E_{qj} internal transient voltage for jth machine, in p.u., which is assumed to be constant

and δ_{j0}, $P_{M_{j0}}$, and X_{E_j0} are the initial values of $\delta_j(t)$, $P_{M_j}(t)$ and $X_{E_j}(t)$, respectively.

From the model shown in (3.117), we can see that system parameters

$$D_j, \quad H_j, \quad T_{M_j}, \quad K_{M_j}, \quad T_{E_j}, \quad K_{E_j}, \quad R_j$$

may be unknown and when a major fault occurs at the transmission line between ith generator and jth generator, the parameter b_{ij} will change. Thus, the model contains parameter uncertainties. Also the power system model contains nonlinearities and interconnections $g_{ij}(x_j, x_j)$. The problem addressed hereafter is phrased as follows:

Design decentralized linear time-invariant feedback control laws

$$P_{c_j}(t) = -K_j x_j(t), \quad j = 1, 2, \dots, N-1,$$

for multimachine power system (3.117) *such that the resulting closed-loop system is transiently stable when a major fault occurs in the system.*

3.5.2 Robust Stabilization

Consider the parameter uncertainties in multimachine power systems, the plant model (3.117) can be generalized as follows:

$$\begin{aligned} \dot{x}_j(t) &= [A_j + \Delta A_j(t)]x_j(t) + [B_j + \Delta B_j(t)]u_j(t) \\ &\quad + \sum_{m=1, m\neq 1}^{N} \{p_{jm}[G_{jm} + \Delta G_{jm}(t)]g_{jm}(x_j, x_j)\}, \quad j = 1, 2, \dots, N-1, \end{aligned} \tag{3.118}$$

where for the jth subsystem we have that: $x_j \in \Re^{n_j}$ is the state, $u_j \in \Re^{m_j}$ is the input, the matrices A_j, B_j and G_{jm} are known real constant matrices of appropriate dimensions that describe the nominal model, $\Delta A_j(\cdot)$, $\Delta B_j(\cdot)$, and $\Delta G_{jm}(\cdot)$ are real time varying parameter uncertainties, and $g_{jm}(x_j, x_j) \in \Re^{l_j}$ is unknown nonlinear vector functions that represent nonlinearities in the ith subsystem and the interactions with other subsystems.

The uncertain matrices $\Delta A_j(t)$, $\Delta B_j(t)$, and $\Delta G_{jm}(t)$ are assumed to be of the following structure:

$$[\Delta A_j(t) \Delta B_j(t)] = L_j F_j(t)[E_{1_j}, E_{2_j}], \tag{3.119}$$

$$\Delta G_{jm}(t) = L_{jm} F_{jm}(t) E_{jm} \tag{3.120}$$

with $F_j(t) \in \Re^{j_m \times m_m}$ and $F_{jm}(t) \in \Re^{j_{Gj} \times m_{Gj}}$ (for all j, m) being unknown matrix functions with Lebesgue measurable elements and satisfying

$$F_j^t(t) F_j(t) \le I_j; \qquad F_{jm}(t) F_{jm}^t(t) \le I_{jm}, \tag{3.121}$$

where L_j, E_{1_j}, E_{2_j}, L_{jm}, and E_{jm} are known real constant matrices with appropriate dimensions.

Remark 3.19 The parameter uncertainty structure in (3.119) has been widely used in the problem of robust stabilization of uncertain systems [25], and can represent parameter uncertainties in many physical systems. The decomposition of parameter uncertainties in the case of a three machine power system will be discussed later on.

The following assumptions concerning the unknown nonlinear vector functions and the matrix E_{2_j} are made:

Assumption 3.12 There exist known constant matrices $\overline{W}_j$ and W_{jm} such that for all $x_j \in \Re^{n_j}$ and $x_j \in \Re^{n_j}$

$$\|g_{jm}(x_j, x_m)\| \leq \|\overline{W}_j x_j(t)\| + \|W_{jm} x_m(t)\|$$

for all j, m and for all $t \geq 0$.

Remark 3.20 If the nonlinear functions $g_{jm}(x_j, x_m)$ satisfy Assumption 3.12, they are Lipschitz bounded nonlinearities. In the power system model (3.117), $g_{jm}(x_j, x_m)$ satisfy Assumption 3.12. A three machine example system will be presented in Sect. 3.5.3 and the detailed analysis will be given.

Assumption 3.13 For all $j = 1, 2, \ldots, N-1$

$$R_j = E_{2_j}^t E_{2_j} \geq 0.$$

Remark 3.21 Assumption 3.13 is made only for simplification of presentation. If Assumption 3.13 does not hold, the results of this section can be easily generalized using the technique similar to that in [25].

The robust stabilization problem for interconnected system (3.118) is now stated as follows:

Robust Stabilization Problem: Design decentralized linear time-invariant feedback control laws $u_j(t) = -K_j x_j(t)$, $j = 1, 2, \ldots, N-1$, for system (2.2) with uncertainties (3.119)–(3.121) such that the resulting closed-loop system is globally uniformly asymptotically stable about the origin for all admissible uncertainties. In this case, the system (3.118) is said to be robustly stabilizable via the decentralized controllers K_j and the closed-loop system is said to be *robustly stable*.

A solution to the robust decentralized stabilization of interconnected system (2.2) depends on the following algebraic Riccati equations

$$\begin{aligned} &A_j^t P_j + P_j A_j + P_j \overline{B}_j \overline{B}_j^t P_j - \nu_j^{-2} B_{P_j}^t R_j^{-1} B_{P_j} + \nu_j^2 E_{1_j}^t E_{1_j} \\ &\quad + \sum_{m=1, m\neq 1}^{N} p_{jm} (\overline{W}_m^t \overline{W}_m + W_{jm}^t W_{jm}) + \tilde{Q}_j = 0, \end{aligned} \tag{3.122}$$

where $j = 1, 2, \ldots, N-1$, $B_{P_j} = B_j^t P_j + \nu_j^2 E_{2_j}^t E_{1_j}$,

$$\overline{B}_j \overline{B}_j^t = \nu_j^{-2} L_j L_j^t + \sum_{j=1, j\neq 1}^{N} p_{jm}[G_{jm}(I - \lambda_{jm}^2 E_{jm}^t E_{jm})^{-1} G_{jm}^t + \lambda_{jm}^{-2} L_{jm} L_{jm}^t] \quad (3.123)$$

and $\nu_j > 0$, $\lambda_{jm} > 0$, $j = 1, 2, \ldots, N-1$ and $m = 1, 2, \ldots, N$, are scaling parameters to be chosen, with λ_{jm} satisfying $\lambda_{jm}^2 E_{jm}^t E_{jm} < I$, $\forall j = 1, 2, \ldots, N-1$ and $m = 1, 2, \ldots, N-1$. $\tilde{Q}_j$ are positive definite matrices.

A main result on the problem of decentralized robust stabilization is stated as follows:

Theorem 3.7 *Consider the multimachine power system* (3.117) *satisfying Assumptions* 3.12 *and* 3.13. *Then, this system is robustly stabilizable via decentralized linear feedback control if there exist positive scaling parameters* ν_j *and* λ_{jm}, $\forall j, m \in \{1, 2, \ldots, N\}$ *such that for any* $j = 1, 2, \ldots, N-1$:

1. $\lambda_{jm}^2 E_{jm}^t E_{jm} < I$, $\forall m \in \{1, 2, \ldots, N\}$; *and*
2. *there exist positive definite solutions* P_j *to* (3.122).

Moreover, a suitable decentralized feedback linear controller is given as follows:

$$u_j(t) = -K_j X_j(t), \quad (3.124)$$

where $K_j = \nu_j^{-2} R_j^{-1}(B_j^t P_j + \nu_j^2 E_{2_j}^t E_{1_j})$.

Proof Combining (3.124) with (3.118) gives a closed-loop system of the form

$$\begin{aligned} \dot{x}_j &= (\overline{A}_j + L_j F_j \overline{E}_j)x_j + \sum_{j=1, j\neq 1}^{N} p_{jm}(G_{ij} + L_{jm} F_{jm} E_{jm}) g_{jm}(x_j, x_j) \\ &= \tilde{A}_j X_j + \sum_{m=1, m\neq 1}^{N} \tilde{G}_{jm} g_{jm}(x_j, x_m), \end{aligned} \quad (3.125)$$

where $\overline{A}_j = A_j - B_j K_j$, $\overline{E}_j = E_{1_j} - E_{2_j} K_j$,

$$\tilde{A}_j = \overline{A}_j + L_j F_j \overline{E}_j, \qquad \tilde{G}_{jm} = p_{jm}[G_{jm} + L_{jm} F_{jm} E_{jm}].$$

From (3.122) and the bounding inequality A from Sect. 9.3.1, it follows that

$$\overline{A}_j^t P_j + P_j \overline{A}_j + P_j \overline{B}_j \overline{B}_j^t P_j + \sum_{m=1, m\neq 1}^{N} p_{jm}(\overline{W}_m^t \overline{W}_m + W_{jm}^t W_{jm}) + \nu_j^2 \overline{E}_j^t \overline{E}_j < 0, \quad (3.126)$$

where $\overline{B}_j$ is as defined in (3.123) and P_j is the positive definite solution to (3.122).

In view of (3.123), we obtain that

$$\overline{A}_j^t P_j + P_j \overline{A}_j + \nu_j^{-2} P_j L_j L_j^t P_j + P_j \left[\sum_{m=1,m\neq 1}^{N} p_{jm} [G_{jm} (I - \lambda_{jm}^2 E_{jm}^t E_{jm})^{-1} G_{jm}^t \lambda_{jm}^{-2} L_{jm} L_{jm}^t] \right] P_j + \sum_{m=1,m\neq 1}^{N} p_{jm} (\overline{W}_m^t \overline{W}_m + W_{jm}^t W_{jm}) + \nu_j^2 \overline{E}_j^t \overline{E}_j < 0.$$

Applying the bounding inequality B from Sect. 9.3.2 to the above inequality gives that

$$\overline{A}_j^t P_j + P_j \overline{A}_j + \nu_j^{-2} P_j L_j L_j^t P_j + \nu_j^2 \overline{E}_j^t \overline{E}_j + \sum_{m=1,m\neq 1}^{N} p_{jm} (\overline{W}_m^t \overline{W}_m + W_{jm}^t W_{jm}) + P_j \left(\sum_{m=1,m\neq 1}^{N} \tilde{G}_{jm} \tilde{G}_{jm}^t \right) P_j < 0$$

and it follows that, by applying the bounding inequality A from Sect. 9.3.1

$$\overline{A}_j^t P_j + P_j \overline{A}_j + \overline{E}_j^t F_j^t(t) L_j^t P_j + P_j L_j F_j(t) \overline{E}_j + \sum_{m=1,m\neq 1}^{N} p_{ij} (\overline{W}_m^t \overline{W}_m + W_{jm}^t W_{jm}) + P_j \left(\sum_{m=1,m\neq 1}^{N} \tilde{G}_{jm} \tilde{G}_{jm}^t \right) P_j < 0.$$

Then, we have

$$\tilde{A}_j^t P_j + P_j \tilde{A}_j + \sum_{m=1,m\neq 1}^{N} p_{jm} (\overline{W}_j^t \overline{W}_j + W_{ji}^t W_{ji}) + P_j \left(\sum_{m=1,m\neq 1}^{N} \tilde{G}_{jm} \tilde{G}_{jm}^t \right) P_j < 0.$$

It follows immediately that there exist positive definite matrices $\tilde{Q}_j$ such that

$$\tilde{A}_j^t P_j + P_j \tilde{A}_j + \sum_{m=1,m\neq 1}^{N} p_{jm} (\overline{W}_m^t \overline{W}_m + W_{jm}^t W_{jm}) + P_j \left(\sum_{m=1,m\neq 1}^{N} \tilde{G}_{jm} \tilde{G}_{jm}^t \right) P_j + \tilde{Q}_j = 0. \tag{3.127}$$

Now, in order to prove the asymptotic stability of the closed loop system (3.125), let the Lyapunov function candidate

$$V(x) = \sum_{j=1}^{N-1} x_j^t P_j x_j,$$

where $x = [x_1^t, x_2^t, \ldots, x_{N-1}^t]^t$. Note that $V(x) > 0$ whenever $x \neq 0$. Then, by using (3.125), we have

$$\frac{d}{dt}V(x) = \sum_{j=1}^{N-1} \left(x_j^t(\tilde{A}_j^t P_j + P_j\tilde{A}_j)x_j + \left[\sum_{m=1,m\neq 1}^{N} \tilde{G}_{jm} g_{jm}(x_j, x_m) \right]^t P_j x_j \right.$$
$$\left. + x_j^t P_j \left[\sum_{m=1,m\neq 1}^{N} \tilde{G}_{jm} g_{jm}(x_j, x_m) \right] \right).$$

Since

$$\sum_{j=1}^{N-1} \sum_{m=1,m\neq 1}^{N} p_{jm}[x_j^t \overline{W}_j^t \overline{W}_j x_j + x_j^t W_{jm}^t W_{jm} x_j - g_{jm}^t g_{jm}]$$
$$= \sum_{j=1}^{N-1} \sum_{m=1,m\neq 1}^{N} p_{jm}[x_j^t \overline{W}_j^t \overline{W}_j x_j + x_j^t W_{jm}^t W_{jm} x_j - g_{jm}^t g_{jm}]$$

it follows that

$$\frac{d}{dt}V(x) = \sum_{m=1}^{N-1} \left(x_j^t(\tilde{A}_j^t P_j + P_j\tilde{A}_j)x_j + \left[\sum_{m=1,m\neq 1}^{N} \tilde{G}_{jm} g_{jm}(x_j, x_j) \right]^t P_j x_j \right.$$
$$\left. + x_j^t P_j \left[\sum_{m=1,m\neq 1}^{N} \tilde{G}_{jm} g_{jm}(x_j, x_j) \right] \right)$$
$$+ \sum_{m=1}^{N-1} \sum_{j=1,j\neq 1}^{N} p_{jm}[x_j^t \overline{W}_j^t \overline{W}_j x_j + x_j^t W_{jm}^t W_{jm} x_j - g_{jm} g_{jm}]$$
$$- \sum_{j=1}^{N-1} \sum_{m=1,m\neq 1}^{N} p_{jm}[x_j^t \overline{W}_j^t \overline{W}_j x_j + x_j^t W_{jm}^t W_{jm} x_j - g_{jm}^t g_{jm}].$$

Introducing $\overline{x}_j = [x_j^t g_{j1} \; \ldots \; g_{jN-1}]^t$, we have

$$\frac{d}{dt}V(x) = \sum_{j=1}^{N-1} \{\overline{x}_j^t \tilde{\Lambda}_j \overline{x}_j\}$$
$$- \sum_{j=1}^{N-1} \sum_{m=1,m\neq 1}^{N} p_{jm}[x_j^t \overline{W}_j^t \overline{W}_j x_j + x_j^t W_{jm}^t W_{jm} x_j - g_{jm}^t g_{jm}],$$

where

$$\tilde{\Lambda}_j = \begin{bmatrix} S_j & P_j\tilde{G}_{j1} & \dots & P_j\tilde{G}_{jN-1} \\ \tilde{G}^t_{j1}P_j & -I & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{G}^t_{jN-1}P_j & 0 & \dots & -I \end{bmatrix},$$

$$S_j = P_j\tilde{A}_j + \tilde{A}^t_j P_j + \sum_{m=1, m\neq 1}^{n} p_{jm}(\overline{W}^t_j\overline{W}_j + W^t_{jm}W_{jm}).$$

Next, taking into account (3.127), the Schur inequality and the fact that

$$\sum_{j=1}^{N-1}\sum_{m=1,m\neq 1}^{N} p_{ij}[x^t_j\overline{W}^t_j\overline{W}_j x_j + x^t_j W^t_{jm}W_{jm}x_j - g^t_{jm}g_{jm}] \geq 0$$

it follows that

$$\frac{d}{dt}V(x) < 0$$

whenever $x \neq 0$. Hence, $V(x)$ is a Lyapunov function for system (3.125) and thus, this system is globally uniformly asymptotically stable for all admissible uncertainties. Therefore, the multimachine power system (3.117) is robustly stabilizable via the decentralized controller (3.124) which concludes the desired result. □

Remark 3.22 The result shown above can be easily extended to the case where dynamic output feedback controls are used [58]. From the result obtained, it is clear that the linear feedback controller (3.124) can ensure the stability of the multimachine power system. The design procedure for the decentralized linear controller (3.124) can be summarized as follows.

1. Formulate the system model (3.117) or (3.118). Find the respective matrices A_j, ΔA_j, B_j, ΔB_j, G_j, and ΔG_j.
2. Find the structure of the parametric uncertainties defined in (3.119)–(3.121).
3. Construct algebraic Riccati equations as given in (3.122) for all j.
4. Select the scaling parameters $\nu_j > 0$ and $\lambda_j > 0$ and find positive definite solution P_j to (3.122). If there exist such kind of P_j, we declare that the algorithm "succeeds" and a robust decentralized controller is found as given in (3.124).
5. If no positive definite solution P_j to (3.122) is found, go back to Step 2 and reformulate the structure of the parametric uncertainties. Repeat Steps 3 and 4. If no "success" is declared after several trials, we declare that the algorithm "fails" and abandon the method.

Remark 3.23 The decentralized controller (3.124) is a linear controller. Compared with nonlinear controllers, linear controllers are of simpler structure and easier to be implemented.

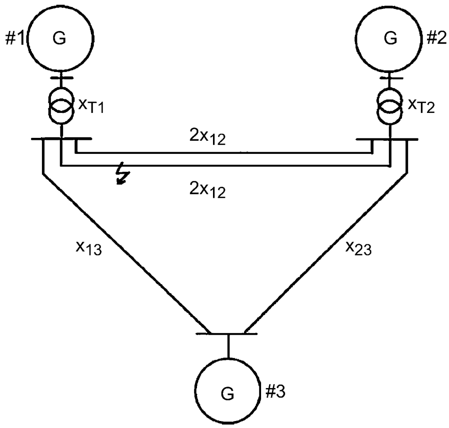


Fig. 3.3 Three-machine example system

3.5.3 Simulation Results

To demonstrate the effectiveness of the developed decentralized control method, a three-machine example system (3.3) is chosen. The system parameters used in the simulation are as follows:

$$\begin{aligned}
&x_{d1} = 1.863, \quad x'_{d1} = 0.257, \quad x_{T1} = 0.129,\\
&T'_{d01} = 6.9 \text{ s}, \quad H_1 = 4 \text{ s}, \quad D_1 = 5, \quad k_{cl} = 1,\\
&x_{d2} = 2.36, \quad x'_{d2} = 0.319, \quad x_{T2} = 0.11,\\
&F_{IP_1} = F_{IP_2} = 0.3, \quad T'_{d02} = 7.96 \text{ s},\\
&H_2 = 5.1 \text{ s}, \quad D_2 = 3, \quad k_{c2} = 1;\\
&T_{M1} = 0.35 \text{ s}, \quad T_{E1} = 0.1 \text{ s}, \quad T_{M2} = 0.35 \text{ s},\\
&T_{E2} = 0.1 \text{ s}, \quad R_1 = R_2 = 0.05,\\
&K_{M1} = K_{E1} = 1.0, \quad K_{M2} = K_{E2} = 1.0 \text{ rad/s},\\
&x_{12} = 0.55, \quad x_{13} = 0.53, \quad x_{23} = 0.6,\\
&\omega_0 = 314.159, \quad x_{\text{ad1}} = x_{\text{ad2}} = 1.712.
\end{aligned}$$

Since generator #3 is an infinite bus, we have $E'_{q3} = 1\angle 0°$.

To simplify the analysis, we only consider the parametric perturbations in G_{jm} and in T_{Mi}. The matrices G_{ij} represent the interconnections and nonlinearities between generators i and j, and uncertainties in parameters T_{M_j} are used to emulate the time constant uncertainties in the high-pressure (HP) and low-pressure (LP) sections. The power system model (3.117) can be rewritten as

$$\begin{aligned}
\dot{x}_1(t) &= (A_1 + \Delta A_1)x_1(t) + B_1 u_1(t)\\
&\quad + [G_{12} + \Delta G_{12}(t)]g_{12}(x_1, x_2) + [G_{13} + \Delta G_{13}(t)]g_{13}(x_1, x_3),
\end{aligned}$$

$$\dot{x}_2(t) = (A_2 + \Delta A_2)x_2(t) + B_2u_2(t) + [G_{21} + \Delta G_{21}(t)]g_{21}(x_2, x_1) + [G_{23} + \Delta G_{23}(t)]g_{23}(x_2, x_3),$$

where $\delta_{12}(t) = \delta_1(t) - \delta_2(t)$, $\delta_{21}(t) = \delta_2(t) - \delta_1(t)$, A_1, A_2, B_1, and B_2 are as in (3.117), and for convenience, we define that for $i = 1, 2$, and $j = 1, 2, 3$, $j \neq 1$

$$\Delta A_j = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu_j(t) & \mu_j(t) \\ 0 & 0 & 0 & 0 \end{bmatrix};$$

$$\mu_j(t) = \frac{1}{T_{M_j}} - \frac{1}{T_{M_j} - \Delta T_{M_j}},$$

$$G_{ij}(t) = \begin{bmatrix} 0 \; \alpha_{ij} \; 0 \; 0 \end{bmatrix}^t,$$

$$g_{ij}(x_j, x_j) = \sin[\delta_j(t) - \delta_j(t)] - \sin(\delta_{i0} - \delta_{j0}),$$

$$\Delta G_{ij}(t) = \begin{bmatrix} 0 \; \Delta\alpha_{ij} \; 0 \; 0 \end{bmatrix}^t.$$

It follows that

$$G_{12}(t) = \begin{bmatrix} 0 \; \alpha_{12} \; 0 \; 0 \end{bmatrix}^t,$$

$$G_{13}(t) = \begin{bmatrix} 0 \; \alpha_{13} \; 0 \; 0 \end{bmatrix}^t,$$

$$g_{12}(x_1, x_2) = \sin[\delta_1(t) - \delta_2(t)] - \sin(\delta_{10} - \delta_{20}),$$

$$g_{13}(x_1, x_3) = \sin[\delta_1(t) - \delta_3(t)] - \sin(\delta_{10} - \delta_{30}),$$

$$G_{21}(t) = \begin{bmatrix} 0 \; \alpha_{21} \; 0 \; 0 \end{bmatrix}^t,$$

$$G_{23}(t) = \begin{bmatrix} 0 \; \alpha_{23} \; 0 \; 0 \end{bmatrix}^t,$$

$$g_{21}(x_2, x_1) = \sin[\delta_2(t) - \delta_1(t)] - \sin(\delta_{20} - \delta_{10}),$$

$$g_{23}(x_2, x_3) = \sin[\delta_2(t) - \delta_3(t)] - \sin(\delta_{20} - \delta_{30}),$$

$$\Delta G_{12}(t) = \begin{bmatrix} 0 \; \Delta\alpha_{12} \; 0 \; 0 \end{bmatrix}^t,$$

$$\Delta G_{13}(t) = \begin{bmatrix} 0 \; \Delta\alpha_{13} \; 0 \; 0 \end{bmatrix}^t,$$

$$\Delta G_{21}(t) = \begin{bmatrix} 0 \; \Delta\alpha_{21} \; 0 \; 0 \end{bmatrix}^t,$$

$$\Delta G_{23}(t) = \begin{bmatrix} 0 \; \Delta\alpha_{23} \; 0 \; 0 \end{bmatrix}^t,$$

where α_{jm} can be defined as the midpoints of $E'_{qi}(t)E'_{qj}(t)B_{ij}\omega_0/2H_j$, and $\Delta\alpha_{ij}$ by variations in $E'_{qi}(t)E'_{qj}(t)B_{jm}\omega_0/2H_j$ from their midpoints. In order to estimate the bounds of the parameters, α_{12}, α_{13}, α_{21}, and α_{23} and their perturbations, $\Delta\alpha_{12}$,

$\Delta\alpha_{13}$, $\Delta\alpha_{21}$, $\Delta\alpha_{23}$, we use the following equation on the electric power

$$\Delta P_{ei}(t) = \sum_{j=1, j\neq 1}^{3} E'_{qi} E'_{qj} B_{ij} \sin[\delta_j(t) - \delta_j(t)] - \sum_{j=1, j\neq 1}^{3} E'_{qi} E'_{qj} B_{ij} \sin(\delta_{i0} - \delta_{j0}).$$

Since there are bounds on the electric power for each generator and on the electric power flow through each transmission line, we have

$$E'_{qi} E'_{qj} B_{ij} \le |\Delta P_{ei}(t)|_{\max}.$$

In this example, $|\Delta P_{e1}(t)|_{\max} = 1.4$ and $|\Delta P_{e2}(t)|_{\max} = 1.5$. It follows that

$$\alpha_{12} = \alpha_{13} = -0.5 \frac{|\Delta P_{e1}(t)|_{\max}\omega_0}{2H_1} = -27.49,$$

$$\alpha_{21} = \alpha_{23} = -0.5 \frac{|\Delta P_{e2}(t)|_{\max}\omega_0}{2H_2} = -23.10,$$

$$|\Delta\alpha_{ij}| \le 0.5 \frac{|\Delta P_{ei}(t)|_{\max}\omega_0}{2H_j}.$$

For $j = 1, 2$ and $m = 1, 2, 3$, $m \neq j$, we have $|\Delta\alpha_{12}| \le 27.49$, $|\Delta\alpha_{13}| \le 27.49$, $|\Delta\alpha_{21}| \le 23.10$, and $|\Delta\alpha_{23}| \le 23.10$.

The structure of parametric uncertainties can be expressed as follows.

- For generator #1:

$$L_1 = \left[0\ 0\ 1.41/|\mu_1(t)|_{\max}\ 0\right]^t,$$

$$F_1(t) = \left[0\ 0\ \frac{-0.707|\mu_1(t)|}{\mu_1(t)|_{\max}}\ \frac{0.707|\mu_1(t)|}{\mu_1(t)|_{\max}}\right],$$

$$E_{11} = \mathrm{diag}\{1\ 1\ 1\ 1\}, \qquad E_{21} = [1\ 1\ 0\ 0]^t$$

$$L_{12} = [0\ |\Delta\alpha_{12}(t)|_{\max}\ 0\ 0]^t,$$

$$F_{12}(t) = \frac{\Delta\alpha_{12}(t)}{|\Delta\alpha_{12}(t)|_{\max}}, \qquad E_{12} = 1,$$

$$L_{13} = [0\ |\Delta\alpha_{13}(t)|_{\max}\ 0\ 0]^t,$$

$$F_{13}(t) = \frac{\Delta\alpha_{13}(t)}{|\Delta\alpha_{13}(t)|_{\max}}, \qquad E_{13} = 1,$$

$$\overline{W}_1 = W_{12} = [1\ 0\ 0\ 0], \qquad W_{13} = [0\ 0\ 0\ 0].$$

- For generator #2, the decomposition is similar. It is clear that robust decentralized controllers for generators #1 and #2 considering the prescribed uncertainties can be found by using the design procedure described in Remark 3.20.

In this example, the Riccati equation (3.122) then becomes

$$A_j^t P_j + P_j A_j + P_j \overline{B}_j \overline{B}_j^t P_j - \nu_j^{-2} P_j B_j B_j^t P_j + \sum_{m=1}^{N} p_{jm} (\overline{W}_j^t \overline{W}_j + W_{jm}^t W_{jm}) + \tilde{Q}_j = 0, \qquad (3.128)$$

where $j = 1, 2,$

$$\overline{B}_j \overline{b}_j^t = \sum_{j=1}^{N} p_{jm} [G_{jm} (I - \lambda_{jm}^2 E_{jm}^t E_{jm})^{-1} G_{jm}^t + \lambda_{jm}^{-2} L_{jm} L_{jm}^t].$$

For generator #1 in the example system, we have

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.625 & 27.48 & 11.781 \\ 0 & 0 & -2.857 & 2.857 \\ 0 & -0.637 & 0 & -10 \end{bmatrix}; \qquad B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \end{bmatrix},$$

$$\Delta A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.635 r_1(t) & 0.635 r_1(t) \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $|r_1(t)| \leq 1$. Let $\nu_1 = 0.02$, $\lambda_{12} = \lambda_{13} = 0.71$, $\tilde{Q}_1 = \mathrm{diag}\{0.001, 0.001, 0.01, 0.01\}$. Solving the Riccati equation (3.128) gives

$$K_1 = \left[k_{\delta_1} \; k_{\omega_1} \; k_{p_1} \; k_{X_1} \right] = \left[191.86 \; 15.16 \; 15.30 \; 6.50 \right].$$

Similarly, for generator #2, we have

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.392 & 20.560 & 9.240 \\ 0 & 0 & -2.857 & 2.857 \\ 0 & -0.637 & 0 & -10 \end{bmatrix}; \qquad B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \end{bmatrix},$$

$$\Delta A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.635 r_2(t) & 0.635 r_2(t) \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $|r_2(t)| \leq 1$. Let $\nu_2 = 0.02, \lambda_{21} = \lambda_{23} = 0.71$, $\tilde{Q}_2 = \mathrm{diag}\{0.001, 0.001, 0.01, 0.01\}$. Solving the Riccati equation (3.128) gives

$$K_2 = \left[k_{\delta_2} \; k_{\omega_2} \; k_{p_2} \; k_{X_2} \right] = \left[262.86 \; 21.43 \; 17.33 \; 7.43 \right].$$

The control laws are as follows:

$$
\begin{aligned}
u_1 &= -k_{\delta_1}[\delta_1(t) - \delta_{10}] - k_{\omega_1}\omega_1(t) \\
&\quad - k_{p_1}[P_{m1}(t) - P_{m10}] - k_{X_1}[X_{E1}(t) - X_{E10}] \\
&= -191.86[\delta_1(t) - \delta_{10}] - 15.16\omega_1(t) \\
&\quad - 15.30[P_{m1}(t) - P_{m10}] - 6.50[X_{E1}(t) - X_{E10}]
\end{aligned}
$$

and

$$
\begin{aligned}
u_2 &= -k_{\delta_2}[\delta_2(t) - \delta_{20}] - k_{\omega_2}\omega_2(t) \\
&\quad - k_{p_2}[P_{m2}(t) - P_{m20}] - k_{X_2}[X_{E2}(t) - X_{E20}] \\
&= -262.86[\delta_2(t) - \delta_{20}] - 21.43\omega_2(t) \\
&\quad - 17.33[P_{m2}(t) - P_{m20}] - 7.43[X_{E2}(t) - X_{E20}].
\end{aligned}
$$

The fault we consider in the simulation is a symmetrical three-phase short circuit fault which occurs on one of the transmission lines between generator #1 and generator #2 with λ being the fraction of the transmission line to the left of the fault. If $\lambda = 0$, the fault is on the bus bar of generator #1, $\lambda = 0.5$ puts the fault in the center point of the transmission line between generator #1 and generator #2, and so on. The fault sequence is as follows.

1. *The system is in pre-fault steady-state.*
2. *A fault occurs at* $t = 0.1$ s.
3. *The fault is removed by opening the breakers of the faulted line at* $t = 0.25$ s.
4. *The transmission lines are restored with the fault cleared at* $t = 1.0$ s.
5. *The system is in post fault-state.*

Three different cases are considered in the simulation. In the first two cases, the fault location is $\lambda = 0.05$.

- *Case 1*. The operating points are

$$
\begin{aligned}
&\delta_{10} = 67.6^\circ, \qquad P_{m10} = 1.2, \qquad V_{t1} = 1.0, \\
&\delta_{20} = 67.7^\circ, \qquad P_{m20} = 1.1, \qquad V_{t2} = 1.0.
\end{aligned}
$$

  The power angles, the real power, and the terminal voltages of the generators #1 and #2 are shown in Figs. 3.4, 3.5 and 3.6, respectively.
- *Case 2*. The operating points are

$$
\begin{aligned}
&\delta_{10} = 24.6^\circ, \qquad P_{m10} = 0.3, \qquad V_{t1} = 0.95, \\
&\delta_{20} = 48.6^\circ, \qquad P_{m20} = 0.9, \qquad V_{t2} = 0.95.
\end{aligned}
$$

  The power angles of the generators #1 and #2 are shown in Fig. 3.7.

Fig. 3.4 Power angle responses ($\lambda = 0.05$)

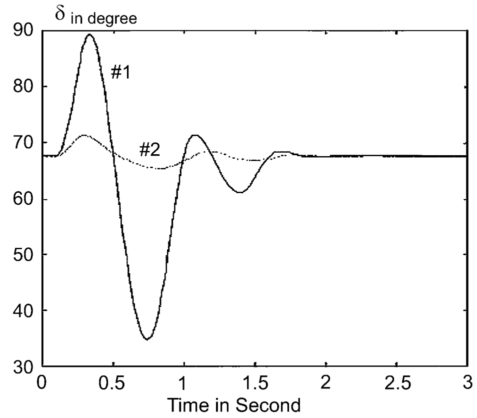


Fig. 3.5 Electrical power responses ($\lambda = 0.05$)

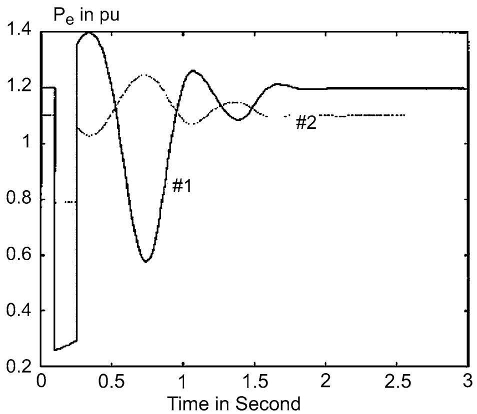
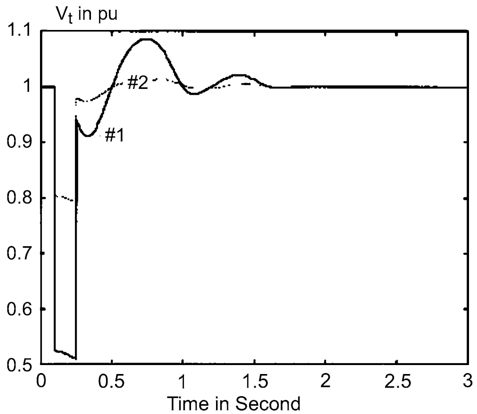


Fig. 3.6 Terminal voltage responses ($\lambda = 0.05$)



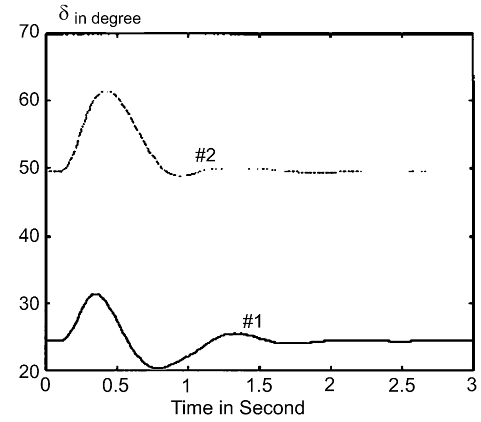


Fig. 3.7 Power angle responses

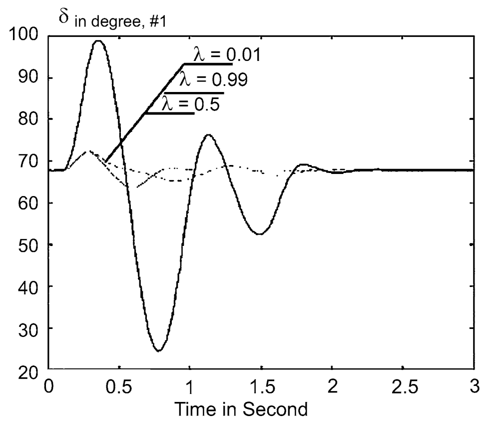


Fig. 3.8 Power angle responses for generator #1

- *Case 3*. We will consider different fault locations. The operating points are

$$\delta_{10} = 67.6°, \qquad P_{m10} = 1.2, \qquad V_{t1} = 1.0,$$
$$\delta_{20} = 67.7°, \qquad P_{m20} = 1.1, \qquad V_{t2} = 1.0.$$

The power angles of the generators #1 and #2 are shown in Figs. 3.8 and 3.9, respectively ($\lambda = 0.01, 0.5, 0.99$).

From the simulation results shown above, it can be seen that despite the interconnections between different generators, nonlinearities in the system, different operating points and different fault locations, under all situations the proposed robust decentralized controller can rapidly damp the oscillation of the system and greatly enhance transient stability of the multimachine power system.

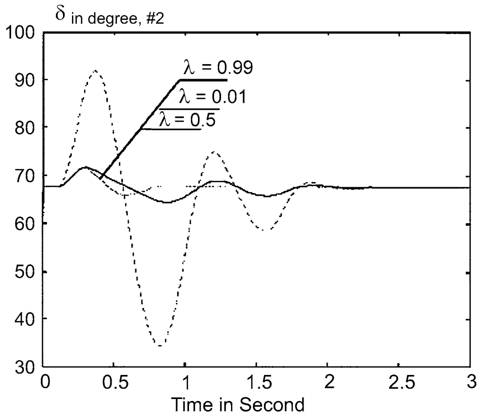


Fig. 3.9 Power angle responses for generator #2

3.6 Decentralized Control with Guaranteed Performance

In what follows, the decentralized excitation control of multimachine power systems is considered. The power system can be modeled as a interconnected system with parameter uncertainty and nonlinear interconnections. The main focus is on the design of a robust decentralized state feedback controller that not only stabilizes the power system but also achieves suboptimal guaranteed cost performance for all admissible variations of generator parameters. Following the results of [38, 39] and references therein, a robust performance analysis result is developed for interconnected systems in terms of a set of linear matrix inequalities (LMIs). The decentralized guaranteed cost control has been solved using an LMI approach. The results shown in this section are given in terms of LMIs which can be solved efficiently using the available LMI tool [9]. Furthermore, a procedure is given to minimize an upper bound of the cost.

3.6.1 Introduction

Power systems are modeled as large-scale nonlinear systems composed of a set of small interconnected subsystems. It is generally impossible to incorporate many feedback loops into the controller design for large-scale interconnected systems and is also too costly even if they can be implemented. These difficulties motivate the development of decentralized control theory where each subsystem is controlled independently on its locally available information.

On the other hand, the operating conditions of power systems are always varying to satisfy different load demands. Control systems are therefore required to have the ability to suppress potential instability and damp the system oscillations that might threaten the system stability as the load demand increases. However, as power systems are large-scale nonlinear systems in nature, the applications of conventional linear control approaches are limited because they can only deal with small disturbances about an operating point and cannot guarantee the system stability when faults or significant changes of operating conditions occur. Since the introduction of differential geometric tools to nonlinear control system design, various nonlinear feedback controllers have been designed to enhance power system stability, see e.g. [5, 57]. Naturally, the robustness issue arises in order to deal with uncertainties which mainly come from the varying transmission line parameters and/or faults. There are some results on decentralized robust control of multimachine power systems, e.g. [5–8, 10–24, 26–40, 42–52, 54]. In particular, in [5–8, 10–24, 26–40, 42–52, 54], the multimachine power system is first compensated via a decentralized nonlinear direct feedback linearization, then a robust decentralized control is applied which guarantees the overall stability of the multimachine power system is the whole working region. Note that the design approach in [54, 57] involves solving a set of parameterized Riccati equations, which is in general a difficult task. Furthermore, only a stabilization problem is addressed in [54, 57].

In any control design, a controller is sought not only to stabilize the system hut also to ensure satisfactory performance of the system. When a quadratic cost is considered for hear systems, the traditional linear quadratic (LQ) design offers an optimal solution. Very recently, [25, 38–57, 59–61], was applied this performance measure in for systems with parameter uncertainty and addressed the problem of guaranteed cost control. The guaranteed cost control is concerned with the design of a state feedback controller so that, for all admissible uncertainties, the closed-loop system is asymptotically stable and an upper bound of the quadratic cost is minimized. The result of the guaranteed cost control is given in terms of a parameterized game-type algebraic Riccati equation which may be difficult to solve in [16, 60] the LQ design has been extended to the decentralized control of large-scale systems without uncertainties. On the other hand, where the subsystems arc treated as if they were decoupled, and, under certain conditions placed on the interconnections, the locally optimal LQ control is obtained and is suboptimal for the overall system. Note that, when uncertainties arise in both the subsystems and interconnections, this passive analysis may have difficulty in guaranteeing the closed-loop stability and may be overly conservative.

3.6.2 Dynamical Model of Multimachine Power System

In the sequel, we refer to the following model parameters:

$\delta_j =$ power angle of the jth generator, in rad $\delta_{ij} = \delta_i - \delta_j$;
$\omega_j =$ relative speed of the jth generator, in rad/s;
$P_{mi0} =$ mechanical input power, in p.u., which is a constant;
$P_{ei} =$ electrical power, in p.u.;
$\omega_0 =$ synchronous machine speed, in rad/s;
$D_i =$ per unit damping constant;
$H_i =$ inertia constant, in seconds;

$E'_{qi} =$ transient EMF in the quadrature axis of the ith generator, in p.u.;
$E_{qi} =$ EMF in the quadrature axis, in p.u.;
$E_{fi} =$ equivalent EMF in the excitation coil, in p.u.;
$T'_{d0i} =$ direct axis transient short circuit time constant, in seconds;
$x_{di} =$ direct axis reactance of the ith generator, in p.u.;
$x'_{di} =$ direct axis transient reactance of the ith generator, in p.u.;
$B_{ij} =$ ith row and jth column element of nodal susceptance matrix at the internal nodes after eliminating all physical buses; in p.u.;
$Q_{ei} =$ reactive power, in p.u.;
$I_{fi} =$ excitation current, in p.u.;
$I_{di} =$ direct axis current, in p.u.;
$I_{qi} =$ quadrature axis current, in p.u.;
$k_{ci} =$ gain of the excitation amplifier, in p.u.;
$u_{fi} =$ input of' the SCR amplifier of the ith generator, in p.u.;
$x_{adi} =$ mutual reactance between the excitation coil and the stator coil of the ith generator, in p.u.;
$x_{Ti} =$ transformer reactance, in p.u.;
$x_{ij} =$ transmission line reactance between the ith generator and the jth generator, in p.u.;
$V_{ti} =$ terminal voltage of the ith generator, in p.u.

A power system consisting of N synchronous generators interconnected through a transmission network can be described by a classical dynamic model (see [3] and [28]). The dynamic model of the ith generator with excitation control is given by the following sets of equations.

- *Mechanical equations*:

$$\dot{\delta}_i = \omega_i, \tag{3.129}$$

$$\dot{\omega}_i = -\frac{D_i}{2H_i}\omega_i + \frac{\omega_0}{2H_i}(P_{mi0} - P_{ei}); \tag{3.130}$$

- *Generator electrical dynamics*:

$$\dot{E}'_{qi} = \frac{1}{T'_{d0i}}(E_{fi} - E_{qi}); \tag{3.131}$$

- *Electrical equations*:

$$E_{qi} = E'_{qi} - (x_{di} - x'_{di})I_{di}, \tag{3.132}$$

$$E_{fi} = k_{ci}u_{fi}, \tag{3.133}$$

$$P_{ei} = \sum_{j=1}^{N} E'_{qi}E'_{qj}B_{ij}\sin(\delta_{ij}), \tag{3.134}$$

$$Q_{ei} = -\sum_{j=1}^{N} E'_{qi} E'_{qj} B_{ij} \cos(\delta_{ij}), \tag{3.135}$$

$$I_{di} = \sum_{j=1}^{N} E'_{qj} B_{ij} \cos(\delta_{ij}), \tag{3.136}$$

$$I_{qi} = \sum_{j=1}^{N} E'_{qj} B_{ij} \sin(\delta_{ij}), \tag{3.137}$$

$$E_{qi} = x_{adi} I_{fi}. \tag{3.138}$$

By using direct feedback linearization (DFL) compensation (3.129)–(3.131), and considering the parametric uncertainties in T'_{d0i} as $\Delta T'_{d0i}$ the following can be obtained:

$$\begin{aligned}\dot{x}_i &= (A_i + \Delta A_i)x_i + (B_i + \Delta B_i)v_{fi} + \sum_{j=1}^{N} p_{1ij}(G_{1ij} + \Delta G_{1ij}) g_{1ij}(x_i, x_j) \\ &\quad + \sum_{j=1}^{N} p_{2ij}(G_{2ij} + \Delta G_{2ij}) g_{2ij}(x_i, x_j),\end{aligned} \tag{3.139}$$

where

$$\begin{aligned}
v_{fi} &= I_{qi} k_{ci} u_{fi} (x_{di} - x'_{di}) I_{qi} I_{di} - P_{mi0} - T'_{d0i} Q_{ei} \omega_i, \\
A_i &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{D_i}{2H_i} & -\frac{\omega_0}{2H_i} \\ 0 & 0 & -\frac{1}{T'_{d0i}} \end{bmatrix}, \qquad B_i = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{T'_{d0i}} \end{bmatrix}, \\
G_{1ij} &= G_{2ij} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad g_{1ij} = \sin(\delta_i - \delta_j), \qquad g_{2ij} = \omega_j, \\
\Delta A_i &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mu_i \end{bmatrix}, \qquad \Delta B_i = \begin{bmatrix} 0 \\ 0 \\ -\mu_i \end{bmatrix}, \\
\Delta G_{1ij} &= \begin{bmatrix} 0 \\ 0 \\ r_{1ij} \end{bmatrix}, \qquad \Delta G_{2ij} = \begin{bmatrix} 0 \\ 0 \\ r_{2ij} \end{bmatrix}, \\
\mu_i &= \frac{1}{T'_{d0i}} - \frac{1}{T'_{d0i} + \Delta T'_{d0i}}, \qquad r_{1ij} = E'_{qi} E'_{qj} B_{ij}, \\
r_{2ij} &= E'_{qi} E'_{qj} B_{ij} \cos(\delta_{ij}),
\end{aligned} \tag{3.140}$$

and the parameters p_{lij}, and p_{2ij}, are constants of either 1 or 0 (if they are 0, this means that the jth subsystem has no connection with the ith subsystem).

Remark 3.24 In (3.139), the parametric uncertainties were considered in generator parameters T'_{d0i}, $i = 1, 2, \ldots, N$, because they vary with load change and changing network topology.

Remark 3.25 Note that E'_{qi}, E'_{qj}, δ_{ij} and B_{ij}, will change when the network parameters and load are changed. For example, B_{ij} will vary when a major fault occurs at the transmission line between the ith and jth generators. Hence, nonlinear uncertain interconnections exist in multimachine power systems. To estimate the bounds of the uncertainties in the interconnections, note that the electrical power P_{ei} of each generator and the electrical power flow through each transmission line are bounded, and the excitation voltage E_{fi} may raise by up to 5 times of the E_{qi} when there is no load in the system. Thus, by considering (3.134) and (3.131), the following may be obtained:

$$|E'_{qi}, E'_{qj} B_{ij}| \leq |P_{ei}|_{\max},$$

$$|\dot{E}'_{qj}| \leq \left| \frac{1}{T'_{d0i}} [E_{fi} - E_{qj}] \right|_{\max} \leq 4 |E_{qj}|_{\max} \frac{1}{|T'_{d0i}|_{\min}}.$$

It also follows that

$$r_{1ij} \leq \frac{4}{|T'_{d0i}|_{\min}} |P_{ei}|_{\max}, \quad r_{2ij} \leq |P_{ei}|_{\max}.$$

It is obvious that the bounds of r_{1ij} and r_{2ij} only depend on generator parameters $|T'_{d0i}|_{\min}$ and $|P_{ei}|_{\max}$.

In this section, the authors are concerned with the design of a decentralized nonlinear feedback controller that will not only enhance the transient stability but also ensure a certain level of performance of the power system in the presence of operating point variations, faults in different locations and changing network parameters. Specifically, the authors will design a robust decentralized controller for the system (3.139) so that, for all admissible uncertainties, the closed-loop interconnected system is asymptotically stable and an upper bound of a specified quadratic cost is minimized. This problem is referred to as a decentralized guaranteed cost control.

Remark 3.26 In [54, 77], a robust stabilization controller has been proposed for the multimachine power system (3.129)–(3.138) and the result involves solving a set of parameterized game-type Riccati equations, which imposes a major difficulty. Furthermore, no performance has been taken into consideration for the controller design.

3.6.3 Guaranteed Cost Controller Design

In this section, the authors present an LMI approach to solve the decentralized guaranteed cost control problem for a class of interconnected nonlinear systems. Before proceeding to address the decentralized controller design, a robust performance analysis, is first presented.

3.6.4 Robust Performance Analysis

Consider the following interconnected large-scale system which consists of N subsystems:

$$\dot{x}_i = (A_i + \Delta A_i)x_i + \sum_{j=1}^{N} p_{1ij}(G_{1ij} + \Delta G_{1ij})g_{1ij}(x_i, x_j) + \sum_{j=1}^{N} p_{2ij}(G_{2ij} + \Delta G_{2ij})g_{2ij}(x_i, x_j), \tag{3.141}$$

where $x_i \in \Re^{n_i}$ is the state of the ith subsystem, A_i, G_{1ij}, and G_{2ij} are real constant matrices of appropriate dimensions, ΔA_i, ΔG_{1ij} and ΔG_{2ij}, are uncertain matrices, $g_{1ij}(x_i, x_j) \in \Re^{r_{ij}}$ and $g_{2ij}(x_i, x_j) \in \Re^{s_{ij}}$ are unknown nonlinear vector functions representing the interconnection between the ith subsystem and the jth subsystem, and the parameters p_{1ij} and p_{2ij} are constants of either 1 or 0 (if they are 0, it means that the jth subsystem has no connection with the ith subsystem).

In this section, the authors consider the following cost performance for the system (3.141):

$$J = \sum_{i=1}^{N} \int_0^{\infty} x_i^t Q_i x_i dt, \tag{3.142}$$

where $Q_i = Q_i^t > 0$, $i = 1, 2, \ldots, N$, are the given weighting matrices of the state. The authors will make the following assumptions on parameter uncertainties and interconnections:

Assumption 3.14

$$\Delta A_i = H_{1i} F_j E_{1i},$$

where $F_j \in \Re^{h_j \times e_j}$ is an unknown matrix function satisfying

$$F_j^t F_j \le I_{e_j}$$

and H_{1i} and E_{1i} are known real constant matrices that structure the uncertainty.

Assumption 3.15

1. Let

$$\Delta G_{1ij} = L_{1ij} F_{1ij} N_{1ij}, \qquad \Delta G_{2ij} = L_{2ij} F_{2ij} N_{2ij},$$

   where $F_{1ij} \in \Re^{\alpha_{1ij} \times \beta_{1ij}}$ and $F_{2ij} \in \Re^{\alpha_{2ij} \times \beta_{2ij}}$ are unknown matrix functions satisfying

$$F_{1ij}^t F_{1ij} \le I_{\beta_{1ij}}, \qquad F_{2ij}^t F_{1ij} \le I_{\beta_{2ij}}$$

   and $L_{1ij}, L_{2ij}, N_{1ij}$ and N_{2ij} are known real constant matrices with appropriate dimensions.
2. There exist known real constant matrices W_{1i}, W_{1ij}, W_{2i}, and W_{2ij}, such that, for all $x_i \in \Re^{n_i}$, $x_j \in \Re^{n_j}$, $i, j = 1, 2, \ldots, N$:

$$\begin{aligned} \|g_{1ij}(x_i, x_j)\| &\le \|W_{1i} x_i\| + \|W_{1ij} x_j\|, \\ \|g_{2ij}(x_i, x_j)\| &\le \|W_{2i} x_i\| + \|W_{2ij} x_j\|. \end{aligned}$$

Introduce the following definition.

Definition 3.1 A set of positive definite real matrices P_i, $i = 1, 2, \ldots, N$, is said to be a quadratic cost matrix set for the system (3.141) and the cost function (3.142), if

$$\begin{aligned} \sum_{i=1}^{N} \Bigg\{ & x_i^t [(A_i + \Delta A_i)^t P_i + P_i (A_i + \Delta A_i)] x_i \\ & + \sum_{j=1}^{N} 2 x_i^t P_i p_{1ij} (G_{1ij} + \Delta G_{1ij}) g_{1ij}(x_i, x_j) \\ & + \sum_{j=1}^{N} 2 x_i^t P_i p_{2ij} (G_{2ij} + \Delta G_{2ij}) g_{2ij}(x_i, x_j) + x_i^t Q_i x_i \Bigg\} < 0 \end{aligned} \tag{3.143}$$

for any nonzero $(x_1, x_2, \ldots, x_N)$ and all admissible uncertainties.

The following result shows that the notion of quadratic cost matrix set defines an upper bound on the cost function (see (3.142)).

Theorem 3.8 *Consider the system* (3.141) *and the cost function* (3.142). *Suppose that $P_i > 0$, $i = 1, 2, \ldots, N$, is a quadratic cost matrix set for the system. Then, the uncertain system is quadratically stable and the cost function satisfies the bound*

$$J \le \sum_{i=1}^{N} x_i^t(0) P_i x_i(0) \tag{3.144}$$

for all admissible uncertainties, where $x_i(0)$ is the initial state of the ith subsystem, $i = 1, 2, \ldots, N$.

Proof Define $V = \sum_{i=1}^{N} V_i = \sum_{i=1}^{N} x_i^t P_i x_i$. Then by taking into account (2.22), we have that along the state trajectory of (3.141),

$$\begin{aligned}\dot{V} = \sum_{i=1}^{N} \Bigg\{ & x_i^t [(A_i + \Delta A_i)^t P_i + P_i (A_i + \Delta A_i)] x_i \\ & + \sum_{j=1}^{N} 2x_i^t P_i p_{1ij} (G_{1ij} + \Delta G_{1ij}) g_{1ij}(x_i, x_j) \\ & + \sum_{j=1}^{N} 2x_i^t P_i p_{2ij} (G_{2ij} + \Delta G_{2ij}) g_{2ij}(x_i, x_j) + x_i^t Q_i x_i \Bigg\} \\ < - \sum_{i=1}^{N} x_i^t Q_i x_i \end{aligned} \tag{3.145}$$

for all nonzero $x = [x_1^t \ \dots \ x_N^t]^t$ and all admissible uncertainties. Hence, the system (3.141) is quadratically stable.

By integrating the inequality (3.145) over $[0, \infty)$ and considering that $V(x(\infty)) = 0$,

$$J = \sum_{i=1}^{N} \int_0^{\infty} x_i^t Q_i x_i dt \le V(x(0)) = \sum_{i=1}^{N} x_i^t(0) P_i x_i(0). \tag{3.146}$$

This completes the proof of the theorem. □

Note that the bound obtained in Theorem 3.8 depends on the initial condition $x_i(0)$. To remove this dependence on the initial condition, there are two approaches, one is the deterministic method [39] and the other is the stochastic approach [38]. In this section, we will adopt the deterministic approach. Suppose that the initial state of the system (3.141) is arbitrary but belongs to the set $S_i \le \{x_i(0) \in \Re^{n_i} : x_i(0) = \Pi_{i0} \nu_i, \nu_i^t \nu_i \le 1\}$. Then, it follows from (3.146) that

$$J \le \sum_{i=1}^{N} \lambda_{\max}(\Pi_{i0}^t P_i \Pi_{i0}), \tag{3.147}$$

where $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue. Hence, in this section, the measure of robust performance considered is as follows:

$$\begin{aligned} J^* \le \inf \Bigg\{ & \sum_{i=1}^{N} \lambda_{\max}(\Pi_{i0}^t P_i \Pi_{i0}) : \ P_i > 0 \text{ is a quadratic} \\ & \text{cost matrix for (2.20) and (2.21)} \Bigg\}. \end{aligned} \tag{3.148}$$

Before proceeding to obtain the main results, the following key lemma will be introduced.

Lemma 3.3 *Given real matrices Y, H and E of appropriate dimensions with Y symmetrical. Then*

$$Y + HFE + E^t F^t H^t < 0$$

for all $F = \mathrm{diag}\{F_1, F_2, \dots, F_k\}$ with $F_j \in \Re^{\alpha_j \times \beta_j}$ satisfying $F^t F \le I$, if there exist some positive scalars γ_j, $j = 1, 2, \dots, k$, such that

$$Y + H\Gamma_\alpha H^t + E^t \Gamma_\beta^{-1} E < 0,$$

where $\Gamma_\alpha = \mathrm{diag}\{\gamma_1 I_{\alpha_1}, \gamma_2 I_{\alpha_2}, \gamma_k I_{\alpha_k}\}$ and $\Gamma_\beta = \mathrm{diag}\{\gamma_1 I_{\beta_1}, \gamma_2 I_{\beta_2}, \dots, \gamma_k I_{\beta_k}\}$.

Proof Note that $HFE = H\Gamma_\alpha^{1/2} F \Gamma_\beta^{-1/2} E$. The desired result then follows by observing that $F^t F \le I$ and

$$\left(F\Gamma_\beta^{-1/2} E - \Gamma_\alpha^{1/2} H^t\right)^t \left(F\Gamma_\beta^{-1/2} E - \Gamma_\alpha^{-1/2} H^t\right)^t \ge 0.$$

The following result provides two sufficient conditions for the existence of quadratic cost matrices satisfying the inequality (3.143). □

Theorem 3.9 *Consider the system* (3.141) *and the cost function* (3.142). *Suppose that there exist a set of matrices $P_i > 0$, $i = 1, 2, \dots, N$, such that*

$$\begin{aligned} &P_i(A_i + \Delta A_i) + (A_i + \Delta A_i)^t P_i \\ &\quad + \sum_{j=1}^N p_{1ij} P_i (G_{1ij} + \Delta G_{1ij})(G_{1ij} + \Delta G_{1ij})^t P_i \\ &\quad + \sum_{j=1}^N p_{2ij} P_i (G_{2ij} + \Delta G_{2ij})(G_{2ij} + \Delta G_{2ij})^t P_i \\ &\quad + \sum_{j=1}^N p_{1ij} (W_{1i}^t + W_{1i} + W_{1ji}^t W_{1ji}) \\ &\quad + \sum_{j=1}^N p_{2ij} (W_{2i}^t W_{2i} + W_{2ji}^t W_{2ji}) + Q_i < 0 \end{aligned} \tag{3.149}$$

for all admissible uncertainties ΔA_i, ΔG_{1ij} and ΔG_{2ij}.

Then the set of matrices P_i is a quadratic cost matrix set for the system (3.141) *and the cost function* (3.142).

Furthermore, (3.149) *holds if there exist some scalars $\varepsilon_i, \gamma_{1ij}$ and γ_{2ij}, $i, j = 1, 2, \dots, N$, such that*

$$\begin{bmatrix} A_i^t + P_i A_i + \tilde{Q}_i + \varepsilon_i E_{1i}^t E_{1i} & P_i H_{1i} & P_i L_{1i} & P_i G_{1i} & P_i L_{2i} & P_i G_{2i} \\ H_{1i}^t P_i & -\varepsilon_i I & 0 & 0 & 0 & 0 \\ L_j^t P_i & 0 & -\Gamma_{1\alpha i} & 0 & 0 & 0 \\ G_{1i}^t P_i & 0 & 0 & -I + N_{1i}^t \Gamma_{1\beta i} N_{1i} & 0 & 0 \\ L_{2i}^t P_i & 0 & 0 & 0 & -\Gamma_{2\alpha i} & 0 \\ G_{2i}^t P_i & 0 & 0 & 0 & 0 & -I + N_{2i}^t \Gamma_{2\beta i} N_{2i} \end{bmatrix} < 0, \quad (3.150)$$

where

$$\begin{aligned} \tilde{Q}_i = \sum_{j=1}^{N} [& p_{1ij}(W_{1i}^t W_{1i} + W_{1ji}^t W_{1ji}) \\ & + p_{2ij}(W_{2i}^t W_{2i} + W_{2ji}^t W_{2ji})] + Q_i, \end{aligned} \quad (3.151)$$

$$\begin{aligned} G_{1i} &= [p_{1i1} G_{1il} \; \dots \; p_{1iN} G_{1iN}], \\ G_{2i} &= [p_{2i1} G_{2il} \; \dots \; p_{2iN} G_{2iN}], \end{aligned} \quad (3.152)$$

$$\begin{aligned} L_{1i} &= [p_{1il} L_{1il} \; \dots \; p_{1iN} L_{1iN}], \\ L_{2i} &= [p_{2il} G_{2il} \; \dots \; p_{2iN} L_{2iN}], \end{aligned} \quad (3.153)$$

$$\begin{aligned} N_{1i} &= [p_{1il} N_{1il} \; \dots \; p_{1iN} N_{1iN}], \\ N_{2i} &= [p_{2il} N_{2il} \; \dots \; p_{2iN} N_{2iN}], \end{aligned} \quad (3.154)$$

$$\begin{aligned} \Gamma_{1\alpha i} &= \mathrm{diag}\{\gamma_{1il} I_{\alpha_{1i}} \; \dots \; \gamma_{1iN} I_{\alpha_{1iN}}\}, \\ \Gamma_{1\beta i} &= \mathrm{diag}\{\gamma_{1il} I_{\beta_{1i}} \; \dots \; \gamma_{1iN} I_{\beta_{1iN}}\}, \end{aligned} \quad (3.155)$$

$$\begin{aligned} \Gamma_{2\alpha i} &= \mathrm{diag}\{\gamma_{2il} I_{\alpha_{2i}} \; \dots \; \gamma_{2iN} I_{\alpha_{2iN}}\}, \\ \Gamma_{2\beta i} &= \mathrm{diag}\{\gamma_{2il} I_{\beta_{2i}} \; \dots \; \gamma_{2iN} I_{\beta_{2iN}}\}. \end{aligned} \quad (3.156)$$

Proof In the light of Assumption 3.15, if (3.149) holds,

$$\begin{aligned} \sum_{i=1}^{N} \Bigg\{ & x_i^t [(A_i + \Delta A_i)^t P_i + P_i (A_i + \Delta A_i)] x_i \\ & + \sum_{j=1}^{N} 2 x_i^t P_i p_{1ij} (G_{1ij} + \Delta G_{1ij}) g_{1ij}(x_i, x_j) \end{aligned}$$

$$+\sum_{j=1}^{N} 2x_i^t P_i p_{2ij}(G_{2ij}+\Delta G_{2ij})g_{2ij}(x_i,x_j)+x_i^t Q_i x_i\Bigg\}$$

$$\begin{aligned}
&\leq \sum_{i=1}^{N} x_j^t \Bigg[(A_i+\Delta A_i)^t P_i + P_i(A_i+\Delta A_i)\\
&\quad+\sum_{j=1}^{N} P_i p_{1ij}(G_{1ij}+\Delta G_{1ij})(G_{1ij}+\Delta G_{1ij})^t P_i\\
&\quad+\sum_{j=1}^{N} P_i p_{2ij}(G_{2ij}+\Delta G_{2ij})(G_{2ij}+\Delta G_{2ij})^t P_i\\
&\quad+\sum_{j=1}^{N} p_{1ij}(W_{1i}^t W_{1i}+W_{1ji}^t W_{1ji})\\
&\quad+\sum_{j=1}^{N} p_{2ij}(W_{2i}^t W_{2i}+W_{2ji}^t W_{2ji})+Q_i\Bigg]x_i<0
\end{aligned}$$

for all nonzero $x=[x_1^t \ \dots \ x_N^t]^t$ and all admissible uncertainties. Hence, $P_i>0$, $i=1,2,\dots,N$, is a set of quadratic cost matrices for the system (3.141) and the cost function (3.142).

Using the Schur complements, (3.149) holds if, and only if,

$$\begin{bmatrix} (A_i+\Delta A_i)^t P_i + P_i(A_i+\Delta A_i)+\tilde{Q}_i & P_i(G_{1i}+\Delta G_{1i}) & P_i(G_{2i}+\Delta G_{2i})\\ (G_{1i}+\Delta G_{1i})^t P_i & -I & 0\\ (G_{2i}+\Delta G_{2i})^t P_i & 0 & -I \end{bmatrix} < 0, \tag{3.157}$$

where

$$\Delta G_{1i}=[p_{1i1}\Delta G_{1i1} \ \dots \ p_{1iN}\Delta G_{1iN}],$$
$$\Delta G_{2i}=[p_{2i1}\Delta G_{2i1} \ \dots \ p_{2iN}\Delta G_{2iN}].$$

That is,

$$\begin{bmatrix} A_i^t P_i + P_i A_i + \tilde{Q}_i & P_i G_{1i} & P_i G_{2i}\\ G_{1i}^t P_i & -I & 0\\ G_{2i}^t P_i & 0 & -I \end{bmatrix} + \begin{bmatrix} P_i H_{1i} & P_i L_{1i} & P_i L_{2i}\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} F_i & 0 & 0\\ 0 & F_{1id} & 0\\ 0 & 0 & F_{2id} \end{bmatrix}\begin{bmatrix} E_{1i} & 0 & 0\\ 0 & N_{1i} & 0\\ 0 & 0 & N_{2i} \end{bmatrix}$$

$$+\begin{bmatrix} E_{1i} & 0 & 0 \\ 0 & N_{1i} & 0 \\ 0 & 0 & N_{2i} \end{bmatrix}^t \begin{bmatrix} F_i & 0 & 0 \\ 0 & F_{1id} & 0 \\ 0 & 0 & F_{2id} \end{bmatrix}^t \begin{bmatrix} P_i H_{1i} & P_i L_{1i} & P_i L_{2i} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^t$$
$$< 0, \tag{3.158}$$

where $F_{1id} = \text{diag}\{F_{1i1}, \dots, F_{1iN}\}$, $F_{2id} = \text{diag}\{F_{2i1}, \dots, F_{2iN}\}$. Using Lemma 3.3, (3.158) holds if there exist some $\varepsilon_i > 0$, $\gamma_{1ij} > 0$ and $\gamma_{2ij} > 0$ such that

$$\begin{bmatrix} M_i & P_i G_{1i} & P_i G_{2i} \\ G_{1i}^t P_i & -I + N_{1i}^t \Gamma_{1\beta i} N_{1i} & 0 \\ G_{2i}^t P_i & 0 & -I + N_{2i}^t \Gamma_{2\beta i} N_{2i} \end{bmatrix} < 0, \tag{3.159}$$

where

$$\begin{aligned} M_i &= A_i^t P_i + P_i A_i + \tilde{Q}_i + \varepsilon_i^{-1} P_i H_{1i} H_{1i}^t P_i \\ &\quad + P_i L_{1i} \Gamma_{1\alpha i}^{-1} L_{1i}^t P_i + P_i L_{2i} \Gamma_{2\alpha i}^{-1} L_{2i}^t P_i + \varepsilon_i E_{1i}^t E_{1i} \end{aligned}$$

and $\Gamma_{1\alpha_j}$, $\Gamma_{1\beta_j}$, $\Gamma_{2\alpha_j}$ and $\Gamma_{2\beta_j}$ are given by (2.37) and (3.156).

By applying the Schur complements again, (3.159) holds if, and only if, (3.150) holds. □

Remark 3.27 Theorem 3.9 provides a sufficient condition for the existence of a set of guaranteed quadratic cost matrices. It gives a suboptimal method for computing the robust performance measure defined in (3.148). In fact, it follows from Theorem 3.9 that

$$\begin{aligned} J_* = \inf \Bigg\{ & \sum_{i=1}^{N} \lambda_{\max}(\Pi_{i0}^t P_i \Pi_{i0}) \mid P_i > 0,\ \varepsilon_i > 0, \\ & \gamma_{1ij} > 0 \text{ and } \gamma_{2ij} > 0 \text{ satisfy (2.29)} \Bigg\}. \end{aligned}$$

Obviously, $J \le J^* \le J_*$. Hence, J_* provides a suboptimal upper bound for the system (2.20) and the cost function (2.21). Note that (2.29) is linear in $\varepsilon_i, \gamma_{1ij}$ and γ_{2ij}, $i, j = 1, 2, \dots, N$, and hence the problem of computing J_* is a standard LMI problem [4].

3.6.5 Guaranteed Cost Controller Design

An LMI approach is presented here to solve the decentralized quadratic guaranteed cost control problem for a class of interconnected nonlinear systems. Consider the following interconnected large-scale system which consists of N subsystems:

$$\dot{x}_i = (A_i + \Delta A_i)(B_i + \Delta B_i)u_i \sum_{j=1}^{N} P_{1ij}(G_{1ij} + \Delta G_{1ij})g_{1ij}(x_i, x_j) + \sum_{j=1}^{N} P_{2ij}(G_{2ij} + \Delta G_{2ij})g_{2ij}(x_i, x_j), \tag{3.160}$$

where $x_i \in \Re^{n_i}$ is the state of the ith subsystem, $u_i \in \Re^{m_i}$ is the control of the ith subsystem, A_i, B_i, G_{1ij} and G_{2ij} are real constant matrices with appropriate dimensions, ΔA_i, ΔB_i, ΔG_{1ij} and ΔG_{2ij} are uncertain matrices, $g_{1ij}(x_i, x_j) \in \Re^{n_{ij}}$ and $g_{2ij}(x_i, x_j) \in \Re^{s_{ij}}$ are unknown nonlinear vector functions representing the interconnection between the ith subsystem and the jth subsystem, and the parameters p_{1ij} and p_{2ij} are constants of either 1 or 0 (if they are 0, it means that the jth subsystem has no connection with the ith subsystem). ΔG_{1ij}, ΔG_{2ij}, $g_{1ij}(x_i, x_j)$ and $g_{2ij}(x_i, x_j)$ satisfy Assumption 3.15 and ΔA_i and ΔB_i, satisfy the following assumption.

Assumption 3.16

$$[\Delta A_i \;\; \Delta B_i] = H_{1i} F_j [E_{1i} \;\; E_{2i}],$$

where $F_j \in \Re^{h_j \times e_j}$ is an unknown matrix function satisfying

$$F_j^t F_j \le I_{e_j}$$

and H_{1i}, E_{1i} and E_{2i} are known real constant matrices with appropriate dimensions.

Remark 3.28 Obviously, the parameter uncertainties and interconnections in the power system (3.139) satisfy Assumptions 3.15 and 3.16.

In what follows, the following cost performance is defined:

$$J = \sum_{j=1}^{N} \int_0^{\infty} (x_j^t Q_j x_j + u_j^t R_j u_j) dt, \tag{3.161}$$

where $Q_j = Q_j^t > 0$ and $R_j = R_j^t > 0$, $j = l, 2, \ldots, N$, are given real constant matrices.

Similar to Definition 3.1, we give the following definition of decentralized state feedback guaranteed cost control.

Definition 3.2 A decentralized controller $u_j = K_j x_j$ is said to be a decentralized state feedback quadratic guaranteed cost controller with a set of cost matrices

$P_i > 0$ for the system (3.160) and (3.161), if

$$\sum_{i=1}^{N} x_i^t\{[A_i + \Delta A_i + (B_i + \Delta B_i)K_i]^t P_i + P_i[A_i + \Delta A_i + (B_i + \Delta B_i)K_i]\}x_i + \sum_{i=1}^{N}\sum_{j=1}^{N} 2x_i^t P_i[p_{1ij}(G_{1ij} + \Delta G_{1ij})g_{1ij}(x_i, x_j) + p_{2ij}(G_{2ij} + \Delta G_{2ij})g_{2ij}(x_i, x_j)] + \sum_{i=1}^{N} x_i^t(Q_i + K_i^t R_i K_i)x_i < 0 \tag{3.162}$$

for all admissible uncertainties.

The following theorem provides the main result of this section.

Theorem 3.10 *Consider the system* (3.160) *satisfying Assumptions* 3.15 *and* 3.16. *Suppose that there exist some real positive scalars* $\varepsilon_i, \gamma_{1ij}, \gamma_{2ij}$ *and some real constant matrices* $X_i = X_i^t > 0$ *and* Y_i, $i, j = 1, 2, \ldots, N$, *such that the following set of linear matrix inequalities* (*LMIs*) *holds*:

$$\begin{bmatrix} \Phi_i & X_i\tilde{Q}_i^{1/2} & Y_i^t & X_iE_{1i}^t & G_{1i} & 0 & G_{2i} & 0 \\ \tilde{Q}_i^{\frac{1}{2}}X_i & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ Y_i & 0 & -R_i^{-1} & 0 & 0 & 0 & 0 & 0 \\ E_{1i}X_i + E_{2i}Y_i & 0 & 0 & -\varepsilon_i I & 0 & 0 & 0 & 0 \\ G_{1i}^t & 0 & 0 & 0 & -I & N_{1i}^t & 0 & 0 \\ 0 & 0 & 0 & 0 & N_{1i} & -\Gamma_{1\beta i} & 0 & 0 \\ G_{2i}^t & 0 & 0 & 0 & 0 & 0 & -I & N_i^t \\ 0 & 0 & 0 & 0 & 0 & 0 & N_{2i} & -\Gamma_{2\beta i} \end{bmatrix} < 0, \tag{3.163}$$

where

$$\Phi_i = A_iX_i + X_iA_i^t + B_iY_i + Y_i^tB_i^t + \varepsilon_i H_{1i}H_{1i}^t + L_{1i}\Gamma_{1\alpha i}L_{1i}^t + L_{2i}\Gamma_{2\alpha i}L_{2i}^t$$

and $\tilde{Q}_i, N_{1i}, N_{2i}, \Gamma_{1\alpha i}, \Gamma_{2\alpha i}, \Gamma_{1\beta i}$ *and* $\Gamma_{2\beta i}$ *are as in* (3.151) *and* (3.154)–(3.156), *respectively. If the above condition is met, there exists a decentralized guaranteed cost controller given by* $u_i = K_ix_i$ *with* $K_i = Y_iX_i^{-1}$, $i = 1, 2, \ldots, N$, *that asymptotically stabilizes the overall closed-loop system and render the performance cost*

J satisfying

$$J \leq \sum_{i=1}^{N} x_i^t(0) X_i^{-1} x_i(0),$$

where $x_i(0), i = 1, 2, \ldots, N$ *is the initial state of the* i*th subsystem.*

Proof The closed-loop system of (3.160) with $u_i = K_i x_i$ is

$$\dot{x}_i = [(A_i + B_i K_i) + (\Delta A_i + \Delta B_i K_i)]x_i + \sum_{j=1}^{N} p_{1ij}(G_{1ij} + \Delta G_{1ij}) g_{1ij}(x_i, x_j) + \sum_{j=1}^{N} p_{1ij}(G_{2ij} + \Delta G_{2ij}) g_{2ij}(x_i, x_j) \tag{3.164}$$

and the corresponding closed-loop cost function is

$$J = \sum_{i=1}^{N} \int_0^{\infty} x_i^t (Q_i + K_i^t R_i K_i) x_i dt. \tag{3.165}$$

On the other hand, premultiply and postmultiply (3.163) by diag$\{X_i^{-1}, I \ldots, I\}$, and let $P_i = X_i^{-1} > 0$, and $K_i = Y_i X_i^{-1}$. Then using the Schur complements, (3.163) holds if, and only if,

$$\begin{bmatrix} \Psi_i & P_i H_{1i} & P_i L_{1i} & P_i G_{1i} & P_i L_{2i} & P_i G_{2i} \\ H_{1i}^t P_i & -\varepsilon_i^{-1} I & 0 & 0 & 0 & 0 \\ L_{1i}^t P_i & 0 & -\Gamma_{1\alpha i}^{-1} & 0 & 0 & 0 \\ G_{1i}^t P_i & 0 & 0 & -I + N_{1i}^t \Gamma_{1\beta i}^{-1} N_{1i} & 0 & 0 \\ L_{2i}^t P_i & 0 & 0 & 0 & -\Gamma_{2\alpha i}^{-1} & 0 \\ G_{2i}^t P_i & 0 & 0 & 0 & 0 & -I + N_{2i}^t \Gamma_{2\beta i}^{-1} N_{2i} \end{bmatrix} < 0, \tag{3.166}$$

where

$$\Psi_i = P_i(A_i + B_i K_i) + (A_i + B_i K_i)^t P_i + \tilde{Q}_i + K_i^t R_i K_i + \varepsilon_i^{-1}(E_{1i} + E_{2i} K_i)^t (E_{1i} + E_{2i} K_i)$$

and $\tilde{Q}_i$ is as in (3.151).

By applying Theorem 3.9 to the closed-loop system (3.164) and the corresponding cost function (3.165), the theorem is established. □

Remark 3.29 Note that (3.163) is linear in $X_i, Y_i, \varepsilon_i, \gamma_{1ij}$ and γ_{2ij}, and can be solved efficiently using the LMI tool [12]. Also, it follows from Theorem 3.10 that, if the LMI (3.163) holds, then the corresponding cost function (3.161) is bounded by $\sum_{i=1}^{N} \lambda_M(\Pi_{i0}^t X_i^{-1} \Pi_{i0})$. For some given constant $\lambda_i > 0$, $\lambda_M(\Pi_{i0}^t X_i^{-1} \Pi_{i0}) < \lambda_i$ if, and only if,

$$\lambda_i I - \Pi_{i0}^t X_i^{-1} \Pi_{i0} > 0$$

which is equivalent to

$$\begin{bmatrix} -\lambda_i I & \Pi_{i0}^t \\ \Pi_{i0} & -X_i \end{bmatrix} < 0. \tag{3.167}$$

Therefore, the problem of minimizing the bound $\sum_{i=1}^{N} \lambda_M(\Pi_{i0}^t X_i^{-1} \Pi_{i0})$ becomes the minimization of $\sum_{i=1}^{N} \lambda_i$ under the LMI constraints of (3.163) and (3.167). This is a parametric LMI problem and can be solved effectively by employing the LMI tool [9].

Remark 3.30 From Theorem 3.10 and the feedback linearized system (3.139), the excitation control input u_{fi} of the power system (3.129)–(3.131) can be obtained by an inverse transform of (3.140):

$$u_{fi} = \frac{1}{k_{ci} I_{qi}} \{\nu_{fi} + P_{mi0} - (x_{di} - x'_{di}) I_{qi} I_{di} + T'_{d0i} Q_{ci} \omega_i\}, \tag{3.168}$$

where $\nu_{fi} = K_i x_i$ with $K_i = Y_i X_i^{-1}$. Note that $I_{qi} = 0$ is not in the normal working region for a generator, so u_{fi} is well defined. On the other hand, in power systems, P_{ei}, Q_{ei} and I_{fi} are readily measurable variables, thus it follows from (3.132), (3.136) and (3.137) that I_{di} and I_{qi} can be calculated by using these available variables. As δ_i and ω_i, $i = 1, 2, \ldots, N$, are also measurable variables, the excitation control (3.168) is practically realizable by only using the local measurements.

3.6.6 Simulation Results

The decentralized guaranteed cost control design proposed in the preceding section is now applied to a three-machine power system as shown in Fig. 3.10. Generator 3 is an infinite bus bar used as the reference ($E'_{q3} = \text{constant} = 1\angle 0°$). The system parameters used in the simulation are given in Table 3.1.

For the purpose of illustration, the authors consider the parametric perturbation as $\Delta T'_{d0i} = 0.1 T'_{d0i}$, $i = 1, 2$, and choose $|P_{e1}|_M = 1.4$ and $|P_{e2}|_M = 1.5$. Thus, the DFL compensated power system model (2.18) can be rewritten as follows:

$$\begin{aligned} \dot{x}_1 = {} & (A_1 + \Delta A_1)x_1 + (B_1 + \Delta B_1)\nu_{f1} \\ & + \Delta G_{112} \sin(\delta_1 - \delta_2) + \Delta G_{211}\omega_1 + \Delta G_{212}\omega_2, \end{aligned} \tag{3.169}$$

$$\begin{aligned} \dot{x}_2 = {} & (A_2 + \Delta A_2)x_2 + (B_2 + \Delta B_2)\nu_{f2} \\ & + \Delta G_{121} \sin(\delta_2 - \delta_1) + \Delta G_{221}\omega_1 + \Delta G_{222}\omega_2, \end{aligned} \tag{3.170}$$

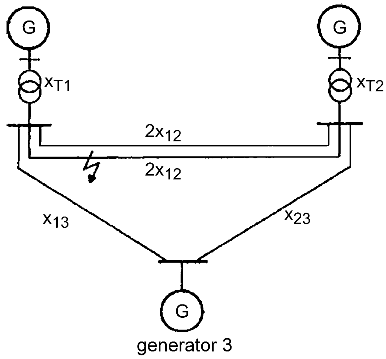


Fig. 3.10 Three-machine power system

Table 3.1 System parameters

	Generator 1	Generator 2	
x_d, p.u.	1.863	2.36	
x_d, p.u.	0.257	0.319	
x_T, p.u.	0.129	0.11	x_{12}, p.u. 0.55
x_{ad}, p.u.	1.712	1.712	x_{13}, p.u. 0.53
T'_{d0}, p.u.	6.9	7.96	x_{23}, p.u. 0.6
H, s	4	5.1	
D, p.u.	5	3	
k_c	1	1	

where

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.625 & -39.27 \\ 0 & 0 & -0.1449 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 \\ 0 \\ 0.1449 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.2941 & -30.8 \\ 0 & 0 & -0.1256 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ 0 \\ 0.1256 \end{bmatrix},$$

$$|\mu_1| \le 0.0132, \quad |T'_{d02}|_{\min} = 7.164 \text{ s},$$

$$|r_{112}| \le 0.7817, \quad |r_{211}| \le 1.4, \quad |r_{212}| \le 1.4,$$

$$|\mu_2| \le 0.0111, \quad |T'_{d01}|_{\min} = 6.21 \text{ s},$$

$$|r_{121}| \le 0.9662, \quad |r_{221}| \le 1.5, \quad |r_{222}| \le 1.5.$$

In the performance index (3.161), the authors set $Q_1 = Q_2 = 0.05I$, $R_1 = 0.002$ and $R_2 = 0.001$. In the light of Remark 3.29, by solving the corresponding LMIs (see (3.163)), the decentralized guaranteed cost controller is obtained as

$$\nu_{f1} = 46.6023(\delta_1 - \delta_{10}) + 48.7572\omega_1 - 245.4968(P_{e1} - P_{m10}), \quad (3.171)$$

$$\nu_{f2} = 59.6959(\delta_2 - \delta_{20}) + 65.0159\omega_2 - 244.7198(P_{e2} - P_{m20}) \quad (3.172)$$

and the minimal upper bound of the cost is 1.7676. Thus, the original excitation control laws for the three-machine power system are as follows:

$$u_{f1} = \frac{1}{I_{q1}}\{\nu_{f1} + P_{m10} - (x_{d1} - x'_{d1})I_{q1}I_{d1} + T'_{d01}Q_{e1}\omega_1\}, \quad (3.173)$$

$$u_{f2} = \frac{1}{I_{q2}}\{\nu_{f2} + P_{m20} - (x_{d2} - x'_{d2})I_{q2}I_{d2} + T'_{d02}Q_{e2}\omega_2\}. \quad (3.174)$$

In the simulation, saturation of synchronous machines is also considered, and so (3.131) becomes

$$\dot{E}'_{qi} = \frac{1}{T'_{d01}}[E_{fi} - E_{qi} - (1 - k_{fi})E'_{qi}], \quad (3.175)$$

where

$$k_{fi} = 1 + \frac{b_j}{a_j}(E'_{qi})^{(n_j - 1)}$$

with

$$a_1 = 0.95, \quad b_1 = 0.051, \quad n_1 = 8.727, \quad (3.176)$$

$$a_2 = 0.935, \quad b_2 = 0.064, \quad n_2 = 10.878. \quad (3.177)$$

The excitation control input limitations are

$$-6 \le E_{fi} = k_{ci}u_{fi} \le 6, \quad i = 1, 2.$$

This example shows the effectiveness of the proposed decentralized control under different operating points, fault locations and transmission-line parameters. The fault under consideration is a symmetrical three-phase short-circuit fault that occurs on one of the transmission lines between generators 1 and 2. The fault location is indexed by a constant λ, which is the fraction of the line to the left of the fault. For example, $\lambda = 0$ means that the fault is on the bus bar of generator 1, whereas $\lambda = 0.5$ indicates that the fault happens midway between generators 1 and 2. The fault sequence under consideration is as follows:

1. *The system is in pre-fault steady state*;
2. *A fault occurs at* $t_0 = 0.1s$;
3. *The fault is removed by opening the circuit breakers of the faulted line at* $t_1 = 0.25\ s$;

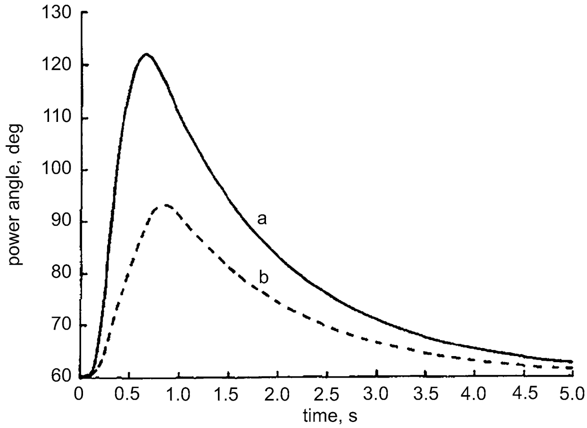


Fig. 3.11 Power angle response of power system: Case 1

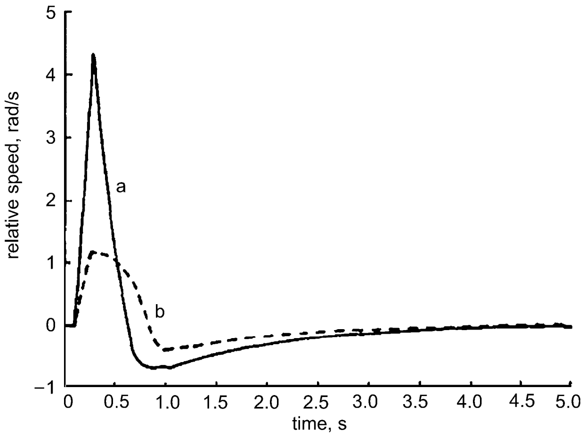


Fig. 3.12 Relative speed response of power system: Case 1

4. *The transmission line is restored with the fault clear at* $t_2 = 1.0\ s$;
5. *The system is in post fault state*.

The system dynamic responses can be tested under the following cases of different operating points, fault locations and network parameters.

1. The operating points are

$$\delta_{10} = 60.78^\circ, \quad P_{m10} = 1.1 \text{ p.u.}, \quad V_{t1} = 1.0 \text{ p.u.}, \tag{3.178}$$

$$\delta_{20} = 60.64^\circ, \quad P_{m20} = 1.0 \text{ p.u.}, \quad V_{t2} = 1.0 \text{ p.u.} \tag{3.179}$$

The fault location is $\lambda = 0.07$. The corresponding closed loop system responses of power angles, relative speeds, real powers and excitation control signals of generators 1 and 2 are shown in Figs. 3.11–3.14.

In particular, the responses of power angles are given in Figs. 3.15 and 3.16 for comparison: in Fig. 3.15, where the fault location is $A = 0.07$, the open-loop system without controller is unstable; in Fig. 3.16, where the fault location

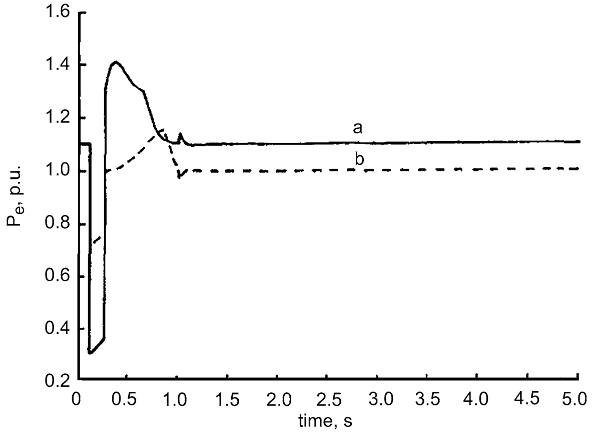


Fig. 3.13 P_e response of power system: Case 1

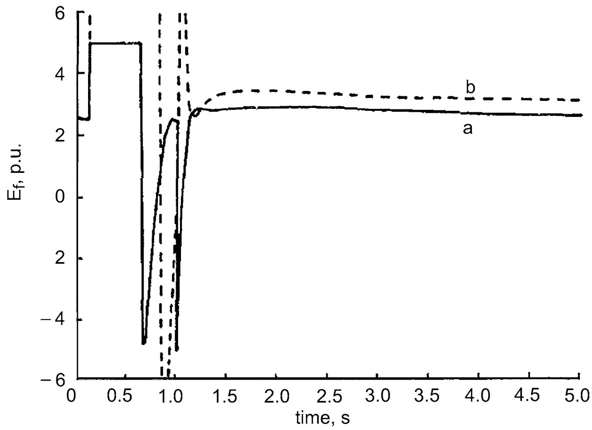


Fig. 3.14 E_f response of power system: Case 1

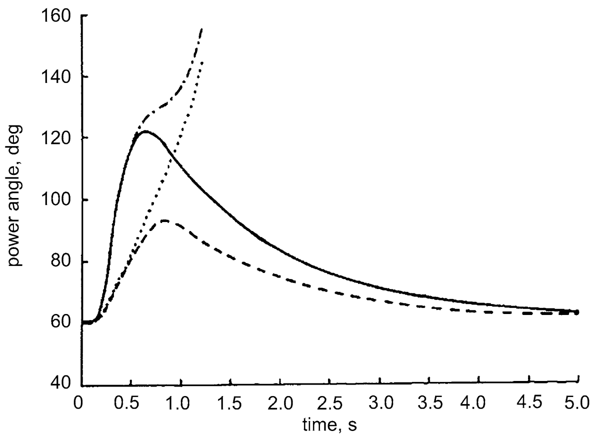


Fig. 3.15 Responses of power angles, controller against no controller: $\lambda = 0.07$

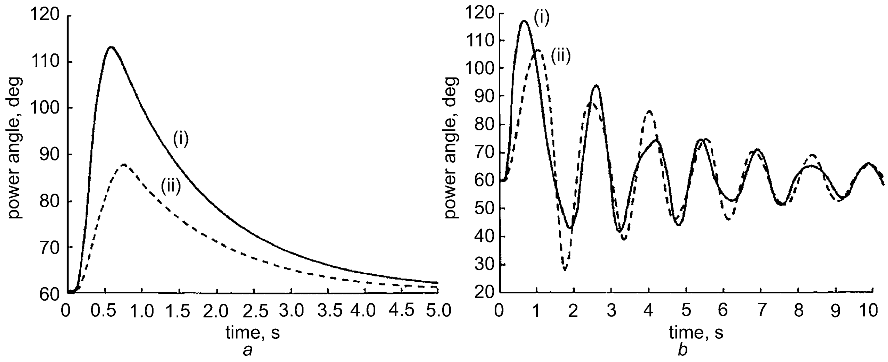


Fig. 3.16 Responses of power angles, controller compared with no controller: $\lambda = 0.09$

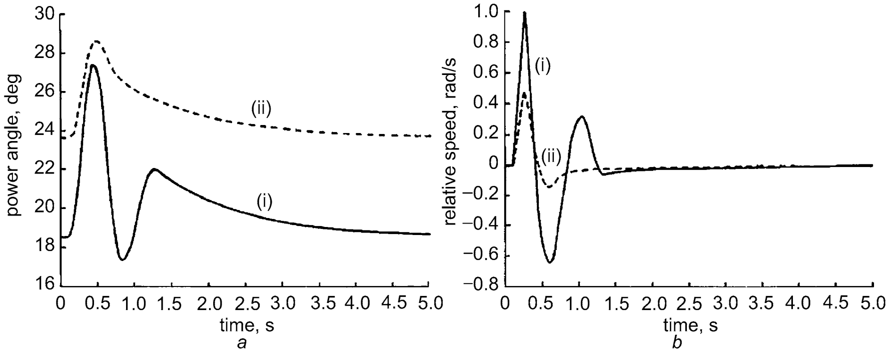


Fig. 3.17 Responses of power system: Case 2

is $\lambda = 0.09$. The open-loop system without controller is stable, but it exhibits significant oscillations. From Figs. 3.15 and 3.16, it is obvious that the proposed controller can enhance the system transient stability and damp out the power angle oscillations.

2. The operating points are

$$\delta_{10} = 18.51°, \quad P_{m10} = 0.3 \text{ p.u.}, \quad V_{t1} = 0.95 \text{ p.u.} \tag{3.180}$$

$$\delta_{20} = 23.68°, \quad P_{m20} = 0.4 \text{ p.u.}, \quad V_{t2} = 0.95 \text{ p.u.} \tag{3.181}$$

The fault locations is $\lambda = 0.1$. The corresponding closed loop system responses of power angles and relative speeds of generators 1 and 2 are shown in Fig. 3.17.

3. The operating points are the same as in Case 1. The corresponding closed-loop system responses of power angles are compared with different fault locations ($\lambda = 0.07, 0.5, 0.95$) in Fig. 3.18.
4. The transmission-line parameters are defined by the following:

$$x_{12} = X_{13} = X_{23} = 0.7. \tag{3.182}$$

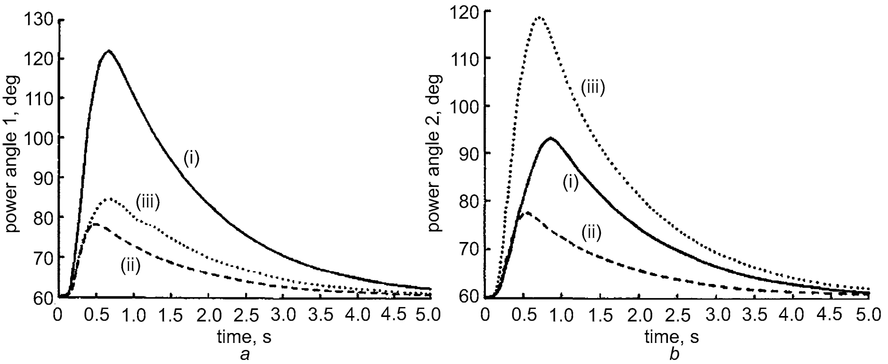


Fig. 3.18 Responses of power angles of the generators 1 and 2: Case 3. (i) $\lambda = 0.07$; (ii) $\lambda = 0.5$; (iii) $\lambda = 0.95$

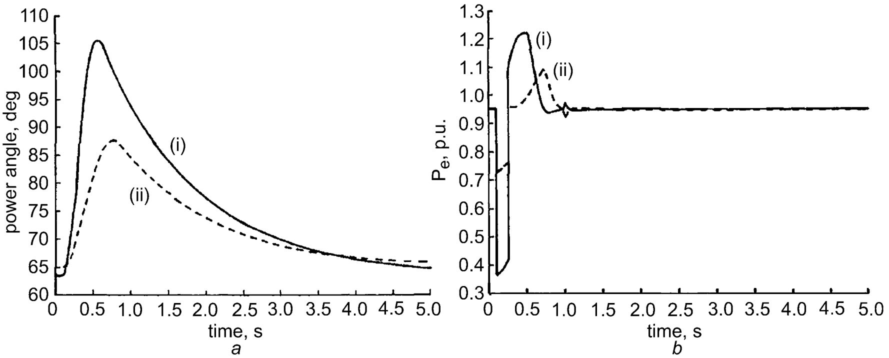


Fig. 3.19 Responses of power system: Case 4. (i) Power angle response; (ii) P_e response

The operating points are as follows:

$$\delta_{10} = 64.08°, \quad P_{m10} = 0.95 \text{ p.u.}, \quad V_{t1} = 10 \text{ p.u.}, \tag{3.183}$$

$$\delta_{20} = 65.33°, \quad P_{m20} = 0.95 \text{ p.u.}, \quad V_{t2} = 1.0 \text{ p.u.} \tag{3.184}$$

The fault location is $\lambda = 0.1$. The corresponding closed loop system responses of power angles and real powers of the generators 1 and 2 are shown in Fig. 3.19.

The simulation results shown here clearly indicate that the proposed controller can enhance the system transient stability and damp out the power angle oscillations in the face of different conditions of operating points, fault locations and transmission parameters.

3.7 Notes and References

In this chapter, a new robust decentralized controller has been proposed to enhance multimachine power system transient stability. The proposed controller is a linear controller that can guarantee system stability over the whole operating region. The controller design procedure is derived. In the design of the controller, the fault location and exact network parameters do not need to be available. The proposed controller uses local measurements through a simple implementation. A three-machine power system is considered as an application example of the theory developed in this chapter. Simulation results show that despite the nonlinear interconnections between generators and significant operating condition variations following the faults, the proposed controller can rapidly damp the system oscillation and greatly enhance the power system transient stability.

Moreover, an LMI-based robust decentralized guaranteed cost control approach has been proposed for multimachine power systems. Our results are given in terms of a set of LMIs which can be solved efficiently by using the available LMI tool. A procedure has been given for the optimization of an upper bound of the performance index. The proposed robust control scheme is demonstrated on a three-machine example power system. Simulation results have shown that the transient stability is greatly enhanced regardless of different operating points, faults in various locations and changing network parameters.

There are several directions of extending the results reported in this chapter. Chief among these is the class of interconnected discrete-time systems, for which there is virtually no results available.

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Chapter 4
Decentralized Systems with Multi-controllers

This chapter looks at particular classes of decentralized systems that incorporate multiple controllers in their basic operation. Three distinct types of these systems are identified: multi-channel time-delay systems, interconnected networked systems and discrete-systems with saturating controllers. In the first two types, the mathematical analysis treats initially with interconnected time-delay systems to develop general delay-dependent stability and stabilization results. Then, several interesting cases are derived. The subsystems are subjected to convex-bounded parametric uncertainties and/or additive feedback gain perturbations. The third type is concerned with stabilization decentralized linear saturating plants. The basic tool is the construction use of appropriate Lyapunov-Krasovskii functionals. We characterize decentralized linear matrix inequalities (LMIs)-based conditions. Resilient decentralized dynamic output-feedback stabilization schemes are designed such that the family of closed-loop feedback subsystems enjoys the delay-dependent asymptotic stability with a prescribed γ-level $\mathcal{L}_2$ gain for each subsystem.

4.1 Introduction

There are many real world systems consisting of coupled units or subsystems which directly interact with each other in a simple and predictable fashion to serve a common pool of objectives. When viewed as a whole, the resulting overall system often displays rich and complex behavior. Typical examples are found in electric power systems with strong interactions, water networks which are widely distributed in space, traffic systems with many external signal or large-space flexible structures, to name a few, which are often termed *large-scale* or *interconnected* systems. It becomes increasingly evidently that the underlying notions of interconnected systems manifest the complexity as an essential and dominating problem in systems theory and practice and that several associated problems cannot be tackled using one-shot approaches. Recent research investigations have revealed [3, 22] that the crucial need for improved methodologies relies on:

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(1) dividing the analysis and synthesis of the overall system into independent or almost independent subproblems,
(2) searching for new ideas of coping with the incomplete information about the system, and
(3) seeking appropriate methods of handling the uncertainties and for dealing with delays.

System complexity frequently leads to severe difficulties that are encountered in the tasks of analyzing, designing, and implementing appropriate control methods. These difficulties arise mainly from the following well-known reasons: *dimensionality; information structure constraints; uncertainty; delays*. Pertinent results can be found in [2, 25, 26, 41, 43, 49–51, 72].

4.2 Decentralized Stabilization of Multi-channel Systems

In this section, we direct attention to a type of decentralized systems described by a class of linear multi-channel time-delay systems with norm-bounded uncertainties and time-varying delays is examined. The objective is to design a class of decentralized dynamic output-feedback controllers to render the closed-loop multi-channel system delay-dependent asymptotically stable with a prescribed disturbance attenuation level.

4.2.1 Introduction

The basic concepts of large scale or interconnected systems have been introduced to deal with the real control problems that cannot be solved using one-shot (centralized) approaches [43, 56–61, 72]. Typical problems arise in the control of water systems which are widely distributed in space, interconnected power systems with strong interactions, traffic systems with different external signals, or large-scale flexible structures. The structures of such systems have led to the development of new ideas for dividing the analysis and synthesis of the overall system into independent (or almost independent) subproblems and for dealing with limited information, uncertainties and time-delays. Therefore in the past few decades, the analysis and design problems of decentralized control for large scale or interconnected systems have been intensively studied [27, 79]. In particular, the linear matrix inequalities (LMIs) framework [8] has appeared to be very attractive to tackle the control and filtering problems of handling interconnected systems [3].

This section develops new results for the problems of decentralized analysis and control synthesis for a class of linear interconnected multi-channel systems. This class includes linear time-delay systems subject to input disturbance and several control agents where the system matrices are allowed to undergo bounded parametric uncertainties. The design objective is to construct robust dynamic output-feedback controllers and derive easily-computable formula for determining the gains. Previous related results are reported in [10, 15, 80] where the main focus has been on delay-free systems using state-feedback. It turns out that the results of [10, 15, 80] are essentially a special case of the approach developed hereafter. We employ a Lyapunov-Krasovskii functional (LKF) approach to developed the closed-loop stabilization conditions and with the aid of a convex optimization framework, LMI-based conditions are obtained.

4.2.2 Problem Statement

We consider a class of linear uncertain systems Σ with N channels and represented by the state-space model:

$$\dot{x}(t) = [A + \Delta A]x(t) + [A_d + \Delta A_d]x(t - \tau(t)) + \sum_{j=1}^{N} B_j u_j(t) + \Gamma w(t), \quad (4.1)$$

$$z(t) = Gx(t) + G_d x(t - \tau(t)) + \Phi w(t), \quad (4.2)$$

$$y_j(t) = [C_j + \Delta C_j]x_j(t), \quad j \in \{1, \ldots, N\}, \quad (4.3)$$

where $x(t) \in \Re^n$ is the state vector, $w(t) \in \Re^r$ is the disturbance input which belongs to $\mathcal{L}_2[0, \infty)$, $z(t) \in \Re^p$ is the controlled output, $u_j(t) \in \Re^{m_j}$ and $y_j(t) \in \Re^{q_j}$ are the control input and the measurement output of channel $j \in \{1, \ldots, N\}$ and τ is an unknown time-delay factor satisfying

$$0 \le \tau(t) \le \varrho, \quad \dot{\tau}(t) \le \mu, \quad (4.4)$$

where the bounds ϱ, μ are known constants in order to guarantee smooth growth of the state trajectories. The matrices $A \in \Re^{n\times n}$, $B_j \in \Re^{n\times m_j}$, $G \in \Re^{p\times n}$, $G_d \in \Re^{p\times n}$, $A_d \in \Re^{n\times n}$, $\Phi \in \Re^{p\times r}$, $\Gamma \in \Re^{n\times r}$, $C_j \in \Re^{q_j\times n_j}$ are real and constants.

Without loss of generality, the following assumptions are made:

Assumption 4.1 There is no unstable fixed modes with respect the triplet A, B_j, C_j.

Assumption 4.2 For every $j \in \{1, \ldots, N\}$, the matrices B_j, C_j have full column rank and full row rank, respectively.

The uncertain matrices ΔA, ΔA_d, ΔC_j are represented by

$$[\Delta A \ \Delta A_d] = E\Delta[M \ N], \qquad \begin{bmatrix} \Delta C_1 \\ \Delta C_2 \\ \vdots \\ \Delta C_N \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_N \end{bmatrix} \Delta F, \quad (4.5)$$

where $E, F, M, N, H_1, \dots, H_N$ are known constant matrices with appropriate dimensions and Δ is an unknown matrix satisfying $\Delta^t \Delta \le I$. The class of systems described by (4.1)–(4.3) subject to delay-pattern (4.4) is frequently encountered in modeling several physical systems and engineering applications including large space structures, multi-machine power systems, cold mills, transportation systems, water pollution management, to name a few [48, 72].

In what follows, we consider the feasible set $\mathcal{C}$ as the set of all linear time-invariant controllers with state-space realization of the form:

$$\begin{aligned} \dot{\hat{x}}_j(t) &= \hat{A}_j \hat{x}_j(t) + \hat{B}_j y_j(t), \\ u_j(t) &= \hat{C}_j \hat{x}_j(t) + \hat{D}_j y_j(t), \quad j = 1, 2, \dots, N \end{aligned} \tag{4.6}$$

where $\hat{x}_j(t) \in \Re^{s_j}$ is the state of the local controller with the order $s_j \le n$ and the matrices $\hat{A}_j \in \Re^{s_j \times s_j}$, $\hat{B}_j \in \Re^{s_j \times q_j}$, $\hat{C}_j \in \Re^{m_j \times s_j}$, $\hat{D}_j \in \Re^{m_j \times qj}$ and are the design parameters. Connecting the controller (4.6) to the system (4.1)–(4.3), we obtain the closed-loop system

$$\begin{aligned} \dot{x}(t) &= \left[A + \Delta A + \sum_{j=1}^{N} B_j \hat{D}_j [C_j + \Delta C_j] \right] x(t) + [A_d + \Delta A_d] x(t - \tau(t)) \\ &\quad + \sum_{j=1}^{N} B_j \hat{C}_j \hat{x}_j(t) + \Gamma w(t), \end{aligned} \tag{4.7}$$

$$\dot{\hat{x}}_j(t) = \hat{B}_j [C_j + \Delta C_j] x(t) + \hat{A}_j \hat{x}_j(t), \quad j \in \{1, \dots, N\}, \tag{4.8}$$

$$z(t) = G x(t) + G_d x(t - \tau(t)) + \Phi w(t). \tag{4.9}$$

For simplicity in exposition, we introduce the following notations

$$\begin{aligned} \hat{x}(t) &= \mathrm{col}\left[\hat{x}_1(t)\ \hat{x}_2(t) \dots \hat{x}_N(t) \right], & w(t) &= \mathrm{col}\left[w_1(t)\ w_2(t) \dots w_N(t) \right], \\ \hat{A} &= \mathrm{diag}\left[\hat{A}_1\ \hat{A}_2 \dots \hat{A}_N \right], & \hat{B} &= \mathrm{diag}\left[\hat{B}_1\ \hat{B}_2 \dots \hat{B}_N \right], \\ \hat{C} &= \mathrm{diag}\left[\hat{C}_1\ \hat{C}_2 \dots \hat{C}_N \right], & \hat{D} &= \mathrm{diag}\left[\hat{D}_1\ \hat{D}_2 \dots \hat{D}_N \right] \end{aligned} \tag{4.10}$$

along with the matrices

$$\begin{aligned} B &= \left[B_1\ B_2 \dots B_N \right], & C &= \mathrm{diag}\left[C_1^t\ C_2^t \dots C_N^t \right]^t, \\ H &= \mathrm{diag}\left[H_1\ H_2 \dots H_N \right], & \Delta C &= \mathrm{diag}\left[\Delta C_1\ \Delta C_2 \dots \Delta C_N \right]. \end{aligned} \tag{4.11}$$

This paves the way to express the closed-loop system (4.7)–(4.9) into the form

$$\begin{aligned} \dot{x}(t) &= \left[A + \Delta A + B \hat{D} [C + \Delta C] \right] x(t) + [A_d + \Delta A_d] x(t - \tau(t)) \\ &\quad + B \hat{C} \hat{x}_j(t) + \Gamma w(t), \end{aligned} \tag{4.12}$$

$$\dot{\hat{x}}(t) = \hat{B} [C + \Delta C] x(t) + \hat{A} \hat{x}(t), \tag{4.13}$$

$$z(t) = G x(t) + G_d x(t - \tau(t)) + \Phi w(t). \tag{4.14}$$

By grouping the unknown controller matrices into one block matrix

$$\mathcal{K} = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \tag{4.15}$$

and introducing the block matrices

$$\begin{aligned} &\widetilde{A} + \Delta\widetilde{A} = \begin{bmatrix} A + \Delta A & 0 \\ 0 & 0 \end{bmatrix}, \qquad \widehat{\Gamma} = \begin{bmatrix} \Gamma \\ 0 \end{bmatrix}, \qquad \widetilde{B} = \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix}, \\ &\widetilde{C} + \Delta\widetilde{C} = \begin{bmatrix} 0 & I \\ C + \Delta C & 0 \end{bmatrix}, \qquad \widetilde{G} = \begin{bmatrix} G \; 0 \end{bmatrix}, \qquad \widehat{\Phi} = \begin{bmatrix} \Phi^t \; 0 \end{bmatrix}^t, \\ &\widetilde{A}_d = \begin{bmatrix} A_d & 0 \\ 0 & 0 \end{bmatrix}, \qquad \Delta\widetilde{A}_d = \begin{bmatrix} \Delta A_d & 0 \\ 0 & 0 \end{bmatrix}, \qquad \Delta\widetilde{C} = \begin{bmatrix} 0 & 0 \\ \Delta C & 0 \end{bmatrix}. \end{aligned} \tag{4.16}$$

We finally write the closed-loop system in the compact-form

$$\begin{aligned} \dot{\xi}(t) &= \mathcal{A}\xi(t) + \mathcal{A}_d\xi(t - \tau(t)) + \widehat{\Gamma}w(t), \\ z(t) &= \widetilde{G}\xi(t) + \widetilde{G}_d\xi(t - \tau(t)) + \widehat{\Phi}w(t), \end{aligned} \tag{4.17}$$

where

$$\begin{aligned} &\xi(t) = \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}, \qquad \hat{E} = \begin{bmatrix} E \\ 0 \end{bmatrix}, \qquad \hat{H} = \begin{bmatrix} 0 \\ H \end{bmatrix}, \\ &\hat{M} = [\, M \; 0 \,], \qquad \hat{N} = [\, N \; 0 \,], \qquad \hat{F} = [\, F \; 0 \,] \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} &\mathcal{A} = \widetilde{A} + \hat{E}\Delta\hat{M} + \widetilde{B}\mathcal{K}[\widetilde{C} + \hat{H}\Delta\hat{F}], \qquad \mathcal{A}_d = \widetilde{A}_d + \hat{E}\Delta\hat{N}, \\ &\widetilde{G}_d = \begin{bmatrix} G_d \; 0 \end{bmatrix}. \end{aligned} \tag{4.19}$$

It must be observed in (4.17) that all the matrices are known except the controller coefficient matrix $\mathcal{K}$.

The problem of interest in this section is to design the decentralized dynamic output-feedback controller (4.6) such that the closed-loop system (4.17) is internally asymptotically stable with $w(t) \equiv 0$ and under zero initial condition, the following condition is satisfied

$$\|z(t)\|_2 \le \gamma \|w(t)\|_2, \quad \forall w(t) \in \mathcal{L}_2[0, \infty). \tag{4.20}$$

4.2.3 Decentralized Stabilization

We adopt a Lyapunov-based approach to design the decentralized controller (4.6). For this purpose, we introduce the Lyapunov-Krasovskii functional (LKF):

$$V(t) = \xi^t(t)\mathsf{P}\xi(t) + \int_{t-\varrho}^{t} \xi^t(s)\mathsf{S}\xi(s)\,ds + \int_{t-\tau(t)}^{t} \xi^t(s)\mathsf{W}\xi(s)\,ds$$
$$+ \varrho \int_{-\varrho}^{0}\int_{t+s}^{t} \dot{\xi}^t(\alpha)\mathsf{R}\dot{\xi}(\alpha)d\alpha\, ds, \qquad (4.21)$$

where $0 < \mathsf{P}$, $0 \leq \mathsf{W}$, $0 < \mathsf{R}$, $0 < \mathsf{S}$ are matrices of appropriate dimensions. The main decentralized stabilization result is established by the following theorem:

Theorem 4.1 *The uncertain nonlinear system (4.17) is robust asymptotically stable and satisfy (4.20), if there exist positive definite matrices $\bar{S}$, $\bar{R}$, $\bar{W}$, real matrices X, Y, and real constants $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, such that the following LMI holds.*

$$\begin{bmatrix} \check{\Sigma}_{11} & \bar{\Sigma}_{12} & \tilde{\Sigma}_{13} & \hat{\Sigma}_{14} & [\tilde{A}X + Y]^t & X\tilde{G}^t & \hat{E} \\ \bullet & \bar{\Sigma}_{22} & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \tilde{\Sigma}_{33} & 0 & X\bar{A}_d{}^t & X\tilde{G}_d^t & 0 \\ \bullet & \bullet & \bullet & \hat{\Sigma}_{44} & \widetilde{\Gamma}^t & \tilde{\Phi}^t & 0 \\ \bullet & \bullet & \bullet & \bullet & -2X + \bar{R} & 0 & \hat{E} \\ \bullet & \bullet & \bullet & \bullet & \bullet & -I & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\varepsilon_1 I \end{bmatrix} < 0, \qquad (4.22)$$

$$\check{\Sigma}_{11} = \widetilde{A}X + Y + [\widetilde{A}X + Y]^t + \bar{S} + \bar{W} + \bar{R} + \varepsilon_1 \hat{M}^t \hat{M} + + \varepsilon_2 \hat{F}^t \hat{F}, \quad \bar{\Sigma}_{12} = \bar{R},$$
$$\hat{\Sigma}_{14} = \widetilde{\Gamma}, \quad \bar{\Sigma}_{22} = -\bar{R} - \bar{S}, \quad \tilde{\Sigma}_{33} = -(1-\mu)\bar{W} + \varepsilon_1 \hat{N}^t \hat{N}, \qquad (4.23)$$
$$\hat{\Sigma}_{44} = -\gamma^2 I, \quad \bar{\mathcal{A}}_d = \widetilde{A}_d, \quad \tilde{\Sigma}_{13} = \bar{\mathcal{A}}_d X + \varepsilon_1 \hat{M}^t \hat{N}.$$

Proof A straightforward computation along the solutions of (4.17) with the help of Lemma 9.9 yields:

$$J = \dot{V}(t) + z^t(t)z(t) - \gamma^2 w(t)w(t)$$
$$= 2\xi^t\mathsf{P}\dot{\xi} + \varrho^2\dot{\xi}^t\mathsf{R}\dot{\xi} - \varrho\int_{t-\varrho}^{t} \dot{\xi}^t(s)\mathsf{R}\dot{\xi}(s)\,ds$$
$$+ \xi^t(t)[\mathsf{S}+\mathsf{W}]\xi(t) - \xi^t(t-\varrho)\mathsf{S}\xi(t-\varrho) - (1-\mu)\xi^t(t-\tau(t))\mathsf{W}\xi(t-\tau(t))$$
$$+ [\widetilde{G}\xi(t) + \widetilde{G}_d\xi(t-\tau(t)) + \widetilde{\Phi}w(t)]^t[\widetilde{G}\xi(t) + \widetilde{G}_d\xi(t-\tau(t)) + \widetilde{\Phi}w(t)]$$
$$- \gamma^2 w(t)w(t)$$
$$\leq 2\xi^t\mathsf{P}[\mathcal{A}\xi(t) + \mathcal{A}_d\xi(t-\tau(t)) + \widetilde{\Gamma}w(t)]$$
$$+ \varrho^2\dot{\xi}^t\mathsf{R}\dot{\xi} - [\xi(t) - \xi(t-\varrho)]^t\mathsf{R}[\xi(t) - \xi(t-\varrho)]$$

$$+\,\xi^t(t)[\mathsf{S}+\mathsf{W}]\xi(t)-\xi^t(t-\varrho)\mathsf{S}\xi(t-\varrho)-(1-\mu)\xi^t(t-\tau(t))\mathsf{W}\xi(t-\tau(t))$$
$$+\,[\widetilde{G}\xi(t)+\widetilde{G}_d\xi(t-\tau(t))+\widetilde{\Phi}w(t)]^t[\widetilde{G}\xi(t)+\widetilde{G}_d\xi(t-\tau(t))+\widehat{\Phi}w(t)]$$
$$-\,\gamma^2 w(t)w(t). \qquad (4.24)$$

Manipulating (4.24), it yields

$$J \le \eta^t(t)\Sigma\eta(t), \qquad (4.25)$$

where $\eta(t)=\mathrm{col}\{\xi(t)\xi(t-\varrho)\xi(t-\tau(t))w(t)\}$, if the matrix inequality

$$\Sigma=\begin{bmatrix}\Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14}\\ \bullet & \Sigma_{22} & 0 & 0\\ \bullet & \bullet & \Sigma_{33} & 0\\ \bullet & \bullet & \bullet & \Sigma_{44}\end{bmatrix}+\begin{bmatrix}\mathcal{A}\\ 0\\ \mathcal{A}_d\\ \widetilde{\Gamma}\end{bmatrix}^t R\begin{bmatrix}\mathcal{A}\\ 0\\ \mathcal{A}_d\\ \widetilde{\Gamma}\end{bmatrix}<0, \qquad (4.26)$$

where

$$\Sigma_{11}=P\mathcal{A}+\mathcal{A}^tP+S+W-R+\widetilde{G}^t\widetilde{G},\quad \Sigma_{12}=R,\quad \Sigma_{13}=P\mathcal{A}_d+\widetilde{G}^t\widetilde{G}_d,$$
$$\Sigma_{14}=P\tilde{\Gamma}+\widetilde{G}^t\tilde{\Phi},\quad \Sigma_{22}=-R-S,\quad \Sigma_{33}=-(1-\mu)W+\widetilde{G}_d^t\widetilde{G}_d, \qquad (4.27)$$
$$\Sigma_{44}=-\gamma^2I+\widehat{\Phi}^t\widehat{\Phi},$$

is feasible. Applying Lemma 9.10, Σ can be changed to Σ_1 as follows:

$$\Sigma_1=\begin{bmatrix}\Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} & \mathcal{A}R\\ \bullet & \Sigma_{22} & 0 & 0 & 0\\ \bullet & \bullet & \Sigma_{33} & 0 & \mathcal{A}_dR\\ \bullet & \bullet & \bullet & \Sigma_{44} & \widetilde{\Gamma}R\\ \bullet & \bullet & \bullet & \bullet & -R\end{bmatrix}<0, \qquad (4.28)$$

where

$$\Sigma_{11}=P\mathcal{A}+\mathcal{A}^tP+S+W-R+\tilde{G}^t\tilde{G},\quad \Sigma_{12}=R,\quad \Sigma_{13}=P\mathcal{A}_d+\tilde{G}^t\tilde{G}_d,$$
$$\Sigma_{14}=P\widetilde{\Gamma}+\widetilde{G}^t\widehat{\Phi},\quad \Sigma_{22}=-R-S,\quad \Sigma_{33}=-(1-\mu)W+\widetilde{G}_d^t\widetilde{G}_d, \qquad (4.29)$$
$$\Sigma_{44}=-\gamma^2I+\widehat{\Phi}^t\widehat{\Phi}.$$

On pre-multiplying and post-multiplying Σ_1 by the diagonal matrix

$$\mathrm{diag}\{P^{-1},P^{-1},P^{-1},I,R^{-1}\}$$

and letting

$$P^{-1}=X,\qquad XSX=\bar{P}_j,\qquad XWX=\bar{Q},\qquad XRX=\bar{R},\qquad R^{-1}=X\bar{R}^{-1}X,$$

it follows from the algebraic inequality

$$X\bar{R}^{-1}X-2X+\bar{R}=(X-\bar{R})\bar{R}^{-1}(X-\bar{R})\ge 0,$$

that

$$-2X + \bar{R} \geq -X\bar{R}^{-1}X,$$

then, the inequality Σ_1 is equivalent to Σ_2 as follows:

$$\Sigma_2 = \begin{bmatrix} \bar{\Sigma}_{11} & \bar{\Sigma}_{12} & \bar{\Sigma}_{13} & \bar{\Sigma}_{14} & X\mathcal{A}^t \\ \bullet & \bar{\Sigma}_{22} & 0 & 0 & 0 \\ \bullet & \bullet & \bar{\Sigma}_{33} & 0 & X\mathcal{A}_d^t \\ \bullet & \bullet & \bullet & \Sigma_{44} & \widetilde{\Gamma}^t \\ \bullet & \bullet & \bullet & \bullet & -2X + \bar{R} \end{bmatrix} < 0, \tag{4.30}$$

where

$$\begin{aligned} &\bar{\Sigma}_{11} = \mathcal{A}X + X\mathcal{A}^t + \bar{S} + \bar{W} + \bar{R} + X\widetilde{G}^t\widetilde{G}X, \quad \bar{\Sigma}_{12} = \bar{R}, \\ &\bar{\Sigma}_{13} = \mathcal{A}_d X + X\widetilde{G}^t\widetilde{G}_d X, \quad \bar{\Sigma}_{14} = \widehat{\Gamma} + X\widetilde{G}^t\widehat{\Phi}, \\ &\bar{\Sigma}_{22} = -\bar{R} - \bar{S}, \quad \bar{\Sigma}_{33} = -(1-\mu)\bar{W} + X\widetilde{G}_d^t\widetilde{G}_d X, \\ &\Sigma_{44} = -\gamma^2 I + \widehat{\Phi}^t\widehat{\Phi}. \end{aligned} \tag{4.31}$$

Applying Lemma 9.10 again, Σ_2 can be changed to Σ_3 as follows:

$$\Sigma_2 = \begin{bmatrix} \hat{\Sigma}_{11} & \bar{\Sigma}_{12} & \hat{\Sigma}_{13} & \hat{\Sigma}_{14} & X\mathcal{A}^t & X\tilde{G}^t \\ \bullet & \bar{\Sigma}_{22} & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \hat{\Sigma}_{33} & 0 & X\mathcal{A}_d^t & X\tilde{G}_d^t \\ \bullet & \bullet & \bullet & \hat{\Sigma}_{44} & \widetilde{\Gamma}^t & \tilde{\Phi}^t \\ \bullet & \bullet & \bullet & \bullet & -2X + \bar{R} & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -I \end{bmatrix} < 0, \tag{4.32}$$

where

$$\begin{aligned} &\hat{\Sigma}_{11} = \mathcal{A}X + X\mathcal{A}^t + \bar{S} + \bar{W} + \bar{R}, \quad \bar{\Sigma}_{12} = \bar{R}, \quad \hat{\Sigma}_{13} = \mathcal{A}_d X, \\ &\hat{\Sigma}_{14} = \widehat{\Gamma}, \quad \bar{\Sigma}_{22} = -\bar{R} - \bar{S}, \quad \hat{\Sigma}_{33} = -(1-\mu)\bar{W}, \\ &\hat{\Sigma}_{44} = -\gamma^2 I. \end{aligned} \tag{4.33}$$

Proceeding further, using the bounding inequality A from Sect. 9.3.1 and considering (4.32), Σ_3 can be manipulated into the form

$$\Sigma_3 = \begin{bmatrix} \tilde{\Sigma}_{11} & \tilde{\Sigma}_{12} & \tilde{\Sigma}_{13} & \hat{\Sigma}_{14} & X\bar{A}^t & X\tilde{G}^t & \hat{E} \\ \bullet & \bar{\Sigma}_{22} & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \tilde{\Sigma}_{33} & 0 & X\bar{A}_d{}^t & X\tilde{G}_d^t & 0 \\ \bullet & \bullet & \bullet & \hat{\Sigma}_{44} & \widetilde{\Gamma}^t & \tilde{\Phi}^t & 0 \\ \bullet & \bullet & \bullet & \bullet & -2X + \bar{R} & 0 & \hat{E} \\ \bullet & \bullet & \bullet & \bullet & \bullet & -I & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\varepsilon_1 I \end{bmatrix} < 0,$$

where

$$\begin{aligned}
&\tilde{\Sigma}_{11} = \bar{\mathcal{A}}X + X\bar{\mathcal{A}}^t + \bar{S} + \bar{W} + \bar{R} + \varepsilon_1 \hat{M}^t \hat{M} + +\varepsilon_2 \hat{F}^t \hat{F}, \quad \bar{\Sigma}_{12} = \bar{R},\\
&\hat{\Sigma}_{14} = \tilde{\Gamma}, \quad \bar{\Sigma}_{22} = -\bar{R} - \bar{S}, \quad \tilde{\Sigma}_{33} = -(1-\mu)\bar{W} + \varepsilon_1 \hat{N}^t \hat{N}, \\
&\hat{\Sigma}_{44} = -\gamma^2 I, \quad \bar{\mathcal{A}} = \widetilde{A} + \widetilde{B}\mathcal{K}\widetilde{C}, \quad \bar{\mathcal{A}}_d = \widetilde{A}_d, \quad \tilde{\Sigma}_{13} = \bar{\mathcal{A}}_d X + \varepsilon_1 \hat{M}^t \hat{N}.
\end{aligned} \tag{4.34}$$

Finally, we denote $\bar{\mathcal{A}}X = \widetilde{A}X + \widetilde{B}\mathcal{K}\widetilde{C}X = \widetilde{A}X + Y$, thus $\mathcal{K} = \widetilde{B}^{-1}YX^{-1}\widetilde{C}^{-1}$. So we can get Σ_3 can be changed to formula (4.22) as desired. □

We now demonstrate the results by numerical simulation.

4.2.4 Simulation Example 4.1

Consider a two-channel linear uncertain systems Σ:

$$A = \begin{bmatrix}
1.0 & -1.1 & -2.1 & -1.0 & 1.9 & 2.1 & 0.3 & -2.1\\
2.0 & -4.9 & -1.1 & 0 & 1.2 & 1.1 & 0.2 & -0.6\\
1.9 & -1.1 & -3.1 & -1.0 & 1.9 & 2.1 & 0.1 & -2.0\\
6.8 & -8.9 & -6.9 & -1.0 & 6.9 & 7.1 & 0.3 & -5.9\\
2.1 & -3.9 & -1.1 & 0 & 0.3 & 1.1 & 0.3 & 0.2\\
-2.0 & 6.8 & 3.1 & 0.2 & -6.9 & -2.1 & -0.8 & 1.1\\
2.5 & 4.7 & -0.1 & -1.0 & -3.9 & 2.1 & -2.9 & -2.0\\
-1.10 & 5.9 & 2.1 & 0.3 & -5.9 & -0.1 & -1.1 & 0.1
\end{bmatrix},$$

$$A_d = \begin{bmatrix}
0.1 & -0.1 & -0.3 & 0 & 0.4 & 0.1 & 0 & -0.1\\
0.1 & -0.5 & -0.1 & 0 & -0.2 & 0.1 & 0.2 & -0.2\\
0.1 & -0.1 & -0.1 & -1.0 & -0.5 & 0.1 & 0.1 & -0.1\\
0.2 & -0.7 & 0.9 & -1.0 & -0.3 & 0.1 & 0.3 & -0.4\\
0.1 & -0.8 & 0.1 & 0 & 0.3 & 0.1 & 0.3 & 0.1\\
-0.1 & 0.38 & 0.1 & 0.2 & -0.9 & -0.1 & -0.8 & 0.1\\
0.2 & 0.7 & 0.1 & -1.0 & -0.9 & 0.1 & -0.9 & -1.0\\
-0.1 & 0.6 & -0.1 & 0.3 & -0.8 & -0.1 & -1.1 & 0.1
\end{bmatrix},$$

$$B_1^t = \begin{bmatrix} -1 & 0 & -1 & 1 & 0 & 0 & -1 & 0 \end{bmatrix},$$

$$B_2^t = \begin{bmatrix} 0 & 0 & 1.9 & 0 & 1 & 0.9 & 1 & 0 \end{bmatrix},$$

$$\Gamma = \begin{bmatrix}
0 & 2.8 & 0 & 0\\
0 & -4.0 & 0 & 0\\
0 & 3.1 & 1.1 & 0\\
0 & -3.0 & 0 & 1.9\\
0 & -3.9 & 0 & 0\\
0.9 & 1.0 & 1.1 & -1\\
0.8 & 2.9 & 0 & 0\\
0.9 & 0 & 2.1 & 0
\end{bmatrix},$$

$$G = \begin{bmatrix} -2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -2 & -1 & 1 & 1 & 1 & 0 & -1 \\ -1 & 0 & 2 & 0 & 0 & 0 & -1 & 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} -2 \; 1 \; 3 \; 0 \; -1 \; 0 \; -1 \; 4 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 1 \; 0 \; -1 \; 0 \; 1 \; 1 \; 0 \; -1 \end{bmatrix},$$

$$\Phi = \mathrm{diag}\begin{bmatrix} 0.1 \; 0.3 \; 0.1 \; 0.4 \end{bmatrix}, \qquad F = 0.6,$$

$$G_d = \begin{bmatrix} 0 & 0.1 & 0 & 0.2 & 0 & 0 & 0 & 0 \\ -0.1 & 0.1 & 0 & 0 & -0.2 & 0 & 0.3 & 0 \\ -0.4 & 0 & -0.1 & 0.2 & 0.3 & 0.4 & 0 & -0.1 \\ -0.1 & 0 & 1 & 0 & 0 & 0 & -0.5 & 0 \end{bmatrix},$$

$$E = \begin{bmatrix} 0.5 \; 0 \; -0.3 \; 0.2 \; 0 \; 0 \; 0 \; 0.4 \end{bmatrix},$$

$$M^t = \begin{bmatrix} 0 \; 0.3 \; -0.1 \; 0 \; 0.2 \; 0 \; 0 \; 0.4 \end{bmatrix},$$

$$N^t = \begin{bmatrix} 0.2 \; 0 \; -0.1 \; 0.1 \; 0 \; 0.2 \; 0 \; 0.3 \end{bmatrix},$$

$$H_1 = \begin{bmatrix} 0.3 \; 0.4 \; -0.1 \; 0.1 \; 0 \; 0.2 \; 0 \; 0.1 \end{bmatrix},$$

$$H_2 = \begin{bmatrix} 0.5 \; 0 \; -0.3 \; 0.2 \; 0.3 \; 0 \; 0.1 \; -0.2 \end{bmatrix},$$

$$\mu = 0.8, \qquad \varrho = 2.3.$$

In implementation, we take the dimensions of the local controllers as $s_1 = 3$, $s_2 = 2$. Taking the advantage of the Matlab LMI Control Toolbox to solve the LMIs (4.22), we obtain a feasible solution as follows:

$$\gamma = 2.28,$$

$$\hat{A}_1 = \begin{bmatrix} -16.21 & -19.45 & -6.11 \\ -3.77 & -15.13 & 8.78 \\ 0.86 & -0.93 & 0.59 \end{bmatrix}, \qquad \hat{B}_1 = \begin{bmatrix} -0.23 \\ 0.12 \\ -0.09 \end{bmatrix},$$

$$\hat{C}_1 = \begin{bmatrix} 4.11 \; 7.89 \; -1.39 \end{bmatrix}, \qquad \hat{D}_1 = 0.47,$$

$$\hat{A}_2 = \begin{bmatrix} 2.21 & 3.65 \\ -6.07 & -5.79 \end{bmatrix}, \qquad \hat{B}_2 = \begin{bmatrix} 1.12 \\ -0.13 \end{bmatrix},$$

$$\hat{C}_2 = \begin{bmatrix} 6.94 \; 8.77 \end{bmatrix}, \qquad \hat{D}_2 = 0.56.$$

4.3 Resilient Stabilization of Interconnected Networked Systems

Networked control systems (NCS) are feedback control systems with network channels in the feedback loop. Two main changes in the control system research directions are the explicit considerations of the interconnections and a renewed emphasis on distributed control systems being closely related to decentralized control of complex large scale systems. Though a variety of structures and models in this framework have been analyzed, a gap remains between decentralized control and control over networks. Decentralized NCS (DNCS) are the control systems with multiple control stations while transmitting control signals through a network, i.e. date signals are transmitted to multiple controllers in the feedback loop. DNCS combine the advantages of the centralized NCS and the decentralized control systems. Such a combination enables to cut unnecessary wiring, reduces the complexity and the overall system cost when designing and implementing control systems. Symmetric composite systems arises in very different real world systems such as industrial manipulators, parallel processes, flexible structure, electric power systems, homogeneous interconnected systems such as seismic cables or in the design of reliable control systems. In practice, controllers are implemented imprecisely because of various reasons determined by digital controller properties or the need for additional tuning of parameters. The need to have a certain degree of freedom in the choice of the controller parameters, i.e. the robustness of stability against perturbations in controller parameters, leads to the requirement to include also uncertainties of the controllers in the control design.

4.3.1 Introduction

Recently, the results dealing with the DNCS design methods are rare. Relevant problems are introduced in [4, 6, 7, 9, 36]. Decentralized stabilization of NCS using periodically time varying local controller is presented in [63], while the reference [73] deals with the synchronization within the DNCS design. Stability of the DNCS is analyzed in [35].

It has been customary to confront with several important issues when dealing with the control of interconnected systems. The first issue is concerned with the practical limitations in the number and the structure of the feedback loops, which motivates decentralized control schemes [72]. The second issue regards the presence of uncertainties both in the subsystems and in the interconnections. The third issue is the impact of time-delays among the subsystems and across the coupling links. The fourth issue has to do with the reliability of the control systems against component failures and/or perturbations in the feedback gain matrices. In this section, we study the robust stability and feedback stabilization problems of a class of linear interconnected continuous time-delay systems, which are frequently encountered to describe propagation, transport phenomena and population dynamics in various engineering and physical applications. Large-scale interconnected system appear in a variety of engineering applications including power systems, large structures and manufacturing systems and for those applications, decentralized control schemes present a practical and effective means for designing control algorithms based on the individual subsystems [72]. Relevant research results on decentralized control of relevance to the present work can be found in [37, 63, 73].

It appears from the existing results that general results pertaining to interconnected time-delay systems are few and restricted, see [36, 46–48, 62] where most of the efforts were centered on matching conditions and were virtually delay-independent. A recent effort was reported recently in [20] where a class of uncertain systems with interconnected and feedback delays has been considered. However, the internal time-delay within the subsystems have not been considered and several bounding inequalities have been included.

It has been recently reported in [3] that the theory of large-scale (interconnected) systems is devoted to the problems due to dimensionality, information structure constraints, uncertainty and delays. Resilient (non-fragile) control methods [28] and [52–55] have added new tools to the task of designing appropriate control algorithms to cope with gain parameter perturbations and controller implementations issues and it is interesting to view these tools as robust re-design algorithms [53]. It is crucial to realize that when dealing with several practical problems arising in power systems, manufacturing systems and irrigation systems, the changes in controller structure and settings might degrade the overall system performance. Thus the important role of resilient (non-fragile) controllers with information structure constraints is underlined when considering large-scale systems [3].

This section develops a resilient decentralized $\mathcal{H}_\infty$ observer-based setting using the reduced-order control design when considering the delay-dependent approach within the framework of the LMIs. An a technical outcome, we develop robust decentralized delay-dependent stability and resilient feedback stabilization methods for a class of linear interconnected continuous-time systems. The subsystems are subjected to convex-bounded parametric uncertainties while time-varying delays occur within the local subsystems and across the interconnections and additive feedback gain perturbations are allowed. In this way, our control design offers decentralized structure and possesses robustness with respect to both parametric uncertainties and gain perturbations. For related results on resilient control, the reader is referred to [53, 54] where it is shown to provide a framework of extended robustness properties.

4.3.2 Problem Formulation

We consider a class of linear systems with unknown nonlinearities **S** of the form:

$$\begin{aligned}\dot{x}(t) &= A_\Delta x(t) + B_o u(t) + \Gamma_\Delta w(t) + c(t,x)\\ &= [A_o + \Delta A]x(t) + B_o u(t) + [\Gamma_o + \Delta\Gamma]w(t) + c(t,x(t)), \qquad (4.35)\end{aligned}$$

$$\begin{aligned}z(t) &= [G_o + \Delta G]x(t) + [\Phi_o + \Delta\Phi]w(t)\\ &= G_\Delta x(t) + \Phi_\Delta w(t), \qquad (4.36)\\ y(t) &= C_o x(t),\end{aligned}$$

where $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^m$ is the control input, $y(t) \in \Re^p$ is the measured output, $w(t) \in \Re^q$ is the disturbance input which belongs to $\mathcal{L}_2[0,\infty)$

and $z(t) \in \Re^q$ is the performance output. The unknown nonlinearities $c(t, x(t))$ are piecewise continuous functions to be specified at the subsystem level. In what follows, we view **S** as structurally composed of n_s coupled subsystems $\mathbf{S}_j$ and modeled by:

$$\dot{x}_j(t) = A_{j\Delta} x_j(t) + B_{jo} u_j(t) + \Gamma_{j\Delta} w_j(t) + \sum_{k=1}^{n_s} F_{jk\Delta} x_k(t) + c_j(t, x), \quad (4.37)$$

$$\begin{aligned} z_j(t) &= G_{j\Delta} x_j(t) + \Phi_{j\Delta} w_j(t), \\ y_j(t) &= C_{jo} x_j(t), \quad j = 1, \ldots, n_s, \\ n = \sum_{j=1}^{n_s} n_j, \quad & m = \sum_{j=1}^{n_s} m_j, \quad p = \sum_{j=1}^{n_s} p_j, \quad q = \sum_{j=1}^{n_s} n_j, \end{aligned} \quad (4.38)$$

where the unknown nonlinearities $c_j(t, x(t))$ are piecewise continuous functions satisfying the global Lipschitz conditions for all $c_j(0, x(0))$ as follows

$$\|c_j(t, x_1(t)) - c_j(t, x_2(t))\| \le \|E_j(x_1(t) - x_2(t))\|, \quad \forall t \ge 0, \quad (4.39)$$

where E_j is a prescribed constant. We further suppose that the structure of the nonlinearities $c_j(t, x(t))$ is in the form

$$c_j(t, x(t)) = e(t, x_j) E x_j(t), \quad e(t, x_j) : \Re^{n+1} \to [-1, 1].$$

The link between the overall system **S** and the collection of subsystems $\mathbf{S}_j$ is provided by

$$\begin{aligned} A_o &= \begin{bmatrix} A_{1o} & F_{12o} & \ldots & F_{1n_so} \\ \vdots & \vdots & \vdots & \vdots \\ F_{n_s1o} & F_{n_s2o} & \ldots & A_{n_so} \end{bmatrix}, \quad B_o = \mathrm{diag}\left[B_{1o}\ B_{2o} \ldots B_{n_so} \right], \\ \Delta A &= \begin{bmatrix} \Delta A_1 & \Delta F_{12} & \ldots & \Delta F_{1n_s} \\ \vdots & \vdots & \vdots & \vdots \\ \Delta F_{n_s1} & \Delta F_{n_s2} & \ldots & \Delta A_{n_s} \end{bmatrix}, \quad C_o = \mathrm{diag}\left[C_{1o}\ C_{2o} \ldots C_{n_so} \right], \\ \Phi_o &= \mathrm{diag}\left[\Phi_{1o}\ \Phi_{2o} \ldots \Phi_{n_so} \right], \quad \Gamma_o = \mathrm{diag}\left[\Gamma_{1o}\ \Gamma_{2o} \ldots \Gamma_{n_so} \right], \\ \Delta\Phi &= \mathrm{diag}\left[\Delta\Phi_1\ \Delta\Phi_2 \ldots \Delta\Phi_{n_s} \right], \quad \Delta\Gamma = \mathrm{diag}\left[\Delta\Gamma_1\ \Delta\Gamma_2 \ldots \Delta\Gamma_{n_s} \right], \\ G_o &= \mathrm{diag}\left[G_{1o}\ G_{2o} \ldots G_{n_so} \right], \quad \Delta G = \mathrm{diag}\left[\Delta G_1\ \Delta G_2 \ldots \Delta G_{n_s} \right]. \end{aligned} \quad (4.40)$$

At the subsystem level, the associated matrices contain parametric uncertainties of the form

$$\begin{bmatrix} A_{j\Delta} & \Gamma_{j\Delta} \\ G_{j\Delta} & \Phi_{j\Delta} \end{bmatrix} = \begin{bmatrix} A_{jo} & \Gamma_{jo} \\ G_{dj} & \Phi_{jo} \end{bmatrix} + \begin{bmatrix} H_{jo} \\ H_{ja} \end{bmatrix} \Delta_{jo}(t) \left[E_{ja}\ E_{jc} \right], \quad (4.41)$$

$$F_{jk\Delta} = F_{jko} + H_{jc} \Delta_{ja}(t) E_{js}, \quad (4.42)$$

where the unknown nonlinearities are bounded in the form

$$c_j^t(t)c_j(t) \le \alpha_j x_j^t(t) E_j^t E_j x_j(t), \quad j = 1, \dots, n_s, \tag{4.43}$$

where α_j are adjustable parameters and the matrices $F_j \in \Re^{n_j \times n_j}$ are real and constant. For $j = 1, \dots, n_s$, $H_{oj}, \dots, E_{sj}$ are known real constant matrices and Δ_{jo}, Δ_{ja} are unknown time-varying real matrices of appropriate dimensions with Lebesgue measurable elements satisfying $\Delta_{jo}^t \Delta_{jo} \le I$, $\Delta_{ja}^t \Delta_{ja} \le I$.

The matrices $A_{jo} \in \Re^{n_j \times n_j}$, $B_{jo} \in \Re^{n_j \times m_j}$, $\Phi_{jo} \in \Re^{q_j \times q_j}$, $\Gamma_{jo} \in \Re^{n_j \times q_j}$, $C_{jo} \in \Re^{p_j \times n_j}$, $G_{jo} \in \Re^{q_j \times n_j}$, $F_{jko} \in \Re^{n_j \times n_k}$ are real and constants. The initial condition $x_j(0) = \phi_{jo} \in \mathcal{L}_2[-\tau_j^*, 0]$, $j \in \{1, \dots, n_s\}$. The constant matrices $A_{jo}, \dots, F_{jko}$ define the nominal state-space model

$$\dot{x}_j(t) = A_{jo} x_j(t) + B_{jo} u_j(t) + \Gamma_{jo} w_j(t) + \sum_{k=1}^{n_s} F_{jko} x_k(t) + c_j, \tag{4.44}$$

$$\begin{aligned} z_j(t) &= G_{jo} x_j(t) + \Phi_{jo} w_j(t), \\ y_j(t) &= C_{jo} x_j(t), \quad j = 1, \dots, n_s, \end{aligned} \tag{4.45}$$

where in uncertain system (4.37)–(4.38) and nominal system (4.44)–(4.45), $x_j(t) \in \Re^{n_j}$ is the state vector, $u_j(t) \in \Re^{m_j}$ is the control input, $y_j(t) \in \Re^{p_j}$ is the measured output, $w_j(t) \in \Re^{q_j}$ is the disturbance input which belongs to $\mathcal{L}_2[0, \infty)$ and $z_j(t) \in \Re^{q_j}$ is the performance output.

The class of systems described by (4.45) is frequently encountered in modeling several physical systems and engineering applications including large space structures, multi-machine power systems, cold mills, transportation systems, water pollution management, to name a few [48, 72].

4.3.3 Resilient Observer-Based Control

In most of the cases, not all subsystem states are available for measurements, we seek a decentralized dynamic output-feedback control using subsystem observers within the network feedback. Consider that one controller-actuator node with a buffer storing the latest sensor signal at the subsystem level. It is customary that new sensor data are compared with the latest data. If a new signal reaches the controller-actuator node, then it is used to compute the control signal, else it is discarded. This yields in a networked resilient observer controller in the form

$$\begin{aligned} \dot{\hat{x}}_j(t) &= A_{jo} \hat{x}_j(t) + B_{jo} u_j(t) + L_{j\Delta}(\bar{y}_j(t_k) - C_{jo} \bar{x}_j(t_k)), \\ u_j(t) &= K_{j\Delta} \hat{x}_j(t), \quad t \in [t_k, t_{k+1}),\ k = 1, 2, \dots, \\ L_{j\Delta} &= L_{jo} + \Delta L_j, \qquad K_{j\Delta} = K_{jo} + \Delta K_j, \end{aligned} \tag{4.46}$$

where $\hat{x}_j(t) \in \Re^{n_j}$ is the observer state of subsystem and the matrices L_{jo}, K_{jo} are the nominal observer gain and the controller feedback gain matrices, respectively. In addition ΔL_j, ΔK_j are additive observer and controller gain matrix uncertainties given by

$$\Delta L_j = N_{jo}\Delta_{tj}(t)M_{jo}, \qquad \Delta K_j = N_{jc}\Delta_{zj}(t)M_{jc}, \tag{4.47}$$

where for $j = 1, \ldots, n_s$, $N_{jo}, \ \ldots, \ M_{jc}$ are known real constant matrices and Δ_{tj}, Δ_{zj} are unknown time-varying real matrices of appropriate dimensions with Lebesgue measurable elements satisfying $\Delta^t_{tj}\Delta_{tj} \le I$, $\Delta^t_{zj}\Delta_{zj} \le I$.

We note in (4.46) that $t_k = k\Delta$, $k > 0$ denotes a sampling instant, Δ is the sampling period and k is an integer. The sampler is equipped with a standard zero order hold in the feedback. The sampled value $y_j(t_k)$ of the output $y_j(t)$ is transmitted through a network channel and the successfully transmitted value is registered in a buffer with $\bar{y}_j(t_k)$ being the output from the buffer and simultaneously represents the input to the observer. Also, $\bar{x}_j(t_k)$ is the observer state copying the whole set of dropped packets appearing in the transmission of $\bar{y}_j(t_k)$. Note that $t_{k+1} \ge t_k + 1$, $k = 1, 2, \ldots$ which corresponds to data packet dropout registered by a buffer and $\bar{y}_j(t_k) = y_j(t_k - \tau_k\Delta - \tau_c)$ where $\tau_k\Delta$ indicates the data packet dropout and τ_c is the network-induced delay. This motivates defining the new time-varying delay $\theta(t) = t - t_k - \tau_k\Delta - \tau_c$ where $1 \le \tau_k \le (t_{k-1} - \tau_{cj})/\Delta$. In the sequel, we consider the number of data packet dropouts to be bounded so that, including the network-induced delays for each subsystem, it satisfies the constraint

$$\theta_m \le \theta(t) \le \theta_M, \tag{4.48}$$

where $\theta_m > 0$, $\theta_M > 0$ are given constants. Therefore, controller (4.46) can be rewritten as

$$\begin{aligned}
\dot{\hat{x}}_j(t) &= A_{jo}\hat{x}_j(t) + B_{jo}u_j(t) + L_{j\Delta}(y_j(t-\theta(t)) - C_{jo}\hat{x}_j(t-\theta(t))),\\
u_j(t) &= K_{j\Delta}\hat{x}_j(t),\\
\hat{x}_j(t) &= 0, \quad t \in [-\theta_M, 0],
\end{aligned} \tag{4.49}$$

while the overall decentralized observer-based controller can be expressed as

$$\begin{aligned}
\dot{\hat{x}}(t) &= A_c\hat{x}(t) + B_o u(t) + (L_o + \Delta L_o)(y(t-\theta(t)) - C_o\hat{x}(t-\theta(t))),\\
u(t) &= (K_o + \Delta K_o)\hat{x}(t),\\
\hat{x}(t) &= 0, \quad t \in [-\theta_M, 0]
\end{aligned} \tag{4.50}$$

with

$$\begin{aligned}
K_o &= \mathrm{diag}\left[\, K_{1o}\ K_{2o}\ \ldots\ K_{n_so} \,\right], \qquad L_o = \mathrm{diag}\left[\, L_{1o}\ L_{2o}\ \ldots\ L_{n_so} \,\right],\\
\Delta K_o &= \mathrm{diag}\left[\, \Delta K_{1o}\ \Delta K_{2o}\ \ldots\ \Delta K_{n_so} \,\right],\\
\Delta L_o &= \mathrm{diag}\left[\, \Delta L_{1o}\ \Delta L_{2o}\ \ldots\ \Delta L_{n_so} \,\right].
\end{aligned} \tag{4.51}$$

Remark 4.1 From the published results on networked-control systems, we note that a single packet transmission is supposed in the feedback loop. It means that in each transmission every control station receives only one packet through the network. It can be understood as multiple data packets simultaneous transmission through parallel network channels, where each channel generally corresponds with a local feedback loop in the DNCS with individual time-varying delays. The availability of Acknowledgement (ACK) about data losses to the sender as well as the communication logics considering dropouts in all local channels if the dropout appears in any local channel. Then, only a single identical time-varying delay can be applied for any channel. It can be considered as a single communication channel with data packet dropouts and communication delays connected within a block diagonal structure of the gain matrix, that is, the sensor-actuator pair structure in the NCS. Such a network feedback architecture enables essential simplification of the DNCS design for the considered class of composite systems. The information structure constraints on only sensor-actuator pairs in the gain matrices is sufficiently justified for symmetric composite systems. Much higher reliability of subsystems than that of the interconnections, an essential simplification of the DNCS design using LMIs, and the design requirement to keep the symmetry in the closed-loop system lead to the preference of decentralized control.

4.3.4 Augmented Closed-Loop System

Define the subsystem error vector $e_j(t) = x_j(t) - \hat{x}_j(t)$ and the corresponding augmented vector $\xi_j(t) = [x_j^t(t)\ e_j^t(t)]^t$. Using (4.37) and (4.46) with some manipulations, we obtain the augmented model as

$$\dot{\xi}_j(t) = \mathsf{A}_{j\Delta}\xi_j(t) + \mathsf{D}_{j\Delta}\xi_j(t-\theta(t)) + \widehat{\Gamma}_{j\Delta}w_j(t) + \sum_{k=1}^{n_s}\mathsf{F}_{jk\Delta}\xi_k(t) + \mathcal{C}_j, \tag{4.52}$$

$$\begin{aligned} z_j(t) &= \mathsf{G}_{j\Delta}\xi_j(t) + \Phi_{j\Delta}w_j(t), \\ y_j(t) &= \mathsf{C}_o\xi_j(t), \end{aligned} \tag{4.53}$$

$$\begin{aligned} \mathsf{A}_{j\Delta} &= \mathsf{A}_{jo} + \Delta\mathsf{A}_j, \qquad \mathsf{D}_{j\Delta} = \mathsf{D}_{jo} + \Delta\mathsf{D}_j, \qquad \Phi_{j\Delta} = \Phi_{jo} + \Delta\Phi_j, \\ \widehat{\Gamma}_{j\Delta} &= \widehat{\Gamma}_j + \Delta\widehat{\Gamma}_j, \qquad \mathsf{G}_{j\Delta} = \mathsf{G}_{jo} + \Delta\mathsf{G}_j, \qquad \mathsf{F}_{jk\Delta} = \mathsf{F}_{jko} + \Delta\mathsf{F}_{jk}, \end{aligned} \tag{4.54}$$

where

$$\begin{aligned} \mathsf{A}_{jo} &= \begin{bmatrix} A_{jo} + B_{jo}K_{jo} & -B_{jo}K_{jo} \\ 0 & A_{jo} \end{bmatrix}, \quad \mathsf{D}_{jo} = \begin{bmatrix} 0 & 0 \\ 0 & -L_{jo}C_{jo} \end{bmatrix}, \\ \widehat{\Gamma}_j &= \begin{bmatrix} \Gamma_{jo} \\ \Gamma_{jo} \end{bmatrix}, \\ \Delta\mathsf{A}_j &= \begin{bmatrix} \Delta A_j + B_{jo}\Delta K_j & -B_{jo}\Delta K_j \\ \Delta A_j & 0 \end{bmatrix} = \begin{bmatrix} H_{jo} \\ H_{jo} \end{bmatrix} \Delta_{jo}(t) \begin{bmatrix} E_{ja} & 0 \end{bmatrix} \end{aligned} \tag{4.55}$$

$$
\begin{aligned}
& + \begin{bmatrix} B_{jo}N_{jc} \\ 0 \end{bmatrix} \Delta_{zj}(t) \begin{bmatrix} M_{jc} & 0 \end{bmatrix} = \hat{H}_{jo}\Delta_{jo}(t)\hat{E}_{ja} + \hat{N}_{jc}\Delta_{zj}(t)\hat{M}_{jc}, \\
\Delta\widehat{\Gamma}_j &= \begin{bmatrix} \Delta\Gamma_j \\ \Delta\Gamma_j \end{bmatrix} = \begin{bmatrix} H_{jo} \\ H_{jo} \end{bmatrix} \Delta_{jo}(t)E_{jc} = \hat{H}_{jo}\Delta_{jo}(t)E_{jc}, \quad \mathsf{C}_{jo} = \begin{bmatrix} C_{jo} & 0 \end{bmatrix}, \\
\Delta\mathsf{D}_j &= \begin{bmatrix} 0 & 0 \\ 0 & -\Delta L_j C_{jo} \end{bmatrix} = \begin{bmatrix} 0 \\ -N_{jo} \end{bmatrix} \Delta_{tj}(t) \begin{bmatrix} 0 & M_{jo}C_{jo} \end{bmatrix} = \hat{N}_{jo}\Delta_{tj}(t)\hat{M}_{jo}, \\
\mathsf{G}_{jo} &= \begin{bmatrix} G_{jo} & 0 \end{bmatrix}, \quad \Delta\mathsf{G}_j = \begin{bmatrix} \Delta G_j & 0 \end{bmatrix} = \begin{bmatrix} H_{ja} & 0 \end{bmatrix} \Delta_{jo}(t)E_{ja} = \tilde{H}_{ja}\Delta_{jo}(t)E_{ja}, \\
\mathsf{F}_{jko} &= \begin{bmatrix} F_{jko} & 0 \\ F_{jko} & 0 \end{bmatrix}, \quad \mathcal{C}_j = \begin{bmatrix} c_j \\ c_j \end{bmatrix}, \\
\Delta\mathsf{F}_{jk} &= \begin{bmatrix} H_{jc} \\ H_{jc} \end{bmatrix} \Delta_{ja}(t) \begin{bmatrix} E_{js} & 0 \end{bmatrix} = \tilde{H}_{jk}\Delta_{ja}(t)\tilde{E}_{jk}.
\end{aligned}
\tag{4.56}
$$

Our objective in this section is to study two main problems: the first problem is the decentralized delay-dependent asymptotic stability by deriving a feasibility testing at the subsystem level so as to guarantee the overall system asymptotic stability. The second problem deals with the resilient decentralized stabilization by developing state-feedback controllers that takes into consideration additive gain perturbations while ensuring that the overall closed-loop system is delay-dependent asymptotically stable.

4.3.5 Delay-Dependent Subsystem Stability

In what follows, we develop new criteria for LMI-based characterization of delay-dependent asymptotic stability and $\mathcal{L}_2$ gain analysis which requires only subsystem information thereby assuring decentralization. The criteria includes some parameter matrices aims at expanding the range of applicability of the developed conditions. We consider the Lyapunov-Krasovskii functional (LKF):

$$
\begin{aligned}
V(t) &= \sum_{j=1}^{n_s} V_j(t), \\
V_j(t) &= \xi_j^t(t)\mathcal{P}_j\xi_j(t) + \int_{-\theta_M}^{0} \xi_j^t(\alpha)\mathcal{W}_j\xi_j(\alpha)d\alpha + \int_{-\theta}^{0} \xi_j^t(\alpha)\mathcal{S}_j\xi_j(\alpha)d\alpha \\
&\quad + \theta_M \int_{-\theta_M}^{0} \int_{t+\sigma}^{t} \dot{\xi}_j^t(s)\mathcal{R}_j\dot{\xi}_j(s)\,ds\,d\sigma,
\end{aligned}
\tag{4.57}
$$

where $0 < \mathcal{P}_j = \mathcal{P}_j^t$, $0 < \mathcal{W}_j = \mathcal{W}_j^t$, $0 < \mathcal{S}_j = \mathcal{S}_j^t$, $0 < \mathcal{R}_j = \mathcal{R}_j^t$, $j \in \{1, \ldots, n_s\}$ are weighting matrices of appropriate dimensions. Introducing the matrices and vector quantities

$$\hat{Q}_j = \begin{bmatrix} (\mathcal{P}_j + \Theta_j)\hat{H}_{jo} & (\mathcal{P}_j + \Theta_j)\hat{N}_{jc} & \Theta_j \hat{N}_{jo} & 0 & 0 & 0 & \phi & 0 \\ \Upsilon_j \hat{H}_{jo} & \Upsilon_j \hat{N}_{jc} & \Upsilon_j \hat{N}_{jo} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{H} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{H} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{H} & 0 & 0 \end{bmatrix},$$

where $\mathcal{H} = \sum_{k=1,k\neq j}^{n_s} \tilde{H}_{kj}$, $\phi = (\mathcal{P}_j + \Theta_j + \Upsilon_j)\hat{H}_{jo}$,

$$\hat{T}_j = \begin{bmatrix} \sigma_1 \hat{E}^t_{jo} & \sigma_1 \hat{M}^t_{jc} & 0 & \sigma_3 \hat{E}^t_{kj} & \sigma_3 \hat{E}^t_{kj} & \sigma_3 \hat{E}^t_{kj} & 0 & \sigma_5 E^t_{ja} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_2 \hat{M}^t_{jo} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Lambda_j = \mathrm{col}[(\mathcal{P}_j + \Theta_j + \Upsilon_j)\widehat{\Gamma}_j,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0],$$

$$\widehat{\mathsf{G}}^t_{jo} = [\mathsf{G}_{jo},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0]^t.$$

for some scalars $\sigma_1 > 0, \ldots, \sigma_6 > 0$ and free-weighting matrices Θ_j, Υ_j, $j = 1, \ldots, n_s$. The following theorems establishes the main design result for subsystem $\mathbf{S}_j$.

Theorem 4.2 *Given the bounds $\theta_m > 0$, $\theta_M > 0$, the family of subsystems described by* (4.52)–(4.56) *is robustly delay-dependent asymptotically stable with $\mathcal{L}_2$-performance bound γ_j if there exist positive-definite matrices $\mathcal{P}_j, \mathcal{W}_j, \mathcal{S}_j, \mathcal{R}_j$, free-weighting matrices Θ_j, Υ_j and scalars $\sigma_1 > 0, \ \ldots, \ \sigma_6 > 0$ satisfying the following LMIs for $j, k = 1, \ldots, n_s$*

$$\Pi_j = \begin{bmatrix} \Psi_{jo} & \Lambda_j & \widehat{\mathsf{G}}^t_{jo} & \widehat{Q}_j & \widehat{T}_j \\ \bullet & -\gamma_j^2 I_j & \Phi^t_{jo} & 0 & 0 \\ \bullet & \bullet & -I_j & 0 & 0 \\ \bullet & \bullet & \bullet & -\Sigma_j & 0 \\ \bullet & \bullet & \bullet & \bullet & -\Sigma_j \end{bmatrix} < 0, \qquad (4.58)$$

where

$$\Psi_{jo} = \begin{bmatrix} \Psi_{ajo} & \Psi_{1jo} & 0 & \Psi_{2jo} & \Psi_{3j} & \Psi_{sjo} \\ \bullet & -\Psi_{cj} & 0 & \Psi_{4jo} & \Psi_{5j} & 0 \\ \bullet & \bullet & -\Psi_{mj} & \Psi_{6j} & 0 & 0 \\ \bullet & \bullet & \bullet & -\Psi_{nj} & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -I_j & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\Psi_{7j} \end{bmatrix}, \tag{4.59}$$

$$\begin{aligned} \Psi_{ajo} &= (\mathcal{P}_j + \Theta_j)\mathsf{A}_{jo} + \mathsf{A}^t_{jo}(\mathcal{P}_j + \Theta_j) + \mathcal{W}_j + \mathcal{S}_j - \mathcal{R}_j \\ &\quad + (n_s - 1)(\mathcal{P}_j + \Theta_j + \Upsilon_j), \\ \Psi_{1jo} &= -\Theta_j + \mathsf{A}^t_{jo}\Upsilon_j, \quad \Psi_{5j} = \Upsilon_j, \quad \Psi_{2jo} = \Theta_j \mathsf{D}_{jo} + \mathcal{R}_j, \\ \Psi_{3j} &= \mathcal{P}_j + \Theta_j, \quad \Psi_{cj} = -\theta_M^2 \mathcal{R}_j + \Upsilon_j + \Upsilon_j^t, \\ \Psi_{7j} &= \mathrm{diag}\left[\mathcal{P}_k^{-1} \; \Theta_k^{-1} \; \Upsilon_k^{-1} \right], \\ \Psi_{4jo} &= \Upsilon_j \mathsf{D}_{jo}, \quad \Psi_{mj} = \mathcal{R}_j + \mathcal{W}_j, \quad \Pi_{6j} = \mathcal{R}_j, \quad \Psi_{nj} = 2\mathcal{R}_j + \mathcal{S}_j, \\ \Psi_{sjo} &= \left[\sum_{k=1, k\neq j}^{n_s} \mathsf{F}^t_{kjo} \; \sum_{k=1, k\neq j}^{n_s} \mathsf{F}^t_{kjo} \; \sum_{k=1, k\neq j}^{n_s} \mathsf{F}^t_{kjo} \right]. \end{aligned} \tag{4.60}$$

Proof A straightforward computation gives the time-derivative of $V_j(t)$ along the solutions of (4.53) with $w(t) \equiv 0$ as:

$$\begin{aligned} \dot{V}_j(t) &= 2\xi_j^t(t)\mathcal{P}_j\dot{\xi}_j(t) + \theta_M^2 \dot{\xi}_j^t(t)\mathcal{R}_j\dot{\xi}_j(t) \\ &\quad - \theta_M \int_{t-\theta_M}^{t} \dot{\xi}_j^t(s)\mathcal{R}_j\dot{\xi}_j(s)ds + \xi_j^t(t)(\mathcal{W}_j + \mathcal{S}_j)\xi_j(t) \\ &\quad - \xi_j^t(t-\theta_M)\mathcal{W}_j\xi_j(t-\theta_M) - \xi_j^t(t-\theta)\mathcal{S}_j\xi_j(t-\theta). \end{aligned} \tag{4.61}$$

Initially, we use the identity

$$\begin{aligned} -\theta_M \int_{-\theta_M}^{0} \dot{\xi}_j^t(s)\mathcal{R}_j\dot{\xi}_j(s)ds &= -\theta_M \int_{t-\theta_M}^{t-\theta} \dot{\xi}_j^t(s)\mathcal{R}_j\dot{\xi}_j(s)ds \\ &\quad - \theta_M \int_{t-\theta}^{t} \dot{\xi}_j^t(s)\mathcal{R}_j\dot{\xi}_j(s)ds. \end{aligned} \tag{4.62}$$

Then apply Jensen's inequality

$$\int_{t-\theta}^{t} \dot{\xi}_j^t(s)\mathcal{R}_j\dot{\xi}_j(s)ds \geq \int_{t-\theta}^{t} \dot{\xi}_j^t(s)ds\mathcal{R}_j \int_{t-\theta}^{t} \dot{\xi}_j(s)ds, \tag{4.63}$$

$$\int_{t-\theta_M}^{t-\theta} \dot{\xi}_j^t(s)\mathcal{R}_j\dot{x}_j(s)ds \geq \int_{t-\theta_M}^{t-\theta} \dot{\xi}_j^t(s)ds\mathcal{R}_j \int_{t-\theta_M}^{t-\theta} \dot{\xi}_j(s)ds. \tag{4.64}$$

By the structural identity

$$\sum_{k=1,k\neq j}^{n_s}\sum_{j=1}^{n_s}\xi_k^t(t)F_{jk\Delta}\xi_k(t)=\sum_{k=1,k\neq j}^{n_s}\sum_{j=1}^{n_s}\xi_j^t(t)F_{kj\Delta}\xi_j(t) \tag{4.65}$$

while invoking the algebraic inequality $X^tZ+Z^tX\leq X^tYX+Z^tY^{-1}Z,\ Y>0$, such that

$$\begin{aligned}
&2\xi_j^t(t)\mathcal{P}_j\sum_{k=1,k\neq j}^{n_s}\mathsf{F}_{jk\Delta}\xi_k(t)\\
&\leq(n_s-1)\xi_j^t(t)\mathcal{P}_j\xi_j(t)+\sum_{k=1,k\neq j}^{n_s}\xi_k^t(t)\mathsf{F}_{jk\Delta}^t\mathcal{P}_j\mathsf{F}_{jk\Delta}\xi_k(t)\\
&\leq(n_s-1)\xi_j^t(t)\mathcal{P}_j\xi_j(t)\\
&\quad+\left(\sum_{k=1,k\neq j}^{n_s}\xi_k^t(t)\mathsf{F}_{jk\Delta}^t\mathcal{P}_j\right)\mathcal{P}_j^{-1}\left(\mathcal{P}_j\sum_{k=1,k\neq j}^{n_s}\mathsf{F}_{jk\Delta}\xi_k(t)\right),
\end{aligned} \tag{4.66}$$

it follows finally that

$$\begin{aligned}
\dot V(t)\leq\sum_{j=1}^{n_s}\Bigg[&2\xi_j^t(t)\mathcal{P}_j\big([\mathsf{A}_{jo}+\Delta\mathsf{A}_j]\xi_j(t)+[\mathsf{D}_{jo}+\Delta\mathsf{D}_j]\xi_j(t-\theta)+\mathcal{C}_j\big)\\
&+\theta_M^2\dot\xi_j^t(t)\mathcal{R}_j\dot\xi_j(t)-(\xi_j(t)-\xi_j(t-\theta))^t\mathcal{R}_j(\xi_j(t)-\xi_j(t-\theta))\\
&-(\xi_j(t-\theta)-\xi_j(t-\theta_M))^t\mathcal{R}_j(\xi_j(t-\theta)-\xi_j(t-\theta_M))\\
&+\xi_j^t(t)(\mathcal{W}_j+\mathcal{S}_j)\xi_j(t)-\xi_j^t(t-\theta_M)\mathcal{W}_jx_j(t-\theta_M)\\
&-\xi_j^t(t-\theta)\mathcal{S}_jx_j(t-\theta)+(n_s-1)\xi_j^t(t)\mathcal{P}_j\xi_j(t)\\
&+\left(\sum_{k=1,k\neq j}^{n_s}\xi_k^t(t)\mathsf{F}_{jk\Delta}^t\mathcal{P}_j\right)\mathcal{P}_j^{-1}\left(\mathcal{P}_j\sum_{k=1,k\neq j}^{n_s}\mathsf{F}_{jk\Delta}\xi_k(t)\right)\Bigg].
\end{aligned} \tag{4.67}$$

Now by adding the zero-value expression

$$\begin{aligned}
0\equiv 2[\xi_j^t(t)\Theta_j+\dot\xi_j^t(t)\Upsilon_j]\Bigg[&-\dot\xi_j(t)+[\mathsf{A}_{jo}+\Delta\mathsf{A}_j]\xi_j(t)+[\mathsf{D}_{jo}+\Delta\mathsf{D}_j]\xi_j(t-\theta)\\
&+\sum_{k=1}^{n_s}\mathsf{F}_{jk\Delta}\xi_k(t)+\mathcal{C}_j\Bigg]
\end{aligned} \tag{4.68}$$

to the right-hand side of (4.67) and setting

$$\zeta_j(t)=\mathrm{col}\{\xi_j(t),\dot\xi_j(t),\xi_j(t-\theta_M),\xi_j(t-\theta),\mathcal{C}_j\}$$

while invoking

$$\begin{aligned}
&2\xi_j^t(t)\Theta_j \sum_{k=1,k\neq j}^{n_s} \mathsf{F}_{jk\Delta}\xi_k(t) \\
&\leq (n_s-1)\xi_j^t(t)\Theta_j\xi_j(t) + \sum_{k=1,k\neq j}^{n_s} \xi_k^t(t)\mathsf{F}_{jk\Delta}^t\Theta_j\mathsf{F}_{jk\Delta}\xi_k(t) \\
&\leq (n_s-1)\xi_j^t(t)\Theta_j\xi_j(t) \\
&\quad + \left(\sum_{k=1,k\neq j}^{n_s} \xi_k^t(t)\mathsf{F}_{jk\Delta}^t\Theta_j\right)\Theta_j^{-1}\left(\Theta_j \sum_{k=1,k\neq j}^{n_s} \mathsf{F}_{jk\Delta}\xi_k(t)\right),
\end{aligned} \tag{4.69}$$

$$\begin{aligned}
&2\xi_j^t(t)\Upsilon_j \sum_{k=1,k\neq j}^{n_s} \mathsf{F}_{jk\Delta}\xi_k(t) \\
&\leq (n_s-1)\xi_j^t(t)\Upsilon_j\xi_j(t) + \sum_{k=1,k\neq j}^{n_s} \xi_k^t(t)\mathsf{F}_{jk\Delta}^t\Upsilon_j\mathsf{F}_{jk\Delta}\xi_k(t) \\
&\leq (n_s-1)\xi_j^t(t)\Upsilon_j\xi_j(t) \\
&\quad + \left(\sum_{k=1,k\neq j}^{n_s} \xi_k^t(t)\mathsf{F}_{jk\Delta}^t\Upsilon_j\right)\Upsilon_j^{-1}\left(\Upsilon_j \sum_{k=1,k\neq j}^{n_s} \mathsf{F}_{jk\Delta}\xi_k(t)\right)
\end{aligned} \tag{4.70}$$

it follows finally that

$$\dot{V}(t) \leq \sum_{j=1}^{n_s} \zeta_j^t(t)\Psi_{j\Delta}\zeta_j(t) \leq 0 \tag{4.71}$$

if the matrix Ψ_j is feasible, where $\Psi_{j\Delta} = \Psi_{jo} + \Delta\Psi_j$ with Ψ_{jo} is the nominal part of $\Psi_{j\Delta}$ by setting $\Delta_{oj} \equiv 0$, $\Delta_{aj} \equiv 0$, $\Delta_{tj} \equiv 0$ and $\Delta_{zj} \equiv 0$ as given by (4.59) and $\Delta\Psi_j$ is given by

$$\Delta\Psi_j = \begin{bmatrix} \Delta\Psi_{aj} & \Delta\Psi_{1j} & 0 & \Delta\Psi_{2j} & 0 & \Delta\Psi_{sj} \\ \bullet & 0 & 0 & \Delta\Psi_{4j} & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & 0 \end{bmatrix}, \tag{4.72}$$

$$
\begin{aligned}
\Delta\Psi_{aj} &= (\mathcal{P}_j + \Theta_j)(\hat{H}_{jo}\Delta_{jo}(t)\hat{E}_{jo} + \hat{N}_{jc}\Delta_{zj}(t)\hat{M}_{jc}) \\
&\quad + (\hat{E}^t_{jo}\Delta^t_{jo}(t)\hat{H}^t_{jo} + \hat{M}^t_{jc}\Delta^t_{zj}(t)\hat{N}^t_{jc})(\mathcal{P}_j + \Theta_j), \\
\Delta\Psi_{1j} &= \hat{E}^t_{jo}\Delta^t_{jo}(t)\hat{H}^t_{jo}\Upsilon_j + \hat{M}^t_{jc}\Delta^t_{zj}(t)\hat{N}^t_{jc}\Upsilon_j, \\
\Delta\Psi_{2j} &= \Theta_j\hat{N}_{jo}\Delta_{tj}(t)\hat{M}_{jo}, \\
\Delta\Psi_{4j} &= \Upsilon_j\hat{N}^t_{jo}\Delta^t_{tj}(t)\hat{M}^t_{jo}, \\
\Delta\Psi_{sj} &= \left[\sum_{k=1,k\neq j}^{n_s}\tilde{E}^t_{kj}\Delta^t_{ka}(t)\tilde{H}^t_{kj} \quad \sum_{k=1,k\neq j}^{n_s}\tilde{E}^t_{kj}\Delta^t_{ka}(t)\tilde{H}^t_{kj} \right. \\
&\qquad \left. \sum_{k=1,k\neq j}^{n_s}\tilde{E}^t_{kj}\Delta^t_{ka}(t)\tilde{H}^t_{kj}\right].
\end{aligned}
\tag{4.73}
$$

Robust asymptotic stability requirement $\dot{V}_j(t)|_{(4.52)} \leq 0$ implies that $\Psi_{j\Delta} < 0$ for all admissible uncertainties satisfying (4.40) and (4.47). Next, considering the $\mathcal{L}_2$-*gain* performance measure $J = \sum_{j=1}^{n_s} J_j$ for any $w_j(t) \in \mathcal{L}_2(0,\infty) \neq 0$ with zero initial condition $x_j(0) = 0$ hence $V(0) = 0$, we have

$$
\begin{aligned}
J_j &= \int_0^\infty \big(z^t_j(s)z_j(s) - \gamma_j^2 w^t_j(s)w_j(s)\big)ds \\
&\leq \int_0^\infty \big(z^t_j(s)z_j(s) - \gamma_j^2 w^t_j(s)w_j(s) + \dot{V}_j(s)|_{(4.52)}\big)ds.
\end{aligned}
\tag{4.74}
$$

Using (4.52) and (4.53), we obtain:

$$
\begin{aligned}
& z^t_j(s)z_j(s) - \gamma_j^2 w^t_j(s)w_j(s) + \dot{V}_j(s)|_{(4.52)} \\
&\quad \leq [\zeta^t_j(s)\ w^t_j(s)]\hat{\Psi}_{j\Delta}[\zeta^t_j(s)\ w^t_j(s)]^t \\
&\quad = \begin{bmatrix} \zeta_j(s) \\ w_j(s) \end{bmatrix}^t \begin{bmatrix} \Psi_{j\Delta} + \hat{\mathsf{G}}^t_{j\Delta}\hat{\mathsf{G}}_{j\Delta} & \hat{\mathsf{G}}^t_{j\Delta}\Phi_{j\Delta} + (\mathcal{P}_j + \Theta_j + \Upsilon_j)\widehat{\hat{\Gamma}}_{j\Delta} \\ \bullet & -\gamma_j^2 I_j + \Phi^t_{j\Delta}\Phi_{j\Delta} \end{bmatrix} \begin{bmatrix} \zeta_j(s) \\ w_j(s) \end{bmatrix}.
\end{aligned}
\tag{4.75}
$$

That $J_j < 0$ for arbitrary $s \in [t,\infty)$ implies for any $w_j(t) \in \mathcal{L}_2(0,\infty) \neq 0$ that

$$
z^t_j(s)z_j(s) - \gamma_j^2 w^t_j(s)w_j(s) + \dot{V}_j(s)|_{(4.52)} < 0.
$$

This leads to $\|z_j(t)\|_2 < \sum_{j=1}^{n_s}\gamma_j\|w(t)_j\|_2$, which assures the desired performance. In terms of (4.58) and considering $\hat{\Psi}_{j\Delta}$ while invoking bounding inequality A from Sect. 9.3.1 with some algebraic manipulations and Schur complements, we obtain LMI (4.58) for some scalars $\sigma_1 > 0,\ \ldots,\ \sigma_6 > 0$ and hence the proof is completed. □

Theorem 4.3 *Given the bounds $\theta_m > 0, \theta_M > 0$, the family of subsystems described by* (4.52)–(4.53) *is delay-dependent asymptotically stabilizable by decentralized output-feedback controller with $\mathcal{L}_2$-performance bound $\gamma_j,\ j = 1,\ldots,n_s$, if there* *exist positive-definite matrices $\mathcal{Y}_j$, $\mathcal{M}_{wj}$, $\mathcal{M}_{rj}$, $\mathcal{M}_{sj}$, any matrices $\mathcal{G}_j$, and scalars $\sigma_1 > 0, \ldots, \sigma_6 > 0$, $\lambda_\Theta, \lambda_\Upsilon$ satisfying the following LMIs for $j = 1, \ldots, n_s$*

$$\widetilde{\Pi}_j = \begin{bmatrix} \widetilde{\Psi}_{jo} & \widetilde{\Lambda}_j & \widetilde{\mathsf{G}}^t_{jo} & \widetilde{Q}_j & \tilde{T}_j \\ \bullet & -\gamma_j^2 I_j & \Phi^t_{jo} & 0 & O_1 \\ \bullet & \bullet & -I_j & O_2 & 0 \\ \bullet & \bullet & \bullet & -\Sigma_j & 0 \\ \bullet & \bullet & \bullet & \bullet & -\Sigma_j \end{bmatrix} < 0, \tag{4.76}$$

where

$$\widetilde{\Lambda}_j = [(1+\lambda_\Theta+\lambda_\Upsilon)\hat{\Gamma}_j, 0, 0, 0, 0, 0]^t, \quad \widetilde{\mathsf{G}}^t_{jo} = [\mathcal{Y}_j \mathsf{G}^t_{jo}, 0, 0, 0, 0, 0]^t,$$

$$\widetilde{Q}_j = \mathcal{Y}_j \hat{Q}_j, \quad \widetilde{T}_j = \mathcal{Y}_j \hat{T}_j.$$

Moreover, the gain matrices are given by $K_j = \mathcal{G}_j \mathcal{Y}_j^{-1}$, $L_j = \mathcal{V}_j \mathcal{Y}_j^{-1} C^\dagger_{jo}$.

Proof Considering LMI (4.58), Letting $\Theta_j = \lambda_\Theta \mathcal{P}_j$, $\Upsilon_j = \lambda_\Upsilon \mathcal{P}_j$ ($\lambda_\Theta, \lambda_\Upsilon$ are any scalars), and applying the congruent transformation

$$\mathsf{T} = \mathrm{diag}[\mathcal{Y}_j,\ \mathcal{Y}_j,\ \mathcal{Y}_j,\ \mathcal{Y}_j,\ \mathcal{Y}_j,\ I_j,\ I_j,\ I_j,\ I_j,\ I_j], \quad \mathcal{Y}_j = \mathcal{P}_j^{-1}$$

we obtain that

$$\Pi_j = \begin{bmatrix} \widetilde{\Psi}_{jo} & \mathcal{Y}_j \Lambda_j & \mathcal{Y}_j \hat{\mathsf{G}}^t_{jo} & \mathcal{Y}_j \hat{Q}_j & \mathcal{Y}_j \widehat{T}_j \\ \bullet & -\gamma_j^2 I_j & \Phi^t_{jo} & 0 & O_1 \\ \bullet & \bullet & -I_j & O_2 & 0 \\ \bullet & \bullet & \bullet & -\Sigma_j & 0 \\ \bullet & \bullet & \bullet & \bullet & -\Sigma_j \end{bmatrix} < 0, \tag{4.77}$$

where

$$\tilde{\Psi}_{jo} = \begin{bmatrix} \tilde{\Psi}_{ajo} & \tilde{\Psi}_{1jo} & 0 & \lambda_\Theta \mathsf{D}_{jo}\mathcal{Y}_j + \mathcal{M}_{rj} & (1+\lambda_\Theta)\mathcal{Y}_j & \tilde{\Psi}_{sjo} \\ \bullet & -\tilde{\Psi}_{cj} & 0 & \lambda_\Upsilon \mathsf{D}_{jo}\mathcal{Y}_j & \lambda_\Upsilon \mathcal{Y}_j & 0 \\ \bullet & \bullet & -\tilde{\Psi}_{mj} & \mathcal{M}_{rj} & 0 & 0 \\ \bullet & \bullet & \bullet & -\tilde{\Psi}_{nj} & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -I_j & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\tilde{\Psi}_{7j} \end{bmatrix}, \tag{4.78}$$

$$\tilde{\Psi}_{ajo} = (1+\lambda_\Theta)\mathsf{A}_{jo}\mathcal{Y}_j + (1+\lambda_\Theta)\mathcal{Y}_j \mathsf{A}^t_{jo} + \mathcal{M}_{wj} + \mathcal{M}_{sj} - \mathcal{M}_{rj} + (n_s - 1)(1+\lambda_\Theta+\lambda_\Upsilon)\mathcal{Y}_j,$$

$$\tilde{\Psi}_{1jo} = -\lambda_\Theta \mathcal{Y}_j + \lambda_\Upsilon \mathcal{Y}_j \mathsf{A}^t_{jo}, \quad \tilde{\Psi}_{cj} = -\theta_M^2 \mathcal{M}_{rj} + \lambda_\Upsilon \mathcal{Y}_j + \lambda_\Upsilon \mathcal{Y}_j^t,$$

$$\tilde{\Psi}_{mj} = \mathcal{M}_{wj} + \mathcal{M}_{rj}, \quad \tilde{\Psi}_{nj} = 2\mathcal{M}_{rj} + \mathcal{M}_{sj}, \quad \tilde{\Psi}_{7j} = \mathrm{diag}\left[\mathcal{Y}_k\ \lambda_\Theta^{-1}\mathcal{Y}_k\ \lambda_\Upsilon^{-1}\mathcal{Y}_k\right],$$

$$\tilde{\Psi}_{sjo} = \left[\sum_{k=1,k\neq j}^{n_s} \mathcal{Y}_k \mathsf{F}^t_{kjo}\ \sum_{k=1,k\neq j}^{n_s} \mathcal{Y}_k \mathsf{F}^t_{kjo}\ \sum_{k=1,k\neq j}^{n_s} \mathcal{Y}_k \mathsf{F}^t_{kjo}\right],$$

$$\mathcal{M}_{wj} = \mathcal{Y}_j \mathcal{W}_j \mathcal{Y}_j, \quad \mathcal{M}_{sj} = \mathcal{X}_j \mathcal{S}_j \mathcal{X}_j, \quad \mathcal{M}_{rj} = \mathcal{Y}_j \mathcal{R}_j \mathcal{Y}_j, \tag{4.79}$$

$$\hat{Q}_j = \begin{bmatrix} (\mathcal{P}_j + \Theta_j)\hat{H}_{jo} & (\mathcal{P}_j + \Theta_j)\hat{N}_{jc} & \Theta_j \hat{N}_{jo} & 0 & 0 & 0 & \phi & 0 \\ \Upsilon_j \hat{H}_{jo} & \Upsilon_j \hat{N}_{jc} & \Upsilon_j \hat{N}_{jo} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{H} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{H} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{H} & 0 & 0 \end{bmatrix}$$

with $\mathcal{H} = \sum_{k=1,k\neq j}^{n_s} \tilde{H}_{kj}$, $\phi = (\mathcal{P}_j + \Theta_j + \Upsilon_j)\hat{H}_{jo}$,

$$\hat{T}_j = \begin{bmatrix} \sigma_1 \hat{E}^t_{jo} & \sigma_1 \hat{M}^t_{jc} & 0 & \sigma_3 \hat{E}^t_{kj} & \sigma_3 \hat{E}^t_{kj} & \sigma_3 \hat{E}^t_{kj} & 0 & \sigma_5 E^t_{ja} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_2 \hat{M}^t_{jo} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$O_1 = [0,\ 0,\ 0,\ 0,\ 0,\ 0,\ \sigma_4 E^t_{jc},\ \sigma_5 E^t_{jc}], \quad O_2 = [0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ H_{ja}],$$
$$\Lambda_j = \mathrm{col}[(\mathcal{P}_j + \Theta_j + \Upsilon_j)\widehat{\Gamma}_j,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0],$$
$$\widehat{\mathsf{G}}^t_{jo} = [\mathsf{G}_{jo},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0]^t,$$
$$\Sigma_j = \mathrm{diag}[\sigma_1 I_j,\ \sigma_1 I_j,\ \sigma_2 I_j,\ \sigma_3 I_j,\ \sigma_3 I_j,\ \sigma_3 I_j,\ \sigma_4 I_j,\ \sigma_5 I_j].$$

Next, let $\mathcal{Y}_j = \left[\begin{smallmatrix} \mathcal{Y}_{1j} & 0 \\ 0 & \mathcal{Y}_{1j} \end{smallmatrix}\right]$, $\mathcal{G}_j = K_j \mathcal{Y}_{1j}$, $\mathcal{V}_j = L_j C_{jo} \mathcal{Y}_{1j}$, we can get LMI (4.76) with (4.79) and therefore the proof is completed. □

4.3.6 Simulation Example 4.2

To illustrate the design procedures developed in Theorem 4.3, we consider a representative water pollution model of two consecutive reaches of the River Nile. This linearized model forms an interconnected system of the type (4.37)–(4.38) for $n_s = 2$ and the following information.

Nominal subsystem matrices

$$A_{1o} = \begin{bmatrix} 1.05 & -0.42 \\ 1.1 & 0 \end{bmatrix}, \quad A_{2o} = \begin{bmatrix} 1 & -0.5 \\ 1.1 & 0.3 \end{bmatrix}, \quad B_{1o} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_{2o} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$C_{1o} = [-1\ 1], \quad C_{2o} = \begin{bmatrix} 0.7\ 1 \end{bmatrix}, \quad G_{1o} = \begin{bmatrix} -1\ 1 \end{bmatrix}, \quad G_{2o} = \begin{bmatrix} 1\ 0.8 \end{bmatrix}.$$

Delay and disturbance parameters

$$\Gamma_{1o} = \begin{bmatrix} 0.2 \\ 1 \end{bmatrix}, \quad \Gamma_{2o} = \begin{bmatrix} 0.5 \\ 0.8 \end{bmatrix}, \quad \Phi_{1o} = 0.02, \quad \Phi_{2o} = 0.03,$$
$$M_{1o} = 0.2, \quad M_{2o} = 0.3, \quad N_{1o} = 0.2, \quad N_{2o} = 0.4,$$
$$M_{1c} = 0.5, \quad M_{2c} = 0.1, \quad N_{1c} = 0.03, \quad N_{2c} = 0.01,$$
$$H_{1o} = \begin{bmatrix} 0.03 \\ 0.03 \end{bmatrix}, \quad H_{1o} = \begin{bmatrix} 0.02 \\ 0.02 \end{bmatrix}, \quad H_{1a} = 0.1, \quad H_{2a} = 0.2,$$
$$H_{21} = 0.3, \quad H_{12} = 0.4,$$
$$E_{1a} = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad E_{2a} = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad E_{1c} = 0.01, \quad E_{2c} = 0.01,$$
$$E_{1o} = 0.2, \quad E_{2o} = 0.2, \quad E_{12} = 0.1, \quad E_{21} = 0.1.$$

Coupling matrices

$$F_{12o} = \begin{bmatrix} -1 & 0 \\ -1 & -0.5 \end{bmatrix}, \quad F_{21o} = \begin{bmatrix} -0.6 & 0 \\ 0.2 & 1 \end{bmatrix}.$$

By selecting

$$\lambda_{\Theta} = 0.01, \quad \lambda_{\Upsilon} = 0.02,$$
$$\sigma_1 = 1.3, \quad \sigma_2 = 1.4, \quad \sigma_3 = 0.9, \quad \sigma_4 = 1.5, \quad \sigma_5 = 1.1,$$

while using the foregoing nominal data and invoking the MATLAB software, we obtain

$$\mathcal{Y}_1 = \begin{bmatrix} 0.0313 & 0.0135 \\ 0.0135 & 0.0868 \end{bmatrix}, \quad \mathcal{Y}_2 = \begin{bmatrix} 0.0266 & 0.0150 \\ 0.0150 & 0.1150 \end{bmatrix},$$
$$\mathcal{M}_{w1} = \begin{bmatrix} 0.0001 & 0.0002 \\ 0.0002 & 0.0013 \end{bmatrix}, \quad \mathcal{M}_{w2} = 10^{-3} \times \begin{bmatrix} 0.0913 & 0.0312 \\ 0.0312 & 0.6294 \end{bmatrix},$$
$$\mathcal{M}_{s1} = \begin{bmatrix} 0.0001 & 0.0002 \\ 0.0002 & 0.0013 \end{bmatrix}, \quad \mathcal{M}_{s2} = 10^{-3} \times \begin{bmatrix} 0.0911 & 0.0312 \\ 0.0312 & 0.6276 \end{bmatrix},$$
$$\mathcal{M}_{r1} = \begin{bmatrix} 0.1210 & 0.0509 \\ 0.0509 & 0.3198 \end{bmatrix}, \quad \mathcal{M}_{r2} = 10^{-3} \times \begin{bmatrix} 0.0991 & 0.0548 \\ 0.0548 & 0.4285 \end{bmatrix},$$
$$\mathcal{G}_1 = \begin{bmatrix} 0.0298 & -0.0978 \end{bmatrix}, \quad \mathcal{G}_2 = \begin{bmatrix} -0.0233 & -0.0297 \end{bmatrix},$$

as feasible solution of the matrix inequalities. These give the following gain matrices:

$$K_1 = \begin{bmatrix} 1.5403 & -1.3657 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -0.7909 & -0.1549 \end{bmatrix},$$
$$L_1 = \begin{bmatrix} -194.0815 \\ 181.7862 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 174.8047 \\ 248.3807 \end{bmatrix}$$

along with the $\mathcal{L}_2$ gain $\gamma = 0.7$ and the maximum of the network-induced delays is 0.1.

4.4 Control of Discrete-Time Systems with Input Saturation

We study decentralized stabilization of discrete time linear time invariant (LTI) systems subject to actuator saturation, using LTI controllers. The requirement of stabilization under both saturation constraints and decentralization impose obvious necessary conditions on the open-loop plant, namely that its eigenvalues are in the closed unit disk and further that the eigenvalues on the unit circle are not decentralized fixed modes. The key contribution of this work is to provide a broad sufficient condition for decentralized stabilization under saturation. Specifically, we show through an iterative argument that stabilization is possible whenever: (1) the open loop eigenvalues are in the closed unit disk, (2) the eigenvalues on the unit circle are not decentralized fixed modes, and (3) these eigenvalues on the unit circle have algebraic multiplicity 1.

4.4.1 Introduction

The result presented here contributes to our ongoing study of the stabilization of decentralized systems subject to actuator saturation. The eventual goal of this study is the design of controllers for saturating decentralized systems that achieve not only stabilization but also high performance. As a first step toward this design goal, we are currently looking for tight conditions on a decentralized plant with input saturation, for the existence of stabilizing controllers. Even this check for the existence of stabilizing controllers turns out to be extremely intricate: we have yet to obtain necessary and sufficient conditions for stabilization, but have obtained a broad sufficient condition, see the results in [29–34, 38–40, 64–77]. This section further contributes to the study of the existence of stabilizing controllers, by describing a analogous sufficient condition for discrete-time decentralized plants.

To motivate and introduce the main result in the section, let us briefly review foundational studies on both decentralized control and saturating control systems. We recall that a necessary and sufficient condition for stabilization of a decentralized system using LTI state-space controllers is given in Wang and Davison's classical work [78]. They obtain that stabilization is possible if and only if all decentralized fixed modes of a plant are in the open left half plane, and give specifications of and methods for finding these decentralized fixed modes. Numerous further characterizations of decentralized stabilization (and fixed modes) have been given, see for instance the work of Corfmat and Morse [12]. In complement, for centralized control systems subject to actuator saturation, not only conditions for stabilization but also practical designs have been obtained, using the low gain and low-high-gain methodology. For a background on the results for centralized systems subject to input saturation we refer to two special issues [5, 69]. Of importance here, we recall that a necessary and sufficient condition for semi global stabilization of LTI plants with actuator saturation is that their open-loop poles are in the closed left half plane. Combining this observation with Wang and Davison's result, one might postulate that stabilization of a saturating linear decentralized control system is possible if and only of (1) the open-loop plant poles are in the closed left half plane (respectively, closed unit disk, for discrete-time systems), and (2) the poles on the imaginary axis (respectively, unit circle) are not decentralized fixed modes. The necessity of the two requirements is immediate, but we have not yet been able to determine whether the requirements are also sufficient. As a first step for continuous-time plants, we showed in [75] that decentralized stabilization under saturation is possible when (1) the plant's open-loop poles are in the CLHP with imaginary axis poles non-repeated, and (2) the imaginary axis poles are not decentralized fixed modes. Here, we develop an analogous result for discrete-time plants, in particular showing that decentralized stabilization under saturation is possible if (1) the plant's open-loop poles are in the closed unit disk with unit-circle poles non-repeated, and (2) the unit circle poles are not decentralized fixed modes.

4.4.2 Problem Formulation

Consider the LTI discrete-time systems subject to actuator saturation,

$$\Sigma\colon \quad \begin{cases} x(k+1) = Ax(k) + \sum_{j=1}^{\nu} B_j \operatorname{sat}(u_j(k)), \\ y_j(k) = C_j x(k), \quad j = 1, \dots, \nu, \end{cases} \tag{4.80}$$

where $x \in \Re^n$ is state, $u_j \in \Re^{m_j}$, $j = 1, \dots, \nu$ are control inputs, $y_j \in \Re^{p_j}$, $j = 1, \dots, \nu$ are measured outputs, and 'sat' denotes the standard saturation element.

Here we are looking for ν controllers of the form,

$$\Sigma\colon \quad \begin{cases} z_j(k+1) = K_j z_i(k) + L_j y_i(k), \quad z_j \in \Re^{s_i}, \\ u_j(k+1) = M_j z_j(k) + N_j y_j(k). \end{cases} \tag{4.81}$$

Let the system (4.80) be given. The semi-global stabilization problem via decentralized control is said to be solvable if for all compact sets W and $S_1, \dots, S_\nu$ there exists ν controllers of the form (4.81) such that the closed loop system is asymptotically stable with the set

$$W \times S_1 \times \cdots \times S_\nu$$

contained in the domain of attraction.

The main objective is to develop necessary and sufficient conditions such that the semi-global stabilization problem via decentralized control is solvable. This objective has not yet been achieved. However, we obtain necessary conditions as well as sufficient conditions which are quite close.

4.4.3 Review Results

Before we tackle the problem introduced in Sect. 4.4.2, let us first review the necessary and sufficient conditions for the decentralized stabilization of the linearized model of the given system Σ,

$$\bar{\Sigma}: \quad \begin{cases} x(k+1) = Ax(k) + \sum_{j=1}^{\nu} B_i u_i(k), \\ y_j(k) = C_j x(k), \quad j = 1, \dots, \nu. \end{cases} \tag{4.82}$$

The decentralized stabilization problem for $\bar{\Sigma}$ is to find LTI dynamic controllers Σ_j, $j = 1, \dots, \nu$, of the form (4.81) such that the poles of the closed loop system are in the desired locations in the open unit disc.

Given system $\bar{\Sigma}$ and controllers Σ_i, defined by (4.82) and (4.81) respectively, let us first define the following matrices in order to provide an easier bookkeeping:

$$\begin{aligned} B &= [B_1 \ \dots \ B_\nu], & C &= [C_1' \ \dots \ C_\nu']', \\ K &= \operatorname{diag}[K_1, \dots, K_\nu], & L &= \operatorname{diag}[L_1, \dots, L_\nu], \\ M &= \operatorname{diag}[M_1, \dots, M_\nu], & N &= \operatorname{diag}[N_1, \dots, N_\nu]. \end{aligned}$$

Definition 4.1 Consider system $\bar{\Sigma}$, $\lambda \in C$ is called a decentralized fixed mode if for all block diagonal matrices H we have

$$\det(\lambda I - A - BHC) = 0.$$

We look at eigenvalues that can be moved by static decentralized controllers. However, it is known that if we cannot move an eigenvalue by static decentralized controllers then we cannot move the eigenvalue by dynamic decentralized controllers either.

Lemma 4.1 *Necessary and sufficient condition for the existence of a decentralized feedback control law for the system $\bar{\Sigma}$ such that the closed loop system is asymptotically stable is that all the fixed modes of the system be asymptotically stable* (*in the unit disc*).

Proof We first establish necessity.

Assume local controllers Σ_i together stabilize $\bar{\Sigma}$ then for any $|\lambda| \geq 1$ there exists a δ such that $(\lambda + \delta)I - K$ is invertible and the closed loop system replacing K with $K - \delta I$ is still asymptotically stable. This choice is possible because if $\lambda I - K$

is invertible obviously we can choose $\delta = 0$. If $\lambda I - K$ is not invertible, by small enough choice of δ we can make sure that $(\lambda + \delta)I - K$ is invertible and the closed loop system replacing K with $K - \delta I$ is still asymptotically stable. But the closed loop system when $K - \delta I$ is in the loop is asymptotically stable. In particular, it can not have a pole in λ. So

$$\det(\lambda I - A - B[M(\lambda I - (K - \delta I))^{-1}L + N]C) \neq 0.$$

Hence the block diagonal matrix

$$S = M(\lambda I - (K - \delta I))^{-1}L + N$$

has the property that

$$\det(\lambda I - A - BSC) \neq 0$$

thus λ is not a fixed mode. Since this argument is true for any λ on or outside the unit disc, this implies that all the fixed modes must be inside the unit disc. This proves the necessity of the Lemma 4.1.

Next, we establish sufficiency. The papers [12, 78] showed that if the decentralized fixed modes of a strongly connected system are stable, we can find a stabilizing controller for the system. However, these papers are based on continuous-time results. For completeness we present the proof for discrete time which is a straightforward modification of [78]. We first claim that decentralized fixed modes are invariant under preliminary output injection. But this is obvious from our necessity proof since a trivial modification shows that no dynamic controller can move a fixed mode. To prove that we can actually stabilize the system, we use a recursive argument. Assume the system has an unstable eigenvalue in μ. Since μ is not a fixed mode there exists N_i such that

$$A + \sum_{j=1}^{\nu} B_j N_j C_j$$

no longer has an eigenvalue in μ. Let k be the smallest integer such that an unstable eigenvalue of A is no longer an eigenvalue of

$$A + \sum_{j=1}^{k} B_j N_j C_j$$

while N_j can be chosen small enough not to introduce additional unstable eigenvalues. Then for the system

$$\left(A + \sum_{j=1}^{k-1} B_j N_j C_j, B_k, C_k\right)$$

an unstable eigenvalue is both observable and controllable. But this implies that there exists a dynamic controller which moves this eigenvalue in the open unit disc without introducing new unstable eigenvalues. Through a recursion, we can move all eigenvalues one-by-one in the open unit disc and in this way find a decentralized controller which stabilizes the system. This proves the sufficiency of the Lemma 4.1. □

4.4.4 Main Results

Here, we present the main results of Sect. 4.4.

Theorem 4.4 *Consider the system Σ. There exists nonnegative integers $s_1, \dots, s_\nu$ such that for any given collection of compact sets $W \subset \Re^n$ and $S_i \subset \Re^{s_i}, i = 1, \dots, \nu$, there exists ν controllers of the form (4.81) such that the origin of the resulting closed loop system is asymptotically stable and the domain of attraction includes $W \times S_1 \times \cdots \times S_\nu$ only if*

- *All fixed modes are in the open unit disc.*
- *All eigenvalues of A are in the closed unit disc.*

Proof There exists an open neighborhood containing the origin for the closed loop system of Σ with the controllers Σ_i is identical to the closed loop system of $\bar{\Gamma}$ with the controllers Σ_i. Hence asymptotic stability of one closed loop system is equivalent to asymptotic stability of the other closed loop system. But then it is obvious from Lemma 4.1 that the first item of Theorem 4.4 is necessary for the existence of controllers of the form (4.81) for $\bar{\Sigma}$ such that the origin of the resulting closed loop system is asymptotically stable.

To prove the necessity of the second item of Theorem 4.4, assume that λ is an eigenvalue of A outside the unit disc with associated left eigenvector p. We obtain:

$$px(k+1) = \lambda px(k) + v(k),$$

where

$$v(k) := \sum_{j=1}^{\nu} pB_i \operatorname{sat}(u_j(k)).$$

Because of the saturation elements, there exists an $\tilde{M} > 0$ such that $|v(k)| \leq \tilde{M}$ for all $k \geq 0$. But then we have

$$px(k) = \lambda^k px(0) + \sum_{j=0}^{k-1} \lambda^{k-1-j} v(j) = \lambda^k (px(0) + S_k), \tag{4.83}$$

where $S_k = \sum_{j=0}^{k-1} v(j) \frac{v(j)}{\lambda^{i+1}}$. We find that

$$|S_k| \leq \tilde{M} \sum_{j=1}^{k} \frac{1}{|\lambda|^j} = \tilde{M} \frac{1 - \frac{1}{|\lambda|^k}}{|\lambda| - 1} < \frac{\tilde{M}}{|\lambda| - 1}$$

and then from (4.83) we find

$$|px(k)| > |\lambda|^k \left(|px(0)| - \frac{\tilde{M}}{|\lambda| - 1} \right) \quad \forall k \geq 1.$$

Hence $|px(k)|$ does not converge to zero independent of our choice for a controller if we choose the initial condition $x(0)$ such that $|px(0)| > \frac{\tilde{M}}{|\lambda|-1}$ because of the fact that $|\lambda| > 1$. However, the system was semi-globally stabilizable and hence there exists a controller which contains this initial condition in its domain of attraction and hence $|px(k)| \to 0$ which yields a contradiction. This proves the second item of Theorem 4.4.

We now proceed to the next theorem which gives a sufficient condition for semi-global stabilizability of (4.80) when the set of controllers given by (4.81) are utilized. □

Theorem 4.5 *Consider the system Σ. There exists nonnegative integers $s_1, \ldots, s_\nu$ such that for any given collection of compact sets $W \subset \Re^n$ and $S_j \subset \Re^{s_j}$, $j = 1, \ldots, \nu$, there exists ν controllers of the form (4.81) such that the origin of the resulting closed loop system is asymptotically stable and the domain of attraction includes $W \times S_1 \times \cdots \times S_\nu$ if*

- *All fixed modes are in the open unit disc.*
- *All eigenvalues of A are in the closed unit disc with those eigenvalues on the unit circle having algebraic multiplicity equal to one.*

To prove this theorem we will exploit the following lemma which follows directly from classical results of eigenvalues and eigenvectors and the results of perturbations of the matrix on those eigenvalues and eigenvectors.

Lemma 4.2 *Let $A_\delta \in \Re^{n\times n}$ be a sequence of matrices parametrized by δ and a matrix $A \in \Re^{n\times n}$ such that $A_\delta \to A$ as $\delta \to 0$. Let A be a matrix with all eigenvalues in the closed unit disc and with p eigenvalues on the unit disc with all of them having multiplicity 1. Also assume that A_δ has all its eigenvalues in the closed unit disc. Let matrix $P > 0$ be such that $A'PA - P \leq 0$ is satisfied. Then for small $\delta > 0$ there exists a family of matrices $P_\delta > 0$ such that*

$$A_\delta' P_\delta A_\delta - P_\delta \leq 0$$

and $P_\delta \to P$ as $\delta \to 0$.

Proof We first observe that there exists a matrix S such that

$$S_\delta^{-1} A_\delta S_\delta = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix},$$

where all eigenvalues of A_{11} are on the unit circle while the eigenvalues of A_{22} are in the open unit disc. Since $A_\delta \to A$ and the eigenvalues of A_{11} and A_{22} are distinct, there exists a parametrized matrix S_δ such that for sufficiently small δ

$$S_\delta^{-1} A_\delta S_\delta = \begin{pmatrix} A_{11,\delta} & 0 \\ 0 & A_{22,\delta} \end{pmatrix},$$

where $S_\delta \to S$, $A_{11,\delta} \to A_{11}$ and $A_{22,\delta} \to A_{22}$ as $\delta \to 0$.

Given a matrix $P > 0$ such that $A' P A - P \leq 0$. Let us define

$$\bar{P} = S' P S = \begin{pmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{12}' & \bar{P}_{22} \end{pmatrix},$$

with this definition we have

$$\begin{pmatrix} A_{11}' & 0 \\ 0 & A_{22}' \end{pmatrix} \bar{P} \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} - \bar{P} \leq 0. \tag{4.84}$$

Next given an eigenvector x_1 of A_{11}, i.e. $A_{11} x_1 = \lambda x_1$ with $|\lambda| = 1$, we have

$$\begin{pmatrix} x_1 \\ 0 \end{pmatrix}^* \left[\begin{pmatrix} A_{11}' & 0 \\ 0 & A_{22}' \end{pmatrix} \bar{P} \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} - \bar{P} \right] \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = 0.$$

Using (4.84), the above implies that

$$\left[\begin{pmatrix} A_{11}' & 0 \\ 0 & A_{22}' \end{pmatrix} \bar{P} \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} - \bar{P} \right] \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = 0.$$

Since all the eigenvalues on the unit disc of $A_{11} \in \Re^{p \times p}$ are distinct we find that the eigenvectors of A_{11} span $\Re^p$ and hence

$$\left[\begin{pmatrix} A_{11}' & 0 \\ 0 & A_{22}' \end{pmatrix} \bar{P} \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} - \bar{P} \right] \begin{pmatrix} I \\ 0 \end{pmatrix} = 0.$$

This results in

$$\left[\begin{pmatrix} A_{11}' & 0 \\ 0 & A_{22}' \end{pmatrix} \bar{P} \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} - \bar{P} \right] \begin{pmatrix} 0 & 0 \\ 0 & V \end{pmatrix} \leq 0.$$

This implies that $A_{11}' \bar{P}_{12} A_{22} - \bar{P}_{12} = 0$ and since eigenvalues of A_{11} are on the unit disc and eigenvalues of A_{22} are inside the unit disc, we find that $\bar{P}_{12} = 0$ because

$$A_{11}' \bar{P}_{12} A_{22} = \bar{P}_{12} \Rightarrow (A_{11}')^k \bar{P}_{12} A_{22}^k = \bar{P}_{12},$$

where k is an arbitrary positive integer. Note that $(A_{11}')^k$ remains bounded while $A_{22}^k \to 0$ as $k \to \infty$. This means that for $k \to \infty$, $\bar{P}_{12} \to 0$ and because $\bar{P}_{12}$ is independent of k, we find that $\bar{P}_{12} = 0$. Next, since A_{22} has all its eigenvalues in the open unit disc, there exists a parametrized matrix $P_{\delta,22}$ such that for δ small enough

$$A_{\delta,22}' P_{\delta,22} A_{\delta,22} - P_{\delta,22} = V \leq 0$$

while $P_{\delta,22} \to P_{22}$ as $\delta \to 0$.

Let $A_{11} = W\Lambda_A W^{-1}$ with Λ_A a diagonal matrix. Because the eigenvectors of A_{11} are distinct and $A_{11,\delta} \to A_{11}$, the eigenvectors of $A_{11,\delta}$ depend continuously on δ for δ small enough and hence there exists a parametrized matrix W_δ such that $W_\delta \to W$ while $A_{11,\delta} = W_\delta \Lambda_{A_\delta} W_\delta^{-1}$ with Λ_{A_δ} diagonal. The matrix $\bar{P}_{11}$ satisfies

$$A'_{11}\bar{P}_{11}A_{11} - \bar{P}_{11} = 0.$$

This implies that $\Lambda_P = W^*\bar{P}_{11}W$ satisfies

$$\Lambda_A^*\Lambda_P\Lambda_A - \Lambda_P = 0.$$

The above equation then shows that Λ_P is a diagonal matrix. We know that

$$\Lambda_{A_\delta}^* \to \Lambda_A.$$

We know that Λ_{A_δ} is a diagonal matrix the diagonal elements of which have magnitude less or equal to one while Λ_P is a positive definite diagonal matrix.

Using this, it can be verified that we have

$$\Lambda_{A_\delta}^*\Lambda_P\Lambda_{A_\delta} - \Lambda_P \le 0.$$

We choose $\bar{P}_{11,\delta}$ as

$$\bar{P}_{11,\delta} = (W_\delta^*)^{-1}\Lambda_P(W_\delta)^{-1}.$$

We can see that this choice of $\bar{P}_{11,\delta}$ satisfies

$$A'_{11,\delta}\bar{P}_{11,\delta}A_{11,\delta} - \bar{P}_{11,\delta} \le 0.$$

It is easy to see that $\bar{P}_{11,\delta} \to \bar{P}_{11}$ as $\delta \to 0$. Then

$$P_\delta = (S_\delta^{-1})'\begin{pmatrix} \bar{P}_{11,\delta} & 0 \\ 0 & \bar{P}_{22,\delta} \end{pmatrix} S_\delta^{-1}$$

satisfies the condition of the lemma. This completes the proof of Lemma 4.2. □

We now show a recursive algorithm that at each step moves at least one eigenvalue on the unit circle in a decentralized fashion while preserving the stability of other modes in the open unit disc in a way that the magnitude of each decentralized feedback control is assured never to exceed $1/n$. The algorithm will consist of at most n steps, and therefore the overall decentralized inputs will not saturate for an appropriate choice of the initial state.

Algorithm

- Step 0: We initialize algorithm at this step. Let $A_0 := A$, $B_{0,ij} := B_j$, $C_{0,i} := C_j$, $n_{j,0} := 0$, $N_{j,\delta}^0 := 0$, $j = 1, \dots, \nu$ and $x_0 := x$. Also let us define $P_0^\varepsilon := \varepsilon P$, where $P > 0$ and satisfies $A'PA - P \le 0$.

- Step m: For the system Σ, we want to design ν parametrized decentralized feedback control laws,

$$\Sigma_j^{m,\varepsilon}: \quad \begin{cases} p_j^m(k+1) = K_{i,\varepsilon}^m p_j^m(k) + L_{j,\varepsilon}^m y_j(k), \\ u_j(k) = M_{j,\varepsilon} p_j^m(k) + N_{j,\varepsilon}^m y_j(k) + v_j^m(k), \end{cases}$$

where $p_j^m \in \Re^{n_{j,m}}$ and if $n_{j,m} = 0$:

$$\Sigma_j^{m,\varepsilon}: \quad u_j(k) = N_{i,\varepsilon}^m y_j(k) + v_j^m(k).$$

The closed loop system consisting of the decentralized controller and the system Σ can be written as

$$\Sigma_{\text{cl}}^{m,\varepsilon}: \quad \begin{cases} x_m(k+1) = A_m^\varepsilon x_m(k) + \sum_{i=1}^{\nu} B_{m,i} v_i^m(k), \\ y_j(k) = C_{m,j} x_m(k), \quad j = 1, \ldots, \nu, \end{cases}$$

where $x_m \in \Re^{n_m}$ with $n_m = n + \sum_{i=1}^{\nu} n_{i,m}$ is given by

$$x_m = \begin{pmatrix} x \\ p_1^m \\ \vdots \\ p_\nu^m \end{pmatrix}$$

we can rewrite u_i as

$$u_i = F_{i,\varepsilon}^m x_m + v_i^m$$

for some appropriate matrix $F_{j,\varepsilon}^m$.

Our objective here is to design the decentralized stabilizers in such a way that they satisfy the following properties:

1. Matrix A_m^ε has all its eigenvalues in the closed unit disc, and eigenvalues on unit circle are distinct.
2. A_m^ε has less eigenvalues on the unit circle than A_{m-1}^ε.
3. There exists a family of matrices P_m^ε such that $P_m^\varepsilon \to 0$ as $\varepsilon \to 0$ and

$$(A_m^\varepsilon)' P_m^\varepsilon A_m^\varepsilon - P_m^\varepsilon \leq 0.$$

   Furthermore, there exists an ε^* such that for $\varepsilon \in (0, \varepsilon^*]$ and $v_i^m = 0$ we have $\|u_i(k)\| \leq \frac{m}{n}$ for all states with $x_m'(k) P_m^\varepsilon x_m(k) \leq n - m + 1$.

- Terminal Step: There exists a value for m, say $l \leq n$, such that A_{11}^ε has all its eigenvalues in the open unit disc, and also property 3 above is satisfied, which means that for ε small enough, $\|u_i\| \leq 1$ for all states with $x_l' P_l^\varepsilon x_l \leq 1$. The decentralized control laws $\Sigma_j^{l,\varepsilon}$, $i = 1, \ldots, l$ together construct our decentralized feedback law for system Σ.

Finally, we show that for an appropriate choice of ε, this recursive algorithm provides a set of decentralized feedbacks which satisfy the requirements of Theorem 4.5. We will first prove properties 1, 2 and 3 listed above by induction. It is easy to see that the initialization step satisfies these properties. We assume that the design in the step m can be done, and then we must show that the design in the step $m+1$ can be done.

Now assume that we are in step $m+1$. The closed loop system $\Sigma_{\rm cl}^{m,\varepsilon}$ has properties (4.80), (4.81) and (4.82). Let λ be an eigenvalue on the unit disc of A_m^ε. We know that λ is not a fixed mode of the closed loop system. Thus there exist $\bar{K}_i$ such that

$$A_m^\varepsilon + \sum_{i=1}^{\nu} B_{m,i}\bar{K}_i C_{m,i}$$

has no eigenvalue at λ. Therefore the determinant of the matrix $\lambda I - A_m^\varepsilon - \delta\sum_{i=1}^{\nu} B_{m,i}\bar{K}_i C_{m,i}$, seen as a polynomial in δ, is non-zero for $\delta = 1$, which implies that it is non-zero for almost all $\delta > 0$. This means that for almost all $\delta > 0$

$$A_m^\varepsilon + \delta\sum_{i=1}^{\nu} B_{m,i}\bar{K}_i C_{m,i}$$

has no eigenvalue at λ. Let j be the largest integer such that

$$A_m^{\varepsilon,\delta} = A_m^\varepsilon + \delta\sum_{i=1}^{j} B_{m,i}\bar{K}_i C_{m,i}$$

has λ as an eigenvalue and the same number of eigenvalues on the unit disc as A_m^ε for small enough δ. This implies that $A_m^{\varepsilon,\delta}$ still has all its eigenvalues in the closed unit disc.

Using Lemma 4.2, we know that there exists a $\bar{P}_m^{\varepsilon,\delta}$ such that

$$(A_m^{\varepsilon,\delta})'\bar{P}_m^{\varepsilon,\delta}A_m^{\varepsilon,\delta} - \bar{P}_m^{\varepsilon,\delta} \le 0$$

while $\bar{P}_m^{\varepsilon,\delta} \to P_m^\varepsilon$ as $\delta \to 0$. Hence for small enough δ

$$x_m'(k)P_m^{\varepsilon,\delta}x_m(k) \le n - m + \frac{1}{2} \quad\Rightarrow\quad x_m'(k)P_m^\varepsilon x_m(k) \le n - m + 1$$

and also for small enough δ we have

$$\|\delta\bar{K}_i x_m\| \le \frac{1}{2n} \quad \forall x_m \text{ such that } x_m' P_m^{\varepsilon,\delta} x_m \le n - m + \frac{1}{2}.$$

We choose $\delta = \delta_\varepsilon$ small enough such that the above two properties hold. Define $K_i^\varepsilon = \delta_\varepsilon \bar{K}_i$, $\bar{P}_m^\varepsilon = \bar{P}_m^{\varepsilon,\delta_\varepsilon}$ and

$$\bar{A}_m^\varepsilon := A_m^\varepsilon + \sum_{i=1}^{j} B_{m,i}K_i^\varepsilon C_{m,i}.$$

By the definition of j, we know that

$$A_m^\varepsilon + \sum_{i=1}^{j+1} B_{m,i} K_i^\varepsilon C_{m,i}$$

either does not have λ as an eigenvalue or has less eigenvalues on the unit circle. This means that

$$(\bar{A}_m^\varepsilon, B_{m,j+1}, C_{m,j+1})$$

has a stabilizable and detectable eigenvalue on the unit circle. Let V be such that

$$VV' = I \quad \text{and} \quad \ker V = \ker\langle C_{m,j+1} | \bar{A}_m^\varepsilon \rangle.$$

Since we might not be able to find a stable observer for the state x_m we actually construct an observer for the observable part of the state Vx_m. Because our triplet has a stabilizable and detectable eigenvalue on the unit disc, the observable part of the state Vx_m must contain at least one eigenvalue on the unit circle that can be stabilized. This motivates the following decentralized feedback law:

$$\begin{aligned}
v_i^m(k) &= K_i^\varepsilon x_m(k) + v_j^{m+1}(k), \quad i = 1, \ldots, j, \\
p(k+1) &= A_s^\varepsilon p(k) + V B_{m,j+1} v_{j+1}^m(k) + K(C_{m,j+1} V' p(k) - y_{j+1}(k)), \\
v_{j+1}^m(k) &= F_\rho p(k) + v_{j+1}^{m+1}(k), \\
v_i^m(k) &= v_j^{m+1}(k), \quad i = j+2, \ldots, \nu.
\end{aligned}$$

Here $p \in \Re^s$ and A_s^ε is such that $A_s^\varepsilon V = V \bar{A}_m^\varepsilon$ and K is chosen such that $A_s^\varepsilon + K C_{m,j+1} V'$ has all its eigenvalues in the open unit disc and does not have any eigenvalues in common with $\bar{A}_k^\varepsilon$. Furthermore F_ρ is chosen in a way that $\bar{A}_m^\varepsilon + B_{m,j+1} F_\rho V$ has at least one less eigenvalue on the unit disc than A_m^ε and still all of its eigenvalues are in the closed unit disc and also $F_\rho \to 0$ as $\rho \to 0$. Defining

$$\bar{x}_{m+1} = \begin{pmatrix} x_m \\ p - V x_m \end{pmatrix},$$

we have

$$\begin{aligned}
\bar{x}_{m+1}(k+1) &= \begin{pmatrix} \bar{A}_m^\varepsilon + B_{m,j+1} F_\rho V & B_{m,j+1} F_\rho \\ 0 & A_s^\varepsilon + K C_{m,j+1} V' \end{pmatrix} \bar{x}_{m+1}(k) \\
&\quad + \sum_{i=1}^{\nu} \bar{B}_{m+1,i} v_j^{m+1}(k), \qquad (4.85) \\
y_i(k) &= \bar{C}_{m+1,i} \bar{x}_{m+1}(k), \quad i = 1, \ldots, \nu,
\end{aligned}$$

where

$$\bar{B}_{m+1,i} = \begin{pmatrix} B_{m,i} \\ -VB_{m,i} \end{pmatrix}, \quad \bar{C}_{m+1,i} = (C_{m,i}\ 0)$$

for $i \neq j+1$ and

$$\bar{B}_{m+1,j+1} = \begin{pmatrix} B_{m,j+1} \\ 0 \end{pmatrix}, \quad \bar{C}_{m+1,j+1} = \begin{pmatrix} C_{m,j+1} & 0 \\ V & I \end{pmatrix}.$$

It is easy to check that the above feedback laws satisfy the properties 1 and 2. What remains is to show that they satisfy property 3. Also we need to show that the control laws can be written in the form mentioned in step m for step $m+1$.

For any ε there exists a $\Re_m^\varepsilon > 0$ with

$$(A_s^\varepsilon + KC_{m,j+1}V')'\Re_m^\varepsilon(A_s^\varepsilon + KC_{m,j+1}V') - \Re_m^\varepsilon < 0$$

such that $\Re_m^\varepsilon \to 0$ as $\varepsilon \to 0$. Because $F_\rho \to 0$ as $\rho \to 0$, for each ε, for small enough ρ we have

$$\|F_\rho e\| < \frac{1}{2n} \quad \forall e \text{ such that } e'\Re_m^\varepsilon e \leq n - m + \frac{1}{2}.$$

Note that $\bar{A}_m^\varepsilon + B_{m,j+1}F_\rho V$ has at least one less eigenvalue on the unit disc than $\bar{A}_m^\varepsilon$ and has all its eigenvalues in the close unit disc. Applying Lemma (4.2), for small ρ we have

$$(\bar{A}_m^\varepsilon + B_{m,j+1}F_\rho V)'\bar{P}_\rho^\varepsilon(\bar{A}_m^\varepsilon + B_{m,j+1}F_\rho V) - \bar{P}_\rho^\varepsilon \leq 0$$

with $\bar{P}_\rho^\varepsilon \to \bar{P}_m^\varepsilon$ as $\rho \to 0$.

Now note that A_m^ε and $A_s^\varepsilon + KC_{m,j+1}V'$ have disjoint eigenvalues we find that for small ρ, the matrices $A_m^\varepsilon + B_{m,j+1}F_\varrho V$ and $A_s^\varepsilon + KC_{m,j+1}V'$ have disjoint eigenvalues since $F_\rho \to 0$ as $\rho \to 0$. But then there exists a $W_{\varepsilon,\rho}$ such that

$$B_{m,j+1}F_\rho + (\bar{A}_m^\varepsilon + B_{m,j+1}F_\rho V)W_{\varepsilon,\rho} - W_{\varepsilon,\rho}(A_s^\varepsilon + KC_{m,j+1}V') = 0$$

while $W_{\varepsilon,\rho} \to 0$ as $\rho \to 0$. Now if we define $\bar{P}_{m+1}^{\varepsilon,\rho}$ to be

$$\bar{P}_{m+1}^{\varepsilon,\rho} = \begin{pmatrix} I & 0 \\ -W'_{\varepsilon,\rho} & I \end{pmatrix} \begin{pmatrix} \bar{P}_\rho^\varepsilon & 0 \\ 0 & \Re_m^\varepsilon \end{pmatrix} \begin{pmatrix} I & -W_{\varepsilon,\rho} \\ 0 & I \end{pmatrix}.$$

We define

$$\bar{A}_{m+1}^{\varepsilon,\rho} = \begin{pmatrix} \bar{A}_m^\varepsilon + B_{m,j+1}F_\rho V & B_{m,j+1}F_\rho \\ 0 & A_s^\varepsilon + KC_{m,j+1}V' \end{pmatrix}.$$

We will have the following properties

$$(\bar{A}_{m+1}^{\varepsilon,\rho})'\bar{P}_{m+1}^{\varepsilon,\rho}\bar{A}_{m+1}^{\varepsilon,\rho} - \bar{P}_{m+1}^{\varepsilon,\rho} \leq 0$$

and

$$\lim_{\rho\to 0} \bar{P}^{\varepsilon,\rho}_{m+1} = \begin{pmatrix} \bar{P}^{\varepsilon}_m & 0 \\ 0 & \Re^{\varepsilon}_m \end{pmatrix}.$$

Now consider $\bar{x}_{m+1}$ such that

$$\bar{x}'_{m+1} \bar{P}^{\varepsilon,\rho}_{m+1} \bar{x}_{m+1} \le n - m.$$

Then with small enough choice of ρ we can have

$$\bar{x}'_m \bar{P}^{\varepsilon}_m x_m \le n - m + \frac{1}{2} \quad \text{and}$$
$$(p - V x_m)' \Re^{\varepsilon}_m (p - V x_m) \le n - m + \frac{1}{2}.$$

Next for each ε we choose $\rho = \rho_\varepsilon$ such that the above holds and we have

$$\| F_\rho V x_m \| < \frac{1}{2n} \quad \forall x_m \text{ such that } x'_m \bar{P}^{\varepsilon}_m x_m \le n - m + \frac{1}{2}.$$

Next we must check the bounds on the inputs in step $m+1$. For $i = 1, \dots, j$, we have

$$\|u_i\| = \| F^m_{i,\varepsilon} x_m + K^{\varepsilon}_i x_m \| \le \frac{m}{n} + \frac{1}{2n} \le \frac{m+1}{n}.$$

For $i = j+1$, we have:

$$\begin{aligned} \|u_i\| &= \| F^m_{i,\varepsilon} x_m + F_{\rho\varepsilon} p \| \\ &= \| F^m_{i,\varepsilon} x_m + F_{\rho\varepsilon} V x_m + F_{\rho\varepsilon}(p - V x_m) \| \\ &\le \frac{m}{n} + \frac{1}{2n} = \frac{m+1}{n}. \end{aligned}$$

Finally, for $i = j+2, \dots, \nu$, we have:

$$\|u_i\| = \| F^m_{i,\varepsilon} x_m \| \le \frac{m}{n} \le \frac{m+1}{n}.$$

Now for $i \ne j+1$ we set $n_{i,m+1} = n_{i,m}$ and for $i = j+1$ we set $n_{i,m+1} = n_{i,m} + s$.

If $n_{i,m} > 0$ we choose

$$p^{m+1}_j = \begin{pmatrix} p^m_i \\ p \end{pmatrix}$$

and if $n_{j,m} = 0$ we choose $p^{m+1}_j = p$. Now we are able to the system in terms of x_{m+1}. We introduce a basis transformation T_{m+1} such that $\bar{x}_{m+1} = T_{m+1} x_{m+1}$. Next, we define

$$p^{\varepsilon}_{m+1} = T'_{m+1} \bar{P}^{\varepsilon,\rho}_{m+1} T_{m+1}.$$

Now for $i = 1, \dots, \nu$ depending on the value of $n_{i,m+1}$ we can rewrite the control laws in the desired form and subsequently the properties 1–3 are obtained.

We know that there exists a value of m, say $l \le n$, such that A_l^ε has all its eigenvalues in the open unit disc. We set $\nu_j^l = 0$ for $j = 1, 2, 3, \dots, l$. Then the decentralized control laws $\Sigma_j^{l,\varepsilon}$, $i = 1, 2, 3, \dots, l$ together represent a decentralized semi global feedback law for the system Σ. In other words, we claim that for any given compact sets $W \subset \Re^n$ and $S_i \subset \Re^{n_j,l}$ for $j = 1, 2, 3, \dots, l$, there exists an ε^*. such that the origin of the closed loop system is exponentially stable for any $0 < \varepsilon < \varepsilon^*$. and the compact set $W \times S_1 \times \cdots \times S_\nu$ is within the domain of attraction. Furthermore for all initial conditions within $W \times S_1 \times \cdots \times S_\nu$, the closed loop system behaves like a linear system, that is the saturation is not activated.

We know that for ε small enough, the set

$$\Omega_1^\varepsilon := \{x_l \in \Re^{n_l} | x_l' P_l^\varepsilon x_l \le 1\}$$

is inside the domain of attraction of the equilibrium point of the closed loop system comprising the given system Σ and the decentralized control laws $\Sigma_j^{l,\varepsilon}$, $i = 1, 2, 3, \dots, l$ because for all initial conditions within Ω_1^ε, it is obvious that $\|u_i\| \le 1, i = 1, 2, 3, \dots, l$ which means that the closed loop system behaves like a linear system, that is the saturation is not activated. Furthermore since all of the eigenvalues of A_l^ε are in the open unit disc, this linear system is asymptotically stable. In addition because of the fact that $P_l^\varepsilon \to 0$ as $\varepsilon \to 0$, we find that $W \times S_1 \times \cdots \times S_\nu$ is inside Ω_1^ε for ε sufficiently small. This concludes that the decentralized control laws $\Sigma^{l,\varepsilon} i$, $i = 1, 2, 3, \dots, l$ are semi-globally stabilizing.

4.5 Notes and References

This chapter has investigated classes of decentralized systems that deploy incorporate multiple controllers in their basic operation. The systems include multi-channel time-delay systems, interconnected networked systems and discrete-systems with saturating controllers. In the first two-types, decentralized delay-dependent stability and stabilization methods were developed for a class of linear interconnection of time-delay plants subjected to convex-bounded parametric uncertainties and coupled with time-delay interconnections. The developed results provide initial step toward further developments around the deployment of multi-controller structures for resolving several issues for decentralized systems. Applications of the foregoing concepts to practical systems can be pursued further following the ideas in [21, 23, 24, 42, 44, 45]. Extensions to time-delay systems offer possibilities along the ideas of [11–19].

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Chapter 5
Decentralized Quantized Control

In this chapter, we address the problem of designing decentralized $\mathcal{H}_\infty$ feedback control for a class of linear interconnected systems with quantized signals in the subsystem control channels. Both continuous- and discrete-time systems are teated. The systems have unknown-but-bounded couplings and interval delays. A decentralized static output-feedback controller is designed at the subsystem level to render the closed-loop system delay-dependent asymptotically stable with guaranteed γ-level. When the local output measurements are quantized, a local output-dependent procedure is developed for updating the quantizer parameters to attain similar asymptotic stability and guaranteed performance of the closed-loop quantized system.

Then, the interesting problem of decentralized feedback control design for a class of linear interconnected discrete-time systems subject to overflow nonlinearities and unknown-but-bounded couplings is subsequently addressed. A decentralized state feedback quantized controller is designed at the subsystem level to render the closed-loop system asymptotically stable. When the local output measurements are available, a decentralized output-feedback quantized controller is developed attain similar asymptotic stability and guaranteed performance of the closed-loop quantized system.

Several special cases of interest are derived and simulation results are provided.

5.1 Decentralized Quantized Control I: Continuous Systems

In conventional feedback control theory, most of data and/or signals are directly processed. In emerging control systems including networks, all signals are transferred through network which eventually gives rise to packet dropouts or data transfer rate limitations [17]. Alternatively, signal processing and signal quantization always exist in computer-based control systems [22], in nanoscale servo control [16] and therefore recent research studies have been reported on the analysis and design problems for control systems involving various quantization methods [5, 8, 19, 29]. In [5], a quantizer taking value in a finite set is defined and then quantized

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feedback stabilization for linear systems is considered. The problem of stabilizing an unstable linear system by means of quantized state feedback, where the quantizer takes value in a countable set, is addressed in [8]. It should be noted that the approach in [5] relies on the possibility of making discrete online adjustments of quantizer parameters which was extended in [21] for more general nonlinear systems with general types of quantizers involving the states of the system, the measured outputs, and the control inputs. Based on [21], stabilization of discrete-time LTI systems with quantized measurement outputs is reported in [29]. Further related results are reported in [33, 34]. In terms of control design, it turns out that the use of output-feedback schemes [7] provides great flexibility in accommodating systems uncertainties. A decentralized $\mathcal{H}_\infty$ feedback control systems with two quantizers was considered in [6].

On another research front, decentralized stability and feedback stabilization of interconnected systems have been the topic of recurring interests and recent relevant results have been reported in [2, 24–28, 31]. In this chapter, we study the problem of decentralized $\mathcal{H}_\infty$ feedback control for a class of linear interconnected continuous-time systems with quantized signals in the subsystem control channel. The system has unknown-but-bounded couplings and interval time-delays. A decentralized static output-feedback controller is designed at the subsystem level using only local variables to render the overall closed-loop system is delay-dependent asymptotically stable with guaranteed γ-level. When the local output measurements are quantized before passing to the controller, we consider the local channel quantizer in a generalized form with a zoom parameter that can be adjusted on-line. We develop a local output-dependent procedure for updating the quantizer parameters to retain the delay-dependent asymptotic stability and guaranteed performance of the closed-loop quantized system.

5.1.1 Problem Statement

We consider a class of linear systems $\mathbf{S}$ structurally composed of n_s coupled subsystems $\mathbf{S}_j$ and the model of the jth subsystem is described by the state-space representation:

$$\begin{aligned}\mathbf{S}_j:\quad \dot{x}_j(t) &= A_j x_j(t) + A_{dj} x_j(t-\tau_j(t)) + B_j u_j(t) + \Gamma_j w_j(t)\\ &\quad + \sum_{k=1,k\neq j}^{n_s} F_{jk} x_k(t) + \sum_{k=1,k\neq j}^{n_s} E_{jk} x_k(t-\eta_{jk}(t)), \end{aligned} \tag{5.1}$$

$$z_j(t) = G_j x_j(t) + G_{dj} x_j(t-\tau_j(t)) + \Phi_j w_j(t), \tag{5.2}$$

$$y_j(t) = C_j x_j(t) + C_{dj} x_j(t-\tau_j(t)) + \Lambda_j w_j(t), \tag{5.3}$$

where for $j \in \{1, \ldots, n_s\}$, $x_j(t) \in \Re^{n_j}$ is the state vector, $u_j(t) \in \Re^{m_j}$ is the control input, $y_j(t) \in \Re^{p_j}$ is the measured output, $w_j(t) \in \Re^{q_j}$ is the disturbance input which belongs to $\mathcal{L}_2[0,\infty)$, $z_j(t) \in \Re^{q_j}$ is the performance output. The matrices $A_j \in \Re^{n_j\times n_j}$, $B_j \in \Re^{n_j\times m_j}$, $A_{dj} \in \Re^{n_j\times n_j}$, $\Phi_j \in \Re^{q_j\times q_j}$, $\Gamma_j \in \Re^{n_j\times q_j}$, $C_j \in$

$\Re^{p_j \times n_j}$, $C_{dj} \in \Re^{p_j \times n_j}$, $G_j \in \Re^{q_j \times n_j}$, $G_{dj} \in \Re^{q_j \times n_j}$, $\Lambda_j \in \Re^{p_j \times q_j}$, $F_{jk} \in \Re^{p_j \times q_j}$ and $E_{jk} \in \Re^{p_j \times q_j}$ are real and constants. The initial condition $\kappa_j \in \mathcal{L}_2[-\varrho_j, 0]$, $j \in \{1, \dots, n_s\}$. The factors $\tau_j, \eta_{jk}, \; j, k \in \{1, \dots, n_s\}$ are unknown time-varying delay factors satisfying

$$\begin{aligned} &0 \le \tau_j(t) \le \varrho_j, \quad \dot{\tau}_j(t) \le \mu_j, \\ &0 \le \eta_{jk}(t) \le \varrho_{jk}, \quad \dot{\eta}_{jk}(t) \le \mu_{jk}, \end{aligned} \tag{5.4}$$

where the bounds $\varrho_j, \varrho_{jk}, \mu_j, \mu_{jk}$ are known constants in order to guarantee smooth growth of the state trajectories. Note in (5.3) that the delay within each subsystem (local delay) and among the subsystems (coupling delay), respectively, are emphasized. The class of systems described by (5.1)–(5.3) subject to delay-pattern (5.4) is frequently encountered in modeling several physical systems and engineering applications including large space structures, multi-machine power systems, cold mills, transportation systems, water pollution management, to name a few [25, 30]. In what follows, we study the problem of decentralized $\mathcal{H}_\infty$ feedback control for a class of linear interconnected continuous-time systems with quantized signals in the subsystem control channel.

5.1.2 Local Quantizer Description

A block-diagram representation of the subsystem model (5.3) under consideration is depicted in Fig. 5.1. In the sequel, we adopt the definition of a local (subsystem) quantizer with general form as introduced in [21]. Let $f_j \in \Re^s$, $j = 1, \dots, n_s$ be the variable being quantized. A *local quantizer* is defined as a piecewise constant function $Q_j : \Re^s \to \mathcal{D}_j$, where $\mathcal{D}_j$ is a finite subset of $\Re^s$. This leads to a partition of $\Re^s$ into a finite number of quantization regions of the form $\{f_j \in \Re^s : Q(f_j) = d_j, d_j \in \mathcal{D}_j\}$. These quantization regions are not assumed to have any particular shape. We assume that there exist positive real numbers M_j and Δ_j such that the following conditions hold:

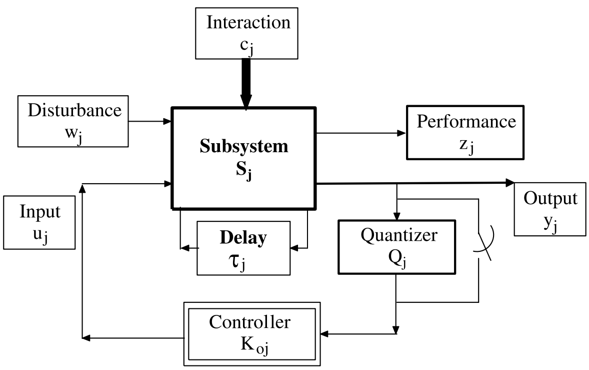


Fig. 5.1 Subsystem model with quantizer

$$1.\quad \text{If } |f_j| \le \mathsf{M}_j \quad \text{then } |Q_j(f_j) - f_j| \le \Delta_j. \tag{5.5}$$

$$2.\quad \text{If } |f_j| > \mathsf{M}_j \quad \text{then } |Q_j(f_j)| > \mathsf{M}_j - \Delta_j. \tag{5.6}$$

We note that condition (5.5) provides a bound on the quantization error when the quantizer does not saturate. Condition (5.6) gives a way to detect the possibility of saturation. In the sequel, M_j and Δ_j will be referred to as the *range of* Q_j and the *quantization error*, respectively. Henceforth, we assume that $Q(x) = 0$ for x in some neighborhood of the origin. The foregoing requirements are met by the quantizer with rectangular quantization regions [5, 19].

In the control strategy to be developed below, we will use local quantized measurements of the form

$$Q_{\mu_j}(f_j) = \mu_j Q_j\left(\frac{f_j}{\mu_j}\right), \tag{5.7}$$

where $\mu_j > 0$ is the subsystem parameter. Observe, at the subsystem level, the extreme case $\mu_j = 0$ is regarded as setting the output of the local quantizer as zero. This local quantizer has the range $\mathsf{M}_j \mu_j$ and the quantization error $\Delta_j \mu_j$. We can view μ_j as a *local zoom* variable: increasing μ_j corresponds to zooming out and essentially generating a new local quantizer with larger range and larger quantization error, whilst decreasing μ_j implies zooming in and obtaining a local quantizer with smaller range and smaller quantization error. We will update μ_j later on depending on the subsystem state (or the subsystem output). In some sense, it can regarded as additional state of the resultant closed-loop subsystem.

5.1.3 Static Output-Feedback Design

In this section, we develop new criteria for LMI-based characterization of decentralized stabilization by local static output-feedback. Initially without quantization (meaning that the switch in Fig. 5.1 is closed), we let the local decentralized static output-feedback has the form

$$u_j(t) = K_{oj} y_j(t), \quad j = 1, \dots, n_s, \tag{5.8}$$

where the gain matrices K_{oj}, $j = 1, \dots, N$ have been selected to guarantee the closed-loop system, composed of (5.1), (5.3) and (5.8), given by

$$\begin{aligned} \dot{x}_j(t) &= \mathcal{A}_j x_j(t) + \mathcal{A}_{dj} x_j(t - \tau_j(t)) + \Omega_j w_j(t) \\ &\quad + \sum_{k=1,k\ne j}^{n_s} F_{jk} x_k(t) + \sum_{k=1,k\ne j}^{n_s} E_{jk} x_k(t - \eta_{jk}(t)), \end{aligned} \tag{5.9}$$

$$\begin{aligned} z_j(t) &= G_j x_j(t) + G_{dj} x_j(t - \tau_j(t)) + \Phi_j w_j(t), \\ \mathcal{A}_j &= A_j + B_j K_{oj} C_j, \qquad \mathcal{A}_{dj} = A_{dj} + B_j K_{oj} C_{dj}, \\ \Omega_j &= \Gamma_j + B_j K_{oj} \Lambda_j \end{aligned} \tag{5.10}$$

is asymptotically stable with disturbance attenuation level γ_j to facilitate further development, we consider the case where the set of output matrices C_j, $j = 1, \ldots, n_s$ are assumed to be of full row rank and $C_j^\dagger$ represents the right-inverse. We consider the Lyapunov-Krasovskii functional (LKF):

$$V(t) = \sum_{j=1}^{n_s} V_j(t),$$

$$\begin{aligned} V_j(t) &= x_j^t(t)\mathcal{P}_j x_j(t) + \int_{-\varrho_j}^{0} x_j^t(\alpha)\mathcal{W}_j x_j(\alpha) d\alpha \\ &\quad + \int_{-\tau_j}^{0} x_j^t(\alpha)\mathcal{S}_j x_j(\alpha) d\alpha \\ &\quad + \varrho_j \int_{-\varrho_j}^{0}\int_{t+\theta}^{t} \dot{x}_j^t(s)\mathcal{R}_j \dot{x}_j(s)\, ds\, d\theta \\ &\quad + \sum_{k=1,k\neq j}^{n_s} \int_{t-\eta_{jk}(t)}^{t} x_k^t(s)\mathcal{Z}_{jk} x_k(s)\, ds \end{aligned} \tag{5.11}$$

where $0 < \mathcal{P}_j = \mathcal{P}_j^t$, $0 < \mathcal{W}_j = \mathcal{W}_j^t$, $0 < \mathcal{S}_j = \mathcal{S}_j^t$, $0 < \mathcal{R}_j = \mathcal{R}_j^t$, $0 < \mathcal{Z}_{jk} = \mathcal{Z}_{jk}^t$, $j, k \in \{1, \ldots, n_s\}$ are weighting matrices of appropriate dimensions.

The following theorems establishes the main design result for subsystem $\mathbf{S}_j$.

Theorem 5.1 *Given the bounds $\varrho_j > 0$, $\mu_j > 0$, $\varrho_{jk} > 0$, $\mu_{jk} > 0$, tuning parameters β_j, σ_j, $j, k = 1, \ldots, n_s$ and let the gain matrices K_{oj} be specified, then the family of subsystems described by (5.9)–(5.10) is delay-dependent asymptotically stable with $\mathcal{L}_2$-performance bound γ_j if there exist positive-definite matrices $\mathcal{P}_j$, $\mathcal{W}_j$, $\mathcal{S}_j$, $\mathcal{R}_j$, $\mathcal{Z}_{jk}$ and free-weighting matrices Θ_j, Υ_j, satisfying the following LMIs for $j, k = 1, \ldots, n_s$*

$$\hat{\Pi}_j = \begin{bmatrix} \Pi_j & \Pi_{vj} & \Pi_{wj} \\ \bullet & -\gamma_j^2 I_j & \Phi_j^t \\ \bullet & \bullet & -I_j \end{bmatrix} < 0, \tag{5.12}$$

where

$$\Pi_j = \begin{bmatrix} \Pi_{oj} & \Pi_{1j} & 0 & \Pi_{2j} & \Pi_{3j} \\ \bullet & \Pi_{aj} & 0 & \Pi_{4j} & \Pi_{5j} \\ \bullet & \bullet & \Pi_{cj} & \Pi_{6j} & 0 \\ \bullet & \bullet & \bullet & \Pi_{mj} & 0 \\ \bullet & \bullet & \bullet & \bullet & \Pi_{nj} \end{bmatrix}, \tag{5.13}$$

$$\begin{aligned} \Pi_{oj} &= (1+\sigma_j)\mathcal{P}_j\left(\mathcal{A}_j + \sum_{k=1,k\neq j}^{n_s} F_{kj}\right) \\ &\quad + \left(\mathcal{A}_j + \sum_{k=1,k\neq j}^{n_s} F_{kj}\right)^t (1+\sigma_j)\mathcal{P}_j \\ &\quad + \mathcal{W}_j + \mathcal{S}_j - \mathcal{R}_j + (n_s - 1)\mathcal{P}_j + \sum_{k=1,k\neq j}^{n_s} \mathcal{Z}_{kj}, \end{aligned}$$

$$\begin{aligned}
\Pi_{1j} &= -\sigma_j \mathcal{P}_j + \Upsilon_j \left(\mathcal{A}_j + \sum_{k=1,k\neq j}^{n_s} F_{kj} \right)^t, \\
\Pi_{2j} &= (1+\sigma_j)\mathcal{P}_j A_{dj} + \mathcal{R}_j, \\
\Pi_{3j} &= \sigma_j \mathcal{P}_j \sum_{k=1,k\neq j}^{n_s} E_{kj}, \quad \Pi_{aj} = \varrho_j^2 \mathcal{R}_j - 2\beta_j \mathcal{P}_j, \\
\Pi_{4j} &= \beta_j \mathcal{P}_j A_{dj}, \quad \Pi_{5j} = \beta_j \mathcal{P}_j \sum_{k=1,k\neq j}^{n_s} E_{kj}, \\
\Pi_{cj} &= -\mathcal{R}_j - \mathcal{W}_j, \quad \Pi_{6j} = \mathcal{R}_j, \\
\Pi_{mj} &= -2\mathcal{R}_j - (1-\mu_j)\mathcal{S}_j, \\
\Pi_{nj} &= -(1-\mu_{kj})\mathcal{Z}_{kj} - \sum_{k=1,k\neq j}^{n_s} E_{kj}^t \mathcal{P}_k E_{kj}, \\
\Pi_{vj} &= \left[\Omega_j^t \mathcal{P}_j \; 0\; 0\; 0\; 0 \right]^t, \quad \Pi_{wj} = \left[G_j^t \; 0\; 0\; G_{dj}^t \; 0 \right]^t .
\end{aligned} \tag{5.14}$$

Proof A straightforward computation gives the time-derivative of $V_j(t)$ along the solutions of (5.10) with $w(t) \equiv 0$ as:

$$\begin{aligned}
\dot{V}_j(t) &= 2x_j^t(t)\mathcal{P}_j \dot{x}_j(t) + \varrho_j^2 \dot{x}_j^t(t) \mathcal{R}_j \dot{x}_j(t) \\
&\quad - \varrho_j \int_{t-\varrho_j}^{t} \dot{x}_j^t(s)\mathcal{R}_j \dot{x}_j(s) ds + x_j^t(t)(\mathcal{W}_j + \mathcal{S}_j) x_j(t) \\
&\quad - x_j^t(t-\varrho_j(t))\mathcal{W}_j x_j(t-\varrho_j(t)) \\
&\quad - (1-\dot{\tau}(t)) x_j^t(t-\tau_j(t))\mathcal{S}_j x_j(t-\tau_j(t)) \\
&\quad + \sum_{k=1,k\neq j}^{n_s} \left[x_k^t(t)\mathcal{Z}_{jk} x_k(t) - (1-\dot{\eta}_{jk}(t)) x_k^t(t-\eta_{jk}(t))\mathcal{Z}_{jk} x_k(t-\eta_{jk}(t)) \right].
\end{aligned} \tag{5.15}$$

Using the identity

$$\begin{aligned}
&-\varrho_j \int_{-\varrho_j}^{0} \dot{x}_j^t(s)\mathcal{R}_j \dot{x}_j(s) ds \\
&\quad = -\varrho_j \int_{t-\varrho_j}^{t-\tau_j(t)} \dot{x}_j^t(s)\mathcal{R}_j \dot{x}_j(s) ds - \varrho_j \int_{t-\tau_j(t)}^{t} \dot{x}_j^t(s)\mathcal{R}_j \dot{x}_j(s) ds
\end{aligned} \tag{5.16}$$

then applying Jensen's inequality

$$\int_{t-\tau_j(t)}^{t} \dot{x}_j^t(s)\mathcal{R}_j \dot{x}_j(s) ds \geq \int_{t-\tau_j(t)}^{t} \dot{x}_j^t(s) ds \mathcal{R}_j \int_{t-\tau_j(t)}^{t} \dot{x}_j^t(s) \dot{x}_j(s) ds, \tag{5.17}$$

$$\int_{t-\varrho_j}^{t-\tau_j(t)} \dot{x}_j^t(s)\mathcal{R}_j \dot{x}_j(s) ds \geq \int_{t-\varrho_j}^{t-\tau_j(t)} \dot{x}_j^t(s) ds \mathcal{R}_j \int_{t-\varrho_j}^{t-\tau_j(t)} \dot{x}_j^t(s) \dot{x}_j(s) ds \tag{5.18}$$

and making use of the following structural identity

$$\sum_{k=1,k\neq j}^{n_s}\sum_{j=1}^{n_s} x_k^t(t)\mathcal{Z}_{jk}x_k(t) = \sum_{k=1,k\neq j}^{n_s}\sum_{j=1}^{n_s} x_j^t(t)\mathcal{Z}_{kj}x_j(t), \tag{5.19}$$

while invoking the algebraic inequality $X^tZ + Z^tX \leq X^tYX + Z^tY^{-1}Z$, $Y > 0$, such that

$$\begin{aligned} & 2x_j^t(t)\sum_{k=1,k\neq j}^{n_s} E_{jk}x_k(t-\eta_{jk}(t)) \\ & \leq (n_s-1)x_j^t(t)\mathcal{P}_j x_j(t) + \sum_{k=1,k\neq j}^{n_s} x_k^t(t-\eta_{jk}(t))E_{jk}^t\mathcal{P}_j E_{jk}x_k(t-\eta_{jk}(t)) \end{aligned} \tag{5.20}$$

it follows that

$$\begin{aligned} \dot{V}(t) \leq \sum_{j=1}^{n_s}\Bigg[& 2x_j^t(t)\mathcal{P}_j\Bigg(\mathcal{A}_j x_j(t) + \mathcal{A}_{dj}x_j(t-\tau_j(t)) + \sum_{k=1,k\neq j}^{n_s} F_{kj}x_j(t)\Bigg) \\ & + \varrho_j^2 \dot{x}_j^t(t)\mathcal{R}_j\dot{x}_j(t) \\ & - (x_j(t)-x_j(t-\tau_j(t)))^t\mathcal{R}_j(x_j(t)-x_j(t-\tau_j(t))) \\ & - (x_j(t-\tau_j(t))-x_j(t-\varrho_j(t)))^t\mathcal{R}_j(x_j(t-\tau_j(t))-x_j(t-\varrho_j(t))) \\ & + x_j^t(t)(\mathcal{W}_j+\mathcal{S}_j)x_j(t) - x_j^t(t-\varrho_j)\mathcal{W}_j x_j(t-\varrho_j) \\ & - (1-\mu_j)x_j^t(t-\tau_j(t))\mathcal{S}_j x_j(t-\tau_j(t)) \\ & + x_j^t(t)\Bigg(\sum_{k=1,k\neq j}^{n_s}\mathcal{Z}_{kj}\Bigg)x_j(t) \\ & + (n_s-1)x_j^t(t)\mathcal{P}_j x_j(t) \\ & + \sum_{k=1,k\neq j}^{n_s} x_k^t(t-\eta_{kj}(t))E_{kj}^t\mathcal{P}_k E_{kj}x_j(t-\eta_{kj}(t)) \\ & - (1-\mu_{kj})x_j^t(t-\eta_{kj}(t))\mathcal{Z}_{kj}x_j(t-\eta_{kj}(t))\Bigg]. \end{aligned} \tag{5.21}$$

Adding the zero-value expression with β_j and σ_j are tunning parameters

$$\begin{aligned} 0 \equiv 2[\sigma_j x_j^t(t)\mathcal{P}_j + \beta_j\dot{x}_j^t(t)\mathcal{P}_j]\Bigg[& -\dot{x}_j(t) + \mathcal{A}_j x_j(t) + \mathcal{A}_{dj}x_j(t-\tau_j(t)) \\ & + \sum_{k=1,k\neq j}^{n_s} F_{jk}x_k(t) + \sum_{k=1,k\neq j}^{n_s} E_{jk}x_k(t-\eta_{jk}(t))\Bigg] \end{aligned} \tag{5.22}$$

to the right-hand side of (5.21) and setting

$$\zeta_j(t) = \mathrm{col}\{x_j(t), \dot{x}_j(t), x_j(t-\varrho_j), x_j(t-\tau_j(t)), x_j(t-\eta_{kj}(t))\}$$

it follows that

$$\dot{V}(t) \leq \sum_{j=1}^{n_s} \zeta_j^t(t) \Pi_j \zeta_j(t) \leq 0 \tag{5.23}$$

if the matrix Π_j given by (5.13) is feasible. Internal asymptotic stability requirement $\dot{V}_j(t)|_{(5.9)} < 0$ implies that $\Pi_j < 0$. Next, considering the $\mathcal{L}_2$-*gain* performance measure $J = \sum_{j=1}^{n_s} J_j$ for any $w_j(t) \in \mathcal{L}_2(0, \infty) \neq 0$ with zero initial condition $x_j(0) = 0$ hence $V(0) = 0$, we have

$$\begin{aligned} J_j &= \int_0^\infty \big(z_j^t(s) z_j(s) - \gamma_j^2 w_j^t(s) w_j(s)\big) ds \\ &\leq \int_0^\infty \big(z_j^t(s) z_j(s) - \gamma_j^2 w_j^t(s) w_j(s) + \dot{V}_j(s)|_{(5.9)}\big) ds. \end{aligned} \tag{5.24}$$

Using (5.10), we obtain:

$$z_j^t(s) z_j(s) - \gamma_j^2 w_j^t(s) w_j(s) + \dot{V}_j(s)|_{(5.9)} \leq [\zeta_j^t(s)\ w_j^t(s)] \hat{\Pi}_j [\zeta_j^t(s)\ w_j^t(s)]^t, \tag{5.25}$$

where $\hat{\Pi}_j$ is given by (5.26). It is readily seen that

$$(z_j^t(s) z_j(s) - \gamma_j^2 w_j^t(s) w_j(s) + \dot{V}_j(s)|_{(5.9)} < 0$$

for arbitrary $s \in [t, \infty)$, which implies for any $w_j(t) \in \mathcal{L}_2(0, \infty) \neq 0$ that $J_j < 0$ leading to $\|z_j(t)\|_2 < \sum_{j=1}^{n_s} \gamma_j \|w(t)_j\|_2$, which assures the desired performance. □

Theorem 5.2 *Given the bounds $\varrho_j > 0$, $\mu_j > 0$, $\varrho_{jk} > 0$, $\mu_{jk} > 0$ and tuning parameters β_j, σ_j, $j, k = 1, \ldots, n_s$. The family of subsystems described by* (5.9)–(5.10) *is delay-dependent asymptotically stabilizable by decentralized static output-feedback controller $u_j(t) = K_{oj} y_j(t)$, $j = 1, \ldots, n_s$ with $\mathcal{L}_2$-performance bound γ_j, $j = 1, \ldots, n_s$ if there exist positive-definite matrices $\mathcal{Y}_j$, $\mathcal{G}_j$, Ψ_{1j}, Ψ_{2j}, Ψ_{3j}, Ψ_{4j}, Ψ_{1kj}, Ψ_{2kj} satisfying the following LMIs for $j = 1, \ldots, n_s$*

$$\widehat{\Xi}_j = \begin{bmatrix} \Xi_j & \Xi_{vj} & \Xi_{wj} \\ \bullet & -\gamma_j^2 I_j & \Phi_j^t \\ \bullet & \bullet & -I_j \end{bmatrix} < 0, \tag{5.26}$$

where

$$\left\{\begin{aligned}
\Xi_j &= \begin{bmatrix} \Xi_{ej} & \Xi_{1j} & 0 & \Xi_{2j} & \Xi_{3j} \\ \bullet & \Xi_{aj} & 0 & \Xi_{4j} & \Xi_{5j} \\ \bullet & \bullet & \Xi_{cj} & \Xi_{6j} & 0 \\ \bullet & \bullet & \bullet & \Xi_{mj} & 0 \\ \bullet & \bullet & \bullet & \bullet & \Xi_{nj} \end{bmatrix}, \\
\Xi_{ej} &= (1+\sigma_j)\left[\left(A_j + \sum_{k=1,k\neq j}^{n_s} F_{kj}\right)\mathcal{Y}_j + B_j\mathcal{G}_j\right] \\
&= (1+\sigma_j)\left[\mathcal{Y}_j\left(A_j + \sum_{k=1,k\neq j}^{n_s} F_{kj}\right)^t + \mathcal{G}_j^t B_j^t\right] \\
&\quad + \Psi_{1j} + \Psi_{2j} - \Psi_{3j} + (n_s-1)\mathcal{Y}_j + \sum_{k=1,k\neq j}^{n_s} \Psi_{1kj}, \\
\Xi_{1j} &= -\sigma_j\mathcal{Y}_j + \beta_j\left[\left(A_j + \sum_{k=1,k\neq j}^{n_s} F_{kj}\right)^t \mathcal{Y}_j + \mathcal{G}_j^t B_j^t\right], \\
\Xi_{2j} &= (1+\sigma_j)A_{dj}\mathcal{Y}_j + \Psi_{3j}, \\
\Xi_{3j} &= \sigma_j \sum_{k=1,k\neq j}^{n_s} E_{kj}\mathcal{Y}_j, \quad \Xi_{aj} = \varrho_j^2\Psi_{3j} - 2\beta_j\mathcal{Y}_j, \\
\Xi_{4j} &= \beta_j A_{dj}\mathcal{Y}_j, \quad \Xi_{5j} = \beta_j \sum_{k=1,k\neq j}^{n_s} E_{kj}\mathcal{Y}_j, \\
\Xi_{cj} &= -\Psi_{1j} - \Psi_{3j}, \quad \Xi_{6j} = \Psi_{3j}, \\
\Xi_{mj} &= -2\Psi_{3j} - (1-\mu_j)\Psi_{2j}, \\
\Xi_{nj} &= -(1-\mu_{kj})\Psi_{1kj} - \sum_{k=1,k\neq j}^{n_s} \Psi_{2kj}, \\
\Xi_{vj} &= \left[\Gamma_j^t + \Psi_{4j}^t \; 0\;0\;0\;0\right]^t, \quad \Xi_{wj} = \left[G_j^t\mathcal{Y}_j \; 0\;0\; G_{dj}^t\mathcal{Y}_j \; 0\right]^t.
\end{aligned}\right. \tag{5.27}$$

Moreover, the local gain matrix is given by $K_j = \mathcal{G}_j\mathcal{Y}_j^{-1}C_j^{\dagger}$.

Proof Applying the congruent transformation

$$\mathsf{T} = \mathrm{diag}[\mathcal{Y}_j,\ \mathcal{Y}_j,\ \mathcal{Y}_j,\ \mathcal{Y}_j,\ \mathcal{Y}_j,\ I_j,\ I_j],\ \mathcal{Y}_j = \mathcal{P}_j^{-1}$$

to LMI (5.12) with (5.13)–(5.14) and using the linearizations

$$\begin{aligned}
&\mathcal{G}_j = K_{oj}C_j\mathcal{Y}_j, \quad \Psi_{1j} = \mathcal{Y}_j\mathcal{W}_j\mathcal{Y}_j, \quad \Psi_{2j} = \mathcal{X}_j\mathcal{S}_j\mathcal{X}_j, \\
&\Psi_{3j} = \mathcal{Y}_j\mathcal{R}_j\mathcal{Y}_j, \quad \Psi_{1kj} = \mathcal{Y}_j\mathcal{Z}_{kj}\mathcal{Y}_j, \quad \Psi_{4j} = B_jK_{oj}\Lambda_j, \\
&\Psi_{2kj} = \mathcal{Y}_jE_{kj}^t\mathcal{W}_kE_{kj}\mathcal{Y}_j
\end{aligned}$$

we readily obtain LMI (5.26) with (5.27) and therefore the proof is completed. □

Remark 5.1 We note that the case of decentralized state feedback control $u_j(t) = K_j x_j(t)$, $j = 1, \ldots, n_s$ can be readily from Theorem 5.1 by setting $C_j \equiv I_j$, $C_{dj} \equiv 0$, $E_{dj} \equiv 0$, $\Lambda_j \equiv 0$ so that the resulting closed-loop system is asymptotically stable with guaranteed $\mathcal{H}_\infty$ performance.

5.1.4 Quantized Output-Feedback Design

Focusing on the availability of quantized local output information (meaning that the switch in Fig. 5.1 is open), we modify the static output feedback (5.8) using the quantized information of y_j as

$$u_j(t) = \mu_j K_{oj} Q_j\left(\frac{y_j(t)}{\mu_j}\right), \quad j = 1, \ldots, n_s. \tag{5.28}$$

For any fixed scalar $\mu_j > 0$, the closed-loop system, composed of (5.1), (5.3) and (5.28) is given by

$$\begin{aligned}
\dot{x}_j(t) &= \mathcal{A}_j x_j(t) + \mathcal{A}_{dj} x_j(t - \tau_j(t)) + c_j(t) + \Omega_j w_j(t) + H_j(\mu_j, y_j),\\
z_j(t) &= G_j x_j(t) + G_{dj} x_j(t - \tau_j(t)) + \Phi_j w_j(t),\\
\mathcal{A}_j &= A_j + B_j K_{oj} C_j, \quad \mathcal{A}_{dj} = A_{dj} + B_j K_{oj} C_{dj},\\
\Omega_j &= \Gamma_j + B_j K_{oj} \Psi_j,\\
H_j(\mu_j, y_j) &= \mu_j B_j K_{oj}\left(Q_j \frac{y_j(t)}{\mu_j} - \frac{y_j(t)}{\mu_j}\right).
\end{aligned} \tag{5.29}$$

Next, we move to examine the stability and desired disturbance attenuation level of the closed-loop system (5.29) in the presence of the quantization error. We employ the LKF (5.11) and consider that the gains K_{oj} are obtained from application of Theorem 5.1. The following theorem establishes the main design result for subsystem $\mathbf{S}_j$.

Theorem 5.3 *Given the bounds $\varrho_j > 0$, $\mu_j > 0$, $\varrho_{jk} > 0$, $\mu_{jk} > 0$ and tuning parameters β_j, σ_j, $j, k = 1, \ldots, n_s$. If the local quantizer M_j is selected large enough with respect to Δ_j while adjusting the local scalar α_j so as to satisfy the inequality*

$$\mathsf{M}_j > 4\Delta_j \frac{\|\mathcal{P}_j B_j K_{oj}\|}{\lambda_m(\Lambda_j)} \|C_j + \alpha_j C_{dj}\|. \tag{5.30}$$

Then, the family of subsystems $\{\mathbf{S}_j\}$ where $\{\mathbf{S}_j\}$ is described by (5.1)–(5.3) is delay-dependent asymptotically stabilizable with $\mathcal{L}_2$-performance bound γ_j by decentralized quantized output-feedback controller (5.28).

Proof Since

$$\frac{y_j(t)}{\mu_j} = \frac{C_j x_j(t) + C_{dj} x_j(t - \tau_j(t))}{\mu_j}$$

is quantized before being passed to the feedback, we obtain by using the properties of local quantizer (5.5) and (5.6) that whenever $|y_j(t)| \leq \mathsf{M}_j \mu_j$, the inequality

$$\left| \frac{y_j(t)}{\mu_j} - Q_j\left(\frac{y_j(t)}{\mu_j}\right) \right| \leq \Delta_j \tag{5.31}$$

holds true. Extending on Theorem 5.1, it follows that

$$\begin{aligned} J_j \leq \int_0^\infty & \big([\zeta_j^t(s)\ w_j^t(s)] \widehat{\Pi}_j [\zeta_j^t(s)\ w_j^t(s)]^t \\ & + 2x_j^t \mathcal{P}_j H_j(\mu_j, y_j) - x_j^t \Lambda_j x_j \big) ds, \end{aligned} \tag{5.32}$$

where $\widehat{\Pi}_j$ corresponds to $\hat{\Pi}_j$ except that $\Pi_{oj} \to \Pi_{oj} + \Lambda_j$ with $\Lambda_j > 0$ being an arbitrary matrix. Proceeding as before, we focus on the integrand in (5.32) while letting $\|x_j(t-\tau_j)\| \leq \alpha_j \|x_j(t)\|$, $\alpha_j > 0$ and manipulating to get

$$\begin{aligned} & [\zeta_j^t(s)\ w_j^t(s)] \widehat{\Pi}_j [\zeta_j^t(s)\ w_j^t(s)]^t + 2x_j^t \mathcal{P}_j H_j(\mu_j, y_j) - x_j^t \Lambda_j x_j \\ & \leq [\zeta_j^t(s)\ w_j^t(s)] \widehat{\Pi}_j [\zeta_j^t(s)\ w_j^t(s)]^t - \frac{1}{2}\lambda_m(\Lambda_j)\left(|x_j| - 4\Delta_j \frac{\|\mathcal{P}_j B_j K_{oj}\|}{\lambda_m(\Lambda_j)} \mu_j \right) \\ & \leq [\zeta_j^t(s)\ w_j^t(s)] \widehat{\Pi}_j [\zeta_j^t(s)\ w_j^t(s)]^t \\ & \quad - \frac{1}{2}\lambda_m(\Lambda_j) \frac{|x_j|}{\|C_j + \alpha_j C_{dj}\|}\left(|y_j| - 4\Delta_j \frac{\|\mathcal{P}_j B_j K_{oj}\|}{\lambda_m(\Lambda_j)} \|C_j + \alpha_j C_{dj}\| \mu_j \right). \end{aligned} \tag{5.33}$$

It follows from (5.30), we can always find a scalar $\beta_j \in (0, 1)$ such that

$$\mathsf{M}_j > 4\Delta_j \frac{\|\mathcal{P}_j B_j K_{oj}\|}{\lambda_m(\Lambda_j)} \|C_j + \alpha_j C_{dj}\| \frac{1}{1-\beta_j}. \tag{5.34}$$

This is equivalent to

$$\frac{1}{1-\beta_j} 4\Delta_j \frac{\|\mathcal{P}_j B_j K_{oj}\|}{\lambda_m(\Lambda_j)} \|C_j + \alpha_j C_{dj}\| \mu_j < \mathsf{M}_j \mu_j. \tag{5.35}$$

Therefore, for any $\mu_j \neq 0$, we can find a scalar $\mu_j > 0$ such that

$$\frac{1}{1-\beta_j} \cdot 4\Delta_j \frac{\|\mathcal{P}_j B_j K_{oj}\|}{\lambda_m(\Lambda_j)} \|C_j + \alpha_j C_{dj}\| \mu_j \leq |y_j| \leq \mathsf{M}_j \mu_j. \tag{5.36}$$

At the extreme case $|y_j| = 0$, we set $\mu_j = 0$ so that the output of the local quantizer is considered zero and therefore (5.36) holds true. This, in turn, implies that we can always select μ_j so that (5.36) is satisfied, (5.33) holds and hence

$$J_j \leq \chi_j^t(t,s) \hat{\Pi}_j \chi_j(t,s) - \frac{1}{2}\beta_j \lambda_m(\Lambda_j) \frac{|x_j|}{\|C_j + \alpha_j C_{dj}\|} |y_j| \tag{5.37}$$

where $\hat{\Pi}_j$ is given by (5.26) for some vector $\chi_j(t,s)$. The rest of the proof follows from Theorem 5.1. □

Remark 5.2 For the case of decentralized state feedback control $u_j(t) = K_j x_j(t)$, $j = 1, \ldots, n_s$, then Theorem 5.3 specializes to the following corollary.

Corollary 5.1 *Given the bounds $\varrho_j > 0$, $\mu_j > 0$. If the local quantizer M_j is selected large enough with respect to Δ_j while adjusting the local scalar α_j so as to satisfy the inequality*

$$\mathsf{M}_j > 4\Delta_j \frac{\|\mathcal{P}_j B_j K_{oj}\|}{\lambda_m(\Lambda_j)}. \tag{5.38}$$

Then, the family of subsystems $\{\mathbf{S}_j\}$ where $\mathbf{S}_j$ is described by (5.1)–(5.3) is delay-dependent asymptotically stabilizable with $\mathcal{L}_2$-performance bound γ_j by decentralized quantized state-feedback controller

$$u_j(t) = \mu_j K_j Q_j\left(\frac{x_j(t)}{\mu_j}\right), \quad j = 1, \ldots, n_s.$$

Remark 5.3 By the mean-value theorem and following [14], it can be shown that $\lambda_m(\mathcal{P}_j)\|x_j\|^2 \le V_j \le \vartheta_j\|\kappa_j\|^2$ where

$$\vartheta_j = \left[\lambda_M(\mathcal{P}_j) + \varrho_j[\lambda_M(\mathcal{Z}_j) + \lambda_M(\mathcal{W}_j)] + 3\varrho_j^2[\lambda_M(A_j^t A_j) + (\lambda_M(A_{dj}^t A_{dj})]\right].$$

Based on the results of [21], we define the local ellipsoids

$$\begin{aligned}
\mathcal{B}_{oj}(\mu_j) &:= \{x_j : x_j^t \mathcal{P}_j x_j \le \lambda_m(\mathcal{P}_j \mathsf{M}_j^2 \mu_j^2\},\\
\mathcal{B}_{sj}(\mu_j) &:= \{x_j : x_j^t \mathcal{P}_j x_j \le \lambda_M(\mathcal{P}_j \mathcal{D}_j^2 \Delta_j^2 (1+\sigma_j)^2 \mu_j^2\},\\
\mathcal{D}_j &:= 2\frac{\|\mathcal{P}_j B_j K_{oj}\|}{\lambda_m(\Lambda_j)}\|C_j + \alpha_j C_{dj}\|.
\end{aligned}$$

In the "zooming-in" stage, it can be inferred that $\mathcal{B}_{sj}(\mu_j) \subset \mathcal{B}_{oj}(\mu_j)$ are invariant regions for system (5.29) given $\sigma_j > 0$. Moreover, all solutions of (5.29) that start in $\mathcal{B}_{oj}(\mu_j)$ enter $\mathcal{B}_{sj}(\mu_j)$ in finite time.

Remark 5.4 The introduction of the local scalar α_j stems from stability consideration of system (5.29) in the light of Razumikhin theory [26]. It is crucial to recognize that it plays a basic role in steering the trajectories of (5.29) towards the final ellipsoid $\mathcal{B}_{sj}(\mu_j)$. This is a distinct feature of quantized time-delay systems.

Remark 5.5 We note in Theorem 5.3 and Corollary 5.1 there are several degrees of freedom to achieve the desired stability with guaranteed performance, particularly since both the off-line gain computation and the on-line quantized feedback are decentralized. This is a salient feature of the developed results of this chapter, which is not shared by several published results [2, 28, 30, 31].

5.1.5 Simulation Example 5.1

To illustrate the theoretical developments, we consider a plant comprised of three chemical reactors. By linearization and time scaling the model matrices in the form of (5.1)–(5.3) have the values:

$$A_j = \begin{bmatrix} -a_{1j} & -1.01 & 0 & 0 \\ -3.2 & -a_{2j} & -12.8 & 0 \\ 6.4 & 0.347 & -a_{3j} & -1.04 \\ 0 & 0.833 & 11.0 & -a_{4j} \end{bmatrix}, \qquad \Gamma_j = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix},$$

$$A_{dj} = \begin{bmatrix} b_{1j} & 0 & 0 & 0 \\ 0 & b_{2j} & 0 & 0 \\ 0 & 0 & b_{3j} & 0 \\ 0 & 0 & 0 & b_{4j} \end{bmatrix}, \qquad \Phi_j = 0.1,$$

$$G_j = [0.1\ 0.2\ 0.4\ 0.3], \qquad G_{dj} = [0.01\ 0\ 0.01\ 0], \qquad \Lambda_j = 0.1,$$

$$B_j^t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$C_j = [10\ 0\ 0\ 0], \qquad C_{dj} = [1\ 0\ 0\ 0],$$

where the values of the parameters are given in Table 5.1. The coupling matrices F_{jk}, E_{jk} are generated randomly. The feasible solution of Theorem 5.3 is found to be

$$\begin{aligned} &\mu_1 = 2, \quad \varrho_1 = 0.775, \quad \gamma_1 = 0.561, \quad \varrho_{12} = 0.819, \quad \varrho_{13} = 0.831, \\ &\mu_{12} = 1.311, \quad \mu_{13} = 1.176, \quad K_{o1}^t = \begin{bmatrix} 7.535 & -3.962 \end{bmatrix}, \\ &\mu_2 = 2, \quad \varrho_2 = 0.775, \quad \gamma_2 = 0.477, \quad \varrho_{21} = 0.921, \quad \varrho_{23} = 0.976, \\ &\mu_{21} = 1.421, \quad \mu_{23} = 1.324, \quad K_{o2}^t = \begin{bmatrix} 1.741 & -10.124 \end{bmatrix}, \\ &\mu_3 = 2, \quad \varrho_3 = 0.775, \quad \gamma_3 = 0.601, \quad \varrho_{31} = 0.819, \quad \varrho_{32} = 0.831, \\ &\mu_{31} = 1.311, \quad \mu_{32} = 1.176, \quad K_{o3}^t = \begin{bmatrix} 3.966 & -4.524 \end{bmatrix}. \end{aligned}$$

Typical simulation results are plotted in Figs. 5.2, 5.3 and 5.4.

Table 5.1 Model parameters

Parameter	S_1	S_2	S_3
a_{1j}	4.931	4.886	4.902
a_{2j}	5.301	5.174	5.464
a_{3j}	32.511	30.645	31.773
a_{4j}	3.961	3.878	3.932
b_{1j}	1.921	1.915	1.908
b_{2j}	1.921	1.914	1.907
b_{3j}	1.878	1.866	1.869
b_{4j}	0.724	0.715	0.706

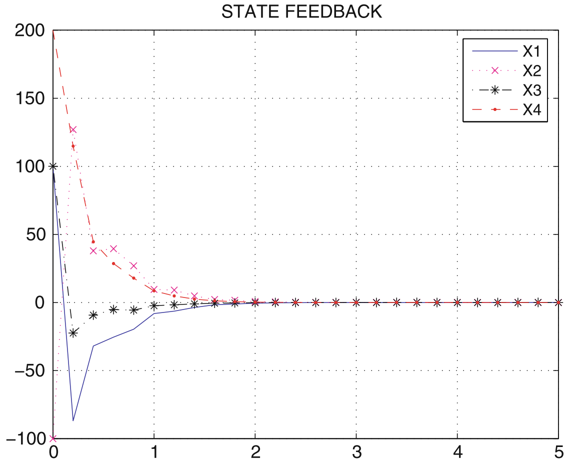


Fig. 5.2 Closed-loop state-trajectories: subsystem 1

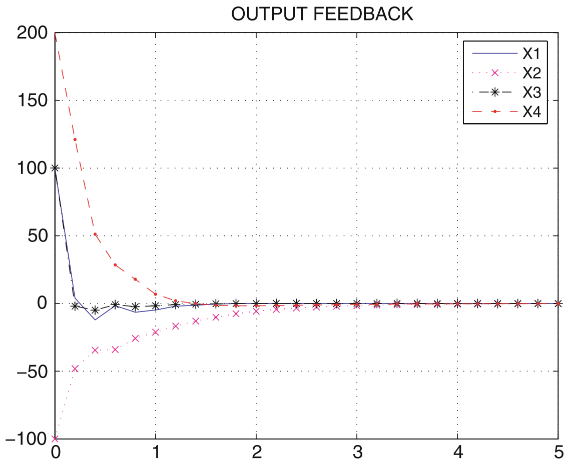


Fig. 5.3 Closed-loop state-trajectories: subsystem 2

5.1.6 Polytopic Systems

When the local subsystems undergo polytopic uncertainties, the model matrices will belong to a real convex bounded polytope of the type

$$\begin{bmatrix} A_j & A_{dj} & B_j & \Gamma_j \\ G_j & G_{dj} & \Lambda_j & \Phi_j \\ C_j & C_{dj} & E_{jk} & F_{jk} \end{bmatrix}$$
$$\in \Pi_\lambda := \left\{ \begin{bmatrix} A_{j\lambda} & A_{dj\lambda} & B_{j\lambda} & \Gamma_{j\lambda} \\ G_{j\lambda} & G_{dj\lambda} & \lambda_{j\lambda} & \Phi_{j\lambda} \\ C_{j\lambda} & C_{dj\lambda} & E_{jk\lambda} & F_{jk\lambda} \end{bmatrix} \right.$$
$$\left. = \sum_{s=1}^{n_s} \lambda_s \begin{bmatrix} A_{js} & A_{djs} & B_{js} & \Gamma_{js} \\ G_{js} & G_{djs} & D_{js} & \Phi_{js} \\ C_{js} & C_{djs} & E_{jks} & F_{jks} \end{bmatrix}, \lambda_s \in \Lambda \right\}, \qquad (5.39)$$

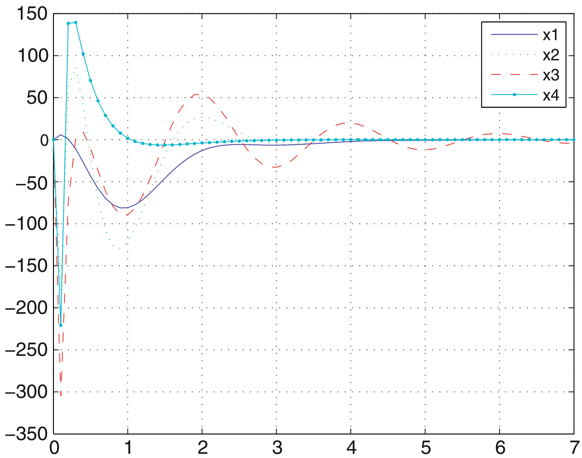


Fig. 5.4 Closed-loop state-trajectories: subsystem 3

where Λ is the unit simplex

$$\Lambda := \left\{ (\lambda_1, \ldots, \lambda_{n_s}) : \sum_{j=1}^{n_s} \lambda_j = 1, \lambda_j \geq 0 \right\}. \tag{5.40}$$

Theorem 5.4 *Given the bounds $\varrho_j > 0$, $\mu_j > 0$, $\varrho_{jk} > 0$, $\mu_{jk} > 0$ and tuning parameters β_j, σ_j, $j,k = 1, \ldots, n_s$. The family of subsystems described by (5.9)–(5.10) with polytopic representation (5.39)–(5.40) is delay-dependent asymptotically stabilizable by decentralized static output-feedback controller $u_j(t) = K_{oj} y_j(t)$, $j = 1, \ldots, n_s$ with $\mathcal{L}_2$-performance bound γ_j, $j = 1, \ldots, n_s$ if there exist positive-definite matrices $\mathcal{Y}_j$, $\mathcal{G}_j$, Ψ_{1j}, Ψ_{2j}, Ψ_{3j}, Ψ_{4j}, Ψ_{1kj}, Ψ_{2kj} satisfying the following LMIs for $s, j = 1, \ldots, n_s$*

$$\widehat{\Xi}_{sj} = \begin{bmatrix} \Xi_{sj} & \Xi_{vsj} & \Xi_{wsj} \\ \bullet & -\gamma_j^2 I_j & \Phi_{sj}^t \\ \bullet & \bullet & -I_j \end{bmatrix} < 0, \tag{5.41}$$

where

$$\Xi_{sj} = \begin{bmatrix} \Xi_{esj} & \Xi_{1sj} & 0 & \Xi_{2sj} & \Xi_{3sj} \\ \bullet & \Xi_{asj} & 0 & \Xi_{4sj} & \Xi_{5sj} \\ \bullet & \bullet & \Xi_{cj} & \Xi_{6sj} & 0 \\ \bullet & \bullet & \bullet & \Xi_{msj} & 0 \\ \bullet & \bullet & \bullet & \bullet & \Xi_{nsj} \end{bmatrix},$$

$$\Xi_{esj} = (1 + \sigma_j)\left[\left(A_{sj} + \sum_{k=1, k\neq j}^{n_s} F_{kjs} \right) \mathcal{Y}_j + B_{sj} \mathcal{G}_j \right]$$

$$= (1 + \sigma_j)\left[\mathcal{Y}_j \left(A_{sj} + \sum_{k=1, k\neq j}^{n_s} F_{kjs} \right)^t + \mathcal{G}_j^t B_{sj}^t \right]$$

$$+\Psi_{1sj}+\Psi_{2sj}-\Psi_{3sj}+(n_s-1)\mathcal{Y}_j+\sum_{k=1,k\neq j}^{n_s}\Psi_{1ksj},$$

$$\begin{aligned}
\Xi_{1sj} &= -\sigma_j\mathcal{Y}_j+\beta_j\left[\left(A_{sj}+\sum_{k=1,k\neq j}^{n_s}F_{kjs}\right)^t\mathcal{Y}_j+\mathcal{G}_j^tB_{js}^t\right],\\
\Xi_{2sj} &= (1+\sigma_j)A_{dsj}\mathcal{Y}_j+\Psi_{3sj},\\
\Xi_{3sj} &= \sigma_j\sum_{k=1,k\neq j}^{n_s}E_{kj}\mathcal{Y}_j,\quad \Xi_{asj}=\varrho_j^2\Psi_{3sj}-2\beta_j\mathcal{Y}_j,\\
\Xi_{4sj} &= \beta_jA_{dsj}\mathcal{Y}_j,\quad \Xi_{5sj}=\beta_j\sum_{k=1,k\neq j}^{n_s}E_{kjs}\mathcal{Y}_j,\\
\Xi_{csj} &= -\Psi_{1j}-\Psi_{3j},\quad \Xi_{6sj}=\Psi_{3sj},\\
\Xi_{msj} &= -2\Psi_{3j}-(1-\mu_j)\Psi_{2sj},\\
\Xi_{nsj} &= -(1-\mu_{kj})\Psi_{1kj}-\sum_{k=1,k\neq j}^{n_s}\Psi_{2ksj},\\
\Xi_{vsj} &= \left[\,\Gamma_{sj}^t+\Psi_{4sj}^t\;0\;0\;0\;0\,\right]^t,\\
\Xi_{wsj} &= \left[\,G_{sj}^t\mathcal{Y}_j\;0\;0\;G_{dsj}^t\mathcal{Y}_j\;0\,\right]^t.
\end{aligned}\tag{5.42}$$

Moreover, the local gain matrix is given by $K_j=\mathcal{G}_j\mathcal{Y}_j^{-1}C_j^{\dagger}$.

5.1.7 Delay-Free Systems

In case of delay-free decentralized systems

$$\mathsf{P}_j\dot{x}_j(t)=A_jx_j(t)+\sum_{k=1,k\neq j}^{n_s}F_{jk}x_k(t)+B_ju_j(t)+\Gamma_jw_j(t),\tag{5.43}$$

$$z_j(t)=G_jx_j(t)+\Phi_jw_j(t),\tag{5.44}$$

$$y_j(t)=C_jx_j(t)+\Lambda_jw_j(t)\tag{5.45}$$

the following result holds:

Theorem 5.5 *Given tuning parameters* $\beta_j,\sigma_j,\ j,k=1,\ldots,n_s$. *The family of subsystems described by* (5.43)–(5.45) *is asymptotically stabilizable by decentralized static output-feedback controller* $u_j(t)=K_{oj}y_j(t),\ j=1,\ldots,n_s$ *with* $\mathcal{L}_2$-*performance bound* $\gamma_j,\ j=1,\ldots,n_s$ *if there exist positive-definite matrices*

$$\mathcal{Y}_j,\quad \mathcal{G}_j,\quad \Psi_{1j},\quad \Psi_{2j},\quad \Psi_{3j},\quad \Psi_{4j},\quad \Psi_{1kj},\quad \Psi_{2kj}$$

satisfying the following LMIs for $j = 1, \dots, n_s$

$$\widehat{\Xi}_j = \begin{bmatrix} \Xi_j & \Xi_{vj} & \Xi_{wj} \\ \bullet & -\gamma_j^2 I_j & \Phi_j^t \\ \bullet & \bullet & -I_j \end{bmatrix}, \tag{5.46}$$

where

$$\begin{cases} \Xi_j = \begin{bmatrix} \Xi_{ej} & \Xi_{1j} \\ \bullet & \Xi_{aj} \end{bmatrix}, \\ \Xi_{ej} = (1+\sigma_j)\left[\left(A_j + \sum_{k=1,k\neq j}^{n_s} F_{kj}\right)\mathcal{Y}_j + B_j\mathcal{G}_j\right] \\ \quad = (1+\sigma_j)\left[\mathcal{Y}_j\left(A_j + \sum_{k=1,k\neq j}^{n_s} F_{kj}\right)^t + \mathcal{G}_j^t B_j^t\right] \\ \qquad + (n_s - 1)\mathcal{Y}_j + \sum_{k=1,k\neq j}^{n_s} \Psi_{1kj}, \\ \Xi_{1j} = -\sigma_j\mathcal{Y}_j + \beta_j\left[\left(A_j + \sum_{k=1,k\neq j}^{n_s} F_{kj}\right)^t \mathcal{Y}_j + \mathcal{G}_j^t B_j^t\right], \\ \Xi_{aj} = -2\beta_j\mathcal{Y}_j, \quad \Xi_{wj} = \left[G_j^t\mathcal{Y}_j \; 0\;0\;0\;0\right]^t, \\ \Xi_{vj} = \left[\Gamma_j^t + \Psi_{4j}^t \; 0\;0\;0\;0\right]^t. \end{cases} \tag{5.47}$$

The local gain matrix is given by $K_j = \mathcal{G}_j\mathcal{Y}_j^{-1}C_j^\dagger$. *Moreover, if the local quantizer* M_j *is selected large enough with respect to* Δ_j *so as to satisfy the inequality*

$$\mathsf{M}_j > 4\Delta_j \frac{\|\mathcal{Y}_j^{-1}B_jK_{oj}\|}{\lambda_m(\Lambda_j)}\|C_j\|. \tag{5.48}$$

Then, the family of subsystems $\{\mathbf{P}_j\}$ *described by* (5.43)–(5.45) *is asymptotically stabilizable with* $\mathcal{L}_2$*-performance bound* γ_j *by decentralized quantized output-feedback controller* (5.28).

Remark 5.6 It is significant to note that Theorem 5.5 provides an improved nominal result over [6] and gives an explicit expression for the quantized output feedback gain. In addition, the result is valid for arbitrary number of subsystems and not restricted to $n_s = 2$ as in [6].

5.2 Decentralized Quantized Control II: Continuous Systems

Quantization in control systems has been an active research topic in recent years, see [11, 13]. Control problems under different types of quantizations in both, linear and nonlinear cases have been examined. The need for quantization arises when digital networks are part of the feedback loop and this eventually gives rise to packet dropouts or data transfer rate limitations [17]. On the other hand, signal processing and signal quantization always exist in computer-based control systems [22] and therefore recent research studies have been reported on the analysis and design problems for control systems involving various quantization methods, see [5, 8, 19, 21, 33, 34] and the references cited therein.

In [5], a quantizer taking value in a finite set is defined and then quantized feedback stabilization for linear systems is considered. In [8], the problem of stabilizing an unstable linear system by means of quantized state feedback, where the quantizer takes value in a countable set is addressed. It should be noted that the approach in [5] relies on the possibility of making discrete on line adjustments of quantizer parameters which was extended in [21] for more general nonlinear systems with general types of quantizers involving the states of the system, the measured outputs, and the control inputs. Recently in [11], a study of quantized and delayed state-feedback control systems under constant bounds on the quantization error and the time-varying delay was reported. Based on [21], stabilization of discrete-time LTI systems with quantized measurement outputs is reported in [29]. Further related results are reported in [33, 34]. On another research front, decentralized stability and feedback stabilization of interconnected systems have been the topic of recurring interests and recent relevant results have been reported in [2, 24–28, 31].

In this section, we develop an approach to the problem of quantized feedback stabilization from a generalized setting by designing a decentralized $\mathcal{H}_\infty$ feedback control for a class of linear interconnected continuous-time systems with unknown-but-bounded couplings and interval delays and where the quantizer has arbitrary form that satisfies a quadratic inequality constraint. An LMI-based method using a decentralized quantized output-feedback controller is designed at the subsystem level to render the closed-loop system delay-dependent asymptotically stable with guaranteed γ-level. It is established that this setting encompasses several special cases of interest including interconnected delay-free systems, single time-delay systems and single systems.

5.2.1 Problem Statement

We consider a class of linear systems $\mathbf{S}$ structurally composed of n_s coupled subsystems $\mathbf{S}_j$ and the model of the jth subsystem is described by the state-space representation:

$$\mathbf{S}_j: \quad \dot{x}_j(t) = A_j x_j(t) + A_{dj} x_j(t-\tau_j(t)) + B_j u_j(t) + c_j(t) + \Gamma_j w_j(t), \tag{5.49}$$

$$\begin{aligned} z_j(t) &= G_j x_j(t) + G_{dj} x_j(t-\tau_j(t)) + \Phi_j w_j(t), \\ y_j(t) &= C_j x_j(t) + C_{dj} x_j(t-\tau_j(t)) + \Psi_j w_j(t), \end{aligned} \tag{5.50}$$

where for $j \in \{1, \ldots, n_s\}$, $x_j(t) \in \Re^{n_j}$ is the state vector, $u_j(t) \in \Re^{m_j}$ is the control input, $y_j(t) \in \Re^{p_j}$ is the measured output, $w_j(t) \in \Re^{q_j}$ is the disturbance input which belongs to $\mathcal{L}_2[0, \infty)$, $z_j(t) \in \Re^{q_j}$ is the performance output. The matrices $A_j \in \Re^{n_j \times n_j}$, $B_j \in \Re^{n_j \times m_j}$, $A_{dj} \in \Re^{n_j \times n_j}$, $\Phi_j \in \Re^{q_j \times q_j}$, $\Gamma_j \in \Re^{n_j \times q_j}$, $C_j \in \Re^{p_j \times n_j}$, $C_{dj} \in \Re^{p_j \times n_j}$, $G_j \in \Re^{q_j \times n_j}$, $G_{dj} \in \Re^{q_j \times n_j}$, $\Psi_j \in \Re^{p_j \times q_j}$ are real and constants. The initial condition $\kappa_j \in \mathcal{L}_2[-\varrho_j, 0]$, $j \in \{1, \ldots, n_s\}$. In the sequel, we treat the interaction term $c_j(t)$ as a piecewise-continuous vector function in its arguments and satisfies the quadratic inequality

$$c_j^t(t)c_j(t) \le \phi_j x_j^t(t)E_j^t E_j x_j(t) + \psi_j x_j^t(t - \tau_j(t))E_{dj}^t E_{dj} x_j(t - \tau_j(t)), \quad (5.51)$$

where $\phi_j > 0$, $\psi_j > 0$ are adjustable bounding parameters. The factors τ_j, $j, k \in \{1, \ldots, n_s\}$ are unknown time-delay factors satisfying

$$0 < \varphi_j \le \tau_j(t) \le \varrho_j, \qquad \dot{\tau}_j(t) \le \eta_j, \quad (5.52)$$

where the bounds τ_j^-, τ_j^+, η_j are known constants in order to guarantee smooth growth of the state trajectories. Note in (5.50) and (5.51) that the delay within each subsystem (local delay) and among the subsystems (coupling delay), respectively, are emphasized. A block-diagram representation of the subsystem model (5.50) is depicted in Fig. 5.2.

The class of systems described by (5.49)–(5.50) subject to delay-pattern (5.52) is frequently encountered in modeling several physical systems and engineering applications including large space structures, multi-machine power systems, cold mills, transportation systems, water pollution management, to name a few [25]. In the course of feedback control design, it is often considered that the process output is passed directly to the controller. A control input signal is generated and in turn passes it directly back to the process. In many applications, it turns out that the interface between the controller and the process features some additional information-processing devices. Of interest in this chapter is the issue of signal quantization.

Our objective in this section is to address a generalized approach to examine the problem of quantized feedback stabilization for a class of linear interconnected continuous-time systems. In this approach, we think of a quantizer as a device that converts a real-valued signal into a piecewise constant one taking on a finite set of values and wherein it is possible to vary some parameters of the quantizer in real time, on the basis of collected data. We seek to design a decentralized $\mathcal{H}_\infty$ feedback control for a class of linear interconnected continuous-time systems with unknown-but-bounded couplings and interval delays

Remark 5.7 In general, the vector $c(t) = \sum_j^{n_s} c_j(t)$ represents the interaction pattern among the subsystems wherein the component vector $c_j(t)$ depends on the current and delayed states of the form $c_j(t) = \sum_{\ell \neq j}^{n_s} A_{j\ell} x_\ell(t) + A_{dj\ell} x_\ell(t - \tau_\ell(t))$.

Under the interconnected structural identity

$$\sum_{j}^{n_s}\sum_{\ell\neq j}^{n_s} A_{j\ell}x_\ell(t)+A_{dj\ell}x_\ell(t-\tau_\ell(t))=\sum_{j}^{n_s}\sum_{j\neq \ell}^{n_s} A_{\ell j}x_j(t)+A_{d\ell j}x_j(t-\tau_j(t))$$

it has been a common practice [24] to rearrange the terms in a convenient way so as to reflect within the jth-subsystem the appropriate components leading to the bounding inequality (5.51) with adjustable bounding parameters ϕ_j, ψ_j. Note in (5.49) and (5.51) that the subsystem delay with local and coupling patterns are emphasized and in numerical simulations, all the subsystems have to be treated simultaneously. An overall feasible solution of system $\mathbf{S}$ is only guaranteed if the feasible solutions of subsystems $\mathbf{S}_j$ are attained. Thus the rationale behind inequality (5.51) is to help in inducing decentralized computations.

The quantizer can be thought of as a coder that generates an encoded signal taking values in a given finite set. By changing the size and relative position of the quantization regions, that is, by modifying the coding mechanism we can learn more about the behavior of the system, without violating the restriction on the type of information that can be communicated to the controller.

5.2.2 A Class of Local Quantizers

In the sequel, we treat a *quantizer* as a device in the control loop that converts a real-valued signal into a piecewise constant one. We adopt the definition of a local (subsystem) quantizer with general form as introduced in [21]. Let $f_j \in \Re^s$, $j = 1, \ldots, n_s$ be the variable being quantized. A *local quantizer* is defined as a piecewise constant function $Q_j : \Re^s \to \mathcal{D}_j$, where $\mathcal{D}_j$ is a finite subset of $\Re^s$. This leads to a partition of $\Re^s$ into a finite number of quantization regions of the form $\{f_j \in \Re^s : Q(f_j) = d_j,\ d_j \in \mathcal{D}_j\}$. These quantization regions are not assumed to have any particular shape.

In the quantized control strategy to be developed below, we will use the local quantization error $\boldsymbol{\Delta}_j(y) = Q_j(y_j) - y_j$ (see Fig. 5.5) based on output measurements such that the following quadratic bounding relation is satisfied:

$$\Delta_j^t(.)\Delta_j(.) \le \alpha_j x_j^t(t)F_j^t F_j x_j(t) + \beta_j x_j^t(t-\tau_j(t))F_{dj}^t F_{dj} x_j(t-\tau_j(t)), \quad (5.53)$$

where $\alpha_j > 0$, $\beta_j > 0$ are adjustable subsystem parameters and the matrices E_j, F_j are arbitrary but constants.

Remark 5.8 It is crucial to recognize that the quadratic bounding relation (5.53) is independent of the structure of the quantizer employed. In fact, it is satisfied by wide class of practically-used quantizers. For example, in case of uniform quantizer [5, 21] for delay-free systems $\tau_j \equiv 0$, we assume that given positive integer M_j (saturation value) and nonnegative real number Σ_j (sensitivity), the quantizer $Q(f_j)$ is

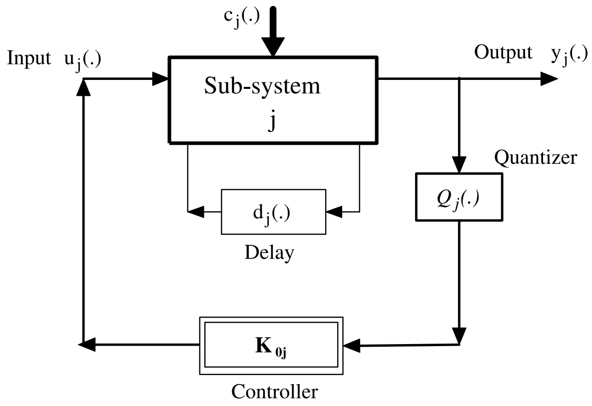


Fig. 5.5 Time-delayed subsystem model with quantizer

defined by:

$$Q(f_j) = \begin{cases} \mathsf{M}_j & \text{if } f_j > (\mathsf{M}_j + \frac{1}{2})\Sigma_j, \\ -\mathsf{M}_j & \text{if } f_j \le -(\mathsf{M}_j + \frac{1}{2})\Sigma_j, \\ [\frac{f_j}{\Sigma_j} + \frac{1}{2}]\mathsf{M}_j & \text{if } -(\mathsf{M}_j + \frac{1}{2})\Sigma_j < f_j \le (\mathsf{M}_j + \frac{1}{2})\Sigma_j. \end{cases}$$

Typical simulation would certainly shows that the uniform quantizer satisfies the quadratic bounding relation (5.53) with $f_j(t) = C_j x_j(t)$. Alternatively, in the case of static logarithmic quantizer [13] for delay-free systems, we assume that given real numbers $\varepsilon_j, \varrho_j \in (0, 1)$, the quantizer $Q(\varepsilon_j)$ is defined by:

$$Q(\varepsilon_j) = \begin{cases} \varrho_j^k \mu_0 & \text{if } \frac{1}{1+\delta_j}\varrho_j^k \mu_0 < \varepsilon_j \le \frac{1}{1-\delta_j}\varrho_j^k \mu_0, \ k = 0, \pm 1, \pm 2, \ldots, \\ 0 & \text{if } \varepsilon_j = 0, \\ -Q(\varepsilon_j) & \text{if } \varepsilon_j < 0, \end{cases}$$

where ϱ_j represents the quantization density at subsystem j and $\delta_j = (1 - \varrho_j)/(1 + \varrho_j)$. Observe that a small ϱ_j corresponds to large δ_j and this implies coarse quantization. Alternatively, a large ϱ_j means small δ_j which leads to coarse quantization. From consideration of the behavior of the static logarithmic quantizer, we reach the conclusion that it satisfies a quadratic bounding relation with $\varepsilon_j(t) = C_j x_j(t)$. Since extension to time-delay systems is quite straightforward hence, we will employ the bounding inequality (5.53) in the subsequent analysis.

5.2.3 Quantized Output-Feedback Design

We develop in this section new criteria for LMI-based characterization of decentralized stabilization by local quantized feedback of the form

$$u_j(t) = K_{oj} Q_j(y_j), \quad j = 1, \ldots, n_s, \tag{5.54}$$

where the gain matrices $K_{oj}, j = 1, \ldots, N$ will be selected to guarantee that the closed-loop system, composed of (5.49)–(5.50), (5.53) and (5.54), given by

$$\begin{aligned}
\dot{x}_j(t) &= \mathcal{A}_j x_j(t) + \mathcal{A}_{dj} x_j(t - \tau_j(t)) + c_j(t) + B_j K_{oj} \Delta_j(y_j) + \Omega_j w_j(t),\\
z_j(t) &= G_j x_j(t) + G_{dj} x_j(t - \tau_j(t)) + \Phi_j w_j(t),\\
\mathcal{A}_j &= A_j + B_j K_{oj} C_j, \quad \mathcal{A}_{dj} = A_{dj} + B_j K_{oj} C_{dj},\\
\Omega_j &= \Gamma_j + B_j K_{oj} \Psi_j
\end{aligned} \tag{5.55}$$

is asymptotically stable with disturbance attenuation level γ_j. To facilitate further development, we consider the case where the set of output matrices C_j, $j = 1, \ldots, n_s$ are assumed to be of full row rank and $C_j^\dagger$ represents the right-inverse. Introduce the local Lyapunov-Krasovskii functional (LKF):

$$\begin{aligned}
V_j(t) &= V_{jo}(t) + V_{ja}(t) + V_{jc}(t) + V_{je}(t) + V_{jm}(t) + V_{jn}(t),\\
V_{jo}(t) &= x_j^t(t) \mathcal{P}_j x(t), \quad V_{ja}(t) = \int_{t-\varphi_j}^{t} x_j^t(s) \mathcal{Q}_j x_j(s)\, ds,\\
V_{jm}(t) &= \varphi_j \int_{-\varphi_j}^{0} \int_{t+s}^{t} \dot{x}_j^t(\alpha) \mathcal{W}_j \dot{x}_j(\alpha) d\alpha\, ds,\\
V_{jn}(t) &= (\varrho_j - \varphi_j) \int_{-\varrho_j}^{-\varphi_j} \int_{t+s}^{t} \dot{x}_j^t(\alpha) \mathcal{S}_j \dot{x}_j(\alpha) d\alpha\, ds,\\
V_{jc}(t) &= \int_{t-\tau_j(t)}^{t} x_j^t(s) \mathcal{Z}_j x_j(s)\, ds,\\
V_{je}(t) &= \int_{t-\varrho_j}^{t} x_j^t(s) \mathcal{R}_j x_j(s)\, ds,
\end{aligned} \tag{5.56}$$

where $0 < \mathcal{P}_j = \mathcal{P}_j^t$, $0 < \mathcal{W}_j = \mathcal{W}_j^t$, $0 < \mathcal{Q}_j = \mathcal{Q}_j^t$, $0 < \mathcal{R}_j = \mathcal{R}_j^t$, $0 < \mathcal{S}_j = \mathcal{S}_j^t$, $0 < \mathcal{Z}_j = \mathcal{Z}_j^t$ are weighting matrices of appropriate dimensions. The main design result is established by the following theorem.

Theorem 5.6 *Given the bounds $\varphi_j > 0$, $\varrho_j > 0$ and $\eta_j > 0$. System (5.49)–(5.50) is delay-dependent asymptotically stable with $\mathcal{L}_2$-performance bound γ if there exist weighting matrices $0 < \mathcal{X}_j$, $\mathcal{Y}_j$, $0 < \Lambda_{mj}$; $m = 1, \ldots, 7$, and scalars $\pi_j > 0$, $\mu_j > 0$, $\sigma_j > 0$, $\nu_j > 0$, $\gamma_j > 0$ satisfying the following LMI*

$$\widetilde{\Pi}_j = \begin{bmatrix} \Pi_{1j} & \Pi_{2j} & \Pi_{4j} \\ \bullet & \Pi_{3j} & 0 \\ \bullet & \bullet & \Pi_{5j} \end{bmatrix} < 0, \tag{5.57}$$

$$\Pi_{1j} = \begin{bmatrix} \Pi_{jo} & 0 & \Pi_{ja} & \Lambda_{2j} & I_j & \Lambda_{6j} \\ \bullet & -\Pi_{jc} & \Lambda_{3j} & 0 & 0 & 0 \\ \bullet & \bullet & -\Pi_{jm} & \Lambda_{3j} & 0 & 0 \\ \bullet & \bullet & \bullet & -\Pi_{jn} & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -I_j & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -I_j \end{bmatrix},$$

$$\Pi_{2j} = \begin{bmatrix} \varphi_j \Pi_{js} & (\varrho_j - \varphi_j)\Pi_{js} & \Lambda_{7j} \\ 0 & 0 & 0 \\ \varphi_j \Pi_{jt} & (\varrho_j - \varphi_j)\Pi_{jt} & \mathcal{X}_j G_{dj}^t \Phi_j \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$\Pi_{3j} = \begin{bmatrix} -\Pi_{jv} & 0 & 0 \\ \bullet & -\Pi_{jw} & 0 \\ \bullet & \bullet & -\gamma_j^2 I_j + \Phi_j^t \Phi_j \end{bmatrix},$$
$$\Pi_{4j} = \begin{bmatrix} \Pi_{41j} \; \Pi_{42j} \; \Pi_{43j} \end{bmatrix}, \tag{5.58}$$
$$\Pi_{41j} = \begin{bmatrix} E_j \mathcal{X}_j & 0 & 0 & 0 & 0 & 0 \\ 0 & E_{dj}\mathcal{X}_j & 0 & 0 & 0 & 0 \end{bmatrix}^t,$$
$$\Pi_{42j} = \begin{bmatrix} F_j \mathcal{X}_j & 0 & 0 & 0 & 0 & 0 \\ 0 & F_{dj}\mathcal{X}_j & 0 & 0 & 0 & 0 \end{bmatrix}^t,$$
$$\Pi_{43j} = \begin{bmatrix} G_j \mathcal{X}_j & 0 & 0 & 0 & 0 & 0 \\ 0 & G_{dj}\mathcal{X}_j & 0 & 0 & 0 & 0 \end{bmatrix}^t,$$
$$\Pi_{5j} = \mathrm{diag}\begin{bmatrix} \pi_j I_j \; \mu_j I_j \; \sigma_j I_j \; \nu_j I_j \; I_j \; I_j \end{bmatrix},$$
$$\Pi_{jo} = \mathcal{A}_j \mathcal{X}_j + \mathcal{X}_j \mathcal{A}_j^t + \Lambda_{1j} + \Lambda_{4j} + \Lambda_{5j} - \Lambda_{2j},$$
$$\Pi_{jc} = \Lambda_{1j} + \Lambda_{3j}, \quad \Pi_{js} = \mathcal{X}_j A_j^t + \mathcal{Y}_j^t B_j^t,$$
$$\Pi_{ja} = \mathcal{A}_{dj}\mathcal{X}_j, \quad \Pi_{jn} = \Lambda_{2j} + \Lambda_{3j} + \Lambda_{5j},$$
$$\Pi_{jm} = (1-\mu_j)\Lambda_{4j} + 2\Lambda_{3j}, \quad \Pi_{jw} = 2\mathcal{X}_j - \Lambda_{3j},$$
$$\Pi_{jv} = 2\mathcal{X}_j - \Lambda_{2j}, \quad \Upsilon_{tj} = \Upsilon_{rj} + \mathcal{X}_j A_{dj}^t.$$

Moreover, the local gain matrix is given by $K_{oj} = \mathcal{Y}_j \mathcal{X}_j^{-1} C_j^{\dagger}$.

Proof A straightforward computation gives the time-derivative of V_j along the solutions of (5.49) with $\omega_j(t) \equiv 0$ as:

$$\begin{aligned} \dot{V}_j(t) &= 2x_j^t(t)\mathcal{P}_j \dot{x}_j(t) + x_j^t(t)[\mathcal{Q}_j + \mathcal{R}_j + \mathcal{Z}_j]x(t) \\ &\quad - x_j^t(t-\varphi_j)\mathcal{Q}_j x_j(t-\varphi_j) \\ &\quad - (1-\dot{\tau}_j)x^t(t-\tau_j(t))\mathcal{Z}x(t-\tau(t)) - x^t(t-\varrho)\mathcal{R}x(t-\varrho) \\ &\quad + \dot{x}_j^t(t)[\varphi_j^2 \mathcal{W}_j + (\varrho_j - \varphi_j)^2 \mathcal{S}_j]\dot{x}_j(t) \\ &\quad - \int_{t-\varphi_j}^{t} \dot{x}_j^t(\alpha)\mathcal{W}_j \dot{x}_j(\alpha)d\alpha \\ &\quad - \int_{t-\varrho}^{t-\varphi_j} \dot{x}^t(\alpha)\mathcal{S}\dot{x}(\alpha)d\alpha \\ &\le 2x_j^t(t)\mathcal{P}_j \dot{x}_j(t) + x_j^t(t)[\mathcal{Q}_j + \mathcal{R}_j + \mathcal{Z}_j]x_j(t) \\ &\quad - x^t(t-\varphi_j)\mathcal{Q}_j x(t-\varphi_j) \end{aligned}$$

$$
\begin{aligned}
&- (1-\mu_j)x^t(t-\tau_j)\mathcal{Z}_j x_j(t-\tau_j) - x_j^t(t-\varrho_j)\mathcal{R}_j x_j(t-\varrho_j) \\
&+ \dot{x}_j^t(t)[\varphi_j \mathcal{W}_j + (\varrho_j - \varphi_j)\mathcal{S}_j]\dot{x}_j(t) \\
&- \varphi_j \int_{t-\varrho_j}^{t} \dot{x}_j^t(\alpha)\mathcal{W}_j \dot{x}_j(\alpha)d\alpha \\
&- (\varrho_j - \varphi_j)\int_{t-\varrho_j}^{t-\varphi_j} \dot{x}_j^t(\alpha)\mathcal{S}_j \dot{x}_j(\alpha)d\alpha.
\end{aligned} \tag{5.59}
$$

Applying the Jenkins's inequality (see Chap. 9), we get

$$
\begin{aligned}
&-\varphi_j \int_{t-\varphi_j}^{t} \dot{x}_j^t(\alpha)\mathcal{W}_j \dot{x}_j(\alpha)d\alpha \\
&\quad \le \begin{bmatrix} x_j(t) \\ x_j(t-\varphi_j) \end{bmatrix}^t \begin{bmatrix} -\mathcal{W}_j & \mathcal{W}_j \\ \bullet & -\mathcal{W}_j \end{bmatrix} \begin{bmatrix} x_j(t) \\ x_j(t-\varphi_j) \end{bmatrix}.
\end{aligned} \tag{5.60}
$$

Similarly,

$$
\begin{aligned}
&-(\varrho_j - \varphi_j)\int_{t-\varrho_j}^{t-\varphi_j} \dot{x}_j^t(\alpha)\mathcal{S}_j \dot{x}_j(\alpha)d\alpha \\
&\quad = -(\varrho_j - \varphi_j)\left[\int_{t-\tau}^{t-\varphi} \dot{x}^t(\alpha)\mathcal{S}\dot{x}(\alpha)d\alpha + \int_{t-\varrho_j}^{t-\tau_j} \dot{x}_j^t(\alpha)\mathcal{S}_j \dot{x}_j(\alpha)d\alpha\right] \\
&\quad \le -(\tau_j - \varphi_j)\left[\int_{t-\tau_j}^{t-\varphi_j} \dot{x}_j^t(\alpha)\mathcal{S}_j \dot{x}_j(\alpha)d\alpha\right] \\
&\qquad - (\varrho_j - \tau_j)\left[\int_{t-\varrho_j}^{t-\tau_j} \dot{x}_j^t(\alpha)\mathcal{S}_j \dot{x}_j(\alpha)d\alpha\right] \\
&\quad \le -\left(\int_{t-\tau_j}^{t-\varphi_j} \dot{x}_j^t(\alpha)d\alpha\right)\mathcal{S}_j\left(\int_{t-\tau_j}^{t-\varphi_j} \dot{x}_j(\alpha)d\alpha\right) \\
&\qquad - \left(\int_{t-\varrho_j}^{t-\tau_j} \dot{x}_j^t(\alpha)d\alpha\right)\mathcal{S}_j\left(\int_{t-\varrho_j}^{t-\tau_j} \dot{x}_j(\alpha)d\alpha\right) \\
&\quad = -[x(t-\varphi_j) - x(t-\tau_j)]^t\mathcal{S}_j[x(t-\varphi_j) - x(t-\tau_j)] \\
&\qquad - [x(t-\tau_j) - x(t-\varrho_j)]^t\mathcal{S}_j[x(t-\tau_j) - x(t-\varrho_j)].
\end{aligned} \tag{5.61}
$$

From (5.56)–(5.61) with Schur complements and incorporating (5.51) and (5.53) via the **S**-procedure, we have

$$
\begin{aligned}
\dot{V}_j(t) &\le \xi_j^t(t)\Xi_j\xi_j(t), \\
\xi_j(t) &= \left[\,\xi_{1j}^t(t)\ \xi_{2j}^t(t)\,\right]^t, \\
\xi_{1j}(t) &= \left[\,x_j^t(t)\ x_j^t(t-\varphi_j)\ x_j^t(t-\tau_j)\,\right]^t, \\
\xi_{2j}(t) &= \left[\,x_j^t(t-\varrho_j)\ c_j^t(t)\ \Delta_j^t(y_j)\,\right]^t,
\end{aligned} \tag{5.62}
$$

where Ξ_j corresponds to $\widetilde{\Pi}_j$ in (5.57) with $G_j \equiv 0$, $G_{dj} \equiv 0$, $\Phi_j \equiv 0$ and Schur complement operations. If $\widetilde{\Pi}_j < 0$ so is $\Xi_j < 0$, leading to $\dot{V}_j(t) \leq -\omega_j \|\xi_j\|^2$. This establishes the internal asymptotic stability.

Next, we consider the performance measure

$$J_j = \int_0^{\infty} \big(z_j^t(s) z_j(s) - \gamma_j^2 w_j^t(s) w_j(s)\big) ds.$$

For any $\omega_j(t) \in \mathcal{L}_2(0, \infty) \neq 0$ and zero initial condition $x(0) = 0$ (hence $V_j(0) = 0$), we have

$$J_j \leq \int_0^{\infty} \big(z_j^t(s) z_j(s) - \gamma_j^2 w_j^t(s) w_j(s) + \dot{V}_j(x)|_{(5.49)}\big) ds,$$

where $\dot{V}_j(x)|_{(5.49)}$ is the Lyapunov derivative along the state trajectories of system (5.49). Proceeding, we get

$$\begin{aligned} z_j^t(s) z_j(s) - \gamma_j^2 w_j^t(s) w_j(s) + \dot{V}_j(s)|_{(5.49)} &= \eta_j^t(s) \widehat{\Xi}_j \eta_j(s), \\ \eta_j(s) &= \big[\, \xi_j^t(s) \;\; w_j^t(s) \,\big]^t, \end{aligned} \tag{5.63}$$

where $\widehat{\Xi}_j$ corresponds to $\widetilde{\Pi}_j$ given by (5.57) by Schur complements. If $\widehat{\Pi}_j < 0$, it is readily seen from (5.63) by Schur complements that

$$z_j^t(s) z_j(s) - \gamma_j^2 w_j^t(s) w_j(s) + \dot{V}_j(s)|_{(5.49)} < 0$$

for arbitrary $s \in [t, \infty)$, which implies for any $\omega_j(t) \in \mathcal{L}_2(0, \infty) \neq 0$ that $J_j < 0$ or equivalently $J = \sum_{j=1}^{n_s} J_j < 0$. This in turn leads to $\|z_j(t)\|_2 < \gamma_j \|\omega_j(t)\|_2$ for all $j = 1, \ldots, n_s$.

To compute that the feedback gains, we apply Schur complements and rewrite $\widehat{\Xi}$ as

$$\widehat{\Pi}_j = \begin{bmatrix} \widehat{\Pi}_{oj} & \widehat{\Pi}_{cj} & \widehat{\Pi}_{vj} \\ \bullet & -\widehat{\Pi}_{sj} & 0 \\ \bullet & \bullet & -\widehat{\Pi}_{wj} \end{bmatrix} < 0, \tag{5.64}$$

$$\widehat{\Pi}_{oj} = \begin{bmatrix} \tilde{\Xi}_{os} & 0 & \mathcal{P}_j A_{dj} & \mathcal{W}_j & \mathcal{P}_j & \mathcal{P}_j B_j K_{oj} \\ \bullet & -\Xi_{cj} & \mathcal{S}_j & 0 & 0 & 0 \\ \bullet & \bullet & -\tilde{\Xi}_{mj} & \mathcal{S}_j & 0 & 0 \\ \bullet & \bullet & \bullet & -\Xi_{nj} & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -I_j & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -I_j \end{bmatrix},$$

$$\widehat{\Pi}_{cj} = \begin{bmatrix} \varphi_j A_s^t & (\varrho_j - \varphi_j) A_s^t & G_j^t \Phi_j + \mathcal{P}_j \Omega_j \\ 0 & 0 & 0 \\ \varphi_j A_{do}^t & (\varrho_j - \varphi_j) A_{do}^t & G_{dj}^t \Phi_j \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\widehat{\Pi}_{sj} = \begin{bmatrix} -\mathcal{W}_j^{-1} & 0 & 0 \\ \bullet & -\mathcal{S}_j^{-1} & 0 \\ \bullet & \bullet & -\gamma^2 + jI + \Phi_j^t \Phi_j \end{bmatrix}, \tag{5.65}$$

$$\widehat{\Pi}_{vj} = \begin{bmatrix} \Pi_{v1j} & \Pi_{v2j} & \Pi_{v3j} \end{bmatrix},$$

$$\Pi_{v1j} = \begin{bmatrix} E_j & 0 & 0 & 0 & 0 & 0 \\ 0 & E_{dj} & 0 & 0 & 0 & 0 \end{bmatrix}^t,$$

$$\Pi_{v2j} = \begin{bmatrix} F_j & 0 & 0 & 0 & 0 & 0 \\ 0 & F_{dj} & 0 & 0 & 0 & 0 \end{bmatrix}^t,$$

$$\Pi_{v3j} = \begin{bmatrix} G_j & 0 & 0 & 0 & 0 & 0 \\ 0 & G_{dj} & 0 & 0 & 0 & 0 \end{bmatrix}^t,$$

$$\tilde{\Xi}_{os} = \mathcal{P}_j \mathcal{A}_j + \mathcal{A}_j^t \mathcal{P}_j + \mathcal{Q}_j + \mathcal{R}_j + \mathcal{Z}_j - \mathcal{W}_j,$$

$$\tilde{\Xi}_{jm} = (1 - \mu_j)\mathcal{Z}_j + 2\mathcal{S}_j.$$

Then we define $\mathcal{X}_j = \mathcal{P}_j^{-1}$, $\pi_j = \phi_j^{-1}$, $\mu_j = \psi_j^{-1}$, $\sigma_j = \alpha_j^{-1}$, $\nu_j = \beta_j^{-1}$ and apply the congruent transformation

$$T_j = \text{diag}\begin{bmatrix} \mathcal{X}_j & \mathcal{X}_j & \mathcal{X}_j & \mathcal{X}_j & I_j & I_j & I_j & I_j & I_j \end{bmatrix}$$

along with the linearizations

$$\Lambda_{1j} = \mathcal{X}_j \mathcal{Q}_j \mathcal{X}_j, \quad \Lambda_{2j} = \mathcal{X}_j \mathcal{W}_j \mathcal{X}_j, \quad \Lambda_{3j} = \mathcal{X}_j \mathcal{S}_j \mathcal{X}_j,$$
$$\Lambda_{4j} = \mathcal{X}_j \mathcal{Z}_j \mathcal{X}_j, \quad \Lambda_{5j} = \mathcal{X}_j \mathcal{R}_j \mathcal{X}_j, \quad \Lambda_{6j} = B_j K_{oj},$$
$$\Lambda_{7j} = \mathcal{X}_j G_j^t \Phi_j + \Gamma_j + B_j K_{oj}, \quad \Pi_{jr} = \mathcal{X}_j^t C_{dj}^t K_{oj}^t B_j^t.$$

Using the algebraic matrix inequalities $-\mathcal{W}_j^{-1} \le -2\mathcal{X}_j + \Lambda_{2j}$, $-\mathcal{S}_j^{-1} \le -2\mathcal{X}_j + \Lambda_{3j}$ in addition to the matrix definitions (5.58), we obtain LMI (5.57) by Schur complements. This concludes the proof. □

5.2.4 *Special Cases*

In the sequel, some special cases are derived to emphasize the generality of our approach. These include nominal delay-free systems, single time-delay systems and single dynamical systems.

5.2.4.1 Delay-Free Systems

First, we consider the class of nominally-linear systems $\mathbf{S}$ structurally composed of n_s coupled subsystems $\mathbf{S}_j$ and the model of the jth subsystem is described by the state-space representation:

$$\mathbf{S}_j: \quad \dot{x}_j(t) = A_j x_j(t) + B_j u_j(t) + c_j(t) + \Gamma_j w_j(t), \tag{5.66}$$

$$\begin{aligned} z_j(t) &= G_j x_j(t) + \Phi_j w_j(t), \\ y_j(t) &= C_j x_j(t) + \Psi_j w_j(t), \end{aligned} \tag{5.67}$$

where for $j \in \{1, \ldots, n_s\}$, the coupling vector $c_j(k)$ is a piecewise-continuous vector function in its arguments and satisfies the quadratic inequality

$$c_j^t(k, ., .)c_j(k, ., .) \leq \phi_j x_j^t(k) E_j^t E_j x_j(k), \tag{5.68}$$

where $\phi_j > 0$ are adjustable bounding parameters and $M_j \in \Re^{n_j \times n_j}$ are constant matrices. We will use local quantized output measurements such that the following quadratic bounding relation is satisfied:

$$\Delta_j^t(.)\Delta_j(.) \leq \alpha_j x_j^t(k) F_j^t F_j x_j(k), \tag{5.69}$$

where $\alpha_j > 0$ are adjustable subsystem parameters. The following corollary stands out:

Corollary 5.2 *System (5.66)–(5.67) is asymptotically stable with $\mathcal{L}_2$-performance bound γ if there exist weighting matrices $0 < \mathcal{X}_j$, $\mathcal{Y}_j$, Λ_{mj}; $m = 1, 2$, and scalars $\pi_j > 0$, $\sigma_j > 0$, $\gamma_j > 0$ satisfying the following LMI*

$$\overline{\Pi}_j = \begin{bmatrix} \overline{\Pi}_{1j} & \overline{\Pi}_{2j} \\ \bullet & \overline{\Pi}_{3j} \end{bmatrix} < 0, \tag{5.70}$$

$$\begin{aligned} \overline{\Pi}_{1j} &= \begin{bmatrix} \overline{\Pi}_{jo} & I_j & \Lambda_{1j} & \Lambda_{2j} \\ \bullet & -I_j & 0 & 0 \\ \bullet & \bullet & -I_j & \mathcal{X}_j G_{dj}^t \Phi_j \\ \bullet & \bullet & \bullet & -\gamma_j^2 I_j + \Phi_j^t \Phi_j \end{bmatrix}, \\ \overline{\Pi}_{2j} &= \left[\Pi_{21j} \ \Pi_{22j} \ \Pi_{23j} \right], \\ \Pi_{21j} &= \left[E_j \mathcal{X}_j \ 0 \ 0 \ 0 \right]^t, \quad \Pi_{22j} = \left[F_j \mathcal{X}_j \ 0 \ 0 \ 0 \right]^t, \\ \Pi_{23j} &= \left[G_j \mathcal{X}_j \ 0 \ 0 \ 0 \right]^t, \quad \overline{\Pi}_{3j} = \mathrm{diag}\left[\eta_j I_j \ \sigma_j I_j \ I_j \right], \\ \overline{\Pi}_{jo} &= \mathcal{A}_j \mathcal{X}_j + \mathcal{X}_j \mathcal{A}_j^t. \end{aligned} \tag{5.71}$$

Moreover, the local gain matrix is given by $K_{oj} = \mathcal{Y}_j \mathcal{X}_j^{-1} C_j^{\dagger}$.

5.2.4.2 Single Time-Delay Systems

In what follows, we consider the single linear time-delay system

$$\dot{x}(t) = Ax(t) + A_d x(t - \tau(t)) + Bu(t) + \Gamma w(t), \tag{5.72}$$

$$\begin{aligned} z_j(t) &= Gx(t) + G_d x(t - \tau(t)) + \Phi w(t), \\ y_j(t) &= Cx(t) + C_d x_j(t - \tau(t)) + \Psi w(t), \end{aligned} \tag{5.73}$$

where $0 < \varphi \leq \tau(t) \leq \varrho$, $\dot{\tau}(t) \leq \eta$. Like before, we will use quantized output measurements such that the following quadratic bounding relation is satisfied:

$$\Delta^t(.)\Delta(.) \leq \alpha x^t(t)F^t F x(t) + \beta x^t(k-\tau(t))F_d^t F_d x(k-\tau(t)), \tag{5.74}$$

where $\alpha > 0$, $\beta > 0$ are adjustable parameters. The following corollary establishes the corresponding design result:

Corollary 5.3 *Given the bounds the bounds $\varphi > 0$, $\varrho > 0$ and $\eta > 0$ then system (5.72)–(5.73) is delay-dependent asymptotically stabilizable by quantized feedback controller $u(t) = K_o y(t)$ with $\mathcal{L}_2$-performance bound γ if there exist weighting matrices $0 < \mathcal{X}$, $\mathcal{Y}$, Θ_m; $m = 1, \ldots, 7$, and a scalar $\gamma > 0$ satisfying the following LMI*

$$\widetilde{\Upsilon} = \begin{bmatrix} \Upsilon_1 & \Upsilon_2 & \Upsilon_4 \\ \bullet & \Upsilon_3 & 0 \\ \bullet & \bullet & \Upsilon_5 \end{bmatrix} < 0, \tag{5.75}$$

$$\Upsilon_1 = \begin{bmatrix} \Upsilon_o & 0 & \Upsilon_a & \Theta_2 & \Theta_6 \\ \bullet & -\Pi_c & \Theta_3 & 0 & 0 \\ \bullet & \bullet & -\Upsilon_m & \Theta_3 & 0 \\ \bullet & \bullet & \bullet & -\Upsilon_n & 0 \\ \bullet & \bullet & \bullet & \bullet & -I \end{bmatrix},$$

$$\Upsilon_2 = \begin{bmatrix} \varphi\Pi_s & (\varrho-\varphi)\Upsilon_s & \Theta_7 \\ 0 & 0 & 0 \\ \varphi\Upsilon_t & (\varrho-\varphi)\Upsilon_t & \mathcal{X}G_d^t\Phi \\ 0 & 0 & 0 \end{bmatrix},$$

$$\Upsilon_3 = \begin{bmatrix} -\Pi_v & 0 & 0 \\ \bullet & -\Pi_w & 0 \\ \bullet & \bullet & -\gamma^2 I + \Phi^t\Phi \end{bmatrix}$$

$$\Upsilon_4 = \begin{bmatrix} \Upsilon_{41} & \Upsilon_{42} \end{bmatrix} \tag{5.76}$$

$$\Upsilon_{41} = \begin{bmatrix} F\mathcal{X} & 0 & 0 & 0 \\ 0 & F_d\mathcal{X} & 0 & 0 \end{bmatrix}^t$$

$$\Upsilon_{42} = \begin{bmatrix} G\mathcal{X} & 0 & 0 & 0 \\ 0 & G_d\mathcal{X} & 0 & 0 \end{bmatrix}^t$$

$$\Upsilon_5 = \mathrm{diag}\begin{bmatrix} \sigma I & \nu I & I & I \end{bmatrix},$$

$$\Upsilon_o = \mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^t + \Theta_1 + \Theta_4 + \Theta_5 - \Theta_2,$$

$$\Upsilon_c = \Theta_1 + \Theta_3, \quad \Upsilon_s = \mathcal{X}A^t + \mathcal{Y}^t B^t$$

$$\Upsilon_a = \mathcal{A}_d\mathcal{X}, \quad \Upsilon_n = \Theta_2 + \Theta_3 + \Theta_5,$$

$$\Upsilon_m = (1-\mu)\Theta_4 + 2\Theta_3, \quad \Upsilon_t = \mathcal{X}A_d^t + \Upsilon_r,$$

$$\Upsilon_v = 2\mathcal{X} - \Theta_2, \quad \Upsilon_w = 2\mathcal{X} - \Theta_3.$$

Moreover, the local gain matrix is given by $K_o = \mathcal{Y}\mathcal{X}^{-1}C^\dagger$.

5.2.4.3 Single Systems

Finally, we consider the single linear system

$$\dot{x}(t) = Ax(t) + Bu(t) + \Gamma w(t), \tag{5.77}$$

$$\begin{aligned} z_j(t) &= Gx(t) + \Phi w(t), \\ y_j(t) &= Cx(t) + \Psi w(t) \end{aligned} \tag{5.78}$$

for which we will use quantized output measurements such that the following quadratic bounding relation is satisfied:

$$\Delta^t(.)\Delta(.) \leq \alpha x^t(t) F^t F x(t), \tag{5.79}$$

where $\alpha > 0$ is an adjustable parameter. The following corollary establishes the corresponding design result:

Corollary 5.4 *System* (5.77)–(5.78) *is asymptotically stabilizable by quantized feedback controller* $u(t) = K_o y(t)$ *with* $\mathcal{L}_2$*-performance bound* γ *if there exist weighting matrices* $0 < \mathcal{X}$, $\mathcal{Y}$, Θ_m; $m = 1, 2$, *and scalar* $\sigma > 0$, $\gamma > 0$ *satisfying the following LMI*

$$\widetilde{\Omega} = \begin{bmatrix} \Omega_1 & \Omega_2 \\ \bullet & \Omega_3 \end{bmatrix} < 0, \tag{5.80}$$

$$\begin{aligned} \Omega_1 &= \begin{bmatrix} \Omega_o & \Theta_1 & \Theta_2 \\ \bullet & -I & 0 \\ \bullet & \bullet & -\gamma^2 I + \Phi^t \Phi \end{bmatrix}, \\ \Omega_2 &= \begin{bmatrix} \Upsilon_{41} \; \Upsilon_{42} \end{bmatrix}, \quad \Omega_3 = \mathrm{diag}\begin{bmatrix} \sigma I \; I \; I \end{bmatrix}, \\ \Omega_{21} &= \begin{bmatrix} F\mathcal{X} \; 0 \; 0 \end{bmatrix}^t, \quad \Omega_{22} = \begin{bmatrix} G\mathcal{X} \; 0 \; 0 \end{bmatrix}^t, \\ \Omega_o &= \mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^t. \end{aligned} \tag{5.81}$$

Moreover, the local gain matrix is given by $K_o = \mathcal{Y}\mathcal{X}^{-1}C^{\dagger}$.

5.2.5 Simulation Example 5.2

For the purpose of illustration, we consider an interconnected system composed of two subsystems having uniform quantizers with the following data:

$$A_1 = \begin{bmatrix} -2.0000 & 0 \\ 0 & -32.5 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} 1.92 & 1.0 \\ 0 & 2.87 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0.7 \; 0.4 \end{bmatrix}, \quad G_{d1} = \begin{bmatrix} 0.1 \; 0.1 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} -2.01 & 1.0 \\ 1.347 & -1.04 \end{bmatrix}, \quad E_{d1} = \begin{bmatrix} -0.02 & -0.01 \\ -0.01 & -0.02 \end{bmatrix},$$

$$F_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad F_{d1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 10 \; 1 \end{bmatrix}, \quad C_{d1} = \begin{bmatrix} 1 \; 0.1 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} -4.3 & 0 \\ 1.0 & -4.0 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 2.0 & 0 \\ 0 & 1.5 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0.5 \; 0.6 \end{bmatrix}, \quad G_{d2} = \begin{bmatrix} 0.2 \; 0.2 \end{bmatrix},$$
$$E_2 = \begin{bmatrix} -3.2 & 1.0 \\ 0.5 & 0 \end{bmatrix}, \quad E_{d2} = \begin{bmatrix} -0.01 & -0.02 \\ -0.02 & -0.01 \end{bmatrix},$$
$$F_2 = \begin{bmatrix} 0.8 & -1.8 \\ 0 & 11.0 \end{bmatrix}, \quad F_{d2} = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.9 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} 0.5 \; 2 \end{bmatrix}, \quad C_{d2} = \begin{bmatrix} 0.8 \; 0.3 \end{bmatrix}.$$

It is found that the feasible solution of LMI (5.57) is attained at

$$\varphi_1 = 0.3, \quad \varrho_1 = 3.89, \quad \varphi_2 = 0.4, \quad \varrho_2 = 3.77,$$
$$K_1 = -0.6729, \quad K_2 = -2.8345, \quad \eta_1 = 1.56, \quad \eta_2 = 1.47.$$

Typical simulation results are shown in Figs. 5.6 and 5.7 for the open-loop response and closed-loop response of both subsystems. Next, by considering the class of interconnected linear systems **S** given by (5.66)–(5.67) and implementing the LMI (5.75), the feasible solution is found to yield the gains

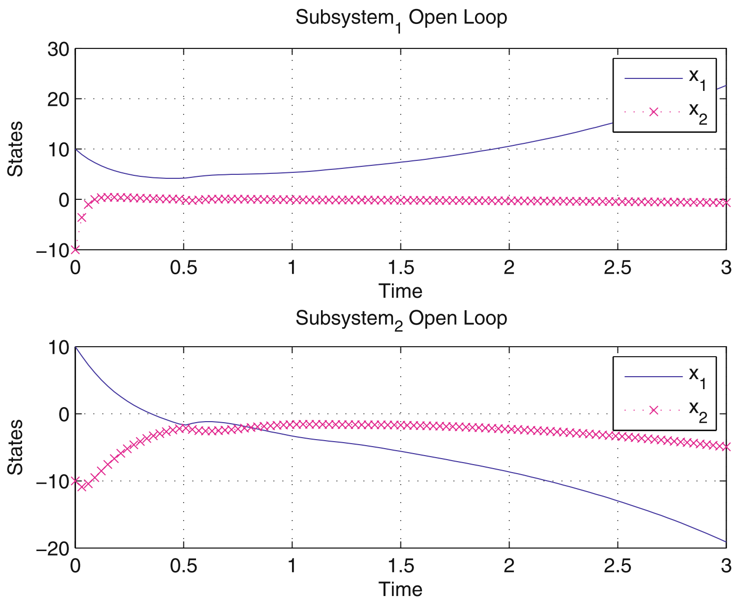


Fig. 5.6 Open-loop response of subsystems 1 and 2

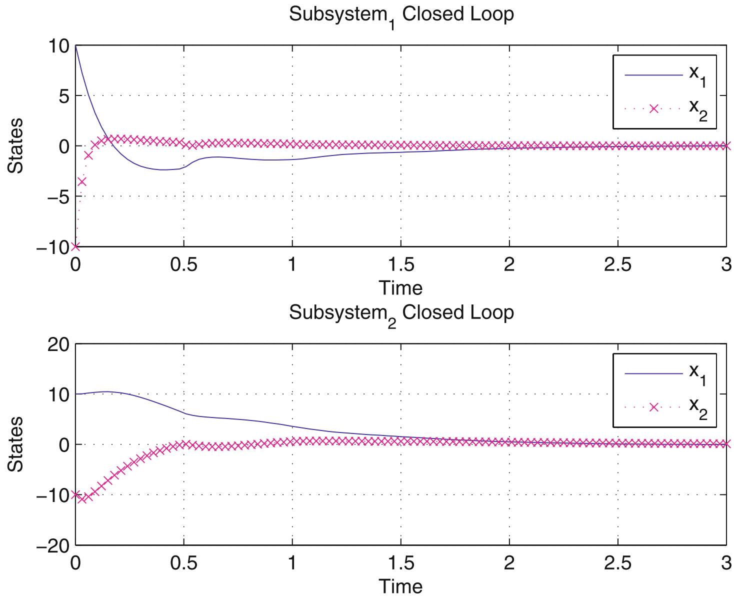
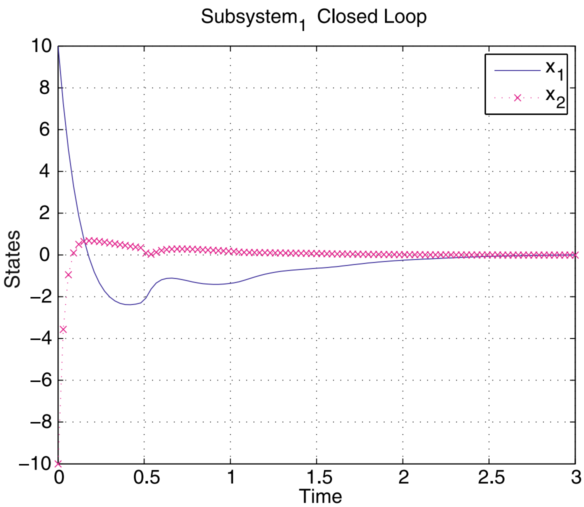


Fig. 5.7 Closed-loop response of subsystems 1 and 2

Fig. 5.8 Closed-loop response of decoupled subsystem 1



$$K_1 = -0.7832, \quad K_2 = -5.9173.$$

The simulation of the closed-loop response of both subsystems are depicted in Figs. 5.8 and 5.9. On implementing the LMI (5.80) for the decoupled subsystem 1, the feasible solution is given by

$$\varphi = 0.5, \quad \varrho = 2.35, \quad \eta = 1.2, \quad K_1 = -0.7832.$$

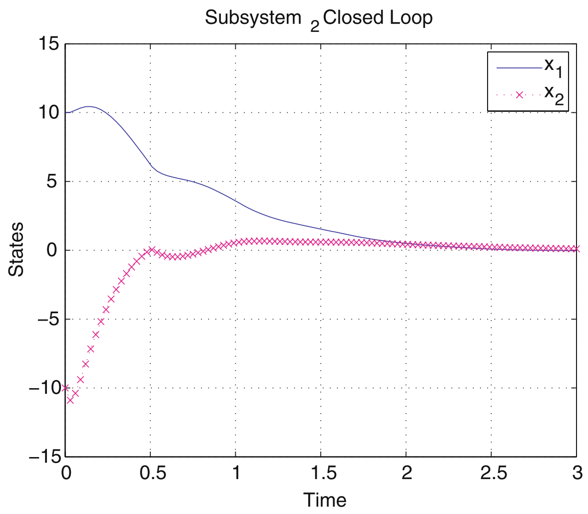


Fig. 5.9 Closed-loop response of decoupled subsystem 2

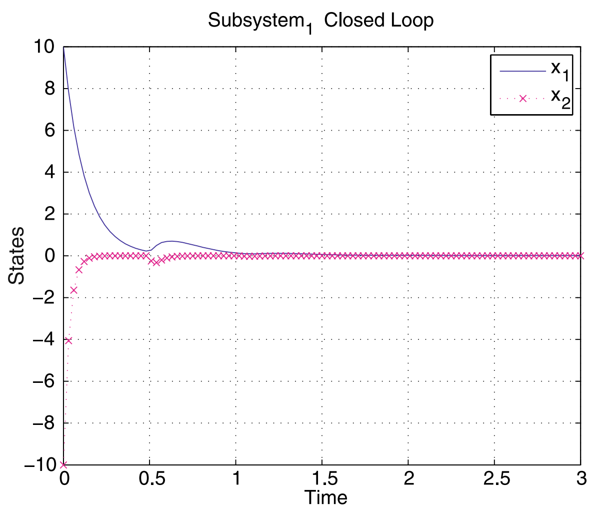


Fig. 5.10 Closed-loop response of single system

The ensuing closed-loop response is plotted in Fig. 5.10. From the ensuing results, it is quite evident that the quantized feedback control system is asymptotically stable for the class of quantizers satisfying the quadratic inequality. This holds true for interconnected time-delay and delay-free systems, single time-delay systems and single systems. The crucial point to record is that the type of quantizer is irrelevant so long as its structure complies with a quadratic inequality. We have observed that the presence of bounding inequalities (5.51) and (5.53) helps in curbing the magnitude of the feedback gains.

5.3 Decentralized Quantized Control I: Discrete Systems

In what follows, we build upon [5, 21] and extend them further to the class of linear interconnected discrete-time systems with unknown-but-bounded couplings and interval time-delays. Specifically, we study the problem of decentralized $\mathcal{H}_\infty$ feedback control for this class of systems where quantized signals exist in the subsystem control channel.

5.3.1 Introduction

It is well known that most of data and/or signals in conventional feedback control theory are processed in a direct manner. In emerging control systems including networks, all signals are transferred through network and this eventually gives rise to packet dropouts or data transfer rate limitations [17]. On the other hand, signal processing and signal quantization always exist in computer-based control systems [22] and therefore recent research studies have been reported on the analysis and design problems for control systems involving various quantization methods [5, 8, 11, 21, 29]. In [5], a quantizer taking value in a finite set is defined and then quantized feedback stabilization for linear systems is considered. The problem of stabilizing an unstable linear system by means of quantized state feedback, where the quantizer takes value in a countable set, is addressed in [8]. It should be noted that the approach in [5] relies on the possibility of making discrete on line adjustments of quantizer parameters which was extended in [21] for more general nonlinear systems with general types of quantizers involving the states of the system, the measured outputs, and the control inputs. In [11], study of quantized and delayed state-feedback control systems under constant bounds on the quantization error and the time-varying delay was reported. Based on [20], stabilization of discrete-time LTI systems with quantized measurement outputs is reported in [29]. Further related results are reported in [33, 34].

On another research front, decentralized stability and feedback stabilization of interconnected systems have been the topic of recurring interests and recent relevant results have been reported in [2, 24–28, 31].

A block-diagram representation of the subsystem model is depicted in Fig. 5.11.

In this regard, an LMI-based decentralized static output-feedback controller (when the switch in Fig. 5.1 is closed) is designed at the subsystem level using only local variables to render the overall closed-loop system is delay-dependent asymptotically stable with guaranteed γ-level and this results provides an important contribution for interconnected discrete systems [2]. When the local output measurements are quantized before passing to the controller (corresponding to opening the switch in Fig. 5.1), we consider the local channel quantizer in a generalized form with a zoom parameter that can be adjusted on-line. We develop a local output-dependent procedure for updating the quantizer parameters to retain

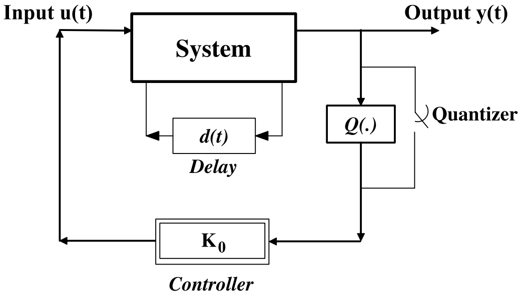


Fig. 5.11 A subsystem quantized model with quantizer

the delay-dependent asymptotic stability and guaranteed performance of the closed-loop quantized system. Several special cases of interest are derived and are shown to provide improved results over the existing literature.

5.3.2 Problem Description

We consider a class of linear systems $\mathbf{S}$ structurally composed of n_s coupled subsystems $\mathbf{S}_j$ and the model of the jth subsystem is described by the state-space representation:

$$x_j(k+1) = A_j x_j(k) + D_j x_j(k - d_j(k)) + B_j u_j(k) + c_j(k) + \Gamma_j w_j(k), \quad (5.82)$$

$$\begin{aligned} z_j(k) &= G_j x_j(k) + L_j x_j(k - d_j(k)) + \Phi_j w_j(k), \\ y_j(k) &= C_j x_j(k) + E_j x_j(k - d_j(k)) + \Psi_j w_j(k), \end{aligned} \quad (5.83)$$

where for $j \in \{1, \ldots, n_s\}$, $x_j(k) \in \Re^{n_j}$ is the state vector, $u_j(k) \in \Re^{m_j}$ is the control input, $y_j(k) \in \Re^{p_j}$ is the control output, $w_j(k) \in \Re^{q_j}$ is the disturbance input which belongs to $\ell_2[0, \infty)$, $z_j(k) \in \Re^{q_j}$ is the performance output and $c_j(k) \in \Re^{n_j}$ is the coupling vector. The matrices $A_j \in \Re^{n_j \times n_j}$, $B_j \in \Re^{n_j \times m_j}$, $D_j \in \Re^{q_j \times n_j}$, $\Phi_j \in \Re^{q_j \times q_j}$, $\Psi_j \in \Re^{p_j \times q_j}$, $\Gamma_j \in \Re^{n_j \times q_j}$, $L_j \in \Re^{q_j \times n_j}$, $G_j \in \Re^{q_j \times n_j}$, $C_j \in \Re^{p_j \times n_j}$, $E_j \in \Re^{p_j \times n_j}$ are real and constants. The initial condition $\kappa_j \in \mathcal{L}_2[-\varrho_j, 0]$, $j \in \{1, \ldots, n_s\}$. In the sequel, we treat $c_j(k)$ as a piecewise-continuous vector function in its arguments and satisfies the quadratic inequality

$$\begin{aligned} & c_j^t(k, ., .) c_j(k, ., .) \\ & \quad \le \phi_j x_j^t(k) M_j^t M_j x_j(k) + \psi_j x_j^t(k - d_j(k)) N_j^t N_j x_j(k - d_j(k)), \end{aligned} \quad (5.84)$$

where $\phi_j > 0$, $\psi_j > 0$ are adjustable bounding parameters and $M_j \in \Re^{n_j \times n_j}$, $N_j \in \Re^{n_j \times n_j}$ are constant matrices. The factors $d_j(k)$, $j \in \{1, \ldots, n_s\}$ are unknown time-delay factors satisfying

$$0 < d_j^- \le d_j(k) \le d_j^+, \quad (5.85)$$

where the bounds d_j^-, d_j^+ are known constants in order to guarantee smooth growth of the state trajectories. Note in (5.82) and (5.84) that the delay within each subsystem (local delay) and among the subsystems (coupling delay) are emphasized.

Assumption 5.1 For all $\theta \in [-d_j^+, 0]$, there exists a scalar $\alpha_j > 0$ such that

$$\|x_j(k - d_j(k))\| \leq \alpha_j \|x_j(k)\|. \tag{5.86}$$

It should be emphasized [23] that (5.86) is not restrictive since we treat α_j as an adjustable parameter at the disposal of the designer who will have the freedom to change it to produce satisfactory system performance.

The class of systems described by (5.82)–(5.83) subject to delay-pattern (5.85) is frequently encountered in modeling several physical systems and engineering applications including large space structures, multi-machine power systems, cold mills, transportation systems, water pollution management, to name a few [25, 27, 28, 30].

5.3.3 Local Quantizers

In the sequel, we adopt the definition of a local (subsystem) quantizer with general form as introduced in [21]. Let $f_j \in \Re^s$, $j = 1, \ldots, n_s$ be the variable being quantized. A *local quantizer* is defined as a piecewise constant function $Q_j : \Re^s \to \mathsf{D}_j$, where D_j is a finite subset of $\Re^s$. This leads to a partition of $\Re^s$ into a finite number of quantization regions of the form $\{f_j \in \Re^s : Q(f_j) = d_j,\ d_j \in \mathsf{D}_j\}$. These quantization regions are not assumed to have any particular shape. We assume that there exist positive real numbers M_j and Δ_j such that the following conditions hold:

$$1.\quad \text{If } |f_j| \leq \mathsf{M}_j \quad \text{then } |Q_j(f_j) - f_j| \leq \Delta_j. \tag{5.87}$$

$$2.\quad \text{If } |f_j| > \mathsf{M}_j \quad \text{then } |Q_j(f_j)| > \mathsf{M}_j - \Delta_j. \tag{5.88}$$

We note that condition (5.87) provides a bound on the quantization error when the quantizer does not saturate. Condition (5.88) gives a way to detect the possibility of saturation. In the sequel, M_j and Δ_j will be referred to as the *range of* Q_j and the *quantization error*, respectively. Henceforth, we assume that $Q(x) = 0$ for x in some neighborhood of the origin. The foregoing requirements are met by the quantizer with rectangular quantization regions [5, 19].

In the control strategy to be developed below, we will use local quantized measurements of the form

$$Q_{\mu_j}(f_j) = \mu_j Q_j\left(\frac{f_j}{\mu_j}\right), \tag{5.89}$$

where $\mu_j > 0$ is an adjustable subsystem parameter.

Remark 5.9 Observe that, at the subsystem level, the extreme case $\mu_j = 0$ is regarded as setting the output of the local quantizer as zero. This local quantizer has the range $\mathsf{M}_j \mu_j$ and the quantization error $\Delta_j \mu_j$. Thus, we can view μ_j as a *local zoom* variable: increasing μ_j corresponds to zooming out and essentially generating a new local quantizer with larger range and larger quantization error, whilst decreasing μ_j implies zooming in and obtaining a local quantizer with smaller range and smaller quantization error. We will update μ_j later on depending on the subsystem state (or the subsystem output). In some sense, it can regarded as additional state of the resultant closed-loop subsystem.

Next, we examine the output-feedback control design.

5.3.4 Static Output-Feedback Design

In this section, we develop new criteria for LMI-based characterization of decentralized stabilization by local static output-feedback. Initially, without quantization, we let the local decentralized static output-feedback has the form

$$u_j(k) = K_{oj} y_j(k), \quad j = 1, \ldots, n_s, \tag{5.90}$$

where the gain matrices K_{oj}, $j = 1, \ldots, N$ have been selected to guarantee the closed-loop system, composed of (5.82)–(5.84) and (5.90), given by

$$x_j(k+1) = \mathcal{A}_j x_j(k) + \mathcal{D}_j x_j(k - d_j(k)) + c_j(k) + \Omega_j w_j(k), \tag{5.91}$$

$$\begin{aligned} z_j(k) &= G_j x_j(k) + L_j x_j(k - d_j(k)) + \Phi_j w_j(k), \\ y_j(k) &= C_j x_j(k) + E_j x_j(k - d_j(k)) + \Psi_j w_j(k), \end{aligned} \tag{5.92}$$

$$\begin{aligned} \mathcal{A}_j &= A_j + B_j K_{oj} C_j, \quad \mathcal{D}_j = D_j + B_j K_{oj} E_j, \\ \Omega_j &= \Gamma_j + B_j K_{oj} \Psi_j \end{aligned} \tag{5.93}$$

is asymptotically stable with disturbance attenuation level γ_j. To facilitate further development, we consider the case where the set of output matrices C_j, $j = 1, \ldots, n_s$ are assumed to be of full row rank and $C_j^\dagger$ represents the right-inverse. Introduce the local Lyapunov-Krasovskii functional (LKF):

$$\begin{aligned} V_j(k) &= x_j^t(k) \mathcal{P}_j x_j(k) + \sum_{m=k-d_j(k)}^{k-1} x_j^t(m) \mathcal{Q}_j x_j(m) \\ &\quad + \sum_{s=2-d_j^+}^{1-d_j^-} \sum_{m=k+s-1}^{k-1} x_j^t(m) \mathcal{Q}_j x_j(m), \end{aligned} \tag{5.94}$$

where $0 < \mathcal{P}_j, 0 < \mathcal{Q}_j$ are weighting matrices of appropriate dimensions.

The following theorem establishes the main design result, without quantization, for subsystem $\mathbf{S}_j$.

Theorem 5.7 *Given the bounds $d_j^- > 0$, $d_j^+ > 0$, $j = 1, \ldots, n_s$, then the family of subsystems $\{\mathbf{S}_j\}$ where $\mathbf{S}_j$ is described by (5.82)–(5.83) is delay-dependent asymptotically stabilizable by decentralized static output-feedback controller $u_j(t) = K_{oj} y_j(t)$ with $\mathcal{L}_2$-performance bound γ_j if there exist matrices*

$$\mathcal{X}_j > 0, \quad \mathcal{G}_j, \quad \Pi_{cj}, \quad \Pi_{sj}, \quad \Pi_{vj}$$

and scalars $\eta_j > 0$, $\mu_j > 0$, $\gamma_j > 0$ *satisfying the following LMIs for* $j = 1, \ldots, n_s$

$$\Pi_j = \begin{bmatrix} \Pi_{1j} & \Pi_{2j} \\ \bullet & \Pi_{3j} \end{bmatrix} < 0, \tag{5.95}$$

$$\Pi_{1j} = \begin{bmatrix} \Pi_{oj} & 0 & 0 & 0 \\ \bullet & -\Pi_{cj} & 0 & 0 \\ \bullet & \bullet & -I_j & 0 \\ \bullet & \bullet & \bullet & -\gamma_j^2 I_j \end{bmatrix},$$

$$\Pi_{2j} = \begin{bmatrix} \mathcal{X}_j G_j & \Pi_{aj} & \mathcal{X}_j M_j^t & 0 \\ \mathcal{X}_j L_j & \Pi_{ej} & 0 & \mathcal{X}_j N_j^t \\ 0 & \mathcal{X}_j & 0 & 0 \\ \Phi_j & \Pi_{wj} & 0 & 0 \end{bmatrix}, \tag{5.96}$$

$$\Pi_{3j} = \begin{bmatrix} -I_j & 0 & 0 & 0 \\ \bullet & -\mathcal{X}_j & 0 & 0 \\ \bullet & \bullet & -\eta_j I_j & 0 \\ \bullet & \bullet & \bullet & -\mu_j I_j \end{bmatrix},$$

$$\Pi_{oj} = -\mathcal{X}_j + d_j^* \Pi_{cj}, \quad \Pi_{aj} = \mathcal{X}_j A_j^t + \mathcal{G}_j B_j^t,$$

$$\Pi_{ej} = \mathcal{X}_j D_j^t + \Pi_{sj} B_j^t, \quad \Pi_{wj} = \mathcal{X}_j \Gamma_j^t + \Pi_{vj} B_j^t.$$

Moreover, the local gain matrix is given by $K_{oj} = \mathcal{G}_j \mathcal{X}_j^{-1} C_j^{\dagger}$.

Proof Let $d_j^* = d_j^+ - d_j^- + 1$. A straightforward computation gives the first-difference of $\Delta V_j(k) = V_j(k+1) - V_j(k)$ along the solutions of (5.82) as:

$$\begin{aligned} \Delta V_j(k) &= [\mathcal{A}_j x_j(k) + \mathcal{D}_j x_j(k - d_j(k)) + c_j(k) + \Omega_j w_j(k)]^t \\ &\quad \times \mathcal{P}_j [\mathcal{A}_j x_j(k) + \mathcal{D}_j x_j(k - d_j(k)) + c_j(k) + \Omega_j w_j(k)] \\ &\quad - x_j^t(k) \mathcal{P}_j x_j(k) + x_j^t(k) \mathcal{Q}_j x_j(k) - x_j^t(k - d_j(k)) \mathcal{Q}_j x_j(k - d_j(k)) \\ &\quad + \sum_{m=k+1-d_j(k+1)}^{k-1} x_j^t(m) \mathcal{Q}_j x_j(m) - \sum_{m=k+1-d_j(k)}^{k-1} x_j^t(m) \mathcal{Q}_j x_j(m) \\ &\quad + (d_j^+ - d_j^-) x_j^t(k) \mathcal{Q}_j x_j(k) - \sum_{m=k+1-d_j^+}^{k-d_j^*} x_j^t(m) \mathcal{Q}_j x_j(m). \end{aligned} \tag{5.97}$$

In order to cast $\Delta V_j(k)$ into a quadratic form, we recall

$$\sum_{m=k+1-d_j(k+1)}^{k-1} x_j^t(m)\mathcal{Q}_j x_j(m)$$
$$= \sum_{m=k+1-d_j^-}^{k-1} x_j^t(m)\mathcal{Q}_j x_j(m) + \sum_{m=k+1-d_j(k+1)}^{k-d_j^-} x_j^t(m)\mathcal{Q}_j x_j(m)$$
$$\leq \sum_{m=k+1-d_j(k)}^{k-1} x_j^t(m)\mathcal{Q}_j x_j(m) + \sum_{m=k+1-d_j^+}^{k-d_j^-} x_j^t(m)\mathcal{Q}_j x_j(m). \tag{5.98}$$

Then using (5.98) into (5.97) and manipulating, we reach

$$\begin{aligned}\Delta V_j(k) \leq{}& [\mathcal{A}_j x_j(k) + \mathcal{D}_j x_j(k-d_j(k)) + c_j(k) + \Omega_j w_j(k)]^t \\ &\times \mathcal{P}_j[\mathcal{A}_j x_j(k) + \mathcal{D}_j x_j(k-d_j(k)) + c_j(k) + \Omega_j w_j(k)] \\ &+ x_j^t(k)[d_j^* \mathcal{Q}_j - \mathcal{P}_j]x_j(k) - x_j^t(k-d_j(k))\mathcal{Q}_j x_j(k-d_j(k)).\end{aligned} \tag{5.99}$$

In terms of the vectors

$$\xi_j(k) = [x_j^t(k)\ x_j^t(k-d_j(k))\ c_j^t(k)\ w_j^t(k)]^t$$

we combine (5.97)–(5.99) with algebraic manipulations using inequalities (5.84) and Schur complements [4] to arrive at:

$$\Delta V_j(k) = \sum_{j=1}^{n_s} \xi_j^t(k)\Xi_j \xi_j(k),$$
$$\Xi_j = \begin{bmatrix} \Xi_{aj} & 0 & 0 & 0 & \mathcal{A}_j^t\mathcal{P}_j \\ \bullet & -\Pi_{cj} & 0 & 0 & \mathcal{D}_j^t\mathcal{P}_j \\ \bullet & \bullet & -I_j & 0 & \mathcal{P}_j \\ \bullet & \bullet & \bullet & 0 & \Omega_j^t\mathcal{P}_j \\ \bullet & \bullet & \bullet & \bullet & -\mathcal{P}_j \end{bmatrix}, \tag{5.100}$$
$$\Xi_{aj} = -\mathcal{P}_j + d_j^*\mathcal{Q}_j + \phi_j M_j^t M_j,$$
$$\Xi_{cj} = \mathcal{Q}_j - \psi_j N_j^t N_j.$$

It is known that the sufficient condition of subsystem internal stability is $\Delta V_j(k) < 0$ when $\omega_j(k) \equiv 0$ which corresponds to deleting the fourth column and row in Ξ_j. This implies that $\Xi_j < 0$ under same requirements.

Next, consider the local performance measure

$$J_j = \sum_{k=0}^{\infty} \big(z_j^t(k)z_j(k) - \gamma^2\omega_j^t(k)\omega_j(k)\big).$$

For any $\omega_j(k) \in \ell_2(0,\infty) \neq 0$ and zero initial condition $x_{jo} = 0$, (hence $V_j(0) = 0$), we have

$$J_j \leq \sum_{k=0}^{\infty} \big[z_j^t(k)z_j(k) - \gamma^2\omega_j^t(k)\omega_j(k) + \Delta V_j(k)|_{(5.91)}\big], \tag{5.101}$$

where $\Delta V_j(k)|_{(5.91)}$ defines the Lyapunov difference along the solutions of system (5.91). On considering (5.93), (5.100) and (5.101), it can easily shown by algebraic manipulations that

$$z_j^t(k)z_j(k) - \gamma^2\omega_j^t(k)\omega_j(k) + \Delta V_j(k)|_{(5.82)} = \chi_j^t(k)\widehat{\Xi}_j\chi_j(k), \quad (5.102)$$

$$\widehat{\Xi}_j = \begin{bmatrix} \Xi_{aj} & 0 & 0 & 0 & G_j & \mathcal{A}_j^t\mathcal{P}_j \\ \bullet & -\Xi_{cj} & 0 & 0 & H_j & \mathcal{D}_j^t\mathcal{P}_j \\ \bullet & \bullet & -I_j & 0 & 0 & \mathcal{P}_j \\ \bullet & \bullet & \bullet & -\gamma_j^2 I_j & \Phi_j & \Omega_j^t\mathcal{P}_j \\ \bullet & \bullet & \bullet & \bullet & -I_j & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\mathcal{P}_j \end{bmatrix} \quad (5.103)$$

for some vector $\chi_j(k)$. It is readily seen that

$$z_j^t(k)z_j(k) - \gamma^2\omega_j^t(k)\omega_j(k) + \Delta V_j(k)|_{(5.82)} < 0$$

for arbitrary $j \in [0, \infty)$, which implies for any $\omega_j(k) \in \ell_2(0, \infty) \neq 0$ that $J_j < 0$. Applying the congruent transformation

$$\mathsf{T} = \mathrm{diag}[\mathcal{X}_j, \mathcal{X}_j, I_j, I_j, I_j, \mathcal{X}_j], \quad \mathcal{X}_j = \mathcal{P}_j^{-1}$$

to (5.103) with Schur complements and using the change of variables

$$\mathcal{G}_j = K_{oj}C_j\mathcal{Y}_j, \quad \Pi_{cj} = \mathcal{X}_j\mathcal{Q}_j\mathcal{X}_j, \quad \Pi_{sj} = \mathcal{X}_j E_j^t K_{oj}^t$$
$$\Pi_{vj} = \mathcal{X}_j\Psi_j^t K_{oj}^t, \quad \eta_j = \phi_j^{-1}, \quad \mu_j = \psi_j^{-1}$$

we readily obtain LMI (5.95) with (5.96) and hence the proof is completed. □

Remark 5.10 It should be emphasized that the LMI variables Π_{cj}, Π_{sj}, Π_{vj} are independent since the matrices E_j, Ψ_j might be singular and thus a unique value of K_{oj} will be produced.

Remark 5.11 We note that the case of decentralized state feedback control $u_j(t) = K_j x_j(t)$, $j = 1, \ldots, n_s$ can be readily obtained from Theorem 5.7 by setting $C_j \equiv I_j$, $E_j \equiv 0$, $\Psi_j \equiv 0$ so that the resulting closed-loop system is asymptotically stable with guaranteed $\mathcal{H}_\infty$ performance.

5.3.5 *Quantized Output-Feedback Design*

Focusing on the availability of quantized local output information, we modify the static output feedback (5.90) using the quantized information of y_j as

$$u_j(k) = \mu_j K_{oj} Q_j\left(\frac{y_j(k)}{\mu_j}\right), \quad j = 1, \ldots, n_s. \quad (5.104)$$

For any fixed scalar $\mu_j > 0$, the closed-loop system, composed of (5.82), (5.84) and (5.104) is given by

$$x_j(k+1) = \mathcal{A}_j x_j(k) + \mathcal{D}_j x_j(k - d_j(k)) + c_j(k) + \Omega_j w_j(k) + H_j(\mu_j, y_j), \tag{5.105}$$

$$z_j(k) = G_j x_j(k) + L_j x_j(k - d_j(k)) + \Phi_j w_j(k), \tag{5.106}$$

$$H_j(\mu_j, y_j) = \mu_j B_j K_{oj}\left(Q_j \frac{y_j(k)}{\mu_j} - \frac{y_j(k)}{\mu_j}\right), \tag{5.107}$$

where $\mathcal{A}_j$, $\mathcal{D}_j$, Ω_j are given by (5.93). Next, we move to examine the stability and desired disturbance attenuation level of the closed-loop system (5.105) in the presence of the quantization error. We employ the LKF (5.94) and consider that the gains K_{oj} are obtained from application of Theorem 5.7. The following theorem establishes the main design result for subsystem $\mathbf{S}_j$.

Theorem 5.8 *Given the bounds $d_j^- > 0$, $d_j^+ > 0$, $j = 1, \ldots, n_s$. If the local quantizer M_j is selected large enough with respect to Δ_j while adjusting the local scalar α_j so as to satisfy the inequality*

$$\mathsf{M}_j > \Delta_j \frac{\|(\mathcal{P}_j + I_j) B_j K_{oj}\|}{\lambda_m(\Lambda_j)} \|C_j + \alpha_j E_j\|. \tag{5.108}$$

Then, the family of subsystems $\{\mathbf{S}_j\}$ where $\mathbf{S}_j$ is described by (5.105)–(5.107) is delay-dependent asymptotically stabilizable with $\mathcal{L}_2$-performance bound γ_j by decentralized quantized output-feedback controller (5.104).

Proof Since

$$\frac{y_j(k)}{\mu_j} = \frac{C_j x_j(k) + E_j x_j(k - d_j(k))}{\mu_j}$$

is quantized before being passed to the feedback channel, we obtain by using the properties of local quantizer (5.87) and (5.88) that whenever $|y_j(k)| \le \mathsf{M}_j \mu_j$, the inequality

$$\left| \frac{y_j(k)}{\mu_j} - Q_j\left(\frac{y_j(k)}{\mu_j}\right)\right| \le \Delta_j \tag{5.109}$$

holds true. Extending on Theorem 5.7, it follows by considering (5.105) and (5.106) that

$$\begin{aligned} J_j \le \sum_{k=0}^{\infty} \Big\{ & \chi_j^t(k) \widetilde{\Xi}_j \chi_j(k) - x_j^t(k) \Lambda_j x_j(k) \\ & + 2H_j^t(\mu_j, y_j)\big[(\mathcal{P}_j \mathcal{A}_j + G_j) x_j(k) + \mathcal{P}_j c_j(k) \\ & + (\mathcal{P}_j \mathcal{D}_j + L_j) x_j(k - d_j(k)) + (\mathcal{P}_j \Omega_j + \Phi_j) w_j(k)\big] \\ & + H_j^t(\mu_j, y_j)(\mathcal{P}_j + I_j) H_j(\mu_j, y_j)\Big\} \\ \le \sum_{k=0}^{\infty} & \big\{\chi_j^t(k) \widetilde{\Xi}_j \chi_j(k) - x_j^t(k) \Lambda_j x_j(k) + \pi_j^t(k) \check{\Xi}_j \pi_j(k)\big\}, \end{aligned} \tag{5.110}$$

where

$$\check{\Xi}_j = \begin{bmatrix} \mathcal{P}_j + I_j & \mathcal{P}_j \mathcal{A}_j + G_j & \mathcal{P}_j \mathcal{D}_j + L_j & \mathcal{P}_j & \mathcal{P}_j \Omega_j + \Phi_j \\ \bullet & 0 & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5.111}$$

$\pi_j(k) = [H_j^t, x_j^t, x_j^t(k - d_j), c_j^t, \omega_j^t]^t$,

where $\widetilde{\Xi}_j$ corresponds to $\widehat{\Xi}_j$ except that $\Xi_{aj} \to \Xi_{aj} + \Lambda_j$ with $\Lambda_j > 0$ being an arbitrary matrix. In view of (5.111), we can express (5.110) for some $\beta_j > 1$ in the form

$$\begin{aligned} J_j &\le \sum_{k=0}^{\infty} \{ \chi_j^t(k) \widetilde{\Xi}_j \chi_j(k) - x_j^t(k) \Lambda_j x_j(k) \\ &\quad + \beta_j^2 H_j^t(\mu_j, y_j)(\mathcal{P}_j + I_j) H_j(\mu_j, y_j) \} \\ &\le \sum_{k=0}^{\infty} \{ \chi_j^t(k) \widetilde{\Xi}_j \chi_j(k) - x_j^t(k) \Lambda_j x_j(k) \\ &\quad + \beta_j^2 \mu_j^2 \Delta_j^2 \| K_{oj}^t B_j^t (\mathcal{P}_j + I_j) B_j K_{oj} \| \} \\ &\le \sum_{k=0}^{\infty} \left\{ \chi_j^t(k) \widetilde{\Xi}_j \chi_j(k) - \frac{\lambda_m(\Lambda_j)}{2} \left[|x_j|^2 \right.\right. \\ &\quad \left.\left. - \left(\beta_j \Delta_j \mu_j \sqrt{\frac{\| K_{oj}^t B_j^t (\mathcal{P}_j + I_j) B_j K_{oj} \|}{\lambda_m(\Lambda_j)}} \right)^2 \right] \right\}. \end{aligned} \tag{5.112}$$

Since the output measurements information are used, we invoke Assumption 5.1 to write

$$|y_j| = \| C_j x_j(k) + E_j x_j(k - d_j(k)) \| \le \| C_j + \alpha_j E_j \| |x_j|$$

and used this inequality into (5.112) to arrive at

$$\begin{aligned} J_j \le \sum_{k=0}^{\infty} &\left\{ \chi_j^t(k) \widetilde{\Xi}_j \chi_j(k) - \frac{\lambda_m(\Lambda_j)}{2 \| C_j + \alpha_j E_j \|^2} \left[|y_j|^2 \right.\right. \\ &\left.\left. - \left(\beta_j \Delta_j \mu_j \| C_j + \alpha_j E_j \| \sqrt{\frac{\| K_{oj}^t B_j^t (\mathcal{P}_j + I_j) B_j K_{oj} \|}{\lambda_m(\Lambda_j)}} \right)^2 \right] \right\}. \end{aligned} \tag{5.113}$$

By virtue of (5.113), we can always find a scalar $\varepsilon_j \in (0, 1)$ such that

$$\mathsf{M}_j > \beta_j \Delta_j \| C_j + \alpha_j E_j \| \sqrt{\frac{\| K_{oj}^t B_j^t (\mathcal{P}_j + I_j) B_j K_{oj} \|}{\lambda_m(\Lambda_j)}} \frac{1}{\sqrt{1 - \varepsilon_j}}. \tag{5.114}$$

This is equivalent to

$$\frac{1}{\sqrt{1-\varepsilon_j}}\beta_j \Delta_j \|C_j + \alpha_j E_j\| \sqrt{\frac{\|K_{oj}^t B_j^t(\mathcal{P}_j + I_j)B_j K_{oj}\|}{\lambda_m(\Lambda_j)}}\mu_j < \mathsf{M}_j. \quad (5.115)$$

Therefore, for any nonzero $|y_j|$, we can find a scalar $\mu_j > 0$ such that

$$\frac{1}{\sqrt{1-\varepsilon_j}}\beta_j \Delta_j \|C_j + \alpha_j E_j\| \sqrt{\frac{\|K_{oj}^t B_j^t(\mathcal{P}_j + I_j)B_j K_{oj}\|}{\lambda_m(\Lambda_j)}}\mu_j \le |y_j| \le \mathsf{M}_j \mu_j. \quad (5.116)$$

At the extreme case $|y_j| = 0$, we set $\mu_j = 0$ so that the output of the local quantizer is considered zero and therefore (5.116) holds true. This, in turn, implies that we can always select μ_j so that (5.116) is satisfied, (5.113) holds and hence

$$J_j \le \sum_{k=0}^{\infty} \left\{ \chi_j^t(k)\widetilde{\Xi}_j \chi_j(k) - \frac{1}{2}\varepsilon_j \lambda_m(\Lambda_j)\frac{|y_j|^2}{\|C_j + \alpha_j C_{dj}\|^2} \right\}. \quad (5.117)$$

The rest of the proof follows from Theorem 5.7. □

Remark 5.12 For the case of decentralized state feedback control $u_j(t) = K_j x_j(t)$, $j = 1, \dots, n_s$, then Theorem 5.8 specializes to the following corollary:

Corollary 5.5 *Given the bounds $d_j^- > 0$, $d_j^+ > 0$, $j = 1, \dots, n_s$. If the local quantizer M_j is selected large enough with respect to Δ_j while adjusting the local scalar α_j so as to satisfy the inequality*

$$\mathsf{M}_j > \Delta_j \frac{\|(\mathcal{P}_j + I_j)B_j K_{oj}\|}{\lambda_m(\Lambda_j)}. \quad (5.118)$$

Then, the family of subsystems $\{\mathbf{S}_j\}$ where $\mathbf{S}_j$ is described by (5.82)–(5.84) is delay-dependent asymptotically stabilizable with $\mathcal{L}_2$-performance bound γ_j by decentralized quantized state-feedback controller

$$u_j(t) = \mu_j K_j Q_j\left(\frac{x_j(t)}{\mu_j}\right), \quad j = 1, \dots, n_s.$$

Remark 5.13 By the mean-value theorem and following [15], it can be shown that $\lambda_m(\mathcal{P}_j)\|x_j\|^2 \le V_j \le \vartheta_j \|\kappa_j\|^2$, where

$$\vartheta_j = \left[\lambda_M(\mathcal{P}_j) + d_j^+ \lambda_M(\mathcal{Q}_j)\right].$$

Based on the results of [21], we define the local ellipsoids

$$\begin{aligned}
\mathcal{B}_{oj}(\mu_j) &:= \{x_j : x_j^t \mathcal{P}_j x_j \le \lambda_m(\mathcal{P}_j \mathsf{M}_j^2 \mu_j^2\}, \\
\mathcal{B}_{sj}(\mu_j) &:= \{x_j : x_j^t \mathcal{P}_j x_j \le \lambda_M(\mathcal{P}_j \mathcal{D}_j^2 \Delta_j^2 (1+\sigma_j)^2 \mu_j^2\}, \\
\mathcal{D}_j &:= \frac{\|(\mathcal{P}_j + I_j)B_j K_{oj}\|}{\lambda_m(\Lambda_j)} \|C_j + \alpha_j E_j\|.
\end{aligned}$$

In the "zooming-in" stage, it can be inferred that $\mathcal{B}_{sj}(\mu_j) \subset \mathcal{B}_{oj}(\mu_j)$ are invariant regions for system (5.107) given $\sigma_j > 0$. Moreover, all solutions of (5.107) that start in $\mathcal{B}_{oj}(\mu_j)$ enter $\mathcal{B}_{sj}(\mu_j)$ in finite time.

Remark 5.14 It is crucial to recognize that the local scalar α_j plays a basic role in steering the trajectories of (5.107) toward the final ellipsoid $\mathcal{B}_{sj}(\mu_j)$. This is a distinct feature of quantized time-delay systems. It should be noted that the parameters $\beta_j,\ j = 1, \ldots, n_s$ are introduced in (5.112) to reach the desired estimates and to handle the interdependence between H_j and $(x_j, x_j(k-d_j), c_j, \omega_j)$. In addition, the parameters $\beta_j,\ j = 1, \ldots, n_s$ can be adjusted to help satisfying (5.108).

Remark 5.15 We note in Theorem 5.8 and Corollary 5.5 there are several degrees of freedom to achieve the desired stability with guaranteed performance, particularly since both the off-line gain computation and the on-line quantized feedback are decentralized. This is a salient feature of the developed results of this chapter, which is not shared by several published results [2, 28, 30, 31].

5.3.6 Special Cases

In the sequel, some special cases are derived to emphasize the generality of our approach. First, we consider the single nominally-linear time-delay system

$$x(k+1) = Ax(k) + Dx(k-d(k)) + Bu(k) + \Gamma w(k), \tag{5.119}$$

$$\begin{aligned} z(k) &= Gx_j(k) + Lx(k-d(k)) + \Phi w(k), \\ y(k) &= Cx_j(k) + Ex(k-d(k)) + \Psi w(k). \end{aligned} \tag{5.120}$$

The factor $d(k)$ is an unknown time-delay satisfying $0 < d^- \le d(k) \le d^+$ where the bounds d^-, d^+ are known constants in order to guarantee smooth growth of the state trajectories. It will be assumed that for all $\theta \in [-d^+, 0]$, there exists a scalar $\alpha > 0$ such that $\|x(k-d(k))\| \le \alpha \|x(k)\|$. The following corollary establishes the corresponding design result:

Corollary 5.6 *Given the bounds* $d^- > 0$, $d^+ > 0$. *Suppose that there exist matrices* $\mathcal{X} > 0, \mathcal{G}, \Pi_c, \Pi_s, \Pi_w$ *and scalar* $\gamma_j > 0$ *satisfying the following LMI*

$$\Sigma = \begin{bmatrix} \Sigma_1 & \Sigma_2 \\ \bullet & \Sigma_3 \end{bmatrix} < 0, \tag{5.121}$$

$$\begin{aligned} \Sigma_2 &= \begin{bmatrix} \mathcal{X}G & \mathcal{X}A^t + \mathcal{G}B^t \\ \mathcal{X}L & \mathcal{X}D^t + \Pi_s B^t \\ 0 & \mathcal{X} \\ \Phi & \Pi_w \end{bmatrix}, \quad \Sigma_3 = \begin{bmatrix} -I & 0 \\ \bullet & -\mathcal{X} \end{bmatrix}, \\ \Sigma_1 &= \begin{bmatrix} -\mathcal{X} + d^* \Pi_c & 0 & 0 & 0 \\ \bullet & -\Pi_c & 0 & 0 \\ \bullet & \bullet & -I & 0 \\ \bullet & \bullet & \bullet & -\gamma^2 I \end{bmatrix} \end{aligned} \tag{5.122}$$

with the gain matrix $K_o = \mathcal{G}\mathcal{X}^{-1}C^{\dagger}$. *Moreover, if the quantizer* M *is selected large enough with respect to* Δ *so as to satisfy the inequality*

$$\mathsf{M} > \Delta \frac{\|(\mathcal{X}^{-1} + I)BK_o\|}{\lambda_m(\Lambda)} \|C + \alpha E\|$$

then system (5.119)–(5.120) *is delay-dependent asymptotically stabilizable by quantized output-feedback controller*

$$u(k) = \mu K_o Q\left(\frac{y(k)}{\mu}\right)$$

with $\mathcal{L}_2$*-performance bound* γ_j.

Next, we consider a class of nominally-linear systems $\mathbf{S}$ structurally composed of n_s coupled subsystems $\mathbf{S}_j$ and the model of the jth subsystem is described by the state-space representation:

$$x_j(k+1) = A_j x_j(k) + B_j u_j(k) + c_j(k) + \Gamma_j w_j(k), \tag{5.123}$$

$$\begin{aligned} z_j(k) &= G_j x_j(k) + \Phi_j w_j(k), \\ y_j(k) &= C_j x_j(k) + \Psi_j w_j(k). \end{aligned} \tag{5.124}$$

Similarly, we treat $c_j(k)$ as a piecewise-continuous vector function in its arguments and satisfies the quadratic inequality

$$c_j^t(k,.,.)c_j(k,.,.) \le \phi_j x_j^t(k) M_j^t M_j x_j(k), \tag{5.125}$$

where $\phi_j > 0$ are adjustable bounding parameters and $M_j \in \Re^{n_j \times n_j}$ are constant matrices. The factors $d_j(k)$, $j \in \{1, \ldots, n_s\}$ are unknown time-delay factors satisfying (5.85). The following corollary stands out:

Corollary 5.7 *Given the bounds* $d_j^- > 0$, $d_j^+ > 0$, $j = 1, \ldots, n_s$. *If there exist matrices* $\mathcal{X}_j > 0$, $\mathcal{G}_j$, Π_{cj}, Π_{sj}, Π_{vj}, Π_{wj} *and scalars* $\eta_j > 0$, $\gamma_j > 0$ *satisfying the following LMIs for* $j = 1, \ldots, n_s$

$$\widehat{\Pi}_j = \begin{bmatrix} \widehat{\Pi}_{1j} & \widehat{\Pi}_{2j} \\ \bullet & \widehat{\Pi}_{3j} \end{bmatrix} < 0, \tag{5.126}$$

$$\widehat{\Pi}_{1j} = \begin{bmatrix} \hat{\Pi}_{oj} & 0 & 0 & 0 \\ \bullet & -\Pi_{cj} & 0 & 0 \\ \bullet & \bullet & -I_j & 0 \\ \bullet & \bullet & \bullet & -\gamma_j^2 I_j \end{bmatrix},$$

$$\widehat{\Pi}_{2j} = \begin{bmatrix} \mathcal{X}_j G_j & \hat{\Pi}_{aj} & \mathcal{X}_j M_j^t \\ \mathcal{X}_j L_j & \Pi_{sj} B_j^t & 0 \\ 0 & \mathcal{X}_j & 0 \\ \Phi_j & \Pi_{wj} & 0 \end{bmatrix}, \tag{5.127}$$

$$\widehat{\Pi}_{3j} = \begin{bmatrix} -I_j & 0 & 0 \\ \bullet & -\mathcal{X}_j & 0 \\ \bullet & \bullet & -\eta_j I_j \end{bmatrix},$$

$$\hat{\Pi}_{oj} = -\mathcal{X}_j + d_j^* \Pi_{cj}, \quad \hat{\Pi}_{aj} = \mathcal{X}_j A_j^t + \mathcal{G}_j B_j^t$$

with the local gain matrix $K_{oj} = \mathcal{G}_j \mathcal{X}_j^{-1} C_j^\dagger$. *Moreover, if the quantizer* M_j *is selected large enough with respect to* Δ_j *so as to satisfy the inequality*

$$\mathsf{M}_j > \Delta_j \frac{\|(\mathcal{X}_j^{-1} + I) B_j K_{oj}\|}{\lambda_m(\Lambda_j)} \|C\|$$

then the family of subsystems $\{\mathbf{S}_j\}$ *where* $\mathbf{S}_j$ *is described by* (5.123)–(5.124) *is asymptotically stabilizable by decentralized static output-feedback controller* $u_j(t) = K_{oj} y_j(t)$ *with* $\mathcal{L}_2$*-performance bound* γ_j.

Finally, we consider the single nominally-linear system

$$x(k+1) = Ax(k) + Bu(k) + \Gamma w(k), \tag{5.128}$$

$$\begin{aligned} z(k) &= Gx(k) + \Phi w(k), \\ y(k) &= Cx(k) + \Psi w(k). \end{aligned} \tag{5.129}$$

The following corollary establishes the corresponding design result:

Corollary 5.8 *Suppose that there exist matrices* $\mathcal{X} > 0$, $\mathcal{G}$, Π_c, Π_s, Π_w *and scalar* $\gamma > 0$ *satisfying the following LMI*

$$\widehat{\Sigma} = \begin{bmatrix} \widehat{\Sigma}_1 & \widehat{\Sigma}_2 \\ \bullet & \widehat{\Sigma}_3 \end{bmatrix} < 0, \tag{5.130}$$

$$\widehat{\Sigma}_2 = \begin{bmatrix} \mathcal{X}G & \mathcal{X}A^t + \mathcal{G}B^t \\ 0 & \Pi_s B^t \\ 0 & \mathcal{X} \\ \Phi & \Pi_w \end{bmatrix}, \quad \widehat{\Sigma}_3 = \begin{bmatrix} -I & 0 \\ \bullet & -\mathcal{X} \end{bmatrix},$$

$$\widehat{\Sigma}_1 = \begin{bmatrix} -\mathcal{X} + d^* \Pi_c & 0 & 0 & 0 \\ \bullet & -\Pi_c & 0 & 0 \\ \bullet & \bullet & -I & 0 \\ \bullet & \bullet & \bullet & -\gamma^2 I \end{bmatrix} \tag{5.131}$$

with the gain matrix $K_o = \mathcal{G}\mathcal{X}^{-1} C^\dagger$. *Moreover, if the quantizer* M *is selected large enough with respect to* Δ *while adjusting the scalar* α *so as to satisfy the inequality*

$$\mathsf{M} > \Delta \frac{\|(\mathcal{X}^{-1} + I) B K_o\|}{\lambda_m(\Lambda)} \|C\|$$

then system (5.128)–(5.129) *is asymptotically stabilizable by quantized output-feedback controller*

$$u(k) = \mu K_o Q\left(\frac{y(k)}{\mu}\right)$$

with $\mathcal{L}_2$-performance bound γ_j.

Remark 5.16 It is significant to note that the results of Corollaries 5.6 through 5.8 establish new designs for quantized output-feedback control. It provides efficient LMI-based results in comparison with [5, 11, 21, 29].

5.3.7 Simulation Example 5.3

For the purpose of illustration, we consider an interconnected system composed of two subsystems having uniform quantizers with the following data:

$$A_1 = \begin{bmatrix} 0.8 & 0 \\ 0.05 & 0.9 \end{bmatrix}, \quad D_1 = \begin{bmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \quad G_1^t = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \quad L_1^t = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 0.1 & 0.01 \\ -0.1 & 0.02 \end{bmatrix}, \quad F_1 = \begin{bmatrix} -0.02 & -0.01 \\ -0.01 & -0.02 \end{bmatrix},$$

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0 & 0.5 & -0.1 \\ 0.1 & 0 & 0.4 \end{bmatrix}, \quad G_2^t = \begin{bmatrix} 1 \\ 0.2 \\ 0.7 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0.5 \\ 1.5 \\ 0.4 \end{bmatrix}, \quad D_2 = \begin{bmatrix} -0.2 & 0.04 & 0.2 \\ -0.4 & -0.15 & 0 \\ 0.1 & 0 & 0.3 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} -0.02 & 0.01 & 0 \\ 0 & 0.1 & 0 \\ -0.02 & 0 & 0.05 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0.03 & 0 & 0.02 \\ 0.1 & 0.2 & 0 \\ -0.01 & 0 & 0.01 \end{bmatrix},$$

$$L_2 = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.1 \end{bmatrix}.$$

It is found that the feasible solution of LMI (5.95) is attained at

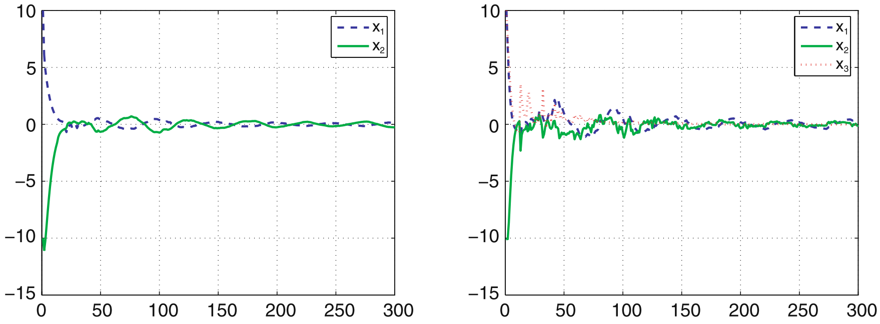


Fig. 5.12 Closed-loop response: subsystem 1 (*left*), subsystem 2 (*right*)

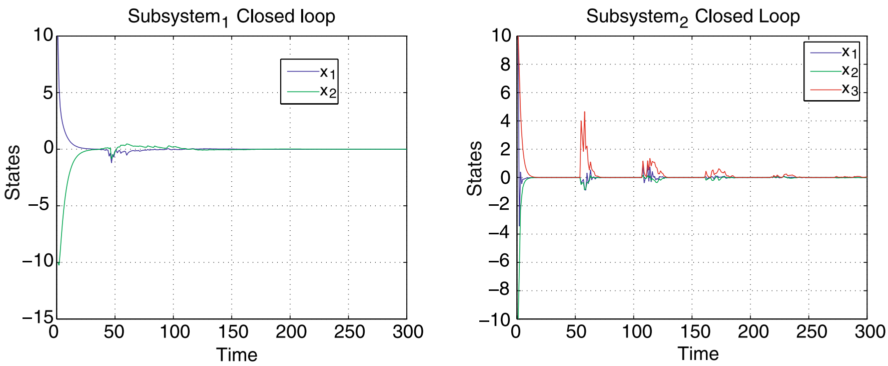


Fig. 5.13 Closed-loop response of decoupled subsystem 1 (*left*), decoupled subsystem 2 (*right*)

$$d_1^- = 10, \quad d_1^+ = 30, \quad d_2^- = 10, \quad d_2^+ = 30,$$
$$K_1 = -0.4023, \quad K_2 = -0.0916.$$

Typical simulation results are shown in Fig. 5.12 for the closed-loop response of both subsystems. Next, by dropping the time-delay factors and considering LMI (5.126) the feasible solution is found to yield the gains

$$K_1 = -0.6653, \quad K_2 = -1.0915.$$

The simulation of the closed-loop response of both subsystems are depicted in Fig. 5.13.

On implementing the LMI (5.121) for subsystem 2, the feasible solution is given by

$$d^- = 20, \quad d_2^+ = 60, \quad K_2 = -1.3391.$$

Finally, the feasible solution of LMI (5.130) for subsystem 1 without delay terms is $K_1 = -0.3039$ and the corresponding closed-loop response is plotted in Fig. 5.14.

From the ensuing results, it is quite evident that the quantized feedback control system is asymptotically stable for the class of quantizers satisfying the quadratic

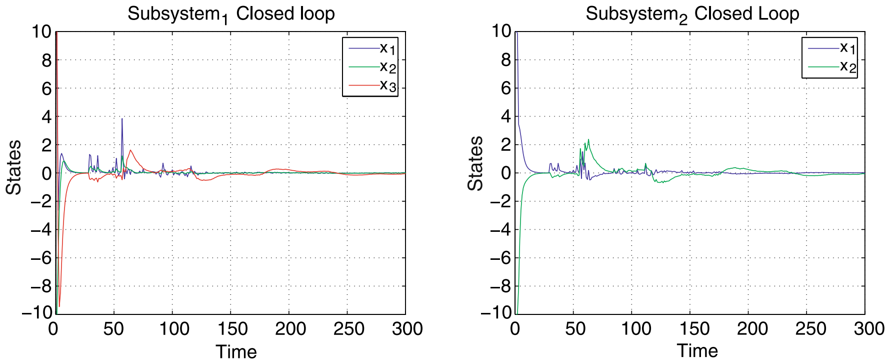


Fig. 5.14 Closed-loop response of single system: time-delay (*left*), delay-free (*right*)

inequality. This is equally true for interconnected time-delay and delay-free systems, single time-delay systems and single systems. The crucial point to record is that the type of quantizer so long as it is satisfies its structure complies with a quadratic inequality. We have observed that the presence of bounding inequalities (5.84) and (5.89) helps in curbing the magnitude of the feedback gains.

5.3.8 Simulation Example 5.4

For the purpose of illustration, we consider an interconnected system composed of two subsystems having uniform quantizers with the following data:

$$A_1 = \begin{bmatrix} 0.75 & -0.20 \\ 0.1 & 0.67 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.21 & 0.14 \\ 0.2 & 0.13 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix}, \quad G_1^t = \begin{bmatrix} 0.4 \\ 1.0 \end{bmatrix}, \quad L_1^t = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 0.3 & 0 & 0.1 \\ -0.1 & -0.2 & 0.02 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 0.2 & -0.02 & -0.1 \\ 0.1 & 0 & -0.2 \end{bmatrix},$$

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.83 & 0 & 0.22 \\ -0.1 & 0.56 & -0.12 \\ 0.23 & -0.20 & 0.4 \end{bmatrix}, \quad G_2^t = \begin{bmatrix} -1 \\ 0.15 \\ 0.57 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 1 \\ -0.5 \\ 0.4 \end{bmatrix}, \quad D_2 = \begin{bmatrix} -0.32 & 0.14 & -0.1 \\ 0.56 & -0.2 & 0.3 \\ 0.1 & -0.4 & 0.24 \end{bmatrix},$$

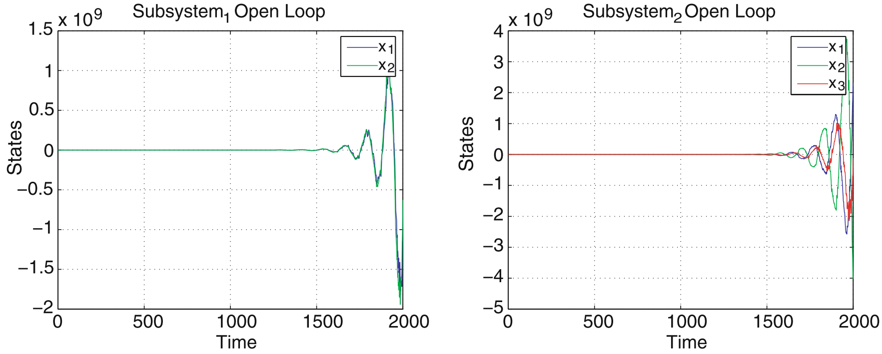


Fig. 5.15 Open-loop response: subsystem 1 (*left*), subsystem 2 (*right*)

$$E_2 = \begin{bmatrix} -0.42 & 0.1 \\ 0 & 0.1 \\ -0.2 & 0.5 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0.43 & 0.02 \\ 0.1 & 0.0 \\ -0.1 & 0.01 \end{bmatrix},$$
$$L_2 = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.1 \end{bmatrix}.$$

As shown in Fig. 5.15, both subsystems are unstable. It is found that the feasible solution of LMI (5.95) is attained at

$$d_1^- = 20, \quad d_1^+ = 30, \quad d_2^- = 50, \quad d_2^+ = 60,$$
$$K_1 = -1.6627, \quad K_2 = 0.3214.$$

The closed-loop response is depicted in Fig. 5.16.

5.4 Decentralized Quantized Control II: Discrete Systems

In conventional feedback control theory, most of data and/or signals are processed in a direct manner. With the emerging control systems including networks, all signals are transferred through network and this eventually gives rise to packet dropouts or data transfer rate limitations [17]. On the other hand, signal processing and signal quantization always exist in computer-based control systems [22] and therefore recent research studies have been reported on the analysis and design problems for control systems involving various quantization methods, see [5, 8, 11, 21, 29] and the references cited therein. In [5], a quantizer taking value in a finite set is defined and then quantized feedback stabilization for linear systems is considered. In [8], the problem of stabilizing an unstable linear system by means of quantized state feedback, where the quantizer takes value in a countable set is addressed. It should

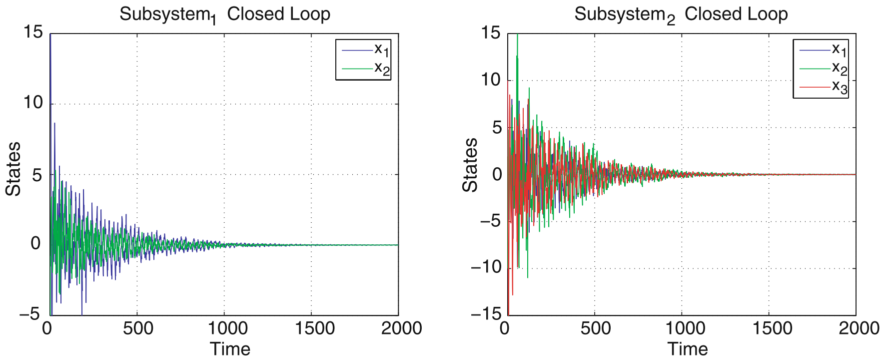


Fig. 5.16 Closed-loop response of decoupled subsystem 1 (*left*), decoupled subsystem 2 (*right*)

be noted that the approach in [5] relies on the possibility of making discrete on line adjustments of quantizer parameters which was extended in [21] for more general nonlinear systems with general types of quantizers involving the states of the system, the measured outputs, and the control inputs. Recently in [11], a study of quantized and delayed state-feedback control systems under constant bounds on the quantization error and the time-varying delay was reported. Based on [21], stabilization of discrete-time LTI systems with quantized measurement outputs is reported in [29]. Further related results are reported in [33, 34]. On another research front, decentralized stability and feedback stabilization of interconnected systems have been the topic of recurring interests and recent relevant results have been reported in [2, 24–28, 31].

5.4.1 Introduction

In this section, we investigate a generalized approach to quantized feedback control in linear discrete-time system. We cast the problem under consideration as the problem of designing a decentralized $\mathcal{H}_\infty$ feedback control for a class of linear interconnected discrete-time systems with quantized signals in the subsystem control channel. The system has unknown-but-bounded couplings and interval time-delays. Within our formulation, we take the quantizer of arbitrary form that satisfies a quadratic inequality constraint in the state and the delayed state. We illustrated the generality of this quantizer structure. Based on quantized output measurements, a decentralized quantized output-feedback controller is designed at the subsystem level to render the overall closed-loop system delay-dependent asymptotically stable with guaranteed γ-level. To further illustrate the generality of the developed approach, it is established that several classes of quantized feedback control systems of interest are readily derived as special cases. These include the classes of interconnected time-delay and delay-free systems, single time-delay systems and single systems.

5.4.2 Problem Statement

We consider a class of nominally-linear time-delay systems $\mathbf{S}$ structurally composed of n_s coupled subsystems $\mathbf{S}_j$ and the model of the jth subsystem is described by the state-space representation:

$$x_j(k+1) = A_j x_j(k) + D_j x_j(k-d_j(k)) + B_j u_j(k) + c_j(k) + \Gamma_j w_j(k), \quad (5.132)$$

$$\begin{aligned} z_j(k) &= G_j x_j(k) + L_j x_j(k-d_j(k)) + \Phi_j w_j(k), \\ y_j(k) &= C_j x_j(k) + H_j x_j(k-d_j(k)), \end{aligned} \quad (5.133)$$

where for $j \in \{1, \dots, n_s\}$, $x_j(k) \in \Re^{n_j}$ is the state vector, $u_j(k) \in \Re^{m_j}$ is the control input, $y_j(k) \in \Re^{p_j}$ is the control output, $w_j(k) \in \Re^{q_j}$ is the disturbance input which belongs to $\ell_2[0, \infty)$, $z_j(k) \in \Re^{q_j}$ is the performance output and $c_j(k) \in \Re^{n_j}$ is the coupling vector. The matrices $A_j \in \Re^{n_j \times n_j}$, $B_j \in \Re^{n_j \times m_j}$, $D_j \in \Re^{q_j \times n_j}$, $\Phi_j \in \Re^{q_j \times q_j}$, $\Gamma_j \in \Re^{n_j \times q_j}$, $L_j \in \Re^{q_j \times n_j}$, $G_j \in \Re^{q_j \times n_j}$, $C_j \in \Re^{p_j \times n_j}$, $E_j \in \Re^{p_j \times n_j}$ are real and constants. The initial condition $\kappa_j \in \mathcal{L}_2[-\varrho_j, 0]$, $j \in \{1, \dots, n_s\}$. In the sequel, we treat $c_j(k)$ as a piecewise-continuous vector function in its arguments and satisfies the quadratic inequality

$$\begin{aligned} & c_j^t(k, ., .) c_j(k, ., .) \\ & \quad \le \phi_j x_j^t(k) M_j^t M_j x_j(k) + \psi_j x_j^t(k-d_j(k)) N_j^t N_j x_j(k-d_j(k)), \end{aligned} \quad (5.134)$$

where $\phi_j > 0$, $\psi_j > 0$ are adjustable bounding parameters and $M_j \in \Re^{n_j \times n_j}$, $N_j \in \Re^{n_j \times n_j}$ are constant matrices. The factors $d_j(k)$, $j \in \{1, \dots, n_s\}$ are unknown time-delay factors satisfying

$$0 < d_j^- \le d_j(k) \le d_j^+, \quad (5.135)$$

where the bounds d_j^-, d_j^+ are known constants in order to guarantee smooth growth of the state trajectories. Note in (5.132) and (5.134) that the subsystem delay with local and coupling patterns are emphasized. The class of systems described by (5.132)–(5.133) subject to delay-pattern (5.135) is frequently encountered in modeling several physical systems and engineering applications including large space structures, multi-machine power systems, cold mills, transportation systems, water pollution management, to name a few [25, 30].

5.4.3 A Class of Local Quantizers

In the sequel, we treat a *quantizer* as a device in the control loop that converts a real-valued signal into a piecewise constant one. We adopt the definition of a local (subsystem) quantizer with general form as introduced in [21]. Let $f_j \in \Re^s$, $j = 1, \dots, n_s$ be the variable being quantized. A *local quantizer* is defined as a piecewise constant function $Q_j : \Re^s \to \mathcal{D}_j$, where $\mathcal{D}_j$ is a finite subset of $\Re^s$. This leads to a partition of $\Re^s$ into a finite number of quantization regions of the form

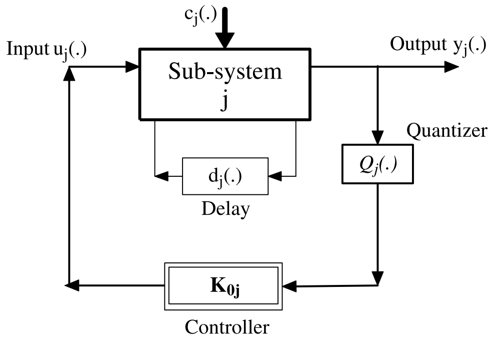


Fig. 5.17 A description of quantized subsystem model

$\{f_j \in \Re^s : Q(f_j) = d_j,\ d_j \in \mathcal{D}_j\}$. These quantization regions are not assumed to have any particular shape.

In the quantized control strategy to be developed below, we will use the local quantization error $\Delta_j(y) = Q_j(y_j) - y_j$ (see Fig. 5.17) based on output measurements such that the following quadratic bounding relation is satisfied:

$$\begin{aligned}\Delta_j^t(.)\Delta_j(.) \leq{} & \alpha_j x_j^t(k) E_j^t E_j x_j(k) \\ & + \beta_j x_j^t(k - d_j(k)) F_j^t F_j x_j(k - d_j(k)),\end{aligned} \tag{5.136}$$

where $\alpha_j > 0$, $\beta_j > 0$ are adjustable subsystem parameters and the matrices E_j, F_j are arbitrary but constants.

It is crucial to recognize that the quadratic bounding relation (5.136) is independent of the structure of the quantizer employed. In fact, it is satisfied by wide class of practically-used quantizers, see Remark 5.8 for further details.

In what follows we seek to design quantized feedback controllers which guarantee the asymptotic stability of the family of subsystems $\mathbf{S}_j$ subject to the structural constraints (5.134)–(5.136).

5.4.4 *Quantized Feedback Design*

In this section, we develop new criteria for LMI-based characterization of decentralized stabilization by local quantized feedback of the form

$$u_j(k) = K_{oj} Q_j(y_j), \quad j = 1, \ldots, n_s, \tag{5.137}$$

where the gain matrices K_{oj}, $j = 1, \ldots, N$ will be selected to guarantee that the closed-loop system, composed of (5.132)–(5.134), (5.136) and (5.137), given by

$$\begin{aligned}x_j(k+1) &= \mathcal{A}_j x_j(k) + \mathcal{D}_j x_j(k - d_j(k)) + c_j(k) + B_j K_{oj} \Delta_j(y_j) + \Gamma_j w_j(k), \\ \mathcal{A}_j &= A_j + B_j K_{oj} C_j, \quad \mathcal{D}_j = D_j + B_j K_{oj} H_j,\end{aligned} \tag{5.138}$$

$$z_j(k) = G_j x_j(k) + L_j x_j(k - d_j(k)) + \Phi_j w_j(k) \tag{5.139}$$

is asymptotically stable with disturbance attenuation level γ_j. To facilitate further development, we consider the case where the set of output matrices C_j, $j =$

$1, \ldots, n_s$ are assumed to be of full row rank and $C_j^\dagger$ represents the right-inverse. Let $d_j^* = d_j^+ - d_j^- + 1$. Introduce the local Lyapunov-Krasovskii functional (LKF):

$$V_j(k) = x_j^t(k)\mathcal{P}_j x_j(k) + \sum_{m=k-d_j(k)}^{k-1} x_j^t(m)\mathcal{R}_j x_j(m)$$
$$+ \sum_{s=2-d_j^+}^{1-d_j^-} \sum_{m=k+s-1}^{k-1} x_j^t(m)\mathcal{R}_j x_j(m), \tag{5.140}$$

where $0 < \mathcal{P}_j$, $0 < \mathcal{Q}_j$ are weighting matrices of appropriate dimensions.

The following theorem establishes the main design result for subsystem $\mathbf{S}_j$.

Theorem 5.9 *Given the bounds* $d_j^- > 0$, $d_j^+ > 0$, $j = 1, \ldots, n_s$, *then the family of subsystems* $\{\mathbf{S}_j\}$ *where* $\mathbf{S}_j$ *is described by* (5.132)–(5.133) *is delay-dependent asymptotically stabilizable by decentralized quantized feedback controller* $u_j(k) = K_{oj} Q_j(y_j)$ *with* $\mathcal{L}_2$*-performance bound* γ_j *if there exist positive-definite matrices* $\mathcal{X}_j$, $\mathcal{G}_j$, Π_{cj}, Π_{sj} *and scalars* $\eta_j > 0$, $\mu_j > 0$, $\sigma_j > 0$, $\nu_j > 0$, $\gamma_j > 0$ *satisfying the following LMIs for* $j = 1, \ldots, n_s$

$$\Pi_j = \begin{bmatrix} \Pi_{1j} & \Pi_{2j} \\ \bullet & \Pi_{3j} \end{bmatrix} < 0, \tag{5.141}$$

$$\Pi_{1j} = \begin{bmatrix} \Pi_{oj} & 0 & 0 & 0 & 0 \\ \bullet & -\Pi_{cj} & 0 & 0 & 0 \\ \bullet & \bullet & -I_j & 0 & 0 \\ \bullet & \bullet & \bullet & -I_j & 0 \\ \bullet & \bullet & \bullet & \bullet & -\gamma_j^2 I_j \end{bmatrix},$$

$$\Pi_{2j} = \begin{bmatrix} \mathcal{X}_j G_j^t & \Pi_{aj} & \mathcal{X}_j M_j^t & 0 & \mathcal{X}_j E_j^t & 0 \\ \mathcal{X}_j L_j^t & \Pi_{ej} & 0 & \mathcal{X}_j N_j^t & 0 & \mathcal{X}_j F_j^t \\ 0 & \mathcal{X}_j & 0 & 0 & 0 & 0 \\ 0 & \Pi_{vj} & 0 & 0 & 0 & 0 \\ \Phi_j^t & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5.142}$$

$$\Pi_{3j} = \begin{bmatrix} -I_j & \Pi_{wj} & 0 & 0 & 0 & 0 \\ \bullet & -\mathcal{X}_j & 0 & 0 & 0 & 0 \\ \bullet & \bullet & -\eta_j I_j & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\mu_j I_j & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\sigma_j I_j & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\nu_j I_j \end{bmatrix},$$

$$\Pi_{oj} = -\mathcal{X}_j + d_j^* \Pi_{cj}, \quad \Pi_{aj} = \mathcal{X}_j A_j^t + \mathcal{G}_j^t B_j^t, \quad \Pi_{vj} = \mathcal{G}_j^t B_j^t,$$
$$\Pi_{ej} = \mathcal{X}_j D_j^t + \Pi_{sj} B_j^t, \quad \Pi_{wj} = \mathcal{X}_j \Gamma_j^t.$$

Moreover, the local gain matrix is given by $K_{oj} = \mathcal{G}_j \mathcal{X}_j^{-1} C_j^\dagger$.

Proof Recall that $d_j^* = d_j^+ - d_j^- + 1$. A straightforward computation gives the first-difference of $\Delta V_j(k) = V_j(k+1) - V_j(k)$ along the solutions of (5.138) with $w_j(k) \equiv 0$ as:

$$\begin{aligned}
\Delta V_j(k) = {} & [\mathcal{A}_j x_j(k) + \mathcal{D}_j x_j(k - d_j(k)) + c_j(k) + B_j K_{oj} \Delta_j(y_j)]^t \mathcal{P}_j \\
& \times [\mathcal{A}_j x_j(k) + \mathcal{D}_j x_j(k - d_j(k)) + c_j(k) + B_j K_{oj} \Delta_j(y_j)] \\
& - x_j^t(k) \mathcal{P}_j x_j(k) + x_j^t(k) \mathcal{R}_j x_j(k) - x_j^t(k - d_j(k)) \mathcal{R}_j x_j(k - d_j(k)) \\
& + \sum_{m=k+1-d_j(k+1)}^{k-1} x_j^t(m) \mathcal{R}_j x_j(m) - \sum_{m=k+1-d_j(k)}^{k-1} x_j^t(m) \mathcal{R}_j x_j(m) \\
& + (d_j^+ - d_j^-) x_j^t(k) \mathcal{R}_j x_j(k) - \sum_{m=k+1-d_j^+}^{k-d_j^*} x_j^t(m) \mathcal{R}_j x_j(m) \qquad (5.143)
\end{aligned}$$

since

$$\begin{aligned}
& \sum_{m=k+1-d_j(k+1)}^{k-1} x_j^t(m) \mathcal{R}_j x_j(m) \\
& = \sum_{m=k+1-d_j^-}^{k-1} x_j^t(m) \mathcal{R}_j x_j(m) + \sum_{m=k+1-d_j(k+1)}^{k-d_j^-} x_j^t(m) \mathcal{R}_j x_j(m) \\
& \le \sum_{m=k+1-d_j(k)}^{k-1} x_j^t(m) \mathcal{R}_j x_j(m) + \sum_{m=k+1-d_j^+}^{k-d_j^-} x_j^t(m) \mathcal{R}_j x_j(m). \qquad (5.144)
\end{aligned}$$

Then using (5.144) into (5.143) and manipulating, we reach

$$\begin{aligned}
\Delta V_j(k) \le {} & [\mathcal{A}_j x_j(k) + \mathcal{D}_j x_j(k - d_j(k)) + c_j(k) + B_j K_{oj} \Delta_j(y_j)]^t \mathcal{P}_j \\
& \times [\mathcal{A}_j x_j(k) + \mathcal{D}_j x_j(k - d_j(k)) + c_j(k) + B_j K_{oj} \Delta_j(y_j)] \\
& + x_j^t(k) [d_j^* \mathcal{R}_j - \mathcal{P}_j] x_j(k) - x_j^t(k - d_j(k)) \mathcal{R}_j x_j(k - d_j(k)). \quad (5.145)
\end{aligned}$$

In terms of the vectors

$$\xi_j(k) = [x_j^t(k),\ x_j^t(k - d_j(k)),\ c_j^t(k),\ \Delta_j^t(y_j)]^t$$

we combine (5.143)–(5.145) with algebraic manipulations using inequalities (5.134) and (5.136) along with Schur complements [4] to arrive at:

$$\varDelta V_j(k) = \xi_j^t(k)\varXi_j\xi_j(k),$$

$$\varXi_j = \begin{bmatrix} \varXi_{aj} & 0 & 0 & 0 & \mathcal{A}_j^t\mathcal{P}_j \\ \bullet & -\varPi_{cj} & 0 & 0 & \mathcal{D}_j^t\mathcal{P}_j \\ \bullet & \bullet & -I_j & 0 & \mathcal{P}_j \\ \bullet & \bullet & \bullet & -I_j & K_{oj}^t B_j^t\mathcal{P}_j \\ \bullet & \bullet & \bullet & \bullet & -\mathcal{P}_j \end{bmatrix}, \tag{5.146}$$

$$\varXi_{aj} = -\mathcal{P}_j + d_j^*\mathcal{R}_j + \phi_j M_j^t M_j + \alpha_j E_j^t E_j,$$

$$\varXi_{cj} = \mathcal{R}_j - \psi_j N_j^t N_j - \beta_j F_j^t F_j.$$

It is known that the sufficient condition of subsystem internal stability is $\varDelta V_j(k) < 0$, hence $\varDelta V(k) = \sum_{j=1}^{n_s}\varDelta V_j(k) < 0$ guaranteeing the internal stability of system **S**.

Next, consider the local performance measure

$$J_j = \sum_{k=0}^{\infty}\big(z_j^t(k)z_j(k) - \gamma^2\omega_j^t(k)\omega_j(k)\big).$$

For any $\omega_j(k) \in \ell_2(0,\infty) \neq 0$ and zero initial condition $x_{jo} = 0$ (hence $V_j(0) = 0$), we have

$$\begin{aligned} J_j &= \sum_{k=0}^{\infty}\big(z_j^t(k)z_j(k) - \gamma^2\omega_j^t(k)\omega_j(k) + \varDelta V_j(k)|_{(5.138)}\big) - \sum_{k=0}^{\infty}\varDelta V_j(k)|_{(5.138)} \\ &\le \sum_{k=0}^{\infty}\big(z_j^t(k)z_j(k) - \gamma^2\omega_j^t(k)\omega_j(k) + \varDelta V_j(k)|_{(5.138)}\big), \end{aligned} \tag{5.147}$$

where $\varDelta V_j(k)|_{(5.138)}$ defines the Lyapunov difference along the solutions of system (5.138). On considering (5.139), (5.146) and (5.147), it can easily shown by algebraic manipulations that

$$z_j^t(k)z_j(k) - \gamma^2\omega_j^t(k)\omega_j(k) + \varDelta V_j(k)|_{(5.138)} = \chi_j^t(k)\widehat{\varXi}_j\chi_j(k), \tag{5.148}$$

$$\widehat{\varXi}_j = \begin{bmatrix} \varXi_{aj} & 0 & 0 & 0 & 0 & G_j^t & \mathcal{A}_j^t\mathcal{P}_j \\ \bullet & -\varPi_{cj} & 0 & 0 & 0 & L_j^t & \mathcal{D}_j^t\mathcal{P}_j \\ \bullet & \bullet & -I_j & 0 & 0 & 0 & \mathcal{P}_j \\ \bullet & \bullet & \bullet & -I_j & 0 & 0 & K_{oj}^t B_j^t\mathcal{P}_j \\ \bullet & \bullet & \bullet & \bullet & -\gamma_j^2 I_j & \varPhi_j^t & \varGamma_j^t\mathcal{P}_j \\ \bullet & \bullet & \bullet & \bullet & \bullet & -I_j & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\mathcal{P}_j \end{bmatrix} \tag{5.149}$$

for some vector $\chi_j(k)$. It is readily seen that

$$z_j^t(k)z_j(k) - \gamma^2\omega_j^t(k)\omega_j(k) + \varDelta V_j(k)|_{(5.138)} < 0$$

for arbitrary $j \in [0,\infty)$, which implies for any $\omega_j(k) \in \ell_2(0,\infty) \neq 0$ that $J_j < 0$ leading $J = \sum_{j=1}^{n_s} J_j < 0$ for the overall system **S**. On applying the congruent transformation

$$\mathsf{T} = \text{diag}[\mathcal{X}_j, \mathcal{X}_j, \mathcal{X}_j, \mathcal{X}_j, I_j, I_j, \mathcal{X}_j], \quad \mathcal{X}_j = \mathcal{P}_j^{-1}$$

to (5.149) with Schur complements and using the change of variables

$$\begin{aligned} &\mathcal{G}_j = K_{oj}\mathcal{X}_j, \quad \Pi_{cj} = \mathcal{X}_j \mathcal{Q}_j \mathcal{X}_j, \\ &\eta_j = \phi_j^{-1}, \quad \mu_j = \psi_j^{-1}, \quad \sigma_j = \alpha_j^{-1}, \quad \nu_j = \beta_j^{-1} \end{aligned}$$

we readily obtain LMI (5.141) with (5.142) and hence the proof is completed. □

5.4.5 Special Cases

In the sequel, some special cases are derived to emphasize the generality of our approach. First, we consider the single nominally-linear time-delay system

$$x(k+1) = Ax(k) + Dx(k-d(k)) + Bu(k) + \Gamma w(k), \tag{5.150}$$

$$\begin{aligned} z(k) &= Gx_j(k) + Lx(k-d(k)) + \Phi w(k), \\ y(k) &= Cx_j(k) + Ex(k-d(k)), \end{aligned} \tag{5.151}$$

where $0 < d^- \leq d(k) \leq d^+$. Let $d^* = d^+ - d^- + 1$. We will use local quantized output measurements such that the following quadratic bounding relation is satisfied:

$$\Delta^t(.)\Delta(.) \leq \alpha x^t(k)E^t Ex(k) + \beta x^t(k-d(k))F^t Fx(k-d(k)), \tag{5.152}$$

where $\alpha > 0$, $\beta > 0$ are adjustable subsystem parameters. The following corollary establishes the corresponding design result:

Corollary 5.9 *Given the bounds $d^- > 0$, $d^+ > 0$, then system (5.150)–(5.151) is delay-dependent asymptotically stabilizable by quantized feedback controller $u(t) = K_o y(t)$ with $\mathcal{L}_2$-performance bound γ if there exist positive-definite matrices $\mathcal{X}$, $\mathcal{G}$, Υ_c, Υ_s and scalars $\sigma > 0$, $\nu > 0$, $\gamma > 0$ satisfying the following LMI*

$$\Upsilon = \begin{bmatrix} \Upsilon_1 & \Upsilon_2 \\ \bullet & \Upsilon_3 \end{bmatrix} < 0, \tag{5.153}$$

$$\Upsilon_1 = \begin{bmatrix} \Upsilon_o & 0 & 0 & 0 \\ \bullet & -\Upsilon_c & 0 & 0 \\ \bullet & \bullet & -I & 0 \\ \bullet & \bullet & \bullet & -\gamma^2 I \end{bmatrix},$$

$$\Upsilon_2 = \begin{bmatrix} \mathcal{X}G^t & \mathcal{X}A^t + \mathcal{G}_j^t B_j^t & \mathcal{X}E^t & 0 \\ \mathcal{X}L^t & \mathcal{X}D^t + \Upsilon_s^t B^t & 0 & \mathcal{X}F^t \\ 0 & \mathcal{G}^t B^t & 0 & 0 \\ \Phi^t & 0 & 0 & 0 \end{bmatrix}, \tag{5.154}$$

$$\Upsilon_3 = \begin{bmatrix} -I & \mathcal{X}\Gamma^t & 0 & 0 \\ \bullet & -\mathcal{X} & 0 & 0 \\ \bullet & \bullet & -\sigma I & 0 \\ \bullet & \bullet & \bullet & -\nu I \end{bmatrix},$$

$$\Upsilon_o = -\mathcal{X} + d^* \Upsilon_c.$$

Moreover, the local gain matrix is given by $K_o = \mathcal{G}\mathcal{X}^{-1}C^\dagger$.

Next, we consider a class of nominally-linear systems $\mathbf{S}$ structurally composed of n_s coupled subsystems $\mathbf{S}_j$ and the model of the jth subsystem is described by the state-space representation:

$$x_j(k+1) = A_j x_j(k) + B_j u_j(k) + c_j(k) + \Gamma_j w_j(k), \tag{5.155}$$

$$\begin{aligned} z_j(k) &= G_j x_j(k) + \Phi_j w_j(k), \\ y_j(k) &= C_j x_j(k), \end{aligned} \tag{5.156}$$

where for $j \in \{1, \dots, n_s\}$, the coupling vector $c_j(k)$ is a piecewise-continuous vector function in its arguments and satisfies the quadratic inequality

$$c_j^t(k,.,.)c_j(k,.,.) \le \phi_j x_j^t(k) M_j^t M_j x_j(k) \tag{5.157}$$

where $\phi_j > 0$ are adjustable bounding parameters and $M_j \in \Re^{n_j \times n_j}$ are constant matrices. We will use local quantized output measurements such that the following quadratic bounding relation is satisfied:

$$\Delta_j^t(.)\Delta_j(.) \le \alpha_j x_j^t(k) E_j^t E_j x_j(k), \tag{5.158}$$

where $\alpha_j > 0$ are adjustable subsystem parameters. The following corollary stands out:

Corollary 5.10 *The family of subsystems* $\{\mathbf{S}_j\}$ *where* $\mathbf{S}_j$ *is described by* (5.155)–(5.156) *is asymptotically stabilizable by decentralized quantized feedback controller* $u_j(k) = K_{oj} Q_j(y_j)$ *with* $\mathcal{L}_2$*-performance bound* γ_j *if there exist positive-definite matrices* $\mathcal{X}_j$, $\mathcal{G}_j$ *and scalars* $\eta_j > 0$, $\sigma_j > 0$, $\gamma_j > 0$ *satisfying the following LMIs for* $j = 1, \dots, n_s$

$$\Theta_j = \begin{bmatrix} \Theta_{1j} & \Theta_{2j} \\ \bullet & \Theta_{3j} \end{bmatrix} < 0, \tag{5.159}$$

$$\Theta_{1j} = \begin{bmatrix} -\mathcal{X}_j & 0 & 0 & 0 \\ \bullet & -I_j & 0 & 0 \\ \bullet & \bullet & -I_j & 0 \\ \bullet & \bullet & \bullet & -\gamma_j^2 I_j \end{bmatrix},$$

$$\Theta_{2j} = \begin{bmatrix} \mathcal{X}_j G_j^t & \mathcal{X}_j A_j^t + \mathcal{G}_j^t B_j^t & \mathcal{X}_j M_j^t & \mathcal{X}_j E_j^t \\ 0 & \mathcal{X}_j & 0 & 0 \\ 0 & \mathcal{G}_j^t B_j^t & 0 & 0 \\ \Phi_j^t & 0 & 0 & 0 \end{bmatrix}, \tag{5.160}$$

$$\Theta_{3j} = \begin{bmatrix} -I_j & \mathcal{X}_j \Gamma_j^t & 0 & 0 \\ \bullet & -\mathcal{X}_j & 0 & 0 \\ \bullet & \bullet & -\eta_j I_j & 0 \\ \bullet & \bullet & \bullet & -\sigma_j I_j \end{bmatrix}.$$

Moreover, the local gain matrix is given by $K_{oj} = \mathcal{G}_j \mathcal{X}_j^{-1} C_j^{\dagger}$.

Finally, we consider the single nominally-linear system

$$x(k+1) = Ax(k) + Bu(k) + \Gamma w(k), \tag{5.161}$$

$$z(k) = Gx(k) + \Phi w(k), \quad y(k) = Cx(k). \tag{5.162}$$

We will use local quantized output measurements such that the following quadratic bounding relation is satisfied:

$$\Delta^t(.)\Delta(.) \le \alpha x^t(k) E^t E x(k), \tag{5.163}$$

where $\alpha > 0$, $\beta > 0$ are adjustable subsystem parameters. The following corollary establishes the corresponding design result:

Corollary 5.11 *System* (5.161)–(5.162) *is asymptotically stabilizable by decentralized quantized feedback controller* $u(k) = K_o Q(y)$ *with* $\mathcal{L}_2$*-performance bound* γ_j *if there exist positive-definite matrices* $\mathcal{X}$, $\mathcal{G}$ *and scalars* $\eta > 0$, $\sigma > 0$, $\gamma > 0$ *satisfying the following LMI*

$$\Sigma = \begin{bmatrix} \Sigma_1 & \Sigma_2 \\ \bullet & \Sigma_3 \end{bmatrix} < 0, \tag{5.164}$$

$$\Sigma_1 = \begin{bmatrix} -\mathcal{X} & 0 & 0 \\ \bullet & -I & 0 \\ \bullet & \bullet & -\gamma_j^2 I_j \end{bmatrix},$$

$$\Sigma_2 = \begin{bmatrix} \mathcal{X} G^t & \mathcal{X} A^t + \mathcal{G}^t B^t & \mathcal{X} E^t \\ 0 & \mathcal{G}^t B^t & 0 \\ \Phi^t & 0 & 0 \end{bmatrix}, \tag{5.165}$$

$$\Sigma_3 = \begin{bmatrix} -I & \mathcal{X}\Gamma^t & 0 \\ \bullet & -\mathcal{X} & 0 \\ \bullet & \bullet & -\eta I \end{bmatrix}.$$

Moreover, the local gain matrix is given by $K_o = \mathcal{G}\mathcal{X}^{-1} C^{\dagger}$.

5.4.6 Simulation Example 5.5

For the purpose of illustration, we consider an interconnected system composed of two subsystems having uniform quantizers with the following data:

$$A_1 = \begin{bmatrix} 0.8 & 0 \\ 0.05 & 0.9 \end{bmatrix}, \quad D_1 = \begin{bmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \quad G_1^t = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \quad L_1^t = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 0.1 & 0.01 \\ -0.1 & 0.02 \end{bmatrix}, \quad F_1 = \begin{bmatrix} -0.02 & -0.01 \\ -0.01 & -0.02 \end{bmatrix},$$

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0 & 0.5 & -0.1 \\ 0.1 & 0 & 0.4 \end{bmatrix}, \quad G_2^t = \begin{bmatrix} 1 \\ 0.2 \\ 0.7 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0.5 \\ 1.5 \\ 0.4 \end{bmatrix}, \quad D_2 = \begin{bmatrix} -0.2 & 0.04 & 0.2 \\ -0.4 & -0.15 & 0 \\ 0.1 & 0 & 0.3 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} -0.02 & 0.01 & 0 \\ 0 & 0.1 & 0 \\ -0.02 & 0 & 0.05 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0.03 & 0 & 0.02 \\ 0.1 & 0.2 & 0 \\ -0.01 & 0 & 0.01 \end{bmatrix},$$

$$L_2 = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.1 \end{bmatrix}.$$

It is found that the feasible solution of LMI (5.141) is attained at

$$d_1^- = 10, \quad d_1^+ = 30, \quad d_2^- = 10, \quad d_2^+ = 30,$$
$$K_1 = -0.4023, \quad K_2 = -0.0916.$$

Typical simulation results are shown in Figs. 5.18, 5.19, 5.20 and 5.21 for the open-loop response and closed-loop response of both subsystems. Next, by dropping the time-delay factors (within the subsystems and across the couplings) and considering LMI (5.141) the feasible solution is found to yield the gains

$$K_1 = -0.6653, \quad K_2 = -1.0915.$$

The simulation of the closed-loop response of both subsystems are depicted in Figs. 5.22 and 5.23. On implementing the LMI (5.153) for subsystem 2, the feasible solution is given by

$$d^- = 20, \quad d_2^+ = 60, \quad K_2 = -1.3391.$$

The ensuing closed-loop response is plotted in Fig. 5.24. Finally, the feasible solution of LMI (5.153) for subsystem 1 without delay terms is $K_1 = -0.3039$ and the corresponding closed-loop response is plotted in Fig. 5.25. From the ensuing re-

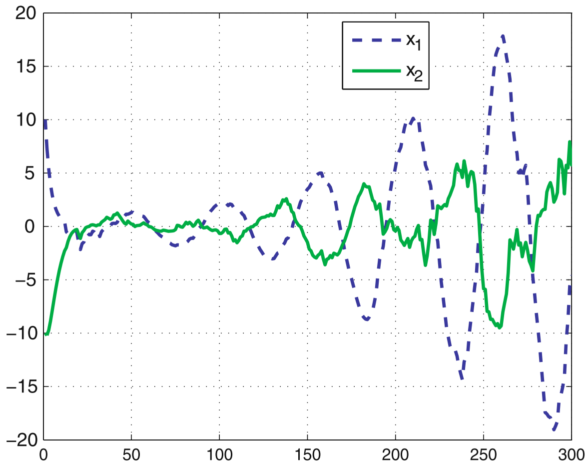


Fig. 5.18 Open-loop response of subsystem 1

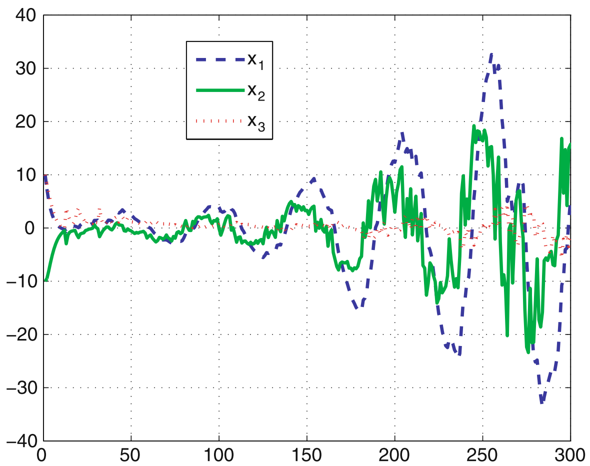


Fig. 5.19 Open-loop response of subsystem 2

sults, it is quite evident that the quantized feedback control system is asymptotically stable for the class of quantizers satisfying the quadratic inequality. This is equally true for interconnected time-delay and delay-free systems, single time-delay systems and single systems. The crucial point to record is that the type of quantizer so long as it is satisfies its structure complies with a quadratic inequality. We have observed that the presence of bounding inequalities (5.134) and (5.136) helps in curbing the magnitude of the feedback gains.

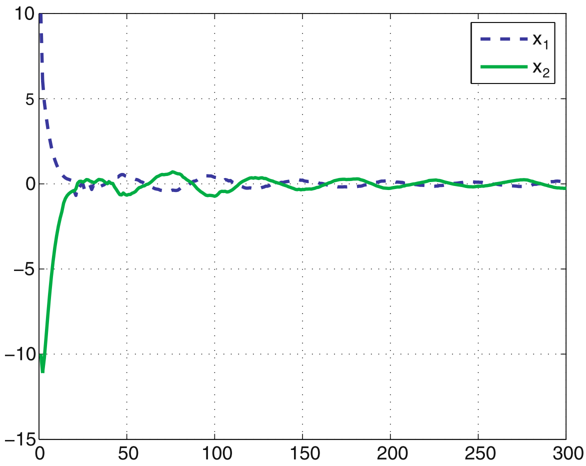


Fig. 5.20 Closed-loop response of subsystem 1

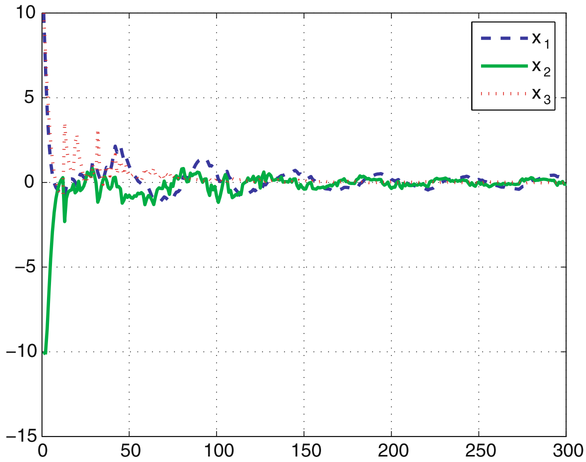


Fig. 5.21 Closed-loop response of subsystem 2

5.5 Interconnected Discrete Systems with Overflow Nonlinearities

In this section, we build upon [3, 10, 18] and extend them further to the class of linear interconnected discrete-time systems with unknown-but-bounded couplings and overflow nonlinearities.

5.5.1 Introduction

In the implementation of discrete-time systems using computer or special-purpose hardware with fixed-point arithmetic, one frequently encounters several kinds of

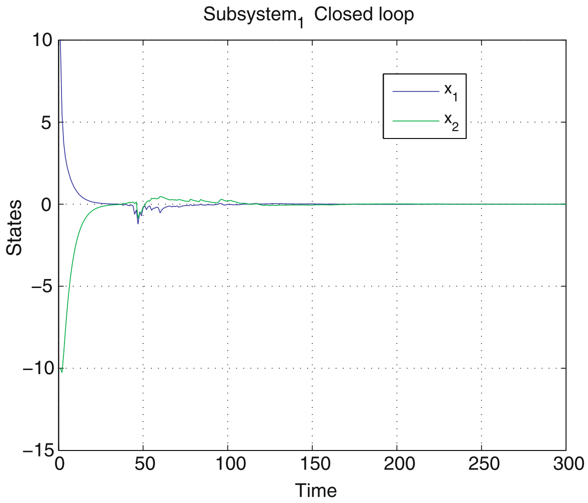


Fig. 5.22 Closed-loop response of decoupled subsystem 1

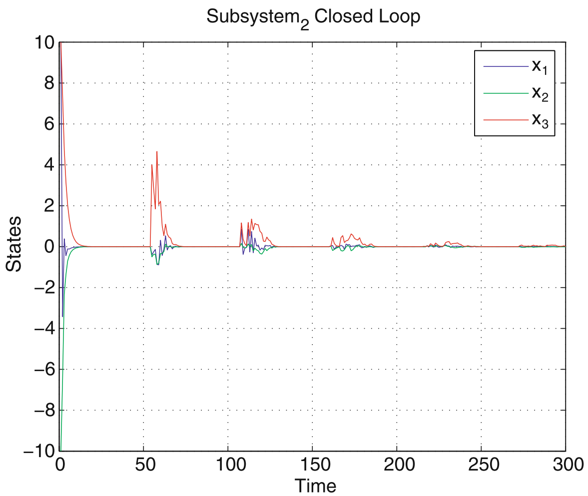


Fig. 5.23 Closed-loop response of decoupled subsystem 2

overflow nonlinearities [1, 3]. On the other hand, quantization effects are present in most control systems, as they heavily rely on digital components, and research on quantized feedback control where a quantizer is regarded as an information coder. The fundamental question of interest is how much information needs to be communicated by the quantizer in order to achieve a certain control objective [5, 8, 9, 11, 12, 32].

When a digital network is present in a feedback system, quantization levels determine the data rate for the transmission of control-related signals and hence the cost for communication [18, 21]. In effect, such overflow nonlinearities and/or quantization may lead to instability in the realized system. An important objective in the design of a discrete-time system is, therefore, to find the values of the system param-

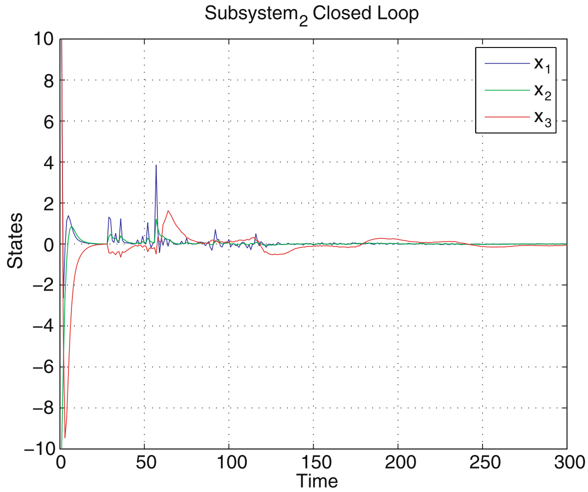


Fig. 5.24 Closed-loop response of single time-delay system

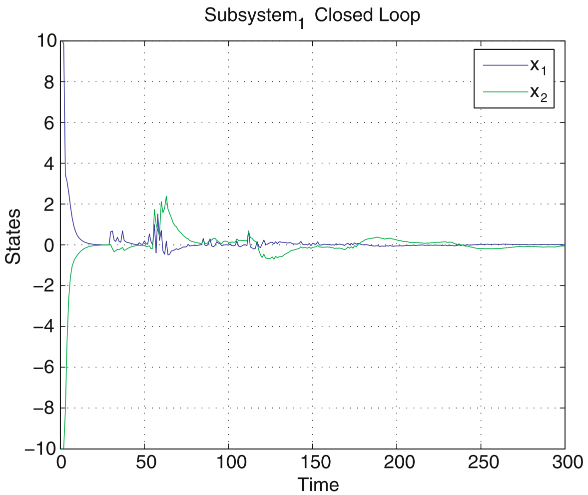


Fig. 5.25 Closed-loop response of single system

eters in the parameter space so that the designed system is globally asymptotically stable. Robust stability analysis of discrete-time systems that include nonlinearities and parameter uncertainties in their physical models is an important problem. So far, very little attention has been paid for the investigation of this problem [3, 10].

On another research front, decentralized stability and feedback stabilization of interconnected systems have been the topic of recurring interests and recent relevant results have been reported in [2, 22, 31].

In what follows, LMI-based decentralized feedback controller is designed at the subsystem level using only local state variables to render the overall closed-loop system asymptotically stable. When the local output measurements are processed to the controller, we develop a set of local observer-based output-feedback controller to guarantee the asymptotic stability of the closed-loop quantized system. Numerical simulations are performed to illustrate the theoretical developments.

5.5.2 Problem Statement

We consider a class of discrete-time systems **S** described by:

$$x(k+1) = Ax(k) + Bu(k), \tag{5.166}$$

where $x(k) \in \Re^n$ is the overall state vector, $u(k) \in \Re^m$ is the control input, $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$ are real and constant matrices and the initial condition $\varphi \in \ell_2[-d_M, 0]$. In what follows, we consider **S** to be structurally composed of n_s coupled subsystems $\mathbf{S}_j$ with the jth subsystem being described by:

$$x_j(k+1) = A_j x_j(k) + B_j u_j(k) + c_j(k), \tag{5.167}$$

where for $j \in \{1, \dots, n_s\}$, $x_j(k) \in \Re^{n_j}$ is the local state vector, $u_j(k) \in \Re^{m_j}$ is the local control input and $c_j(k) \in \Re^{n_j}$ is the coupling vector. The matrices $A_j \in \Re^{n_j\times n_j}$, $B_j \in \Re^{n_j\times m_j}$ are real and constants such that

$$A = \mathrm{blockdiag}[A_1,\ A_2,\ \dots,\ A_{n_s}],$$
$$B = \mathrm{blockdiag}[B_1,\ B_2,\ \dots,\ B_{n_s}].$$

In the sequel, we treat $c_j(k)$ as a piecewise-continuous vector function in its arguments and satisfies the quadratic inequality

$$c_j^t(k,.,.)c_j(k,.,.) \le \sum_{m=1,m\ne j}^{n_s} x_j^t(k) E_{mj} x_j(k), \tag{5.168}$$

where the matrices $E_{mj} \in \Re^{n_j\times n_j}$ are constants. The rationale behind the quadratic inequality (5.168) is to preserve the decentralized information structure constraints and it has global nature as the right-hand side depends on all local states.

The class of systems described by (5.166) subject to delay-pattern is frequently encountered in modeling several physical systems and engineering applications including large space structures, multi-machine power systems, cold mills, transportation systems, water pollution management, to name a few [22, 30]. Our objective in this work is to design appropriate control signals at the subsystem level to stabilize the overall system (5.166).

5.5.3 Local Static Control Function

In order to cope with the effects of quantization and/or overflow nonlinearities, we consider that the controller generates the signal

$$u_j(k) = K_j f_j(x_j(k)), \tag{5.169}$$

where $f_j(x_j(k))$ is the controller function at time k such that the deviation from the nominal linear case

$$e_j(x_j(k)) = f_j(x_j(k)) - x_j(k)$$

is assumed to satisfy the bounding inequality

$$\sigma_o x_j^t(k) x_j(k) \le e_j^t(x_j(k)) e_j(x_j(k)) \le \sigma_q x_j^t(k) x_j(k). \tag{5.170}$$

Remark 5.17 It should be noted that in the event of either magnitude truncation or roundoff, $f_j(x_j(k))$ turns out to have the form

$$\sigma_q = \begin{cases} 1, & \text{for magnitude truncation,} \\ 2, & \text{for roundoff,} \end{cases} \tag{5.171}$$

$$\sigma_o = \begin{cases} 0, & \text{for zeroing or saturation,} \\ -1/3, & \text{for triangle,} \\ -1, & \text{for two's complement.} \end{cases} \tag{5.172}$$

Note that the overall control function, to be applied to the overall system (5.166) is given by

$$\begin{aligned} u(k) &= Kf(x(k)) = [u_1^t(k), \ldots, u_{n_s}^t(k)]^t, \\ f(x(k)) &= [f_1^t(x_1(k)), \ldots, f_{n_s}^t(x_{n_s}(k))]^t. \end{aligned} \tag{5.173}$$

5.5.4 Closed-Loop Stabilization

The closed loop subsystem of (5.167) and (5.169) is given by

$$\begin{aligned} x_j(k+1) &= [A_j + B_j K_j] x_j(k) + B_j K_j e_j(k) + c_j(k) \\ &= A_{cj} x_j(k) + B_j K_j e_j(k) + c_j(k). \end{aligned} \tag{5.174}$$

To examine the stability of (5.174), we consider the following quadratic Lyapunov function

$$V(k) = \sum_{j=1}^{n_s} V_j(k) = \sum_{j=1}^{n_s} x_j^t(k) \mathcal{P}_j x_j(k), \quad \mathcal{P}_j > 0.$$

The following theorem summarizes the main stabilization result.

Theorem 5.10 *Given scalars β_j, δ_j, $j = 1, \ldots, n_s$, the overall system (5.166) is asymptotically stable if there exist matrices $\mathcal{X}_j > 0$, $\mathcal{Y}_j$ and scalars α_j, $j = 1, \ldots, n_s$ satisfying the following LMI*

$$\Omega_j = \begin{bmatrix} -\mathcal{X}_j & 0 & 0 & 0 & \Pi_{aj} & \mathcal{X}_j \\ \bullet & -(\delta_j - \beta_j)I_j & \mathcal{Y}_j & 0 & 0 & 0 \\ \bullet & \bullet & -2\mathcal{X}_j & 0 & B_j^t & 0 \\ \bullet & \bullet & \bullet & -\alpha_j I_j & I_j & 0 \\ \bullet & \bullet & \bullet & \bullet & -\mathcal{X}_j & \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\Pi_{cj} \end{bmatrix} < 0, \quad (5.175)$$

$$\Pi_{aj} = \mathcal{X}_j A_j^t + \mathcal{Y}_j^t B_j^t,$$
$$\Pi_{cj} = \left[\alpha_j \sum_{m=1, m\neq j}^{n_s} E_{mj} + \delta_j \sigma_q - \beta_j \sigma_o \right]^{-1}. \quad (5.176)$$

Moreover the gain matrix is given by $K_j = \mathcal{Y}_j \mathcal{X}_j^{-1}$.

Proof Evaluating the first difference $\Delta V(k) = \sum_{j=1}^{n_s} \Delta V_j(k)$, $\Delta V_j(k) = V_j(k+1) - V_j(k)$ along the solutions of (5.174) to yield

$$\begin{aligned} \Delta V_j(k) &= x_j^t(k+1)\mathcal{P}_j xj(k+1) - x_j^t(k)\mathcal{P}_j x_j(k) \\ &= \left[x_j^t(k)A_{cj}^t + e_j^t(k)K_j^t B_j^t + c_j^t(k)\right]\mathcal{P}_j\left[A_{cj}x_j(k) + B_j K_j e_j(k) + c_j(k)\right] \\ &\quad - x_j^t(k)\mathcal{P}_j x_j(k). \end{aligned} \quad (5.177)$$

In terms of

$$\xi_j(k) = \text{col}[x_j(k), e_j(k), e_j K_j, c_j(k)],$$
$$g_j(k) = \text{col}[A_j^t \mathcal{P}_j + K_j^t B_j^t \mathcal{P}_j, 0, \mathcal{P}_j B_j^t \mathcal{P}_j, \mathcal{P}_j],$$
$$\Upsilon_j = \begin{bmatrix} -\mathcal{P} & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 \\ \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & \bullet & 0 \end{bmatrix}$$

we employ Schur complements to express (5.177) in the form

$$\begin{aligned} \Delta V_j(k) &= \xi_j^t(k)\Upsilon_j \xi_j(k) + g_j(k)\mathcal{P}_j^{-1} g_j^t(k) \\ &= \zeta_j^t(k)\hat{\Upsilon}_j \zeta_j(k), \end{aligned}$$
$$\hat{\Upsilon}_j = \begin{bmatrix} -\mathcal{P}_j & 0 & 0 & 0 & A_j^t \mathcal{P}_j + K_j^t B_j^t \mathcal{P}_j \\ \bullet & 0 & 0 & 0 & 0 \\ \bullet & \bullet & 0 & 0 & \mathcal{P}_j B_j^t \mathcal{P}_j \\ \bullet & \bullet & \bullet & 0 & \mathcal{P}_j \\ \bullet & \bullet & \bullet & \bullet & -\mathcal{P}_j \end{bmatrix} \quad (5.178)$$

for some $\zeta_j(k)$. On invoking the structural identity

$$\sum_{j=1}^{n_s} \sum_{m=1, m\neq j}^{n_s} x_m^t(k) E_{mj} x_m(k) = \sum_{j=1}^{n_s} \sum_{m=1, m\neq j}^{n_s} x_j^t(k) E_{jm} x_j(k) \quad (5.179)$$

and considering the constraints (5.168) for some scalars α_j with (5.178), the internal stability requirement $\Delta V(k) = \sum_{j=1}^{n_s} \Delta V_j(k) < 0$ with some algebraic manipulations implies that

$$\tilde{\Upsilon}_j = \begin{bmatrix} -\mathcal{P}_j + \mathcal{N}_{aj} & 0 & 0 & 0 & A_j^t \mathcal{P}_j + K_j^t B_j^t \mathcal{P}_j \\ \bullet & 0 & 0 & 0 & 0 \\ \bullet & \bullet & 0 & 0 & B_j^t \mathcal{P}_j \\ \bullet & \bullet & \bullet & -\alpha_j I_j & \mathcal{P}_j \\ \bullet & \bullet & \bullet & \bullet & -\mathcal{P}_j \end{bmatrix} < 0,$$

$$\mathcal{N}_{aj} = \alpha_j \sum_{m=1, m\neq j}^{n_s} E_{mj}. \tag{5.180}$$

Under the congruent transformation

$$T = \text{diag}[\mathcal{P}_j, I_j, I_j, I_j, \mathcal{P}_j], \quad \mathcal{X}_j = \mathcal{P}_j^{-1}$$

and the change of variable $\mathcal{Y}_j = K_j \mathcal{X}_j$, the condition $\tilde{\Upsilon}_j < 0$ is equivalent to

$$\widetilde{\Upsilon}_j = \begin{bmatrix} -\mathcal{X}_j + \mathcal{X}_j \mathcal{N}_{aj} \mathcal{X}_j & 0 & 0 & 0 & \mathcal{X}_j A_j^t + \mathcal{Y}_j^t B_j^t \\ \bullet & 0 & 0 & 0 & 0 \\ \bullet & \bullet & 0 & 0 & B_j^t \\ \bullet & \bullet & \bullet & -\alpha_j I_j & I_j \\ \bullet & \bullet & \bullet & \bullet & -\mathcal{X}_j \end{bmatrix} < 0. \tag{5.181}$$

Now the bounding constraints (5.170) with (5.172) for some scalars β_j, δ_j can be written as

$$\begin{aligned} &\beta_j\big[-\sigma_o x_j^t(k) xj(k) + e_j^t(x_j(k)) e_j(x_j(k))\big] \le 0, \\ &\delta_j\big[\sigma_q x_j^t(k) xj(k) - e_j^t(x_j(k)) e_j(x_j(k))\big] \le 0. \end{aligned} \tag{5.182}$$

Equivalently stated

$$\begin{aligned} &\beta_j\big[-\sigma_o x_j^t(k) \mathcal{P}_j \mathcal{X}_j \mathcal{X}_j \mathcal{P}_j xj(k) + e_j^t(x_j(k)) e_j(x_j(k))\big] \le 0, \\ &\delta_j\big[\sigma_q x_j^t(k) \mathcal{P}_j \mathcal{X}_j \mathcal{X}_j \mathcal{P}_j xj(k) - e_j^t(x_j(k)) e_j(x_j(k))\big] \le 0. \end{aligned} \tag{5.183}$$

It follows from (5.181) and (5.183) that the stability requirement becomes

$$\begin{aligned} &\begin{bmatrix} -\Pi_{oj} & 0 & 0 & 0 & \Pi_{aj} \\ \bullet & 0 & 0 & 0 & 0 \\ \bullet & \bullet & 0 & 0 & B_j^t \\ \bullet & \bullet & \bullet & -\alpha_j I_j & I_j \\ \bullet & \bullet & \bullet & \bullet & -\mathcal{X}_j \end{bmatrix} < 0, \\ &\Pi_j = \mathcal{X}_j - \mathcal{X}_j \mathcal{N}_{aj} \mathcal{X}_j - \mathcal{X}_j(\delta_j \sigma_q - \beta_j \sigma_o)\mathcal{X}_j, \\ &\Pi_{aj} = \mathcal{X}_j A_j^t + \mathcal{Y}_j^t B_j^t. \end{aligned} \tag{5.184}$$

By taking into consideration

$$e_j^t(k) K_j^t \mathcal{X}_j K_j e_j(k) = e_j^t(k) \mathcal{Y}_j^t K_j e_j(k)$$

the result can be cast into the form

$$\Delta V(k) = \sum_{j=1}^{n_s} \eta_j^t(k) \Omega_j \eta_j(k) \tag{5.185}$$

for some vector $\eta_j(k)$ and Ω_j is given by (5.175). Subject to the condition of the theorem, it follows that $\Delta V(k) < 0$ and hence we conclude that system (5.166) is asymptotically stable. □

5.5.5 Local Dynamic Control Function

An alternative method to handle the effects of quantization and/or overflow nonlinearities, we consider a class of discrete-time systems **S** described by:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k), \end{aligned} \tag{5.186}$$

where $y(k) = \text{col}[y_1(k), \ldots, y_{n_s}(k)] \in \Re^n$ is the overall output vector and $C = \text{blockdiag}[C_1,\ C_2,\ \ldots,\ C_{n_s}]$. The jth subsystem $\mathbf{S}_j$ is described by:

$$\begin{aligned} x_j(k+1) &= A_j x_j(k) + B_j u_j(k) + c_j(k), \\ y_j(k) &= C_j x_j(k). \end{aligned} \tag{5.187}$$

Let the jth controller generates the signal using the observer-based scheme

$$\begin{aligned} x_{cj}(k+1) &= A_j x_{cj}(k) + L_j[y_j(k) - C_j x_{cj}(k)], \\ u_j(k) &= K_j f_j(x_{cj}(k)), \end{aligned} \tag{5.188}$$

where L_j, K_j are the controller gain matrices and $f_j(x_j(k))$ is the controller function at time k such that the deviation from the nominal case

$$g_j(x_{cj}(k)) = f_j(x_{cj}(k)) - x_{cj}(k)$$

is assumed to satisfy the bounding inequality

$$\sigma_o x_{cj}^t(k) x_{cj}(k) \le g_j^t(x_{cj}(k)) g_j(x_{cj}(k)) \le \sigma_q x_{cj}^t(k) x_{cj}(k). \tag{5.189}$$

Define the signal $\hat{x}_j = x_j - x_{cj}$, then from (5.187) and (5.188), we obtain

$$\begin{aligned} z_j(k+1) &= \mathcal{Z}_j z_j(k) + \mathcal{B}_j g_j(z_j(k)) + \mathcal{C}_j(k), \\ z_j(k) &= \begin{bmatrix} x_j(k) \\ \hat{x}_j \end{bmatrix}, \quad \mathcal{B}_j = \begin{bmatrix} B_j K_j \\ 0 \end{bmatrix}, \quad \mathcal{C}_j = \begin{bmatrix} c_j(k) \\ c_j(k) \end{bmatrix}, \\ \mathcal{Z}_j &= \begin{bmatrix} A_{cj} & -B_j K_j \\ 0 & \hat{A}_j \end{bmatrix}, \\ \hat{A}_j &= A_j - L_j C_j, \quad A_{cj} = A_j + B_j K_j. \end{aligned} \tag{5.190}$$

Invoking the separation principle paves the way to determine the unknown gain matrices in two independent and consecutive stages. In the first stage, we determine the controller gain K_j by applying Theorem 5.10. During the second stage, we proceed to determine the observer gain L_j by selecting the quadratic Lyapunov function

$$V_c(k) = \sum_{j=1}^{n_s} V_{cj}(k) = \sum_{j=1}^{n_s} \hat{x}_j^t(k)\mathcal{S}_j\hat{x}_j(k), \quad \mathcal{S}_j > 0$$

and evaluating the first difference $\varDelta V_c(k) = \sum_{j=1}^{n_s} \varDelta V_{cj}(k)$, $\varDelta V_{cj}(k) = V_{cj}(k+1) - V_{cj}(k)$ along the solutions of (5.190) to yield

$$\begin{aligned}\varDelta V_{cj}(k) = {} & \big[\hat{x}_j^t(k)\hat{A}_j^t + c_j^t(k)\big]\mathcal{S}_j\big[\hat{A}_j\hat{x}_j(k) + c_j(k)\big] \\ & - \hat{x}_j^t(k)\mathcal{S}_j\hat{x}_j(k).\end{aligned} \tag{5.191}$$

By parallel development to the foregoing section, the following theorem summarizes the main stabilization result.

Theorem 5.11 *Given the matrices $\mathcal{P}_j$, K_j and scalars β_j, δ_j, $j = 1, \ldots, n_s$, the overall system (5.186) is asymptotically stable if there exist matrices $\mathcal{S}_j > 0$, $\mathcal{R}_j$ and scalars α_j, $j = 1, \ldots, n_s$ satisfying the following LMI*

$$\varLambda_j = \begin{bmatrix} \varLambda_{1j} & \varLambda_{2j} \\ \bullet & \varLambda_{3j} \end{bmatrix} < 0, \tag{5.192}$$

$$\begin{aligned}
\varLambda_{1j} &= \begin{bmatrix} \varLambda_{oj} & 0 & 0 \\ \bullet & -(\delta_j - \beta_j)I_j & 0 \\ \bullet & \bullet & \varLambda_{cj} \end{bmatrix}, \\
\varLambda_{2j} &= \begin{bmatrix} 0 & \varLambda_{aj} & 0 \\ 0 & K_j^t B_j^t \mathcal{P}_j & 0 \\ \varLambda_{ej} & K_j^t B_j^t \mathcal{P}_j & C_j^t \mathcal{R}_j^t \end{bmatrix}, \\
\varLambda_{3j} &= \begin{bmatrix} -\alpha_j I_j + \mathcal{S}_j & \mathcal{P}_j & 0 \\ \bullet & -\mathcal{P}_j & 0 \\ \bullet & \bullet & -I_j \end{bmatrix}, \\
\varLambda_{oj} &= \mathcal{P}_j^{-1} + \mathcal{N}_{aj} + (\delta_j\sigma_q - \beta_j\sigma_o)I_j, \\
\varLambda_{aj} &= A_j^t\mathcal{P}_j + K_j^t B_j^t \mathcal{P}_j, \\
\varLambda_{cj} &= A_j^t\mathcal{S}_j A_j + C_j^t\mathcal{R}_j^t A_j + A_j^t\mathcal{R}_j C_j, \\
\varLambda_{ej} &= C_j^t\mathcal{R}_j^t + \mathcal{R}_j C_j, \quad \mathcal{N}_{aj} = \alpha_j \sum_{m=1, m\neq j}^{n_s} E_{mj}.
\end{aligned} \tag{5.193}$$

Moreover the gain matrix is given by $L_j = \mathcal{S}_j^{-1}\mathcal{R}_j$.

5.5.6 Simulation Example 5.6

To illustrate the theoretical developments, we consider a plant comprised of three reactors connected in tandem. By linearization and time scaling, the model matrices in the form of (5.187) have the values:

Table 5.2 Model parameters

Parameter	S_1	S_2	S_3
a_{1j}	4.931	4.886	4.902
a_{2j}	5.301	5.174	5.464
a_{3j}	32.511	30.645	31.773
a_{4j}	3.961	3.878	3.932
b_{1j}	1.219	1.345	1.297
b_{2j}	1.453	1.362	1.245
b_{3j}	0.764	0.805	0.696
b_{4j}	0.524	0.615	0.603

$$A_j = \begin{bmatrix} -a_{1j} & -1.01 & 0 & 0 \\ -3.2 & -a_{2j} & -12.8 & 0 \\ 6.4 & 0.347 & -a_{3j} & -1.04 \\ 0 & 0.833 & 11.0 & -a_{4j} \end{bmatrix}, \quad B_j = \begin{bmatrix} b_{1j} \\ 0 \\ b_{2j} \\ b_{3j} \end{bmatrix},$$
$$C_j = [b_{4j}\ 0\ 0\ 0],$$

where the values of the parameters are given in Table 5.2.

The coupling matrices $E_{mj} \in \Re^{n_j \times n_j}$, $m = 1, \ldots, 3$ are generated randomly and classified into two distinct cases:

1. The elements of E_{mj} have values in the range $[0.01,\ 0.7)$ corresponding to weak coupling.
2. The elements of E_{mj} have values in the range $[0.7,\ 1.9)$ corresponding to strong coupling.

The feasible solution of Theorem 5.10 for the case of weak coupling is found to be

$$K_1 = \left[7.535\ -3.962\ -0.935\ 0.007\right],$$
$$K_2 = \left[1.741\ -10.124\ -1.045\ 0.015\right],$$
$$K_3 = \left[3.966\ -4.524\ -1.104\ 0.021\right]$$

and the associated trajectories are plotted in Fig. 5.26. On the other hand, the feasible solution of Theorem 5.11 for the case of weak coupling is found to be

$$L_1^t = \left[0.535\ -0.223\ -0.035\ 0.007\right],$$
$$L_2^t = \left[1.034\ -0.145\ -0.045\ 0.005\right],$$
$$L_3^t = \left[0.911\ -1.105\ -0.804\ 0.019\right]$$

and the associated trajectories are plotted in Fig. 5.27. Turning to the case of strong coupling, the corresponding results are summarized below

$$K_1 = \left[7.535\ -3.962\ -0.935\ 0.007\right],$$
$$K_2 = \left[1.741\ -10.124\ -1.045\ 0.015\right],$$
$$K_3 = \left[3.966\ -4.524\ -1.104\ 0.021\right],$$
$$L_1^t = \left[0.535\ -0.223\ -0.035\ 0.007\right],$$

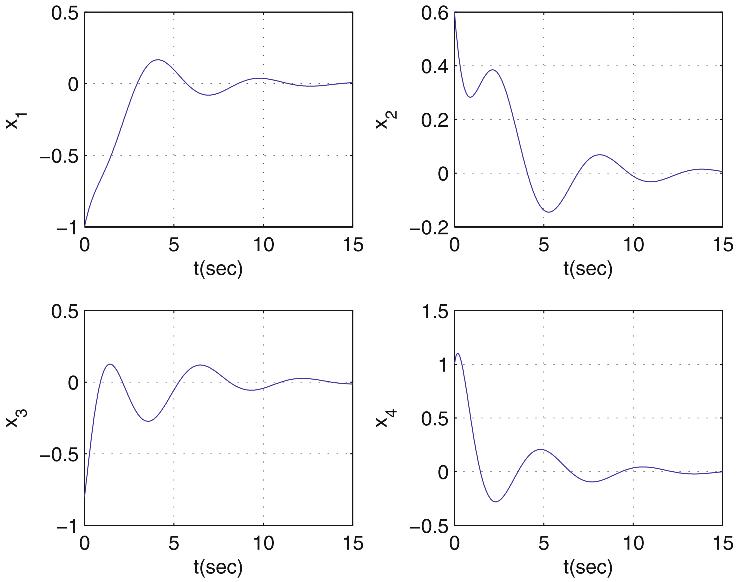


Fig. 5.26 Closed-loop state-trajectories—weak coupling

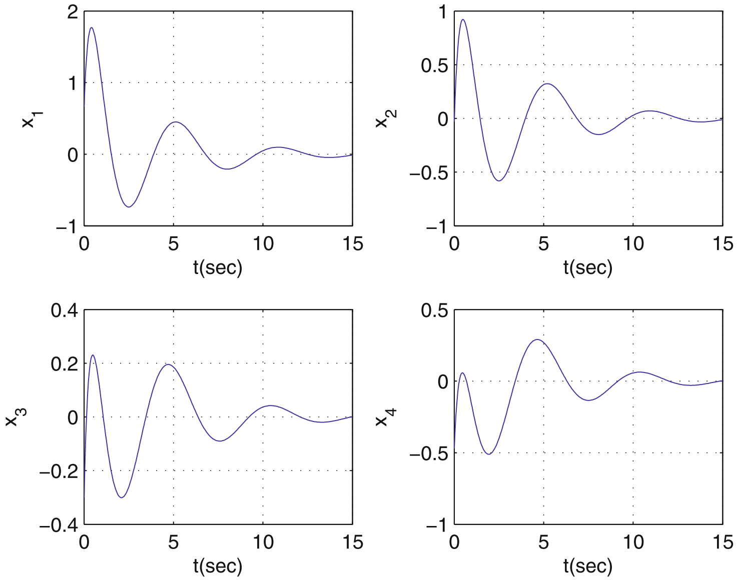


Fig. 5.27 Closed-loop observer-based state-trajectories—weak coupling

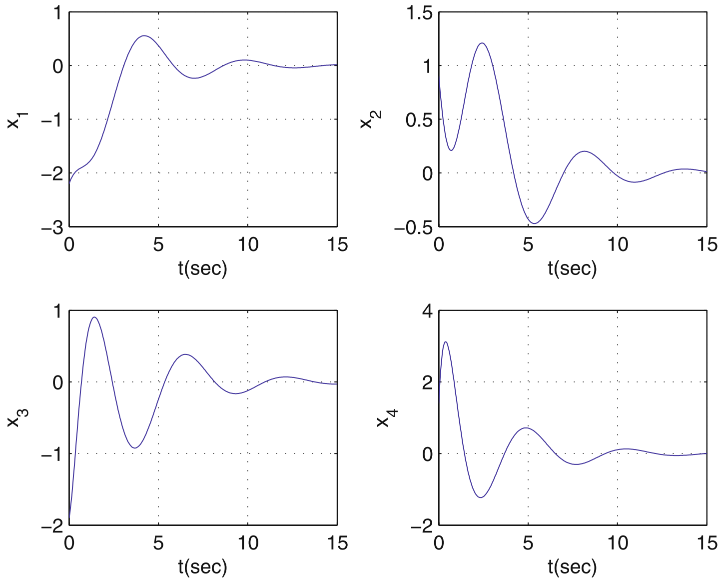


Fig. 5.28 Closed-loop state-trajectories—strong coupling

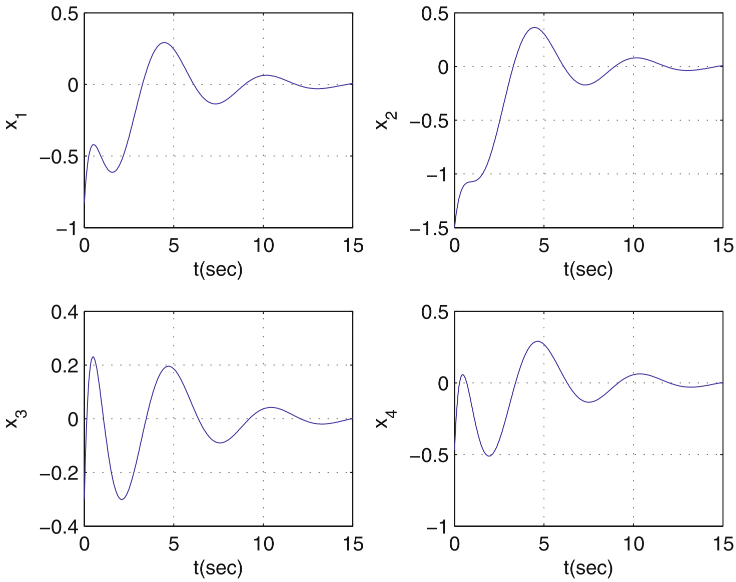


Fig. 5.29 Closed-loop observer-based state-trajectories—strong coupling

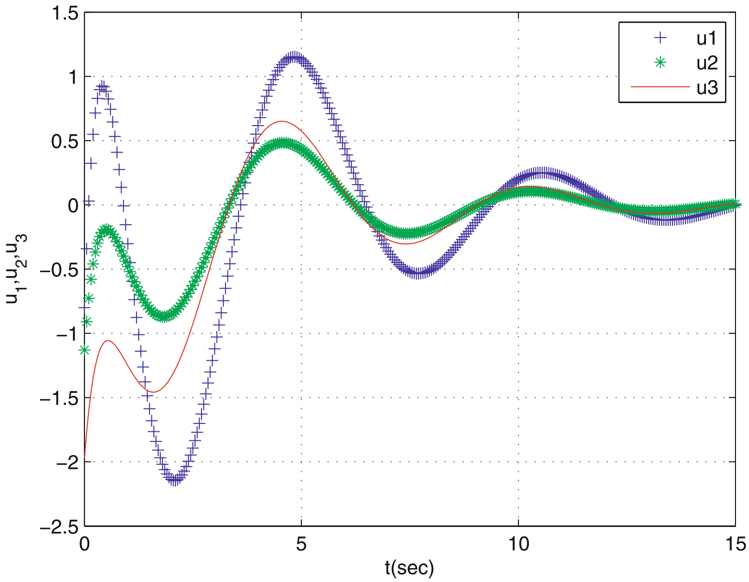


Fig. 5.30 Closed-loop control input trajectories—strong coupling

$$L_2^t = \left[1.034\ -0.145\ -0.045\ 0.005\right],$$
$$L_3^t = \left[0.911\ -1.105\ -0.804\ 0.019\right]$$

and the associated trajectories are plotted in Figs. 5.28 and 5.29. It is readily seen, as expected, the observer-based feedback control is more effective that the state feedback control. The observer-based feedback control trajectories are depicted in Fig. 5.30.

5.6 Notes and References

This chapter has fully examined the problem of designing decentralized $\mathcal{H}_\infty$ feedback control for a class of linear interconnected continuous-time and discrete-time systems with quantized signals in the subsystem control channels. The system under consideration has unknown-but-bounded couplings with adjustable local parameters and interval time-delays. Complete design of a decentralized output-feedback controller using local information (either continuous or quantized) is attained to render the closed-loop system is delay-dependent asymptotically stable with guaranteed γ-level.

Next, a general approach to quantized decentralized $\mathcal{H}_\infty$ feedback control of linear continuous-time or discrete-time systems where the quantizer has arbitrary form that satisfies a quadratic inequality constraint is developed and an LMI-based method is designed at the subsystem level to render the closed-loop system delay-dependent asymptotically stable with guaranteed γ-level.

Finally, for a class of linear interconnected discrete-time systems with quantized signals in the subsystem control channels, the problem of designing decentralized feedback control with overflow nonlinearities is treated. The system under consideration has unknown-but-bounded couplings with adjustable local parameters.

There are ample of extensions to the results of this chapter. This includes, but limited to, quantized filtering and quantized dynamic output-feedback within the decentralized framework. Indeed, dealing with nonlinear interconnected systems is an attractive area that calls for serious work. Promising results can be derived along the methods of Chaps. 2 and 3.

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Chapter 6
Decentralized Control of Traffic Networks

In this chapter, we direct our attention to the development of decentralized-control methods for large-scale traffic networks systems. Loosely speaking, large-scale traffic networks including computer and communication networks, freeway systems can be modeled as graphs in which a set of nodes (with storing capacities) are connected through a set of links (where traffic delays and transport costs may be incurred) that cannot be loaded above their traffic capacities. Traffic flows may vary over time. Then the nodes, that is, the decision makers acting at the nodes) may be requested to modify the traffic flows to be sent to their neighboring nodes.

6.1 Introduction

Traffic networks are engineering systems characterized by advanced technological importance. The following are typical applications:

- computer networks extending over large geographical areas;
- store-and-forward packet switching communication networks;
- large-scale freeway systems;
- reservoir networks in multi reservoir systems;
- queuing networks in manufacturing systems.

A fundamental challenge to communication networking has been the increased complexity to meet the explosive demand of applications, which brings about more technical issues to be solved. By increasing the number of nodes in a network, they may in general have fewer links to communicate to each other, directly. Therefore the routing problem, which deals with determining a route for packets from source nodes to destination nodes through other nodes emerges as one of the main challenging problems. This problem becomes more compounded with the presence of other issues such as delays patterns, packet losses, and bandwidth limitations that are crucial in selecting the intermediate nodes for routing. Early routing algorithms, such as those implemented in ARPANET, were based on finding the shortest path from the

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source node to the destination node [1–7]. When the network becomes crowded, link congestion arises thereby leading to poor performance of these algorithms. This in turn alleviates the issue of link congestion for routing decisions, that is the message flow rate on a link is related to the capacity of that link and optimal routing can be achieved by minimizing the total delay [7, 35]. The relevance of optimal networks for a class of mobile networks was reported in [9, 10]. In [3], a dynamic model of the network was proposed and a centralized routing controller was developed based on minimization of the total queuing delay. The queuing delay is a primary source of delay in routing and is defined as the total time that messages have to spend in the queue. This delay is obtained by integration of the queue length during the routing period. In [1], robust centralized as well as decentralized routing control strategies were introduced for networks with a fixed topology based on the minimization of the worst-case queuing length, which is related to the queuing delays. In deriving the queuing dynamics, the fluid flow conservation principle is frequently employed, wherein each state of the subsystem (node) represents a queue corresponding to a given destination node.

Maximizing the utility is another issue in routing. In [29], the shortest path routing algorithm for TCP/AQM networks was investigated which also maximizes the link utility. The problem of delay-constrained routing was addressed in recent years in [17] where a routing-based admission control mechanism considering an end-to-end delay for IP traffic flows was introduced.

By increasing size of networks makes the number of different possible paths from one node to another increase significantly. Therefore, it is virtually impossible to implement a centralized controller. Centralized controllers are also vulnerable to failures in the network and introduce a large communication overhead on the network. Specially, when the nodes are distributed in a large area, the communication between each node and the centralized controller enforces a costly communication overhead with noticeable delays. Thus, decentralized controllers which can be implemented locally at individual nodes are desirable for reasons of practical implementation. In [15], other types of delays, namely transmission delay, propagation delay, and processing delay were also considered in dynamic model of the network flow and a decentralized controller was proposed that guarantees the boundedness of the queue length and the delays.

Extending on the results of [1, 5], the objective of this chapter is to develop improved routing strategies based on minimization of the worst-case queuing length which is also minimizes the congestion and packet loss [15].

Based on continuous-time version of the queuing model presented in [5], we adopt the $\mathcal{H}_\infty$ performance criterion to form an optimal control scheme so as to maintain the robust performance of the routing strategy in the presence of multiple unknown time-varying delays. The resulting optimization problem is then reformulated as a linear objective minimization problem involving Linear Matrix Inequality (LMI) constraints. In the present work, both centralized and decentralized solution strategies are developed. In the centralized case, a refined LMI specifications facilitated the inclusion of several physical constraints imposed on the queuing model.

6.2 A Model of Communication Networks

In what follows, we consider a communication network (CN) as a directed graph (N, L), consisting of N, a set of n nodes and L, a set of ℓ oriented links. Each node receives messages from both from outside the network and the upstream nodes within the network. Each message has a destination node $d \in \mathsf{N}$ which is absorbed as soon as it arrives at that node. Messages can arrive to a node as their final destination. Alternatively, arrive as transition in which case, are put into a queue to be sent out to a downstream node. It is assumed that the network is "*connected*", that is, each node of the network must be reachable from all other nodes. When all the nodes are source as well as destination, at each node $j \in \mathsf{N}$ there will be $n-1$ queues in which messages are stored for all destinations, $1, 2, j-1, j+1, \dots, n$. Based on the fluid flow conservation principle, The dynamics of CN can be expressed by the following model:

$$\dot{q}_j^d(t) = \sum_{k \in \Sigma_u(j), k \neq d} f_{kj}^d(t - \tau_{kj}^d(t)) + r_j^d(t) - \sum_{k \in \Sigma_d(j)} f_{jk}^d(t), \tag{6.1}$$

where

q_j^d: message queue length at node j destined to node d
$\Sigma_u(j)$: set of upstream neighbors of node j
$\Sigma_d(j)$: set of downstream neighbors of node j
$f_{kj}^d(t)$: input traffic flow routed from node $k \in \Sigma_u(j)$ to node j destined to node d
$f_{jm}^d(t)$: output traffic flow routed from node j to node $m \in \Sigma_d(j)$ destined to node d
$r_j^d(t)$: external input flow entering node j destined to node d
$\tau_{kj}^d(t)$: total unknown time-varying and bounded delay in transmitting, propagating, and processing of messages (including identifying the destination, inserting in the queue and routing computation) with destination d routed from node k to node j.

It must be emphasized that the delays defined in (6.1) are assumed to be unknown and time-varying and this is clearly more realistic for traffic network applications. When dealing the routing problem, some physical characteristics impose constraints on a traffic network model. A typical set of constraints can be given as

$$f_{jk}^d \geq 0, \tag{6.2}$$

$$0 \leq q_j^d(t) < q_{\max_j}^d, \tag{6.3}$$

$$\sum_{d \in N^j} \leq c_{jk}, \quad j \in N, \ k \in \Sigma_d(j), \tag{6.4}$$

where $\mathsf{N}^j = \mathsf{N} \setminus \{j\}$. The first two constraints (6.2) and (6.3 are frequently termed non-negativity constraints based on physical reasons. Constraint (6.3) implies that the queuing length cannot exceed the buffer size $q^d_{\max}$. Finally, the capacity constraint (6.4) states that the total flow in each link cannot exceed the capacity of that link denoted by c_{jk}.

6.3 Problem Formulation

Routing problem in CN is concerned with adjusting the output flow of each queue, $f^d_{jk}(t)$, according to the network traffic information, such that certain objective functions are minimized. To formulate the routing problem, we introduce

$$\begin{aligned} x(t) &= \mathrm{col}\{q^d_j(t)\} \in \Re^{n(n-1)}, \\ w(t) &= \mathrm{col}\{r^d_j(t)\} \in \Re^{n(n-1)}, \\ u(t) &= \mathrm{col}\{f^d_j(t)\} \in \Re^{n(n-1)}, \quad j, d = 1, 2, \ldots, n. \end{aligned} \tag{6.5}$$

Since the input flows for these queue are due to its upstream neighbors, they are in turn the output flows of these nodes after some delays. The time-varying delay functions associated with these nodes are not known *a priori* and are different from one another due to differences in the traffic load in each link and other network uncertainties. Define

$$\begin{aligned} u(t - \tau(t)) &= \mathrm{col}\{f^d_{kj}(t - \tau^d_{kj}(t))\} \in \Re^{\ell(n-1)}, \\ \tau(t) &= \mathrm{col}\{\tau^d_{kj}(t)|\}, \quad k, j, d = 1, 2, \ldots, n. \end{aligned} \tag{6.6}$$

Thus (6.1) can be rewritten compactly as

$$\dot{x}(t) = Bu(t) + Du(t - \tau(t)) + \Gamma w(t), \tag{6.7}$$

$$z(t) = Cx(t), \tag{6.8}$$

where $z(t)$ is the regulated output, C is a weight matrix that can be defined according to the queues priorities, $B \in \Re^{n(n-1)\times\ell(n-1)}$ and $D \in \Re^{n(n-1)\times\ell(n-1)}$ represent the network connectivity (downstream and upstream nodes, respectively). Actually, $\{B_{jk}\ (D_{jk})\}$ is equal to -1 (1), if the flow u_j is a downstream (upstream) flow of node j and is zero otherwise. Moreover, $\Gamma = I_{n(n-1)\times n(n-1)}$. For several technical reasons, we consider that the delay $\tau(t)$ is an unknown differentiable function satisfying

$$0 \le \tau(t) \le \varrho, \quad 0 < \dot{\tau} \le \mu, \quad \forall t \ge 0. \tag{6.9}$$

Inequality (6.9) ensures smooth flow across the traffic network.

6.3.1 A Network Example

For the purpose of illustration, we consider the network shown in Fig. 6.1 which consists of 4 nodes. All messages are routed to one destination, namely node 4, thereby leading to one queue at each node. In terms of

$$u(t) = \begin{bmatrix} f_{12}^4(t) \\ f_{13}^4(t) \\ f_{14}^4(t) \\ f_{23}^4(t) \\ f_{34}^4(t) \end{bmatrix}, \quad u(t-\tau(t)) = \begin{bmatrix} f_{12}^4(t-\tau_{12}^4(t)) \\ f_{13}^4(t-\tau_{13}^4(t)) \\ f_{14}^4(t-\tau_{14}^4(t)) \\ f_{23}^4(t-\tau_{23}^4(t)) \\ f_{34}^4(t-\tau_{34}^4(t)) \end{bmatrix}.$$

Corresponding to this example, $\Gamma = I_{3\times 3}$ and the matrices B and D are given by

$$B = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

Remark 6.1 It should be emphasized that the time-delays are indeed a major source of instability for the entire network. Classical control theory have failed to sufficiently address stability and performance issues of time-delayed systems. Complications do arise in practical situations when there is no *a priori* knowledge about transmitting, propagating, and processing delays. Furthermore, the time delay functions vary according to the traffic flow and other stimuli and disturbances in the network. Based on a given traffic flow characteristics the amount of delay may increase substantially, Stabilizing a time delayed system having multiple **fast** time-varying delay functions is still an open area of research.

In what follows, we adopt the $\mathcal{H}_\infty$ criterion in addressing the $\mathcal{H}_\infty$ routing algorithm based on state feedback controller of the form $u(t) = Kx(t)$, that simultaneously guarantees stability of the network model (6.7) in presence of time-varying delays and minimizes the *worst-case queuing length* due to the external inputs.

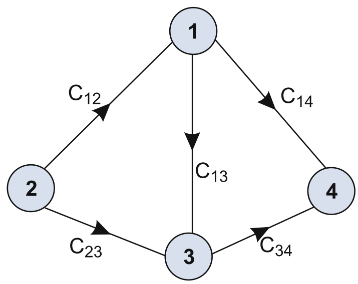


Fig. 6.1 A sample network topology

Specifically, the routing problem can be cast into the following optimization problem

$$\min_{\gamma>0} J(w) = \int_0^{\infty} (z^t(s)z(s) - \gamma^2 w^t(s)w(s))\, ds < 0. \tag{6.10}$$

Using the objective function (6.10), the messages are routed such that the network is simultaneously stabilized subject to unknown transmitting, propagating, and processing delays $\tau(t)$, and the queuing length, x, is minimized subject to presence of the external input w. Note that by minimizing the worst case queuing length one can actually accomplish a measure of minimum queuing delay.

6.4 Centralized Routing Controller

In what follows, we seek to design a centralized $\mathcal{H}_\infty$ state feedback controller for the network model (6.7)–(6.8) subject to the constraints (6.2)–(6.4). We adopt a Lyapunov-based approach to establish the stability conditions of the time-delayed closed-loop network without physical constraints and with prescribed performance criteria. The resulting conditions will be cast into the framework of LMIs. Then we impose the physical constraints (6.2)–(6.4) to refine the LMI feasibility conditions.

6.4.1 Delay-Dependent $\mathcal{H}_\infty$ Unconstrained Control Design

Since the traffic network routing model (6.7)–(6.8) represents a time-delayed system with unknown time-varying parameters (delays), we proceed to construct an appropriate Lyapunov-Krasovskii functional (LKF) to derive delay-dependent stability and stabilization conditions. It must be noted that despite the fact that the total delay of messages are not known in advance, nevertheless without loss of generality the delays can be assumed to satisfy the bounding inequalities (6.9). Under the state feedback $u = Kx$, the closed-loop traffic model becomes

$$\begin{aligned} \dot{x}(t) &= BKx(t) + DKx(t-\tau) + \Gamma w(t), \\ z(t) &= Cx(t). \end{aligned} \tag{6.11}$$

The following theorem establishes a basis for an $\mathcal{H}_\infty$ state feedback design in the unconstrained case.

Theorem 6.1 *Consider the traffic network model* (6.7)–(6.8) *with the delay pattern* (6.9) *and* $w \in \mathcal{L}_2[0,\infty)$. *Given the bounds* ϱ, μ, γ, *there exists a centralized state-feedback controller of the form* $u = Kx$, *such that the closed-loop system* (6.11) *is asymptotically stable with* $\mathcal{L}_2$*-gain less than* γ, *if there exist matrices*

$\mathsf{X} > 0$, Y, Ψ_1, Ψ_2, Ψ_3 *satisfying the LMI*

$$\begin{bmatrix} \Pi_1 & 0 & \Psi_1 & \Gamma & \mathsf{X}C^t & \varrho Y^t B^t \\ \bullet & -\Pi_2 & -\Psi_1 & 0 & 0 & 0 \\ \bullet & \bullet & -\Pi_3 & 0 & 0 & \varrho Y^t D^t \\ \bullet & \bullet & \bullet & -\gamma^2 I & 0 & \varrho \Gamma^t \\ \bullet & \bullet & \bullet & \bullet & -I & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -2\mathsf{X} + \Psi_1 \end{bmatrix} < 0, \tag{6.12}$$

where

$$\Pi_1 = B\mathsf{Y} + \mathsf{Y}^t B^t - \Psi_1 + \Psi_2 + \Psi_3,$$
$$\Pi_2 = \Psi_1 + \Psi_2, \quad \Pi_3 = 2\Psi_1 + (1-\mu)\Psi_2. \tag{6.13}$$

The state feedback gain is given by $K = \mathsf{Y}\mathsf{X}^{-1}$.

The proof is provided in Sect. 6.7.

Remark 6.2 The success of the centralized routing controller hinges upon the validity that the external input flows to nodes $w \in \mathcal{L}_2$. In principle, one can employ an appropriate shaping filter, decoding or interpolation techniques to ensure the satisfaction of this condition without removing some information from the input signal. By this way, the queuing length is guaranteed to remain in $\mathcal{L}_2$ thereby assuring that the boundedness of the queuing length of the proposed routing methodology is supported for a bounded input flow w.

Remark 6.3 It must be noted that Theorem 6.1 yields delay-dependent stability condition that is less-conservative and has wider operational range than the one developed in [1]. The main reason for this is the controller gain requires the feasibility of a single strict LMI (6.12) as opposed to three non-strict LMIs in [1]. Moreover, the number of LMI variables are fewer thereby reducing the computational burden.

6.4.2 Delay-Dependent $\mathcal{H}_\infty$ Design

In what follows, we impose the physical constraints of the network and recast the solution of the routing problem as LMI feasibility conditions. By including the constraints, we proceed for determining a complete solution to the robust dynamic routing problem. For this purpose, we invoke the following result established in [13]:

Lemma 6.1 *The linear time-delay system*

$$\dot{x}(t) = Ax(t) + Dx(t - \tau(t)), \quad A \in \Re^{n\times n}, \ D \in \Re^{n\times n}$$

is non-negative if and only if the off-diagonal elements of A are non-negative and all of the elements of D are non-negative.

The main result is summarized by the following theorem:

Theorem 6.2 *Consider the dynamical queuing model* (6.11). *A constrained centralized routing controller with* $\mathcal{H}_\infty$ *performance is obtained by solving the minimization problem*

$$\min_{0<\mathsf{X},\mathsf{Y},\Psi_1,\Psi_2,\Psi_3} \gamma \tag{6.14}$$

$$\textit{subject to inequalities} \quad I_1,\ I_{2k},\ I_3,\ I_4,\ I_5,\ I_6. \tag{6.15}$$

The proof is provided in Sect. 6.7.

Next, we generalize our results to the case of large-scale traffic networks.

6.5 Decentralized Traffic Routing Control

An appealing methodology for dealing with large-scale systems including traffic networks is the decentralized control approach [19], whereby all the analysis and design tasks are performed at the subsystem level. In the sequel, we appropriately modify the robust centralized routing control strategy and develop it in the form of a decentralized control scheme. Specifically, the traffic routing problem is reformulated such that each node in the network is treated as a subsystem requiring only its own local information to route the received messages while ensuring that a global performance index is optimized and desired specifications are satisfied.

In this regard, robust control of large scale time-delay systems has witnessed an intense research activity in the past few years to develop decentralized stabilizing controllers for constant time-delay systems. Stabilization of a class of time-varying large-scale systems subject to time-varying delays was investigated in [20, 22, 23]. In this section the dynamic model of the traffic flow is modified for design and implementation of an improved robust decentralized routing control strategy for unknown and fast time-varying delays. The properties of the developed decentralized robust control scheme and a brief discussion on its complexity and scalability are also provided.

6.5.1 Decentralized Dynamic Model

As mentioned previously, in routing problem, each node of the traffic network is considered as a subsystem that includes all its queues corresponding to different destinations. Consequently, the decentralized dynamic model of the traffic flow at each node, or subsystem is given by

$$\dot{x}(t) = B_j u_j(t) + \sum_{m\in\Sigma(j)} D_{d_{jm}} u_m(t-\tau_{jm}(t)) + \Gamma_j w_j(t), \tag{6.16}$$

where

- $x_j = \mathrm{col}\{q_j^d(t)\} \in \Re^n$ for $d = 1, \dots, n$ denotes the queue length in node j for different destinations,
- $u_j(t) = \mathrm{col}\{f_{jm}^d(t)\} \in \Re^\ell$, denotes the flows sent from node j,
- $\tau_{jm}(t)$ denotes the *unknown but bounded time-varying* total-delay in transmission, propagation, and processing,
- $w_j(t) \in \Re^n$ denotes the external input flow for node j.

The matrices B_j, $D_{d_{jm}}$ and Γ_j are defined for the node j similar to that in Sect. 6.5.

Assumption 6.1 The delays $\tau_{jm}(t)$ are unknown differentiable functions that for all $t \geq 0$ satisfy

$$\begin{aligned} &0 \leq \varphi_{jm} \leq \max\{\tau_{jm}(t)\} \leq \varrho_{jm}, \\ &\max\{|\dot{\tau}_{jm}(t)|\} \leq \mu_{jm}, \quad \mu_{jm} > 0. \end{aligned} \tag{6.17}$$

For simplicity, we consider that the delay between two nodes in both directions are the same, that is $\tau_{mj} = \tau_{jm}$.

At each node (subsystem) the traffic routing problem is to determine an $\mathcal{H}_\infty$ state feedback controller $u_j = K_j x_j$ such that the following global objective function is minimized:

$$J(w) = \int_0^\infty [z^t(j)z(t) - \gamma^2 w^t(t)w(t)]\,ds < 0, \quad \gamma > 0, \tag{6.18}$$

where the vectors z and w are defined previously.

6.5.2 Decentralized Robust Routing Controller: Unconstrained Case

Under the state feedback controller $u_j = K_j x_j$, the closed-loop node model becomes:

$$\begin{aligned} \dot{x}_j(t) &= B_j K_j x_j(t) + \sum_{m \in \Sigma(j)} D_{d_{jm}} K_m x_m(t - \tau_{jm}(t)) + \Gamma_j w_j(t), \\ z_j(t) &= C_j x_j(t) \end{aligned} \tag{6.19}$$

with the delay pattern (6.17). Toward our goal, we select the following Lyapunov-Krasovskii Functional:

$$\begin{aligned} V_j t &= x_j^t(t) P_j x_j(t) \\ &\quad + \sum_{m \in \Sigma(j)} \int_{t-\varrho_{jm}}^t e^{a_j(s-t)} x_j^t(s) S_{1jm} x_j(s) ds \end{aligned}$$

$$+ \sum_{m\in\Sigma(j)} \int_{t-\varrho_{jm}}^{t} e^{a_j(s-t)} x_j^t(s) S_{2jm} x_j(s) ds$$

$$+ \sum_{m\in\Sigma(j)} \int_{t-\tau_{jm}}^{t} e^{a_j(s-t)} x_j^t(s) E_{jm} x_j(s) ds$$

$$+ \sum_{m\in\Sigma(j)} \varrho_{jm} \int_{-\varrho_{jm}}^{0} \int_{t+\theta}^{t} e^{a_j(s-t)} \dot{x}_j^t(s) R_{1jm} \dot{x}_j(s) ds d\theta$$

$$+ \sum_{m\in\Sigma(j)} \delta_{jm} \int_{-\varphi_{jm}}^{-\varrho_{jm}} \int_{t+\theta}^{t} e^{a_j(s-t)} \dot{x}_j^t(s) R_{2jm} \dot{x}_j(s) ds d\theta, \qquad (6.20)$$

where $0 < P_j$, S_{1jm}, S_{21jm}, E_{jm}, R_{1jm}, R_{2jm} are weighting matrices with appropriate dimensions. The following theorem establishes LMI-based delay-dependent sufficient stability conditions for the decentralized time-delay system (6.19) and provides control design conditions for constructing an $\mathcal{H}_\infty$ state feedback routing controller.

Theorem 6.3 *The traffic network model* (6.16) *with* $w_j \in \mathcal{L}_2[0,\infty)$ *is exponentially stabilizable by the decentralized state feedback controllers of the form* $u_j = K_j x_j$, *with an* $\mathcal{L}_2$*-gain less than* γ_j, *if given scalars* $0 < a_j$, *there exist matrices* $0 < X_j$, Y_j, $\overline{S}_{1rj}$, $\overline{S}_{2rj}$, $\overline{E}_{rj}$, $\overline{N}_{ajr}$, $\overline{N}_{cjr}$, $\overline{S}_{ajr}$, $\overline{S}_{cjr}$ *for* $j = 1, \ldots, n$ *such that the following LMI is satisfied*

$$\Pi_j = \begin{bmatrix} \Pi_{j11} & \Pi_{j12} & \Pi_{j13} & \Pi_{j14} & \Pi_{j15} & \Pi_{j16} \\ \bullet & \Pi_{j22} & \Pi_{j23} & \Pi_{j24} & \Pi_{j25} & \Pi_{j26} \\ \bullet & \bullet & \Pi_{j33} & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \Pi_{j44} & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \Pi_{j55} & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \Pi_{j66} \end{bmatrix}, \qquad (6.21)$$

where

$$\Pi_{j11} = B_j Y_j + Y_j^t B_j^t + \sum_{r\in\Sigma_d(j)} (\overline{S}_{1rj} + \overline{S}_{2rj} + \overline{E}_{rj} + a_j X_j)$$

$$+ \sum_{r\in\Sigma_d(j)} (\overline{N}_{ajr} + \overline{N}_{ajr}^t),$$

$$\Pi_{j12} = [\Pi_{12i}\ \Pi_{12r}\ \ldots], \quad r \in \Sigma_d(j),$$

$$\Pi_{j12i} = \sum_{r\in\Sigma_d(j)} (-2\overline{N}_{ajr} - 2\overline{S}_{ajr} + \overline{N}_{cjr}^t),$$

$$\Pi_{j12r} = P B_j Y_j,$$

$$\Pi_{j13} = \sum_{r\in\Sigma_d(j)} (\overline{N}_{ajr} + \overline{S}_{ajr}),$$

$$\Pi_{j14} = \sum_{r \in \Sigma_d(j)} \overline{S}_{ajr},$$

$$\Pi_{j15} = \left[\sum_{r \in \Sigma_d(j)} \overline{N}_{ajr} \sum_{r \in \Sigma_d(j)} \overline{S}_{ajr} \right],$$

$$\Pi_{j16} = \left[P\Gamma_j \; C_j^t \; Y_j^t B_j^t \sum_{r \in \Sigma_d(j)} (\varrho_{rj}^2 R_{1rj} + \delta_{rj}^2 R_{2rj}) \right],$$

$$\Pi_{j22} = \mathrm{diag}[\Pi_{j22j} \; \Pi_{j22r}], \quad r \in \Sigma_d(j),$$

$$\Pi_{j22j} = \sum_{r \in \Sigma_d(j)} (-2\overline{N}_{cij} - 2\overline{N}_{cjr}^t - 2\overline{S}_{cir} - 2\overline{S}_{cir}^t),$$

$$\Pi_{j22r} = -(1-\mu_{jm})\overline{E}_{jm} - 2\overline{N}_{cjm} - 2\overline{N}_{cjm}^t - 2\overline{S}_{cjm} - 2\overline{S}_{cjm}^t, \tag{6.22}$$

$$\Pi_{j23} = \begin{bmatrix} \sum_{r \in \Sigma_d(j)} (\overline{N}_{cjr} + \overline{S}_{cjr}) \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$\Pi_{j24} = \begin{bmatrix} \sum_{r \in \Sigma_d(j)} \overline{S}_{cjr} \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$\Pi_{j25} = \begin{bmatrix} \sum_{r \in \Sigma_d(j)} \overline{N}_{cjr} & \sum_{r \in \Sigma_d(j)} \overline{S}_{cjr} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix},$$

$$\Pi_{j26} = \begin{bmatrix} 0 & 0 & \Pi_{j261} \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix},$$

$$\Pi_{j261} = Y_j^t B_j^t \sum_{r \in \Sigma_d(j)} (\underline{h}_{ir}^2 R_{1jr} + \delta_{jr} R_{2jr}), \tag{6.23}$$

$$\Pi_{j33} = \sum_{r \in \Sigma_d(j)} \overline{S}_{1rj}, \quad \Pi_{j44} = -\sum_{r \in \Sigma_d(j)} \overline{S}_{2rj},$$

$$\Pi_{j55} = \mathrm{diag}\left[-\sum_{r \in \Sigma_d(j)} e^{-a_r \underline{h}_{jr}} \overline{R}_{1jr} - \sum_{r \in \Sigma_d(j)} e^{-a_r \delta_{jr}} \overline{R}_{2jr} \right], \tag{6.24}$$

$$\Pi_{j66} = \begin{bmatrix} -\gamma_j^2 & 0 & \Gamma_j^t \\ \bullet & -I & 0 \\ \bullet & \bullet & \Pi_{j661} \end{bmatrix},$$

$$\Pi_{j661} = \sum_{r \in \Sigma_d(j)} (\varrho_{ir}^2 \overline{R}_{1jr} + \delta_{jr}^2 \overline{R}_{2jr}) - 2X_j. \tag{6.25}$$

Moreover, the robust decentralized state feedback controller gain is given by $K_i j = X_j^{-1} Y_j$.

The details of the proof are given in Sect. 6.7.

Remark 6.4 It must be emphasized that Theorem 6.3 provides new results in the context of decentralized control of large-scale systems with time-varying delays in comparison to the available literature. In part, it builds upon the parametrization technique developed in [24]. Considering the case $a_j \equiv 0$, the results reduce to decentralized asymptotic stabilization which out performs the results reported in [8, 15, 22, 23].

6.5.3 Decentralized Robust Routing Controller: Constrained Case

Extending on the centralized case, we can similarly deal with the physical constraints imposed at each node to arrive at the constrained decentralized routing problem. The additional LMI conditions are now derived by parallel development to (6.38) through (6.51). The main result is summarized by the following theorem:

Theorem 6.4 *Consider the dynamical queuing model* (6.19). *A constrained decentralized routing controller with* $\mathcal{H}_\infty$ *performance is obtained by solving the minimization problem*

$$\text{for } j = 1, \ldots, n \qquad \min_{0 < X_j, Y_j, \overline{S}_{1rj}, \overline{S}_{2rj}, \overline{E}_{rj}, \overline{N}_{ajr}, \overline{N}_{cjr}, \overline{S}_{ajr}, \overline{S}_{cjr}} \gamma \tag{6.26}$$

$$\text{subject to inequalities} \quad I_{j1},\ I_{j2k},\ I_{j3},\ I_{j4},\ I_{j5},\ I_{j6}. \tag{6.27}$$

The proof is provided in Sect. 6.7.

Remark 6.5 It must be noticed that by increasing the number of nodes in a traffic network, the number of possible paths from a given source node to a given destination will significantly increase. It is therefore expected that a centralized routing controller algorithm will be vulnerable to failures and introduces a huge communication overhead on the network. On the other hand, decentralized controllers that can be constructed locally at individual nodes are highly desirable for practical purposes and implementation.

Given n is the number of nodes, k is the number of destination nodes, and ℓ is the number of links, the developed routing method requires five $(n-1)k \times (n-1)k$ and $(l-1)k \times (n-1)k$ unknown matrices should be determined for the feasibility of LMI conditions in (6.12). Alternatively, the developed decentralized routing method for each node there are nine unknown matrices to be determined so as to satisfy the LMI conditions in (6.21). Therefore, there would be a total $8(n-k)$ unknown matrices with dimension $k \times k$, $8k$ unknown matrices with dimension $(k-1) \times (k-1)$, $n-k$ unknown matrices with dimension $m_i k \times k$, and k unknown matrices with dimension $m_i(k-1) \times (k-1)$ are required. Even though the number of the unknown matrices in the decentralized control scheme is higher than the centralized controller method, given that the dimensions of matrices are lower than the centralized case, the LMI technique can attain a solution to the decentralized scheme much faster and more efficiently. Moreover, the implementation of our decentralized controller is computationally less expensive when compared to our centralized method. In effect, for *non crowded* networks, the centralized method is more desirable due to their optimality (vis-à-vis use of the full information set) and accuracy. However, by increasing the number of nodes in the network it becomes difficult and sometimes even impossible (due to ill-conditioning and curse of dimensionality) to design and implement the centralized controller. Consequently, the decentralized method is more suitable and appropriate for these large-scale traffic networks.

6.6 Simulation Results

To evaluate the performance of our proposed traffic network routing control strategies, simulations are conducted in this section on two examples, The results obtained by applying our proposed decentralized routing strategy is compared with that of the centralized scheme as well as another conventional optimal control scheme, namely Model Predictive Control (MPC) method. All simulations are done on a MATLAB Simulink 7.0 platform.

6.6.1 Simulation Example 6.1

Consider the network shown in Fig. 6.2. The capacity of each link is also indicated in the figure, the unit of which is kbit/sec. All the nodes are assumed to be sources as well as destinations. From Fig. 6.2 it follows that there are 6 queues (the states of the system) and 8 output flows for these queues (the input signals). The initial values of all the states are set to 0. The external input is considered to be the following pulse function:

$$w(t) = \begin{cases} 1 \text{ kbit/sec}, & 0 < t < 50, \\ 0, & \text{otherwise} \end{cases}$$

and the delay is assumed to be 5 sec for each input flow.

Fig. 6.2 Network topology of Example 1

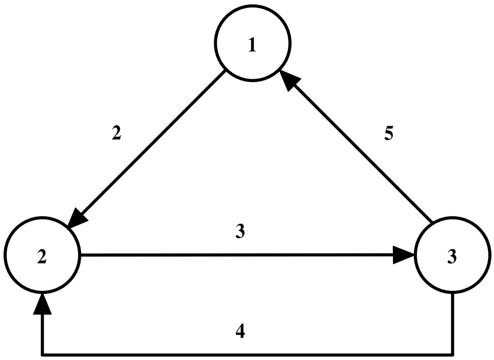
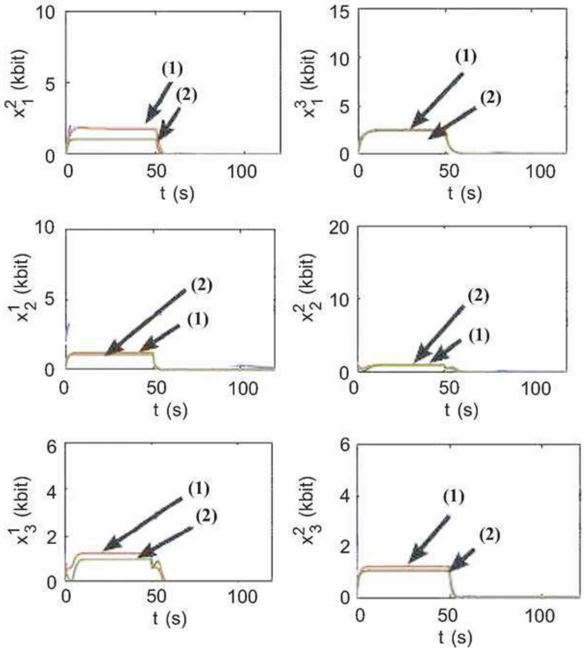


Fig. 6.3 The response of the queueing lengths: *(1)*—centralized $\mathcal{H}_\infty$ controller, *(2)*—the decentralized $\mathcal{H}_\infty$ controller



The results obtained by using our proposed decentralized and centralized $\mathcal{H}_\infty$ controllers for the queue lengths are shown in Fig. 6.3 (where x_i^d denotes the queue length of node i destined to node d) and for the flow links in Fig. 6.4, respectively. These figures clearly demonstrate the smooth behavior of the $\mathcal{H}_\infty$ controller, where no *a priori* knowledge about the delays, except their upper bound, was assumed. It can be stated that overall the decentralized control performance is comparable with that of the centralized method, However, since there are only few (3) nodes in the network, the difference is not quite significant.

We are also interested in investigating the effects of jitter on the performance of our proposed centralized and decentralized routing strategies. Therefore, the follow-

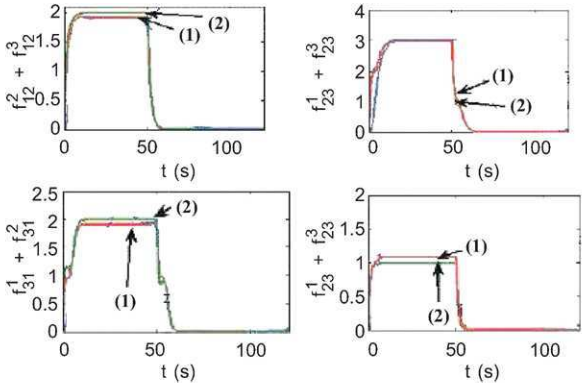


Fig. 6.4 The response of output flows: *(1)*—centralized $\mathcal{H}_\infty$ controller, *(2)*—the decentralized $\mathcal{H}_\infty$ controller

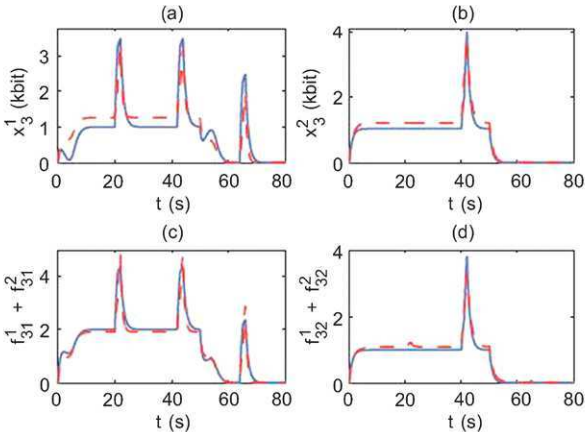


Fig. 6.5 (**a**) and (**b**): Queuing lengths of node 3: centralized $\mathcal{H}_\infty$ controller (*solid*) and decentralized $\mathcal{H}_\infty$ controller (*dashed*), (**c**) and (**d**): Link flows for node 3: centralized $\mathcal{H}_\infty$ controller (*solid*) and the decentralized $\mathcal{H}_\infty$ controller (*dashed*)

ing input signal is added to the above signal w for node 3:

$$\acute{w}_3^1(t) = \begin{cases} pp, & 20k < t < 23k, \\ 0, & \text{otherwise}, \end{cases}$$
$$\acute{w}_3^2(t) = \begin{cases} pp, & 40k < t < 43k, \\ 0, & \text{otherwise}, \end{cases} \tag{6.28}$$

where pp is a random signal with a Poisson distribution and rate of 2.5. Figures 6.5(a) and (b) illustrate the queuing length of node 3 and Figs. 6.5(c) and (d) show their corresponding flow link, for the centralized and decentralized methods, respectively. By considering the results obtained in Fig. 6.5, it can be concluded that the effects of the jitter do not deteriorate the stability of our routing strategies and the additional disturbances are attenuated after a short transient time.

6.6.2 Simulation Example 6.2

Consider the network that is shown in Fig. 6.6 which is adopted from [18]. The capacity of each link, with unit of k bit/sec, is also depicted in the figure. The destination nodes are 7 and 10 and overall there are 17 queues (the states of the system) and 35 output flows for these queues (the input signals). The initial values of all the states are set to 0. The external input is considered as a poison distribution with the flow rate λ for 70 sec. For each input flow the delay is taken as a fast time-varying function $3 + 5|\sin(100t)|$ sec which is considered to be *unknown* to the controllers.

Each simulation is run for 100 sec. For $\lambda = 0.2$, the results obtained by using our proposed decentralized and centralized $\mathcal{H}_\infty$ controllers for the queue lengths of nodes 1–3 are shown in Fig. 6.7. As can be seen from this figure our proposed

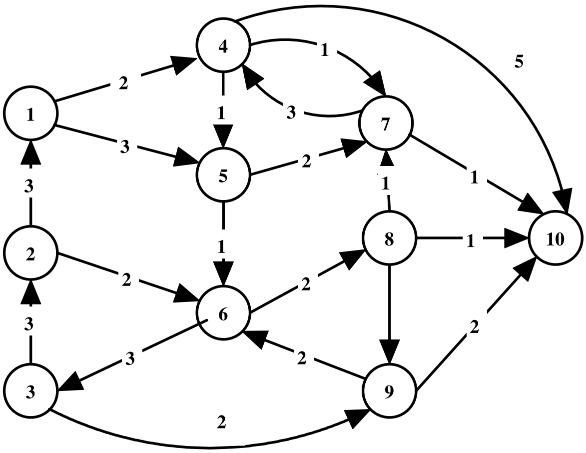


Fig. 6.6 The network topology of Example 2

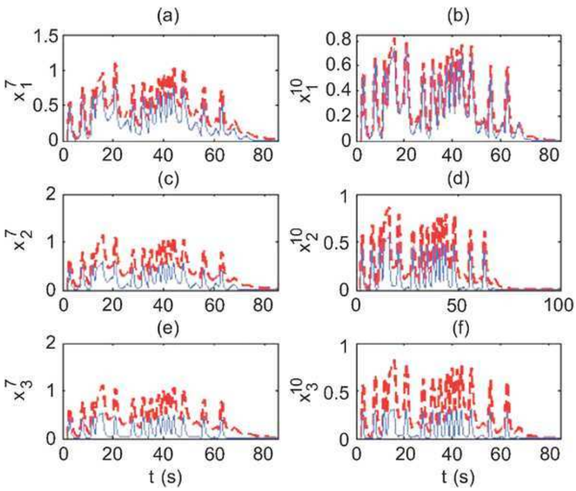


Fig. 6.7 Queuing lengths of nodes 1–3: centralized $\mathcal{H}_\infty$ controller (*solid*) and the decentralized $\mathcal{H}_\infty$ controller (*dashed*)

Table 6.1 Queuing length for different external flow rates

λ (kbits/sec)	0.1	0.2	0.3	0.4
External input	187	289	425	561
Queuing delay for the $\mathcal{H}_\infty$ centralized method	211.18	438.26	644.07	847.02
Queuing delay for the $\mathcal{H}_\infty$ decentralized method	220.05	615.02	720.89	955.81

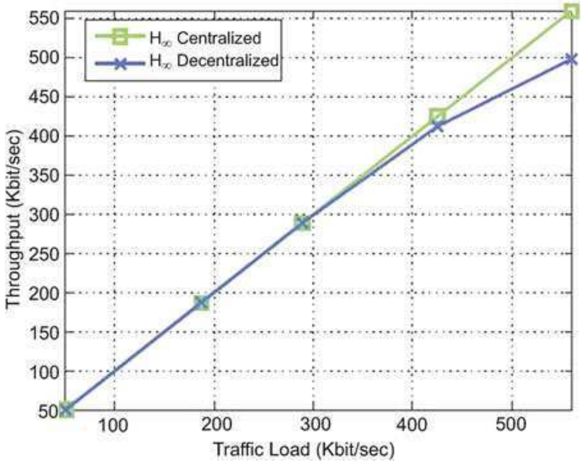


Fig. 6.8 Throughput comparison between the centralized and decentralized controllers

control schemes satisfy the physical constraints and the closed-loop system behaves robustly in presence of fast time-varying delays. Similar results are also obtained for nodes 4–9, but these results are omitted due to space limitations. Note that the time derivative upper bound of the delay is taken as $d = 200$ and the upper bound of the delay is $h = 8$. Table 6.1 shows the resulting queuing delays for different input flow rates. It should be noted that our proposed routing algorithms can provide acceptable performance for higher values of the transmission, propagation and processing delays. By invoking the stability results obtained in Theorems 4.1 and 5.1, the maximum delay upper bound for which the routing controller can maintain its acceptable performance is found to be $h = 10$ for the decentralized controller, and is found to be $h = 8$ for the centralized controller.

When the network traffic is heavy, congestion occurs and packets are dropped in the network, which can cause a decrease in the throughput performance. Figure 6.8 depicts the performance of our proposed algorithms under different traffic loads. Indeed, by increasing λ to 0.3 the decentralized method loses 3% of its packets while for $\lambda = 0.4$ this loss increased to 11%.

Generally, it can be concluded that the decentralized control strategy could fairly compete with the performance of the centralized method. However, as the number of nodes increases, one may not be able to easily solve the corresponding high dimensional LMI conditions associated with the centralized control strategy due to ill-conditioning and/or reductions in the size of the feasibility regions. On the other hand, the decentralized strategy is scalable and would provide an acceptable performance even in crowded networks.

6.7 Proofs

6.7.1 Proof of Theorem 6.1

Proof Introduce the Lyapunov-Krasovskii functional (LKF):

$$\begin{aligned} V(t) = x^t(t)\mathcal{P}x(t) + \int_{t-\varrho}^{t} x^t(\alpha)\mathcal{W}x(\alpha)d\alpha \\ + \int_{t-\tau}^{t} x^t(\alpha)\mathcal{S}x(\alpha)d\alpha \\ + \varrho \int_{t-\varrho}^{t} \int_{t+\sigma}^{t} \dot{x}^t(s)\mathcal{R}\dot{x}(s)\, ds\, d\theta, \end{aligned} \tag{6.29}$$

where $0 < \mathcal{P}$, $0 < \mathcal{W}$, $0 < \mathcal{S}$, $0 < \mathcal{R}$ are weighting matrices of appropriate dimensions. A straightforward computation gives the time-derivative of $V(t)$ along the solutions of (6.11) yields:

$$\begin{aligned} J &= \dot{V}(t) + z^t(t)z(t) - \gamma^2 w^t(t)w(t) \\ &= 2x^t(t)\mathcal{P}\dot{x}(t) + \varrho^2\dot{x}^t(t)\mathcal{R}\dot{x}(t) \\ &\quad - \varrho \int_{t-\varrho}^{t} \dot{x}^t(s)\mathcal{R}\dot{x}(s)ds + x^t(t)(\mathcal{W}+\mathcal{S})x(t) \\ &\quad - x^t(t-\varrho)\mathcal{W}x(t-\varrho) - (1-\dot{\tau})x^t(t-\tau)\mathcal{S}x(t-\tau) \\ &\quad + x^t(t)C^tCx(t) - \gamma^2 w^t(t)w(t). \end{aligned} \tag{6.30}$$

Using the identity

$$-\varrho \int_{t-\varrho}^{t} \dot{x}^t(s)\mathcal{R}\dot{x}(s)ds = -\varrho \int_{t-\varrho}^{t-\tau} \dot{x}^t(s)\mathcal{R}\dot{x}(s)ds - \varrho \int_{t-\tau}^{t} \dot{x}^t(s)\mathcal{R}\dot{x}(s)ds, \tag{6.31}$$

then applying Jensen's inequality

$$\begin{aligned} \int_{t-\tau}^{t} \dot{x}^t(s)\mathcal{R}\dot{x}(s)ds &\ge \int_{t-\tau}^{t} \dot{x}^t(s)ds\mathcal{R} \int_{t-\tau}^{t} \dot{x}(s)ds \\ &\quad + \int_{t-\varrho}^{t-\tau} \dot{x}^t(s)\mathcal{R}\dot{x}(s)ds \end{aligned} \tag{6.32}$$

$$\ge \int_{t-\varrho}^{t-\tau} \dot{x}(s)ds\mathcal{R} \int_{t-\varrho}^{t-\tau} \dot{x}(s)ds. \tag{6.33}$$

Then it follows that

$$\begin{aligned} J &\leq 2x^t(t)\mathcal{P}\big[BKx(t) + DKx(t-\tau) + \Gamma w(t)\big] \\ &\quad - (x(t) - x(t-\tau))^t \mathcal{R}(x(t) - x(t-\tau)) \\ &\quad - (x(t-\tau) - x(t-\varrho))^t \mathcal{R}(x(t-\tau) - x(t-\varrho)) \\ &\quad + x^t(t)(\mathcal{W} + \mathcal{S})x(t) - x^t(t-\varrho)\mathcal{W}x(t-\varrho) \\ &\quad - (1-\mu)x^t(t-\tau)\mathcal{S}x(t-\tau) + \varrho^2\dot{x}^t(t)\mathcal{R}\dot{x}(t) \\ &\quad + x^t(t)C^tCx(t) - \gamma^2 w^t(t)w(t). \end{aligned} \tag{6.34}$$

Setting

$$\zeta(t) = \mathrm{col}\{x(t), x(t-\varrho), \xi_j(t-\tau), w(t)\}$$

while expanding $\dot{x}(t)$ and using Schur complements, it follows that

$$J \leq \zeta^t(t)\Psi\zeta(t) \leq 0 \tag{6.35}$$

if the matrix inequality

$$\Psi = \begin{bmatrix} \Psi_o & 0 & \mathcal{R} & \mathcal{P}\Gamma & C^t & \varrho K^t B^t \\ \bullet & -\mathcal{R} - \mathcal{W} & \mathcal{R} & 0 & 0 & 0 \\ \bullet & \bullet & -\Psi_a & 0 & 0 & \varrho K^t D^t \\ \bullet & \bullet & \bullet & -\gamma^2 I & 0 & \varrho K^t \Gamma^t \\ \bullet & \bullet & \bullet & \bullet & -I & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\varrho\mathcal{R}^{-1} \end{bmatrix} < 0 \tag{6.36}$$

is feasible, where

$$\begin{aligned} \Psi_o &= \mathcal{P}BK + K^tB^t\mathcal{P} + \mathcal{W} + \mathcal{S} - \mathcal{R}, \\ \Psi_a &= 2\mathcal{R} + (1-\mu)\mathcal{S}. \end{aligned} \tag{6.37}$$

Applying the congruent transformation

$$T = \mathrm{diag}\{\mathsf{X},\ \mathsf{X},\ \mathsf{X},\ I,\ I,\ I\}, \quad \mathsf{P}^{-1} = \mathsf{X}.$$

Then using the change of variables

$$\mathsf{XRX} = \Psi_1, \quad \mathsf{XSX} = \Psi_2, \quad \mathsf{WRX} = \Psi_3, \quad K\mathsf{X} = \mathsf{Y}$$

along with the algebraic inequality $(\Psi_1 - \mathsf{X})\Psi_1^{-1}(\Psi_1 - \mathsf{X}) \geq 0$, we obtain LMI (6.12) as desired. Setting $w \equiv 0$, one gets the LMI condition implying $\dot{V} < 0$ which in turn guarantees the asymptotic stability of the closed-loop system (6.11). Thus, we conclude that the closed-loop system is asymptotically stable with $\mathcal{L}_2$-gain less than γ and this completes the proof. □

6.7.2 Proof of Theorem 6.2

Proof Considering the capacity constraint (6.4), it can be readily expressed in view of (6.5) as

$$G_k u < c_k, \quad k = 1, 2, \ldots, \ell \tag{6.38}$$

which depends on the control input u for some weighting matrix G_k. By considering the following ellipsoid for a selected $\omega > 0$

$$\Xi = \{x(t) | x^t(t) \mathsf{X}^{-1} x(t) \leq \omega, \ \mathsf{X} = \mathsf{X}^t > 0\}. \tag{6.39}$$

When the LMI (6.21) is feasible, it follows from $V(t)$ of (6.29) that

$$x^t(t) X^{-1} x_j(t) \leq V(t).$$

By integrating J in (6.30) over the period $0 \rightarrow t$ with $V(0) = 0$ subject to Theorem 6.1, it is not difficult to show that $J < 0$ corresponds to

$$\begin{aligned} V(t) &\leq -\int_0^t z^t(t) z(t)\, dt + \int_0^t \gamma w^t(t) w(t)\, dt \\ &\leq \gamma \int_0^\infty w^t(t) w(t)\, dt := \gamma W, \end{aligned} \tag{6.40}$$

where W is the upper bound energy of the external input $w(t)$. This shows that $x(t)$ belongs to an invariant set Ξ for all $t > 0$, if

$$\gamma W < \omega. \tag{6.41}$$

By using the controller gain $K = \mathsf{Y}\mathsf{X}^{-1}$, we write the state-feedback controller $u = \mathsf{Y}\mathsf{X}^{-1} x$, which leads to express (6.38) as

$$G_k \mathsf{Y}\mathsf{X}^{-1} x < c_k. \tag{6.42}$$

Equivalently

$$x^t (G_k \mathsf{Y}\mathsf{X}^{-1})^t (G_k \mathsf{Y}\mathsf{X}^{-1}) x < c_k^2. \tag{6.43}$$

Combining (6.39) and (6.43), we have

$$(G_k \mathsf{Y}\mathsf{X}^{-1})^t (\omega / c_k^2)(G_k \mathsf{Y}\mathsf{X}^{-1}) < \mathsf{X}^{-1}. \tag{6.44}$$

By applying the Schur complement, the capacity constraints can be cast into the following LMI conditions

$$I_1 := \gamma < \omega / W, \tag{6.45}$$

$$I_{2k} := \begin{bmatrix} \mathsf{X} & \mathsf{Y}^t G_k^t \\ \bullet & c_k^2 / \omega \end{bmatrix} \geq 0, \quad k = 1, \ldots, \ell. \tag{6.46}$$

Following parallel development, the constraint on the queue buffer size in (6.3) can be written as

$$Q_{dj}x < x_{\max_{dj}}, \tag{6.47}$$

where $x_{\max_{dj}} = q^d_{\max_j}$. Using (6.39) and (6.47), we have

$$Q^t_{dj}(\omega/x^2_{\max_{dj}})Q_{dj} < \mathsf{X}^{-1} \tag{6.48}$$

which is equivalent by the Schur complements to the LMI

$$I_3 := \begin{bmatrix} \mathsf{X} & Q_{dj} \\ \bullet & c_k^2/x_{\max_{dj}}/\omega \end{bmatrix} \geq 0. \tag{6.49}$$

It remains to look at the non-negativity constraint (6.2). Applying Lemma 6.1 to the closed-loop system (6.11) requires that the off-diagonal entries of BK and all entries of DK to be non-negative. For simplicity in exposition, we take $0 < \mathsf{X}$ in Theorem 6.1 to be diagonal. Considering $K = \mathsf{Y}\mathsf{X}^{-1}$, this requirement is translated to

$$\begin{aligned} I_4 &:= (B\mathsf{Y})_{jm} \geq 0, \quad j \neq m, \\ I_5 &:= (D\mathsf{Y})_{jm} \geq 0, \quad j, m = 1, \ldots, n. \end{aligned} \tag{6.50}$$

Observe that meeting the non-negativity condition $x \geq 0$ implies that the second non-negativity condition $u \geq 0$ can be easily satisfied by imposing $K_{jm} > 0$. Consequently, by using $K = \mathsf{Y}\mathsf{X}^{-1}$ with $0 < \mathsf{X}$ being diagonal matrix, the constraint (6.2) is satisfied if the following LMI condition holds

$$I_6 := \mathsf{Y}_{jm} > 0, \quad j, m = 1, \ldots, \ell(n-1). \tag{6.51}$$

The result of Theorem 6.1 in light of inequalities (6.45), (6.46) and (6.49)–(6.51) leads to the desired result. □

6.7.3 *Proof of Theorem 6.3*

Proof In the sequel, we introduce the following

- $\Sigma_u(j) \equiv$ the set whose elements are the upstream nodes of the node j,
- $\Sigma_d(j) \equiv$ the set whose elements are the downstream nodes of the node j

and the zero-value equations

$$\sum_{r\in\Sigma_d(j)} \left[x^t_j(t)2N^t_{ajr} + x^t_j(t-\tau_{rm})2N^t_{cjr}\right] \\ \times \left[x_j(t) - x_j(t-\tau_{rm}) - \int_{t-\tau_{rm}}^t \dot{x}(s)ds\right]$$

$$
\begin{aligned}
&= \sum_{r \in \Sigma_d(j)} \left[x_j^t(t)(-N_{ajr}^t) + x_j^t(t-\varrho_{rm})(-N_{cjr}^t)\right] \\
&\quad \times \left[x_j(t) - x_j(t-\tau_{rm}) - \int_{t-\varrho_{rm}}^{t} \dot{x}(s)ds\right] \\
&= \sum_{r \in \Sigma_d(j)} \left[x_j^t(t-\varphi_{rm})2S_{ajr}^t + x_j^t(t-\tau_{rm})2S_{cjr}^t\right] \\
&\quad \times \left[x_j(t-\varphi_{rm}) - x_j(t-\tau_{rm}) - \int_{t-\tau_{rm}}^{t-\varphi_{rm}} \dot{x}(s)ds\right] \\
&= \sum_{r \in \Sigma_d(j)} \left[x_j^t(t-\varphi_{rm})(-S_{ajr}^t) + x_j^t(t-\tau_{rm})(-S_{cjr}^t)\right] \\
&\quad \times \left[x_j(t-\varphi_{rm}) - x_j(t-\tau_{rm}) - \int_{t-\tau_{rm}}^{t-\varphi_{rm}} \dot{x}(s)ds\right] \qquad (6.52)
\end{aligned}
$$

for some free weighting matrices $N_{ajr},\ N_{cjr},\ S_{ajr}$ and S_{cjr}. Now to ensure an exponential stability, the following inequality should be preserved

$$
W_j := \frac{d}{dt}V_j(t) + a_j V_j(t) - b w_j^t(t) w_j(t) \leq 0 \qquad (6.53)
$$

for some scalars $a_j > 0, b_j > 0$. A straightforward mathematical manipulation computation using Schur complements yields

$$
\begin{aligned}
W_j &\leq 2x_j^t(t)P_j\dot{x}_j(t) + a_i x_i^t(t)Px_i(t) - bw^t(t)w(t) \\
&\quad + \sum_{m \in \Sigma_u(j)} \varrho_{jm}^2 \dot{x}_j^t(t) R_{1jm} \dot{x}_j(t) \\
&\quad + \sum_{m \in \Sigma_u(j)} \delta_{jm}^2 \dot{x}_j^t(t) R_{2jm} \dot{x}_j(t) \\
&\quad - \sum_{m \in \Sigma_u(j)} \varrho_{jm} e^{-a_i \varrho_{jm}} \int_{t-\varrho_{jm}}^{t} \dot{x}_j^t(s) R_{1jm} \dot{x}_j(s)ds \\
&\quad - \sum_{m \in \Sigma_u(j)} \delta_{ij} e^{-a_i \delta_{jm}} \int_{t-\varphi_{jm}}^{t} \dot{x}_j^t(s) R_{2jm} \dot{x}_j(s)ds \\
&\quad + \sum_{m \in \Sigma_u(j)} x_j^t(t)[S_{1jm} + S_{2jm} + E_{jm}]x_j(t) \\
&\quad - \sum_{m \in \Sigma_u(j)} x_j^t(t-\varrho_{jm}) S_{1jm} x_j(t-\varrho_{jm}) \\
&\quad - \sum_{m \in \Sigma_u(j)} x_j^t(t-\varphi_{jm}) S_{2jm} x_j(t-\varphi_{jm})
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{m \in \Sigma_u(j)} (1-\mu_{jm}) x_j^t(t-\tau_{jm}) E_{jm} x_j(t-\tau_{jm}) \\
&= \chi_j(t) \Pi_j \chi_j(t)
\end{aligned} \tag{6.54}
$$

for some vector χ_j and

$$
\Pi_j = \begin{bmatrix}
\pi_{j11} & \pi_{j12} & \pi_{j13} & \pi_{j14} & \pi_{j15} & \pi_{j16} \\
\bullet & \pi_{j22} & \pi_{j23} & \pi_{j24} & \pi_{j25} & \pi_{j26} \\
\bullet & \bullet & \pi_{j33} & \pi_{j34} & \pi_{j35} & \pi_{j36} \\
\bullet & \bullet & \bullet & \pi_{j44} & \pi_{j45} & \pi_{j46} \\
\bullet & \bullet & \bullet & \bullet & \pi_{j55} & \pi_{j56} \\
\bullet & \bullet & \bullet & \bullet & \bullet & \pi_{j66}
\end{bmatrix}, \tag{6.55}
$$

where

$$
\begin{aligned}
\pi_{j11} &= P_j B_j K_j + K_j^t B_j^t P_j + \sum_{r \in \Sigma_d(j)} (S_{1rj} + S_{2rj} + E_{rj} + a_j P) \\
&\quad + \sum_{r \in \Sigma_d(j)} (N_{ajr} + N_{ajr}^t), \\
\pi_{j12} &= [\pi_{j12s}\ \pi_{j12r}], \quad r \in \Sigma_d(j), \\
\pi_{j12s} &= \sum_{r \in \Sigma_d(j)} (-2N_{air} - 2S_{ajr} + N_{cjr}^t), \\
\pi_{j12r} &= P B_j K_j, \\
\pi_{j13} &= \sum_{r \in \Sigma_d(j)} (N_{ajr} + S_{ajr}), \\
\pi_{j14} &= \sum_{r \in \Sigma_d(j)} S_{ajr}, \\
\pi_{j15} &= \left[\sum_{r \in \Sigma_d(j)} N_{ajr} \ \sum_{r \in \Sigma_d(j)} S_{ajr} \right], \\
\pi_{j16} &= \left[P \Gamma_j \ C_j^t \ K_j^t B_j^t \sum_{r \in \Sigma_d(j)} (\underline{h}_{rj}^2 R_{1rj} + \delta_{rj}^2 R_{2rj}) \right], \\
\pi_{j22} &= \mathrm{diag}[\pi_{j22s}\ \pi_{j22r}], \quad r \in \Sigma_d(j), \\
\pi_{j22s} &= \sum_{r \in Q(i), i} (-2N_{cjr} - 2N_{cjr}^t - 2S_{cjr} - 2S_{cjr}^t), \\
\pi_{j22r} &= (1-\mu_{jm}) E_{jm} - 2N_{cjm} - 2N_{cij}^t - 2S_{cjm} - 2S_{cjm}^t.
\end{aligned} \tag{6.56}
$$

Also,

$$\pi_{j23} = \begin{bmatrix} \sum_{r\in\Sigma_d(j)}(N_{cjr} + S_{cjr}) \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$\pi_{j24} = \begin{bmatrix} \sum_{r\in\Sigma_d(j)} S_{cjr} \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$\pi_{j25} = \begin{bmatrix} \sum_{r\in\Sigma_d(j)} N_{cjr} & \sum_{r\in\Sigma_d(j)} S_{cjr} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, \tag{6.57}$$

$$\pi_{j26} = \begin{bmatrix} 0 & 0 & \pi_{j261} \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix},$$

$$\pi_{j261} = K_j^t B_j^t \sum_{r\in\Sigma_d(j)} (\varrho_{jr} R_{1jr} + \delta_{jr} R_{2jr})$$

and

$$\pi_{j33} = -\sum_{r\in\Sigma_d(j)} S_{1rj}, \quad \pi_{j34} = \pi_{35} = \pi_{36} = 0,$$

$$\pi_{j44} = -\sum_{r\in\Sigma_d(j)} S_{2rj}, \quad \pi_{j45} = \pi_{46} = 0,$$

$$\pi_{j55} = \mathrm{diag}\Bigg[-\sum_{r\in\Sigma_d(j)} e^{-a_r \underline{h}_{ir}} R_{1jr} - \sum_{r\,\Sigma_d(j)} e^{-a_r \delta_{jr}} R_{2jr}\Bigg],$$

$$\pi_{j56} = [0\;0\;0], \tag{6.58}$$

$$\pi_{j66} = \begin{bmatrix} -b_j & 0 & \pi_{j661} \\ \bullet & -I & 0 \\ \bullet & \bullet & \pi_{j662} \end{bmatrix},$$

$$\pi_{j661} = \Gamma_j^t \Bigg(\sum_{r\in\Sigma_d(j)} \varrho_{jr}^2 R_{1jr} + \delta_{jr}^2 R_{2jr}\Bigg),$$

$$\pi_{j662} = -\sum_{r\in\Sigma_d(j)} (\varrho_{ir}^2 R_{1jr} + \delta_{jr}^2 R_{2jr}). \tag{6.59}$$

To determine the controller gains K_j, $j = 1, 2, \ldots, m$, we per- and post-multiply the LMI (6.55) with diag$[I, I, I, \ldots, \Lambda_j]$, where

$$\Lambda_j = \left[\sum_{r \in \Sigma_d(j)} (\varrho_{jr}^2 R_{1jr} + \delta_{jr}^2 R_{2jr}) \right]^{-1}.$$

Then pre- and post-multiply the resulting LMI by diag$[X_j, X_j, X_j, \ldots, I, I, X_j]$, $X_j = P_j^{-1}$ with $X_j K_j = Y_j$ and invoking the inequality $-ZFZ \leq F^{-1} - 2Z$, the LMI (6.21) is obtained. Setting $\gamma_j^2 = b_j$ and subject to the theorem, the network model (6.19) is exponentially stabilizable by the decentralized state feedback controllers of the form $u_j = K_j x_j$, with an $\mathcal{L}_2$-gain less than γ_j and the proof is completed. □

6.7.4 Proof of Theorem 6.4

Proof The capacity constraint for each subsystem can be defined as

$$G_{kj} u_j < c_{kj}, \quad kj = 1, \ldots, \ell j, \; j = 1, \ldots, n. \tag{6.60}$$

By considering the following ellipsoid for a selected $\omega_j > 0$

$$\Xi_j = \{x_j(t) | x_j^t(t) X^{-1} x_j(t) \leq \omega_j, \; X_j = X_j^t > 0\}. \tag{6.61}$$

When the LMI (6.12) is feasible and proceeding in line with Theorem 6.2, it follows that the capacity constraints can be cast into the following LMI conditions

$$I_{j1} := \gamma_j < \omega_j / W_j, \tag{6.62}$$

$$I_{2jk} := \begin{bmatrix} X_j & Y^t G_{jk}^t \\ \bullet & c_{kj}^2 / \omega_j \end{bmatrix} \geq 0, \quad k = 1, \ldots, \ell. \tag{6.63}$$

In a similar way, the constraint on the queue buffer size in (6.3) at the node j can be cast into the LMI format

$$I_{j3} := \begin{bmatrix} X_j & Q_{djj} \\ \bullet & c_{kj}^2 / x_{\max_{dj}} / \omega_j \end{bmatrix} \geq 0. \tag{6.64}$$

The non-negativity constraint (6.2) can be stated as

$$I_{j4} := (B_j Y_j)_{pq} \geq 0, \quad p \neq q, \tag{6.65}$$

$$I_{j5} := (D_j Y_j)_{pq} \geq 0, \quad p, q = 1, \ldots, n, \tag{6.66}$$

$$I_{j6} := Y_{j_{pq}} > 0, \quad p, q = 1, \ldots, \ell(n-1), \; j = 1, \ldots, n. \tag{6.67}$$

The result of Theorem 6.2 in light of inequalities (6.62)–(6.67) leads to the desired result. □

6.8 Discrete-Time Dynamic Routing

In this section, we direct attention to the presentation of decentralized controller for dynamic routing in multi-destination large-scale data communication networks using discrete-time format. Although this section is the discrete counterpart of the previous one, the analysis and topics are treated in a distinct way.

An important problem in the operation of data communication networks is the routing of messages. Typically, a data communication network consists of many nodes which are connected through a number of links. The routing problem is to direct messages from one node to another, through such links. until they reach their desired destination.

Generally speaking, the amount of messages entering a network at various nodes may vary from time to time, a dynamic routing strategy, which can adopt to such variations, is required. In addition, it is often the case that the number of nodes in a network is large; in this case the vast number of different possible paths from one node to another, makes it virtually impossible to implement a centralized controller. As we noted before, centralized controllers are vulnerable to failures in the network and introduce a large communication overhead on the network. Decentralized controllers, which can be implemented locally at individual nodes, and which require a minimum amount of information from the other nodes, are desirable to implement in practice.

Early routing algorithms, such as those implemented in the ARPANET [25] and TYMNET [34], were based on finding the *shortest path* from the initial node to the destination node [7, 16]. In these algorithms, the *length* of a path is usually taken to be proportional to the message flow rate on that path. Most of these algorithms could be implemented in a decentralized way in the sense that the computations can be done locally; however, for dynamic routing, they require excessive information transfer inside the network (all the nodes must be informed about the changing link lengths). Other algorithms have been proposed to improve the network performance by minimizing a cost function related to the *link congestion* (message flow rate on a link relative to the capacity of that link) [11, 26, 35]. For dynamic routing, however, these algorithms also require excessive information exchange.

6.8.1 Routing Algorithms

A routing algorithm which uses distributed computation has been introduced in [12]; this algorithm considers an optimal routing problem to minimize a generic measure of link flows (that is, a more general definition of link congestion which is also related to the *total delay*, is minimized) and produces a solution which converges to the optimal solution under certain assumptions. Various computational and practical aspects of similar algorithms have also been extensively discussed [6]. A key assumption of these algorithms is the so-called *quasi-static assumption* which states that the external traffic arrival rates for each origin-destination pair is stationary over time. Therefore, although these algorithms may be successful when the external traffic arrival rates remain approximately constant over time, they may fail to produce satisfactory results when the rates change appreciably within a relatively short time. Furthermore, in general,this class of algorithms (i.e. the shortest path algorithms and the algorithms based on link congestion minimization) are suitable only when the total traffic arrival rate into a network is small compared to its total capacity; i.e. such algorithms usually require separate flow control algorithms to deal with situations where congestion occur [33].

The problem of determining a routing controller which minimizes a measure of the total queue length of the network was studied by [27]; it may be argued that such a measure reflects the overall performance better than a measure based on link congestion. Furthermore, algorithms based on minimizing such measures can also work well under congestion. A conceptual algorithm is presented in [27] to compute the optimal control for minimizing a measure of the queue length of a network; however, it was stated that the implementation of such an algorithm may not be possible to achieve in the most general case due to the computational complexity required and a number of other problems. A number of decentralized routing algorithms based on minimizing a measure of the queue length were reported by a number of researchers for certain special cases. In [30] a decentralized routing algorithm is presented based on minimizing a measure of the queue length and the total travel time (transportation networks rather than communication networks were considered). The algorithm, however, is valid only for single destination networks. Multi-destination networks were considered within the same context by [30]; however, in this case it was assumed that the total flow rate entering a node is constant. A decentralized controller for the general case was proposed by [4, 14]; the structure of this controller was motivated by the structure of an optimal controller which minimizes a measure of the total queue length. Although this controller may not produce optimal performance in general. it is easy to design and simple to implement.

A different approach for designing a routing controller was later undertaken in [14]; an off-line optimization approach was proposed to determine a decentralized controller for a network whose dynamics can be modeled as a continuous-time system. In the present book, this approach is extended to the discrete-time case. This case is important from a practical point of view, since the implementation of routing control strategies is done generally in discrete-time. The dynamic model used in this book (which is a discrete-time version of the model developed by [14, 15]) can incorporate processing delays. The basic aim of the introduced approach is to maximize the magnitude of each external message arrival rate which may occur in the system, without violating any constraints on the system. It is shown that this problem can be formulated as a linear programming problem, and can be solved off-line. The proposed controller is decentralized in the sense that all the on-line computations are done locally at the nodes without any information transfer from the other nodes. This controller guarantees stability and clears the queues of the network in the absence of external message arrivals. It also keeps the queue lengths bounded in the presence of external message arrival rates, which do not exceed a certain maximum rate. The controller also avoids looping of messages. Furthermore, it can cope with congestion, that is, no separate flow control algorithms are needed.

6.8.2 Network Dynamics and Assumptions

Consider a data communication network consisting of N nodes. The nodes are connected through directed links on which messages can be transmitted. Each node receives messages from both upstream nodes inside the network and from outside the network. The node on which a message enters the network is said to be the *source node* of this message. Each message also has another node, called the *destination node*, associated with it. Messages are absorbed as soon as they arrive at their destination. Messages arriving at a node other than their destination are put into a queue (or *buffer*) and eventually are sent out to a down-stream node.

Assumption 6.2 For simplicity in exposition, it is assumed that

1. The rate of messages being sent out from one node to another (the control signals) are updated at discrete periodic instants, which is the usual case in practice. Without loss of generality this period is taken to be unity.
2. A *processing delay* occurs at each node before an arriving message can be put into a queue and then sent out. Such a delay necessarily occurs in practical systems. This delay is the total time needed for tasks such as receiving a message from an upstream node or from outside the network, identifying its destination, placing it into the appropriate queue, and performing necessary calculations for routing. Some of these tasks may eventually be handled in parallel. Note that each type of message, associated with a particular origin-destination pair at the node. may have a different processing delay.
3. For technical ease, initially the processing delay times are considered fixed; however, it will be later shown that the main results remain valid even if these delay times are time-varying, but bounded. It is to be noted that the delay due to waiting at the input to a transmission link (that is, the *queuing delay*) is separately considered in the model. In the following formulation, the processing delay for a particular message is the time it spends in the $p_i^{\ell k}$ queue, and the queuing delay is the time it spends in the $m^l k_i$ queue.
4. The *propagation delay* along a link (that is, the transport time of a signal on any link) may be included in the processing delay of the node at the receiving end. The *transmission delay* (that is, the time between starting and ending the transmission of an individual message), on the other hand, is taken care of in two different ways in our model. If the message can be sent out from the present node immediately after it is processed, then this delay is included in the processing delay of the transmitting node. If, on the other hand, the message (or at least a part of it) has to wait in the $m_\ell k_j$ queue (possibly due to congestion in the out-going links) then at least a part of the transmission delay is included in the queuing delay. The transmission delay is, of course, dependent on the message length; but it is bounded as long as the message lengths are bounded.

5. The control signals are updated synchronously throughout the network.

In view of the forgoing assumptions, the queue dynamics at node j for messages with source ℓ and destination k can be described as follows:

$$q^{\ell k}(t+1) = q_j^{\ell k}(t) + \delta_{j\ell} f_j^k(t) + \sum_{m\in\mathcal{U}(j)} u_{mj}^{\ell k}(t) - \sum_{j\in\mathcal{D}(j)} u_{jm}^{lk}(t), \quad (6.68)$$

$$m_j^{lk}(t+1) = m_j^{\ell k}(t) + \delta_{j\ell} f_i^k(t - r_j^{\ell k}) + \sum_{m\in\mathcal{U}(j)} u_{mj}^{lk}(t - r_j^{\ell k}) - \sum_{m\in\mathcal{D}(j)} u_{jm}^{lk}(t), \quad (6.69)$$

where

- $m_j^{\ell k}(t)$ is the volume of processed messages with source i and destination k, waiting at node j at time t, $j \in \bar{N}^k$, $\ell \in \bar{N}^k$, where $\bar{N} \leq \{1, 2, \ldots, N\}$, $\bar{N}^k \leq \bar{N} \backslash \{k\}$ (the queuing delay is the time a message spends in this queue).
- $q_j^{\ell k}(t) \leq p_j^{\ell k}(t) + m_j^{\ell k}(t)$ is the total volume of messages with source j and destination k, either being processed or waiting at node j at time t, $j \in \bar{N}^k$, $\ell \in \bar{N}^k$, $k \in \bar{N}$.
- $f_j^k(t)$ is the arrival rate of messages with destination k, entering the network at node i at time t, $j \in \bar{N}^k$, $k \in \bar{N}$.
- $u_{jm}^{\ell k}(t)$ is the flow rate of messages with source ℓ and destination k, sent out from node j to the downstream node m along the link i to j at time t, $j \in \mathcal{D}(i)$, $i \in \bar{N}^k$, $l \in \bar{N}^k$, $k \in \bar{N}$.
- $r_j^{\ell k}$ is the processing delay at node j for messages with source ℓ and destination k, $j \in \bar{N}^k$, $\ell \in \bar{N}^k$, $k \in \bar{N}$.
- $\mathcal{V}(j)$ and $\mathcal{D}(j)$ are respectively the sets of adjacent upstream and downstream nodes of node j, $j \in \bar{N}$; that is,

$$\mathcal{V}_j := \{m \mid \text{there exists a link from } m \text{ to } j\},$$
$$\mathcal{D}_j := \{m \mid \text{there exists a link from } j \text{ to } m\}$$

and

$$\delta_{j\ell} \leq \begin{cases} 1 & \text{if } j = l, \\ 0 & \text{if } i \neq l. \end{cases}$$

Instead of either (6.68) or (6.69), we could also use:

$$p_j^{lk}(t+1) = p_j^{\ell k}(t) + \delta_{j\ell}[f_j^k(t) - f_j^k(t - r_j^{\ell k})] + \sum_{m\in\mathcal{U}(j)} [u_{mj}^{\ell k}(t) - u_{mj}^{\ell k}(t - r_j^{\ell k})], \quad (6.70)$$

where

- $p_j^{\ell k}(t)$ is the volume of messages with source ℓ and destination k, being processed at node j at time t, $j \in \bar{N}^k$, $\ell \in \bar{N}^k$, $k \in \bar{N}$ (the processing delay is the time a message spends in this queue).

In the sequel, we let $q \leq \{q_j^{\ell k}\}$ be the $N(N\ell)^2$ dimensional vector of total volume of messages, $m \leq \{m_j^{\ell k}\}$ be the $N(N-\ell)^2$ dimensional vector of volume of processed messages, $p \leq \{p_j^{\ell k}\}$ be the $N(N\ell)^2$ dimensional vector of volume of messages presently being processed, where m, p, and q have the same index structure. Let $f \leq \{f_j^k\}$ be the $N(N\ell)$ dimensional vector of external message arrival rates, $u \leq \{u_{jm}^{\ell k}\}$ be the $L(N\ell)^2$ dimensional vector of message flow rates along the links, where L is the total number of links of the network, and let

$$\tilde{f}(t) \leq \{\tilde{f}_j^{\ell k}\} \leq \mathbf{F} f(t) + \mathbf{G}_j u(t) \tag{6.71}$$

be an $N(N-\ell)^2$ dimensional vector which has the same index structure as q, where F is a matrix consisting of a 1 corresponding to $\tilde{f}_j^{\ell k}$ on the column corresponding to f_j^k ($j \in \bar{N}^k$, $k \in \bar{N}$), and 0's elsewhere, and $\mathbf{G}_j$, is a matrix consisting of 1's corresponding to $u_{mj}^{\ell k}$'s with $m \in \mathcal{U}(j)$ on the row corresponding to $\tilde{f}_j^{\ell k}$ ($j \in \bar{N}^k$, $l \in \bar{N}^k$, $k \in \bar{N}$), and 0's elsewhere. Then model (6.68)–(6.70) can be compactly written as:

$$q(t+1) = q(t) + \mathbf{F} f(t) + \mathbf{G} u(t), \tag{6.72}$$

$$m(t+1) = m(t) + \sum_{r=1}^{r_{\max}} \mathbf{D}_r \tilde{f}(t-r) - \mathbf{G}_o u(t), \tag{6.73}$$

$$p(t+1) = p(t) + \tilde{f}(t) - \sum_{r=1}^{r_{\max}} \mathbf{D}_r \tilde{f}(t-r), \tag{6.74}$$

where $\mathbf{G}_o$ is a matrix consisting of 1's corresponding to $u_{jm}^{\ell k}$'s with $j \in \mathcal{D}(j)$ on the row corresponding to $m_j^{\ell k}$ ($j \in \bar{N}^k$, $\ell \in \bar{N}^k$, $k \in \bar{N}$), and 0's elsewhere, $\mathbf{G} \leq \mathbf{G}_j - \mathbf{B}_o$, $r_{\max} \leq \max_{j \in \bar{N}^k, \ell \in \bar{N}^k, k \in \bar{N}} \{r_j^{\ell k}\}$ and $\mathbf{D}_r$ ($r = 1, 2, \ldots, r_{\max}$) are diagonal matrices containing a l at the diagonal position corresponding to $m_j^{\ell k}$ if $r_j^{\ell k} = r$ and containing a 0 otherwise ($j \in \bar{N}^k$, $\ell \in \bar{N}^k$, $k \in \bar{N}$).

The objective of the routing control is to determine a suitable controller for u (that is to control the flow rates along the links) to regulate the queue lengths q and m in the presence of f. To reflect practical issues, certain *routing control constraints* must be recalled:

1. The message arrival rates into the network must be non-negative:

$$f_j^k(t) \geq 0, \quad \forall t \geq t_0, \ \forall k \in \bar{N}.$$

2. The queue length of messages being processed at a node cannot be negative:

$$p_j^{\ell k}(t) \geq 0, \quad \forall t \geq t_0, \ \forall j \in \bar{N}^k, \ \forall \ell \in \bar{N}^k, \ \forall k \in \bar{N}.$$

3. The queue length of processed messages cannot be negative:

$$m_j^{\ell k}(t) \geq 0, \quad \forall t \geq t_0,\ \forall j \in \bar{N}^k,\ \forall \ell \in \bar{N}^k,\ \forall k \in \bar{N}.$$

4. The message flow rates cannot be negative:

$$u_{jm}^{\ell k}(t) \geq 0 \quad \forall m \in \mathcal{D}(j),\ \forall t \geq t_0,\ \forall j \in \bar{N}^k,\ \forall \ell \in \bar{N}^k,\ \forall k \in \bar{N}.$$

5. The total message flow rate along a link cannot exceed the capacity $c_{jm} > 0$ of that link:

$$\sum_{k\in\bar{N}} \sum_{\ell\in\bar{N}^k} u_{jm}^{\ell k}(t) \leq c_{jm}, \quad \forall t \geq t_0,\ \forall m \in \mathcal{D}(j),\ \forall i \in \bar{N},$$

where t_0 denotes the initial time.

Remark 6.6 It is interesting to note that forgoing constraint 1 is satisfied naturally since no one can insert a negative amount of message volume into the network. Constraint 2 is also automatically satisfied as long as constraints 1 and 4 are satisfied, since

$$p_j^{\ell k}(t) = \sum_{\tau=t-r_j^{\ell k}}^{t-1} \tilde{f}_j^{\ell k}(\tau). \tag{6.75}$$

Note that constraints 2 and 3 imply $q_j^{\ell k} \geq 0$, $\forall t > t_0$, $\forall j \in \bar{N}^k$, $\forall \ell \in \bar{N}^k$, $\forall k \in \bar{N}$. A control strategy must therefore be chosen such that constraints 3–5 are satisfied at all times.

Remark 6.7 It must be noted that, in a practical situation, there may also be constraints imposed on the volume of messages that can be processed or buffered, that is, upper constraints on $p_j^{\ell k}$ and $m_j^{\ell k}$. We assume that such limits are sufficiently high so that such constraints are not violated. This assumption is justified, since in view of the available computing power, storage restrictions, that is, constraints on $p_j^{\ell k}$ and $m_j^{\ell k}$ in our case are usually less important than transmission capacity constraints, that is, constraints on $u_{jm}^{\ell k}$ in our case.

By dropping out that processing delays, the queue dynamics can be described as:

$$q(t+1) = q(t) + \mathbf{F} f(t) + \mathbf{G} u(t), \tag{6.76}$$

$$m(t) = q(t), \tag{6.77}$$

$$p(t) = 0. \tag{6.78}$$

Definition 6.1 A network with N nodes is said to be *connected* if $\forall k \in \bar{N}$ and $\forall \ell \in \bar{N}^k$, there exists a set $\{j_1, j_2, \ldots, j_m\} \subset \bar{N}^k$ such that $j_1 = 1$, $j_2 \in \mathcal{D}(j_1)$, $j_3 \in \mathcal{D}(j_2)$, $\ldots$, $j_m \in \mathcal{D}(j_m)$.

6.8.3 Routing Control Algorithm

First we consider a decentralized controller of the form:

$$u(t) = \hat{u}(t), \quad t \geq t_0, \tag{6.79}$$

where

$$\hat{u}(t) = \Theta\left[m(t) + \sum_{r=1}^{r_{\max}} \mathbf{D}_r \tilde{f}(t-r)\right], \quad t \geq t_0 \tag{6.80}$$

and Θ is an $L(N-1)^2 \times N(N-1)^2$ dimensional matrix which has the following properties:

1. All elements of Θ are non-negative;
2. $\mathbf{G}_o\Theta = \mathbf{I}$ where $\mathbf{I}$ denotes the identity matrix;
3. For any $N(N-\ell)^2 \times N(N\ell)^2$ dimensional diagonal matrix $\hat{\mathbf{D}}$, all eigenvalues of $G_j\Theta\hat{D}$ are contained in the interior of the disk centered at the origin with a radius equal to the maximum singular value of $\hat{\mathbf{D}}$;
4. Θ is a decentralized feedback matrix in the sense that $u = \Theta q$ implies $u_{jm}^{\ell k} = h_{jm}^{\ell k} q_j^{\ell k}$ (where $h_{jm}^{\ell k} \geq 0$ indicates the appropriate element of Θ) $\forall m \in \mathcal{D}(j)$, $\forall j \in \bar{N}^k$, $\forall \ell \in \bar{N}^k$, $\forall k \in \bar{N}$.

Remark 6.8 We observe that property 1 is needed to ensure that the calculated link flows are non-negative. Property 2 is required to ensure that all the messages received at a node (except those whose destination is the present node) are eventually sent to at down-stream node and that the calculated downstream link flows are not larger than the volume of the processed messages present at that node. Property 3 is a sufficient condition to guarantee system stability; and property 4 is required to achieve decentralization.

Necessary and sufficient conditions for the existence of a matrix Θ satisfying the above properties are given by the following lemma:

Lemma 6.2 *Consider a network described by* (6.72)–(6.74); *then there exists a matrix Θ satisfying properties* 1–4 *if and only if the network is connected.*

Proof To prove the **if** part, we will construct a specific Θ and show that it satisfies properties 1–4. Assuming that the network is connected and define a non-cyclic path

$$j = j_0, \ j_1, \ j_2, \ \ldots, \ j_m = k \tag{6.81}$$

from each node j to each other node k in the network. These paths are defined such that there exists a link with positive capacity from each node j_m to the next node

j_{m+1} on the path, and such that if the path from j to k is defined as above, then the path from j_m to k for any j_m on the above path is

$$j_m, \ j_{m+1}, \ j_{m+2}, \ \ldots, \ j_{m_m} = k.$$

Introduce

$$h_{jm}^{\ell k} \le \begin{cases} 1 & \text{if } m \text{ is the first node on the path from } j \text{ to } k, \\ & \forall m \in \mathcal{D}(j), \ \forall j \in \bar{N}^k, \ \forall \ell \in \bar{N}^k, \ \forall k \in \bar{N}, \\ 0 & \text{otherwise} \end{cases}$$

and let $h_{jm}^{\ell k}$ be the element of Θ which relates $q_j^{\ell k}$ to $u_{jm}^{\ell k}$ in the equation $u = \Theta q$ ($m \in \mathcal{D}(j)$, $j \in \bar{N}^k$, $\ell \in \bar{N}^k$, $k \in \bar{N}$); let all other elements of Θ be zero.

Clearly Θ satisfies the foregoing properties 1 and 4. To show that Θ satisfies property 2, consider the relation $\hat{q} = \mathbf{G}_o u$, where $\hat{q}$ has the same index structure as q, and let $u = \Theta q$. Then, for each $j \in \bar{N}^k$, $\ell \in \bar{N}^k$, $k \in \bar{N}$, we obtain:

$$\hat{q}_j^{\ell k} = \sum_{m \in \mathcal{D}(j)} u_{jm}^{lk} = \sum_{m \in \mathcal{D}(j)} h_{jm}^{\ell k} q_j^{lk} = q_j^{\ell k}$$

since for each triple (j, ℓ, k), there exists exactly $m \in \mathcal{D}(j)$ such that $h_{jm}^{\ell k} = 1$ and $h_{jm}^{\ell k} = 0$ for all other $m \in \mathcal{D}(j)$. The above relation therefore implies that $\mathbf{G}_o \Theta = \mathbf{I}$.

Next to show that Θ satisfies property 3, consider the equations $\hat{q} = \mathbf{G}_j u$, $u = \Theta \bar{q}$, $\bar{q} = \hat{\mathbf{D}} q$, where $\hat{\mathbf{D}}$ is a diagonal matrix and $\hat{q}$ and $\bar{q}$ have the same index structure as q. Then, for each $j \in \bar{N}^k$, $\ell \in \bar{N}^k$, $k \in \bar{N}$, we obtain:

$$\bar{q}_j^{\ell k} = \sum_{m \in \mathcal{V}(j)} u_{mj}^{\ell k} = \mathcal{V}(j) h_{mj}^{\ell k} d_m^{\ell k} q_m^{\ell k},$$

where $d_j^{\ell k}$ the diagonal element of $\hat{\mathbf{D}}$ corresponding to $q_i^{\ell k}$ in $\bar{q} = \hat{\mathbf{D}} q$ ($j \in \bar{N}^k, \ell \in \bar{N}^k, k \in \bar{N}$). Observe that $j \notin \mathcal{U}(i)$; it follows from the above equation that $\mathbf{G}_j \Theta \hat{\mathbf{D}}$ consists of 0's on the diagonal. Furthermore, if the following condition (called condition A): $k \in \mathcal{D}(j)$ and $h_{jk}^{\ell k} = 1$, for some (j, ℓ, k) holds, then all non-diagonal elements on the column of $\mathbf{G}_j \Theta \hat{\mathbf{D}}$ which corresponds to $q_j^{\ell k}$ are zero; if condition A does not hold, there exists exactly one $m \in \mathcal{D}(j)$, $(m \neq k)$ for which $h_{jm}^{\ell k} = 1$, and in this case the sum of the absolute values of the non-diagonal elements on the column of $\mathbf{G}_j \Theta \hat{\mathbf{D}}$ which corresponds to $q_j^{\ell k}$ is equal to $|d_j^{\ell k}|$ ($j \in \bar{N}^k, \ell \in \bar{N}^k, k \in \bar{N}$). Therefore, by Gershgorin's theorem, the eigenvalues of $\mathbf{G}_j \Theta \hat{\mathbf{D}}$ are contained in a closed disk of radius equal to the maximum singular value of $\hat{\mathbf{D}}$, centered at the origin on the complex plane. To complete the proof, it needs to show that the eigenvalues of $\mathbf{G}_j \Theta \hat{\mathbf{D}}$ cannot be on the boundary of this disk. We do this by contradiction. Without loss of generality assume that the maximum singular value of $\hat{\mathbf{D}}$ is one, and that $\mathbf{G}_j \Theta \hat{\mathbf{D}}$ has an eigenvalue at $\exp(j\theta)$ for some $\theta \in [0, 2\pi)$, where $j = \sqrt{-1}$.

Then this implies that there must exist scalars a_i^{lk} $(i \in \bar{N}^k, \ell \in \bar{N}^k, k \in \bar{N})$ at least one being non-zero, such that

$$\sum_{k\in\bar{N}} \sum_{\ell\in\bar{N}^k} \sum_{j\in\bar{N}^k} a_j^{\ell k} \beta_j^{\ell k} = 0,$$

where $\beta_j^{\ell k}$ denotes the row of $\exp(m\theta)\mathbf{I} - \mathbf{G}_j \Theta \hat{\mathbf{D}}$ which corresponds to $\hat{q}_j^{\ell k}$ $(j \in \bar{N}^k$, $\ell \in \bar{N}^k$, $k \in \bar{N})$. Note that $\beta_j^{\ell k}$ contains $\exp(m\theta)$ at the position corresponding to $q_j^{\ell k}$. If $k \in \mathcal{D}(j)$ and $h_{jm}^{\ell k} = 1$, then all the other rows contain a zero at the same position and hence we must have $a_j^{\ell k} = 0$. If $k \notin \mathcal{D}(j)$ or $h_{jk}^{\ell k} = 0$, then there exists a path

$$j = j_0,\ j_1,\ j_2,\ \ldots,\ j_{m-1},\ j_m = k$$

from j to k, as constructed in (6.80), such that $\beta_{j_{m+1}}^{\ell k}$ contains $d_j^{\ell k}$ at the position corresponding to $q_{j_m}^{\ell k}$; at this position $\beta_{j_m}^{\ell k}$ contains $\exp(m\theta)$, and all other $\beta_j^{\ell k}$'s contain a zero at this same position, which gives

$$a_{j_m}^{\ell k} = a_{j_{m+1}}^{\ell k} d_j^{\ell k} \exp(-m\theta).$$

Since, by the above argument, $a_{j_{m-1}}^{\ell k} = 0$, by extending the above procedure, we conclude that

$$a_j^{\ell k} \le a_{j_0}^{\ell k} = a_{j_1}^{\ell k} = a_{j_2}^{\ell k} = \cdots = a_{j_{m-1}}^{\ell k} = 0.$$

On repeating the above argument for all j, ℓ, k, we conclude that $a_j^{\ell k} = 0, \forall j \in \bar{N}^k, \forall \ell \in \bar{N}^k, \forall k \in \bar{N}$, which is a contradiction. Hence, the eigenvalues of $\mathbf{G}_j \Theta \hat{\mathbf{D}}$ are contained in the interior of the above considered disk, which completes the proof.

To prove the **only if** part, we will show that if the given network is not connected then it is not possible to satisfy both properties 2 and 3 at the same time. If the given network is not connected, then there exists at least one pair (s, r) for which there exists no path from s to r. Let $\hat{q} = \{\hat{q}_i^{lk}\}$ be such that

$$\hat{q}_i^{lk} = \begin{cases} 1 & \text{if } \ell = s \text{ and } k = r \text{ and } s \sim j, \\ 0 & \text{otherwise,} \end{cases}$$

where $s \sim j$ means that either $j = s$ or there exists a path from r to j or from j to s. Note that $\hat{q}_s^{sr} = 1$ and hence $\hat{q} \ne 0$. On noting that the equation $u = \mathbf{G}^t q$ gives

$$\hat{u}_{jm}^{\ell k} = \begin{cases} q_m^{\ell k} - q_j^{\ell k} & \text{if } m \ne k, \\ -q_j^{\ell k} & \text{if } m = k \end{cases}$$

it is seen that $\mathbf{G}^t q = 0$. Therefore, $\text{rank}(\mathbf{G}) < N(N\ell)^2$, which means that $\mathbf{I} + \mathbf{G}\Theta$ must have at least one eigenvalue at one for any Θ. Assuming that Θ satisfies property 2, this implies that $\mathbf{G}_j \Theta$ must have at least one eigenvalue at one, which implies that property 3 is not satisfied for $\hat{\mathbf{D}} = \mathbf{I}$. Therefore, if the given network is not connected, there can exist no Θ which will satisfy properties 2 and 3 which concludes the proof. □

Remark 6.9 It is important to note that the quantity inside the square brackets in (7) is the vector of total message volume which can be sent out at time t; i.e. the messages are sent out as they become available. Therefore, a knowledge of the processing delays r_i^{lk} is not needed for the actual implementation of the controller. Furthermore, since the only information required to compute the control u_{ij}^{lk} $(j \in \mathcal{D}, l \in \bar{N}, k \in \bar{N}^i)$ is related to the queue lengths at node i $(i \in \bar{N})$, the controller can be implemented in a decentralized way at each node, that is, no information transfer between the nodes is needed and the calculations can be carried out at individual nodes.

In order to calculate the controls and to direct messages accordingly, the controller at a particular node needs to know the destination and the source of each message that is present at that node. Therefore, as a message enters the network, two *marks*, one indicating its source and the other indicating its destination, must be added to this message.

Definition 6.2 Given a connected network with N nodes described by (6.72)–(6.74), let

$$w_r(t) = \tilde{f}(t-r), \quad r = 1, 2, \ldots, r_{\max},$$
$$x = \left[m^t \; w_1^t \; \ldots \; w_{r_{\max}}^t\right]^t,$$

where $N^* \le (r_{\max} + 1)N(N-1)^2$, and assume that $f(t) = 0$ for all $t \ge t_0$, and that a feedback controller (possibly nonlinear, time-varying) is applied. Then the closed loop system is said to be *globally asymptotically stable* if for all $t_0 \in \mathbf{Z}$ and for all $x(t_0) \in \Re_+^{N^*}$,

$$\lim_{t \to \infty} x(t) = 0,$$

where $\mathbf{Z}$ denotes the set of integers and $\Re_+^n$ denotes the set of n-dimensional real vectors with non-negative entries.

The following two theorems present some important properties of the controller (6.79). The first result shows that routing control constraints (b), (c), and (d) are always satisfied for any external message arrival rates into the network, and that the queue length m of the processed messages is driven immediately to zero using controller (6.79).

Theorem 6.5 *Consider a connected network with N nodes described by* (6.72)–(6.74). *Assume that the controller described by* (6.79) *is applied where the matrix Θ satisfies the properties* 1–4. *Then for all $t_0 \in \mathbf{Z}$, for all $x(t_0) \in \Re_+^{N^*}$, and for all $f(t) \in \Re_+^{N(N-1)}$;*

(i) $m_i^{lk}(t) \geq 0$, $\forall t \geq t_0$, $\forall i \in \bar{N}^k$, $\forall l \in \bar{N}^k$, $\forall k \in \bar{N}$; *in particular*, $m(t) = 0$, $\forall t \geq t_0 + 1$;
(ii) $p_i^{lk}(t) \geq 0$, $\forall t \geq t_0$, $\forall i \in \bar{N}^k$, $\forall l \in \bar{N}^k$, $\forall k \in \bar{N}$; *and*
(iii) $u_{ij}^{lk}(t) \geq 0$, $\forall t \geq t_0$, $\forall j \in \mathcal{D}(i)$, $\forall i \in \bar{N}^k$, $\forall l \in \bar{N}^k$, $\forall k \in \bar{N}$.

Proof To prove this theorem, we will use the system (6.72)–(6.74), the controller (6.79) and (6.80) and the properties 1–4 of the controller. Substituting (6.79) and (6.80) into (6.72) and using property 2, gives $m(t+l) = 0$, which implies that $m(t) = 0$, $\forall t \geq t_0 + 1$. Hence, given $m(t_0) \geq 0$, this implies that (i) holds.

Note that (6.79) and (6.80) implies

$$u(t) = \Theta\left[m(t) + \sum_{r=1}^{r_{\max}} \mathbf{D}_r w_r(t) \right].$$

Therefore, since Θ satisfies (P1) and $x(t_0) \in \Re_+^{N^*}$ this implies that $u_{ij}^{lk}(t_0) \geq 0$, $\forall j \in \mathcal{D}(i)$, $\forall i \in \bar{N}^k$, $\forall l \in \bar{N}^k$, $\forall k \in \bar{N}$. This, together with $f(t_0) \in \Re_+^{N(N-1)}$, implies that $w_1(t_0+1) = \tilde{f}(t_0) \in \Re_+^{N(N-1)^2}$. On noting now that

$$w_r(t+1) = w_{r-1}(t), \quad r = 2, 3, \ldots, r_{\max},$$

it follows that $x(t0 + l) \in \Re_+^{N^*}$. On repeating this argument, it is concluded that (iii) holds.

Given that $\tilde{f}(t_0 - r) = w_r(t_0) \in \Re_+^{N(N-1)^2}$ $(r = 1, 2, \ldots, r_{\max},\ f(t) \in \Re_+^{N(N-1)}$ $\forall t \geq t_0)$, and that (iii) holds, (ii) now directly follows from property 4. □

The following result shows that the system (6.72)–(6.74) controlled by controller (6.79) is stable, that the queue length m of messages is immediately driven to zero, and that the queue lengths p and q asymptotically become equal to a constant for the case when the external message arrival rates are constant.

Theorem 6.6 *Given a connected network with N nodes described by* (6.72) *and* (6.74), *assume that, for some $t_0 \in \mathbf{Z}$, $f(t) = f_\infty \in \Re_+^{N(N-1)}$ is a constant vector, $\forall t \geq t_0$, and that controller* (6.79) *is applied, where the matrix Θ satisfies properties* 1–4; *then*

(a) *the closed-loop system is globally asymptotically stable, and*
(b) *for all $t_0 \in \mathbf{Z}$, for all $x(t_0) \in \Re_+^{N^*}$, and for all $f_\infty \in \Re_+^{N(N-1)}$*:

$$m(\infty) \leq \lim_{t\to\infty} m(t) = 0; \quad \textit{in particular}, \quad m(t) = 0, \quad \forall t \geq t_0 + 1,$$

$$p(\infty) \leq \lim_{t\to\infty} p(t) = -(\mathbf{R})(\mathbf{B}\Theta)^{-1}\mathbf{E} f_\infty,$$

$$q(\infty) \leq \lim_{t\to\infty} q(t) = -(\mathbf{R})(\mathbf{B}\Theta)^{-1}\mathbf{E} f_\infty,$$

$$u(\infty) \leq \lim_{t\to\infty} m(t) = -(\Theta)(\mathbf{B}\Theta)^{-1}\mathbf{E} f_\infty,$$

where $\mathbf{R}$ *is a diagonal matrix containing* r_i^{lk} *at the diagonal position corresponding to* q_i^{lk} $(i \in \bar{N}^k,\ l \in \bar{N}^k,\ k \in \bar{N})$.

Proof To prove part (a), we will show that the eigenvalues of the (linear time invariant) closed-loop system (6.72) and (6.74) under the controller (6.79) and (6.80) are contained in the interior of the unit disk. The closed-loop system under the controller (6.79) and (6.80) is described as

$$x(t+1) = \mathbf{F}x(t) + \mathbf{G}f(t),$$

where

$$\mathbf{F} = \begin{bmatrix} 0 & 0 & 0 & \cdots & \cdots & 0 \\ \mathbf{B}_i\Theta & \mathbf{B}_i\Theta\mathbf{D}_1 & \mathbf{B}_i\Theta\mathbf{D}_2 & \cdots & \cdots & \mathbf{B}_i\Theta\mathbf{D}_{r_{\max}} \\ 0 & \mathbf{I} & 0 & \cdots & \cdots & 0 \\ \vdots & & \ddots & & & \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & 0 & \mathbf{I} & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{G} \leq \begin{bmatrix} 0 \\ \mathbf{F} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}.$$

Note that the non-zero eigenvalues λ of $\mathbf{F}$ must satisfy

$$\det[\lambda^{r_{\max}}\mathbf{I} - \mathbf{B}_i\Theta\hat{\mathbf{D}}_\lambda] = 0, \tag{6.82}$$

where

$$\hat{\mathbf{D}}_\lambda \leq \sum_{r=1}^{r_{\max}} \lambda^{r-1}\mathbf{D}_r$$

is a diagonal matrix with diagonal elements $d_i^{lk} = \lambda^{r_i^{lk}-1}$. Assume now that $|\lambda| \geq 1$; then, by (P3), the eigenvalues of $\mathbf{B}_i\Theta\hat{\mathbf{D}}_\lambda$ are contained in the interior of the disk with a radius $|\lambda|^{r_{\max}-1}$. However, since $|\lambda|^{r_{\max}} \geq |\lambda|^{r_{\max}-1}$ for $|\lambda| \geq 1$ this comes in contradiction to the foregoing analysis. It is therefore concluded that the eigenvalues of $\mathbf{F}$ are contained in the interior of the unit disk, which proves global asymptotic stability.

To prove part (b), we will use part (i) if Theorem 6.5 above, and the system dynamics and the controller. Recall that from the proof of Theorem 6.5 that

$$m(t) = 0 \quad \forall t \geq t_0 + 1,$$

from which the desired result for $m(\infty)$ also follows.

Since the closed-loop system is globally asymptotically stable and $f(t)$ is constant,

$$\begin{aligned} w_{r_{\max}}(\infty) &= w_{r_{\max}-1}(\infty) = \cdots = w_2(\infty) = w_1(\infty) \\ &= \mathbf{B}_i u(\infty) + \mathbf{F} f_\infty \leq w_\infty \end{aligned}$$

and

$$u(\infty) = \Theta\left[m(\infty) + \sum_{r-1}^{r_{\max}} \mathbf{D}_r w_r(\infty)\right] = \Theta w_\infty.$$

Hence,

$$w_\infty = \mathbf{B}_i \Theta w_\infty + \mathbf{E} f_\infty$$

or

$$w_\infty = (\mathbf{I} - \mathbf{B}_i \Theta)^{-1} \mathbf{E} f_\infty := -(\mathbf{B}\Theta)^{-1} \mathbf{E} f_\infty,$$

where the invertibility of $\mathbf{B}\Theta = \mathbf{B}_i\Theta - \mathbf{I}$ is guaranteed by property 3. Therefore, we obtain

$$u(\infty) = \Theta w_\infty = -\Theta(\mathbf{B}\Theta)^{-1} \mathbf{E} f_\infty.$$

Note that $q(t) = m(t) + p(t)$, and from (6.75)

$$p(t) = \sum_{r=1}^{r_{\max}} \bar{\mathbf{D}}_r w_r(t), \tag{6.83}$$

where $\bar{\mathbf{D}}_r$ is a diagonal matrix whose diagonal element corresponding to p_i^{lk} is given by

$$d_{i,r}^{lk} = \begin{cases} 1 & \text{if } r \le r_i^{lk}, \\ 0 & \text{otherwise} \end{cases} \quad \forall i \in \bar{N}^k,\ \forall l \in \bar{N}^k,\ \forall k \in \bar{N}.$$

Therefore, we obtain

$$q(\infty) = p(\infty) = \sum_{r=1}^{r_{\max}} \bar{\mathbf{D}}_r w_\infty = -\mathbf{R}(\mathbf{B}\Theta)^{-1} \mathbf{E} f_\infty$$

which is the desired result. □

To summarize, the controller given by (6.79) guarantees the non-negativity constraints (routing control constraints (b)–(d)), and stabilizes the network dynamics assuming that the link capacity constraints (routing control constraint (e)) are not violated. In particular, it drives the queue length m of the processed messages immediately to zero. It will not guarantee, in general, that the routing control constraint (e) is satisfied. In order to satisfy this constraint, we modify the controller as follows:

$$u(t) = \tilde{\Gamma}(t)\hat{u}(t), \quad t \ge t_0, \tag{6.84}$$

where $\hat{u}(t)$ is defined as in (6.80), and $\tilde{\Gamma}(t)$ is a diagonal matrix with the diagonal entry

$$\gamma_{ij} = \begin{cases} 1, & \text{if } \sum_{k\in\bar{N}^i} \sum_{l\in\bar{N}^k} \bar{u}_{ij}^{lk}(t) \le c_{ij}, \\ \frac{c_{ij}}{\sum_{k\in\bar{N}^i} \sum_{l\in\bar{N}^k} \bar{u}_{ij}^{lk}(t)}, & \text{if } \sum_{k\in\bar{N}^i} \sum_{l\in\bar{N}^k} \bar{u}_{ij}^{lk}(t) > c_{ij} \end{cases}$$

corresponding to u_{ij}^{lk} $(j \in \mathcal{D}(i),\ i \in \bar{N}^k,\ l \in \bar{N}^k,\ k \in \bar{N})$.

Remark 6.10 It is important to stress that controller (6.84) will guarantee both the link capacity and the non-negativity constraints hold. Furthermore, a knowledge of processing delays r_i^{lk} will not be needed for the actual implementation of the controller. In addition, the controller (6.84) can be implemented in a decentralized way at each node, and it will be shown in the sequel, that it will also recover the steady-state properties achieved by the previous controller (6.79), if the external message arrival rates are sufficiently small. The maximum magnitude of the external message arrival rates for which the steady-state properties of controller (6.79) can generally be recovered.

6.8.4 Selection of the Feedback Matrix

In what follows, we consider the problem of choosing an appropriate feedback matrix Θ for controller (6.84), so that it can recover the steady-state properties of controller (6.79), for as large as possible external message arrival rates. To achieve this. let us consider the condensed system corresponding to

$$z(t+1) = z(t) + f(t) + \hat{\mathbf{B}} v(t), \tag{6.85}$$

where f is defined in the paragraph following equation (6.68), $z = \{z_i^k\}$ is the $N(N-l)$ dimensional vector of *condensed system states* which has the same index structure as f, $v = \{v_{ij}^k\}$ is the $L(N-l)$ dimensional *condensed control input vector*, $\hat{\mathbf{B}}$ is a matrix consisting of 1's corresponding to v_{ji}^k elements with $j \in \mathcal{U}(i)$, -1's corresponding to v_{ij}^k elements with $j \in \mathcal{D}(i)$ on the row corresponding to z_i^k $(i \in \bar{N}^k, k \in \bar{N})$, and zeros elsewhere. Also consider the following constraints

$$\sum_{k \in \bar{N}^i} v_{ij}^k \le c_{ij}, \quad \forall j \in \mathcal{D}(i),\ \forall i \in \bar{N}$$

which can be written compactly as

$$\mathbf{M} v \le c,$$

where $c \le \{c_{ij}\}$ is the L dimensional vector of link capacities and $\mathbf{M}$ is an $L \times L(N-1)$ dimensional matrix consisting of 1's corresponding to the v_{ij}^k elements on the row corresponding to c_{ij} $(j \in \mathcal{D}(i),\ i \in \bar{N}^k,\ k \in \bar{N})$, and zeros elsewhere. Here, and in the sequel, '$\le$' (respectively '$\ge$') means that each component of the vector on the left is less than or equal to (respectively greater than or equal to) the corresponding component of the vector on the right.

A search for an $L(N-l) \times N(N-l)$ dimensional matrix $\mathbf{H}$ is needed, such that the steady-state flow

$$\bar{v} = \mathbf{H} \hat{f} \tag{6.86}$$

will satisfy the non-negativity and link-capacity constraints, and such that the magnitude of the external message arrival rate is maximized for any

$$\hat{f} \in \{\hat{f}_{ik} | \hat{f}_{ik} = (0, \ldots, 0, \tilde{f}_i^k, 0, \ldots, 0)^t,\ i \in \bar{N}^k,\ k \in \bar{N}\}. \tag{6.87}$$

Observe that, in case of multiple solutions to the above problem, we select one of the solutions which minimizes the total rate of link flows inside the network.

This problem can be stated in two stages as follows.

(a) **Initial stage**: Find $\bar{v}_{ik} \in \Re_+^{L(N-1)}$ to maximize $\tilde{f}_i^k > 0$ subject to:

$$\hat{f}_{ik} + \hat{\mathbf{B}}\bar{v}_{ik} = 0 \tag{6.88}$$

and

$$\mathbf{M}\bar{v}_{ik} \le c \tag{6.89}$$

for $i \in \bar{N}^k$, $k \in \bar{N}$.

(b) **Final stage**: If multiple solutions exists to the primary stage, choose the solution which minimizes $\|\bar{v}_{ik}\|_1$, where $\|(\cdot)\|_1$ denotes the one-norm of $(\cdot)$, defined as the sum of the absolute values of all the elements of $(\cdot)$.

Then the column of **H** which corresponds to $\tilde{f}_i^k$ in (6.86) is given by:

$$\frac{1}{(\tilde{f}_i^k)_{\max}} \bar{v}_{ik},$$

where $(\tilde{f}_i^k)_{\max}$ is the maximum value of $\tilde{f}_i^k$.

Remark 6.11 Note that the above optimization problem must be repeated for all origin-destination pairs i–k. The primary stage effectively minimizes a *global cost function* by maximizing the magnitude of maximum possible external traffic arrival rates. In the ease of multiple solutions to the primary stage, the secondary stage, on the other hand, minimizes a *user cost function* by routing messages through a minimum number of nodes, and in doing so this also avoids any looping problems (see Lemma 6.3 below). The second stage may also have multiple solutions; in this case any one of these solutions can be chosen without effecting either the global objective (that is, maximizing the maximum possible external arrival rates) or the user objective (that is, minimizing the number of nodes a message passes through).

The above two-stage optimization problem can be formulated as a single-stage linear programming (LP) problem as follows:

Let $\hat{b}_{ik}$ denote the row of $\hat{\mathbf{B}}$ which corresponds to z_i^k in (6.85), let $\hat{\mathbf{B}}_{ik}^*$ denote the matrix $\hat{\mathbf{B}}$ with $\hat{b}_{ik}$ removed, and let ϵ be a positive scalar. Then the LP problem can be stated as follows:

$$\max_y J_{ik}(y),$$

where

$$J_{ik} \leq -\hat{b}_{ik}y - \epsilon\|y\|_1 = -(\hat{b}_{ik} + \begin{bmatrix} \epsilon \ \epsilon \ \dots \ \epsilon \end{bmatrix})y$$

subject to

$$y \in \Re_+^{L(N-1)}, \tag{6.90}$$

$$\hat{\mathbf{B}}_{ik}^* y = 0 \tag{6.91}$$

and

$$\mathbf{M}y \leq c. \tag{6.92}$$

The following result shows that a solution to the original two-stage optimization problem may be obtained by solving the above one-stage LP problem.

Lemma 6.3 *Let $\epsilon^* \leq 1/L_{\mathbf{M}}^{ik}$, where $L_{\mathbf{M}}^{ik}$ is the maximum number of links in any non-cyclic path from node i to node k. Then, if $\epsilon < \epsilon^*$, any solution y^* to the above LP problem is also a solution to the primary stage of the optimization problem. Furthermore, if $\epsilon < \epsilon^*$, if y is any other solution to the primary stage of the optimization problem, then $\|y^*\|_1 \leq \|y\|_1$; i.e. y^* is also a solution the secondary stage of the optimization problem.*

Proof In what follows and for simplicity in exposition, we will use a different notation to denote the links; we will number the links as $1, 2, \dots, L$ and refer to them by their number (i.e. we will use v_l^k instead of v_{ij}^k, where l is the number of the link from node i to node j). We will also denote the objective function of the primary stage of the original optimization problem by J_{ik}^0. Note that, by using the notation of the LP problem, $J_{ik}(y) = -\hat{b}_{ik}y$.

Next note that any solution to the original optimization problem and any solution to the LP problem has the property that $y_l^\kappa = 0$, $\forall l$, $\forall \kappa \neq k$ and the property that $y_l^k = 0$ unless link l is on a non-cyclic path from node i to node k. Let $p_1, p_2, \dots, p_m$ be all possible paths from node i to node k. Let l_j $(j = 1, 2, \dots, m)$ be a link on path p_j, but not on any other path $p_1, \dots, p_{j-1}, p_{j+1}, \dots, p_m$ (if the original network does not allow this, one can always replace a single link $i - -j$ with a number of parallel links such that each link belongs to a different path). Then, by using constraints (16b) the objective function of the primary stage of the original optimization problem can be written as:

$$J_{ik}^0 = \sum_{j-1}^{m} y_{l_j}^k$$

and the objective function of the LP problem can be written as

$$J_{ik}(y) = \sum_{j=0}^{m} (1 - \epsilon L_j^{ik}) y_{l_j}^k,$$

where L_j^{ik} denotes the number of links on path p_j, where only active constraints are (6.90) and (6.92). Therefore, if $\epsilon < 1/\max(L_{\mathbf{M}}^{ik})$, then any y which maximizes J_{ik} also maximizes J_{ik}^0. Furthermore, since the weight in J_{il} of a path which involves more links is greater than the weight of a path which involves less links, the LP problem will also minimize $\|y\|_1$ among all solutions to the primary stage of the original problem. □

It is quite clear that $L_{\mathbf{M}}^{ik} \le L$, the number of links of the network. Therefore, we can solve the above LP problem for any $\epsilon \in (0, 1/L)$ (or for any $\epsilon \in (0, 1/L_{\mathbf{M}}^{ik})$, if $L_{\mathbf{M}}^{ik}$ is known) to find a solution to the original two-stage optimization problem.

Now denote the optimal solution to this LP problem by y_{ik}^* and the optimal value of $J_{ik}(y)$ by J_{ik}^*. Then the column $\mathbf{H}$ which corresponds $\tilde{f}_i^k$ ($i \in \bar{N}, k \in \bar{N}$) in (6.86) is given by:

$$\frac{1}{J_{ik}^*} y_{ik}^*$$

and

$$(\tilde{f}_i^k)_{\max} = J_{ik}^*.$$

Lemma 6.4 *Consider a network with N nodes described by* (6.72)–(6.74) *which has the associated equation* (6.85). *Then*:

1. *There exists a matrix* $\mathbf{H}$ *such that*
   (a) *All elements of* $\mathbf{H}$ *are non-negative, and*
   (b) $\mathbf{I} + \hat{\mathbf{B}}\mathbf{H} = 0$, *where* $\hat{\mathbf{B}}$ *is given in* (6.72)–(6.74),
   if and only if the given network is connected.
2. *If all link capacities* c_{ij}, $j \in \mathcal{D}(i)$, $i \in \bar{N}$, *then all elements of* $\mathbf{H}$ *and* $(\tilde{f}_i^k)_{\max}$, $i \in \bar{N}^k$, $k \in \bar{N}$, *are bounded.*

Remark 6.12 The forgoing lemma, whose proof is straightforward, ensures that: (a) there exists a solution to the above optimization problem (and thus to the LP problem) if and only if the given network is connected, and (b) the solution is bounded if the link capacities are finite.

Given that matrix $\mathbf{H}$ is constructed as above, we obtain a feedback matrix Θ as follows:

Algorithm 1 Let $\bar{h}_{ij}^{lk}$ be the element of H which relates f_l^k to v_{ij}^k in $v = \mathbf{H}f$ ($j \in \mathcal{D}(i)$, $i \in \bar{N}^k$, $l \in \bar{N}^k$, $k \in \bar{N}$), and let

$$h_{ij}^{lk} \le \begin{cases} \frac{\bar{h}^{lk}}{\bar{s}_i^{lk}} & \text{if } \bar{s}_i^{lk} > 0, \\ \frac{1}{\bar{n}_i} & \text{if } \bar{s}_i^{lk} = 0, \end{cases}$$

where $\bar{s}_i^{lk} \leq \sum_{j\in\mathcal{D}(i)} \bar{h}_{ij}^{lk}$ and $\bar{n}_i$ is the number of elements of $\mathcal{D}(i)$. To construct Θ, let h_{ij}^{lk} be the element of Θ which relates q_i^{lk} to u_{ij}^{lk} in $u = \Theta q$ ($j \in \mathcal{D}(i)$, $i \in \bar{N}^k$, $l \in \bar{N}^k$, $k \in \bar{N}$), and let all other elements of Θ be zero.

Observe that, any feedback matrix Θ must satisfy properties 1–4 in order to obtain a valid controller. To show that Θ as constructed above satisfies these properties, we have the following

Lemma 6.5 *Given a connected network with N nodes described by* (6.72)–(6.74), *let Θ be constructed by Algorithm* 1. *Then Θ satisfies properties* 1–4.

Proof The fact that Θ satisfies properties 1 and 4 is readily obvious from the construction of Θ. Now to show that Θ satisfies property 2, consider $\hat{q} = \mathbf{B}_o u$, $u = \Theta q$ which gives:

$$\hat{q}_i^{lk} = \sum_{j\in\mathcal{D}(i)} u_{ij}^{lk} = \sum_{j\in\mathcal{D}(i)} h_{ij}^{lk} q_i^{lk} = q_i^{lk},$$

since $\sum_{j\in\mathcal{D}(i)} h_{ij}^{lk} = 1$, $\forall i \in \bar{N}^k$, $\forall l \in \bar{N}^k$, $\forall k \in \bar{N}$ by construction.

Next to show that Θ satisfies property 3, we first establish that for each $l \in \bar{N}$, Θ defines a path

$$i = i_0,\ i_i,\ i_2,\ \ldots,\ i_{m-1},\ i_m = k$$

from each $i \in \bar{N}$ to each $k \in \bar{N}\backslash\{i, l\}$, in the sense that $i_{j+1} \in \mathcal{D}(i_j)$ and $h_{i_j i_{j+1}}^{lk} > 0$ for each successive pair i_j, i_{j+1} on the path. Note that in order to satisfy the constraint (6.88) of the optimization problem, $\mathbf{H}$ must define at least one such path from each $l \in \bar{N}$ to each $k \in \bar{N}^l$. Thus, for each $l \in \bar{N}$, Θ defines at least one path from l to each $k \in \bar{N}^l$. If $i \notin \{l, k\}$ is on such a path, then Θ also defines a path from i to k for l; if i is not on such a path, then $h_{ij}^{lk} = 1/\bar{n}_i > 0$, $\forall j \in \mathcal{D}(i)$. Thus if $k \in \mathcal{D}(i)$ or if there exists a $j \in \mathcal{D}(i)$ which is on a path from l to k, then a path is established from i to k for l; if neither of these two conditions are satisfied, then $h_{j_1 j_2}^{lk} = l/\bar{n}_{j_1} > 0$, $\forall j_2 \in \mathcal{D}(j_1)$, $\forall j_1 \in \mathcal{D}(i)$. On continuing this procedure (since the network is connected and has a finite number of nodes), we eventually reach either node k, or a node which is on a path from l to k, and hence the desired result is established. The rest of the proof is now directly obtained by the foregoing lemmas. □

6.8.5 Some Properties

Now consider the controller described by (6.88), where the feedback matrix Θ is given by Algorithm 1. Given that routing control constraint (a) is satisfied, the following theorem shows that this controller satisfies routing control constraints (b)–(e):

Theorem 6.7 *Consider a connected network with N nodes described by* (6.72)–(6.74). *Assume that the controller described by* (6.79) *is applied where the matrix Θ is constructed by Algorithm* 1. *Then for all $t_0 \in \mathbf{Z}$, for all $x(t_0) \in \Re_+^{N^*}$, and for all $f(t) \in \Re_+^{N(N-1)}$,*

1. *Non-negativity constraints* (*routing control constraints* (b), (c), *and* (d)) *are satisfied for all $t \geq t_0$,*
2. *Link capacity constraints* (*routing control constraint* (e)) *are satisfied for all $t \geq t_0$.*

Proof 1. Given $x(t_0) \in \Re_+^{N^*}$ and that Θ satisfies property 1, (6.80) implies that $\hat{u}(t_0) \geq 0$. By (6.84) then implies that $0 \leq u(t_0) \leq \hat{u}(t_0)$. Using this result in (6.73) we obtain $m(t_0 + 1) \geq 0$. Since we also have $f(t_0) \geq 0$, we also obtain $x(t_0 + 1) \geq 0$. By repeating this argument, we obtain $m(t) \geq 0$ and $u(t) \geq 0$ for all $t \geq t_0$. Routing control constraints (e) and (d) are thus satisfied. Given that $x(t) \geq 0$, $\forall t \geq t_0$ (as established above), the desired result for routing control constraint (b) follows from (6.83).

2. Immediately follows from (6.84). □

Before proceeding further, we need to introduce the following definition:

Definition 6.3 Consider a connected network with N nodes described by (6.72)–(6.74) and assume that a controller is applied to this network. We say that *the messages are directed around a loop* if for some $k \in \bar{N}$, and for some set of nodes $\{i_1, i_2, \ldots, im\} \subset \bar{N}^k$ such that $i_{j+1} \in \mathcal{D}(i_j)$, $j = l, 2, \ldots, m-1$, $i_1 \in \mathcal{D}(i_m)$, and $i_j \neq i_l$, for $j \neq l$, we have:

$$u_{i_1 i_2}^{lk}(t_1) > 0, \quad u_{i_2 i_3}^{lk} > 0, \quad \ldots, \quad u_{i_{m-1} i_m}^{lk}(t_{m-1}) > 0 \tag{6.93}$$

for some $l \in \bar{N}^k$, $t_1 \geq t_0$, $t_2 \geq t_0$, $\ldots$, and $t_{m-1} \geq t_0$, and:

$$u_{i_m i_1}^{lk}(t) > 0 \tag{6.94}$$

for some $t \geq t_0$. If the above condition never happens, then we say that *the messages are not directed around a loop*.

Remark 6.13 Note that the above definition is more general than the usual message looping criteria, which considers looping of individual messages alone. According to the above definition, a loop is formed when messages belonging to a particular origin-destination pair are transmitted around a loop (whether they are the same messages or not). Therefore, avoiding the above defined looping is stronger than avoiding individual message looping.

That the proposed controller (6.84) does not direct messages around a loop, is shown by the following theorem:

Theorem 6.8 *Consider a connected network with N nodes described by* (6.72)–(6.74)*. Assume that the controller described by* (6.84) *is applied where the matrix Θ is constructed by Algorithm* 1*. Furthermore, assume that $x(t_0) = 0$, and that $f(t) \in \Re_+^{N(N-1)}$, $\forall t \geq t_0$, for some $t_0 \in \mathbf{Z}$. Then the messages are not directed around a loop.*

Proof We will use contradiction to prove this theorem. Given (6.93), assume that (6.94) also holds. This, however. can happen only if

$$h_{i_1 i_2}^{lk} > 0, \quad h_{i_2 i_3}^{lk} > 0, \quad \ldots, \quad h_{i_m i_1}^{lk} > 0, \tag{6.95}$$

$$h_{i_m i_1}^{lk} > 0, \tag{6.96}$$

where h_{ij}^{lk} denotes appropriate element of Θ.

If $h_{i_j i_{j+1}}^{lk} > 0$, for any $j \in 1, 2, \ldots, m-1$, then either $\bar{h}_{i_j i_{j+1}}^{lk} > 0$ or $\bar{s}_{i_j}^{lk} = 0$, where $\bar{h}_{ij}^{lk}$ and $\bar{s}_i^{lk}$ are defined in Algorithm 1. Similarly, at $h_{i_m i_1}^{lk} > 0$, then either $\bar{h}_{i_m i_1}^{lk} > 0$ or $\bar{s}_{i_j}^{lk} = 0$. If $\bar{s}_{i_j}^{lk} = 0$, for any $j \in \{1, 2, \ldots, m\}$, then no messages with source l and destination k can reach node i_j under controller (6.84); thus $u_{i_j s}^{lk} = 0, \forall s \in \mathcal{D}(i)$, which contradicts (6.95)–(6.96). On the other hand, if $\bar{h}_{i_1 i_2}^{lk} > 0$, $\bar{h}_{i_2 i_3}^{lk} > 0, \ldots, \bar{h}_{i_{m-1} i_m}^{lk} > 0$, and $\bar{h}_{i_m i_1}^{lk} > 0$ results from a solution of the initial stage of the optimization problem, then there must exist another solution of the same stage, where all the elements of $\mathbf{H}$ remains the same, except that $\bar{h}_{i_1 i_2}^{lk}, \bar{h}_{i_2 i_3}^{lk}, \ldots, \bar{h}_{i_{m-1} i_m}^{lk}$, and $\bar{h}_{i_m i_1}^{lk}$ are reduced by $\min(\min_j(\bar{h}_{i_j i_{j+1}}^{lk}), \bar{h}_{i_m i_1}^{lk})$, in which $\bar{h}_{i_j i_{j+1}}^{lk}$ becomes zero for at least one $j \in \{1, 2, \ldots, m-1\}$ or $\bar{h}_{i_m i_1}^{lk}$ becomes zero. By the final stage, we then choose the latter solution over the former one; hence situation (6.95)–(6.96) will never arise. □

Remark 6.14 It is evident from the above proof that, if $x(t_0) \neq 0$, $x(t_0) \in \Re_+^{N^*}$, then only messages which are present at a node, which is not on a path defined by the matrix $\mathbf{H}$ from their source node to their destination node at time t_0 may be routed around a loop. In particular, since the source node is always on a path defined by the matrix $\mathbf{H}$, messages which enter the network at time $t \geq t_0$, are never directed around a loop.

That the controller (6.84) can recover the steady-state properties of the controller described by (6.79) is established by the following theorem provided that the message arrival rates do not exceed the maximum rates given by the optimization problem.

Theorem 6.9 *Consider a connected network with N nodes described by* (6.72)–(6.74)*. Assume that the controller described by* (6.84) *is applied where the matrix Θ is constructed by Algorithm* 1*. Then*

1. *The closed-loop system is globally asymptotically stable.*

2. *Assume that* $x(t_0) = 0$ *for some* $t_0 \in \mathbf{Z}$ *and that*

$$f(t) = f_\infty = (0, \dots, 0, f_i^k, 0, \dots, 0)^t \quad \forall t \geq t_0 \tag{6.97}$$

for some $i \in \bar{N}^k$, $k \in \bar{N}$; *then for all constant* $f_i^k \in [0, (\tilde{f}_i^k)_{\max}]$,

$$m(t) = 0, \quad \forall t \geq t_0$$

and $p(t)$, $q(t)$, *and* $u(t)$ *remain bounded for all* $t \geq t_0$.

3. *Assume that* $x(t_0) = 0$ *for some* $t_0 \in \mathbf{Z}$ *and that* $f(t)$ *is given by* (6.97); *then*

$$p(\infty) = q(\infty) = -\mathbf{R}(\mathbf{B}\Theta)^{-1}\mathbf{E} f_\infty \tag{6.98}$$

and

$$u(\infty) = -\Theta(\mathbf{B}\Theta)^{-1}\mathbf{E} f_\infty \tag{6.99}$$

for all constant $f_i^k \in [0, (\bar{f}_i^k)_{\max}]$.

Proof To prove part 1, we need to establish that, under the hypothesis, $\sum_{i,l,k} |q_i^{lk}|$ cannot increase, where $\sum_{i,l,k}(\cdot) \leq \sum_{k \in \bar{N}} \sum_{l \in \bar{N}^k} \sum_{i \in \bar{N}^k}(\cdot)$. Then we will show that the desired result holds if $q(t)$ converges to zero. Finally, we will use contradiction to show that under the hypothesis, $q(t)$ must indeed converge to zero.

Now assume that $f(t) = 0$, $\forall t \geq t_0$; then from (6.68) we have:

$$\begin{aligned}
\sum_{i \in \bar{N}^k} q_i^{lk}(t+1) &= \sum_{i \in \bar{N}^k} q_i^{lk} + \sum_{i \in \bar{N}^k} \Bigg[\sum_{j \in \mathcal{U}(i)} u_{ji}^{lk}(t) - \sum_{j \in \mathcal{D}(i)} u_{ij}^{lk}(t) \Bigg] \\
&= \sum_{i \in \bar{N}^k} q_i^{lk}(t) + \sum_{i \in \bar{N}^k} \sum_{j \in \mathcal{D}(i) \setminus k} u_{ij}^{lk}(t) - \sum_{i \in \bar{N}^k} \sum_{j \in \mathcal{D}(i)} u_{ij}^{lk}(t) \\
&= \sum_{i \in \bar{N}^k} q_i^{lk}(t) - \sum_{i \in \mathcal{U} k} u_{ik}^{lk}(t), \quad \forall l \in \bar{N}^k, \forall k \in \bar{N}.
\end{aligned}$$

This means that

$$\begin{aligned}
\sum_{i,l,k} q_i^{lk}(t+1) &= \sum_{i,l,k} q_i^{lk}(t) - \sum_{k \in \bar{N}} \sum_{l \in \bar{N}^k} \sum_{i \in \mathcal{U}(k)} u_{ik}^{lk}(t) \\
&= \sum_{i,l,k} q_i^{lk}(t) - \sum_{i \in \bar{N}} \sum_{l \in \bar{N}} \sum_{k \in \mathcal{D}(i) \setminus l} u_{ik}^{lk}(t).
\end{aligned}$$

Taking into account that the controller (6.84) satisfies the non-negativity constraints, this implies that $\sum_{i,l,k} q_i^{lk}$ (or equivalently $\sum_{i,l,k} |q_i^{lk}|$) not increase. Assume now that $q(t)$ converges to zero. Then, since $0 \leq m(t) \leq q(t), m(t)$ also converges to zero. Furthermore, since $w_r(t) \in \Re_+^{N(N-1)}$ $\forall t \geq t_0$ (follows from the fact that $x(t) \geq 0$, $\forall t \geq t_0$, which was shown earlier) and $q(t) = m(t) + p(t)$, where $p(t)$ is given by (6.68), this implies that $w_r(t)$ $(t = l, 2, \ldots, r_{\max})$ also converge to zero. Therefore, we obtain that $\lim_{t\to\infty} x(t) = 0$, which is the desired result.

Assume now that $q(t)$ does not converge to zero. Then there exists $k \in \bar{N}$, $\ell \in \bar{N}^k$, and a set $S \subset \bar{N}^k$ such that

$$\lim_{t\to\infty} \sum_{i\in S} q_i^{lk}(t) > 0,$$

$$\lim_{t\to\infty} \sum_{i\in S} \sum_{j\in\mathcal{D}(i)\setminus S} u_{ij}^{lk} = 0.$$

This, however, can only be true if there exists $i \in S$ such that there exists no path from i to k for l which is a contradiction. It was established in before that Θ defines a path from each $i \in \bar{N}^k$ to each $k \in \bar{N}$ for each $\ell \in \bar{N}^k$. Thus, the desired result is attained.

To prove part 2, we will refer to (6.68)–(6.69). Assume that $x(t_0) = 0$, and

$$f(t) = (0, \ldots, 0, f_l^k, 0, \ldots, 0)^t, \quad \forall t \geq t_0$$

where $f_l^k \in [0, (\tilde{f}_l^k)_{\max}]$.

An immediate result of Theorem 6.8 is that $u_{il}^{lk}(t) = 0, \forall t \geq t_0, \forall i \in \mathcal{U}(l)$, which implies that

$$q_l^{lk}(t+1) = q_l^{lk} + f_l^k(t) - \sum_{j\in\mathcal{D}(i)} u_{ij}^{lk}(t).$$

Now since $x(t_0) = 0$, and $u_{il}^{lk}(t) = 0, \forall t \geq t_0, \forall i \in \mathcal{U}(l)$, this implies that

$$u_{lj}^{lk}(t) = h_{lj}^{lk} f(t - r_l^{lk}) = \begin{cases} h_{lj}^{lk} f_l^k, & \forall t \geq t_0 + r_l^{lk}, \\ 0, & \forall t \in [t_0, t_0 + r_l^{lk}) \end{cases} \tag{6.100}$$

provided that $\sum_{s,r} u_{lj}^{sr}(t) \leq c_{lj}, \forall j \in \mathcal{D}(l)$. However, since $f_s^r(t) = 0, \forall(s,r) \neq (l,k)$, $u_{lj}^{sr} = 0, \forall(s,r) \neq (l,k)$ and since $f_l^k \leq (\tilde{f}_l^k)_{\max}$, $u_{lj}^{lk} \leq c_{lj}$, this implies that

$$m_l^{lk}(t) = 0, \quad \forall t \geq t_0, \ \forall t \geq t_0 \tag{6.101}$$

and

$$q_l^{lk}(t+1) = \begin{cases} q_l^{lk}(t), & \forall t \geq t_0 + r_l^{lk}, \\ q_l^{lk}(t) + f_l^k, & \forall t \in [t_0, t_0 + r_l^{lk}). \end{cases} \tag{6.102}$$

Now consider a node i, $i \neq k$, which is on a path from l to k for l. Then

$$q_i^{lk}(t+1) = q_i^{lk}(t) + \sum_{j\in\mathcal{U}(i)} u_{ji}^{lk}(t) - \sum_{j\in\mathcal{D}(i)} u_{ij}^{lk}.$$

Since $x(t_0)=0$, this implies that $u_{ij}^{lk}(t)=h_{ij}^{lk}\sum_{s\in\mathcal{U}(i)}u_{si}^{lk}(t-r_i^{lk}), \forall t\geq t_0$, provided that $\sum_{s,r}u_{ij}^{sr}(t)=u_{ij}^{lk}\leq c_{ij}, \forall j\in\mathcal{D}(i)$. Note that, by Theorem 4, u_{ji}^{lk} does not depend on $q_i^{lk}, \forall j\in\mathcal{U}(i)$. Furthermore, since $f_l^k\leq(\tilde{f}_l^l)_{\max}$, then $u_{ij}^{lk}(t), j\in\mathcal{D}(i)$ is structured such that

$$0\leq\sum_{j\in\mathcal{D}(i)}u_{ij}^{lk}(t)\leq q_i^{lk}(t)$$

from which we obtain

$$q_i^{lk}(t+1)=\lambda q_i^{lk}+\sum_{j\in\mathcal{U}(i)}u_{ji}^{lk},$$

where $\lambda\in[0,1]$. Therefore, the overall system is described by a set of first order linear time-invariant systems, with non-negative real eigenvalues none of which exceed one, cascaded together. This implies that no overshoot in the system's response occurs, that is

$$u_{ij}^{lk}(t)\leq u_{ij}^{lk}(\infty)\leq c_{ij}, \quad \forall j\in\mathcal{D}(i).$$

It is concluded therefore that:

$$q_i^{lk}(t)\leq q_i^{lk}(\infty)=r_i^{lk}\sum_{j\in\mathcal{U}}u_{ji}^{lk}(\infty), \quad \forall t\geq t_0, \tag{6.103}$$

$$m^{lk}(t)=0, \quad \forall t\geq t_0 \tag{6.104}$$

and

$$u_{ij}^{lk}(t)\leq u_{ij}^{lk}(\infty)=h_{ij}^{lk}\sum_{s\in\mathcal{U}(i)}u_{si}^{lk}(\infty), \quad \forall t\geq t_0. \tag{6.105}$$

Combining equations (6.100)–(6.102) with (6.103)–(6.105), and on noting that:

$$q_i^{si}(t)=0, \quad \forall t\geq t_0, \forall i\in\bar{N}^r, \forall(s,r)\neq(l,k), \tag{6.106}$$

$$m_i^{sr}(t)=0, \quad \forall t\geq t_0, \forall i\in\bar{N}^r, \forall(s,r)\neq(l,k) \tag{6.107}$$

and

$$u_{ij}^{sr}(t)=0, \quad \forall t\geq t_0, \ \forall j\in\mathcal{D}(i), \ \forall i\in\bar{N}^r, \ \forall(s,r)\neq(l,k) \tag{6.108}$$

we conclude that

$$m(t)=0, \quad \forall t\geq t_0$$

and that $q(t)$, $p(t)(\leq q(t)-m(t))$, $u(t)$ remain bounded.

To prove part 3, we extend on the proof of part 2 and the assumption that r_i^{lk} is constant for all $i\in\bar{N}^k$, $l\in\bar{N}^k$, $k\in\bar{N}$. From (6.100)–(6.102), we obtain

$$q_l^{lk}(\infty)=r_l^{lk}f_l^k, \tag{6.109}$$

$$m_l^{lk}(\infty) = 0 \tag{6.110}$$

and

$$u_{ij}^{lk}(\infty) = h_{lj}^{lk} f_l^k, \quad \forall j \in \mathcal{D}(l). \tag{6.111}$$

Combining (6.100)–(6.111) and on noting that $p(t) = q(t) - m(t)$, we obtain (6.100)–(6.102), which is the desired result. □

Remark 6.15 If $f_i^k > (\tilde{f}_i^k)_{\max}$, then there exists no controller which can keep the queue lengths bounded.

Remark 6.16 So far we have assumed that the control signals are updated synchronously throughout the network and that the processing delay times remain fixed. However, note that the proofs of Theorems 6.7–6.9 remain valid whether not the control Signals are updated synchronously at different nodes. In fact, the control signals may be updated using different periods at different nodes. Furthermore, the proofs of Theorems 6.7–6.8 and the proof of part 1 of Theorem 6.9 remain valid as long as the processing delay times are non-negative and bounded (not necessarily fixed). For part 2 of Theorem 6.9, if the processing delay decreases from one sampling instant to the next, then the queue of length of processed messages may become positive, The rest of this part (that $m(t)$, $p(t)$, $q(t)$, and $u(t)$ remain bounded), however, also remains valid for the case of time varying but non-negative and bounded processing delay times.

The following result is obtained, which summarizes the results obtained for the more general case of asynchronous operation with possibly time varying processing delay times.

Corollary 6.1 *Consider a connected network with N nodes whose dynamics at each node is described by* (1*a*) *and* (1*b*), *where the time-scale for each node may be different*, *that is*, *the unit of t* (*the sampling period in the actual time scale*) *in* (6.68) *and* (6.69) *may be different from each node. The processing delay times may be time-varying*, *but are assumed to be bounded*, *i.e. we assume that there exists* $(r_i^{lk})_{\max} \geq 0$, *such that* $0 \leq (r_i^{lk})_{\max}$, $\forall t \geq t_0$, $\forall i \in \bar{N}^k$, $\forall \ell \in \bar{N}^k$, $\forall k \in \bar{N}$. *Assume that the following controller is applied at each node*:

$$u^{lk} = \gamma_{ij}(t)\hat{u}^{lk}, \quad \forall t \geq t_0, \tag{6.112}$$

where $\gamma_{ij}(t)$ is given following (6.84) *and*

$$\hat{u}_{ij}^{lk}(t) = h_{ij}^{lk} g_i^{lk}(t), \quad \forall t \geq t_0, \tag{6.113}$$

where $g_i^{lk}(t)$ is the of total volume of messages with source l and destination k which can be sent out from node i at time t and h_{ij}^{lk} is the appropriate element of Θ, where the matrix Θ is constructed by Algorithm 1. *Moreover, assume that $x(t_0) \in \Re_+^{N^*}$ and that $f(t) \in \Re_+^{N(N-1)}$, $\forall t \geq t_0$, for some $t_0 \in \mathbf{Z}$. Then*

1. *Routing control constraints* (a)–(e) *are satisfied.*
2. *Assume that* $x(t_0)$; *then the messages are not directed around a loop. If* $x(t_0) \neq 0$, *then only messages which are present at a node, which is not on a path defined by the matrix* **H** *from their source node to their destination node at time* t_0 *may be routed around a loop. In particular, messages which enter the network at time* $t \geq t_0$ *are never directed around a loop.*
3. *The closed-loop system is globally asymptotically stable.*
4. *Assume that* $x(t_0) = 0$ *and that* $f(t)$ *is given by* (6.97), *then for all constant* $f_i^k \in [0, (\tilde{f}_i^k)_{\max}]$, $m(t)$, $p(t)$, $q(t)$, *and* $u(t)$ *remain bounded for all* $t \geq t_0$.
5. *Assume that* $x(t_0) = 0$, *that* $f(t)$ *is given by* (6.97), *and that* r_i^{lk} *is constant,* $\forall t \geq t_0$, $\forall i \in \bar{N}^k$, $\forall \ell \in \bar{N}^k$, $\forall k \in \bar{N}$; *then* $m(t) = 0$, $\forall t \geq t_0 + 1$, *and* $p(t)$, $q(t)$ *and* $u(t)$ *converge to the steady state values given in* (6.98)–(6.99).

Remark 6.17 If the control signals are updated synchronously throughout the network, then controller (6.112)–(6.113) is equivalent to controller (6.84). However, controller (6.112)–(6.113) can also be used in the case of asynchronous operation. Furthermore, (6.112)–(6.113) are more suitable from an implementation point of view, since the implementation of the proposed controller will be done locally at individual nodes.

Remark 6.18 Assume that $f(t)$ belongs to the class used for the optimization; then Θ, constructed by Algorithm 1, allows the maximum possible magnitudes on each individual component of $f(t)$ to occur, such that all of the above properties (given by Theorems 6.7–6.9) hold.

The actual external message arrival rates into a network may, of course, not belong to the class (14) used for the optimization; they may in fact be time-varying. In addition, the initial conditions may be non-zero. For a constant vector $\hat{f} = \{\hat{f}_i^k\} \in \Re_+^{N(N-1)}$ and for a scalar $\phi > 0$, let us define:

$$\Omega_\phi^{\hat{f}} = \{f(t) = \{f_i^k(t)\} \in \Re_+^{N(N-1)} \mid 0 \leq f_i^k(t) \leq \phi \hat{f}_i^k,\ \forall t \geq t_0, \forall i \in \bar{N}^k, \forall k \in \bar{N}\}.$$

The following corollary, stated without proof, ensures that the proposed controller (6.112)–(6.113) can keep the queue lengths bounded. without violating the constraints (b)–(e), for all non-negative initial conditions and for all (possibly time-varying) non-negative external message arrival rates, provided only that the magnitude of these rates is not too large. This property is true whether or not these arrival rates belong to the class used for the optimization.

Corollary 6.2 *Consider a connected network with* N *nodes whose dynamics at each node is described by* (6.68)–(6.69), *where the time-scale for each node may be different. The processing delay times may be time-varying, but are assumed to be bounded. Assume that the controller described by* (6.112)–(6.113) *is applied where the matrix* Θ *is constructed by Algorithm* 1. *Then for all constant* $\hat{f} \in \Re_+^{N(N-1)}$, *for* *all* $t_0 \in \mathbf{Z}$ *and for all constant* $x_0 \in \Re_+^{N^*}$, *there exists a scalar* $\phi > 0$, *such that for all* $f(t) \in \Omega_\phi^{\hat{f}}$, $m(t)$, $p(t)$, $q(t)$, *and* $u(t)$ *remain bounded* $\forall t \geq t_0$.

It can also be shown that if, in addition to the external message arrival rates, the magnitude of the initial conditions is also sufficiently small, then the controller (6.112)–(6.113) also drives the queue length m of the processed messages immediately to zero.

Corollary 6.3 *Consider a connected network with N nodes whose dynamics at each node is described by* (6.68)–(6.69), *where the time-scale for each node may be different. The processing delay times may be time-varying, but are assumed to be bounded. Assume that the controller described by* (6.112)–(6.113) *is applied where the matrix* Θ *is constructed by Algorithm* 1. *Then for all constant* $\bar{f} \in \Re_+^{N(N-1)}$, *for all constant* $\hat{x}_0 \in \Re_+^{N^*}$, *and for all* $t_0 \in \mathbf{Z}$, *there exist scalars* $\phi_x > 0$ *and* $\phi_f > 0$, *such that for all initial conditions* $x(t_0) \leq \phi_x \hat{x}_0$, $x(t_0) \geq 0$ *and for all* $f(t) \in \Omega_{\phi_f}^{\hat{f}}$, $m_i^{lk}(t_0 + 1) = 0$, $\forall i \in \bar{N}^k$, $\forall l \in \bar{N}^k$, $\forall k \in \bar{N}$, *and* $m(t)$, $p(t)$, $q(t)$, *and* $u(t)$ *remain bounded for all* $t \geq t_0$. *Furthermore, if processing delay time* r_i^{lk} *is constant,* $\forall t \geq t_0$, $\forall i \in \bar{N}^k$, $\forall l \in \bar{N}^k$, $\forall k \in \bar{N}$, *then* $m(t) = 0$, $\forall t \geq t_0 + 1$, *and* $p(t), q(t)$ *and* $u(t)$ *converge to the steady-state values given in* (6.98)–(6.99).

Remark 6.19 In the foregoing analysis, the following assumptions were made:

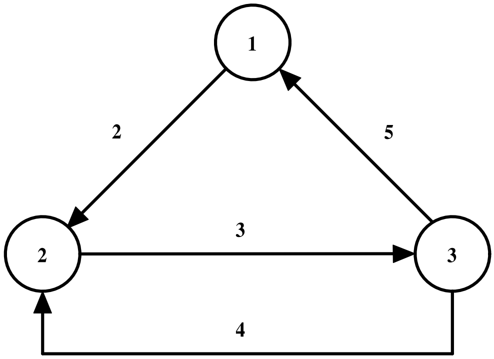
(i) the rate of messages being sent out from one node to another (the control signals) are updated at discrete periodic instants;
(ii) processing delay times are fixed;
(iii) the control signals are updated synchronously throughout the network;
(iv) upper constraints on the volume of messages that can be processed or buffered are not violated.

It was shown that (iii) can be removed; in fact control signals may be updated with different periods at different nodes. We have also shown that (ii) is not a necessary assumption, as long as processing delay times remain bounded. This latter assumption can be justified as long as the arrival rate of messages remain bounded since any message can be processed in a finite time. Assumption (i) represents the usual case in a practical implementation [14, 15] for the case when the control signals are updated continuously. Finally, (iv) can be justified, since for today‘s computers, storage restrictions are usually less important than transmission capacity constraints.

6.8.6 Simulation Examples

In this section we consider the network shown in Fig. 6.9. The capacity of each link is indicated by a number associated with each link in the figure. The processing delays are assumed to be $r_i^{lk} = 5$ ($i \in \bar{N}^k$, $\ell \in \bar{N}^k$, $k \in \bar{N}$, $N = 3$).

Fig. 6.9 Example network



Routing Controller Obtained. The maximum external message arrival rates are obtained by solving the LP problem and are given as follows:

$$(\tilde{f}_1^2)_{\max} = (\tilde{f}_1^3)_{\max} = 2, \quad (\tilde{f}_2^1)_{\max} = (\tilde{f}_2^3)_{\max} = 3,$$
$$(\tilde{f}_3^1)_{\max} = 5, \quad (\tilde{f}_3^2)_{\max} = 6$$

and the non-zero elements of the feedback matrix Θ obtained from Algorithm 1 are therefore given by:

$$\begin{aligned} h_{12}^{12} &= h_{12}^{13} = h_{12}^{23} = h_{12}^{32} = h_{23}^{13} = h_{23}^{21} = h_{23}^{23} \\ &= h_{23}^{31} = h_{31}^{21} = h_{31}^{31} = 1, \\ h_{31}^{12} &= h_{32}^{12} = \frac{1}{2}, \quad h_{31}^{32} = \frac{1}{3}, \quad h_{32}^{32} = \frac{2}{3}, \end{aligned}$$

where $\bar{f}$ is defined by (6.70). The controller (6.79) (equivalently (6.112)–(6.113)), with the choice of Θ as above, then produces the following controls:

1. For link 1 to 2:

$$\begin{cases} u_{12}^{12}(t) = \gamma_{12}(t)\phi_1^{12}(t), \\ u_{12}^{13}(t) = \gamma_{12}(t)\phi_1^{13}(t), \\ u_{12}^{23}(t) = \gamma_{12}(t)\phi_1^{23}(t), \\ u_{12}^{32}(t) = \gamma_{12}(t)\phi_1^{32}(t), \end{cases}$$

   where

$$\gamma_{12}(t) = \begin{cases} 1 & \text{if } \hat{U}_{12}(t) \le 2, \\ \frac{2}{\hat{U}_{12}} & \text{if } \hat{U}_{12}(t) > 2 \end{cases}$$

   and

$$\hat{U}_{12}(t) = \phi_1^{12}(t) + \phi_1^{13}(t) + \phi_1^{23}(t) + \phi_1^{32}(t).$$

2. For link 2 to 3:

$$\begin{cases} u_{23}^{13}(t) = \gamma_{23}(t)\phi_2^{13}(t), \\ u_{23}^{21}(t) = \gamma_{23}(t)\phi_2^{21}(t), \\ u_{23}^{23}(t) = \gamma_{23}(t)\phi_2^{23}(t), \\ u_{23}^{31}(t) = \gamma_{23}(t)\phi_2^{31}(t), \end{cases}$$

where

$$\gamma_{23}(t) = \begin{cases} 1 & \text{if } \hat{U}_{23}(t) \le 3, \\ \frac{3}{\hat{U}_{23}} & \text{if } \hat{U}_{23}(t) > 3 \end{cases}$$

and

$$\hat{U}_{23}(t) = \phi_2^{13}(t) + \phi_2^{21}(t) + \phi_2^{23}(t) + \phi_2^{31}(t).$$

3. For link 3 to 1:

$$\begin{cases} u_{31}^{12}(t) = \gamma_{31}(t)\phi_3^{31}(t), \\ u_{31}^{21}(t) = \gamma_{31}(t)\phi_3^{13}(t), \\ u_{31}^{31}(t) = \gamma_{31}(t)\phi_3^{31}(t), \\ u_{31}^{32}(t) = \gamma_{31}(t)\phi_3^{32}(t), \end{cases}$$

where

$$\gamma_{31}(t) = \begin{cases} 1 & \text{if } \hat{U}_{31}(t) \le 5, \\ \frac{5}{\hat{U}_{31}} & \text{if } \hat{U}_{31}(t) > 5 \end{cases}$$

and

$$\hat{U}_{31}(t) = \frac{1}{2}\phi_3^{12}(t) + \phi_3^{21}(t) + \phi_3^{31}(t) + \frac{1}{3}\phi_3^{32}(t).$$

4. For link 3 to 2:

$$\begin{cases} u_{32}^{12}(t) = \gamma_{32}(t)\phi_3^{12}(t), \\ u_{32}^{21}(t) = \gamma_{32}(t)\phi_3^{13}(t), \\ u_{32}^{31}(t) = \gamma_{32}(t)\phi_3^{23}(t), \\ u_{32}^{32}(t) = \gamma_{32}(o)\phi_3^{32}(t), \end{cases}$$

where

$$\gamma_{32}(t) = \begin{cases} 1 & \text{if } \hat{U}_{32}(t) \le 4, \\ \frac{4}{\hat{U}_{32}} & \text{if } \hat{U}_{32}(t) > 4 \end{cases}$$

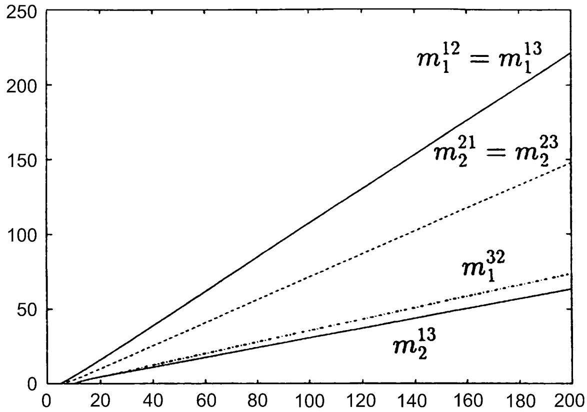


Fig. 6.10 m_i^{lk} versus time for case 1

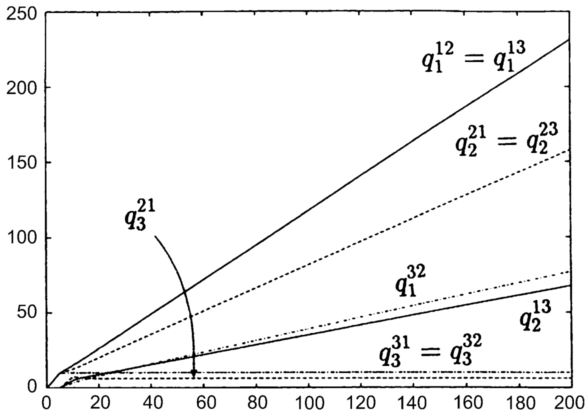


Fig. 6.11 q_i^{lk} versus time for case 1

and

$$\hat{U}_{32}(t) = \frac{1}{2}\phi_3^{12}(t) + \frac{2}{3}\phi_3^{32}(t).$$

This controller is now applied to the simulation example for the following cases:

Case 6.1 $f_i^k(t) = 2, \forall t \geq 0, \forall i \in \bar{N}^k, x(0) = 0$. In this case, certain components of m and q are unbounded as shown in Figs. 6.10 and 6.11. The components which are not shown in the figures remain at zero $\forall t \geq 0$. The total message flow rates along the links:

$$u_{ij}^t \leq \sum_{k \in \bar{N}^i} \sum_{l \in \bar{N}^k} u_{ij}^{lk}, \quad j \in \mathcal{D}(i),\ i \in \bar{N}$$

are shown in Fig. 6.12. The capacities of the links l-to-2 and 2-to-3 are saturated. In fact, in this case, there is no solution to the routing problem which keeps the queue lengths bounded.

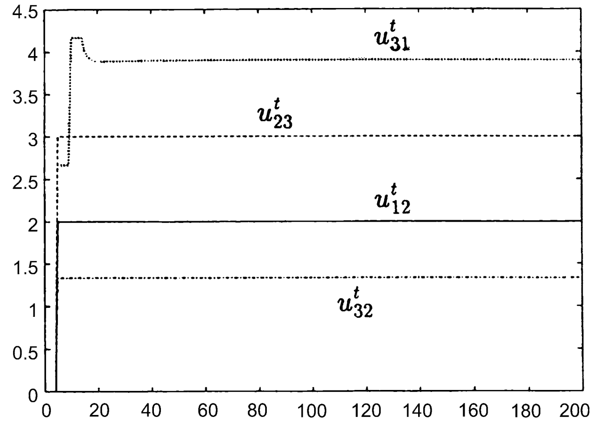


Fig. 6.12 u^t_{ij} versus time for case 1

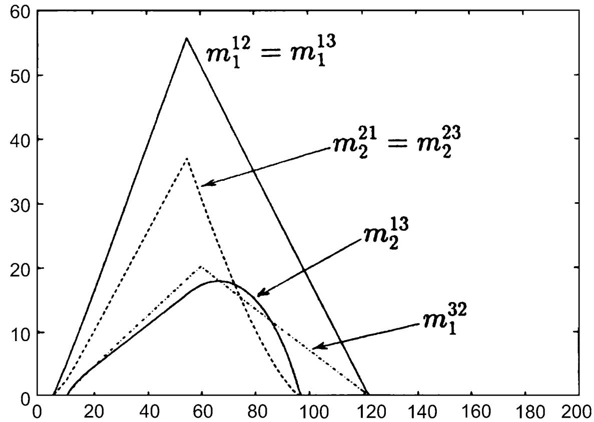


Fig. 6.13 m_i^{lk} versus time for case 2

Case 6.2 $x(0) = 0$,

$$f_i^k(t) = \begin{cases} 2, & 0 \le t < 50, \\ 0, & t \ge 50 \end{cases} \qquad \forall i \in \bar{N}^k,\ \forall k \in \bar{N}.$$

Certain components of m and q build-up during the first 50 sampling periods. However, they start decreasing immediately after the external message arrivals are shut-off, as shown in Figs. 6.13 and 6.14.

All the queues are cleared within 76 sampling periods after shut-off. The total message flow rates along the links are shown in Fig. 6.15.

Case 6.3 $f_i^k(t) = \frac{1}{2}$, $\forall t \ge 0$, $\forall i \in \bar{N}^k$, $\forall k \in \bar{N}$, $x(0) = 0$. In this case $m(t)$ remains at zero $\forall t > 0$; $q(t)$ converges to a constant steady-state value as shown in Fig. 6.16. The total message flow rates are shown in Fig. 6.17.

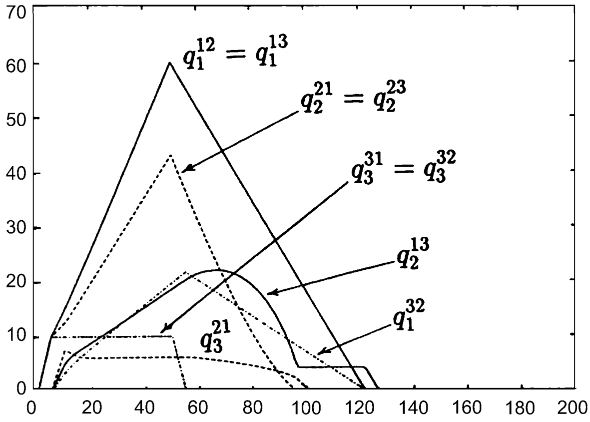


Fig. 6.14 q_i^{lk} versus time for case 2

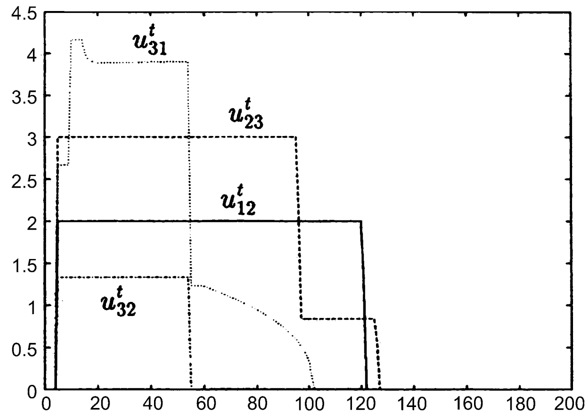


Fig. 6.15 u_{ij}^t versus time for case 2

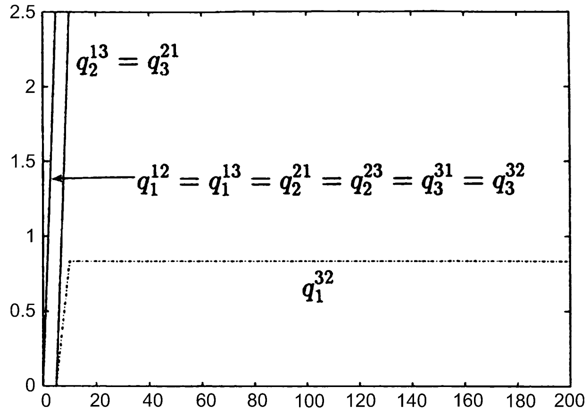


Fig. 6.16 q_i^{lk} versus time for case 3

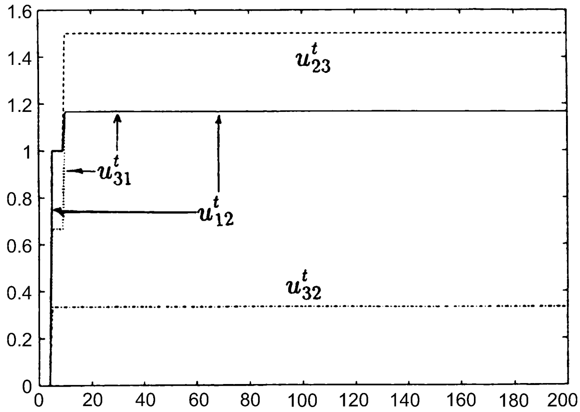


Fig. 6.17 u^t_{ij} versus time for case 3

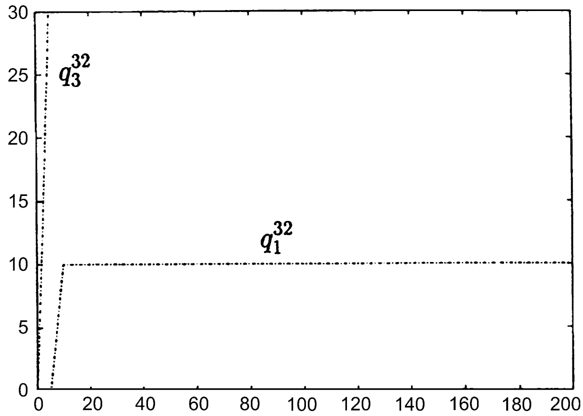


Fig. 6.18 q^{lk}_i versus time for case 4

Case 6.4 $f^2_3(t) = 6 = (\tilde{f}^2_3)_{\max}$, $f^k_i(t) = 0$, $\forall(i,k) \neq (3,2)$, $\forall t \geq 0$, $x(0) = 0$. In this case $m(t)$ remains at zero $\forall t \geq 0$; and $q(t)$ converges to a constant steady state value as shown in Fig. 6.18. The individual message flow rates are shown in Fig. 6.19. Note that the steady-state values of u^{32}_{12} and u^{32}_{32} are equal to the link capacities c_{l2} and c_{32} respectively.

Case 6.5 $f^2_3(t) = 10 > (\tilde{f}^2_3)_{\max}$, $f^k_i(t) = 0$, $\forall(i,k) \neq (3,2)$, $\forall t \geq 0$, $x(0) = 0$. In this case, m and q are unbounded as shown in Figs. 6.20 and 6.21, and the link capacities of the links l-to-2, 3-to-1, and 3-to-2 are saturated, as shown in Fig. 6.22. In this example, since $f^2_3(t) = 10 > (\tilde{(}f)^2_3)_{\max} = 6$, there is no solution to the routing problem which keeps the queue lengths bounded.

Case 6.6 $f^2_1(t) = (\tilde{f}^2_1)_{\max} = 2$, $f^3_2(t) = (\tilde{f}^3_2)_{\max} = 3$, $f^1_3(t) = (\tilde{f}^1_3)_{\max} = 5$, $f^3_1(t) = f^1_2(t) = f^2_3 t = f^k_i(t) = 0$, $\forall t \geq 0$, $x(0) = 0$. In this case, $m(t)$ remains at zero $\forall t \geq 0$, and $q(t)$ converges to a constant steady-state value as shown in

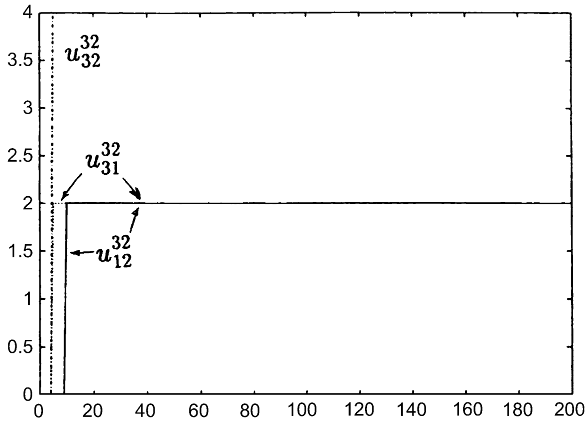


Fig. 6.19 u_{ij}^{lk} versus time for case 4

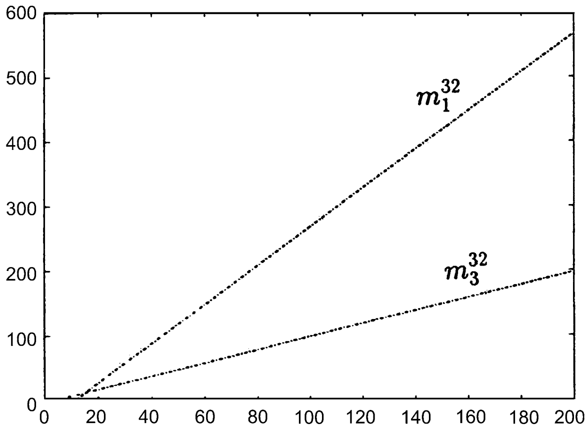


Fig. 6.20 m_i^{lk} versus time for case 5

Fig. 6.23. The message flow rates $u_{12}^{12}(t)$, $u_{23}^{23}(t)$, and u_{31}^{31} converge to the link capacities c_{12}, c_{23}, and c_{31} respectively as shown in Fig. 6.24. Other message flow rates remain at zero.

Case 6.7 $f(t) = 0$, $\forall t \geq 0$, $m_i^{lk}(0) = 1$, $\forall i \in \bar{N}^k$, $\forall l \in \bar{N}^k$, $\forall k \in \bar{N}$, $w_1(0) = w_2(0) = \cdots = w_5(0)$. In this case, all the queues of the processed messages are cleared within 2 sampling periods as shown in Table 6.2, and the total queue lengths are cleared within 7 sampling periods as shown in Table 6.3. The total message flow rates along the links are shown in Table 6.4. Between $t = 2$ and $t = 5$ there is no flow of messages along the links; the messages are processed at different nodes during this period.

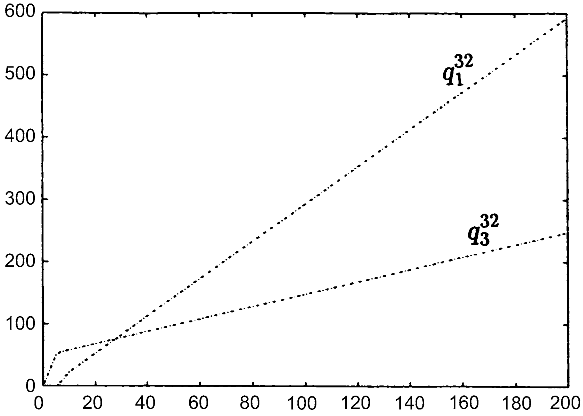


Fig. 6.21 q_i^{lk} versus time for case 5

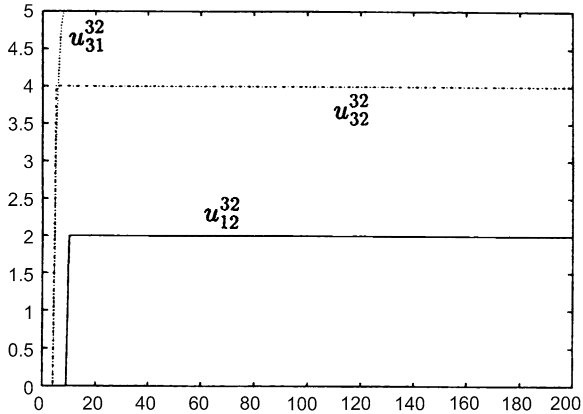


Fig. 6.22 u_{ij}^{lk} versus time for case 5

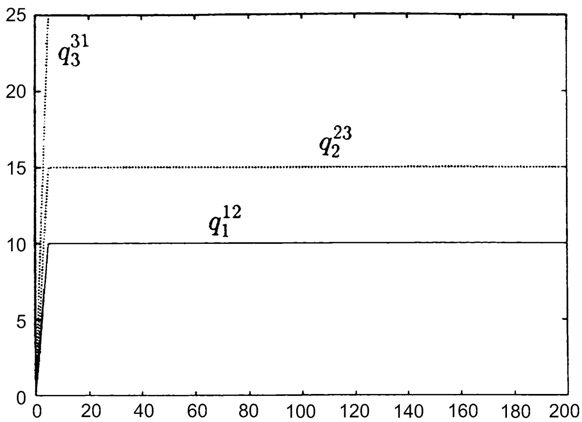


Fig. 6.23 q_i^{lk} versus time for case 6

Fig. 6.24 u_{ij}^{lk} versus time for case 6

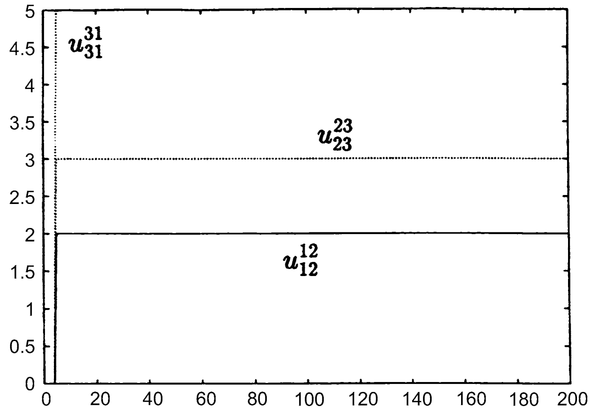


Table 6.2 m_i^{lk} versus time for case 7

	$m_1^{12}=m_1^{13}=m_1^{23}=m_1^{32}$	$m_2^{13}=m_2^{21}=m_2^{23}=m_2^{31}$	$m_3^{12}=m_3^{21}=m_3^{31}=m_3^{32}$
$t=0$	1.0000	1.0000	1.0000
$t=1$	0.5000	0.2500	0.0000
$t\geq 2$	0.0000	0.0000	0.0000

Table 6.3 q_i^{lk} versus time for case 7

	q_1^{12}	$q_1^{13}=q_1^{23}$	q_1^{32}	$q_2^{13}=q_2^{23}$	$q_2^{21}=q_2^{31}$	$q_3^{12}=q_3^{32}$	$q_3^{21}=q_3^{31}$
$t=0$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$t=1$	1.0000	0.5000	0.8333	0.7500	0.2500	0.0000	0.7500
$t=2,3,4,5$	0.5000	0.0000	0.3333	1.0000	0.0000	0.0000	1.0000
$t=6$	0.0000	0.0000	0.0000	0.5000	0.0000	0.0000	0.2500
$t\geq 7$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 6.4 u_{ij}^t versus time for case 7

	u_{12}^t	u_{23}^t	u_{31}^t	u_{32}^t
$t=0$	2.0000	3.0000	2.8333	1.1667
$t=1$	2.0000	1.0000	0.0000	0.0000
$t=2,3,4$	0.0000	0.0000	0.0000	0.0000
$t=5$	0.8333	1.0000	1.5000	0.0000
$t=6$	0.0000	1.0000	0.5000	0.0000
$t\geq 7$	0.0000	0.0000	0.0000	0.0000

6.9 Notes and References

The main contribution of this chapter lies is the methodology of solving the centralized and decentralized routing problems by incorporating queuing dynamics and physical constraints that exist in the traffic network. The transmitting, propagating, and processing delays considered in the dynamics of the network were assumed to be unknown and fast time-varying. By employing the $\mathcal{H}_\infty$ robust control strategy, the developed routing schemes will guarantee the desired routing performance in the presence of unknown *fast time-varying* delays and other network uncertainties through the minimization of the *worst-case queuing length*. It is worth noting that since the proposed decentralized routing controller can be implemented locally at each individual node, it is therefore scalable to large and crowded traffic networks.

For routing in multi-destination data communication networks, a decentralized controller were presented where the dynamic model developed to describe the network dynamics incorporates processing delays. The controller design involves an optimization problem, which can be conveniently solved off-line. The developed controller has the property that it allows the maximum possible magnitude on each external message arrival rate of the system to occur, without violating any constraints on the system. Further interesting results and applications can been developed by applying the ideas of [16, 28, 31, 32, 36].

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Chapter 7
Decentralized Control of Markovian Jump Systems

This chapter deals with systems having Markovian jump parameters. There are basically two types of models. The first type describes interconnected systems with Markovian jump parameters for which the problems of stochastic stability and stabilization are examined and a set of feedback controls is conveniently developed. In the second type, we deal with systems with Markov chains exhibiting slow-fast separation. An appropriate averaging and aggregation technique is developed for this purpose. Under state feedback policies, the $\mathcal{H}_\infty$ control design for large scale jump linear systems where the form process admits strong and weak interactions. Through an analysis that covers both finite and infinite horizon cases and using averaging and aggregation techniques, an aggregate jump linear system of considerably smaller order has been obtained, along with a corresponding (compatible) cost function. This reduced-order (aggregate) problem is another piecewise-deterministic $\mathcal{H}_\infty$ control problem, and based on the solution of this problem, we obtain the asymptotic limit of the optimal performance level for the full-order system, as well as an approximate controller that can asymptotically achieve any desired performance level for the full-order system.

7.1 Control for Markovian Jump Systems

In this section, we examine the problems of stochastic stability and stabilization for a class of interconnected systems with Markovian jump parameters. The jumping parameters are treated as continuous-time, discrete-state Markov process. The purpose is to design decentralized state feedback controller such that stochastic quadratic stability and a prescribed $\mathcal{H}_\infty$-performance are guaranteed. Next, the robust $\mathcal{H}_\infty$-control problem for linear interconnected systems with Markovian jump parameters and parametric uncertainties is studied. The parametric uncertainties are assumed to be real, time-varying and norm-bounded that appear in the state matrix. Both cases of finite-horizon and infinite-horizon are analyzed. We establish that the decentralized control problem for interconnected Markovian jump systems with and without

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uncertain parameters can be essentially solved in terms of the solutions of a finite set of coupled differential (or algebraic) Riccati equations. Extension of the developed results to the case of uncertain jumping rates is provided. Further interesting results and applications can been developed by applying the ideas of [26, 29–31, 60].

7.1.1 Introduction

Dynamical systems subject to frequent unpredictable structural changes can be conveniently modeled as piecewise deterministic systems, where the underlying dynamics are represented by different forms depending on the value of an associated Markov chain process. An important class of such systems is the jump linear systems. Research into class of systems and their applications span several decades. For some representative prior work on this general topic, we refer the reader to [1–4, 6–12, 14–21, 32–48] and the references therein.

When the plant modeling uncertainty or external disturbance uncertainty is of major concern in control systems, robust control theory provides tractable design tools using the time domain and the frequency domain; see [9] and the references cited therein.

On the other hand, problems of decentralized control and stabilization of interconnected systems are receiving considerable interests [11–18] where most of the effort are focused on dealing with the interaction patterns and performing the control analysis and design on the subsystem level.

The purpose of this section is to develop criteria of stochastic stability and stabilization of a class of linear interconnected systems with Markovian jump parameters. The jumping parameters are treated as continuous-time, discrete-state Markov process. First, the notion of stochastic decentralized stability is introduced and an appropriate LMI-based criterion is developed. Based thereon, the purpose is to design decentralized state feedback controller such that stochastic quadratic stability and a prescribed $\mathcal{H}_\infty$-performance are guaranteed. Next, the robust $\mathcal{H}_\infty$-control problem for linear interconnected systems with Markovian jump parameters and parametric uncertainties is studied. The parametric uncertainties are assumed to be real, time-varying and norm-bounded that appear in the state matrix. Both cases of finite-horizon and infinite-horizon are analyzed. We establish that the decentralized control problem for interconnected Markovian jump systems with and without uncertain parameters can be essentially solved in terms of the solutions of a finite set of coupled differential (or algebraic) Riccati equations.

7.1.2 Problem Statement

Given a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ where Ω is the sample space, $\mathcal{F}$ is the algebra of events and $\mathbf{P}$ is the probability measure defined on $\mathcal{F}$. We consider a class of uncertain systems with Markovian jump parameters described by:

$$\dot{x}(t) = A(t, \eta_t)x(t) + B(t, \eta_t)u(t) + \Gamma(t, \eta_t)w(t),$$
$$x_o = \phi, \quad \eta_o = i, \quad t \in [0, \mathcal{T}], \tag{7.1}$$
$$z(t) = G(t, \eta_t)x(t) + F(t, \eta_t)u(t) \tag{7.2}$$

which we will recognize in the sequel as an interconnection of n_s coupled uncertain subsystems and modeled in state-space form by:

$$\dot{x}_j(t) = A_j(t, \eta_t)x_j(t) + B_j(t, \eta_t)u_j(t) + \Gamma_j(t, \eta_t)w_j(t) + g_j(t),$$
$$x_o = \phi, \quad \eta_o = i, \quad t \in [0, \mathcal{T}], \quad j \in \{1, \dots, n_s\}, \tag{7.3}$$
$$g_j(t) = \sum_{k=1, j\neq k}^{n_s} A_{jk}(t, \eta_t)x_k(t), \tag{7.4}$$
$$y_j(t) = x_j(t), \tag{7.5}$$
$$z_j(t) = G_j(t, \eta_t)x_j(t) + F_j(t, \eta_t)u_j(t), \tag{7.6}$$

where for (7.1)–(7.2) x, u, w satisfy

$$x = (x_1^t, \dots, x_{n_s}^t)^t, \quad u = (u_1^t, \dots, u_{n_s}^t)^t,$$
$$w = (w_1^t, \dots, w_{n_s}^t)^t, \quad z = (z_1^t, \dots, z_{n_s}^t)^t.$$

For (7.3)–(7.6) with $j \in \{1, \dots, n_s\}$, $x_j(t) \in \Re^{n_j}$ is the state vector; $u_j(t) \in \Re^{m_j}$ is the control input, $w_j(t) \in \Re^{q_j}$ is the disturbance input which belongs to $\mathcal{L}_2[0, \mathcal{T}]$; $y_j(t) \in \Re^{p_j}$ is the measured output and $z_j(t) \in \Re^{r_j}$ is the controlled output which belongs to $\mathcal{L}^2[(\Omega, \mathcal{F}, \mathbf{P}), [0, \mathcal{T}]]$. From now onwards, the notations **Lss** and **Css** refer, respectively, to the original large-scale system (7.1)–(7.2) and composite subsystem representation (7.3)–(7.6). An important identity that links both representations is expressed as [22]:

$$\sum_{j=1}^{n_s}\left\{\sum_{k=1}^{n_s} A_{jk}(t, \eta_t)x_k(t)\right\} = \sum_{k=1}^{n_s}\left\{\sum_{j=1}^{n_s} A_{kj}(t, \eta_t)x_j(t)\right\}. \tag{7.7}$$

The main difference in the underlying treatment of both representations is the explicit modeling of interconnections among subsystems as represented by the vector $g_j(t) \in \Re^{n_j}$ which in effect designates an interaction input to the jth subsystem. For various technical and operational factors, it is considered convenient to deal with **Css** instead of **Lss** and hence, in the remaining part of this work, we will base the analysis and design on the subsystem level. This implies that we will closely examine the role of interactions on the system behavior.

With reference to either **Css** or **Lss**, the random form process $\{\eta_t, t \in [0, \mathcal{T}]\}$ is a homogeneous, finite-state Markovian process with right continuous trajectories and taking values in a finite set $\mathcal{S} = \{1, 2, \dots, s\}$ with transition probability from mode i

at time t to mode r at time $t+\delta$, $i, r \in \mathcal{S}$:

$$p_{ir} = Pr(\eta_{t+\delta} = r \mid \eta_t = i) = \begin{cases} \alpha_{ij}\delta + o(\delta), & \text{if } i \neq r, \\ 1 + \alpha_{ij}\delta + o(\delta), & \text{if } i = r \end{cases} \tag{7.8}$$

with transition probability rates $\alpha_{ij} \geq 0$ for $i, r \in \mathcal{S}$, $i \neq r$ and

$$\alpha_{ii} = -\sum_{m=1, m\neq i}^{s} \alpha_{im}, \tag{7.9}$$

where $\delta > 0$ and $\lim_{\delta \downarrow 0} o(\delta)/\delta = 0$. The set $\mathcal{S}$ comprises the various operation modes of system (7.3)–(7.6) and for each possible value $\eta_t = i$, $i \in \mathcal{S}$, we will denote the matrices of subsystem j associated with mode i by $A_j(t, \eta_t) := A_j(t, i)$, $B_j(t, \eta_t) := B_j(t, i)$, $\Gamma_j(t, \eta_t) := \Gamma_j(t, i)$, $A_{jk}(t, \eta_t) := A_{jk}(t, i)$, $G_j(t, \eta_t) := G_j(t, i)$, $F_j(t, \eta_t) := F_j(t, i)$ where $A_j(t, i)$, $B_j(t, i)$, $\Gamma_j(t, i)$, $A_{jk}(t, i)$, $G_j(t, i)$, $F_j(t, i)$ are known, real, time-varying, piecewise-continuous between each jump, bounded matrices of appropriate dimensions describing the nominal system.

Distinct from (7.3)–(7.6) is the free nominal jump subsystem:

$$\dot{x}_j(t) = A_j(t, \eta_t)x_j(t) + g_j(t), \quad x_o = \phi, \quad \eta_o = i, \quad i \in \mathcal{S}, \quad \in \{1, \dots, n_s\} \tag{7.10}$$

and the nominal jump subsystem

$$\dot{x}_j(t) = A_j(t, \eta_t)x_j(t) + B_j(t, \eta_t)u_j(t) + g_j(t), \quad x_o = \phi, \quad \eta_o = i, \quad i \in \mathcal{S}, \quad j \in \{1, \dots, n_s\} \tag{7.11}$$

for which we have the following

Definition 7.1 The free nominal jump subsystem (7.10) is said to be *stochastically decentrally stable* if for all finite initial state $\phi \in \Re^n$, for all interaction inputs $g_j(t) \in \Re^n$, $j \in \{1, \dots, n_s\}$ and initial mode $\eta_o \in \mathcal{S}$

$$\int_0^\infty \mathcal{E}\{||x_j(t, g_j, \phi)||^2\}\, dt < +\infty, \quad j \in \{1, \dots, n_s\}. \tag{7.12}$$

Lemma 7.1 *Consider the free nominal jump subsystem* (7.10). *For any matrix* $Q_j(i) = Q_j^t(i) > 0$, $i \in \mathcal{S}$, *if there exist matrices* $P_j(i) = P_j^t(i) > 0$, $i \in \mathcal{S}$, $j \in \{1, \dots, n_s\}$, *satisfying*

$$P_j(i)A_j(i) + A_j^t(i)P_j(i) + \sum_{m=1, i\neq m}^{n_s} \alpha_{im} P_j(m) + P_j(i)\left\{\sum_{m=1, j\neq m}^{n_s} A_{jm}(i)A_{jm}^t\right\}P_j(i) + (n_s - 1)I + Q_j(i) = 0 \tag{7.13}$$

then the subsystem (7.10) *is stochastically decentrally stable.*

Proof Let $\Im^x[.]$ and $\Im_j^x[.]$ be the infinitesimal operators, respectively, of the processes $\{x(t), \eta_t\}$ and $\{x_j(t), \eta_t\}$ associated with the **Lss** and **Css** representations at the point $\{t, x, \eta_t\}$ [21] and let $V_j = V_j(t, x, \eta_t)$. Then,

$$\Im_j^x[V_j] = \frac{\partial V_j}{\partial t} + \dot{x}^t(t)\frac{\partial V_j}{\partial x}\bigg|_{\eta_t=i} + \sum_{k=1}^{n_s} \alpha_{ik} V_j(t, x, k, i) \tag{7.14}$$

and

$$V(t, x, \eta_t) := \sum_{j=1}^{n_s} V_j(t, x, \eta_t), \quad \Im^x[V(t, x, \eta_t)] := \sum_{j=1}^{n_s} \Im_j^x[V_j(t, x, \eta_t)]. \tag{7.15}$$

For $\eta_t = i$, $i \in \mathcal{S}$, and $V_j(t, x, \eta_t) = x_j^t(t)P_j(t, i)x_j(t)$, we get from (7.14)–(7.15):

$$\begin{aligned}
\Im^x[V] &:= \sum_{j=1}^{n_s} \Im_j^x[V_j] \\
&= \sum_{j=1}^{n_s} \Bigg\{ \dot{x}_j^t(t)P_j(t, i)x_j(t) + x_j^t(t)\dot{P}_j(t, i)x_j(t) + x_j^t(t)P_j(t, i)\dot{x}_j(t) \\
&\quad + \sum_{m=1}^{n_s} \alpha_{im} x_j^t(t)P_j(t, m)x_j(t) \Bigg\} \\
&= \sum_{j=1}^{n_s} \Bigg\{ x_j^t(t)[\dot{P}_j(t, i) + A_j^t(t, i)P_j(t, i) + P_j(t, i)A_j(t, i)]x_j(t) \\
&\quad + x_j^t(t)P_j(t, i) \sum_{m=1, j\neq m}^{n_s} A_{jm}(t, \eta_t)x_m(t) \\
&\quad + \sum_{m=1, j\neq m}^{n_s} x_m^t(t)A_{jm}^t(t, \eta_t)P_j(t, i)x_j^t(t) \\
&\quad + \sum_{m=1}^{n_s} \alpha_{im} x_j^t(t)P_j(t, m)x_j(t) \Bigg\}.
\end{aligned} \tag{7.16}$$

Making use of identity (7.7), applying the inequality

$$\Sigma_1\Sigma_3\Sigma_2 + \Sigma_2^t\Sigma_3^t\Sigma_1^t \leq \alpha^{-1}\Sigma_1\Sigma_1^t + \alpha\Sigma_2^t\Sigma_2, \quad \forall \alpha > 0 \tag{7.17}$$

for any real matrices Σ_1, Σ_2 and Σ_3 with appropriate dimensions and $\Sigma_3^t \Sigma_3 \le I$, and using the equality $\sum_{k=1, j\neq k}^{n_s} I = (n_s - 1)I$, it follows from (7.16) that

$$\begin{aligned}
\Im^x[V] \le \sum_{j=1}^{n_s} \Bigg\{ & x_j^t(t) \Bigg[\dot{P}_j(t,i) + A_j^t(t,i) P_j(t,i) + P_j(t,i) A_j(t,i) \\
& + \sum_{m=1}^{n_s} \alpha_{im} P_j(t,m) \Bigg] x_j(t) + x_j^t(t) P_j(t,i) \\
& \times \Bigg[\sum_{k=1, j\neq k}^{n_s} A_{jk}(t,\eta_t) A_{jk}^t(t,\eta_t) \Bigg] P_j(t,i) x_j(t) + \sum_{k=1, j\neq k}^{n_s} x_k^t(t) x_k(t) \Bigg\} \\
\le \sum_{j=1}^{n_s} \Bigg\{ & x_j^t(t) \Bigg[\dot{P}_j(t,i) + A_j^t(t,i) P_j(t,i) + P_j(t,i) A_j(t,i) \\
& + \sum_{m=1}^{n_s} \alpha_{im} P_j(t,m) \Bigg] x_j(t) + x_j^t(t) P_j(t,i) \Bigg[\sum_{k=1, j\neq k}^{n_s} A_{jk}(t,\eta_t) A_{jk}^t(t,\eta_t) \Bigg] \\
& \times P_j(t,i) x_j(t) + x_j^t(t) [(n_s - 1) I] x_j(t) - x_j^t(t) x_j(t) \Bigg\} \\
:= \sum_{j=1}^{n_s} & \Im_j^x[V_j].
\end{aligned} \tag{7.18}$$

As $t \to \infty$, it follows that $P_j(t,i) \to P_j(i)$, $\dot{P}_j(t,i) \to 0$. By using (2.13), inequality (2.17) reduces to:

$$\Im_j^x[V_j] \le -x_j^t(t) Q_j(i) x_j(t) - x_j^t(t) x_j(t) \tag{7.19}$$

and therefore

$$\mathcal{E}\big\{ \Im_j^x[V_j] \big\} < 0 \tag{7.20}$$

which in the light of Definition 7.1 completes the proof. □

Remark 7.1 In [18], it has been established that, for subsystem (7.10) with $j = 1$, the terms “stochastically stable”, “exponentially mean-square stable”, and “asymptotically mean-square stable”, are equivalent, and any of them can imply “almost surely asymptotically stable”. Extending on these results, we have introduced Definition 7.1 to suit **Lss** and **Css** representations. In the sequel, we will use for subsystem (7.10) with $j \in \{1, \ldots, n_s\}$, the equivalent terms “stochastically decentrally stable”, “exponentially mean-square decentrally stable”, and “asymptotically mean-square decentrally stable”, interchangeably.

Definition 7.2 The nominal jump subsystem (7.11) is said to be *stochastically decentrally stabilizable* if for all finite initial state $\phi \in \Re^n$, for all interaction inputs $g_j(t) \in \Re^n$, $j \in \{1, \dots, n_s\}$ and initial mode $\eta_o \in \mathcal{S}$ there exists a linear constant feedback gain $K_j^*(t, \eta_t)$, $j \in \{1, \dots, n_s\}$, such that the decentralized control law

$$u_j(t) = -K_j^*(t, \eta_t)x_j(t), \quad j \in \{1, \dots, n_s\}, \ \|K_j^*(t, \eta_t)\| < \infty \tag{7.21}$$

ensures that the resulting closed-loop subsystem is stochastically decentrally stable.

By similarity to Lemma 7.1, we have the following result for the stochastic decentralized stabilizability of subsystem (7.11).

Lemma 7.2 *Consider the nominal jump subsystem* (7.11). *For any matrix* $Q_j(i) = Q_j^t(i)^t > 0, i \in \mathcal{S}$, *if there exist matrices* $P_j(i) = P_j^t(i)^t > 0, i \in \mathcal{S}, j \in \{1, \dots, n_s\}$, *satisfying*

$$\begin{aligned} &P_j(i)\bar{A}_j(i) + \bar{A}_j^t(i)P_j(i) + \sum_{m=1}^{n_s} \alpha_{im} P_j(m) \\ &\quad + P_j(i)\left\{\sum_{k=1, j\neq k}^{n_s} A_{jk}(i)A_{ij}^t\right\}P_j(i) + (n_s - 1)I + Q_j(i) = 0, \quad \forall i \in \mathcal{S}, \end{aligned} \tag{7.22}$$

where

$$\bar{A}_j(i) = A_j(i) - B_j(i)K_j^*(i) \tag{7.23}$$

then the subsystem (7.11) *is stochastically decentrally stabilizable.*

Remark 7.2 Both Lemma 7.1 and Lemma 7.2 show that the stochastic stabilizability of the nominal jump subsystem is related to the existence of positive-definite solutions to a set of **s** coupled algebraic Riccati equations. Equivalently stated, the stochastic stabilizability of the interconnected nominal jump system amounts to the existence of positive-definite solutions to a coupled set of $n_s \times s$ algebraic Riccati equations.

In the sequel, associated with Css (7.3)–(7.6) such that $j \in \{1, \dots, n_s\}$, we consider the stabilization problem of Lss (7.1)–(7.2) with $\mathcal{H}_\infty$ performance using decentralized state-feedback controllers of the type (7.21) under the assumption of perfect state information. The objective is to design a decentralized feedback controller $\mathcal{G}_j(t, \eta_t)$ such that, for all nonzero $w(t) \in \mathcal{L}_2$

$$\|z_j(t)\|_{E_2} := \mathcal{E}\left[\int_0^{\mathcal{T}} z_j^t(t)z_j(t)dt\right]^{1/2} < \gamma\|w_j(t)\|_2, \quad j \in \{1, \dots, n_s\}, \tag{7.24}$$

where $\gamma > 0$ is a prescribed level of disturbance attenuation. When system (7.3)–(7.6) under the action of the controller $\mathcal{G}_j(t, \eta_t)$ satisfies condition (2.23), the interconnected controlled system is said to have $\mathcal{H}_\infty$-performance over the horizon $[0, \mathcal{T}]$.

Two distinct cases arise:

(1) The finite-horizon case in which the system (7.3)–(7.6) with $j \in \{1, \dots, n_s\}$ under the decentralized feedback controller $\mathcal{G}_j(t, \eta_t)$ has performance (7.24) over a given horizon $[0, \mathcal{T}]$;
(2) The infinite-horizon case in which the system (7.3)–(7.6) with $j \in \{1, \dots, n_s\}$ under the decentralized feedback controller $\mathcal{G}_j(t, \eta_t)$ is stochastically decentrally stable and has performance (7.24) over a given horizon $[0, \infty]$.

Under perfect state-information, we consider the following system description for $\eta_t = i$, $i \in \mathcal{S}$

$$\begin{aligned} \dot{x}_j(t) &= A_j(t,i)x_j(t) + B_j(t,i)u_j(t) + \Gamma_j(t,i)w_j(t) + g_j(t) \\ &\quad + \sum_{k=1, j\neq k}^{n_s} A_{jk}(t,i)x_k(t), \quad x_o = \phi, \ t \in [0, \mathcal{T}], \end{aligned} \tag{7.25}$$

$$g_j(t) = \sum_{k=1, j\neq k}^{n_s} A_{jk}(t, \eta_t)x_k(t), \tag{7.26}$$

$$y_j(t) = x_j(t), \tag{7.27}$$

$$z_j(t) = G_j(t,i)x_j(t) + F_j(t,i)u_j(t) \tag{7.28}$$

and make the following assumptions:

Assumption 7.1 For all $i \in \mathcal{S}$ on $[0, \mathcal{T}]$ and for all $j \in \{1, \dots, n_s\}$,

$$F_j^t(t,i)F_j(t,i) = R(t,i), \quad R_j(t,i) = R^t(t,i) > 0.$$

Assumption 7.2 For all $i \in \mathcal{S}$ and for all $j \in \{1, \dots, n_s\}$,

(1) $\{A_j(i), B_j(i)\}$ is stochastically decentrally stabilizable,
(2) $\{C_j(i), A_j(i)\}$ is decentrally observable.

Remark 7.3 Assumption 7.1 ensures that the $\mathcal{H}_\infty$-control problem for system (7.25)–(7.28) is nonsingular and corresponds to the standard assumption in $\mathcal{H}_\infty$-control theory fr linear systems without jump parameters. Assumption 7.2 guarantees the existence of a decentralized stabilizing controller for system (7.25)–(7.28) subject to (7.8)–(7.9). The term "decentrally" is used to emphasize that the underlying condition is satisfied on the subsystem level.

7.1.3 $\mathcal{H}_\infty$-State Feedback Controller

In this section, we consider the design of a decentralized $\mathcal{H}_\infty$-state feedback controller for system (7.25)–(7.28) subject to (7.8)–(7.9). First we treat the design problem on a finite horizon.

Theorem 7.1 *Consider the system* (7.25)–(7.28) *subject to* (7.8)–(7.9) *and* $j \in \{1, \dots, n_s\}$. *Then, for a given* $\gamma > 0$, *there exists a decentralized state-feedback controller* $u_j(t)$ *of the type* (7.21) *such that*

$$\|z_j(t)\|_{E_2} < \gamma \|w_j(t)\|_2$$

for all nonzero $w(t) \in \mathcal{L}_2[0, \mathcal{T}]$, *if the following set of* $n_s \times s$ *coupled differential Riccati equations*:

$$\begin{aligned}
&\dot{P}_j(t,i) + P_j(t,i)A_j(t,i) + A_j^t(t,i)P_j(t,i) + (n_s - 1)I + G_j^t(t,i)G_j(t,i) \\
&\quad + \sum_{m=1, i\neq m}^{n_s} \alpha_{im} P_j(t,m) \\
&\quad + P_j(t,i)\left\{ \sum_{k=1, j\neq k}^{n_s} A_{jk}(t,i)A_{t,jk}^t + \gamma^{-2}\Gamma_j(i)\Gamma_j^t(t,i) \right. \\
&\quad \left. - B_j(t,i)R_j^{-1}(t,i)B_j^t(t,i) \right\} P_j(t,i) = 0, \\
&\quad P_j(\mathcal{T}) = 0, \quad i \in \mathcal{S}, \quad t \in [0, \mathcal{T}], \quad j \in \{1, \dots, n_s\}
\end{aligned} \tag{7.29}$$

has a solution $P_j(t,i), i \in \mathcal{S}, j \in \{1, \dots, n_s\}$ *on* $[0, \mathcal{T}]$. *Moreover, the decentralized controller is given by*:

$$\begin{aligned}
u_j(t) &= -K_j^*(t, \eta_t)x_j(t), \\
K^*(t, \eta_t) &= R_j^{-1}(t, \eta_t)[B_j^t(t, \eta_t)P_j(t, \eta_t) + F_j^t(t, \eta_t)G_j(t, \eta_t)], \\
& t \in [0, \mathcal{T}], \quad \eta_t = i \in \mathcal{S}, \quad j \in \{1, \dots, n_s\}.
\end{aligned} \tag{7.30}$$

Proof Let

$$\mathcal{J}(x_j) := \mathcal{E}\left\{ \int_0^{\mathcal{T}} z_j^t(t)z_j(t) - \gamma^2 w_j^t(t)w_j(t)\,dt \right\} \tag{7.31}$$

and let $\Im^x[.]$ be the infinitesimal operator of the process $\{x_j(t), \eta_t\}$ for system (7.25)–(7.28) at the point $\{t, x, \eta_t\}$ as given by (7.11). For $\eta_t = i, i \in \mathcal{S}$, and

$V_j(t,x,\eta_t) = x_j^t(t)P_j(t,i)x_j(t)$, we obtain:

$$\begin{aligned}
\Im^x[V] &:= \sum_{j=1}^{n_s} \Im_j^x[V_j] \\
&= \sum_{j=1}^{n_s} \Bigg\{ \dot{x}_j^t(t)P_j(t,i)x_j(t) + x_j^t(t)\dot{P}_j(t,i)x_j(t) + x_j^t(t)P_j(t,i)\dot{x}_j(t) \\
&\quad + \sum_{m=1}^{n_s} \alpha_{im} x_j^t(t)P_j(t,m)x_j(t) \Bigg\} \\
&= \sum_{j=1}^{n_s} \Bigg\{ x_j^t(t)\big\{\dot{P}_j(t,i) + A_j^t(t,i)P_j(t,i) + P_j(t,i)A_j(t,i)\big\}x_j(t) \\
&\quad + \big\{x_j^t(t)P_j(t,i)B_j(t,i)u_j(t) + u_j^t(t)B_j^t(t,i)P_j(t,i)x_j(t)\big\} \\
&\quad + \big\{x_j^t(t)P_j(t,i)\Gamma_j(t,i)w_j(t) + w_j^t(t)\Gamma_j^t(t,i)P_j(t,i)x_j(t)\big\} \\
&\quad + x_j^t(t)P_j(t,i) \sum_{k=1, j\neq k}^{n_s} A_{jk}(t,\eta_t)x_k(t) \\
&\quad + \sum_{k=1, j\neq k}^{n_s} x_k^t(t)A_{jk}^t(t,\eta_t)P_j(t,i)x_j^t(t) \\
&\quad + \sum_{m=1}^{n_s} \alpha_{im} x_j^t(t)P_j(t,m)x_j(t) \Bigg\}.
\end{aligned} \tag{7.32}$$

It follows from (7.3)–(7.4) that

$$\begin{aligned}
\mathcal{J}(x) &:= \sum_{j=1}^{n_s} \mathcal{J}(x_j) \\
&= \sum_{j=1}^{n_s} \Bigg\{ \mathcal{E}\Bigg[\Bigg\{ \int_0^{\mathcal{T}} z_j^t(t)z_j(t) \\
&\quad + x_j^t(t)\big\{\dot{P}_j(t,i) + A_j^t(t,i)P_j(t,i) + P_j(t,i)A_j(t,i)\big\}x_j(t) \\
&\quad + \big\{x_j^t(t)P_j(t,i)\Gamma_j(t,i)w_j(t) + w_j^t(t)\Gamma_j^t(t,i)P_j(t,i)x_j(t)\big\} \\
&\quad + \big\{x_j^t(t)P_j(t,i)B_j(t,i)u_j(t) + u_j^t(t)B_j^t(t,i)P_j(t,i)x_j(t)\big\} \\
&\quad + x_j^t(t)P_j(t,i) \sum_{k=1, j\neq k}^{n_s} A_{jk}(t,\eta_t)x_k(t)
\end{aligned}$$

$$+ \sum_{k=1, j\neq k}^{n_s} x_k^t(t) A_{jk}^t(t,\eta_t) P_j(t,i) x_j^t(t)$$

$$+ \sum_{m=1}^{n_s} \alpha_{im} x_j^t(t) P_j(t,m) x_j(t) - \gamma^2 w_j^t(t) w_j(t) \Big\} dt \Big]$$

$$- \mathcal{E}\Big[\int_0^{\mathcal{T}} \Im^x [x_j^t(t) P_j(t,i) x_j(t)] dt \Big] \Big\}. \tag{7.33}$$

The substitution of (7.28) into (7.5) with standard manipulations yields:

$$\mathcal{J}(x) := \sum_{j=1}^{n_s} \Big\{ \mathcal{E}\Big\{ \int_0^{\mathcal{T}} \Big[x_j^t(t)[\dot{P}_j(t,i) + A_j^t(t,i) P_j(t,i) + P_j(t,i) A_j(t,i)$$

$$+ \sum_{m=1}^{n_s} \alpha_{im} P_j(t,m)$$

$$+ P_j(t,i)\big[\gamma^{-2} \Gamma_j(t,i) \Gamma_j^t(t,i) - B_j(t,i) R_j^{-1}(t,i) B_j^t(t,i)\big] P_j(t,i)$$

$$+ G_j^t(t,i) G_j(t,i) x_j(t)$$

$$+ \big\{ u_j^t(t) + x_j^t(t)[F_j^t(t,i) G_j(t,i) + P_j(t,i) B_j(t,i)] R_j^{-1}(t,i) \big\} R_j(t,i)$$

$$\times \big\{ u_j(t) + x_j^t(t)[G_j^t(t,i) F_j(t,i) + B_j^t(t,i) P_j(t,i)] R_j^{-1}(t,i) \big\}$$

$$- \gamma^2 \big\{ w_j^t(t) - \gamma^{-2} x_j^t(t) P_j(t,i) \Gamma_j^t(t,i) \big\}$$

$$\times \big\{ w_j(t) - \Gamma_j^t(t,i) P_j(t,i) x_j(t,i) \big\}$$

$$+ x_j^t(t) P_j(t,i) \sum_{k=1, j\neq k}^{n_s} A_{jk}(t,\eta_t) x_k(t)$$

$$+ \sum_{k=1, j\neq k}^{n_s} x_k^t(t) A_{jk}^t(t,\eta_t) P_j(t,i) x_j^t(t) \Big] dt \Big\}$$

$$- \mathcal{E}\Big\{ \int_0^{\mathcal{T}} \Im^x [x_j^t(t) P_j(t,i) x_j(t)] dt \Big\} \Big\}. \tag{7.34}$$

Using inequality (7.17), (7.34) reduces to

$$\mathcal{J}(x) \le \sum_{j=1}^{n_s} \Big\{ \mathcal{E}\Big\{ \int_0^{\mathcal{T}} \Big[x_j^t(t)[\dot{P}_j(t,i) + A_j^t(t,i) P_j(t,i) + P_j(t,i) A_j(t,i)$$

$$+ \sum_{m=1}^{n_s} \alpha_{im} P_j(t,m) + G_j^t(t,i) G_j(t,i)$$

$$+ P_j(t,i)\big[\gamma^{-2}\Gamma_j(t,i)\Gamma_j^t(t,i) - B_j(t,i)R_j^{-1}(t,i)B_j^t(t,i)\big]P_j(t,i)$$
$$+ P_j(t,i)\left\{\sum_{k=1, j\neq k}^{n_s} A_{jk}(t,\eta_t)A_{jk}^t(t,\eta_t)\right\}P_j(t,i) + (n_s-1)Ix_j(t)$$
$$+ \big\{u_j^t(t) + x_j^t(t)[F_j^t(t,i)G_j(t,i) + P_j(t,i)B_j(t,i)]R_j^{-1}(t,i)\big\}R_j(t,i)$$
$$\times \big\{u_j(t) + x_j^t(t)[G_j^t(t,i)F_j(t,i) + B_j^t(t,i)P_j(t,i)]R_j^{-1}(t,i)\big\}$$
$$-\gamma^2\big\{w_j^t(t) - \gamma^{-2}x_j^t(t)P_j(t,i)\Gamma_j^t(t,i)\big\}\big\{w_j(t) - \Gamma_j^t(t,i)P_j(t,i)x_j(t,i)\big\}$$
$$\left.\left. - x_j^t(t)x_j(t)\right]dt\right\} - \mathcal{E}\left\{\int_0^{\mathcal{T}} \Im^x[x_j^t(t)P_j(t,i)x_j(t)]dt\right\}\Bigg\}. \tag{7.35}$$

On using the Dynkin formula [31]

$$\sum_{j=1}^{n_s}\left\{\mathcal{E}\left\{\int_0^{\mathcal{T}} \Im^x[x_j^t(t)P_j(t,i)x_j(t)]dt\right\}\right\}$$
$$= \sum_{j=1}^{n_s}\big\{\mathcal{E}[x_j^t(\mathcal{T})P_j(\mathcal{T},i)x_j(\mathcal{T})] - \mathcal{E}[x_j^t(0)P_j(0,i)x_j(0)]\big\}$$

together with the facts that $x_j(0) = 0$ and $P_j(\mathcal{T}, i) = 0$, we choose the decentralized controller $u_j(t)$ as that of (3.2) with $P_j(t,i)$ satisfying (7.30) and hence we get from (7.35) the inequality:

$$\mathcal{J}(x) \le \sum_{j=1}^{n_s}\left\{-\gamma^2\mathcal{E}\left\{\int_0^{\mathcal{T}} [w_j^t(t) - \gamma^{-2}x_j^t(t)P_j(t,i)\Gamma_j^t(t,i)][w_j(t)\right.\right.$$
$$\left.\left. - \Gamma_j^t(t,i)P_j(t,i)x_j(t,i)] + \gamma^{-2}x_j^t(t)x_j(t)dt\right\}\right\}$$
$$:= \sum_{j=1}^{n_s}\mathcal{J}(x_j) < 0 \tag{7.36}$$

and the proof is completed. □

For the infinite-horizon case, the main result is established by the following theorem.

Theorem 7.2 *Consider the system* (7.25)–(7.28) *subject to* (7.8)–(7.9) *and* $j \in \{1,\ldots,n_s\}$. *Then, for a given* $\gamma > 0$, *there exists a decentralized state-feedback controller* $u_j(t)$ *such that the interconnected closed-loop system is stochastically decentrally stable and*

$$\|z_j(t)\|_{E_2} < \gamma\|w_j(t)\|_2$$

for all nonzero $w(t) \in \mathcal{L}_2[0, \infty]$, *if the following set of* $n_s \times s$ *coupled algebraic Riccati equations*:

$$P_j(i)A_j(i) + A_j^t(i)P_j(i) + \sum_{m=1, i\neq m}^{n_s} \alpha_{im} P_j(m) + (n_s - 1)I + G_j^t(i)G_j(i)$$
$$+ P_j(i)\left\{\sum_{k=1, j\neq k}^{n_s} A_{jk}(i)A_{jk}^t + \gamma^{-2}\Gamma_j(i)\Gamma_j^t(i) - B_j(i)R_j^{-1}(i)B_j^t(i)\right\}P_j(i) = 0,$$
$$i \in \mathcal{S}, \quad t \in [0, \mathcal{T}], \quad j \in \{1, \dots, n_s\} \tag{7.37}$$

has a solution $P_j(i) = P_j^t(i)$, $i \in \mathcal{S}$, $j \in \{1, \dots, n_s\}$. *Moreover, the decentralized controller is given by*:

$$u_j(t) = -K_j^*(\eta_t)x_j(t),$$
$$K^*(\eta_t) = R_j^{-1}(\eta_t)[B_j^t(\eta_t)P_j(\eta_t) + F_j^t(\eta_t)G_j(\eta_t)],$$
$$t \in [0, \infty], \quad \eta_t = i \in \mathcal{S}, \quad j \in \{1, \dots, n_s\}. \tag{7.38}$$

Proof In terms of the closed-loop system matrix

$$\begin{aligned}\bar{A}_j(i) &= A_j(i) - B_j(i)K_j^*(i) \\ &= A_j(i) - B_j(i)R_j^{-1}(i)[B_j^t(i)P_j(i) + F_j^t(i)G_j(i)]\end{aligned} \tag{7.39}$$

we rewrite (7.36) as

$$P_j(i)\bar{A}_j(i) + \bar{A}_j^t(i)P_j(i) + (n_s - 1)I + G_j^t(i)G_j(i) + \sum_{m=1, i\neq m}^{n_s} \alpha_{im} P_j(m)$$
$$+ P_j(i)\left\{\sum_{k=1, j\neq k}^{n_s} A_{jk}(i)A_{jk}^t + \gamma^{-2}\Gamma_j(i)\Gamma_j^t(i) - B_j(i)R_j^{-1}(i)B_j^t(i)\right\}P_j(i)$$
$$+ P_j(i)B_j^t(i)R_j^{-1}(i)F_j^t(i)G_j(i) + G_j^t(i)F_j(i)R_j^{-1}(i)B_j^t(i)P_j(i) = 0,$$
$$i \in \mathcal{S}, \quad j \in \{1, \dots, n_s\}. \tag{7.40}$$

Since $P_j(i) = P_j^t(i) > 0$, $i \in \mathcal{S}$, $j \in \{1, \dots, n_s\}$ and the n_s-pairs $\{C_j(i), A_j(i)\}$, $i \in \mathcal{S}$, $j \in \{1, \dots, n_s\}$ are decentrally observable, the stochastic stability of the interconnected closed-loop systems follows from the results of [31]. The $\mathcal{H}_\infty$-performance $\|z_j(t)\|_{E_2} < \gamma\|w_j(t)\|_2$ for all nonzero $w_j(t) \in \mathcal{L}^2[0, \infty]$, can be readily obtained in the manner of Theorem 7.1. □

Remark 7.4 Theorem 7.1 and Theorem 7.2 establish sufficient solvability conditions for the $\mathcal{H}_\infty$-control problem of the interconnected system (2.1)–(2.3) over the finite-horizon and infinite-horizon cases, respectively. The resulting conditions are expressed in terms $n_s \times s$ coupled differential and algebraic Riccati equations, respectively. It should be noted that when $\eta_t = 1$ and $A_{jk} \equiv 0$, Theorem 7.1 and Theorem 7.2 recover the standard results of $\mathcal{H}_\infty$-control problems of single linear systems.

7.1.4 Robust $\mathcal{H}_\infty$-Control Results

In this section, we consider the design of a decentralized robust $\mathcal{H}_\infty$ feedback controller for the interconnected system (7.3)–(7.6) with uncertain parameters. In this case, the state-space model is given by:

$$S_j:\quad \dot{x}_j(t) = [A_j(t,\eta_t) + \Delta A_j(t,\eta_t)]x_j(t) + B_j(t,\eta_t)u_j(t) + \Gamma_j(t,\eta_t)w_j(t) + \sum_{k=1, j\neq k}^{n_s} A_{jk}(t,\eta_t)x_k(t), \quad x_o = \phi,\ \eta_o = i,\ t \in [0,\mathcal{T}], \tag{7.41}$$

$$y_j(t) = x_j(t), \tag{7.42}$$

$$z_j(t) = G_j(t,\eta_t)x_j(t) + F_j(t,\eta_t)u_j(t), \tag{7.43}$$

where for $j \in \{1,\dots,n_s\}$, $x_j(t)$, $u_j(t)$, $w_j(t)$, $y_j(t)$, $z_j(t)$ and $A_j(t,\eta_t)$, $B_j(t,\eta_t)$, $\Gamma_j(t,\eta_t)$, $G_j(t,\eta_t)$, $F_j(t,\eta_t)$ are the same as in (7.3)–(7.6) and $\Delta A_j(t,\eta_t)$ is a real, time-varying matrix function representing the norm-bounded parameter uncertainty. The admissible parameter uncertainties are assumed to be modeled in the form:

$$\Delta A_j(t,\eta_t) = M_j(t,\eta_t)\Delta_j(t,\eta_t)N_j(t,\eta_t), \tag{7.44}$$

where for $\eta_t = i$ and $j \in \{1,\dots,n_s\}$, $M_j(t,\eta_t) \in \Re^{n_j \times \beta_j}$ and $N_j(t,\eta_t) \in \Re^{\beta_j \times n_j}$ are known real, time-varying, piecewise-continuous matrices between each jump, which designates the way the uncertain parameters in $\Delta_j(t,\eta_t)$ affects the nominal matrix $A_j(t,\eta_t)$ with $\Delta_j(t,\eta_t)$, $\eta_t = i \in \mathcal{S}$ being an unknown, time-varying matrix function satisfying

$$\|\Delta_j(t,\eta_t)\|_2 \le 1, \quad \eta_t = i \in \mathcal{S}, \tag{7.45}$$

where the elements of $\Delta_j(t,\eta_t)$ are Lebesgue measurable for any $\eta_t = i \in \mathcal{S}$. In the infinite-horizon case, $t \to \infty$, the matrices $M_j(t,\eta_t)$, $\Delta_j(t,\eta_t)$, $N_j(t,\eta_t)$ are constants with $\eta_t = i \in \mathcal{S}$ and will be denoted by $M_j(i)$, $\Delta_j(i)$, $N_j(i)$, respectively.

In the sequel, we consider the problem of robust state-feedback control of the uncertain, interconnected Markovian jumping system (7.41)–(7.43). Our purpose is to design a decentralized feedback controller $\mathcal{G}_j(t,\eta_t)$

$$u_j(t) = -K_j(t,\eta_t)x_j(t), \quad j \in \{1,\dots,n_s\}, \tag{7.46}$$

where $\|K_j(t,\eta_t)\| < \infty$ such that, for all nonzero $w_j(t) \in \mathcal{L}^2[0,\infty)$ and for all parametric uncertainties satisfying (7.45)–(7.46)

$$\|z_j(t)\|_{E_2} < \gamma \|w_j(t)\|_2, \quad j \in \{1,\ldots,n_s\}, \tag{7.47}$$

where $\gamma > 0$ is a prescribed level of disturbance attenuation.

When system (7.41)–(7.43) under the action of the controller (7.45) satisfies condition (7.47), the interconnected controlled system is said to have $\mathcal{H}_\infty$-performance over the horizon $[0,\mathcal{T}]$. We now establish some stochastic stability properties.

Distinct from (7.41)–(7.43) is the free uncertain nominal jump subsystem:

$$\begin{aligned} S_{jo}\colon \quad \dot{x}_j(t) &= [A_j(t,\eta_t) + \Delta A_j(t,\eta_t)]x_j(t) + \sum_{k=1, j\neq k}^{n_s} A_{jk}(t,\eta_t)x_k(t), \\ x_o = \phi, \quad \eta_o &= i, \quad i \in \mathcal{S}, \quad j \in \{1,\ldots,n_s\} \end{aligned} \tag{7.48}$$

and the uncertain nominal jump subsystem

$$\begin{aligned} S_{ju}\colon \quad \dot{x}_j(t) &= [A_j(t,\eta_t) + \Delta A_j(t,\eta_t)]x_j(t) + B_j(t,\eta_t)u_j(t) \\ &\quad + \sum_{k=1, j\neq k}^{n_s} A_{jk}(t,\eta_t)x_k(t), \\ x_o = \phi, \quad \eta_o &= i, \quad i \in \mathcal{S}, \quad j \in \{1,\ldots,n_s\} \end{aligned} \tag{7.49}$$

for which we have the following

Definition 7.3 The free nominal jump subsystem (7.48) is said to be *robustly stochastically decentrally stable* if for all finite initial state $\phi \in \Re^n$, for all $j,k \in \{1,\ldots,n_s\}$, initial mode $\eta_o \in \mathcal{S}$, and for all admissible uncertainties satisfying (7.44)–(7.45)

$$\int_0^\infty \mathcal{E}\{\|x_j(t,\phi)\|^2\}dt < +\infty, \quad j \in \{1,\ldots,n_s\}.$$

Lemma 7.3 *Consider the free uncertain nominal jump subsystem* (7.48). *Then, the following statements are equivalent*:

(a) *subsystem* (7.48) *is robustly stochastically decentrally stable*;
(b) *for any matrix* $\Phi_j(i) = \Phi_j^t(i)^t > 0$, $i \in \mathcal{S}$ *and a scalar* $\mu_j(i) > 0$, $i \in \mathcal{S}$ *there exist matrices* $\Pi_j(i) = \Pi_j^t(i)^t > 0$, $i \in \mathcal{S}$ *for all* $j \in \{1,\ldots,n_s\}$, *satisfying*

$$\begin{aligned} &\Pi_j(i)A_j(i) + A_j^t(i)\Pi_j(i) + \Pi_j(i)\left\{\sum_{m=1, j\neq m}^{n_s} A_{jm}(i)A_{jm}^t\right\}\Pi_j(i) \\ &\quad + (n_s - 1)I + \Phi_j(i) + \sum_{m=1, i\neq m}^{n_s} \alpha_{im}\Pi_j(m) \\ &\quad + \mu_j^{-1}(i)\Pi_j(i)M_j(t,\eta_t)M_j^t(t,\eta_t)\Pi_j(i) + \mu_j(i)N_j^t(t,\eta_t)N_j(t,\eta_t) = 0. \end{aligned} \tag{7.50}$$

Proof Let (7.50) have a solution $\Pi_j(i) = \Pi_j^t(i)^t > 0, i \in \mathcal{S}$ and $j \in \{1, \ldots, n_s\}$. For the class of admissible uncertainties $\Delta_j(t, \eta_t)$ satisfying (7.45) and for $\eta_t = i \in \mathcal{S}$, we get from inequality (7.17)

$$\begin{aligned}&\mu_j^{-1}(i)\Pi_j(i)M_j(i)M_j^t(i)\Pi_j(i) + \mu_j(i)N_j^t(i)N_j(i)\\&\quad \geq N_j^t(i)\Delta_j^t(i)M_j^t(i)\Pi_j(i) + \Pi_j(i)M_j(i)\Delta_j(i)N_j(i).\end{aligned} \tag{7.51}$$

It follows from (7.50)–(7.51) that

$$\begin{aligned}&\Pi_j(i)[A_j(i) + M_j(i)\Delta_j(i)N_j(i)] + [A_j(i) + M_j(i)\Delta_j(i)N_j(i)]^t\Pi_j(i)\\&\quad + \Pi_j(i)\left\{\sum_{m=1, j\neq m}^{n_s} A_{jm}(i)A_{jm}^t\right\}\Pi_j(i)\\&\quad + (n_s - 1)I + \Phi_j(i) + \sum_{m=1, i\neq m}^{n_s} \alpha_{im}\Pi_j(m) \leq 0\end{aligned} \tag{7.52}$$

holds for all admissible uncertainties $\Delta_j(t, \eta_t)$ satisfying (7.45). The equality condition (7.50) is readily obtained from application of Lemma 7.1. □

Extending on Definition 7.3, we introduce the following

Definition 7.4 The nominal jump subsystem (7.49) is said to be *stochastically decentrally stabilizable* if for all finite initial state $\phi \in \Re^n$, initial mode $\eta_o \in \mathcal{S}$, and for all admissible uncertainties satisfying (4.4)–(4.5) there exists a linear feedback gain $K_j^*(t, \eta_t)$, $j \in \{1, \ldots, n_s\}$ that is constant for each value of $\eta_t \in \mathcal{S}$ such that the decentralized control law

$$u_j(t) = -K_j(t, \eta_t)x_j(t), \quad j \in \{1, \ldots, n_s\}, \tag{7.53}$$

where $\|K_j(t, \eta_t)\| < \infty$, ensure that the resulting closed-loop subsystem is robustly stochastically decentrally stable.

By similarity to Lemma 7.3, we have the following result for the stochastic decentralized stabilizability of subsystem (7.49).

Lemma 7.4 *Consider the uncertain nominal jump subsystem* (7.49). *Then*, *the following statements are equivalent*:

(a) *the subsystem* (7.49) *is robustly stochastically decentrally stabilizable by a decentralized control law*

$$u_j(t) = -K_j(t, i)x_j(t), \quad i \in \mathcal{S}, \; j \in \{1, \ldots, n_s\};$$

(b) *for any matrix* $\Phi_j(i) = \Phi_j^t(i)^t > 0$, $i \in \mathcal{S}$ *and a scalar* $\mu_j(i) > 0$, $i \in \mathcal{S}$, *there exist matrices* $\Pi_j(i) = \Pi_j^t(i)^t > 0$, $i \in \mathcal{S}$ *for all* $j \in \{1, \dots, n_s\}$, *satisfying*

$$\begin{aligned}
&\Pi_j(i)\tilde{A}_j(i) + \tilde{A}_j^t(i)\Pi_j(i) + \sum_{m=1}^{n_s} \alpha_{im}\Pi_j(m) \\
&\quad + \Pi_j(i)\left\{\sum_{k=1, j\neq k}^{n_s} A_{jk}(i)A_{ij}^t\right\}\Pi_j(i) + (n_s - 1)I + \Phi_j(i) \\
&\quad + \mu_j^{-1}(i)\Pi_j(i)M_j(i)M_j^t(i)\Pi_j(i) + \mu_j(i)N_j^t(i)N_j(i) = 0, \quad \forall i \in \mathcal{S}, \quad (7.54)
\end{aligned}$$

where

$$\tilde{A}_j(i) = A_j(i) - B_j(i)K_j(i). \tag{7.55}$$

Proof Follows by parallel development to Lemma 7.3 and using Lemma 7.2. □

We now focus attention on the controller design. More specifically, the objective is to design a robust decentralized state-feedback controller $\mathcal{G}_j(t, \eta_t)$ such that:

(1) In the finite-horizon case , the system (7.41)–(7.45) with $j \in \{1, \dots, n_s\}$ under the decentralized feedback controller $\mathcal{G}_j(t, \eta_t)$ has performance (7.47) over a given horizon $[0, \mathcal{T}]$; or
(2) In the infinite-horizon case in which the system (7.41)–(7.45) with $j \in \{1, \dots, n_s\}$ under the decentralized feedback controller $\mathcal{G}_j(t, \eta_t)$ is stochastically decentrally stable and has performance (7.47) over a given horizon $[0, \infty]$.

The main results are established by the following theorems for the cases of finite-horizon and infinite-horizon cases, respectively.

Theorem 7.3 *Consider system* (7.41)–(7.43) *subject to* (7.4)–(7.6) *and* $j \in \{1, \dots, n_s\}$. *Then, for a given* $\gamma > 0$, *there exists a decentralized state-feedback controller* $u_j(t)$ *such that*

$$\|z_j(t)\|_{E_2} < \gamma \|w_j(t)\|_2$$

for all nonzero $w(t) \in \mathcal{L}^2[0, \mathcal{T}]$, *and for all admissible uncertainties satisfying* (7.45)–(7.46) *if for a given scalar* $\mu_j(i) > 0$, $i \in \mathcal{S}$, *the following set of* $n_s \times s$ *coupled differential Riccati equations*:

$$\begin{aligned}
&\dot{\Pi}_j(t,i) + \Pi_j(t,i)A_j(t,i) + A_j^t(t,i)\Pi_j(t,i) + (n_s - 1)I + G_j^t(t,i)G_j(t,i) \\
&\quad + \sum_{m=1, i\neq m}^{n_s} \alpha_{im}\Pi_j(t,m) + \Pi_j(t,i)\left\{\sum_{k=1, j\neq k}^{n_s} A_{jk}(t,i)A_{t,jk}^t\right. \\
&\quad \left. + \gamma^{-2}\Gamma_j(i)\Gamma_j^t(t,i) - B_j(t,i)R_j^{-1}(t,i)B_j^t(t,i)\right\}\Pi_j(t,i)
\end{aligned}$$

$$+\mu_j^{-1}(i)\Pi_j(t,i)M_j(t,i)M_j^t(t,i)\Pi_j(t,i)+\mu_j(i)N_j^t(t,i)N_j(t,i)=0,$$
$$\Pi_j(\mathcal{T})=0, \quad i\in\mathcal{S}, \quad t\in[0,\mathcal{T}], \quad j\in\{1,\dots,n_s\} \tag{7.56}$$

has a solution $P_j(t,i), i\in\mathcal{S}, j\in\{1,\dots,n_s\}$ *on* $[0,\mathcal{T}]$. *Moreover, the decentralized controller is given by*:

$$u_j(t)=-K_j(t,\eta_t)x_j(t),$$
$$K(t,\eta_t)=R_j^{-1}(t,\eta_t)[B_j^t(t,\eta_t)\Pi_j(t,\eta_t)+F_j^t(t,\eta_t)G_j(t,\eta_t)],$$
$$t\in[0,\mathcal{T}], \quad \eta_t=i\in\mathcal{S}, \quad j\in\{1,\dots,n_s\}. \tag{7.57}$$

Proof Let (7.16) have a solution $\Pi_j(i)=\Pi_j^t(i)^t>0, i\in\mathcal{S}$ and $j\in\{1,\dots,n_s\}$. For the class of admissible uncertainties $\Delta_j(t,\eta_t)$ satisfying (4.5) and for $\eta_t=i\in\mathcal{S}$, and proceeding like Lemma 7.3, we have:

$$\dot{\Pi}_j(t,i)+\Pi_j(t,i)[A_j(t,i)+\Delta A_j(t,i)]+[A_j(t,i)+\Delta A_j(t,i)]^t\Pi_j(t,i)$$
$$+(n_s-1)I+G_j^t(t,i)G_j(t,i)+\sum_{m=1,i\neq m}^{n_s}\alpha_{im}\Pi_j(t,m)$$
$$+\Pi_j(t,i)\left\{\sum_{k=1,j\neq k}^{n_s}A_{jk}(t,i)A_{t,jk}^t\right.$$
$$\left.+\gamma^{-2}\Gamma_j(i)\Gamma_j^t(t,i)-B_j(t,i)R_j^{-1}(t,i)B_j^t(t,i)\right\}\Pi_j(t,i)=0,$$
$$\Pi_j(\mathcal{T})=0, \quad i\in\mathcal{S}, \quad t\in[0,\mathcal{T}], \quad j\in\{1,\dots,n_s\} \tag{7.58}$$

for all admissible uncertainties $\Delta_j(t,\eta_t)$, $\eta_t=i\in\mathcal{S}$ satisfying (7.45)–(7.46). It follows from Theorem 7.1 that

$$\|z_j(t)\|_{E_2}<\gamma\|w_j(t)\|_2$$

and the proof is completed. □

Theorem 7.4 *Consider system* (7.41)–(7.43) *subject to* (7.4)–(7.6) *and* $j\in\{1,\dots,n_s\}$. *Then, for a given* $\gamma>0$, *there exists a decentralized state-feedback controller* $u_j(t)$ *such that the interconnected closed-loop system is stochastically decentrally stable and*

$$\|z_j(t)\|_{E_2}<\gamma\|w_j(t)\|_2$$

for all nonzero $w(t)\in\mathcal{L}^2[0,\infty]$, *and for all admissible uncertainties satisfying* (7.45)–(7.46) *if for a given scalar* $\mu_j(i)>0$, $i\in\mathcal{S}$, *if the following set of* $n_s\times s$ *coupled algebraic Riccati equations*:

$$\Pi_j(i)A_j(i)+A_j^t(i)\Pi_j(i)+\sum_{m=1,i\neq m}^{n_s}\alpha_{im}\Pi_j(m)+(n_s-1)I+G_j^t(i)G_j(i)$$
$$+\Pi_j(i)\left\{\sum_{k=1,j\neq k}^{n_s}A_{jk}(i)A_{jk}^t+\gamma^{-2}\Gamma_j(i)\Gamma_j^t(i)-B_j(i)R_j^{-1}(i)B_j^t(i)\right\}\Pi_j(i)$$

$$+\mu_j^{-1}(i)\Pi_j(i)M_j(i)M_j^t(i)\Pi_j(i)+\mu_j(i)N_j^t(i)N_j(i)=0,$$
$$i\in\mathcal{S},\quad t\in[0,\mathcal{T}],\quad j\in\{1,\dots,n_s\} \tag{7.59}$$

has a solution $\Pi_j(i)=\Pi_j^t(i)>0$, $i\in\mathcal{S}$, $j\in\{1,\dots,n_s\}$. *Moreover, the decentralized controller is given by*:

$$\begin{aligned}u_j(t)&=-K_j(\eta_t)x_j(t),\\ K(\eta_t)&=R_j^{-1}(\eta_t)[B_j^t(\eta_t)\Pi_j(\eta_t)+F_j^t(\eta_t)G_j(\eta_t)],\\ &\qquad t\in[0,\infty],\quad \eta_t=i\in\mathcal{S},\quad j\in\{1,\dots,n_s\}.\end{aligned} \tag{7.60}$$

Proof It can easily established following similar procedure to Theorem 7.4 with the help of Theorem 7.2. □

Remark 7.5 Using the convex optimization techniques over linear matrix inequalities [5], the existence of scaling parameters $\mu_j(i)>0$, $i\in\mathcal{S}$, $j\in\{1,\dots,n_s\}$ can be conveniently checked out.

Remark 7.6 It can easily shown from Theorem 7.3 and Theorem 7.4 that the $n_s\times s$ differential Riccati equations (7.56) and the $n_s\times s$ algebraic Riccati equations (7.59) are the sufficient stochastic stability conditions for the following $\mathcal{H}_\infty$-control problem without parametric uncertainties over the finite-horizon and the infinite-horizon, respectively:

$$\begin{aligned}\tilde{S}_j:\quad \dot{x}_j(t)&=A_j(t,\eta_t)x_j(t)+B_j(t,\eta_t)u_j(t)\\ &\quad+\left[\Gamma_j(t,\eta_t),\ \frac{\gamma}{\sqrt{\mu_j(\eta_t)}}M_j(t,\eta_t)\right]\tilde{w}_j(t)\\ &\quad+\sum_{k=1,j\neq k}^{n_s}A_{jk}(t,\eta_t)x_k(t),\quad x_0=0,\ \eta_0=i,\ t\in[0,\mathcal{T}],\end{aligned} \tag{7.61}$$

$$\tilde{z}(t)=\begin{bmatrix}\sqrt{\mu_j(\eta_t)}N_j(t,\eta_t)\\ G_j(t,\eta_t)\end{bmatrix}x_j(t)+\begin{bmatrix}0\\ F_j(t,\eta_t)\end{bmatrix}u_j(t),\quad \eta_j=i\in\mathcal{S}, \tag{7.62}$$

where

$$\tilde{w}_j(t)=\begin{bmatrix}w_j(t)\\ \gamma^{-1}\sqrt{\mu_j(\eta_t)}\Delta(t,\eta_t)N(t,\eta_t)\end{bmatrix},\quad \tilde{z}_j(t)=\begin{bmatrix}\sqrt{\mu_j(\eta_t)}N(t,\eta_t)x_j(t)\\ z_j(t)\end{bmatrix}. \tag{7.63}$$

It is readily seen that

$$\|\tilde{z}_j(t)\|_{E_2}<\|z_j(t)\|_{E_2}$$

and hence we conclude that if we solve the $\mathcal{H}_\infty$-control problem for system (7.61)–(7.63) with (7.4)–(7.6), then we can also solve the robust $\mathcal{H}_\infty$-control problem for system (7.41)–(7.43) with (7.4)–(7.6) using the same controller.

Remark 7.7 Extension of the developed robustness results to the case where the jumping rates are subject to uncertainties. Specifically, we consider the transition probability from mode i at time t to mode j at time $t+\delta$, $i, j \in \mathcal{S}$ to be:

$$\begin{aligned} p_{ij} &= Pr(\eta_{t+\delta} = j \mid \eta_t = i) \\ &= \begin{cases} (\alpha_{ij} + \Delta\alpha_{ij})\delta + o(\delta), & \text{if } i \neq j, \\ 1 + (\alpha_{ij} + \Delta\alpha_{ij})\delta + o(\delta), & \text{if } i = j \end{cases} \end{aligned} \tag{7.64}$$

with transition probability rates $(\alpha_{ij} + \Delta\alpha_{ij}) \geq 0$ for $i, j \in \mathcal{S}, i \neq j$ and

$$\alpha_{ii} + \Delta\alpha_{ii} = -\sum_{m=1, m\neq i}^{s} (\alpha_{im} + \Delta\alpha_{im}). \tag{7.65}$$

We assume that the uncertainties $\Delta\alpha_{ij}$ satisfies

$$\|\Delta\alpha_{ij}\| \leq \beta_{ij}, \quad \forall i, j \in \mathcal{S}, \tag{7.66}$$

where β_{ij} are known scalars $\forall i, j \in \mathcal{S}$.

In line of Theorem 7.3 and Theorem 7.4, we have the following robustness results:

Theorem 7.5 *Consider system* (7.41)–(7.43) *subject to* (7.64)–(7.65) *and* $j \in \{1, \ldots, n_s\}$. *Then, for a given* $\gamma > 0$, *there exists a decentralized state-feedback controller* $u_j(t)$ *such that*

$$\|z_j(t)\|_{E_2} < \gamma \|w_j(t)\|_2$$

for all nonzero $w(t) \in \mathcal{L}^2[0, \mathcal{T}]$, *and for all admissible uncertainties satisfying* (4.4)–(4.5) *and* (4.26) *if for a given scalar* $\mu_j(i) > 0$, $i \in \mathcal{S}$, *the following set of* $n_s \times s$ *coupled differential Riccati equations*:

$$\begin{aligned} &\dot{\Pi}_j(t,i) + \Pi_j(t,i)A_j(t,i) + A_j^t(t,i)\Pi_j(t,i) + (n_s - 1)I + G_j^t(t,i)G_j(t,i) \\ &\quad + \Pi_j(t,i)\left\{ \sum_{k=1, j\neq k}^{n_s} A_{jk}(t,i)A_{t,jk}^t + \gamma^{-2}\Gamma_j(i)\Gamma_j^t(t,i) \right. \\ &\quad \left. - B_j(t,i)R_j^{-1}(t,i)B_j^t(t,i) \right\} \Pi_j(t,i) \\ &\quad + \mu_j^{-1}(i)\Pi_j(t,i)M_j(t,i)M_j^t(t,i)\Pi_j(t,i) + \mu_j(i)N_j^t(t,i)N_j(t,i) \end{aligned}$$

$$+ \sum_{m=1, i \neq m}^{n_s} (\alpha_{im} + \beta_{im}) \Pi_j(t, m) = 0,$$

$$\Pi_j(\mathcal{T}) = 0, \quad i \in \mathcal{S}, \quad t \in [0, \mathcal{T}], \quad j \in \{1, \ldots, n_s\} \tag{7.67}$$

has a solution $P_j(t, i), i \in \mathcal{S}, j \in \{1, \ldots, n_s\}$ *on* $[0, \mathcal{T}]$. *Moreover, the decentralized controller is given by*:

$$u_j(t) = -K_j(t, \eta_t) x_j(t),$$

$$K(t, \eta_t) = R_j^{-1}(t, \eta_t)[B_j^t(t, \eta_t) \Pi_j(t, \eta_t) + F_j^t(t, \eta_t) G_j(t, \eta_t)],$$

$$t \in [0, \mathcal{T}], \quad \eta_t = i \in \mathcal{S}, \quad j \in \{1, \ldots, n_s\}. \tag{7.68}$$

Proof It can be derived using similar arguments to Theorem 7.3. □

Theorem 7.6 *Consider system* (7.41)–(7.43) *subject to* (7.64)–(7.65) *and* $j \in \{1, \ldots, n_s\}$. *Then, for a given* $\gamma > 0$, *there exists a decentralized state-feedback controller* $u_j(t)$ *such that the interconnected closed-loop system is stochastically decentrally stable and*

$$\|z_j(t)\|_{E_2} < \gamma \|w_j(t)\|_2$$

for all nonzero $w(t) \in \mathcal{L}^2[0, \infty]$, *and for all admissible uncertainties satisfying* (4.4)–(4.5) *and* (4.26) *if for a given scalar* $\mu_j(i) > 0$, $i \in \mathcal{S}$, *if the following set of* $n_s \times s$ *coupled algebraic Riccati equations*:

$$\Pi_j(i) A_j(i) + A_j^t(i) \Pi_j(i)$$

$$+ \sum_{m=1, i \neq m}^{n_s} (\alpha_{im} + \beta_{im}) \Pi_j(m) + (n_s - 1) I + G_j^t(i) G_j(i)$$

$$+ \Pi_j(i) \left\{ \sum_{k=1, j \neq k}^{n_s} A_{jk}(i) A_{jk}^t + \gamma^{-2} \Gamma_j(i) \Gamma_j^t(i) - B_j(i) R_j^{-1}(i) B_j^t(i) \right\} \Pi_j(i)$$

$$+ \mu_j^{-1}(i) \Pi_j(i) M_j(i) M_j^t(i) \Pi_j(i) + \mu_j(i) N_j^t(i) N_j(i) = 0,$$

$$i \in \mathcal{S}, \quad t \in [0, \mathcal{T}], \quad j \in \{1, \ldots, n_s\} \tag{7.69}$$

has a solution $\Pi_j(i) = \Pi_j^t(i) > 0$, $i \in \mathcal{S}$, $j \in \{1, \ldots, n_s\}$. *Moreover, the decentralized controller is given by*:

$$u_j(t) = -K_j(\eta_t) x_j(t),$$

$$K(\eta_t) = R_j^{-1}(\eta_t)[B_j^t(\eta_t) \Pi_j(\eta_t) + F_j^t(\eta_t) G_j(\eta_t)],$$

$$t \in [0, \infty], \quad \eta_t = i \in \mathcal{S}, \quad j \in \{1, \ldots, n_s\}. \tag{7.70}$$

Proof It can be carried out by parallel development to Theorem 7.4. □

7.2 Mode-Dependent Decentralized Stability and Stabilization

7.2.1 Introduction

Problems of decentralized stability and stabilization of interconnected systems are receiving considerable interests [13, 24, 26, 27, 52], where most of the effort are focused on dealing with the interaction patterns. When the interconnected system involves delays, related studies are reported in [23, 49–51]. In [23], the stability and stabilization of interconnected systems are considered where uncertainties are assumed to satisfy the matching conditions.

In this section, we consider a wide class of continuous-time, interconnected jumping time-delay systems with mode-dependent interval delays. This class of systems has not been fully investigated in the literature. The importance of these systems stems from the fact it encompasses other numerous classes of interests [31]. In particular, we consider hereafter uncertain interconnected systems in which full state-measurements are not available and the delays occur both within the subsystems and in the interaction patterns. In the present work, the objective is to design decentralized linear feedback controllers based on state and dynamic-output schemes to guarantee the robust stabilization and robust $\mathcal{H}_\infty$ performance. Since Lyapunov theory is the main vehicle in stability analysis, the resulting conditions are only sufficient. In our work, we construct an appropriate Lyapunov-Krasovskii functional and introduce some parameters as manipulative factors to reduce the degree of conservativeness. The main contributions of this chapter are the constructive use of linear matrix inequalities as a vehicle to solve both the mode-dependent decentralized stochastic stability and stabilization problems with $\mathcal{H}_\infty$ performance.

7.2.2 Problem Statement

Given a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ where Ω is the sample space, $\mathcal{F}$ is the algebra of events and $\mathbf{P}$ is the probability measure defined on $\mathcal{F}$. Let the random form process $\{\eta_t, t \in [0, \mathcal{T}]\}$ be a homogeneous, finite-state Markovian process with right continuous trajectories and taking values in a finite set $\mathcal{S} = \{1, 2, \dots, s\}$ with generator $\Im = (\alpha_{ij})$ and transition probability from mode i at time t to mode j at time $t + \delta$, $i, j \in \mathcal{S}$:

$$\begin{aligned} p_{ij} &= Pr(\eta_{t+\delta} = j \mid \eta_t = i) \\ &= \begin{cases} \alpha_{ij}\delta + o(\delta), & \text{if } i \neq j, \\ 1 + \alpha_{ij}\delta + o(\delta), & \text{if } i = j \end{cases} \end{aligned} \tag{7.71}$$

with transition probability rates $\alpha_{ij} \geq 0$ for $i, j \in \mathcal{S}$, $i \neq j$ and $\alpha_{ii} = -\sum_{m=1, m\neq i}^{s} \alpha_{im}$ where $\delta > 0$ and $\lim_{\delta \downarrow 0} o(\delta)/\delta = 0$. The set $\mathcal{S}$ comprises the various operational modes of the system under study.

We consider a class **S** of nonlinear time-delay systems with Markovian jump parameters and bounded-uncertainties composed of n_s coupled subsystems S_j described over the space $(\Omega, \mathcal{F}, \mathbf{P})$ by:

$$\begin{aligned} S_j\colon \quad \dot{x}_j(t) &= A_j(\eta_t)x_j(t) + E_j(\eta_t)x_j(t-\tau_j(\eta_t)) + \Gamma_j(\eta_t)w_j(t) \\ &\quad + B_j(\eta_t)u_j(t) + g_j(\eta_t), \end{aligned} \tag{7.72}$$

$$y_j(t) = C_j(\eta_t)x_j(t) + D_j(\eta_t)x_j(t-\tau_j(\eta_t)) + \Phi_j(\eta_t)w_j(t), \tag{7.73}$$

$$\begin{aligned} z_j(t) &= G_j(\eta_t)x_j(t) + L_j(\eta_t)x_j(t-\tau_j(\eta_t)) + \Psi_j(\eta_t)w_j(t), \\ x_j(t) &= \kappa_j(t), \quad t \in [\tau_{Mj}, 0], \end{aligned} \tag{7.74}$$

where $j \in \{1, \dots, n_s\}$, $x_j(t) \in \Re^{n_j}$ is the state vector $u_j(t) \in \Re^{m_j}$ is the control input $w_j(t) \in \Re^{q_j}$ is the disturbance input $y_j(t) \in \Re^{p_j}$ is the measured output $z_j(t) \in \Re^{r_j}$ is the controlled output and $\tau_j(\eta_t)$ are unknown mode-dependent time-delays within known ranges in order to guarantee smooth growth of the state trajectories. The time-varying delay $\tau_j(\eta_t)$ is unknown mode-dependent and satisfies $\tau_{mj} \le \tau_j(\eta_t) \le \tau_{Mj}$ with the bounds τ_{mj}, τ_{Mj} being known for every $j \in \{1, \dots, n_s\}$. The initial condition is $\kappa_j(.) \in \mathcal{L}_2[-\tau_{Mj}, 0]$, $j \in \{1, \dots, n_s\}$. The function $g_j : \mathsf{Z}_+ \times \Re^n \times \Re^n \to \Re^{n_j}$ is a piecewise-continuous mode-dependent vector function in its arguments and represents the interaction of other subsystems to subsystem j. In this chapter, it is assumed that g_j satisfies the quadratic inequality

$$\begin{aligned} g_j^t(\eta_t)g_j^t(\eta_t) &\le \phi_j^2(\eta_t)x_j^t(t)F_j^tF_jx_j(t) \\ &\quad + \psi_j^2(\eta_t)x_j^t(t-\tau_j(\eta_t))H_j^tH_jx_j(t-\tau_j(\eta_t)), \end{aligned} \tag{7.75}$$

where $\phi_j(\eta_t) > 0$, $\psi_j(\eta_t) > 0$ are mode-dependent bounding parameters and $\widetilde{G}_j$, $\widetilde{G}_{dj}$ are appropriate constant matrices. In the sequel, we let $\xi_j(k) = [x_j^t(k)\ x_j^t(t - \tau_j(\eta_t))\ g_j^t(\eta_t)]^t$. Then (7.75) can be conveniently written as

$$\xi_j^t \begin{bmatrix} -\phi_j^2(i)\, F_j^tF_j & 0 & 0 \\ \bullet & -\psi_j^2(i)H_{dj}^tH_j & 0 \\ \bullet & \bullet & I_j \end{bmatrix} \xi_j \le 0. \tag{7.76}$$

For each possible value $\eta_t = i$, $i \in \mathcal{S}$, we will denote the system matrices of (S_j) associated with mode i by

$$\begin{aligned} &A_j(\eta_t) := A_j(i), \quad E_j(\eta_t) := E_j(i), \quad \Gamma_j(\eta_t) := \Gamma_j(i), \\ &B_j(\eta_t) := B_j(i), \quad C_j(\eta_t) = C_j(i), \\ &D_j(\eta_t) := D_j(i), \quad \Phi(\eta_t) := \Phi(i), \quad G_j(\eta_t) := G_j(i), \\ &L_j(t, \eta_t) := L_j(i), \end{aligned} \tag{7.77}$$

where $A_j(i)$, $E_j(i)$, $C_j(i)$, $D_j(i)$, $G_j(i)$, $L_j(i)$, $B_j(i)$, $\Phi_j(i)$, $\Gamma_j(i)$ are known real constant matrices of appropriate dimensions representing the nominal subsystem (without uncertainties and interactions):

$$
\begin{aligned}
S_j^n\colon \quad \dot{x}_j(t) &= A_{oj}(i)x_j(t) + E_{oj}(i)x_j(t-\tau_j(i)) + \Gamma_j(i)w_j(t) \\
&\quad + B_{oj}(i)u_j(t) + g_j(i), \\
y_j(t) &= C_{oj}(i)x_j(t) + D_{oj}(i)x_j(t-\tau_j(i)) + \Phi_j(i)w_j(t), \\
z_j(t) &= G_{oj}(i)x_j(t) + L_{oj}(i)x_j(t-\tau_j(i)) + \Psi_j(i)w_j(t).
\end{aligned}
\tag{7.78}
$$

In the sequel, we assume $\forall j \in \{l, \dots, n_s\}$ that the n_s-pairs $(A_{oj}(i), B_{oj}(i))$ and $(A_{oj}(i), C_{oj}(i))$ are stabilizable and detectable, respectively. Let $\mathsf{X}(t,\kappa)$ denotes the state trajectory of the interconnected system (7.72)–(7.74) from the initial condition $\kappa \equiv [\kappa_1^t, \dots, \kappa_{n_s}^t]^t$. We have the following

Definition 7.5 System (7.78) with $u_j \equiv 0$ is said to be *stochastically stable* if there exists a constant $\mathcal{G}(\eta_o, \kappa_j) > 0$ such that for all finite initial vector function $\kappa_j \in \Re^n$ defined on the interval $[-\tau_{Mj}, 0]$, $j \in \{1, \dots, n_s\}$ and initial mode $\eta_o = i \in \mathcal{S}$.

$$
\mathsf{E}\left[\int_0^\infty \{\|\mathsf{X}(t,\kappa_j)\|^2\}\,dt \,\middle|\, \eta_o, \kappa(s), s \in [-\tau_{Mj}, 0]\right] \le \mathcal{G}(\eta_o, \kappa_j).
$$

Definition 7.6 (7.78) is said to be *stochastically stable with a disturbance attenuation* γ_j if for zero initial vector function $\kappa_j \equiv 0$ and initial mode $\eta_o = i \in \mathcal{S}$ the following inequality holds

$$
\|z(t)\|_{E_2} \stackrel{\Delta}{=} \mathsf{E}\left[\int_0^\infty z_j^t(t) z_j(t)\,dt\right]^{1/2} < \gamma_j \|w_j(t)\|_2
$$

for all $0 \ne w(t) \in \mathcal{L}_2[0,\infty)$, where $\gamma_j > 0$ is a prescribed level of disturbance attenuation and $\|.\|_{E_2}$ denotes the norm in $\mathcal{L}_2((\Omega, \mathcal{F}, \mathbf{P}), [0,\infty))$.

7.2.3 Local Subsystem Stability

In the sequel, we introduce

$$
\tau_{aj} = \frac{1}{2}(\tau_{Mj} + \tau_{mj}), \qquad \delta_j = \frac{1}{2}(\tau_{Mj} - \tau_{mj}) = \tau_{Mj} - \tau_{aj} = \tau_{aj} - \tau_{mj}.
$$

Remark 7.8 It is readily evident that the case $\delta_j = 0$ leads to $\tau_{Mj} = \tau_{mj}$ which corresponds to constant delay. The case $\tau_{mj} = 0$ yields $\delta_j = \tau_{aj} = \frac{1}{2}\tau_{Mj}$.

Theorem 7.7 *Given the delay bounds τ_{mj}, τ_{Mj}. If there exist positive-definite matrices $\mathcal{P}_j(i)$, $\mathcal{Q}_j(i)$, $\mathcal{R}_j$, $\mathcal{Z}_j$, $\mathcal{W}_j$, $\mathcal{X}_j$ and scalars $\gamma_j > 0$, $\varepsilon_j(i) > 0$, $\lambda_j(i) > 0$, $i \in \mathcal{S}$ satisfying*

$$
\Sigma_j(i) = \begin{bmatrix} \Sigma_{aj}(i) & \Sigma_{cj}(i) \\ \bullet & \Sigma_{oj}(i) \end{bmatrix} + \sigma_j(\tau_{aj}^2 \mathcal{Z}_j + 2\delta_j \mathcal{W}_j)\sigma_j^t < 0, \tag{7.79}
$$

$$\sum_{m=1}^{n_s} \alpha_{im} \mathcal{Q}_j(m) < \mathcal{R}_j, \tag{7.80}$$

$$\Sigma_{aj}(i) = \begin{bmatrix} \Sigma_{1j}(i) & \mathcal{P}_j(i)E_{oj}(i) & \mathcal{Z}_j & \mathcal{P}_j(i) \\ \bullet & -\Sigma_{2j} & \frac{1}{\delta_j}\mathcal{W}_j & 0 \\ \bullet & \bullet & -\Sigma_{3j} & 0 \\ \bullet & \bullet & \bullet & -I_j \end{bmatrix},$$

$$\Sigma_{cj}(i) = \begin{bmatrix} \mathcal{P}_j(i)\Gamma_j(i) & G^t_{oj}(i) & F^t_j & 0 \\ 0 & L^t_{oj}(i) & 0 & H^t_j \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\sigma_j = \mathrm{col}\big[\, A^t_{oj}(i)\; E^t_{oj}(i)\; 0\; 0\; \Gamma^t_j(i)\; 0\; 0\; 0 \,\big], \tag{7.81}$$

$$\Sigma_{aj}(i) = \begin{bmatrix} -\gamma_j^2 I & \Psi^t_j(i) & 0 & 0 \\ \bullet & -I & 0 & 0 \\ \bullet & \bullet & -\lambda_j(i)I & 0 \\ \bullet & \bullet & \bullet & -\varepsilon_j(i)I \end{bmatrix},$$

$$\Sigma_{1j}(i) = \mathcal{P}_j(i)A_{oj}(i) + A^t_{oj}(i)\mathcal{P}_j(i) + \sum_{m=1}^{n_s} \alpha_{im}\mathcal{P}_j(m) + \mathcal{Q}_j(i) + \mathcal{X}_j + \tau_{aj}\mathcal{R}_j - \mathcal{Z}_j,$$

$$\Sigma_{3j} = \mathcal{Q}_j(i) + \frac{1}{\delta_j}\mathcal{W}_j + \mathcal{Z}_j, \qquad \Sigma_{2j} = \mathcal{X}_j + \frac{1}{\delta_j}\mathcal{W}_j$$

then the free nominal jump subsystem (7.78) *is stochastically stable with a disturbance attenuation* γ_j.

Proof For each $\eta_j = i \in \mathcal{S},\ j \in \{1, \dots, n_s\}$ we construct a stochastic Lyapunov-Krasovskii functional as follows

$$V_j(x_j, \eta_t, t) = V_{1j}(x_j, \eta_t, t) + V_{2j}(x_j, \eta_t, t) + V_{3j}(x_j, \eta_t, t) + V_{4j}(x_j, \eta_t, t), \tag{7.82}$$

$$V_{1j}(x_j, \eta_t, t) = x^t_j(t)\mathcal{P}_j(i)x_j(t),$$

$$V_{2j}(x_j, \eta_t, t) = \int_{t-\tau_{aj}}^{t} x^t_j(s)\mathcal{Q}_j(i)x_j(s)ds + \int_{t-\tau_j(t)}^{t} x^t_j(s)\mathcal{X}_j x_j(s)ds + \int_{\tau_{aj}}^{0}\int_{t-\beta_j}^{t} x^t_j(s)\mathcal{R}_j x_j(s)ds d\beta_j, \tag{7.83}$$

$$V_{3j}(x_j, \eta_t, t) = 2\delta_j \int_{t-\tau_{aj}+\delta_j}^{t} x^t_j(s)\mathcal{W}_j x_j(s)ds$$

$$+ \int_{t-\tau_{aj}-\delta_j}^{t-\tau_{aj}+\delta_j} \int_{s}^{t-\tau_{aj}+\delta_j} x_j^t(s)\mathcal{W}_j x_j(s)ds,$$

$$V_{4j}(x_j, \eta_t, t) = \tau_{aj} \int_{t-\tau_{aj}}^{t} \int_{s}^{t} \dot{x}_j^t(\alpha)\mathcal{Z}_j \dot{x}_j(\alpha)d\alpha ds.$$

In terms of the weak infinitesimal operator $\Im_j^x[.]$ of the process $\{x_j(t), \eta_t,\ t \geq 0\}$ for system (7.78) with $w_j(.) \equiv 0$ at the point and $\{x_j(t), \eta_t\}$ is given by [21]:

$$\begin{aligned}
\Im_j^x[V_j] \leq\ & x_j^t(t)\mathcal{P}_j(i)\dot{x}_j(t) + \dot{x}_j^t(t)\mathcal{P}_j(i)x_j(t) + \sum_{m=1}^{n_s} \alpha_{im} x_j^t(t)\mathcal{P}_j(m)x_j(t) \\
& + x_j^t(t)(\mathcal{Q}_j(i) + \mathcal{X}_j)x_j(t) \\
& - x_j^t(t-\tau_{aj})\mathcal{Q}_j(i)x_j(t-\tau_{aj}) - x_j^t(t-\tau_j)\mathcal{X}_j x_j(t-\tau_j) \\
& + \int_{t-\tau_{aj}}^{t} x_j^t(s) \sum_{m=1}^{n_s} \alpha_{im} \mathcal{Q}_j(m)x_j(s)ds \\
& + \tau_{aj} x_j^t(t)\mathcal{R}_j x_j(t) - \int_{t-\tau_{aj}}^{t} x_j^t(s)\mathcal{R}_j x_j(s)ds + 2\delta_j \dot{x}_j^t(t)\mathcal{W}_j \dot{x}_j^t(t) \\
& - \int_{t-\tau_{aj}-\delta_j}^{t-\tau_{aj}+\delta_j} \dot{x}_j^t(s)\mathcal{W}_j \dot{x}_j^t(s)ds + \tau_{aj}^2 \dot{x}_j^t(t)\mathcal{Z}_j \dot{x}_j(t) \\
& - \int_{t-\tau_{aj}}^{t} \dot{x}_j^t(\alpha)\mathcal{Z}_j \dot{x}_j(\alpha)d\alpha.
\end{aligned} \tag{7.84}$$

During the range $\tau_{aj} < \tau_j(t)$, the following inequality holds

$$\begin{aligned}
- \int_{t-\tau_{aj}-\delta_j}^{t-\tau_{aj}+\delta_j} \dot{x}_j^t(s)\mathcal{W}_j \dot{x}_j^t(s)ds \leq\ & - \int_{t-\tau_j(t)}^{t-\tau_{aj}} \dot{x}_j^t(s)\mathcal{W}_j \dot{x}_j(s)ds \\
& - \frac{1}{\tau_j(t) - \tau_{aj}} \left[\int_{t-\tau_j}^{t-\tau_{aj}} \dot{x}_j(s)ds \right]^t \mathcal{W}_j \left[\int_{t-\tau_j}^{t-\tau_{aj}} \dot{x}_j(s)ds \right] \\
& - \frac{1}{\tau_{aj} - \tau_{mj}} \big[x_j(t-\tau_j(t)) - x_j(t-\tau_{aj}) \big]^t \\
& \times \mathcal{W}_j \big[x_j(t-\tau_j) - x_j(t-\tau_{aj}) \big].
\end{aligned} \tag{7.85}$$

When $\tau_{aj} > \tau_j(t)$, we get

$$\begin{aligned}
- \int_{t-\tau_{aj}-\delta_j}^{t-\tau_{aj}+\delta_j} \dot{x}_j^t(s)\mathcal{W}_j \dot{x}_j^t(s)ds \leq\ & - \int_{t-\tau_{aj}}^{t-\tau_j(t)} \dot{x}_j^t(s)\mathcal{W}_j \dot{x}_j(s)ds \\
& - \frac{1}{\tau_{aj} - \tau_{mj}} \left[\int_{t-\tau_{aj}}^{t-\tau_j(t)} \dot{x}_j(s)ds \right]^t \mathcal{W}_j \left[\int_{t-\tau_{aj}}^{t-\tau_j(t)} \dot{x}_j(s)ds \right]
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{\tau_{aj} - \tau_{mj}} \big[x_j(t - \tau_j(t)) - x_j(t - \tau_{aj}) \big]^t \\
&\times \mathcal{W}_j \big[x_j(t - \tau_j(t)) - x_j(t - \tau_{aj}) \big]. \qquad (7.86)
\end{aligned}
$$

After some algebraic manipulations, it can be shown that

$$
\begin{aligned}
&- \int_{t-\tau_{aj}-\delta_j}^{t-\tau_{aj}+\delta_j} \dot{x}_j^t(s) \mathcal{S}_j \dot{x}_j^t(s) ds \le -\frac{1}{\delta_j} x_j^t(t - \tau_{aj}) \mathcal{S}_j x_j(t - \tau_{aj}) \\
&+ \frac{2}{\delta_j} x_j^t(t - \tau_{aj}) \mathcal{S}_j x_j(t - \tau_j) - \frac{1}{\delta_j} x_j^t(t - \tau_j(t)) \mathcal{S}_j x_j(t - \tau_j). \qquad (7.87)
\end{aligned}
$$

According to Lemma 9.12 in Chap. 9, we have for any $i \in \mathcal{S}$ that

$$
\begin{aligned}
\Im_j^x[V_j(t, x, \eta_t)] &\le \xi_j^t(t) \Xi_j(i) \xi_j(t) \\
&+ \int_{t-\tau_{aj}}^{t} x_j^t(s) \left[\sum_{m=1}^{n_s} \alpha_{im} \mathcal{Q}_j(m) - \mathcal{R}_j \right] x_j^t(s) ds < 0, \qquad (7.88)
\end{aligned}
$$

$$
\begin{aligned}
\Xi_j(i) &= \begin{bmatrix} \Xi_{1j}(i) & \Xi_{2j}(i) & \mathcal{Z}_j & \mathcal{P}_j(i) \\ \bullet & \Xi_{3j} & \frac{1}{\delta_j}\mathcal{S}_j & 0 \\ \bullet & \bullet & \Xi_{4j} & 0 \\ \bullet & \bullet & \bullet & 0 \end{bmatrix} \\
&+ \begin{bmatrix} A_{oj}^t(i) \\ E_{oj}^t(i) \\ 0 \\ 0 \end{bmatrix} (\tau_{aj}^2 \mathcal{Z}_j + 2\delta_j \mathcal{S}_j) \begin{bmatrix} A_{oj}^t(i) \\ E_{oj}^t(i) \\ 0 \\ 0 \end{bmatrix}^t, \qquad (7.89)
\end{aligned}
$$

$$
\xi_j^t(t) = \Big[x_j^t(t) \; x_j^t(t - \tau_j(t)) \; x_j^t(t - \tau_{aj}) \; g_j^t \Big]^t, \qquad (7.90)
$$

where

$$
\begin{aligned}
\Xi_{1j}(i) &= \mathcal{P}_j(i) A_{oj}(i) + A_{oj}^t(i) \mathcal{P}_j(i) \\
&\quad + \sum_{m=1}^{n_s} \alpha_{im} \mathcal{P}_j(m) + \mathcal{Q}_j(i) + \mathcal{X}_j + \tau_{aj} \mathcal{R}_j, \\
\Xi_{2j}(i) &= \mathcal{P}_j(i) E_{oj}(i), \qquad (7.91) \\
\Xi_{3j} &= -\mathcal{X}_j - \frac{1}{\delta_j} \mathcal{W}_j, \quad \Xi_{4j} = -\mathcal{Q}_j(i) - \frac{1}{\delta_j} \mathcal{W}_j - \mathcal{Z}_j.
\end{aligned}
$$

By resorting to the **S**-procedure, Lemma 9.13, inequalities (7.76) and (7.88) can be rewritten together as

$$\Im_j^x[V_j(t,x,\eta_t)] \leq \xi_j^t(t)\widehat{\Xi}_j(i)\xi_j(t) + \int_{t-\tau_{aj}}^{t} x_j^t(s)\left[\sum_{m=1}^{n_s}\alpha_{im}\mathcal{Q}_j(m) - \mathcal{R}_j\right]x_j(s)ds < 0, \quad (7.92)$$

$$\widehat{\Xi}_j(i) = \begin{bmatrix} \widehat{\Xi}_{1j}(i) & \Xi_{2j}(i) & \mathcal{Z}_j & \mathcal{P}_j(i) \\ \bullet & \widehat{\Xi}_{3j} & \frac{1}{\delta_j}\mathcal{W}_j & 0 \\ \bullet & \bullet & \Xi_{4j} & 0 \\ \bullet & \bullet & \bullet & -I_j \end{bmatrix} + \begin{bmatrix} A_{oj}^t(i) \\ E_{oj}^t(i) \\ 0 \\ 0 \end{bmatrix}(\tau_{aj}^2\mathcal{Z}_j + 2\delta_j\mathcal{W}_j)\begin{bmatrix} A_{oj}^t(i) \\ E_{oj}^t(i) \\ 0 \\ 0 \end{bmatrix}^t, \quad (7.93)$$

$$\widehat{\Xi}_{1j}(i) = \Xi_{1j}(i) + \phi^2 F_j^t F_j, \quad \widehat{\Xi}_{3j}(i) = \Xi_{3j}(i) + \psi^2 H_j^t H_j. \quad (7.94)$$

We can show that $\widehat{\Xi}_j(i) < 0$. By $\sum_{m=1}^{n_s}\alpha_{im}\mathcal{Q}_j(m) < \mathcal{R}_j$, we have $\Im_j^x[V_j] < 0$ for all $\xi_j \neq 0$ and $\Im_j^x[V_j] \leq 0$ for all ξ_j. Following [38], we reach $\Im_j^x[V_j] \leq -\omega_j V_j(t,x,i)$, $\omega_j > 0$ and by Dynkin's formula [21], one has

$$\mathsf{E}\left[\int_0^\infty \Im_j^x[V_j]dt\right] = \mathsf{E}[V_j(x_j,i,t)|_{t=\infty}] - V_j(x_j,i,t)|_{t=0} \geq 0.$$

Consequently there exists a scalar ν_j such that $\mathsf{E}[V_j(t,x,i)] \leq \nu_j\|\kappa_j\|_{\tau_{Mj}}^2$ and this leads to

$$\lim_{T\to\infty}\left\{\int_0^T \|x_j(t)\|^2 dt|_{\eta_o}, x_j(s) = \kappa_j(s), s \in [\tau_{Mj}, 0]\right\} \leq \nu_j\mathsf{E}\|\kappa_j\|_{\tau_{Mj}}^2.$$

On considering this and Definition 7.5, system (7.78) with is stochastically stable for any time delay $\tau_j(t)$. With some manipulations using (7.83) and (7.94), we obtain:

$$\begin{aligned} \mathcal{J}(x_j) &= \mathsf{E}\left\{\int_0^\infty \left[z^t(t)z(t) - \gamma^2 w^t(t)w(t) + \Im_j^x[V_j] - \Im_j^x[V_j]\right]dt\right\} \\ &\leq \mathsf{E}\left\{\int_0^\infty \left[z^t(t)z(t) - \gamma^2 w^t(t)w(t) + \Im_j^x[V_j]\right]dt\right\} \\ &\leq \mathsf{E}\left\{\int_0^\infty \zeta_j^t\Sigma_j(i)\zeta_j dt\right\}, \end{aligned} \quad (7.95)$$

where $\Sigma_j(i)$ is given by (7.79) and $\zeta_j^t = [\xi_j^t \quad w_j^t]$. Hence, by Schur complement with $\lambda_j = 1/\phi_j$ and $\varepsilon_j = 1/\psi_j$, we conclude the results of Theorem 7.7 for all $t \geq 0$, $\mathcal{J}(x_j) < 0$. This completes the proof. □

7.2.4 $\mathcal{H}_\infty$ State-Feedback Synthesis

The problem of $\mathcal{H}_\infty$ state-feedback control could be phrased as follows: *Given subsystem* (7.78), *determine a local state-feedback control law*

$$u_j(t) = K_j(\eta_t)x_j(t), \quad \eta_t = i \in \mathcal{S} \tag{7.96}$$

which guarantees that $\mathcal{H}_\infty$ *performance measure is bounded by* γ_j *for all* $w(t) \in \mathcal{L}_2[0,\infty]$.

Applying the controller (7.96) to system (7.72), we obtain the closed-loop subsystem

$$\begin{aligned} S_j^n\colon \quad \dot{x}_j(t) &= A_{sj}(i)x_j(t) + E_{oj}(i)x_j(t-\tau_j(i)) + \Gamma_j(i)w_j(t) + g_j(i), \\ z_j(t) &= G_{oj}(i)x_j(t) + L_{oj}(i)x_j(t-\tau_j(i)) + \Psi_j(i)w_j(t). \\ A_{sj}(i) &= A_j + B_{oj}(i)K_j(i). \end{aligned} \tag{7.97}$$

Extending on Theorem 7.7, the main design result is summarized by the following theorem.

Theorem 7.8 *Given the delay bounds* τ_{mj}, τ_{Mj}. *If there exist positive-definite matrices* $\mathcal{Y}_j(i)$, $\mathcal{G}_j(i)$, $\Lambda_{1j}(i)$, $\Lambda_{2j}(i)$, $\Lambda_{3j}(i)$, $\Lambda_{4j}(i)$, $\Lambda_{5j}(i)$ *and scalars* $\gamma_j > 0$, $\varepsilon_j(i) > 0$, $\lambda_j(i) > 0$, $i \in \mathcal{S}$ *satisfying*

$$\Pi_j(i) = \begin{bmatrix} \Pi_{aj}(i) & \Pi_{cj}(i) \\ \bullet & \Pi_{oj}(i) \end{bmatrix} < 0, \qquad \sum_{m=1}^{n_s} \alpha_{im}\Lambda_{1j}(m) < \Lambda_{4j}(i), \tag{7.98}$$

$$\begin{aligned} \Pi_{aj}(i) &= \begin{bmatrix} \Pi_{1j}(i) & \Pi_{5j}^t(i) & \Lambda_{3j}(i) & I_j & \Gamma_j(i) \\ \bullet & -\Pi_{2j} & \frac{1}{\delta_j}\Lambda_{2j} & 0 & 0 \\ \bullet & \bullet & -\Pi_{3j} & 0 & 0 \\ \bullet & \bullet & \bullet & -I_j & 0 \\ \bullet & \bullet & \bullet & \bullet & -\gamma_j^2 I \end{bmatrix}, \\ \Pi_{cj}(i) &= \begin{bmatrix} \mathcal{Y}_j(i)G_{oj}^t(i) & \mathcal{Y}_j(i)F_j^t & 0 & \Pi_4(i) & \Pi_4(i) \\ \mathcal{Y}_j(i)G_{oj}^t(i) & 0 & \mathcal{Y}_j(i)H_j^t & \Pi_{5j}(i) & \Pi_{5j}(i) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \Psi_j(i) & 0 & 0 & \Gamma_j^t(i) & 0 \end{bmatrix}, \\ \Pi_{oj}(i) &= \begin{bmatrix} -I & 0 & 0 & 0 & 0 \\ \bullet & -\lambda_j(i)I & 0 & 0 & 0 \\ \bullet & \bullet & -\varepsilon_j(i)I & 0 & 0 \\ \bullet & \bullet & \bullet & -\Pi_{6j}(i) & 0 \\ \bullet & \bullet & \bullet & \bullet & -\Pi_{7j}(i) \end{bmatrix}, \end{aligned} \tag{7.99}$$

$$\Pi_{1j}(i) = A_{oj}(i)\mathcal{Y}_j(i) + \mathcal{Y}_j(i)A_{oj}^t(i) + B_{oj}(i)\mathcal{G}_j(i) + \mathcal{G}_j^t(i)B_{oj}^t(i)$$

$$+\sum_{m=1}^{n_s}\alpha_{im}\mathcal{Y}_j(m)+\Lambda_{1j}(i)+\Lambda_{5j}(i)+\tau_{aj}\Lambda_{4j}(i),$$

$$\Pi_{3j}=\Lambda_{1j}(i)+\frac{1}{\delta_j}\Lambda_{2j}(i)+\Lambda_{3j}(i),$$

$$\Pi_{2j}=\Lambda_{5j}(i)+\frac{1}{\delta_j}\Lambda_{2j}(i),\quad \Pi_{4j}=\mathcal{Y}_j(i)A^t_{oj}(i)+\mathcal{G}^t_j(i)B^t_{oj}(i),$$

$$\Pi_{5j}(i)=\mathcal{Y}_j(i)E^t_{oj}(i),$$

$$\Pi_{6j}(i)=\tau^2_{aj}[\mathcal{Y}^t_j(i)+\mathcal{Y}_j(i)-\Lambda_{3j}(i)],$$

$$\Pi_{7j}(i)=1/2\delta_j^{-1}[\mathcal{Y}^t_j(i)+\mathcal{Y}_j(i)-\Lambda_{2j}(i)]$$

then the jump subsystem (7.97) is stochastically stable with a disturbance attenuation γ_j. Moreover, the controller gain is given by $\mathcal{K}_j(i)=\mathcal{G}_j(i)\mathcal{Y}_j^{-1}(i)$.

Proof Using $A_{sj}(i)$ instead of $A_{oj}(i)$ with Schur complements, we rewrite LMI (7.79) into the form:

$$\widehat{\Sigma}_j(i)=\begin{bmatrix}\widehat{\Sigma}_{aj}(i) & \widehat{\Sigma}_{cj}(i)\\ \bullet & \widehat{\Sigma}_{oj}(i)\end{bmatrix}<0,\quad \sum_{m=1}^{n_s}\alpha_{im}\mathcal{Q}_j(m)<\mathcal{R}_j, \tag{7.100}$$

$$\widehat{\Sigma}_{aj}(i)=\begin{bmatrix}\Sigma_{sj}(i) & \mathcal{P}_j(i)E_{oj}(i) & \mathcal{Z}_j & \mathcal{P}_j(i) & \mathcal{P}_j(i)\Gamma_j(i)\\ \bullet & -\Sigma_{2j} & \frac{1}{\delta_j}\mathcal{W}_j & 0 & 0\\ \bullet & \bullet & -\Sigma_{3j} & 0 & 0\\ \bullet & \bullet & \bullet & -I_j & 0\\ \bullet & \bullet & \bullet & \bullet & -\gamma_j^2 I\end{bmatrix},$$

$$\widehat{\Sigma}_{cj}(i)=\begin{bmatrix}G^t_{oj}(i) & F^t_j & 0 & A^t_{sj}(i) & A^t_{sj}(i)\\ 0 & L^t_{oj}(i) & H^t_j & E^t_{oj}(i) & E^t_{oj}(i)\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ \Psi_j(i) & 0 & 0 & \Gamma^t_j(i) & \Gamma^t_j(i)\end{bmatrix}, \tag{7.101}$$

$$\widehat{\Sigma}_{oj}(i)=\begin{bmatrix}-I & 0 & 0 & 0 & 0\\ \bullet & -\lambda_j(i)I & 0 & 0 & 0\\ \bullet & \bullet & -\varepsilon_j(i)I & 0 & 0\\ \bullet & \bullet & \bullet & -\tau_{aj}^{-2}\mathcal{Z}_j^{-1} & 0\\ \bullet & \bullet & \bullet & \bullet & -1/2\delta_j^{-1}\mathcal{W}_j^{-1}\end{bmatrix},$$

$$\Sigma_{sj}(i)=\mathcal{P}_j(i)A_{sj}(i)+A^t_{sj}(i)\mathcal{P}_j(i)+\sum_{m=1}^{n_s}\alpha_{im}\mathcal{P}_j(m)$$
$$+\mathcal{Q}_j(i)+\mathcal{X}_j+\tau_{aj}\mathcal{R}_j.$$

Recall that the algebraic inequality

$$(I - MZ)Z^{-1}(I - ZM^t) = Z^{-1} - M^t - M - MZM^t > 0$$
$$\Rightarrow \quad -Z^{-1} < MZM^t - M^t - M$$

holds for any matrices M, $Z > 0$. Applying the congruent transformation

$$\mathsf{T} = \mathrm{diag}[\mathcal{Y}_j(i), \mathcal{Y}_j(i), \mathcal{Y}_j(i), I_j, I_j, I_j, I_j, I_j, I_j, I_j, I_j], \quad \mathcal{Y}_j(i) = \mathcal{P}_j^{-1}(i)$$

to $\widehat{\Sigma}_j(i)$ along with the change of variables

$$\mathcal{G}_j(i) = \mathcal{K}_j(i)\mathcal{Y}_j(i), \quad \Lambda_{1j}(i) = \mathcal{Y}_j(i)\mathcal{Q}_j(i)\mathcal{Y}_j(i), \quad \Lambda_{2j}(i) = \mathcal{Y}_j(i)\mathcal{W}_j\mathcal{Y}_j(i),$$
$$\Lambda_{3j}(i) = \mathcal{Y}_j(i)\mathcal{Z}_j\mathcal{Y}_j(i),$$
$$\Lambda_{4j}(i) = \mathcal{Y}_j(i)\mathcal{R}_j\mathcal{Y}_j(i), \quad \Lambda_{5j}(i) = \mathcal{Y}_j(i)\mathcal{X}_j\mathcal{Y}_j(i)$$

with some mathematical manipulations, we finally reach LMI (7.98). □

7.2.5 Dynamic Output-Feedback Control

Consider the dynamic output-feedback control

$$\begin{aligned}\dot{\hat{x}}_j(t) &= A_{oj}(i)\hat{x}_j(t) + B_{oj}(i)u_j(t) + K_{oj}(i)[y_j(t) - C_{oj}(i)\hat{x}_j(t)],\\ u_j(t) &= K_{cj}(i)\hat{x}_j(t),\end{aligned} \tag{7.102}$$

where $K_{oj}(i),\ K_{cj}(i)$ are the unknown mode-dependent observer and control gain matrices. Applying the dynamic controller (7.102) to the linear system (7.78), we obtain the closed-loop system and associated matrices

$$e_j(t) = \left[x_j^t(t)\ x_j^t(t) - \hat{x}_j^t(t)\right]^t, \tag{7.103}$$

$$\begin{aligned}\dot{e}_j(t) &= \mathcal{A}_j(i)e_j(t) + \hat{E}_{oj}(i)e_j(t - \tau_j(t)) + \hat{c}_j(t) + \hat{\Gamma}_j(i)w_j(t),\\ z_j(t) &= \hat{G}_{oj}(i)e_j(t) + \hat{L}_{oj}(i)e_(t - \tau_j(t)) + \Psi_j(i)w_j(t),\end{aligned} \tag{7.104}$$

where

$$\begin{aligned}\mathcal{A}_j(i) &= \begin{bmatrix} A_{oj}(i) + B_{oj}(i)K_{cj}(i) & -B_{oj}(i)K_{cj}(i) \\ 0 & A_{oj}(i) - K_{oj}(i)C_{oj}(i) \end{bmatrix},\\ \hat{\Gamma}_j(i) &= \begin{bmatrix} \Gamma_j(i) \\ \Gamma_j(i) - K_{oj}(i)\Phi_j(i) \end{bmatrix},\\ \hat{G}_{oj}(i) &= \begin{bmatrix} G_{oj}(i)\ 0 \end{bmatrix}, \quad \hat{L}_{oj}(i) = \begin{bmatrix} L_{oj}(i)\ 0 \end{bmatrix},\\ \hat{c}_j(t)(i) &= \begin{bmatrix} c_j(t)(i) \\ c_j(t)(i) \end{bmatrix}, \quad \hat{E}_{oj}(i) = \begin{bmatrix} E_{oj}(i) & 0 \\ 0 & E_{oj}(i) - K_{oj}(i)D_{oj}(i) \end{bmatrix}.\end{aligned} \tag{7.105}$$

It follows from Theorem 7.7 and (7.100) that the closed-loop system (7.104) is asymptotically stable with γ_j disturbance attenuation level if there exist matrices $\widetilde{\mathcal{P}}_j(i)$, $\widetilde{\mathcal{Q}}_j(i)$, $\widetilde{\mathcal{R}}_j$, $\widetilde{\mathcal{Z}}_j$, $\widetilde{\mathcal{W}}_j$, $\widetilde{\mathcal{X}}_j$, $i \in \mathcal{S}$ and scalars $\hat{\gamma}_j > 0$, $\hat{\varepsilon}_j > 0$, $\hat{\lambda}_j > 0$ satisfying the LMI

$$\widetilde{\Sigma}_j(i) = \begin{bmatrix} \widetilde{\Sigma}_{j1}(i) & \widetilde{\Sigma}_{j2}(i) \\ \bullet & \widetilde{\Sigma}_{j3}(i) \end{bmatrix} < 0, \quad \sum_{m=1}^{n_s} \alpha_{im} \widetilde{\mathcal{Q}}_j(m) < \widetilde{\mathcal{R}}_j, \tag{7.106}$$

$$\widetilde{\Sigma}_{j1}(i) = \begin{bmatrix} \widetilde{\Sigma}_{sj}(i) & \widetilde{\mathcal{P}}_j(i)\hat{E}_{oj}(i) & \widetilde{\mathcal{Z}}_j & \widetilde{\mathcal{P}}_j(i) & \widetilde{\mathcal{P}}_j(i)\hat{\Gamma}_j(i) \\ \bullet & -\widetilde{\Sigma}_{2j} & \frac{1}{\delta_j}\widetilde{\mathcal{W}}_j & 0 & 0 \\ \bullet & \bullet & -\widetilde{\Sigma}_{3j} & 0 & 0 \\ \bullet & \bullet & \bullet & -I_j & 0 \\ \bullet & \bullet & \bullet & \bullet & -\hat{\gamma}_j^2 I \end{bmatrix},$$

$$\widetilde{\Sigma}_{j2}(i) = \begin{bmatrix} \hat{G}^t_{oj}(i) & \hat{F}^t_j & 0 & \mathcal{A}^t_{sj}(i) & \mathcal{A}^t_{sj}(i) \\ 0 & \hat{L}^t_{oj}(i) & \hat{H}^t_j & \widetilde{E}^t_{oj}(i) & \widetilde{E}^t_{oj}(i) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \Psi_j(i) & 0 & 0 & \hat{\Gamma}^t_j(i) & \hat{\Gamma}^t_j(i) \end{bmatrix}, \tag{7.107}$$

$$\widetilde{\Sigma}_{j3}(i) = \begin{bmatrix} -I & 0 & 0 & 0 & 0 \\ \bullet & -\hat{\lambda}_j I & 0 & 0 & 0 \\ \bullet & \bullet & -\varepsilon_j I & 0 & 0 \\ \bullet & \bullet & \bullet & -\tau_{aj}^{-2}\widetilde{\mathcal{Z}}_j^{-1} & 0 \\ \bullet & \bullet & \bullet & \bullet & -1/2\delta_j^{-1}\widetilde{\mathcal{W}}_j^{-1} \end{bmatrix},$$

$$\begin{aligned} \widetilde{\Sigma}_{sj}(i) &= \widetilde{\mathcal{P}}_j(i)\widetilde{\mathcal{A}}_j(i) + \widetilde{\mathcal{A}}^t_{sj}(i)\widetilde{\mathcal{P}}_j(i) \\ &\quad + \sum_{m=1}^{n_s} \alpha_{im}\widetilde{\mathcal{P}}_j(m) + \widetilde{\mathcal{P}}_j(i) + \widetilde{\mathcal{Q}}_j(i) + \widetilde{\mathcal{X}}_j + \tau_{aj}\widetilde{\mathcal{R}}_j, \end{aligned} \tag{7.108}$$

$$\widetilde{\Sigma}_{3j} = \widetilde{\mathcal{Q}}_j(i) + \frac{1}{\delta_j}\widetilde{\mathcal{W}}_j + \widetilde{\mathcal{Z}}_j, \ \widetilde{\Sigma}_{2j} = \widetilde{\mathcal{X}}_j + \frac{1}{\delta_j}\widetilde{\mathcal{W}}_j,$$

where

$$\begin{aligned} \widetilde{\mathcal{P}}_j(i) &= \begin{bmatrix} \mathcal{P}_{jo}(i) & \mathcal{P}_{jc}(i) \\ 0 & \mathcal{P}_{jc}(i) \end{bmatrix}, \quad \widetilde{\mathcal{Q}}_j(i) = \begin{bmatrix} \mathcal{Q}_{jo}(i) & \mathcal{Q}_{jc}(i) \\ 0 & \mathcal{Q}_{jc}(i) \end{bmatrix}, \\ \widetilde{\mathcal{X}}_j &= \begin{bmatrix} \mathcal{X}_{jo} & \mathcal{X}_{jc} \\ 0 & \mathcal{X}_{jc} \end{bmatrix}, \\ \widetilde{\mathcal{R}}_j &= \begin{bmatrix} \mathcal{R}_{jo} & \mathcal{R}_{jc} \\ 0 & \mathcal{R}_{jc} \end{bmatrix}, \quad \widetilde{\mathcal{Z}}_j = \begin{bmatrix} \mathcal{Z}_{jo} & \mathcal{Z}_{jc} \\ 0 & \mathcal{Z}_{jc} \end{bmatrix}, \quad \widetilde{\mathcal{W}}_j = \begin{bmatrix} \mathcal{W}_{jo} & \mathcal{W}_{jc} \\ 0 & \mathcal{W}_{jc} \end{bmatrix}. \end{aligned} \tag{7.109}$$

The main design result is summarized by the following theorem.

Theorem 7.9 *Given the delay bounds* τ_{mj}, τ_{Mj}. *If there exist positive-definite matrices* $\mathcal{Y}_{jo}(i), \mathcal{Y}_{jc}(i), \mathcal{G}_{ja}(i), \mathcal{G}_{jo}(i), \mathcal{G}_{jc}(i), \mathcal{G}_{js}(i), \Upsilon_{11j}(i), \Upsilon_{12j}(i), \Upsilon_{21j}(i), \Upsilon_{22j}(i), \Upsilon_{31j}(i), \Upsilon_{32j}(i), \Upsilon_{41j}(i), \Upsilon_{42j}(i), \Upsilon_{51j}(i), \Upsilon_{52j}(i)$ *and scalars* $\gamma_j > 0$, $\varepsilon_j(i) > 0$, $\lambda_j(i) > 0$, $i \in \mathcal{S}$ *satisfying*

$$\Theta_j(i) = \begin{bmatrix} \Theta_{aj}(i) & \Theta_{cj}(i) \\ \bullet & \Theta_{oj}(i) \end{bmatrix} < 0, \quad \sum_{m=1}^{n_s} \alpha_{im} \Upsilon_{1j}(m) < \Upsilon_{4j}(i), \tag{7.110}$$

$$\Theta_{aj}(i) = \begin{bmatrix} \Theta_{1j}(i) & \Theta_{5j}(i) & \Upsilon_{3j}(i) & I_j & \hat{\Gamma}_j(i) \\ \bullet & -\Theta_{2j}(i) & \frac{1}{\delta_j}\Upsilon_{2j} & 0 & 0 \\ \bullet & \bullet & -\Theta_{3j} & 0 & 0 \\ \bullet & \bullet & \bullet & -I_j & 0 \\ \bullet & \bullet & \bullet & \bullet & -\gamma_j^2 I \end{bmatrix},$$

$$\Theta_{cj}(i) = \begin{bmatrix} \mathcal{Y}_j(i)G_{oj}^t(i) & \mathcal{Y}_j(i)F_j^t & 0 & \Theta_4(i) & \Theta_4(i) \\ \mathcal{Y}_j(i)G_{oj}^t(i) & 0 & \mathcal{Y}_j(i)H_j^t & \Theta_{5j}(i) & \Theta_{5j}(i) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \Psi_j(i) & 0 & 0 & \Gamma_j^t(i) & 0 \end{bmatrix},$$

$$\Theta_{oj}(i) = \begin{bmatrix} -I & 0 & 0 & 0 & 0 \\ \bullet & -\lambda_j(i)I & 0 & 0 & 0 \\ \bullet & \bullet & -\varepsilon_j(i)I & 0 & 0 \\ \bullet & \bullet & \bullet & -\Theta_{6j}(i) & 0 \\ \bullet & \bullet & \bullet & \bullet & -\Theta_{7j}(i) \end{bmatrix},$$

$$\Theta_{1j}(i) = \begin{bmatrix} \Theta_{11j}(i) & \Theta_{12j}(i) \\ 0 & \Theta_{13j}(i) \end{bmatrix},$$

$$\Theta_{5j}(i) = \begin{bmatrix} E_{oj}(i)\mathcal{Y}_{jo}(i) & E_{oj}(i)\mathcal{Y}_{jc}(i) \\ 0 & E_{oj}(i)\mathcal{Y}_{jc}(i) - \mathcal{G}_{ja}(i) \end{bmatrix},$$

$$\begin{aligned} \Theta_{11j}(i) &= A_{oj}(i)\mathcal{Y}_{jo}(i) + \mathcal{Y}_{jo}(i)A_{oj}^t(i) + B_{oj}(i)\mathcal{G}_{jo}(i) + \mathcal{G}_{jo}^t(i)B_{oj}^t(i) \\ &\quad + \sum_{m=1}^{n_s} \alpha_{im}\mathcal{Y}_{jo}(m) + \Upsilon_{11j}(i) + \Upsilon_{51j}(i) + \tau_{aj}\Upsilon_{41j}(i), \\ \Theta_{12j}(i) &= A_{oj}(i)\mathcal{Y}_{jo}(i) + B_{oj}(i)\mathcal{G}_{jo}(i) - \mathcal{G}_{js}(i) + \sum_{m=1}^{n_s} \alpha_{im}\mathcal{Y}_{jc}(m) \\ &\quad + \Upsilon_{12j}(i) + \Upsilon_{52j}(i) + \tau_{aj}\Upsilon_{42j}(i), \\ \Theta_{13j}(i) &= A_{oj}(i)\mathcal{Y}_{jc}(i) - \mathcal{G}_{jc}(i) + \sum_{m=1}^{n_s} \alpha_{im}\mathcal{Y}_{jc}(m) \\ &\quad + \Upsilon_{12j}(i) + \Upsilon_{52j}(i) + \tau_{aj}\Upsilon_{42j}(i), \end{aligned} \tag{7.111}$$

$$\Theta_{2j}(i) = \begin{bmatrix} \Upsilon_{51j}(i) + \frac{1}{\delta_j}\Upsilon_{21j}(i) & \Upsilon_{52j}(i) + \frac{1}{\delta_j}\Upsilon_{22j}(i) \\ 0 & \Upsilon_{52j}(i) + \frac{1}{\delta_j}\Upsilon_{22j}(i) \end{bmatrix},$$

$$\Theta_{4j}(i) = \begin{bmatrix} \Theta_{41j}(i) & \Theta_{42j}(i) \\ 0 & \Theta_{43j}(i) \end{bmatrix},$$

$$\Theta_{3j}(i) = \begin{bmatrix} \Upsilon_{11j}(i) + \frac{1}{\delta_j}\Upsilon_{21j}(i)\Upsilon_{31j}(i) & \Upsilon_{122j}(i) + \frac{1}{\delta_j}\Upsilon_{22j}(i)\Upsilon_{32j}(i) \\ 0 & \Upsilon_{12j}(i) + \frac{1}{\delta_j}\Upsilon_{22j}(i)\Upsilon_{32j}(i) \end{bmatrix},$$

$$\Theta_{41j}(i) = \mathcal{Y}_{jo}(i)A^t_{oj}(i) + \mathcal{G}^t_{jo}(i)B^t_{oj}(i),$$

$$\Theta_{22j}(i) = A_{oj}(i)\mathcal{Y}_{jo}(i) + B_{oj}(i)\mathcal{G}_{jo}(i) - \mathcal{G}_{js}(i),$$

$$\Theta_{43j}(i) = A_{oj}(i)\mathcal{Y}_{jc}(i) - \mathcal{G}_{jc}(i),$$

$$\Theta_{6j}(i) = \begin{bmatrix} \Theta_{61j}(i) & \Theta_{62j}(i) \\ 0 & \Theta_{62j}(i) \end{bmatrix}, \quad \Theta_{7j}(i) = \begin{bmatrix} \Theta_{71j}(i) & \Theta_{72j}(i) \\ 0 & \Theta_{72j}(i) \end{bmatrix},$$

$$\Theta_{61j}(i) = \tau^2_{aj}[\mathcal{Y}^t_{jo}(i) + \mathcal{Y}_{jo}(i) - \Upsilon_{31j}(i)],$$

$$\Theta_{62j}(i) = \tau^2_{aj}[\mathcal{Y}^t_{jc}(i) + \mathcal{Y}_{jc}(i) - \Upsilon_{32j}(i)],$$

$$\Theta_{71j}(i) = 1/2\delta_j^{-1}[\mathcal{Y}^t_{jo}(i) + \mathcal{Y}_{jo}(i) - \Upsilon_{21j}(i)],$$

$$\Theta_{72j}(i) = 1/2\delta_j^{-1}[\mathcal{Y}^t_{jc}(i) + \mathcal{Y}_{jc}(i) - \Upsilon_{22j}(i)]$$

then the jump subsystem (7.104) *is stochastically stable with a disturbance attenuation* γ_j. *Moreover, the controller gains are given by* $\mathcal{K}_{cj}(i) = \mathcal{G}_{jo}(i)\mathcal{Y}^{-1}_{jo}(i)$, $\mathcal{K}_{oj}(i) = \mathcal{G}_{jc}(i)\mathcal{Y}^{-1}_{jc}(i)C^{\dagger}_{oj}(i)$.

Proof Following parallel development to Theorem 7.8 and applying the congruent transformation

$$\widetilde{\mathsf{T}} = \mathrm{diag}[\widetilde{\mathcal{Y}}_j(i), \widetilde{\mathcal{Y}}_j(i), \widetilde{\mathcal{Y}}_j(i), I_j, I_j, I_j, I_j, I_j, I_j, I_j, I_j],$$
$$\widetilde{\mathcal{Y}}_j(i) = \widetilde{\mathcal{P}}^{-1}_j(i) = \begin{bmatrix} \mathcal{Y}_{jo}(i) & \mathcal{Y}_{jc}(i) \\ 0 & \mathcal{Y}_{jc}(i) \end{bmatrix}$$

to $\widetilde{\Sigma}_j(i)$ along with the change of variables

$$\mathcal{G}_{jo}(i) = \mathcal{K}_{cj}(i)\mathcal{Y}_{jo}(i), \quad \mathcal{G}_{js}(i) = B_{oj}(i)K_{cj}(i)\mathcal{Y}_{jc}(i),$$

$$\mathcal{G}_{jc}(i) = K_{oj}(i)C_{oj}(i)\mathcal{Y}_{jc}(i),$$

$$\Upsilon_{1j}(i) = \widetilde{\mathcal{Y}}_j(i)\widetilde{\mathcal{Q}}_j(i)\widetilde{\mathcal{Y}}_j(i) = \begin{bmatrix} \Upsilon_{11j} & \Upsilon_{12j} \\ 0 & \Upsilon_{12j} \end{bmatrix},$$

$$\Upsilon_{2j}(i) = \mathcal{Y}_j(i)\mathcal{W}_j\mathcal{Y}_j(i)\begin{bmatrix} \Upsilon_{21j} & \Upsilon_{22j} \\ 0 & \Upsilon_{22j} \end{bmatrix},$$

$$\Upsilon_{3j}(i) = \mathcal{Y}_j(i)\mathcal{Z}_j\mathcal{Y}_j(i) = \begin{bmatrix} \Upsilon_{31j} & \Upsilon_{32j} \\ 0 & \Upsilon_{32j} \end{bmatrix},$$

$$\Upsilon_{4j}(i) = \mathcal{Y}_j(i)\mathcal{R}_j\mathcal{Y}_j(i) = \begin{bmatrix} \Upsilon_{41j} & \Upsilon_{42j} \\ 0 & \Upsilon_{42j} \end{bmatrix},$$

$$\Upsilon_{5j}(i) = \mathcal{Y}_j(i)\mathcal{X}_j\mathcal{Y}_j(i) = \begin{bmatrix} \Upsilon_{51j} & \Upsilon_{52j} \\ 0 & \Upsilon_{52j} \end{bmatrix}, \quad \mathcal{G}_{ja}(i) = K_{oj}(i)D_{oj}(i)\mathcal{Y}_{jc}(i)$$

with some mathematical manipulations, we finally reach LMI (7.98). □

7.2.6 Simulation Example 7.1

To illustrate the design procedures developed in Theorems 7.7–7.9, we consider a representative model composed of three subsystems ($n_s = 2$) and two-operating conditions $\mathcal{S} = 1, 2$. This model has a mode-switching generator

$$\Im = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

and the following data:

$$A_1(1) = \begin{bmatrix} -3 & 1 & 0 \\ 0.3 & -2.5 & -4 \\ -0.1 & 0.3 & -3.8 \end{bmatrix}, \quad E_1(1) = \begin{bmatrix} 0.1669 & 0.0802 & 1.6820 \\ -0.8162 & -0.9373 & 0.5936 \\ 2.0941 & 0.6357 & 0.7902 \end{bmatrix},$$

$$L_1(1) = \begin{bmatrix} 0.01\ 0.02\ 0.01 \end{bmatrix}, \quad \Psi_1(1) = 0.2, \quad \Gamma_1(1) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

$$G_1(1) = \begin{bmatrix} 0.5\ {-0.1}\ 1 \end{bmatrix}, \quad F_1(1) = 0.04 \times I_3, \quad H_1(1) = 0.01 \times I_3,$$

$$A_2(1) = \begin{bmatrix} -5 & 0 & 0 \\ 0.3 & -3.5 & 2 \\ -1.0 & 0.1 & -8 \end{bmatrix}, \quad E_2(1) = \begin{bmatrix} -1.1 & 0.1 & 1 \\ -0.5 & -2.0 & 0.8 \\ 2.1 & 0 & 0.4 \end{bmatrix},$$

$$G_2(1) = \begin{bmatrix} 0\ 0.2\ 1 \end{bmatrix},$$

$$L_2(1) = \begin{bmatrix} 0.01\ 0.01\ 0.01 \end{bmatrix}, \quad \Psi_2(1) = 0.1, \quad \Gamma_2(1) = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix},$$

$$F_2(1) = 0.04 \times I_3, \quad H_2(1) = 0.01 \times I_3,$$

$$A_1(2) = \begin{bmatrix} -2.5 & 0.5 & -0.1 \\ 0.1 & -3.5 & 0.3 \\ -0.1 & 1 & -2 \end{bmatrix}, \quad E_1(2) = \begin{bmatrix} 0.1053 & -0.1948 & -0.6855 \\ -0.1586 & 0.0.0755 & -0.2684 \\ 0.8709 & -0.5266 & -1.1883 \end{bmatrix},$$

$$L_1(2) = \begin{bmatrix} 0.02\ 0.01\ 0.01 \end{bmatrix}, \quad \Psi_1(2) = 0.5, \quad \Gamma_1(2) = \begin{bmatrix} -0.5 \\ 0.6 \\ 0 \end{bmatrix},$$

$$G_1(2) = \begin{bmatrix} 0 \; 1 \; 0.6 \end{bmatrix}, \quad F_1(2) = 0.09 \times I_3, \quad H_1(2) = 0.025 \times I_3,$$

$$A_2(2) = \begin{bmatrix} -2.0 & 0 & -0.1 \\ 0.1 & -5 & 0.7 \\ 0.1 & 1.2 & -2.4 \end{bmatrix}, \quad E_2(2) = \begin{bmatrix} 0.1 & -1 & -0.5 \\ -1.5 & 0.02 & -0.3 \\ -0.4 & 0.6 & -1.2 \end{bmatrix},$$

$$G_2(2) = \begin{bmatrix} 0.1 \; 0.1 \; 1.75 \end{bmatrix},$$

$$L_2(2) = \begin{bmatrix} 0.01 \; 0.04 \; 0.02 \end{bmatrix}, \quad \Psi_2(2) = 0.3, \quad \Gamma_2(2) = \begin{bmatrix} 0.4 \\ 0.5 \\ -1.2 \end{bmatrix},$$

$$F_2(2) = 0.09 \times I_3, \quad H_2(2) = 0.025 \times I_3.$$

Initial simulation of the open-loop trajectories, depicted in Fig. 7.1, shows that both subsystems are unstable. For the purpose of stabilization, we employ the Matlab LMI-solver for numerical solution with the input matrices

$$B_1(1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B_1(2) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$B_2(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_2(2) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is found that the feasible solution of the LMIs (7.79)–(7.81) is attained at

$$\begin{aligned} \textit{Subsystem } 1: \quad & \tau_{m1} = 0.1, \quad \tau_{M1} = 0.7, \\ & \gamma_1 = 0.8336, \quad \varepsilon_1(1) = 7.9588, \quad \varepsilon_1(2) = 7.9588, \\ & \lambda_1(1) = 7.9588, \quad \lambda_1(2) = 7.9588, \\ & \mathcal{K}_1(1) = \begin{bmatrix} -1.2242 & -0.0048 & -2.0974 \\ -0.4566 & -1.5360 & 2.5571 \end{bmatrix}, \end{aligned}$$

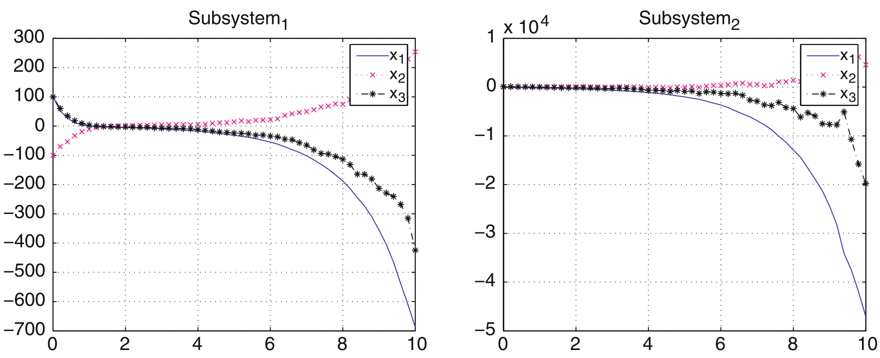

Fig. 7.1 Plots of open-loop trajectories: subsystem 1 (*left*) and subsystem 2 (*right*)

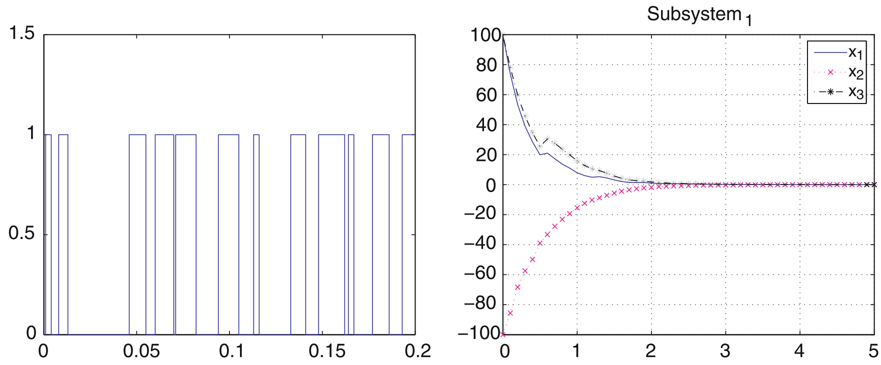

Fig. 7.2 Plots of subsystem 1: switching signal (*left*) and state trajectories (*right*)

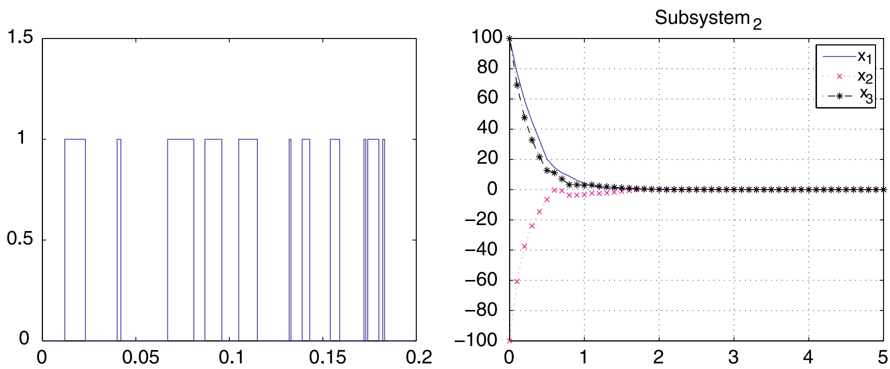

Fig. 7.3 Plots of subsystem 2: switching signal (*left*) and state trajectories (*right*)

$$\mathcal{K}_1(2) = \begin{bmatrix} -0.0701 & -0.9263 & -3.4063 \\ -1.9252 & -0.9824 & -0.4140 \end{bmatrix},$$

$$\textit{Subsystem } 2\text{:} \quad \tau_{m2} = 0.2, \quad \tau_{M2} = 0.8,$$

$$\gamma_2 = 0.8423, \quad \varepsilon_2(1) = 1.6782, \quad \varepsilon_2(2) = 1.6782,$$

$$\lambda_2(1) = 1.6782, \quad \lambda_2(2) = 1.6782,$$

$$\mathcal{K}_2(1) = \begin{bmatrix} -4.9939 & -2.6350 & 0.9172 \\ -2.2851 & -7.8264 & -1.0395 \end{bmatrix},$$

$$\mathcal{K}_2(2) = \begin{bmatrix} 6.7707 & -3.5949 & -7.9858 \\ -3.8214 & 2.0099 & -2.7224 \end{bmatrix}.$$

The corresponding closed-loop subsystem trajectories under state-feedback clarify that the subsystems are asymptotically stable under abrupt changes between modes. The ensuing plots are given in Figs. 7.2 and 7.3. In Fig. 7.4, the behavior of the switching signal is presented. Turning attention to the dynamic output-feedback case

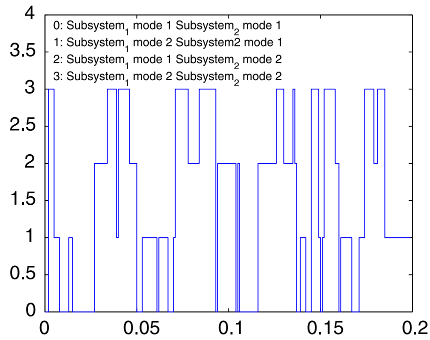

Fig. 7.4 Pattern of switching behavior

and using the Matlab LMI-solver with the output matrices

$$C_1(1) = \begin{bmatrix} 1 & 0.5 & 1 \\ 0 & 1 & 1.5 \end{bmatrix}, \quad C_1(2) = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix},$$

$$\Phi_1(1) = \begin{bmatrix} 0.1 \\ 0.5 \end{bmatrix}, \quad \Phi_1(2) = \begin{bmatrix} 0 \\ 0.3 \end{bmatrix},$$

$$C_2(1) = \begin{bmatrix} 0.4 & 0 & 1 \\ 1 & 0.5 & 0.5 \end{bmatrix}, \quad C_2(2) = \begin{bmatrix} 0.5 & 1.5 & 0 \\ 1 & 0 & 0.5 \end{bmatrix},$$

$$\Phi_2(1) = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, \quad \Phi_2(2) = \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix},$$

$$D_1(1) = \begin{bmatrix} 0.1 & 0.2 & 0 \\ 0.4 & 0.5 & 0.5 \end{bmatrix}, \quad D_1(2) = \begin{bmatrix} 0.2 & 0.4 & 0.3 \\ 0.1 & 0 & 0.3 \end{bmatrix},$$

$$D_2(1) = \begin{bmatrix} 0.2 & 0.1 & 0.1 \\ 0.2 & 0.3 & 0.4 \end{bmatrix}, \quad D_2(2) = \begin{bmatrix} 0 & 0.5 & 0.3 \\ 0.2 & 0.2 & 0.1 \end{bmatrix}$$

the results of numerical simulation yield the feasible solution of the LMIs (7.79)–(7.81), (7.110)–(7.111) as

$$\begin{aligned} \textit{Subsystem}\ 1: \quad & \tau_{m1} = 0.1, \quad \tau_{M1} = 0.7, \\ & \gamma_1 = 0.8336, \quad \varepsilon_1(1) = 7.9588, \quad \varepsilon_1(2) = 7.9588, \\ & \lambda_1(1) = 7.9588, \quad \lambda_1(2) = 7.9588, \\ & \mathcal{K}_{c1}(1) = \begin{bmatrix} -1.2242 & -0.0048 & -2.0974 \\ -0.4566 & -1.5360 & 2.5571 \end{bmatrix}, \\ & \mathcal{K}_{c1}(2) = \begin{bmatrix} -0.0701 & -0.9263 & -3.4063 \\ -1.9252 & -0.9824 & -0.4140 \end{bmatrix}, \\ & \mathcal{K}^t_{o1}(1) = \begin{bmatrix} 0.0488 & 0.1996 & 0.0318 \\ -0.2248 & 0.6650 & 0.0433 \end{bmatrix}, \end{aligned}$$

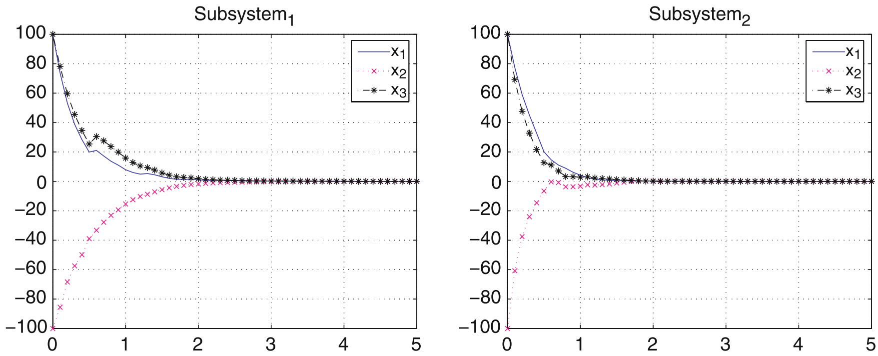

Fig. 7.5 Plots of output-feedback trajectories: subsystem 1 (*left*) and subsystem 2 (*right*)

$$\mathcal{K}^t_{o1}(2) = \begin{bmatrix} 0.8419 & 0.4566 & 0.9497 \\ 0.6341 & 0.3460 & 0.6976 \end{bmatrix},$$

$$\textit{Subsystem } 2: \quad \tau_{m2} = 0.2, \quad \tau_{M2} = 0.8,$$

$$\gamma_2 = 0.8423, \quad \varepsilon_2(1) = 1.6782, \quad \varepsilon_2(2) = 1.6782,$$

$$\lambda_2(1) = 1.6782, \quad \lambda_2(2) = 1.6782,$$

$$\mathcal{K}_{c2}(1) = \begin{bmatrix} -4.9939 & -2.6350 & 0.9172 \\ -2.2851 & -7.8264 & -1.0395 \end{bmatrix},$$

$$\mathcal{K}_{c2}(2) = \begin{bmatrix} 6.7707 & -3.5949 & -7.9858 \\ -3.8214 & 2.0099 & -2.7224 \end{bmatrix},$$

$$\mathcal{K}^t_{o2}(1) = \begin{bmatrix} 0.1312 & -0.1555 & 0.0246 \\ -0.4663 & 0.5364 & -0.0383 \end{bmatrix},$$

$$\mathcal{K}^t_{o2}(2) = \begin{bmatrix} 0.4195 & -0.5346 & 0.4667 \\ -0.2311 & 0.1960 & -0.3335 \end{bmatrix}.$$

The corresponding closed-loop subsystem trajectories under dynamic output-feedback illustrate that the jump subsystems are asymptotically stable under abrupt changes between modes. The ensuing plots are given in Fig. 7.5.

7.3 $\mathcal{H}_\infty$ Control by Averaging and Aggregation

In this section, we follow a different route to provide a complete study of the $\mathcal{H}_\infty$ optimal control problem for linear interconnected systems with Markovian jump parameters. We consider that the form process admits strong and weak interactions. Under perfect state measurements, for both finite and infinite horizon cases, we proceed to construct an aggregate jump linear system from the original jump linear system, which has a considerably smaller state space for the Markov chain process and eventually is independent of the singular perturbation parameter $\varepsilon > 0$. Taking into account the nature of this smaller order aggregate jump linear system, a threshold level $\bar{\gamma}$ will be defined, which is shown to constitute an upper bound for the optimal performance level of the overall system, $\gamma^*(\varepsilon)$, as the parameter ε approaches 0. In particular, this bound is shown to be exact in the infinite horizon case. In both cases, an approximate controller is constructed from the solution to the aggregate problem that can achieve any desired level of performance for the full-order system for sufficiently small values of ε.

7.3.1 Introduction

In control theory, systems subject to frequent unpredictable structural changes can be adequately modeled as piecewise deterministic systems, where the system dynamics take on different forms depending on the value of an associated Markov chain process, which is known as form or indicant process associated with the controlled system. In the linear case, these systems are also known as jump linear systems. Such a system model is useful particularly since it allows the decision maker to cope adequately with the discrete events that disrupt and/or change significantly the normal operation of a system by using the knowledge of their occurrence and the statistical information on the rate at which these events take place. Research into this class of systems and their applications into manufacturing management span several decades, with some representative books and papers in this area being [1, 6, 10–12, 14, 18, 20, 21, 55–58].

Research into the control of piecewise-deterministic systems in the presence of unknown (continuous) disturbances has been initiated, in [53] and [8] for jump linear systems, and in [4] for nonlinear systems. This is the paradigm of $\mathcal{H}_\infty$ optimal control [2, 9, 60], where there is an additional (discrete) element, which is the stochastic (piecewise constant) Markov process disturbance, that causes structural changes. A complete set of solutions for this class of problems, under perfect state and imperfect state measurements, and in both finite and infinite horizon cases has been presented in [53] based on properties of zero-sum differential games with piecewise deterministic dynamics [3]. This hybrid $\mathcal{H}_\infty$ control formulation, like its deterministic counterpart, leads to design of a robust controller which, by the small gain theorem, stabilizes a class of systems centered around the nominal system. It is particularly useful for systems whose parameters are either difficult to identify, or even simply time-varying within a significant bounded set. Compared with a single robust controller design intended to stabilize the overall jump linear system, this approach removes unnecessary conservatism by utilizing the statistical information about the disrupting discrete events. It is noted that a limited version of this problem was studied in the perfect state measurements case in [8].

In real applications, a formulation of the foregoing type eventually leads to a very high dimensional state for the Markov chain, which makes it computationally infeasible or extremely sensitive to small inaccuracies. One way of coping with (and alleviating) this difficulty, would be to group different Markov chain states into several separate sets, based on some temporal decomposition, and define a new (approximate) problem on these individual sets. More precisely, we consider the situation where the form process exhibits a two-time-scale behavior, thus admitting strong and weak interactions among its states, with the separation quantified in terms of a (small) singular perturbation parameter, $\varepsilon > 0$. Modeling of such Markov chain processes as well as their optimal control was conducted earlier by [7, 54], upon which the material of this section is built around. The derivation hereafter is based on averaging and essentially leads to an aggregate jump linear system. In effect, this system has a considerably smaller state space for its form process and is also independent of $\varepsilon > 0$. Using this smaller-order aggregate jump linear system, a threshold level is defined which asymptotically upper bounds the achievable performance level of the overall system as the parameter ε decreases to 0.

7.3.2 Problem Formulation

The class of jump linear systems under consideration is described by:

$$\dot{x} = A(\theta(t))x + B(\theta(t))u + D(\theta(t))w; \quad x(0) = x_0, \tag{7.112}$$

where $x \in \Re^n$ is the system state vector; $u \in \Re^p$ is the control input; $w \in \Re^q$ is the disturbance input; $\theta(t)$ is a finite-state Markov process defined on the state space $S = \{1, \ldots, \sigma\}$ with the infinitesimal generator matrix

$$\Lambda = (\lambda_{ij})_{\sigma \times \sigma}$$

and an initial distribution $\pi_{\cdot} = [\pi_{01}, \ldots, \pi_{0\sigma}]$. The underlying probability space is the triple (Ω, F, P). The initial condition x_0 is available to the controller, but it is not fixed a priori and is determined as part of the disturbance policy. In what follows, $\theta(t)$ defines the so-called form or indicant process, which determines the form of the system at time t. The system state x, inputs u and w, each belong to appropriate $\mathcal{L}_2$-Hilbert spaces H_x, H_u and H_w respectively, defined on the time interval $[0, \infty]$. Let $\mathbb{E}$ denote the expectation with respect to the underlying probability space.

When the form process is large-scale, it is quite natural to think of the large number of states to be grouped into different collections of states, based on whether the interaction between any two states is weak or strong. Occurrence of such a phenomenon is expressed mathematically by taking the probability transition rate matrix Λ in an appropriate singularly perturbed form [7, 54]:

$$\Lambda = \Lambda_s + (1/\varepsilon)\Lambda_f, \tag{7.113}$$

where $\Lambda_s := (\lambda_{ij}^{(s)})_{\sigma \times \sigma}$, and $\Lambda_f := (\lambda_{ij}^{(f)})_{\sigma \times \sigma}$ are probability transition rate matrices corresponding to, respectively, weak interactions and strong interactions within the form process. The scalar ε is a small positive number, whose inverse provides a measure of the order separation between the weak and strong interactions. The states of the form process can always be properly arranged so that the generator matrix Λ_f takes the following structure:

$$\Lambda_f = \begin{bmatrix} \Lambda^{(f1)} & \dots & 0_{\sigma_1 \times \sigma_{\bar{\sigma}}} & 0_{\sigma_1 \times \sigma_t} \\ \vdots & \ddots & \vdots & \vdots \\ 0_{\sigma_{\bar{\sigma}} \times \sigma_1} & \dots & \Lambda^{(f\bar{\sigma})} & 0_{\sigma_{\bar{\sigma}} \times \sigma_t} \\ \Lambda^{(ft1)} & \dots & \Lambda^{(ft\bar{\sigma})} & \Lambda^{(ft)} \end{bmatrix}, \tag{7.114}$$

where the matrices $\Lambda^{(fi)}$, $i = 1, \dots, \bar{\sigma}$ are infinitesimal generators, with the ith one corresponding to a positive recurrent Markov chain with σ_i states; the matrix $\Lambda^{(ft)}$ is of dimensions $\sigma_t \times \sigma_t$ and is Hurwitz; and the dimensions of the matrices $\Lambda^{(fti)}$, $i = 1, \dots, \bar{\sigma}$ are defined accordingly. Clearly, $\sigma = \sum_{i=1}^{\bar{\sigma}} \sigma_i + \sigma_t$. This effectively partitions the set S into $\bar{\sigma}$ recurrent (or ergodic) sets,

$$S_i := \left\{ \sum_{j=1}^{i-1} \sigma_j + 1, \dots, \sum_{j=1}^{i} \sigma_j \right\}, \quad i = 1, \dots, \bar{\sigma}$$

and a transient set,

$$S_t := \left\{ \sum_{j=1}^{\bar{\sigma}} \sigma_j + 1, \dots, \sigma \right\}.$$

From now onwards, each recurrent set and their associated form systems will be called (collectively) a recurrent group; the transient set and its associated form systems will be called the transient group. Compatible with this structure, the generator matrix Λ_s is partitioned as:

$$\Lambda_s = \begin{bmatrix} \Lambda^{(s11)} & \dots & \Lambda^{(s1\bar{\sigma})} & \Lambda^{(s1t)} \\ \vdots & \ddots & \vdots & \vdots \\ \Lambda^{(s\bar{\sigma}1)} & \dots & \Lambda^{(s\bar{\sigma}\bar{\sigma})} & \Lambda^{(s\bar{\sigma}t)} \\ \Lambda^{(st1)} & \dots & \Lambda^{(st\bar{\sigma})} & \Lambda^{(stt)} \end{bmatrix}, \tag{7.115}$$

where the superscript of the subblock matrices indicates the transitions between the indexed recurrent sets or the transient set.

The control input u is generated by a control policy μ according to

$$u(t) = \mu(t, x_{[0,t]}, \theta_{[0,t]}) \tag{7.116}$$

where $\mu : H_x \times \Omega \to H_u$ is piecewise continuous in t and Lipschitz continuous in x and measurable in θ, further satisfying the given causality condition. Let us denote the class of all admissible controllers by M.

The initial condition x_0 and the input w are determined by the disturbance policy $\delta := (\delta_0, \nu)$, according to:

$$x_0 = \delta_0(\theta(0)), \tag{7.117}$$

$$w(t) = \nu(t, x_{[0,t]}, \theta_{[0,t]}), \tag{7.118}$$

where $\delta_0 : S \to \Re^n$, and $\nu : H_x \times \Omega \to H_w$ is piecewise continuous in t and Lipschitz continuous in x and measurable in θ, further satisfying the given causality condition. Let us denote the class of all admissible disturbance policies δ by D.

Following [3], we focus on the upper value of an associated zero-sum game, where the disturbance is allowed to act after the controller policy has been selected, the disturbance policy can in fact be restricted (without any loss of generality) to one that only depends on state history of the form process, $\theta_{[0,t]}$, but not on the state trajectory of the continuous system, $x_{[0,t]}$. Observe that the causal dependence of the disturbance on the form process is a crucial assumption, which cannot be dispensed with.

We adopt for this system the infinite-horizon quadratic performance index:

$$L(\mu,\delta) = \mathsf{E}\left\{\int_0^\infty (|x(t)|^2_{Q(\theta(t))} + |u(t)|^2)dt\right\}; \quad Q(\cdot) \geq 0, \tag{7.119}$$

in which context the $\mathcal{H}_\infty$ optimal control problem is to find the infimum over all admissible controllers (belonging to M) of the following "squared" worst-case $\mathcal{L}_2$ gain:

$$J(\mu,\delta) := \sup_{\delta\in D} \frac{L(\mu,\delta)}{\mathsf{E}\{\int_0^\infty |w(t)|^2 dt + |x_0|^2_{Q_0(\theta(0))}\}}.$$

Let us denote the optimal performance level by $\gamma^*(\varepsilon)$, that is:

$$\inf_{\mu\in M} \sup_{\delta\in D} \frac{L(\mu,\delta)^{1/2}}{(\mathsf{E}\{\int_0^\infty |w(t)|^2 dt + |x_0|^2_{Q_0(\theta(0))}\})^{1/2}} := \gamma^*(\varepsilon). \tag{7.120}$$

The objective of the controller design is then to find control policies that guarantee a performance level within a given neighborhood of the optimal one.

This $\mathcal{H}_\infty$ optimal control problem is known to be closely related to a class of zero-sum differential games for the jump linear system (7.112), with the following γ-parametrized cost function:

$$J_\gamma(\mu,\delta) = L(\mu,\delta) - \gamma^2 \mathsf{E}\left\{\int_0^\infty (|w(t)|^2)dt + |x_0|^2_{Q_0(\theta(0))}\right\}, \tag{7.121}$$

where the control μ is the minimizer and the disturbance δ is the maximizer [53]. The threshold $\gamma^*(\varepsilon)$ is then the "smallest" value of $\gamma > 0$ such that the above game admits a finite upper value, which is necessarily zero.

Remark 7.9 It is significant to note that for each fixed $\varepsilon > 0$, the complete solution to the above stochastic $\mathcal{H}_\infty$ control problem has been obtained in [53], the computation of the quantity $\gamma^*(\varepsilon)$ and that of a corresponding $\mathcal{H}_\infty$-optimal or sub-optimal controller for small values of ε presents serious difficulties due to numerical ill-conditioning and high dimensionality. To remedy this, an averaging and aggregation technique is proposed, following the results of [7, 54], which leads to an aggregated jump linear system that is independent of $\varepsilon > 0$. The solution to the H^∞ control problem for the aggregated jump linear system will then be used to construct near-optimal solutions to the original stochastic H^∞ control problem.

To proceed with the study, three basic assumptions are recalled and are quite natural in the present context.

Assumption 7.3 Matrix functions $Q_0(i)$ are positive definite for all $i \in S$.

Assumption 7.4 The initial probability distribution of the form process satisfies $\pi_{oi} > 0$ for all $i \in S$.

Assumption 7.5 The pairs $(A(i), Q(i))$ are observable for each $i \in S$.

Remark 7.10 Assumption 7.3 guarantees a strictly concave cost term for the initial state of the form system, which is to be selected by the disturbance. Assumption 7.4 then says that any state of the form process has a positive probability of being visited at any time $t \in [0, \infty)$. Depending on whether the form process is recurrent or not, the probability of visiting some states may diminish to zero as $t \to \infty$.

Remark 7.11 Looking at the solution to the full-order problem in the infinite-horizon case, for a fixed $\varepsilon > 0$, we let the pair $(A(\theta), B(\theta))$ be stochastically stabilizable and Assumptions 7.3–7.5 hold. In view of the results of [53], the optimal performance level $\gamma^*(\varepsilon)$ for the full-order problem is finite. For every γ larger than this quantity, the associated zero-sum differential game has a zero upper value. A control policy that guarantees this zero upper value, which is then a control policy that guarantees an H^∞ performance level of γ, is given by

$$\mu^*_\gamma(x(t), \theta(t)) = -B'(\theta(t))\bar{Z}(\theta(t); \varepsilon)x(t), \tag{7.122}$$

where $Z(i; \varepsilon)$, $i = 1, \dots, \sigma$ are positive-definite solutions to the following set of coupled generalized algebraic Riccati inequalities (GARI's):

$$\begin{aligned} &A'(i)\bar{Z}(i) + \bar{Z}(i)A(i) - \bar{Z}(i)(B(i)B'(i)(1/\gamma^2)D(i)D'(i))\bar{Z}(i) + Q(i) \\ &\quad + \sum_{j=1}^{\sigma} \lambda_{ij}\bar{Z}(j) \le 0; \quad i = 1, \dots, \sigma. \end{aligned} \tag{7.123}$$

Furthermore, these solutions satisfy the spectral radius condition:

$$\bar{Z}(i; \varepsilon) < \gamma^2 Q_0(i); \quad i = 1, \dots, \sigma. \tag{7.124}$$

On the other hand, for any γ less than the threshold $\gamma^*(\varepsilon)$, either the set of coupled GARI's (7.123) does not admit any nonnegative definite solutions, and in particular, the following set of coupled generalized algebraic Riccati equations (GARE's) does not admit any nonnegative definite solutions:

$$A'(i)Z(i) + Z(i)A(i) - Z(i)(B(i)B'(i) - (1/\gamma^2)D(i)D'(i))Z(i) + Q(i) + \sum_{j=1}^{\sigma} \lambda_{ij} Z(j) \leq 0; \quad i = 1, \ldots, \sigma, \tag{7.125}$$

or the solution to the above sets of coupled GARI's and GARE's do not satisfy the spectral radius condition (7.124). Hence, for these values of γ, the upper value of the associated zero-sum differential game is infinity.

Now turning attention to the finite-horizon time-varying problem, we take the time interval of interest to be $[0, t_f]$, and the system matrices A, B, D, Q, Λ_s, Λ_f to be possibly time dependent, in addition to them being dependent on the form process $\theta(t)$. The cost function associated with this system, (7.119), is then replaced by the following finite-horizon quadratic function:

$$L(\mu, \delta) = \mathsf{E}\left\{\int_0^{t_f} (|x(t)|^2_{Q(t,\theta(t))} + |u(t)|^2)dt + |x(t_f)|^2_{Q_{f(\theta(t_f))}}\right\}, \\ Q(.,.) \geq 0, \quad Q_f(\cdot) \geq 0. \tag{7.126}$$

The $\mathcal{H}_\infty$ optimal control problem is again the minimization of the worst-case $\mathcal{L}_2$ gain, for which the optimal performance level is again denoted by $\gamma^*(\varepsilon)$ as defined in (7.120), with L replaced by its expression given by (7.126) above.

By similarity to the infinite-horizon case, this finite-horizon $\mathcal{H}_\infty$ control problem is closely related to a class of zero-sum differential game problems indexed by $\gamma > 0$, with the game kernel defined as in (7.121), where the function $L(\mu, \delta)$ is as defined in (7.126), and with the norm on w now defined on $[0, t_f]$. The optimal performance level $\gamma^*(\varepsilon)$ is again the "smallest" value of γ such that the associated zero-sum differential game admits a finite upper value, which is necessarily zero.

In this case, in addition to Assumptions 7.3 and 7.4, we need the following one:

Assumption 7.6 The matrices $A(t,i)$, $B(t,i)$, $D(t,i)$, $Q(t,i)$ are piecewise continuous in t, and the generator matrices $\Lambda_s(t)$, $\Lambda_f(t)$ are piecewise continuously differentiable in t, for each $i \in S$.

It follows from [53], under Assumptions 7.3, 7.4 and 7.6, for each fixed $\varepsilon > 0$ and $\gamma > \gamma^*(\varepsilon)$, there exists a controller that guarantees the $\mathcal{H}_\infty$ performance level γ for the full-order problem. The controller is then given by

$$\mu^*_\gamma(t, x(t), \theta(t)) = -B'(t, \theta(t))Z(t, \theta(t); \varepsilon)x(t), \tag{7.127}$$

where $Z(t, i; \varepsilon)$, $i = 1, \ldots, \sigma$ are nonnegative definite solutions to the following set of coupled generalized Riccati differential equations (GRDE's):

$$\dot{Z}(t,i) + A'(t,i)Z(t,i) + Z(t,i)A(t,i) - Z(t,i)(B(t,i)B'(t,i)$$
$$- (1/\gamma^2)D(t,i)D'(t,i))Z(t,i) + Q(t,i) + \sum_{j=1}^{\sigma} \lambda_{ij} Z(t,j) = 0, \tag{7.128}$$
$$Z(t_f,i) = Q_f(i), \quad i = 1,\dots,\sigma.$$

These solutions satisfy a spectral radius condition at time 0:

$$Z(0,i;\varepsilon) < \gamma^2 Q_0(i); \quad i = 1,\dots,\sigma. \tag{7.129}$$

For any γ less than the threshold $\gamma^*(\varepsilon)$, however, either the set of coupled GRDE's (7.128) has a conjugate point on the interval $[0,t_f]$, or the spectral radius condition (7.129) is violated. In either case, the upper value of the associated differential game is unbounded.

For both the finite and infinite horizon cases, we seek to use averaging and aggregation techniques to obtain an ε-independent aggregate jump linear system, so as to obtain ε-independent suboptimal controllers that guarantee any achievable performance level for the full-order problem for sufficiently small values of ε.

7.3.3 Results in the Infinite-Horizon Case

In this section, we study the problem formulated in the previous section in the infinite-horizon case, under the working Assumptions 7.3–7.5. The aggregation of the form process with the transition rate matrix given as (7.113) has already been studied extensively in [7, 54]. The aggregated Markov chain process, $\theta_A(t)$, of the form process, $\theta(t)$, constitutes the form process of the aggregate jump linear system and can be constructed as follows:

Let $\pi^{(fi)}$ be a σ_i-dimensional probability row vector that constitutes the invariant distribution corresponding to the probability transition rate matrix $\Lambda^{(fi)}$, $i = 1,\dots,\bar{\sigma}$, that is,

$$\pi^{(fi)}\Lambda^{(fi)} = 0_{1\times\sigma_i}; \quad \pi^{(fi)}1_{\sigma_i\times 1} = 1; \quad i = 1,\dots,\bar{\sigma}.$$

Define a $(\sum_{i=1}^{\bar{\sigma}} \sigma_i)$-dimensional row vector $\pi^{(f)}$ by lining up all the $\pi^{(fi)}$'s:

$$\pi^{(f)} := \left[\pi^{(f1)} \ \dots \ \pi^{(f\bar{\sigma})}\right]$$

where the jth element of $\pi^{(f)}$ is denoted by $\pi_j^{(f)}$. Define a $(\sigma_t \times \bar{\sigma})$-dimensional matrix P_T:

$$P_T = \left(P_{ij}^{(T)}\right)_{\sigma_t\times\bar{\sigma}} := -\Lambda^{(ft)^{-1}}\left[\Lambda^{(ft1)}1_{\sigma_1\times 1} \ \dots \ \Lambda^{(ft\bar{\sigma})}1_{\sigma_{\bar{\sigma}}\times 1}\right], \tag{7.130}$$

where each element $P_{ij}^{(T)}$ is the conditional probability of ending up in the jth recurrent set given that the process has started in ith state of the transient set. Thus, the aggregated process θ_A takes values in the aggregate set:

$$S_A := \{1, \dots, \bar{\sigma}\}.$$

The infinitesimal generator for θ_A is given by

$$\Lambda_A = (\lambda_{ij}^{(A)})_{\bar{\sigma}\times\bar{\sigma}} := \begin{bmatrix} \pi^{(f1)}\Lambda^{(s11)}1_{\sigma_1\times 1} & \dots & \pi^{(f1)}\Lambda^{(s1\bar{\sigma})}1_{\sigma_{\bar{\sigma}}\times 1} \\ \vdots & \ddots & \vdots \\ \pi^{(f\bar{\sigma})}\Lambda^{(s\bar{\sigma}1)}1_{\sigma_1\times 1} & \dots & \pi^{(f\bar{\sigma})}\Lambda^{(s\bar{\sigma}\bar{\sigma})}1_{\sigma_{\bar{\sigma}}\times 1} \end{bmatrix} + \begin{bmatrix} \pi^{(f1)}\Lambda^{(s1t)} \\ \vdots \\ \pi^{(f\bar{\sigma})}\Lambda^{(s\bar{\sigma}t)} \end{bmatrix} P_T, \tag{7.131}$$

where the ijth element of the first matrix is the transition rate from ith recurrent set to jth recurrent set directly, $i \neq j$; and the ijth element of the second matrix is the transition rate from ith recurrent set to jth recurrent set via the transient set, $i \neq j$.

The initial distribution for the aggregated process θ_A is

$$\pi_{A0} = \left[\sum_{l\in S_1}\pi_{0l} \ \dots \ \sum_{l\in S_{\bar{\sigma}}}\pi_{0l}\right] + [\pi_{0_{\sigma-\sigma t+1}} \ \dots \ \pi_{0_\sigma}]P_T, \tag{7.132}$$

where the ith element of the first vector is the probability of the process θ starting in the ith recurrent set; and the ith element of the second vector is the probability of the process θ to start in the transient set and land in the ith recurrent set the first instant θ leaves the transient set.

The underlying probability space for this aggregate Markov chain is denoted by the triple (Ω_A, F_A, P_A), and the expectation under this probability space by E_A.

To arrive at the form systems associated with each state of the aggregated Markov chain, we now make the following observation. As ε approaches zero, jumps of the form process within any recurrent set occur at much higher frequency than any of the time constants of any individual form system. Thus, within a short time interval of length $O(\sqrt{\varepsilon})$, the form process may visit every state within a recurrent set many times while the state of the form system moves a very short distance. The average time spent in the states among a recurrent group is proportional with the corresponding elements of the invariant distribution $\pi^{(f)}$. Thus, it should be intuitively expected that the value function for the zero-sum differential game, if it exists, is asymptotically identical on every recurrent set as $\varepsilon \to 0$. By formally setting $\varepsilon = 0$, the solution to the set of coupled GARI's (2.18) are such that

$$Z(j_1) = Z(j_2); \quad \text{for all } j_1, j_2 \in S_i, \ i = 1, \dots, \bar{\sigma}.$$

We will prove shortly that this is indeed the case as $\varepsilon \to 0$. Given this structure of the solution to the GARI's (7.123), the corresponding control policy and the worst-case disturbance input will be of the form

$$M(\theta(t))Z(\theta(t))x(t),$$

where $M(\cdot)$ is a matrix function of appropriate dimensions that is generally different for each value of the form process.

The above observation leads to the following form of the aggregated system:

$$\dot{x}_A = A_A(\theta_A(t))x_A + B_A(\theta_A(t))u_A + D_A(\theta_A(t))w_A; \quad x_A(0) = x_0, \tag{7.133}$$

where

$$A_A(i) := \sum_{l \in S_i} \pi_l^{(f)} A(l); \quad i = 1, \ldots, \bar{\sigma}, \tag{7.134}$$

$$B_A(i) := \left(\sum_{l \in S_i} \pi_l^{(f)} B(l)B'(l) \right)^{-1/2}; \quad i = 1, \ldots, \bar{\sigma}, \tag{7.135}$$

$$D_A(i) := \left(\sum_{l \in S_i} \pi_l^{(f)} D(l)D'(l) \right)^{-1/2}; \quad i = 1, \ldots, \bar{\sigma}. \tag{7.136}$$

The form of averaging for the matrices B_A and D_A is also intuitive from the fact that the control and the disturbance policies are generally switching policies whose second moments are penalized in the cost function. A direct consequence of this formulation is that the inputs into the aggregate system do not in general correspond to those for the original system. The control input and the disturbance input are generated by the following causal mappings:

$$u_A(t) = \mu_A(t, x_{A[0,t]}, \theta_{A[0,t]}), \tag{7.137}$$

$$(x_0, w_A(t)) = (\delta_{A0}(\theta_A(0)), \nu_A(t, x_{A[0,t]}, \theta_{A[0,t]})). \tag{7.138}$$

The general cost function associated with this system is defined to be:

$$\begin{aligned} &J_{A\gamma}(\mu_A, \delta_A) \\ &\quad = E_A\{(x_A' Q_A(\theta_A(t))x_A + u_A' u_A - \gamma^2 w_A' w_A)dt - \gamma^2 |x_0|^2_{Q_{A0}(\theta_A(0))}\}, \end{aligned} \tag{7.139}$$

where

$$Q_A(i) := \sum_{l \in S_i} \pi_l^{(f)} Q(l); \quad i = 1, \ldots, \bar{\sigma}, \tag{7.140}$$

$$Q_{A0}(i) := \sum_{l \in S_i} \pi_l^{(f)} Q_0(l); \quad i = 1, \ldots, \bar{\sigma}. \tag{7.141}$$

This class of zero-sum differential games for the aggregate jump linear system (7.133) and (7.131) is obviously closely related to the $\mathcal{H}_\infty$ optimal control problem:

$$\inf_{\mu_A} \sup_{\nu_A} \left(\frac{E_A\{\int_0^\infty (x_A' Q_A(\theta_A(t))x_A + u_A' u_A)dt\}}{E_A\{\int_0^\infty w_A' w_A dt + |x_0|^2_{Q_{A0}(\theta_A(0))}\}} \right)^{1/2}.$$

Remark 7.12 The aggregate problem above provides a fairly good approximation to the original full-order problem for $t > 0$. At the initial time $t = 0$, however, there is an information loss for the disturbance through the aggregation process since the disturbance can choose the initial state x_0 depending only on $\theta_A(0)$, and not on $\theta(0)$. Thus, the spectral radius condition associated with the aggregate problem is not directly relevant to the solution of the original problem except in some special cases, as it will be shown in the sequel.

For this reduced, average $\mathcal{H}_\infty$ control problem, we introduce two regularity conditions:

Assumption 7.7 The pair $(A_A(\theta_A), B_A(\theta_A))$ is stochastically stabilizable.

Assumption 7.8 The pair $(A_A(i), Q_A(i))$ are observable for each $i \in S_A$.

A related set of coupled GARE's is also introduced:

$$A'_A(i)Z_A(i) + Z_A(i)A_A(i) - Z_A(i)(B_A(i)B'_A(i) - (1/\gamma^2)D_A(i)D'_A(i))Z_A(i) + Q_A(i) + \sum_{j=1}^{\bar{\sigma}} \lambda_{ij}^{(A)} Z_A(j) = 0; \quad i = 1, \dots, \bar{\sigma}. \tag{7.142}$$

By the results of [53], the optimal performance level for the aggregate H^∞ control problem is given by the following infimum:

$$\gamma_A^* := \inf\{\gamma > 0 : \text{There exists a set of nonnegative definite solutions } Z_A(i),\ i \in S_A,\ \text{to the set of coupled GARE's (7.142), such that } Z_A(i) < \gamma^2 Q_{A0}(i) \text{ for all } i \in S_A\}.$$

This threshold level is finite under the working Assumption 7.7.

Next we introduce a set of spectral radius conditions:

$$Z_A(i) < \gamma^2 Q_0(j); \quad \forall_j \in S_i,\ i = 1, \dots, \bar{\sigma}, \tag{7.143}$$

$$\sum_{i=1}^{\bar{\sigma}} P_{\tilde{l}i}^{(T)} Z_A(i) < \gamma^2 Q_0(l); \quad \forall l \in S_t,\ \tilde{l} = l - \sigma + \sigma_t. \tag{7.144}$$

Define the quantity $\bar{\gamma} > 0$:

$$\bar{\gamma} := \inf\{\gamma > 0 : \text{There exists a set of nonnegative definite solutions } Z_A(i),\ i \in S_A,\ \text{to the set of coupled GARE's (7.142) that further satisfies the set of spectral radius conditions (7.143)–(7.144)}\}. \tag{7.145}$$

This quantity will be shown to be the asymptotic limit of $\gamma^*(\varepsilon)$ as $\varepsilon \to 0$. Under Assumption 7.7, $\bar{\gamma}$ is finite.

Remark 7.13 The quantities $\bar{\gamma}$ and γ_A^* are generally different. They are equal if $Q_0(\dot{})$ remains constant on each recurrent set S_i, $i = 1, \dots, \bar{\sigma}$; and

$$\sum_{i=1}^{\bar{\sigma}} P_{\tilde{l}i}^{(T)} Q_{A0}(i) < Q_0(l); \quad \forall l \in S_t,\ \tilde{l} = l - \sigma + \sigma_t.$$

In this special case, the optimal performance level for the aggregate problem is the asymptotic limit of the optimal performance level for the full-order system as $\varepsilon \to 0$.

For every $\gamma > \bar{\gamma}$, under Assumptions 7.7 and 7.8, it can be shown, by following lines of reasoning similar to that of [53], that there exists a set of positive definite solutions $\bar{Z}_A(i)$, $i \in S_A$ to the following set of coupled GARI's:

$$\begin{aligned} &A'_A(i)\bar{Z}_A(i) + \bar{Z}_A(i)A_A(i) - \bar{Z}_A(i)(B_A(i)B'_A(i) - (1/\gamma^2 D_A(i)D'_A(i))\bar{Z}_A(i) \\ &+ Q_A(i) + \sum_{j=1}^{\bar{\sigma}} \lambda_{ij}^{(A)} \bar{Z}_A(j) \le 0; \quad i = 1, \dots, \bar{\sigma} \end{aligned} \tag{7.146}$$

which also satisfy the spectral radius conditions (2.37). Furthermore, the following jump linear system is mean-square stable:

$$\begin{aligned} \dot{x} &= (A_A(\theta_A) - (B_A(\theta_A)B'_A(\theta_A) - (1/\gamma^2)D_A(\theta_A)D'_A(\theta_A))\bar{Z}_A(\theta_A))x \\ &=: A_{FA}(\theta_A)x. \end{aligned} \tag{7.147}$$

Let us define $\Delta_A(i)$ to be the residue to the set of coupled GARI's (7.146):

$$\begin{aligned} \Delta_A(i) := &-A'_A(i)\bar{Z}_A(i) - \bar{Z}_A(i)A_A(i) + \bar{Z}_A(i)(B_A(i)B'_A(i) \\ &- (1/\gamma^2)D_A(i)D'_A(i))\bar{Z}_A(i) \\ &- Q_A(i) - \sum_{j=1}^{\bar{\sigma}} \lambda_{ij}^{(A)} \bar{Z}_A(j) \ge 0; \quad i = 1, \dots, \bar{\sigma}. \end{aligned}$$

In terms of the matrices $\bar{Z}_A(i)$, $i \in S_A$, the following approximate control policy is introduced for the full-order system:

$$\mu_a^*(t, x(t), \theta(t)) = \begin{cases} -B'(i)\bar{Z}_A(j)x(t); & \theta(t) = i \in S_j, \\ K(i)x(t); & \theta(t) = i \in S_t, \end{cases} \tag{7.148}$$

where the p-by-n dimensional matrix function $K(\cdot)$ can be fixed arbitrarily, and its specific choice does not affect the overall performance of the full-order problem.

Now in order to relate the solution of the set of coupled GARI's (7.123) for the full-order problem to the solution of (7.146) for the aggregate problem, let us consider the following reparametrization of the former. Define

$$\bar{Z}_a(i) := \sum_{l\in S_i} \pi_l^{(f)} \bar{Z}(l); \quad i = 1,\ldots,\bar{\sigma}, \tag{7.149}$$

$$\bar{Z}_d(l) := (1/\varepsilon)(\bar{Z}(l) - \bar{Z}_a(i)); \quad l \in S_i,\ i = 1,\ldots,\bar{\sigma}, \tag{7.150}$$

$$\bar{Z}_d(l) := (1/\varepsilon)\left(\bar{Z}(l) - \sum_{i=1}^{\bar{\sigma}} P_{\tilde{l}i}^{(T)} \bar{Z}_a(i)\right); \quad l \in S_t,\ \tilde{l} = l - \sigma + \sigma_b. \tag{7.151}$$

Obviously, the matrices $\bar{Z}_a(i)$ correspond to the quadratic kernel for the average value function for the ith recurrent group; the matrices $\bar{Z}_d(l), l \in S_i$ for some $i \in S_A$, then correspond to the quadratic kernel for the $O(\varepsilon)$ deviation between the value function associated with the form process state l and the average value function of the recurrent group where the state l lies in. For any state l within the transient set S_t, the quadratic kernel of the value function at $\varepsilon = 0$ will be shown shortly to be $\sum_{i=1}^{\bar{\sigma}} P_{\tilde{l}i}^{(T)} \bar{Z}_a(i)$. Thus, the matrix $\bar{Z}_d(l)$ corresponds to the quadratic kernel for the $O(\varepsilon)$ deviation from this asymptotic value.

An independent parametrization can be formed by using a subset of the above matrices:

$$\bar{Z}_a(i), \quad i \in S_A; \qquad \bar{Z}_d(l), \quad l \in S_i\left\{\sum_{j=1}^{i}\sigma_j\right\},\ i \in S_A; \qquad \bar{Z}_d(l), \quad l \in S_t \tag{7.152}$$

since we have the obvious linear dependence, for each $i = 1,\ldots,\bar{\sigma}$:

$$\sum_{l\in S_i} \pi_l^{(f)} \bar{Z}_d(l) = 0_{n\times n}. \tag{7.153}$$

Now introduce the following matrices, for $i = 1,\ldots,\bar{\sigma}$:

$$\bar{\Lambda}^{(fi)} = (\bar{\lambda}_{lk}^{(fi)})_{(\sigma_i-1)\times(\sigma_i-1)} := \left(\lambda_{lk}^{(fi)} - \frac{\pi_k^{(fi)}}{\pi_{\sigma_i}^{(fi)}}\lambda_{l\sigma_i}^{(fi)}\right)_{(\sigma_i-1)\times(\sigma_i-1)}. \tag{7.154}$$

Clearly, these matrices are, respectively, the transition rate matrices $\Lambda^{(fi)}$, $i = 1,\ldots,\bar{\sigma}$, deflated of the zero eigenvalue. They are Hurwitz by the following lemma:

Lemma 7.5 *Let $\Lambda = (\lambda_{ij})_{p\times p}$ be the probability transition rate matrix corresponding to a continuous-time positive recurrent finite state Markov chain. Let π be the stationary probability distribution with respect to this Markov chain. Then, the matrix*

$$\bar{\Lambda} = (\bar{\lambda}_{lk})_{(p-1)\times(p-1)} := \left(\lambda_{lk} - \frac{\pi_k}{\pi_p}\lambda_{lp}\right)_{(p-1)\times(p-1)}$$

is Hurwitz.

Proof It is well known that the matrix Λ has one and only one zero eigenvalue, and its remaining eigenvalues are in the open left half of the complex plane. Define a matrix

$$T = \begin{bmatrix} 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \\ -\frac{\pi_1}{\pi_p} & \dots & -\frac{\pi_{p-1}}{\pi_p} & 1 \end{bmatrix}.$$

It is easy to verify that

$$T^{-1}\Lambda T = \begin{bmatrix} \bar{\Lambda} & * \\ 0_{1\times(p-1)} & 0 \end{bmatrix},$$

where $*$ denotes some constant matrix. This implies that the eigenvalues of $\bar{\Lambda}$ are precisely the eigenvalues of Λ except the zero eigenvalue. Hence, the lemma is proven. □

Using the solution to the aggregate GARI's (7.146), we now attempt to solve the full-order GARI's (7.123) in the following special case:

$$\begin{aligned} &A'(l)\bar{Z}(l) + \bar{Z}(l)A(l) - \bar{Z}(l)(B(l)B'(l) - (1/\gamma^2)D(l)D'(l))\bar{Z}(l) + Q(l) \\ &\quad + \sum_{j=1}^{\sigma} \lambda_{lj}\bar{Z}(j) + \Delta_A(i) = 0; \quad l \in S_i,\ i = 1, \dots, \bar{\sigma}, \end{aligned} \tag{7.155}$$

$$\begin{aligned} &A'(l)\bar{Z}(l) + \bar{Z}(l)A(l) - \bar{Z}(l)(B(l)B'(l) - (1/\gamma^2)D(l)D'(l))\bar{Z}(l) + Q(l) \\ &\quad + \sum_{j=1}^{\sigma} \lambda_{lj}\bar{Z}(j) = 0; \quad l \in S_t. \end{aligned} \tag{7.156}$$

In terms of the parametrization (7.152), the set of coupled GARE's (7.155)–(7.156) can be equivalently written with some algebraic manipulations, as

$$\begin{aligned} &A'_A(i)\bar{Z}_a(i) + \bar{Z}_a(i)A_A(i) - \bar{Z}_a(i)(B_A(i)B'_A(i) - (1/\gamma^2)D_A(i)D'_A(i))\bar{Z}_a(i) \\ &\quad + Q_A(i) + \sum_{j=1}^{\bar{\sigma}} \lambda_{ij}^{(A)}\bar{Z}_a(j) + \Lambda_A(i) + \varepsilon\eta_a(\varepsilon, i) = 0; \quad i = 1, \dots, \bar{\sigma}, \end{aligned} \tag{7.157}$$

$$\begin{aligned} &A'(l)\bar{Z}_a(i) + \bar{Z}_a(i)A(l) - \bar{Z}_a(i)(B(l)B'(l) - (1/\gamma^2)D(l)D'(l))\bar{Z}_a(i) + Q(l) \\ &\quad + \sum_{j=1}^{\bar{\sigma}}\sum_{k\in S_j} \lambda_{lk}^{(s)}\bar{Z}_a(j) + \sum_{k\in S_t}\lambda_{lk}^{(s)}\sum_{j=1}^{\bar{\sigma}} P_{(k-\sigma+\sigma_t)j}^{(T)}\bar{Z}_a(j) \\ &\quad + \sum_{j=1}^{\sigma_i-1} \bar{\lambda}_{\tilde{l}j}^{(fi)}\bar{Z}_d\left(j + \sum_{l=1}^{i-1}\sigma_l\right) + \Delta_A(i) + \varepsilon\eta_d(\varepsilon, l) = 0; \\ &\quad l \in S_i\left\{\sum_{j=1}^{i}\sigma_j\right\},\ i = 1, \dots, \bar{\sigma},\ \tilde{l} = l - \sum_{j=1}^{i-1}\sigma_j, \end{aligned} \tag{7.158}$$

$$
\begin{aligned}
&A'(l)\sum_{i=1}^{\bar\sigma} P_{\tilde l i}^{(T)}\bar Z_a(i) + \sum_{i=1}^{\bar\sigma} P_{\tilde l i}^{(T)}\bar Z_a(i)A(l) - \left(\sum_{i=1}^{\bar\sigma} P_{\tilde l i}^{(T)}\bar Z_a(i)\right)(B(l)B'(l)\\
&\quad - (1/\gamma^2)D(l)D'(l))\left(\sum_{i=1}^{\bar\sigma} P_{\tilde l i}^{(T)}\bar Z_a(i)\right) + Q(l) + \sum_{i=1}^{\bar\sigma}\sum_{j\in S_i}\lambda_{lj}^{(s)}\bar Z_a(i)\\
&\quad + \sum_{j\in S_t}\lambda_{lj}^{(s)}\sum_{i=1}^{\bar\sigma} P_{(j-\sigma\times\sigma_t)i}^{(T)}\bar Z_a(i) + \sum_{i=1}^{\bar\sigma}\sum_{j\in S_i}\lambda_{lj}^{(s)}\bar Z_d(i)\\
&\quad + \sum_{j\in S_t}\lambda_{li}^{(f)}\bar Z_d(j) + \varepsilon\eta_d(\varepsilon,l) = 0;\quad l\in S_t,\ \tilde l = l-\sigma+\sigma_t,
\end{aligned}
\tag{7.159}
$$

where the terms η_a's and η_d's are bounded and analytic functions of the parametrization (7.152). The first set above, that is (7.157), is obtained by taking the weighted average of each recurrent group of GARE's (7.155)–(7.156) with respect to the invariant distribution for that recurrent group. The second and third sets, (7.158) and (7.159), are obtained by direct substitution of relationships (7.150), (7.151) and (7.153) into the corresponding equations of (7.155)–(7.156).

The following theorem establishes that $\bar\gamma$ as the asymptotic limit of $\gamma^*(\varepsilon)$ and the suboptimality of the control policy μ_a^*, given by (7.148).

Theorem 7.10 *Consider the infinite-horizon H^∞ control problem for jump linear system* (7.112), (7.121) *formulated in Sect.* 7.3.2. *Let Assumptions* 7.3–7.5, 7.7 *and* 7.8 *hold. Then,*

1. *The optimal performance level $\gamma^*(\varepsilon)$ asymptotically converges to $\bar\gamma$ as $\varepsilon\to 0^+$, that is,*

$$
\lim_{\varepsilon\to 0^+}\gamma^*(\varepsilon) = \bar\gamma,
$$

where the quantity $\bar\gamma$ defined by (7.145) *is finite.*

2. *For each $\gamma > \bar\gamma$, there exists an $\varepsilon_\gamma > 0$ such that the set of coupled GARI's* (2.18) *admits a set of positive definite solutions for $\varepsilon\in(0,\varepsilon_\gamma]$, which can further be approximated by*

$$
\bar Z(l;\varepsilon) = \bar Z_A(i) + O(\varepsilon);\quad l\in S_i,\ i=1,\dots,\bar\sigma, \tag{7.160}
$$

$$
\bar Z(l;\varepsilon) = \sum_{i=1}^{\bar\sigma} P_{\tilde l i}^{(T)}\bar Z_A(i) + O(\varepsilon);\quad l\in S_t,\ \tilde l = l-\sigma+\sigma_b, \tag{7.161}
$$

where $\bar Z_A(i)$, $i=1,\dots,\bar\sigma$ are solutions to the set of coupled GARI's (7.146).

3. *For each $\gamma > \bar\gamma$, there exists an $\tilde\varepsilon_\gamma > 0$ such that the approximate controller, μ_a^*, achieves the performance level γ for the full-order system, for $\varepsilon\in(0,\tilde\varepsilon_\gamma]$.*

Proof Under the working Assumptions 7.7 and 7.8, the quantity $\bar\gamma$ is finite by results of [53]. Now fix any $\gamma > \bar\gamma$. At $\varepsilon = 0$, the set of coupled GARE's (7.157)–(7.158) admit a triangular structure such that a set of solutions can be obtained as follows. First, the matrices $Z_a(i), i \in S_A$ can be solved from the set of coupled GARE's (7.157) independently of the rest of the equations as

$$\bar{Z}_a(i) = \bar{Z}_A(i); \quad i = 1, \dots, \bar{\sigma}. \tag{7.162}$$

Then, the matrices $\bar{Z}_d(l)$, $l \in S_i\{\sum_{j=1}^{i} \sigma_j\}$, $i \in S_A$, can be solved from the set of coupled GARE's (7.158) by the fact that the matrices $\bar{\Lambda}^{(fi)}$, $i \in S_A$ are Hurwitz. Let us denote these solutions by

$$\bar{Z}_d(l) = \bar{Z}_{d0}(l); \quad l \in S_i\left\{\sum_{j=1}^{i} \sigma_j\right\}, \ i \in S_A. \tag{7.163}$$

Last, the matrices $\bar{Z}_d(l)$, $l \in S_t$, can be solved from the set of coupled GARE's (7.159) by the fact that the matrix $\Lambda^{(ft)}$ is Hurwitz. Let us denote the solutions here by

$$\bar{Z}_d(l) = \bar{Z}_{d0}(l); \quad l \in S_t. \tag{7.164}$$

For any symmetric matrix M, let $\mathrm{vec}(M)$ denote the vector whose elements are the lower triangular elements of M. In simple mathematical terms:

$$\mathrm{vec}(M) := \begin{bmatrix} m_{11} \\ m_{21} \\ m_{22} \\ m_{31} \\ \vdots \\ m_{nm} \end{bmatrix}, \quad M = \begin{bmatrix} m_{11} & m_{12} & \dots & m_{1n} \\ m_{21} & m_{22} & \dots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \dots & m_{nm} \end{bmatrix}.$$

For any matrix M, let $\overrightarrow{M}$ denote the stacked up vector of column vectors of M. In simple mathematical terms:

$$\overrightarrow{M} := \left[m_1' \ m_2' \ \dots \ m_n'\right]' [m_1 \ m_2 \ \dots \ m_n].$$

In terms of this notation, we can view the set of coupled GARE's (7.157)–(7.159) as a multidimensional nonlinear algebraic equation:

$$\chi(\varepsilon, \xi) = 0,$$

where

$$\xi := \left[\xi_a' \ \xi_{d1}' \ \dots \ \xi_{d\bar{\sigma}}' \ \xi_{dt}'\right]',$$

$$\xi_a := \left[\mathrm{vec}(\bar{Z}_a(1)) \ \dots \ \mathrm{vec}(\bar{Z}_a(\bar{\sigma}))'\right]',$$

$$\xi_{di} := \left[\overrightarrow{\bar{Z}_d\left(\sum_{j=1}^{i=1}\sigma_j + 1\right)}' \quad \dots \quad \overrightarrow{\bar{Z}_d\left(\sum_{j=1}^{i}\sigma_j - 1\right)}' \right]'; \quad i = 1, \dots, \bar{\sigma},$$

$$\xi_{dt} := \left[\overrightarrow{\bar{Z}_d(\sigma - \sigma_t + 1)}' \quad \dots \quad \overrightarrow{\bar{Z}_d(\sigma)}' \right]'.$$

This equation admits a solution at $\varepsilon = 0$, which is denoted by ξ_0.

It is easy to see that the Jacobian of χ with respect to ξ at $(0, \xi_0)$ is:

$$\begin{bmatrix} F(A_{AF}(1), \dots, A_{AF}(\bar{\sigma}), \Lambda_A) & 0_{(\bar{\sigma}\frac{n(n+1)}{2}) \times (\sigma_1 n^2)} & \dots & 0_{(\bar{\sigma}\frac{n(n+1)}{2}) \times (\sigma_{\bar{\sigma}} n^2)} & 0_{(\bar{\sigma}\frac{n(n+1)}{2}) \times (\sigma_t n^2)} \\ * & \bar{\Lambda}^{(f1)} \otimes I_{n^2} & \dots & 0_{\sigma n^2 \times (\sigma_{\bar{\sigma}} n^2)} & 0_{\sigma_1 n^2 \times (\sigma_t n^2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \dots & \bar{\Lambda}^{(f\bar{\sigma})} \otimes I_{n^2} & 0_{\sigma_{\bar{\sigma}} n^2 \times (\sigma_t n^2)} \\ * & * & \dots & * & \bar{\Lambda}^{(ft)} \otimes I_{n^2} \end{bmatrix},$$

where the operator $\otimes$ denotes the Kronecker product, $*$ denotes any constant matrix term not of any relevance to the analysis to follow, and the matrix $F(A_{AF}(1), \dots, A_{AF}(\bar{\sigma}), \Lambda_A)$ is the Jacobian matrix of the left-hand-side (LHS) of (7.146) with respect to

$$\left[\text{vec}(\bar{Z}_a(1))' \ \dots \ \text{vec}(\bar{Z}_a(\bar{\sigma}))'\right]',$$

evaluated at $\bar{Z}_a(i) = \bar{Z}_A(i)$, $i \in S_i$. By the mean-square stability of the system (7.147), this matrix is Hurwitz (see Lemma 7.5 of [53]). Hence, the Jacobian matrix $\frac{\partial \chi}{\partial \xi}(0, \xi)$ is Hurwitz.

This establishes the conditions for the application of a standard Implicit Function Theorem, which then implies that the set of coupled GARE's (7.157) admits a set of solutions for sufficiently small $\varepsilon > 0$, which can be approximated by

$$\bar{Z}_a(i) = \bar{Z}_A(i) + O(\varepsilon); \quad i = 1, \dots, \bar{\sigma},$$

$$\bar{Z}_d(l) = \bar{Z}_{d0}(l) + O(\varepsilon); \quad l \in S_i \left\{ \sum_{j=1}^{i} \sigma_j, \right\} i \in S_A,$$

$$\bar{Z}_d(l) = \bar{Z}_{d0}(l) + O(\varepsilon); \quad l \in S_t.$$

Furthermore, we conclude that the set of coupled GARI's (7.123) admits a set of positive definite solutions, which can be approximated as (7.160). These matrices satisfy the spectral radius condition (7.124) for sufficiently small $\varepsilon > 0$, since the matrices $\bar{Z}_A(i)$, $i \in S_A$, satisfy the spectral radius condition (7.143)–(7.144). This then establishes statement 2 of the theorem.

For statement 3, we first substitute the control policy μ_a^* into the full-order system (7.112), as well as the cost function (7.121), which results in a single-person maximization problem with respect to the disturbance δ. For any fixed $\gamma > \bar{\gamma}$, the maximum value is zero if the following set of coupled GARE's admits positive definite solutions that further satisfy the spectral radius condition (7.124):

$$A_c'(l)\bar{Z}(l) + \bar{Z}(l)A_c(l) + (1/\gamma^2)\bar{Z}(l)D(l)D'(l)\bar{Z}(l) + \bar{Z}_A(i)B(l)B'(l)\bar{Z}_A(i) + Q(l) + \sum_{j=1}^{\sigma} \lambda_{lj}\bar{Z}(j) + \Delta_A(i) = 0; \quad l \in S_i,\ i = 1, \ldots, \bar{\sigma}, \tag{7.165}$$

$$A_c'(l)\bar{Z}(l) + \bar{Z}(l)A_c(l) + (1/\gamma^2)\bar{Z}(l)D(l)D'(l)\bar{Z}(l) + Q(l) + K'(l)K(l) + \sum_{j=1}^{\sigma} \lambda_{lj}\bar{Z}(j) = 0; \quad l \in S_t, \tag{7.166}$$

where

$$A_c(l) = \begin{cases} A(l) - B(\ell)B'(\ell)\bar{Z}_A(i); & \ell \in S_i,\ i = 1, \ldots, \bar{\sigma}, \\ A(l) + B(\ell)K(\ell); & \ell \in S_t. \end{cases}$$

Following a line of reasoning that is similar to that used in proving statement 2, we can again apply the Implicit Function Theorem to show that the above holds for $\varepsilon \in (0, \tilde{\varepsilon}]$, for some $\tilde{\varepsilon} > 0$. This then establishes statement 3 of the theorem. Statements 2 and 3 together imply that $\bar{\gamma}$ is no smaller than the lim sup of the optimal performance level $\gamma^*(\varepsilon)$ as $\varepsilon \to 0^+$. In order to show statement 1, it is then sufficient to prove that the upper value of differential game with kernel J_γ, as defined in (7.121), is strictly larger than 0 for any $\gamma < \bar{\gamma}$. This proof is much easier to carry out by utilizing the counterpart of this theorem in the finite horizon case; hence, we relegate this part of the proof to Sect. 7.3.6. This completes the proof of this theorem. □

A byproduct of Theorem 7.10 and its proof, which would also be of independent interest, is the result given in the following corollary to Theorem 7.10.

Corollary 7.1 *The full-order jump linear system* (7.112) *is stochastically stabilizable for sufficiently small* $\varepsilon > 0$, *if the aggregate jump linear system* (7.133) *is stochastically stabilizable.*

7.3.4 Results in the Finite Horizon Case

In what follows, we present the counterpart of the results in the previous section in the finite horizon case. First, we define the aggregate Markov chain process $\theta_A(t)$ with state space S_A, whose probability transition rate matrix $\Lambda_A(t)$ is exactly as defined in (7.131), but is now time varying, where the probability vectors $\pi^{(fi)}$, $i \in S_A$ and the matrix P_T are time dependent as well. The aggregate jump linear system is then the same as (7.133), with the system matrices A_A, B_A and D_A being possibly time-dependent. The control input, initial state and the disturbance input are again generated by the causal mapping (7.137)–(7.138). Motivated by the solution structure to the infinite horizon case, we make the following assumption on the terminal cost function:

Assumption 7.9 The terminal weighting matrices $Q_f(i)$, $i \in S$ are of the following structure:

$$Q_f(i) = Q_{Af}(j) + \varepsilon Q_{fd}(i); \quad \forall i \in S_j,\ j = 1, \dots, \bar{\sigma},$$

$$Q_f(i) = \sum_{j=1}^{\sigma} P_{\tilde{i}j}^{(T)}(t_f) Q_{Af}(j); \quad \forall i \in S_t,\ \tilde{i} = i - \sigma + \sigma_t,$$

where $Q_{Af}(\cdot) \geq 0$, and $Q_{fd}(\cdot)$ are symmetric.

The finite-horizon cost function associated with this aggregate system is:

$$J_{A\gamma}(\mu_A, \delta_A) = E_A\left\{\int_0^{t_f} (x_A' Q_A(t, \theta_A(t)) x_A + u_A' u_A - \gamma^2 w_A' w_A) dt + |x(t_f)|^2_{Q_{Af(\theta_A(t_f))}} - \gamma^2 |x_0|^2_{Q_{A0(\theta_A(0))}}\right\}, \tag{7.167}$$

where Q_A and Q_{A0} are as defined in (7.140)–(7.141).

For this finite-horizon aggregate H^∞ control problem, we introduce the set of coupled GRDE's:

$$\begin{aligned}&\dot{Z}_A(t,i) + A_A'(t,i) Z_A(t,i) + Z_A(i) A_A(t,i) - Z_A(t,i)(B_A(t,i)B'A(t,i)\\&\quad - (1/\gamma^2 D_A(t,i) D_A'(t,i)) Z_A(t,i) + Q_A(t,i) + \sum_{j=1}^{\bar{\sigma}} \lambda_{ij}^{(A)}(t) Z_A(t,j) = 0;\end{aligned} \tag{7.168}$$

$$Z_A(t_f, i) = Q_{Af}(i), \quad i = 1, \dots, \bar{\sigma}.$$

The counterpart of the spectral radius condition (7.143)–(7.144) is:

$$Z_A(0,i) < \gamma^2 Q_0(j); \quad \forall j \in S_i,\ i = 1, \dots, \bar{\sigma}, \tag{7.169}$$

$$\sum_{i=1}^{\bar{\sigma}} P_{\tilde{l}i}^{(T)}(0) Z_A(0,i) < \gamma^2 Q_0(l); \quad \forall l \in S_t,\ \tilde{l} = l - \sigma + \sigma_t. \tag{7.170}$$

The counterpart of (7.145), which we again denote by $\bar{\gamma}$, is defined as:

$$\begin{aligned}\bar{\gamma} := \inf\{\gamma > 0 :\ &\text{There exists a set of nonnegative definite solutions } Z_A(t,i),\\ &i \in S_A,\ \text{on the interval } [0, t_f] \text{ to the set of coupled GRDE's (7.168) that}\\ &\text{further satisfies the set of spectral radius conditions (7.169)–(7.170)}\}.\end{aligned} \tag{7.171}$$

This quantity is different in general from the optimal performance level for the aggregate problem. But, in the special case delineated in Remark 7.13, it is exactly the $\mathcal{H}_\infty$ performance level of the aggregate problem.

For a fixed $\gamma > \bar{\gamma}$, we introduce the approximate control policy for the full-order system, which is the counterpart of (7.148)

$$\mu_a^*(t, x(t), \theta(t)) = \begin{cases} -B'(t,i)Z_A(t,j)x(t); & \theta(t) = i \in S_j, \\ K(t,i)x(t); & \theta(t) = i \in S_t, \end{cases} \tag{7.172}$$

where the p-by-n dimensional matrix function $K(.,.)$ can be fixed arbitrarily, and its specific selection does not affect the overall performance of the full-order problem.

By similarity to the infinite horizon case, let us consider the following independent reparametrization of the set of coupled GRDE's (7.128):

$$Z_a(t,i) := \sum_{l \in S_i} \pi_l^{(f)}(t) Z(t,l); \quad i = 1, \ldots, \bar{\sigma}, \tag{7.173}$$

$$Z_d(t,l) := (1/\varepsilon)(Z(t,l) - Z_a(t,i)); \quad l \in S_i \left\{ \sum_{j=1}^{i} \sigma_j \right\}, \ i = 1, \ldots, \bar{\sigma}, \tag{7.174}$$

$$Z_d(t,l) := (1/\varepsilon)\left(Z(t,l) - \sum_{i=1}^{\bar{\sigma}} P_{\tilde{l}i}^{(T)}(t) Z_a(t,i) \right); \quad l \in S_t, \ \tilde{l} = l - \sigma + \bar{\sigma}. \tag{7.175}$$

Define the matrices $\bar{A}^{(fi)}(t)$, $i = 1, \ldots, \bar{\sigma}$, as in (7.154). They are Hurwitz for each fixed $t \in [0, t_f]$.

In terms of the parametrization (7.173)–(7.175), the set of coupled GRDE's (7.128) can be equivalently written, after lengthy but straightforward algebraic manipulations, as

$$\begin{aligned} &\dot{Z}_a(i) + A'_A(i) Z_a(i) + Z_a(i) A_A(i) - Z_a(i)(B_A(i) B'_A(i) \\ &\quad - (1/\gamma^2) D_A(i) D'_A(i)) Z_a(i) + Q_A(i) \\ &\quad + \sum_{j=1}^{\bar{\sigma}} \lambda_{ij}^{(A)} Z_a(j) + \varepsilon \tilde{\eta}_a(\varepsilon, i) = 0; \end{aligned} \tag{7.176}$$

$$Z_a(t_f, i) = Q_{Af}(i) + \varepsilon \bar{Q}_{Adf}(i), \quad i = 1, \ldots, \bar{\sigma},$$

$$\begin{aligned} &\varepsilon \dot{Z}_d(l) + \dot{Z}_a(i) + A'(l) Z_a(i) + Z_a(i) A(l) - Z_a(i)(B(l) B'(l) \\ &\quad - (1/\gamma^2) D(l) D'(l)) Z_a(i) + \sum_{j=1}^{\bar{\sigma}} \sum_{k \in S_j} \lambda_{lk}^{(s)} Z_a(j) \end{aligned}$$

$$+\sum_{k\in S_t}\lambda_{lk}^{(s)}\sum_{j=1}^{\bar{\sigma}}P_{(k-\sigma+\sigma_t)j}^{(T)}Z_a(j)+\sum_{j=1}^{\sigma_i-1}\bar{\lambda}_{\tilde{l}j}^{(fi)}Z_d\left(j+\sum_{l=1}^{i-1}\sigma_l\right) \quad (7.177)$$

$$+Q(l)+\varepsilon\tilde{\eta}_d(\varepsilon,l)=0;$$

$$Z_d(t_f,l)=\bar{Q}_{fd}(l), \quad l\in S_i\left\{\sum_{j=1}^{i}\sigma_j\right\},\ i=1,\dots,\bar{\sigma},\ \tilde{l}=l-\sum_{j=1}^{i-1}\sigma_j,$$

$$\varepsilon\dot{Z}_d(l)+\frac{d}{dt}\left(\sum_{i=1}^{\bar{\sigma}}P_{\tilde{l}i}^{(T)}(t)Z_a(t,i)\right)+A'(l)\sum_{i=1}^{\bar{\sigma}}P_{\tilde{l}i}^{(T)}Z_a(i)+\sum_{j=1}^{\bar{\sigma}}P_{\tilde{l}i}^{(T)}Z_a(i)A(l)$$

$$+Q(l)-\left(\sum_{i=1}^{\bar{\sigma}}P_{\tilde{l}i}^{(T)}Z_a(i)\right)(B(l)B'(l)-(1/\gamma^2)D(l)D'(l))\left(\sum_{i=1}^{\bar{\sigma}}P_{\tilde{l}i}^{(T)}Z_a(i)\right)$$

$$+\sum_{i=1}^{\bar{\sigma}}\sum_{j\in S_i}\lambda_{lj}^{(s)}Z_a(i)+\sum_{j\in S_t}\lambda_{lj}^{(s)}\sum_{i=1}^{\bar{\sigma}}P_{(j-\sigma+\sigma_t)i}^{(T)}Z_a(i)+\sum_{i=1}^{\bar{\sigma}}\sum_{j\in S_i}\lambda_{lj}^{(f)}Z_d(j)$$

$$+\sum_{i\in S_t}\lambda_{li}^{(f)}Z_d(i)+\varepsilon\tilde{\eta}_d(\varepsilon,l)=0; \quad (7.178)$$

$$Z_d(t_f,l)=\bar{Q}_{fd}(l), \quad l\in S_t,\ \tilde{l}=l-\sigma+\sigma_t, \quad (7.179)$$

where the dependence on time t has been suppressed for the sake of simplicity, the terms $\tilde{\eta}_a$'s and $\tilde{\eta}_d$'s are bounded and analytic functions of the parametrization (7.173)–(7.175), and the terminal values are defined as:

$$\bar{Q}_{Adf}(i):=\sum_{j\in S_i}\pi_j^{(f)}(t_f)Q_{fd}(j), \quad i=1,\dots,\bar{\sigma}, \quad (7.180)$$

$$\bar{Q}_{fd}(l):=Q_{fd}(l)-\bar{Q}_{Afd}(i),$$

$$l\in S_i\left\{\sum_{j=1}^{i}\sigma_j\right\},\ i=1,\dots,\bar{\sigma},\ \tilde{l}=l-\sum_{j=1}^{i-1}\sigma_j, \quad (7.181)$$

$$\bar{Q}_{fd}(l):=Q_{fd}(l)-\sum_{i=1}^{\bar{\sigma}}P_{\tilde{l}i}^{(T)}(t_f)\bar{Q}_{Adf}(i), \quad l\in S_t,\ \tilde{l}=l-\sigma+\sigma_t. \quad (7.182)$$

The following theorem, which is the counterpart of Theorem 7.10, is now stated:

Theorem 7.11 *Consider the finite-horizon H^∞ control problem for jump linear system* (7.112)–(7.118) *and* (7.126) *formulated in Sect.* 7.3.2. *Let Assumptions* 7.3, 7.4, 7.6 *and* 7.9 *hold. Then*:

1. *The optimal performance level* $\gamma^*(\varepsilon)$ *is asymptotically bounded above by* $\bar{\gamma}$ *as* $\varepsilon \to 0^+$, *that is,*

$$\limsup_{\varepsilon \to 0^+} \gamma^*(\varepsilon) \le \bar{\gamma},$$

where the quantity $\bar{\gamma}$ *defined by* (7.171) *is finite.*
2. *For each* $\gamma > \bar{\gamma}$, *there exists an* $\varepsilon_\gamma > 0$ *such that set of coupled GRDE's* (7.128) *admits a set of nonnegative definite solutions on the interval* $[0, t_f]$ *for* $\varepsilon \in (0, \varepsilon_\gamma]$, *which can be approximated by*

$$Z(t, l; \varepsilon) = Z_A(t, i) + O(\varepsilon); \quad t \in [0, t_f],\ l \in S_i,\ i = 1, \ldots, \bar{\sigma}, \tag{7.183}$$

$$Z(t, l; \varepsilon) = \sum_{i=1}^{\bar{\sigma}} P_{\tilde{l}i}^{(T)}(t) Z_A(t, i) + O(\varepsilon);$$
$$t \in [0, t_f],\ l \in S_t,\ \tilde{l} = l - \sigma + \sigma_b, \tag{7.184}$$

where $Z_A(t, i), i = 1, \ldots, \bar{\sigma}$ *are solutions to the set of coupled GRDE's* (7.168). *Furthermore, these matrices satisfy the spectral radius condition* (7.129).
3. *For each* $\gamma > \bar{\gamma}$, *there exists an* $\tilde{\varepsilon} > 0$ *such that the approximate controller,* μ_a^*, *defined by* (7.172), *achieves the performance level* γ *for the full-order system, for* $\varepsilon \in (0, \tilde{\varepsilon}_\gamma]$.

Proof The quantity γ is finite since at $\gamma = \infty$, the solution to GRDE's (7.128) is always bounded on any finite interval. Now fix any $\gamma > \bar{\gamma}$, and set $\varepsilon = 0$. The set of coupled GRDE's (7.176)–(7.178) again admits a triangular structure such that a set of solutions can be obtained as described below. First, the matrix functions $Z_a(t, i)$, $i \in S_A$ can be solved from the set of coupled GRDE's (7.176)–(7.178) independently of the rest of the equations, as

$$Z_a(t, i) = Z_A(t, i); \quad t \in [0, t_f],\ i = 1, \ldots, \bar{\sigma}. \tag{7.185}$$

Then, the matrix functions $Z_d(t, l), l \in S_i\{\sum_{j=1}^{i} \sigma_j\}$ and $i \in S_A$ can be solved from the set of coupled GARE's (7.176)–(7.178) by the fact that the matrices $\bar{\Lambda}^{(fi)}(t), i \in S_A$, are Hurwitz for each fixed $t \in [0, t_f]$. Last, the matrix functions $Z_d(t, l), l \in S_t$, can be solved from the set of coupled GARE's (7.176)–(7.178) by the fact that the matrix $\Lambda^{(ft)}(t)$ is Hurwitz for each fixed $t \in [0, t_f]$.

The Jacobian matrix of the LHS of (7.176)–(7.178) with respect to $\xi^{(f)}$ evaluated at $\varepsilon = 0$ is

$$\begin{bmatrix} 0_{\bar{\sigma}\frac{n(n+1)}{2} \times (\sigma_1 n^2)} & \cdots & 0_{(\bar{\sigma}\frac{n(n+1)}{2}) \times (\sigma_{\bar{\sigma}} n^2)} & 0_{(\bar{\sigma}\frac{n(n+1)}{2}) \times (\sigma_t n^2)} \\ \bar{\Lambda}^{(f1)} \otimes I_{n^2} & \cdots & 0_{(\sigma_1 n^2) \times (\sigma_{\bar{\sigma}} n^2)} & 0_{(\sigma_1 n^2) \times (\sigma_t n^2)} \\ \vdots & \ddots & \vdots & \vdots \\ * & \cdots & \bar{\Lambda}^{(f\bar{\sigma})} \otimes I_{n^2} & 0_{(\sigma_{\bar{\sigma}} n^2) \times (\sigma_t n^2)} \\ * & \cdots & * & \bar{\Lambda}^{(ft)} \otimes I_{n^2} \end{bmatrix},$$

where

$$\xi^{(f)} := \left[\xi'_{d1} \ \dots \ \xi'_{d\bar{\sigma}} \ \xi'_{dt}\right]',$$

$$\xi_{di} := \left[\overrightarrow{Z_d\left(t, \sum_{j=1}^{i=1}\sigma_j + 1\right)}' \ \dots \ \overrightarrow{Z_d\left(\sum_{j=1}^{i}\sigma_j - 1\right)}'\right]'; \quad i = 1, \dots, \bar{\sigma},$$

$$\xi_{dt} := \left[\overrightarrow{\bar{Z}_d(t, \sigma - \sigma_t + 1)}' \ \dots \ \overrightarrow{\bar{Z}_d(t, \sigma)}'\right]'.$$

Obviously, this Jacobian matrix is Hurwitz for any $t \in [0, t_f]$. This sets up the conditions for the application of the Implicit Function Theorem [19], which then implies that the set of coupled GRDE's (7.176)–(7.178) admits a set of solutions on $[0, t_f]$ for sufficiently small $\varepsilon > 0$. Furthermore, we conclude that the set of coupled GRDE's (7.128) admits a set of nonnegative definite solutions on $[0, t_f]$, which can be approximated as (7.183)–(7.184). These matrices satisfy the spectral radius condition (7.129) for sufficiently small $\varepsilon > 0$, since the matrices $Z_A(0, i)$, $i \in S_A$, satisfy the spectral radius condition (7.169)–(7.170). This then establishes statement 2 of the theorem.

For statement 3, we first substitute the control policy μ_a^* into the full-order system (7.112), as well as the cost function (7.121), which results in a single-person maximization problem with respect to the disturbance δ. A similar application of the Implicit Function Theorem then establishes the result. Statements 2 and 3 together imply statement 1. This completes the proof of the theorem. □

7.3.5 Simulation Example 7.2

In what follows, we consider a numerical example in the infinite-horizon case to illustrate the results of Sect. 2.2.1.

Consider the H^∞ optimal control problem for a jump linear system where the form process is defined on a state space of $S = \{1, 2, 3, 4, 5\}$, and the system matrices are specified as follows:

$$A(1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad B(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad D(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$$

$$Q(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad Q_0(1) = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix};$$

$$A(2) = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}; \quad B(2) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}; \quad D(2) = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix};$$

$$Q(2) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}; \quad Q_0(2) = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix};$$

$$A(2) = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}; \quad B(2) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}; \quad D(2) = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix};$$

$$Q(2) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}; \quad Q_0(2) = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix};$$

$$A(3) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad B(3) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad D(3) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix};$$

$$Q(3) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \quad Q_0(3) = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix};$$

$$A(4) = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}; \quad B(4) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad D(4) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix};$$

$$Q(4) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}; \quad Q_0(4) = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix};$$

$$A(5) = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}; \quad B(5) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; \quad D(5) = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix};$$

$$Q(5) = \begin{bmatrix} 2 & 1 \\ 1 & 1/2 \end{bmatrix}; \quad Q_0(5) = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix}.$$

The infinitesimal generator of the Markov chain is of the standard singularly perturbed form:

$$\Lambda = \lambda_s + (1/\varepsilon)\lambda_f,$$

where

$$\lambda_f = \begin{bmatrix} -2 & 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 1 & -3 \end{bmatrix}; \quad \lambda_s = \begin{bmatrix} -3 & 0 & 1 & 0 & 2 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus the state space S of the form process can be decomposed into two recurrent sets: $S_1 := \{1, 2\}$ and $S_2 := \{3, 4\}$, and a transient set: $S_t := \{5\}$.

Using a particular search algorithm, we can compute the optimal performance level $\gamma^*(\varepsilon)$ for different values of $\varepsilon > 0$, which are listed in Table 7.1. As $\varepsilon \to 0^+$, we observe that the complexity of the computation increases from 14 million flops in the case of $\varepsilon = 0.1$ to 744 million flops in the case of $\varepsilon = 0.001$. The aggregate form process takes values in the set $S_A := \{1, 2\}$. The parameter matrices of the

Table 7.1 Optimal performance levels for the full-order system for different values of ε

ε	0.1	0.05	0.01	0.001
$\gamma^*(\varepsilon)$	3.467	3.498	3.584	2.622

aggregate jump linear system are obtained as follows:

$$A_A(1) = \begin{bmatrix} 0.6667 & -0.3333 \\ 1.333 & 0 \end{bmatrix}; \quad B_A(1) = \begin{bmatrix} 0.7045 & -0.4127 \\ -0.4127 & 0.9109 \end{bmatrix};$$

$$D_A(1) = \begin{bmatrix} 1.274 & 0.2098 \\ 0.2098 & 1.903 \end{bmatrix}; \quad Q_A(1) = \begin{bmatrix} 1.667 & 0.6667 \\ 0.6667 & 1.667 \end{bmatrix};$$

$$Q_{A0}(1) = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix}; \quad A_A(2) = \begin{bmatrix} -0.5 & 1 \\ 0.5 & 0.5 \end{bmatrix}; \quad B_A(2) = \begin{bmatrix} 0.7071 & 0 \\ 0 & 0.7071 \end{bmatrix};$$

$$D_A(2) = \begin{bmatrix} 0.6325 & 0.3162 \\ 0.3162 & 0.9487 \end{bmatrix}; \quad Q_A(2) = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}; \quad Q_{A0}(2) = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix}.$$

Again using a particular search algorithm, we can compute the threshold level $\bar{\gamma}$ to be

$$\bar{\gamma} = 3.630,$$

where the computational complexity is 0.893 million flops. For this problem, the specification of the initial cost term falls into the special case described in Remark 7.13. Hence, the quantity $\bar{\gamma}$ is actually equal to the H^∞ optimal performance level for the aggregate problem. Subsequently, we conclude that the optimal performance $\gamma^*(\varepsilon)$ for the full-order system converges to the optimal performance for the aggregate problem as $\varepsilon \to 0^+$, which corroborates the results of Theorem 7.10 and the observation made in Remark 7.13.

Selecting a desired performance level $\gamma = 3.65$, we can compute the positive-definite solutions to the set of coupled GARI's for the aggregate problem, which are given as

$$Z_1 = \begin{bmatrix} 16.71 & 6.627 \\ 6.627 & 3.996 \end{bmatrix}; \quad Z_1 = \begin{bmatrix} 4.719 & 3.162 \\ 3.162 & 4.102 \end{bmatrix}.$$

The approximate controller that asymptotically guarantees the performance level 3.65 is given by

$$\mu_a^*(x(t), \theta(t)) = \begin{cases} [-6.627 \ -3.996]x(t); & \theta(t) = 1, \\ [-10.08 \ -2.632]x(t); & \theta(t) = 2, \\ [-3.162 \ -4.102]x(t); & \theta(t) = 3, \\ [-4.719 \ -3.162]x(t); & \theta(t) = 4, \\ 0; & \theta(t) = 5. \end{cases}$$

The performance levels achieved by this controller, when applied to the full-order system with different values of ε, are listed in Table 7.2. We see that as ε decreases to zero, this approximate controller achieves the desired performance level 3.65 for the overall system.

This example clearly illustrates the effectiveness of the aggregation and averaging design procedure. For the $\varepsilon = 0.001$case, this procedure effectively reduces

Table 7.2 Performance levels attained by μ_a^* for the full-order system for different values of ε

ε	0.1	0.05	0.01	0.001
$\gamma^*(\varepsilon)$	3.751	3.673	3.637	3.631

the computational complexity 800 fold. Even for a not-so-small value of ε, such as $\varepsilon = 0.1$, the procedure still achieves a 15 fold reduction in computational complexity.

7.3.6 Appendix

In this appendix, we continue with the proof of Theorem 7.10, and complete the verification of the fact that $\bar{\gamma}$ is asymptotically less than or equal to the optimal performance of the full-order system. Let us fix a $\gamma < \bar{\gamma}$. Then, either (a) the set of coupled GARE's (2.36) does not admit any nonnegative definite solutions; or (b) the set of coupled GARE's (2.36) admits a set of minimal positive definite solutions $Z_A(i)$, $i \in S_A$, but the set of spectral radius conditions (7.143)–(7.144) is violated.

In either of these two cases, we can find a $t_1 > 0$ such that the following set of coupled GRDE's:

$$\begin{aligned}
&\dot{Z}_A(t,i;t_1) + A'_A(i)Z_A(t,i;t_1) + Z_A(t,i;t_1)A_A(i) - Z_A(t,i;t_1)(B_A(i)B'A(i) \\
&\qquad - (1/\gamma^2 D_A(i)D'_A(i))Z_A(t,i;t_1) + Q_A(i) + \sum_{j=1}^{\bar{\sigma}} \lambda_{ij}^{(A)}(t)Z_A(t,j;t_1) = 0; \\
&Z_A(t_1,i;t_1) = 0_{n\times n}, \quad i = 1,\ldots,\bar{\sigma}
\end{aligned}$$

admits a set of nonnegative definite solutions $Z_A(t,i;t_1)$, $i \in S_A$, on $[0,t_1]$ and the spectral radius condition (7.169)–(7.170) is violated. Since γ is strictly less than $\bar{\gamma}$, we can choose t_1 such that, either a matrix $\gamma^2 Q_0(j_0) - Z_A(0,i_0;t_1)$, for some $j_0 \in S_{i_0}$ and some $i_0 \in S_A$; or a matrix $\gamma^2 Q_0(j_0) - \sum_{i=1}^{\bar{\sigma}} P_{\tilde{j}_0 i}^{(T)}(0)Z_A(0,i;t_1)$, for some $j_0 \in S_t$ and $\tilde{j}_0 = j_0 - \sigma + \sigma_t$, has at least one negative eigenvalue. Then, by an application of the Implicit Function Theorem, as in the proof of Theorem 2.2, it can be shown that the following set of coupled GRDE's admits a set of nonnegative definite solutions on $[0,t_1]$ for sufficiently small $\varepsilon > 0$:

$$\begin{aligned}
&\dot{Z}(t,i;t_1) + A'(i)Z(t,i;t_1) + Z(t,i;t_1)A(i) - Z(t,i;t_1)(B(i)B'(i) \\
&\qquad - (1/\gamma^2 D_A(i)D'(i))Z(t,i;t_1) + Q(i) + \sum_{j=1}^{\bar{\sigma}} \lambda_{ij} Z(t,j;t_1) = 0; \\
&Z(t_1,i;t_1) = 0_{n\times n}, \quad i = 1,\ldots,\bar{\sigma}.
\end{aligned}$$

Furthermore, there exists an $i_0 \in S$ such that the matrix $\gamma^2 Q_0(i_0) - Z(0,i_0;t_1)$ has at least one negative eigenvalue for sufficiently small values of ε. Let $\bar{x}_0$ be the nonzero vector associated with this negative eigenvalue. Introduce a particular disturbance policy $\bar{\delta} = (\bar{\delta}_0, \bar{\nu})$ given by

$$x_0 = \bar{\delta}(\theta(0)) = \begin{cases} 0_{n\times 1}; & \theta(0) \in S\{i_0\}, \\ \bar{x}_0; & \theta(0) = i_0, \end{cases}$$

$$w(t) = \bar{\nu}(t, x(t), \theta(t)) = \begin{cases} 0_{n\times 1}; & t \in (t_1, \infty), \\ -B'(\theta(t))Z(t, \theta(t); t_1)x(t); & t \in [0, t_1]. \end{cases}$$

Finally, note the following series of inequalities:

$$\begin{aligned} &\inf_{\mu\in M} \sup_{\delta\in D} J_\gamma(\mu, \delta) \\ &\quad \geq \inf_{\mu\in M} \sup_{\delta\in D} E\left\{ \int_0^{t_1} (|x|^2_{Q(\theta(t))} + |u|^2 - \gamma^2|w|^2)dt - \gamma^2|x_0|^2_{Q(\theta(0))} \right\} \\ &\quad \geq \sup_{\delta=\bar{\delta}} \inf_{\mu\in M} E\left\{ \int_0^{t_1} (|x|^2_{Q(\theta(t))} + |u|^2 - \gamma^2|w|^2)dt - \gamma^2|x_0|^2_{Q(\theta(0))} \right\} \\ &\quad \geq -\pi_{0i_0}|\bar{x}_0|^2_{\gamma^2 Q_0(i_0) - Z(0, i_0; t_1)} > 0. \end{aligned}$$

This implies that the upper value of the game is strictly positive. Hence, $\gamma < \gamma^*(\varepsilon)$ for sufficiently small values of ε. This completes the proof of Theorem 7.10.

7.4 Notes and References

In this chapter, we have investigated the problems of mode-dependent decentralized stochastic stability and stabilization with $\mathcal{H}_\infty$ performance for a class of continuous-time interconnected jumping time-delay systems. The jumping parameters are governed by a finite state Markov process and the delays are unknown time-varying and mode-dependent within interval. The interactions among subsystems satisfy quadratic bounding constraints. We have established mode-dependent local stability behavior by employing an improved Lyapunov-Krasovskii functional at the subsystem level and express the stability conditions in terms of linear matrix inequalities (LMIs). We have developed a class of local decentralized state-feedback and a class of dynamic observer-based controllers to render the closed-loop interconnected jumping system stochastically stable.

Next we looked at deterministic systems with slow-fast Mark chains and developed decentralized polices based on averaging and aggregation. Results presented here could be extended in several directions. The first, conceptually simple extension would be to go from a single singular perturbation parameter (as in this chapter) to multiple parameters, representing multiple temporal decompositions. Though conceptually easy, this extension requires cumbersome notation for a precise description of the underlying decomposition, which we have therefore refrained from doing in this chapter. Another parallel study would be the investigation of time-scale decomposition for jump linear systems whose form systems (and not the form process) exhibit a two-time-scale behavior. A more challenging problem would be the one where both the form process and the form systems are singularly perturbed. This is a topic that is currently under study.

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Chapter 8
Decentralized Adaptive Control

Decentralized adaptive control design for a class of large-scale interconnected nonlinear systems with unknown interconnections is considered in the sequel. The motivation behind this work is to develop decentralized control for a class of large-scale systems which do not satisfy the matching condition requirement. To this end, large-scale nonlinear systems transformable to the decentralized strict feedback form are considered. Coordinate-free geometric conditions under which any general interconnected nonlinear system can be transformed to this form are obtained. The interconnections are assumed to be bounded by polynomial-type nonlinearities. However, the magnitudes of the nonlinearities are unknown. Global stability and asymptotic regulation are established using classical Lyapunov techniques. The controller is shown to maintain robustness for a wide class of systems obtained by perturbations in the dynamics of the original system. Furthermore, appending additional subsystems does not require controller redesign for the original subsystems. Finally, the scheme is extended to the model reference tracking problem where global uniform boundedness of the tracking error to a compact set is established.

A digital redesign of the analogue model-reference-based decentralized adaptive tracker is proposed for the sampled-data large scale system consisting of N interconnected linear subsystems, so that the system output will follow any trajectory specified at sampling instants. This may not be presented by the analytic reference initially. It will be shown in the sequel that the proposed decentralized controller induces a good robustness on the decoupling of the closed loop controlled system. The adaptation of the analogue controller gain is derived by using the model-reference adaptive control theory based on Lyapunov's method. In this chapter, it is shown that using the sampled-data decentralized adaptive control system it is theoretically possible to asymptotically track the desired output with a desired performance. It is assumed that all the controllers share their prior information and the principal result is derived when they cooperate implicitly. Based on the prediction-based digital redesign methodology, the optimal digital redesigned tracker for the sampled-data decentralized adaptive control systems is proposed.

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8.1 Introduction

The decentralized control schemes, different from the classic centralized information structure, have been considered with significant interests for the control of interconnected systems in recent years. The main objectives of decentralized control are to find some feedback laws for adapting the intersections from the other subsystems where no state information is transferred. The advantage of decentralized control design is to reduce the complexity and therefore allows the control implementation to be more feasible.

In the last two decades, the decentralized stabilization of large-scale interconnected systems has received considerable attention [5, 22]. But in the above-mentioned studies the interconnected patterns are free of time delays. As we know, time delays are frequently encountered in various engineerings and can be a cause of instability [20]. There are instances where delays in the interconnections for many physical systems must be included in the model to account for transmission or information delays. However, few results on the decentralized stabilization of large-scale interconnected systems with delays in the interconnection patterns have been reported in the literature. In [23], the control problem of interconnected time-delay systems is considered and the controller is designed based on the knowledge of the uncertainties bounds. Further elaboration and extensions are found in [23, 24, 26]. The problem of decentralized adaptive control for nonlinear interconnected uncertain systems with time-delays is important and challenging in both theory and practice.

In this section, we will consider a class of time-varying nonlinear large-scale systems subject to multiple time-varying delays in the interconnections. The interconnections satisfy the so-called matching condition, and the uncertainties are bounded by nonlinear functions that are partly known. Adaptive state feedback control strategy is proposed and controllers obtained are independent of time delays. Based on Lyapunov stability theorem, it has been shown that the proposed controllers can render the closed-loop system globally uniformly ultimately bounded stable. The result is also applied to stabilize a class of interconnected time delay systems whose nominal systems are linear. Finally, several examples are included to illustrate the theoretic results developed in the sequel.

8.2 System Formulation

Consider the nonlinear time-varying composite system S with multiple delays in interconnections defined by N interconnected subsystems S_i, $i = 1, 2, \ldots, N$:

$$\begin{aligned} S_i\colon\quad \dot{x}_i &= f_i(x_i, t) + g_i(x_i, t)ui \\ &\quad + \sum_{j=1}^{N} H_{ij}(x_i, x_j, x_i[t - d_{ij}(t)], x_j[t - d_{ij}(t)], t), \end{aligned} \tag{8.1}$$

where $x_i \in \Re^{n_i}$ and $u_i \in \Re^{m_i}$ represent the state and control vectors, respectively, of the subsystem S_i, $f_i(x_i, t)$ and $g_i(x_i, t)$ are assumed to be known functions with appropriate dimensions, $H_{ij}(x_i, x_j, x_i[t - d_{ij}(t)], x_j[t - d_{ij}(t)], t)$ is an uncertain nonlinear interconnection, which indicates the interconnections among the current states and the delayed states of system S_i and S_j, while $d_{ij}(t)$ is the bounded time-varying delay and differentiable satisfying

$$0 \le d_{ij}(t) \le \bar{d}_{ij} \le \infty, \quad \dot{d}_{ij}(t) \le d_{ij}^* < 1,$$

where $\bar{d}_{ij}$, d_{ij}^* are positive scalars, and the initial condition is given as follows

$$x_i(t) = \Omega_i(t), \quad t \in [t_0 - \bar{d}_{ij}, t_0],\ i = 1, 2, \ldots, N, \tag{8.2}$$

where $\Omega_i(t)$ is a continuous initial function. The following assumptions are imposed on system (8.1).

Assumption 8.1 Interconnection $H_{ij}(x_i, x_j, x_i[t - d_{ij}(t)], x_j[t - d_{ij}(t)], t)$ satisfies the so-called matching condition, that is,

$$\begin{aligned} &H_{ij}(x_i, x_j, x_i[t - d_{ij}(t)], x_j[t - d_{ij}(t)], t) \\ &\quad = g_i(x_i, t)\tilde{H}_{ij}(x_i, x_j, x_i[t - d_{ij}(t)], x_j[t - d_{ij}(t)], t), \end{aligned}$$

where uncertain part $\tilde{H}_{ij}(x_i, x_j, x_i[t - d_{ij}(t)], x_j[t - d_{ij}(t)], t)$ is bounded by

$$\begin{aligned} &\left\| \sum_{j=1}^{N} \tilde{H}_{ij}(x_i, x_j, x_i[t - d_i j(t)], x_j[t - d_{ij}(t)], t) \right\| \\ &\quad \le \sum_{j=1}^{N} \sum_{s=1}^{p_{ij}} \alpha_{ijs} U_{ijs}(x_j) + \sum_{j=1}^{N} \sum_{l=1}^{q_{ij}} \beta_{ijl} W_{ijl}(x_j[t - d_{ij}(t)]) \\ &\quad = \sum_{j=1}^{N} \alpha_{ij}^t U_{ij}(x_j) + \sum_{j=1}^{N} \beta_{ij}^t W_{ij}(x_j[t - d_{ij}(t)]) \end{aligned} \tag{8.3}$$

in which functions $U_{ijs}(\cdot)$ and $W_{ijl}(\cdot)$ are known, p_{ij} and q_{ij} are proper known scalars, α_{ijs} and β_{ijl} are unknown scalars, and

$$\begin{aligned} \alpha_{ij} &= (\alpha_{ij1}, \alpha_{ij2}, \ldots, \alpha_{ijp_{ij}})^t, \quad U_{ij}(\cdot) = [U_{ij1}(\cdot), U_{ij2}(\cdot), \ldots, U_{ijp_{ij}}(\cdot)]^t, \\ \beta_{ij} &= (\beta_{ij1}, \beta_{ij2}, \ldots, \beta_{ijq_{ij}})^t, \quad W_{ij}(\cdot) = [W_{ij1}(\cdot), W_{ij2}(\cdot), \ldots, W_{ijq_{ij}}(\cdot)]^t. \end{aligned} \tag{8.4}$$

Remark 8.1 The scalars p_{ij}, q_{ij} and functions $U_{ijs}(\cdot)$, $W_{ijl}(\cdot)$ are chosen according to the structure of functions $\tilde{H}_{ij}$. In the existing literature investigating the decentralized control problem of an interconnected system, matching conditions were often assumed and many practical systems satisfy Assumption 8.1. In the matching parts condition (8.3) is imposed, in which the interconnections $U_{ij}(\cdot)$ and $W_{ij}(\cdot)$ can be nonlinear functions. In the existing papers investigating the time delay interconnected systems, the interconnections are often assumed to be bounded by a known or unknown first-order polynomial. Therefore our assumption on the interconnections is less conservative.

Assumption 8.2 There exist continuous function $k_i(x_i)$, positive function $V_i(x_i, t)$ and functions γ_{i1}, γ_{i2}, and γ_{i3} of class κ (zero at zero, positive, and increasing) such that for all x_i and t, the following inequalities hold

$$\begin{aligned} &(1) \quad \gamma_{i1}(\|x_i\|) \le V_i(x_i, t)\gamma_{i2}(\|x_i\|), \\ &(2) \quad \frac{\partial V_i(x_i,t)}{\partial t} + \frac{\partial V_i(x_i,t)}{\partial x_i}[f_i(x_i,t) - g_i(x_i,t)k_i(x_i)] \le -\gamma_{i3}(\|x_i\|). \end{aligned} \tag{8.5}$$

Remark 8.2 Assumption 8.2 guarantees that the nominal subsystems of system (8.1) are stabilizable using state feedback. If the nominal subsystems are exponentially stabilizable using state feedback, Assumption 8.2 is also satisfied.

For the purpose of illustration, we introduce an example of system (8.1) as follows

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1^3 e^{4|x_1|} + u_1 + \delta_{11}x_1^2 + \delta_{12}x_1 + \delta_{13}x_1x_2 + \delta_{14}x_2x_1e^{|x_1|}, \\ \dot{x}_2 &= -x_2 + u_2 + \delta_{21}x_1^2 + \delta_{22}x_2(t-2) + \delta_{23}x_1x_2 + \delta_{24}x_2x_1(t-0.5), \end{aligned}$$

where δ_{1j} and δ_{2j} $(j = 1, 2, 3, 4)$ are bounded parameters that may be time varying, and the bounds are not known. We will show how to obtain p_{ij}, q_{ij} and U_{ijs}, W_{ijl}. For this system the interconnections satisfy the matching condition, and

$$\begin{aligned} &\sum_{j=1}^{2} \tilde{H}_{1j}(x_i, x_j, x_i[t - d_{ij}(t)], x_j[t - d_{ij}(t)], t) \\ &\quad = \delta_{11}x_1^2 + \delta_{12}x_1 + \delta_{13}x_1x_2 + \delta_{14}x_2x_1e^{|x_1|} \\ &\quad \le |\delta_{11}|x_1^2 + |\delta_{12}||x_1| + |\delta_{131}|x_1^2 + |\delta_{132}|x_2^2 + |\delta_{141}|x_2^2 + |\delta_{142}|x_1^2e^{2|x_1|} \\ &\quad \le \alpha_{111}|x_1| + \alpha_{112}x_1^2 + \alpha_{113}x_1^2e^{2|x_1|} + \alpha_{121}x_2^2, \end{aligned}$$

where $\alpha_{111}, \alpha_{112}, \alpha_{113}$ and α_{121} are unknown positive scalars. From Assumption 8.4 we can see $p_{11} = 3$, $p_{12} = 1$, $q_{11} = q_{12} = 0$, and select

$$\begin{aligned} &U_{11}(\cdot) = [|x_1| \; x_1^2 \; x_1^2e^{2|x_1|}], \quad U_{12}(\cdot) = x_2^2, \\ &W_{11}(\cdot) = 0, \quad W_{12}(\cdot) = 0. \end{aligned}$$

In the same way the following bounds can also be obtained as

$$\begin{aligned} &U_{21}(\cdot) = x_1^2, \quad U_{22}(\cdot) = x_2^2, \\ &W_{21}(\cdot) = x_1^2(t-0.5), \quad W_{22}(\cdot) = |x_2(t-2)|. \end{aligned}$$

We see that the example satisfies Assumption 8.1. Assumption 8.2 also holds since the two nominal subsystems are completely controllable and can be easily stabilized with Lyapunov function $V_1 = x_1^2$ and $V_2 = x_2^2$. In what follows a decentralized adaptive feedback controller will be constructed to stabilize this class of interconnected systems.

8.2.1 Decentralized Feedback Control for Nonlinear Systems

In this section, we will propose a decentralized state feedback controller that can render the closed-loop system stable in the sense of uniform ultimate boundedness.

Theorem 8.1 *For system* (8.1) *satisfying Assumptions* 8.1 *and* 8.2, *if the following inequality*

$$\sum_{i=1}^{N}\left\{-\gamma_{i3}(\|x_i\|)+\sum_{j=1}^{N}\epsilon_{ij}\|U_{ij}(x_j)\|^2+\sum_{j=1}^{N}\nu_{ij}\|W_{ij}(x_j)\|^2\right\}\le-\gamma(\|x\|) \tag{8.6}$$

is satisfied, where ϵ_{ij} *and* ν_{ij} *are positive scalars*, $\gamma(\cdot)$ *is a class* κ *function, then the feedback control law*

$$u_i=-k_i(x_i)-\theta_i g_i(x_i,t)^t\frac{\partial V_i(x_i,t)^t}{\partial x_i} \tag{8.7}$$

with adaptive law

$$\dot{\theta}_i=\frac{1}{2}\Gamma_i\left\|g_i(x_i,t)^t\frac{\partial V_i(x_i,t)^t}{\partial x_i}\right\|^2-\Gamma_i\eta_i\theta_i, \tag{8.8}$$

where $V_i(x_i,t)$ *satisfies Assumption* 8.5, Γ_i *and* η_i *are positive scalars*, *will render the closed-loop system uniformly ultimately bounded stable.*

Proof Define a Lyapunov function candidate for the closed-loop system as follows

$$\begin{aligned}\tilde{V}(x,\theta,t)&=\sum_{i=1}^{N}\bar{V}_i(x,\theta,t)\\&=\sum_{i=1}^{N}\left[V_i(x_i,t)+\sum_{j=1}^{N}\nu_{ij}\int_{t-d_{ij}(t)}^{t}\|W_{ij}[x_j(\xi)]\|^2d\xi+\Gamma_i^{-1}(\theta_i-\tilde{\theta}_i)^2\right].\end{aligned} \tag{8.9}$$

$\tilde{\theta}_i$ is defined as follows

$$\tilde{\theta}_i=\sum_{j=1}^{N}\frac{\|\alpha_{ij}\|^2}{4\epsilon_{ij}}+\sum_{j=1}^{N}\frac{\|\beta_{ij}\|^2}{4\nu_{ij}(1-d_{ij}^*)}, \tag{8.10}$$

where adaptive scalar θ_i is used to estimate $\tilde{\theta}_i$. Then by taking the derivative of $V(\cdot)$ along the trajectories of the closed-loop system, we obtain

$$\begin{aligned}\frac{d\tilde{V}_i(x,\theta,t)}{dt}&=\sum_{i=1}^{N}\frac{d\bar{V}(x,\theta,t)}{dt}\\&\le\sum_{i=1}^{N}\left\{\frac{\partial V_i(x_i,t)}{\partial t}+\frac{\partial V_i(x_i,t)}{\partial x_i}\dot{x}_i+\sum_{j=1}^{N}[\nu_{ij}\|W_{ij}[x_j(t)]\|^2\right.\\&\qquad\left.-\nu_{ij}(1-d_{ij}^*)\|W_{ij}(x_j[t-d_{ij}(t)])\|^2]+2\Gamma_i^{-1}(\theta_i-\tilde{\theta}_i)\frac{d\theta_i}{dt}\right\}.\end{aligned}$$

Applying (8.3), (8.5), and (8.7), we obtain

$$
\begin{aligned}
\frac{d\bar{V}(x,\theta,t)}{dt} \le \sum_{i=1}^{N}\Bigg\{ & \frac{\partial V_i(x_i,t)}{\partial t} + \frac{\partial V_i(x_i,t)}{\partial x_i}[f_i(x_i,t) - g_i(x_i,t)k_i(x_i)] \\
& - \theta_i \left\| g_i(x_i,t)^t \frac{\partial V_i(x_i,t)^t}{\partial x_i} \right\|^2 + 2\Gamma_i^{-1}(\theta_i - \tilde{\theta}_i)\frac{d\theta_i}{dt} \\
& + \sum_{j=1}^{N}[\nu_{ij}\|W_{ij}(x_j)(t)]\|^2 - \nu_{ij}(1-d_{ij}^*)\|W_{ij}(x_j[t-d_{ij}(t)])\|^2 \\
& + \sum_{j=1}^{N}\left\| g_i(x_i,t)^t \frac{\partial V_i(x_i,t)^t}{\partial x_i} \right\| \alpha_{ij}^t U_{ij}[x_j(t)] \\
& + \sum_{j=1}^{N}\left\| g_i(x_i,t)^t \frac{\partial V_i(x_i,t)^t}{\partial x_i} \right\| \beta_{ij}^t W_{ij}[x_j(t-d_{ij(t)})]\Bigg\} \\
\le \sum_{i=1}^{N}\Bigg\{ & -\gamma_{i3}(\|x_i\|) - \theta_i \left\| g_i(x_i,t)^t \frac{\partial V_i(x_i,t)^t}{\partial x_i} \right\|^2 \\
& + 2\Gamma_i^{-1}(\theta_i - \tilde{\theta}_i)\frac{d\theta_i}{dt} \\
& + \sum_{j=1}^{N}[\nu_{ij}\|W_{ij}[x_j(t)]\|^2 - \nu_{ij}(1-d_{ij}^*)\|W_{ij}(x_j[t-d_{ij}(t)])\|^2] \\
& + \sum_{j=1}^{N}\left\| g_i(x_i,t)^t \frac{\partial V_i(x_i,t)^t}{\partial x_i} \right\| \alpha_{ij}^t U_{ij}[x_j(t)] \\
& + \sum_{j=1}^{N}\left\| g_i(x_i,t)^t \frac{\partial V_i(x_i,t)^t}{\partial x_i} \right\| \beta_{ij}^t W_{ij}[x_j(t-d_{ij(t)})]\Bigg\}.
\end{aligned} \tag{8.11}
$$

Since

$$
\begin{aligned}
& \sum_{i=1}^{N}\sum_{j=1}^{N}\left\| g_i(x_i,t)^t \frac{\partial V_i(x_i,t)^t}{\partial x_i} \right\| \alpha_{ij}^t U_{ij}[x_j(t)] \\
& \quad \le \sum_{i=1}^{N}\sum_{j=1}^{N}\epsilon_{ij}\|U_{ij}[x_j(t)]\|^2 + \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\|\alpha_{ij}\|^2}{4\epsilon_{ij}}\left\| g_i(x_i,t)^t \frac{\partial V_i(x_i,t)^t}{\partial x_i} \right\|^2
\end{aligned}
$$

and

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\left\|g_i(x_i,t)^t\frac{\partial V(x_i,t)^t}{\partial x_i}\right\|\beta_{ij}^t W_{ij}(x_j[t-d_{ij}(t)])$$
$$\le \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\|\beta_{ij}\|^2}{4\nu_{ij}(1-d_{ij}^*)}\left\|g_i(x_i,t)^t\frac{\partial V(x_i,t)^t}{\partial x_i}\right\|^2$$
$$+\sum_{i=1}^{N}\sum_{j=1}^{N}\nu_{ij}(1-d_{ij}^*)\|W_{ij}(x_j[t-d_{ij}(t)])\|^2$$

so we further obtain

$$\frac{d\bar{V}(x,\theta,t)}{dt}\le\sum_{i=1}^{N}\Bigg\{-\gamma_{i3}(\|x_i\|)+2\Gamma_i^{-1}(\theta_i-\tilde{\theta}_i)\frac{d\theta_i}{dt}$$
$$+\sum_{i=1}^{N}\epsilon_{ij}\|U_{ij}[x_j(t)]\|^2+\sum_{i=1}^{N}\nu_{ij}\|W_{ij}[x_j(t)]\|^2$$
$$+\sum_{j=1}^{N}\frac{\|\alpha_{ij}\|^2}{4\epsilon_{ij}}\left\|g_i(x_i,t)^t\frac{\partial V_i(x_i,t)^t}{\partial x_i}\right\|^2$$
$$+\sum_{j=1}^{N}\frac{\|\beta_{ij}\|^2}{4\nu_{ij}(1-d_{ij}^*)}\left\|g_i(x_i,t)^t\frac{\partial V_i(x_i,t)^t}{\partial x_i}\right\|^2$$
$$-\theta_i\left\|g_i(x_i,t)^t\frac{\partial V_i(x_i,t)^t}{\partial x_i}\right\|^2\Bigg\}. \tag{8.12}$$

By using (8.10), we can get

$$\frac{d\bar{V}(x,\theta,t)}{dt}\le\sum_{i=1}^{N}\Bigg\{-\gamma_{i3}(\|x_i\|)+\sum_{i=1}^{N}\epsilon_{ij}\|U_{ij}[x_j(t)]\|^2$$
$$+\sum_{i=1}^{N}\nu_{ij}\|W_{ij}[x_j(t)]\|^2+2\Gamma_i^{-1}(\theta_i-\tilde{\theta}_i)\frac{d\theta_i}{dt}$$
$$+\tilde{\theta}\left\|g_i(x_i,t)^t\frac{\partial V_i(x_i,t)^t}{\partial x_i}\right\|^2-\theta_i\left\|g_i(x_i,t)^t\frac{\partial V_i(x_i,t)^t}{\partial x_i}\right\|^2\Bigg\}.$$

Applying (8.6) and (8.8), one obtains

$$\frac{d\bar{V}(x,\theta,t)}{dt}\le-\gamma(\|x\|)-\sum_{i=1}^{N}2\eta_i(\theta_i-\tilde{\theta}_i)\theta_i$$
$$\le-\gamma(\|x\|)-\sum_{i=1}^{N}\{2\eta_i\theta_i^2-2\eta_i|\theta_i||\tilde{\theta}_i|\}-\gamma(\|x\|)$$
$$-\sum_{i=1}^{N}\eta_i\theta_i^2+\sum_{i=1}^{N}\eta_i\tilde{\theta}_i^2. \tag{8.13}$$

From (8.10), we know that $\tilde{\theta}_i$ is bounded, so the closed-loop system is uniformly ultimately bounded stable based on Lyapunov stability theory. □

Remark 8.3 From (8.13), we know that one can obtain the upper bound on the steady state as small as desired by decreasing the value of η_i. So the system designers can turn the size of the residual set by adjusting properly parameter η_i. To obtain good transient performance, we should choose the function $k_i(x_i)$ properly. In practical systems, the function $k_i(x_i)$ should be selected to render function $\gamma_{i3}(x_i)$ positive enough, so that function $\gamma(\|x\|)$ is sufficiently positive. The good transient performance will be obtained based on (8.13).

Remark 8.4 In Theorem 8.1, the key problem is how to get positive function $V_i(x_i,t)$ and $k_i(x_i)$ to obtain $-\gamma_{i3}(x_i)$ satisfying inequality (8.6). We should confirm the class of $-\gamma_{i3}(x_i)$ according to the given functions $U_{ij}(\cdot)$ and $W_{ij}(\cdot)$ first, then select proper $V_i(x_i,t)$ and $k_i(x)$ to satisfy inequality (8.5). Particularly, if $U_{ij}(x_j)=U_j(x_j)$ and $W_{ij}(x_j)=W_j(x_j)$ for all $i\in[1,N]$, we can select $\epsilon_j=\epsilon_{ij}$, $\nu_j=\nu_{ij}$, then if the following inequality

$$-\gamma_{i3}(\|x_i\|)+N\epsilon_i\|U_i(x_i)\|^2+N\nu_i\|W_i(x_i)\|^2<0$$

is satisfied, inequality (8.6) will hold.

8.3 Application to Decentralized Control

Let us consider the following class of interconnected systems with time delays

$$S_i:\quad \dot{x}_i=A_i(t)x_i+B_i(t)u_i(t)+B_i(t)\sum_{j=1}^{N}\tilde{H}_{ij}(x_j,x_j[t-d_{ij}(t)],t), \tag{8.14}$$

where $A_i(t)$ and $B_i(t)$ are linear time-varying matrices, while the interconnections satisfy the following inequalities

$$\begin{aligned}\left\|\sum_{j=1}^{N}\tilde{H}_{ij}(x_j,x_j[t-d_{ij}(t)],t)\right\| &\le \sum_{j=1}^{N}\sum_{s=1}^{p_{ij}}\alpha_{ijs}\|x_j\|^s+\sum_{j=1}^{N}\sum_{l=1}^{q_{ij}}\beta_{ijl}\|x_j[t-d_{ij}(t)]\|^l\\ &=\sum_{j=1}^{N}\alpha_{ij}^tU_{ij}(\|x_j\|)+\sum_{j=1}^{N}\beta_{ij}^tW_{ij}(\|x_j[t-d_{ij}(t)]\|)\end{aligned}$$

in which p_{ij} and q_{ij} are known scalars representing the highest order of $\|x_j\|$ and $\|x_j(t-d_{ij}(t))\|$, respectively, the parameters α_{ijs} and β_{ijl} are unknown scalars similar to those of system (8.1). Based on Theorem 8.1, we will propose decentralized feedback controllers for system (8.14) to render the closed-loop system stable in the sense of uniform ultimate boundedness.

Now, we introduce the following standard assumption:

Assumption 8.3 There exists a positive parameter matrix $\sigma_i(t)$ satisfying the following Riccati inequality holds

$$\dot{P}_i(t) + A_i(t)^t P_i(t) + P_i(t)A_i(t) - P_i(t)B_i(t)\sigma_i(t)B_i(t)^t P_i \leq -Q_i(t), \quad (8.15)$$

where $P_i(t)$ and $Q_i(t)$ are positive matrices satisfying

$$\lambda_{\min}[P_i(t)] > a_i, \qquad \lambda_{\min}[Q_i(t)] > a_i,$$

where a_i is a sufficiently small positive scalar.

Corollary 8.1 *When system (8.14) satisfies the above two inequalities, the following adaptive feedback controller will render the closed-loop system stable in the sense of uniform ultimate boundedness.*

$$u_i = -\frac{1}{2}\sigma_i(t)B_i(t)^t P_i(t)x_i(t) - \theta_i B_i^t \frac{\partial V_i(x_i)^t}{\partial x_i}, \quad (8.16)$$

where $\sigma_i(t)$ and $P_i(t)$ satisfy Assumption 8.3, $V_i(x_i) = \sum_{k=1}^{h_i}(1/k)(x_i^t P_i x_i)^k$, $h_i = \max\{p_{ji}, q_{ji}\}$ $(j \in [1..N])$, and θ_i is an adaptive parameter whose adaptive law is

$$\dot{\theta}_i = -\frac{1}{2}\Gamma_i \left\| B_i(t)^t \frac{\partial V_i(x_i)^t}{\partial x_i} \right\|^2 - \Gamma_i \eta_i \theta_i \quad (8.17)$$

in which Γ_i and η_i are positive scalars.

Proof Based on Theorem 8.1, define Lyapunov function for system (8.14) as follows

$$\begin{aligned}
\tilde{V}(x,\theta,t) &= \sum_{i=1}^{N} \bar{V}_i(x,\theta,t) \\
&= \sum_{i=1}^{N} \left\{ \sum_{k=1}^{h_i} \frac{1}{k}(x_i^t P_i x_i)^k \right. \\
&\quad \left. + \sum_{i=1}^{N}\sum_{k=1}^{q_{ij}} \nu_{ij} \int_{t-d_i j(t)}^{t} \|x_j(\xi)\|^{2k} d\xi + \Gamma_i^{-1}(\theta_i - \tilde{\theta}_i)^2 \right\},
\end{aligned}$$

where

$$\tilde{\theta}_i = \sum_{i=1}^{N} \left(\frac{\|\alpha_{ij}\|^2}{4\epsilon_{ij}} + \frac{\|\beta_{ij}\|^2}{4(1-d_{ij}^*)\nu_{ij}} \right).$$

Then by taking the derivative of $\tilde{V}(\cdot)$ along the trajectories of the closed-loop system, similar to the proof of Theorem 2.1, we have

$$\begin{aligned}
\dot{\tilde{V}}(x,\theta,t) &= \sum_{i=1}^{N} \bar{V}_i(x,\theta,t) \\
&\leq \sum_{i=1}^{N} \left[\sum_{k=1}^{h_i} (x_i^t P_i x_i)^{k-1} [-x_i^t Q_i x_i - (\theta_i - \tilde{\theta}_i)\|2B_i^t P_i x_i\|^2] \right.
\end{aligned}$$

$$+\sum_{i=1}^{N}\left(\sum_{k=1}^{q_{ij}}\nu_{ij}\|x_j\|^{2k}+\sum_{k=1}^{p_{ij}}\epsilon_{ij}\|x_j\|^{2k}\right)$$
$$\left.+2\Gamma_i^{-1}(\theta_i-\tilde{\theta}_i)\dot{\theta}_i\right].$$

By applying (8.17), we can obtain

$$\dot{\tilde{V}}(x,\theta,t)\le\sum_{i=1}^{N}\left[-\sum_{k=1}^{h_i}(x_i^tP_ix_i)^{k-1}(x_i^tQ_ix_i)\right.$$
$$\left.+\sum_{i=1}^{N}\left(\sum_{k=1}^{q_{ij}}\nu_{ij}\|x_j\|^{2k}+\sum_{k=1}^{p_{ij}}\epsilon_{ij}\|x_j\|^{2k}\right)-\eta_i\theta_i^2+\eta_i\|\tilde{\theta}_i\|^2\right].$$

On selecting the parameters

$$\nu_j=\max\nu_{ij},\quad \epsilon_j=\max\epsilon_{ij},\quad \text{for } i\in[1..N]$$

the following inequality holds

$$\sum_{i=1}^{N}\left\{-\sum_{k=1}^{h_i}(x_i^tP_ix_i)^{k-1}(x_i^tQ_ix_i)+\sum_{j=1}^{N}\left(\sum_{k=1}^{q_{ij}}\nu_{ij}\|x_j\|^{2k}+\sum_{k=1}^{p_{ij}}\epsilon_{ij}\|x_j\|^{2k}\right)\right\}$$
$$\le\sum_{i=1}^{N}\left\{-\sum_{k=1}^{h_i}(x_i^tP_ix_i)^{k-1}(x_i^tQ_ix_i)+\sum_{j=1}^{N}\left(\sum_{k=1}^{h_j}\nu_j\|x_j\|^{2k}+\sum_{k=1}^{h_j}\epsilon_j\|x_j\|^{2k}\right)\right\}$$
$$=\sum_{i=1}^{N}\sum_{k=1}^{h_i}\left[-(x_i^tP_ix_i)^{k-1}(x_i^tQ_ix_i)+N\nu_i\|x_i\|^{2k}+N\epsilon_i\|x_i\|^{2k}\right].$$

Then we have that

$$\dot{\tilde{V}}(x,\theta,t)\le\sum_{i=1}^{N}\sum_{k=1}^{h_i}\{-\lambda_{\min}(P_i)^{k-1}\lambda_{\min}(Q_i)\|x_i\|^{2k}+N\nu_i\|x_i\|^{2k}$$
$$+N\epsilon_i\|x_i\|^{2k}\}-\sum_{i=1}^{N}(\eta_i\theta_i^2-\eta_i\|\tilde{\theta}_i\|^2).$$

Based on Assumption 8.3, there exist ν_j and ϵ_j small enough to render the following inequality satisfied

$$-\lambda_{\min}^{k-1}(P_i)\lambda_{\min}(Q_i)+N\nu_i+N\epsilon_i\le-a_i^k+N\nu_i+N\epsilon_i=-\Pi_i<0,$$

where the scalar $\Pi_i>0$. Further we can obtain

$$\dot{\tilde{V}}(x,\theta,t)\le-\sum_{i=1}^{N}h_i\Pi_i\|x_i\|^{2k}-\sum_{i=1}^{N}(\eta_i\theta_i^2-\eta_i\|\tilde{\theta}_i\|^2).$$

In view of the Lyapunov stability theorem, the proposed feedback controller (8.39) with adaptive law (8.40) can render the closed-loop system uniformly ultimately bounded stable. □

For the case of interconnected systems, we have the following corollary.

Corollary 8.2 *For system (8.14) with A_i and B_i being a constant matrix, the adaptive feedback controller*

$$u_i = -\frac{1}{2}\sigma_i B_i^t P_i x_i - \theta_i B_i^t \frac{\partial V(x_i)^t}{\partial x_i} \tag{8.18}$$

with adaptive law

$$\dot{\theta}_i = \frac{1}{2}\Gamma_i \left\| B_i^t \frac{\partial V(x_i)^t}{\partial x_i} \right\|^2 - \Gamma_i \eta_i \theta_i \tag{8.19}$$

will render the closed-loop system uniformly ultimately bounded stable. In (8.41) and (8.42), $\Gamma_i > 0$ and $\eta_i > 0$ and

$$V(x_i) = \sum_{k=1}^{h_i} \frac{1}{k}(x_i^t P_i x_i)^k, \quad h_i = \max\{p_{ji}, q_{ji}\}\ (j \in [1..N]).$$

σ_i and P_i are the positive scalar and the positive matrix, respectively, satisfying the following inequality

$$A_i^t P_i + P_i A_i - \sigma_i P_i B_i B_i^t P_i = -Q_i < 0. \tag{8.20}$$

It is known that if (8.20) holds there always exists scalar a_i satisfying $\lambda_{\min}(P_i) > a_i$ and $\lambda_{\min}(Q_i) > a_i$, which means Assumption 8.3 is satisfied. The proof is quite similar to that of Corollary 8.1.

Remark 8.5 In the existing literature, system (8.14) were considered based on Riccati inequalities and linear matrix inequalities with the interconnections known or bounded by a known linear function. However, in the present set-up, the interconnections may be bounded by high-order polynomial. Furthermore, we adopt adaptive method, and do not have to know the bounds.

In what follows, we will present two examples to demonstrate the validity of the foregoing results.

8.3.1 Simulation Example 8.1

Consider the following nonlinear interconnected system with time delays

$$\begin{aligned}
\dot{x}_1 &= \begin{pmatrix} \dot{x}_{11} \\ \dot{x}_{12} \end{pmatrix} = \begin{pmatrix} -x_{11} \\ 2x_1^3 2 \end{pmatrix} + \begin{pmatrix} 0 \\ x_{12} \end{pmatrix} u_1 \\
&\quad + \begin{pmatrix} 0 \\ \delta_{11} x_1^2 2 x_{21} + \delta_{12} x_1 2^2 |x_{22}|^{1/2} \\ +\delta_{13} x_{12} |x_{11}(t - 0.5|\sin(t)|)|^{1/2} x_{21}(t - 0.25|\sin(t)|) \end{pmatrix}, \\
\dot{x}_2 &= \begin{pmatrix} \dot{x}_{21} \\ \dot{x}_{22} \end{pmatrix} = \begin{pmatrix} -x_{21} \\ \dot{x}_{22}^2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_2 \\
&\quad + \begin{pmatrix} 0 \\ \delta_{21} |x_{11}|^{1/2} x_{21} + \delta_{22} x_{12} |x_{22}|^{1/2} + \delta_{23} x_{12}(t-2) |x_{22}(t-1)|^{1/2} \end{pmatrix},
\end{aligned} \tag{8.21}$$

where δ_{ij} ($i = 1, 2$; $j = 1, 2, 3$) are bounded unknown parameters. The interconnections satisfy matching condition with the following bounds

$$\begin{aligned}
&U_{11}(x_1) = x_{12}^2, \quad W_{11}(x_1[t - d_{11}(t)]) = |x_{11}[t - 0.5|\sin(t)|]|, \\
&U_{12}(x_2) = [x_{21}^2 \; |x_{22}|]^t, \quad W_{12}(x_1[t - d_{12}(t)]) = x_{21}^2[t - 0.25|\sin(t)|]. \\
&U_{21}(x_1) = [|x_{11}| \; x_{12}^2]^t, \quad W_{21}(x_1[t - d_{21}(t)]) = x_{12}^2(t - 2), \\
&U_{22}(x_2) = [x_{21}^2 \; |x_{22}|]^t, \quad W_{22}(x_2[t - d_{22}(t)]) = x_{22}(t - 1).
\end{aligned}$$

Taking

$$\begin{aligned}
&V_1(x_1, t) = x_{11}^2 + x_{12}^2, \quad V_2(x_2, t) = x_{21}^4 + x_{22}^2, \quad k_1(x_1) = 3x_{12}^2, \\
&k_2(x_2) = x_{22}^2 + 2x_{22},
\end{aligned} \tag{8.22}$$

it follows that

$$\begin{aligned}
&\frac{\partial V_1^t(x_1)}{\partial t} + \frac{\partial V_1^t(x_1)}{\partial x_1}[f_1(x_1, t) - g_1(x_1, t)k_1(x_1)] \\
&= -2x_{11}^2 + 4x_{12}^4 - 6x_{12}^4 = -2x_{11}^2 + 4x_{12}^4 \\
&= \gamma_{13}(x_1)\frac{\partial V_2(x_2)^t}{\partial t} + \frac{\partial V_2(x_2)^t}{\partial x_2}[f_2(x_2, t) - g_1(x_2, t)k_2(x_2)] \\
&= -4x_{21}^2 + 2x_{22}^3 - 2x_{22}^3 - 4x_{22}^2 = -4x_{21}^4 - 4x_{22}^2 = \gamma_{23}(x_i)
\end{aligned}$$

is satisfied. For inequality (8.6), we have

$$\begin{aligned}
&\sum_{i=1}^{N}\left\{-\gamma_{i3}(\|x_i\|) + \sum_{j=1}^{N}\epsilon_{ij}\|U_{ij}(x_j)\|^2 + \sum_{j=1}^{N}\nu_{ij}\|W_{ij}(x_j)\|^2\right\} \\
&= \sum_{i=1}^{N} -2x_{11}^2 - 4x_{12}^4 - 4x_{21}^4 - 4x_{22}^2 + \epsilon_{11}x_{12}^4 \\
&\quad + \epsilon_{12}(x_{21}^4 + x_{22}^2) + \epsilon_{21}(x_{11}^2 + x_{12}^4) + \epsilon_{22}(x_{21}^4 + x_{22}^2) + \nu_{11}x_{11}^2 \\
&\quad + \nu_{12}x_{21}^4 + \nu_{21}x_{12}^4 + \nu_{22}x_{22}^4.
\end{aligned}$$

On selecting $\epsilon_{ij} = \nu_{ij} = 0.1$ ($i = 1, 2$, $j = 1, 2$), it is easy to see that (8.6) is satisfied.

Based on Theorem 8.1, the feedback controllers are found to be

$$\begin{aligned}
u_1 &= -3x_{12}^2 - 2\theta_1 x_{12}^2, \\
u_2 &= -x_{22}^2 - 2x_{22} - 2\theta_2 x_{22}
\end{aligned}$$

with the adaptive law

$$\begin{aligned}
\dot{\theta}_1 &= \Gamma_1\left\|g_1(x_1, t)\frac{\partial V_1^t(x_1, t)}{\partial x_1}\right\|^2 - \Gamma_1\eta_1\theta_1 = 2x_{12}^4 - 0.01\theta_1, \\
\dot{\theta}_2 &= \Gamma_2\left\|g_2(x_2, t)\frac{\partial V_2^t(x_2, t)}{\partial x_2}\right\|^2 - \Gamma_2\eta_2\theta_2 = 2x_{22}^2 - 0.01\theta_2.
\end{aligned}$$

For simulation we give the following initial conditions:

$$\theta_1(0) = 1, \quad \theta_2(0) = 1,$$
$$x_1(t) = [3\ 2]^t, \quad x_2(t) = [1\ -1]^t, \quad t \in [t_0 - 2, t_0].$$

The simulations are done via the Simulink toolbox in MATLAB 6.5. We use the fixed step size 0.01 and ode4 (Runge Kutta). When the unknown parameters $\delta_{ij} = 1$ ($i = 1, 2$; $j = 1, 2, 3$), the simulation result is shown in Fig. 8.1.

From the figure, we can see that the adaptive controllers render the closed-loop system uniformly ultimately bounded stable. Based on Theorem 2.1, the controllers are obtained without the knowledge of the bounds of the interconnections, which means that the bounds of interconnections can be arbitrary. Further, let us make simulations when the controller and initial conditions are the same, but $\delta_{ij} = 5$ and $\delta_{ij} = 10$. The state response trajectories are shown in Figs. 8.2 and 8.3, respectively. From the figures we can see the controllers render the corresponding system stable, which further shows that the proposed controllers are valid and the conclusions are feasible.

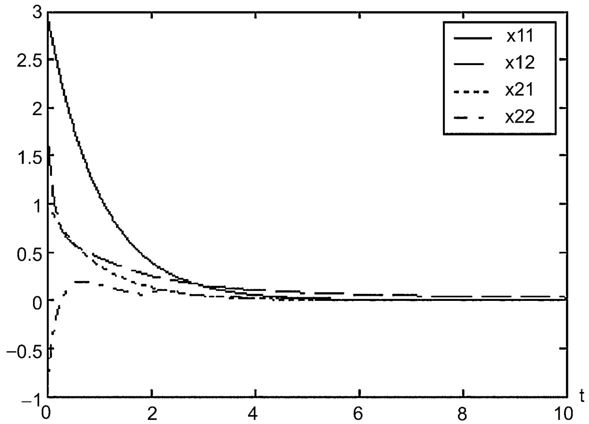


Fig. 8.1 State response curves of system (8.21) with $\delta_{ij} = 1$

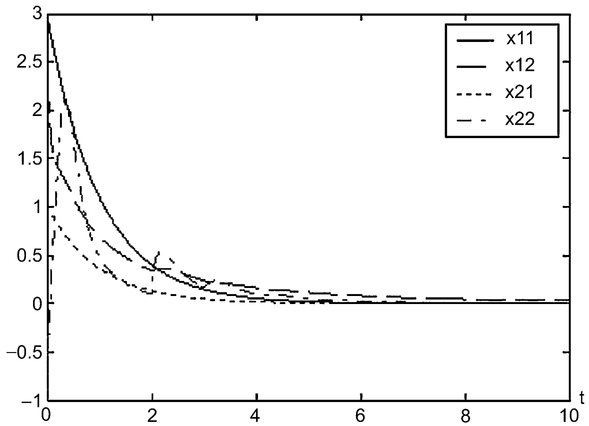


Fig. 8.2 The states response curves of system (8.21) with $\delta_{ij} = 5$

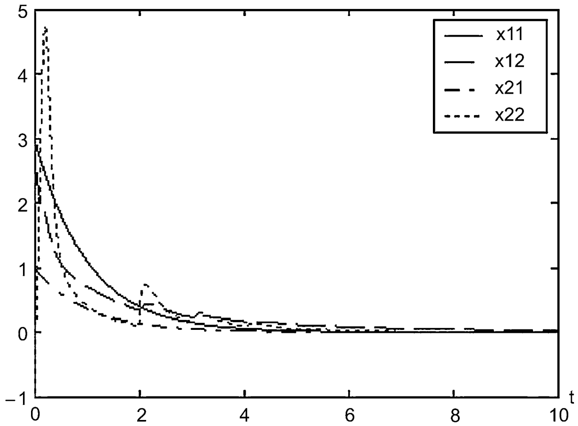


Fig. 8.3 The states response curves of system (8.21) with $\delta_{ij} = 10$

8.3.2 Simulation Example 8.2

Consider the following interconnected time delay system

$$\begin{aligned}\dot{x}_1 &= \begin{bmatrix} -1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ &\quad + \begin{bmatrix} 0 \\ \delta_{11}x_{11} + \delta_{12}x_{22}^2 + \delta_{13}x_{12}x_{21}(t-0.5) \\ +\delta_{14}x_{11}^2[t-0.25|\sin(t)|] \end{bmatrix}, \\ \dot{x}_2 &= \begin{bmatrix} -2 & 0 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ &\quad + \begin{bmatrix} 0 \\ \delta_{21}x_{21} + \delta_{22}x_{12}^2 \\ +\delta_{23}x_{12}x_{22}[t-0.5|\sin(t)|] + \delta_{24}x_{12}^2(t-1) \end{bmatrix},\end{aligned} \tag{8.23}$$

where δ_{ij} $(i = 1, 2;\ j = 1, 2, 3, 4)$ are unknown scalars. The interconnections satisfy matching conditions, and the bounds are given by

$$\begin{aligned} &U_{11}(x_1) = [|x_{11}|\ x_{12}^2]^t, \quad W_{11}(x_1[t-d_{11}(t)]) = x_{11}^2[t-0.25|\sin(t)|], \\ &U_{12}(x_2) = x_{22}^2, \quad W_{12}(x_1[t-d_{12}(t)]) = x_{21}^2(t-0.5), \\ &U_{21}(x_1) = x_{12}^2, \quad W_{21}(x_1[t-d_{21}(t)]) = x_{12}^2(t-1), \\ &U_{22}(x_2) = |x_{21}|, \quad W_{22}(x_2[t-d_{22}(t)]) = x_{22}[t-0.25|\sin(t)|]. \end{aligned}$$

For Assumption 8.3, if we select

$$Q_1 = \begin{bmatrix} 2 & -3 \\ -3 & 10 \end{bmatrix}, \quad \sigma_1 = 12I, \qquad Q_2 = \begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix}, \quad \sigma_2 = 14I$$

then it is easy to see that $P_1 = P_2 = I$ is a feasible solution. Based on Corollary 8.1 or Corollary 8.2, we have the following feedback controller

$$u_1 = -6x_{12}(t) - 2\theta_1[x_{12} + x_{12}(x_{11}^2 + x_{12}^2)],$$
$$u_2 = -7x_{22}(t) - 2\theta_2[x_{22} + x_{22}(x_{21}^2 + x_{22}^2)]$$

with adaptive law

$$\dot{\theta}_1 = 4[x_{12} + x_{12}(x_{11}^2 + x_{12}^2)]^2 - 0.1\theta_1,$$
$$\dot{\theta}_2 = 4[x_{22} + x_{22}(x_{21}^2 + x_{22}^2)]^2 - 0.1\theta_2$$

The initial conditions are selected as follows:

$$\theta_1(0) = 10, \quad \theta_2(0) = -5,$$
$$x_1(t) = [8\ 5]^t, \quad x_2(t) = [3\ 1]^t, \quad t \in [t_0 - h,\ t_0].$$

The simulation circumstance is the same of Example 8.1. When the unknown parameters $\delta_{ij} = 1, 5, 10$ ($i = 1, 2;\ j = 1, 2, 3, 4$); respectively, the simulation results are shown in Figs. 8.4, 8.5 and 8.6. From the ensuing results, we see that the

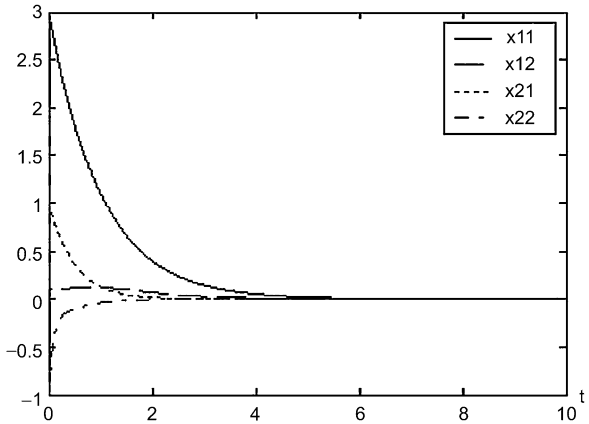


Fig. 8.4 The states response curves of system (8.23) with $\delta_{ij} = 1$

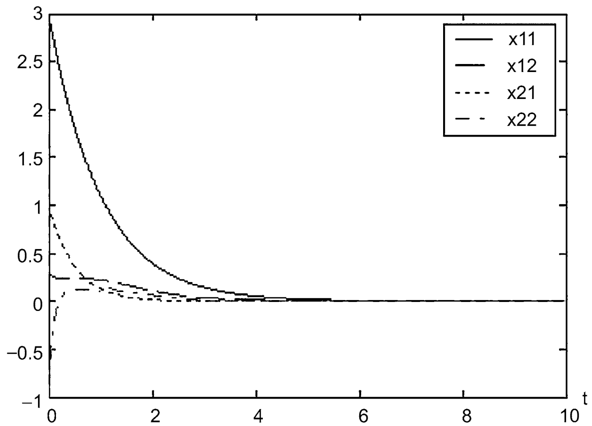


Fig. 8.5 The states response curves of system (8.23) with $\delta_{ij} = 5$

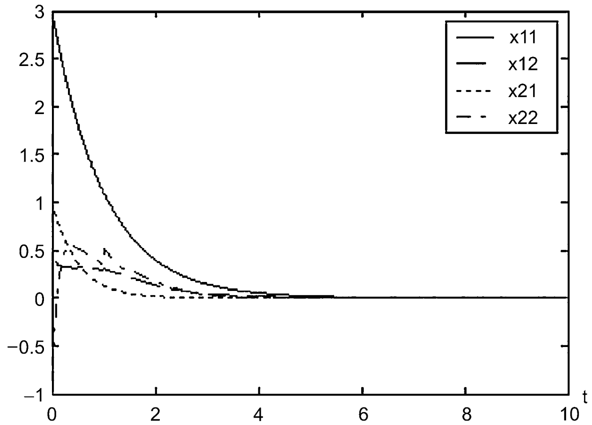


Fig. 8.6 The states response curves of system (8.23) with $\delta_{ij} = 10$

controllers render the closed-loop systems stable in the sense of uniform ultimate boundedness, which further verifies that Corollaries 8.1 and 8.2 are feasible.

8.4 Adaptive Techniques for Interconnected Nonlinear Systems

Many physical systems are composed of interconnections of lower-dimensional subsystems. Furthermore, information transfer among the subsystems may have a high cost associated with it, if it is not at all impossible. Much effort has been focused on the application of decentralized control for large-scale interconnected systems. A decentralized control structure naturally alleviates the computational burden associated with a centralized control scheme. Furthermore, the extension problem (i.e., inclusion of additional subsystems to the system) is more easily handled through decentralization. Earlier works on decentralized control were focused on control of large-scale linear systems. However, most physical systems are inherently nonlinear. For most practical applications, the linear control design is nicely applicable to linearized models of interconnected systems. However, this only guarantees stability in a region about the operating point and possibly degradation in performance and instability over a large domain of operation.

Most of the literature on decentralized and adaptive control of interconnected nonlinear systems [10, 15–19, 25, 27, 43] is focused on systems with first-order bounded interconnections. These results cannot guarantee stability when the interconnections between the subsystems are of higher orders [38]. In [6], decentralized adaptive controllers for robotic manipulators are designed under the assumption that the nonlinear interconnection terms due to Coriolis and centrifugal effects are slowly time varying. This assumption fails to hold, however, for high-speed trajectory tracking and could possibly result in instability. The first result on global decentralized stabilization of large-scale systems with higher-order interconnections is by Shi and Singh [38, 39]. The interconnections are assumed to be bounded by a pth order polynomial in states. Adaptation of control gains is utilized to relax the requirement of explicit knowledge of the polynomial.

A major structural restriction imposed on the system in all these schemes is that the uncertainties and interconnections are in the range space of the input matrix, which is basically the strict matching condition. Global stabilization for systems with mismatched uncertainties is not possible using these schemes. Recent advances in the area of centralized adaptive nonlinear control have resulted in the recursive design of control laws for systems which do not satisfy the strict matching requirement [20, 21]. Motivated by these advancements in the area of centralized nonlinear control, we seek to investigate the possible extension of the class of large-scale nonlinear systems for which decentralized control laws can be developed.

In this section, we extend the strict matching requirement to a class of large-scale nonlinear systems with parametric and nonparametric uncertainties. These are termed as systems of the *decentralized strict feedback form*. In Sect. 8.4.2, we characterize this class of large-scale systems. Geometric conditions are stated under which any interconnected large scale nonlinear system can be transformed into this form via a parameter independent decentralized diffeomorphism. The interconnections are assumed to be bounded by an unknown pth-order polynomial in states. In Sect. 8.4.3, an adaptive decentralized control is developed using a stepwise design procedure. The design is motivated by the decentralized design in [38, 39] and integrator backstepping along with the tuning function design for centralized control [20, 21]. The scheme is proven to guarantee global stabilization and regulation properties. The developed control is robust to perturbations in the system dynamics. Furthermore, no redesign is required for controllers of the original subsystems if additional subsystems are appended to the system. In Sect. 8.4.4, the scheme is extended to a model reference tracking problem, where uniform boundedness of the tracking error is guaranteed.

8.4.1 A Class of Interconnected Nonlinear Systems

We consider a large-scale nonlinear system comprised of N interconnected subsystems with the interconnections being linear in the unknown parameters. The ith subsystem is given as

$$\dot{\zeta}_i = f_{i0}(\zeta_i) + \sum_{j=1}^{p_i} \theta_{ij} f_{ij}(\zeta) + g_{i0}(\zeta_i)u_i, \quad 1 \le i \le N \tag{8.24}$$

where $\zeta_i \in \Re^{n_i}$ is the state vector for the ith subsystem, $\zeta \in \Re^{n_1+\cdots+n_N}$ is the state vector for the overall system, $u_i(t) \in \Re$ is the control input, and $\theta_i = [\theta_{i1}, \ldots, \theta_{ip_i}]^t$ is a vector of unknown constant parameters for the ith subsystem. The vector fields f_{i0}, f_{ij}, and g_{i0} are assumed to be smooth with $f_{i0}(0) = 0$, $f_{ij}(0) = 0$ and $g_{i0} \neq 0$.

We make the following assumption for the isolated subsystems.

Assumption 8.4 Assume that the isolated subsystems

$$\dot{\zeta}_i = f_{i0}(\zeta_i) + g_{i0}(\zeta_i)u_i \tag{8.25}$$

are globally input-to-state linearizable. Equivalently, there exists an output function $h_i(\zeta_i)$ such that the isolated subsystem has a relative degree n_i with respect to h_i. Sufficient conditions under which this assumption is satisfied are stated in [11].

In what follows, we define the notion of the *degree of mismatch for interconnected systems*.

Definition 8.1 Assume that the isolated subsystems are exactly externally feedback linearizable. Let κ_i denote the smallest integer such that:

1. $L^k_{f_{ij}} h_i(\zeta_i) \equiv 0$, $1 \le k \le \kappa_i - 1$, $\forall j \in [1..p_i]$;
2. $L^{k_i}_{f_{ij}} h_i(\zeta_i) \ne 0$, for at least one $j \in [1..p_i]$, where $f_{ij}(\zeta)$ are the vector fields in (8.24) for the ith subsystem corresponding to the interconnections. Then, the degree of mismatch (ρ_i) for the ith subsystem is defined as $\rho_i = n_i - \kappa_i$.

The decentralized scheme proposed in the sequel is applicable to interconnected nonlinear systems which are transformable using a global parameter independent, decentralized transformation $w_i = [w_{i1}, \dots, w_{i,n_i}]^t = \phi_i(\zeta_i)$ to the following decentralized strict feedback form:

$$
\begin{aligned}
\dot{w}_{i1} &= w_{i2}, \\
&\ \vdots \\
\dot{w}_{i,\kappa_i-1} &= w_{i,\kappa_i}, \\
\dot{w}_{i,\kappa_i} &= w_{i,\kappa_i+1} + \theta_i^t \gamma_{i0}(w_{j1}, \dots, w_{j,\kappa_j} | 1 \le j \le N), \\
\dot{w}_{i,\kappa_i+1} &= w_{i,\kappa_i+2} + \theta_i^t \gamma_{i1}(w_{j1}, \dots, w_{j,\kappa_j}, w_{i,\kappa_i+1} | 1 \le j \le N), \\
&\ \vdots \\
\dot{w}_{i,n_i-1} &= w_{i,n_i} + \theta_i^t \gamma_{i,n_i-\kappa_i-1}(w_{j1}, \dots, w_{j,\kappa_j}, w_{i,\kappa_i+1}, \dots, w_{i,n_i-1} | 1 \le j \le N), \\
\dot{w}_{i,n_i} &= v_i(w_i) + \theta_i^t \gamma_{i,n_i-\kappa_i}(w_{j1}, \dots, w_{j,\kappa_j}, w_{i,\kappa_i+1}, \dots, w_{i,n_i} | 1 \le j \le N) \\
&\quad + \delta_i(w_i) u_i, \quad 1 \le i \le N
\end{aligned}
\tag{8.26}
$$

with $v_i(0) = 0$; $\gamma_{ij}(0) = 0$, $\delta_i(w_i) \ne 0\ \forall w_i \in \Re^{n_i}$, $1 \le i \le N$, $0 \le j \le n_i - \kappa_i$.

Remark 8.6 A primary objective is decentralized control, it is desirable for the transformation for each subsystem to utilize the states local to that subsystem. Assuming that all the states of a subsystem are available for feedback to the controller corresponding to that subsystem, a decentralized control designed in the transformed coordinates still maintains its decentralized structure in the original coordinates. Thus, the terminology decentralized transformation is justified. The degree of mismatch defines the separation of the interconnections from the control. The mismatch between the interconnections (and hence uncertainties) and the control is larger for larger ρ_i, resulting in increased complexity in the control design.

Remark 8.7 The class of systems (8.26) illustrates a tradeoff between the degree of mismatch and the complexity of the interconnections. The larger the degree of mismatch ρ_i for the jth subsystem, the fewer are the number of states of that subsystem appearing in the interconnections.

Given that the isolated subsystem (8.25) is input-state linearizable, the degree of mismatch ρ_i of each subsystem stays the same irrespective of the choice of the feedback linearizing coordinate transformation for the isolated subsystem. This is made precise in the following theorem, the proof of which is relegated to Sect. 8.5.

Theorem 8.2 *The degree of mismatch for each subsystem of a large-scale interconnected system given by* (8.24) *is invariant with respect to the choice of the input-state linearizing transformation for the isolated system.*

To proceed further, the following distributions and codistributions for the i subsystem:

$$g^{ik} = \text{span}\{g_{i0}, ad_{f_{i0}} g_{i0}, \ldots, ad^k_{f_{i0}} g_{i0}\}, \quad 1 \le i \le N,\ 0 \le k \le n_i - 1, \tag{8.27}$$

$$\Omega^{ij} = \text{span}\{dL^{n_i-1}_{f_{i0}} h_i, \ldots, dL^{n_i-j}_{f_{i0}} h_i\}, \quad 1 \le j \le n_i, \tag{8.28}$$

$$W^{ij} = \{v \in \Re^{n_i} : \langle w_i, v\rangle = 0,\ \forall w_i \in \Omega^{ij}\}, \tag{8.29}$$

where $h_i(\zeta_i)$is the output function for which (8.25) has relative degree n_i.

The necessary and sufficient conditions for the existence of a decentralized diffeomorphism transforming (8.24) into (8.26) are now stated in the following proposition.

Proposition 8.1 *Under Assumption* 8.4, *a global decentralized diffeomorphism* $w_i = \phi_i(\zeta_i)$; $1 \le i \le N$, *transforming* (8.24) *into* (8.26), *exists if and only if the following two conditions hold globally.*

1. $[X,\ f_{il}] \in g^{ij}$, $\forall X \in g^{ij}$, $1 \le l \le p_i$, $0 \le j \le \rho_i$, *where* ρ_i *is the degree of mismatch for the* i*th subsystem.*
2. $[Y,\ f_{i\rho_i}] \in g^{i\rho_i}$, $\forall Y \in W^{k\rho_k}$, $1 \le l \le p_i$, $1 \le i, k \le N$.

A proof of this proposition is given in Sect. 8.5.

8.4.2 Decentralized Adaptive Design

In what follows, the decentralized control is designed for system (8.26), which obtained from (8.24) through a decentralized transformation. For notational simplicity, we assume uniform degree of mismatch $(n - \kappa)$ and same number of states (n) for each subsystem. Denoting the first $\kappa = n - \rho$ states of the ith subsystem by y_i, $1 \le i \le N$ and the remaining ρ states by x_i, the transformed system is rewritten in terms of $(y_i^t,\ x_i^t)^t = w_i$ as

$$
\begin{aligned}
\dot{y}_i &= \begin{bmatrix} \dot{y}_{i1} \\ \vdots \\ \dot{y}_{i,\kappa-1} \\ \dot{y}_{i\kappa} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ 0 & 0 & & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_{i1} \\ \vdots \\ \dot{y}_{i,\kappa-1} \\ \dot{y}_{i\kappa} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \{x_{i1} + \theta_i^t \gamma_{i0}(y_1, \dots, y_N)\} \\
&= A_i y_i + B_i [x_{i1} + \theta_i^t \gamma_{i0}(y_1, \dots, y_N)], \qquad (8.30)
\end{aligned}
$$

$$
\begin{aligned}
\dot{x}_{i1} &= x_{i2} + \theta_i^t \gamma_{i1}(y_1, \dots, y_N, x_{i1}), \\
\dot{x}_{i2} &= x_{i3} + \theta_i^t \gamma_{i2}(y_1, \dots, y_N, x_{i1}, x_{i2}), \\
&\ \vdots \\
\dot{x}_{i,n-\kappa-1} &= \dot{x}_{i,n-\kappa} + \theta_i^t \gamma_{i,n-\kappa-1}(y_1, \dots, y_N, x_{i1}, \dots, x_{i,n-\kappa-1}), \\
\dot{x}_{i,n-\kappa} &= v_i(y_i, x_i) + \theta_i^t \gamma_{i,n-\kappa}(y_1, \dots, y_N, x_{i1}, \dots, x_{i,n-\kappa}) \\
&\quad + \delta_i(y_i, x_i) u_i, \quad 1 \le i \le N.
\end{aligned} \qquad (8.31)
$$

A further assumption is needed for the unstructured uncertainties in the interconnections γ_{ij} in (8.30) and (8.31).

Assumption 8.5 The nonlinear interconnection terms in γ_{ij} in (8.30) and (8.31) are bounded by polynomial-type nonlinearities in yl, $1 \le l \le N$, that is,

$$
\begin{aligned}
&\|\theta_i^t \gamma_{ij}(y_1, \dots, y_N, x_{i1}, \dots, x_{ij}) - \theta_i^t \gamma_{ij}(0, \dots, 0, x_{i1}, \dots, x_{ij})\| \\
&\quad \le \sum_{k=1}^{p_{ij}} \sum_{l=1}^{N} \eta_{ij}^k (x_{i1}, \dots, x_{ij}) \zeta_{ilj}^k \|y_l\|^k, \quad 0 \le j \le n-\kappa, \ 1 \le i \le N
\end{aligned} \qquad (8.32)
$$

with η_{i0}^k being a prescribed functional term. Note that the parametric uncertainty θ_i can be lumped with the unknown coefficients ζ_{ilj}^k.

Remark 8.8 When the subsystems have an unequal number of states and a degree of mismatch is notationally more cumbersome than insightful and can be handled, as shown in the sequel. Moreover, in several practical situations, including power systems, the system is generally comprised of dynamically similar interconnected subsystems.

Motivated by the decentralized design in [39] and integrator backstepping along with the tuning function design for centralized control [20, 21] a systematic design procedure is developed. Consider initially the problem of regulating the states to a desired set-point. The extension to the state tracking problem is considered subsequently.

Step 0: Define

$$p = \max_{1\le i\le N; 0\le j\le n-\kappa} \{p_{ij}\} \tag{8.33}$$

and consider the ith subsystem given above. Start with the zeroth sub-subsystem, given by (8.30), with x_{i1} as the virtual control input. Since (A_i, B_i) is a controllable pair, there exists a solution $0 < P_i^t = P_i$ to the algebraic Riccati equation (ARE)

$$A_i^t P_i + P_i A_i - 2\alpha_i P_i B_i B_i^t P_i + Q_i = 0 \tag{8.34}$$

with $\alpha_i > 0$ and $0 < Q_i = Q_i$. Now, choose appropriate Q_i and α_i to solve (8.34) for P_i. Following [39], the decentralized control law x_{i1} for the zeroth sub-subsystem of the ith subsystem is given by

$$x_{i1} = -\alpha_i B_i^t P_i y_i - \hat{\beta}_i B_i^t P_i y_i \{1 + (y_i^t y_i)^{p-1}\} = r_{i1}(y_i, \hat{\beta}_i), \tag{8.35}$$

where p is given by (8.33) and $\hat{\beta}_i$ is a time-varying adaptation gain. Since x_{i1} is not the control, we define

$$z_{i1} = x_{i1} - r_{i1}(y_i, \hat{\beta}_i)$$

and consequently

$$\dot{y}_i = A_i y_i + B_i\{z_{i1} + r_{i1}(y_i, \hat{\beta}_i) + \theta_i^t \gamma_{i0}(y_1, \dots, y_N)\}, \quad 1 \le i \le N. \tag{8.36}$$

Let β_i^* be the desired value of the control gain $\hat{\beta}_i$ to counteract the effect of the interconnections.

Now introduce the following composite Lyapunov function for the zeroth sub-subsystems of the overall system:

$$V_0 = \sum_{i=1}^{N} \left\{ \sum_{k=1}^{p} (y_i^t \; P_i y_i)^k + \Gamma_i^{-1} (\hat{\beta}_i - \beta_i^*)^2 \right\} \tag{8.37}$$

where $\Gamma_i > 0$ is a weighting factor. By differentiating V_0 along the trajectories of system (8.36), we obtain

$$\begin{aligned} V_0 = \sum_{i=1}^{N} \Bigg[\sum_{k=1}^{p} & k(y_i^t P_i y_i)^{k-1} + \Big(y_i^t (A_i P_i + P_i A_i - 2\alpha_i P_i B_i B_i^t P_i) y_i \\ & + 2 y_i^t P_i B_i z_{i1} + 2(B_i^t P_i y_i)\big\{ -\hat{\beta}_i \beta_i^t P_i y_i [1 + (y_i^t y_i)^{p-1}] \\ & + \theta_i^t \gamma_{i0}(y_1, \dots, y_N) \big\} \Big) + 2\Gamma_i^{-1} (\hat{\beta}_i - \beta_i^*) \dot{\hat{\beta}}_i \Bigg]. \end{aligned}$$

Using (8.34) and bounds from (8.32), we get

$$\begin{aligned} \dot{V}_0 \le \sum_{i=1}^{N} \Bigg[& \sum_{k=1}^{p} \{ 2k (y_i^t \; P_i y_i)^{k-1} y_i^t P_i B_i z_{i1} - k \lambda_{\min}^{k-1}(P_i) \lambda_{\min}(Q_i) \| y_i \|^{2k} \} \\ & - 2\beta_i^* \| B_i^t P_i y_i \|^2 [1 + \| y_i \|^{2(p-1)}] \sum_{k=1}^{p} k \lambda_{\min}^{k-1}(P_i) \| y_i \|^{2(k-1)} \end{aligned}$$

$$+ 2(\hat{\beta}_i - \beta_i^*)\left(\Gamma_i^{-1}\dot{\hat{\beta}}_i - \|B_i^t P_i y_i\|^2[1 + \|y_i\|^{2(p-1)}]\sum_{k=1}^{p} k(y_i^t P_i y_i)^{(k-1)}\right)$$

$$+ 2\sum_{k=1}^{p} k\lambda_{\min}^{k-1}(P_i)\|y_i\|^{2(k-1)}\|B_i^t P_i y_i\| \sum_{k_1=1}^{p_{i0}}\sum_{j=1}^{N} \varsigma_j^{1/2}\varsigma_j^{-(1/2)}\zeta_{ij0}^{k_1}\|y_j\|^{k_1}\Bigg], \tag{8.38}$$

where $\varsigma_j > 0$ is introduced as a degree of freedom [39].

Utilizing the standard algebraic inequalities

$$2ab \leq a^2 + b^2, \tag{8.39}$$

$$\left(\sum_{k=1}^{p} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{p} a_k^2\right)\left(\sum_{k=1}^{p} b_k^2\right) \tag{8.40}$$

the last term of (8.38) can be written as

$$\sum_{i=1}^{N}\left\{2\sum_{k=1}^{p} k\lambda_{\max}^{k-1}(P_i)\|y_i\|^{2(k-1)}\|B_i^t P_i y_i\| \sum_{k_1=1}^{p_{i0}}\sum_{j=1}^{N} \varsigma_j^{1/2}\varsigma_j^{-(1/2)}\zeta_{ij0}^{k_1}\|y_j\|^{k_1}\right\}$$
$$\leq \sum_{i=1}^{N}\left\{\varsigma^* d_2\|B_i^t P_i y_i\|^2 \sum_{k=1}^{p}\lambda_{\max}^{2(k-1)}(P_i)\|y_i\|^{4(k-1)} + \varsigma_i^{-1}\sum_{k=1}^{p_{i0}} d_{ik0}\|y_i\|^{2k}\right\}, \tag{8.41}$$

where

$$d_2 = p_{i0}\sum_{k=1}^{p} k^2 = \frac{1}{6}p_{i0}p(p+1)(2p+1), \tag{8.42}$$

$$\varsigma^* = \sum_{j=1}^{N}\varsigma_j, \tag{8.43}$$

$$d_{ik0} = \sum_{j=1}^{N}(\zeta_{ji0}^{k})^2. \tag{8.44}$$

Since $d_{ik0} > 0$, then

$$\sum_{k=1}^{p_{i0}} d_{ik0}\|y_i\|^{2k} \leq \sum_{k=1}^{p} d_{ik0}\|y_i\|^{2k},$$

where the last $p - p_{i0}$ terms can be taken as zero. The following adaptation for $\hat{\beta}_i$ is appropriately used:

$$\dot{\hat{\beta}}_i = \Gamma_i\|B_i^t P_i y_i\|^2[1 + \|y_i\|^{2(p-1)}]\sum_{k=1}^{p} k(y_i^t P_i y_i)^{(k-1)} = \tau_{i0}(y_i). \tag{8.45}$$

Using (8.41) and (8.45), $\dot{V}_0$ can be written as

$$\begin{aligned}
\dot{V}_0 \le & \sum_{i=1}^{N}\sum_{k=1}^{p} 2k(y_i^t \; P_i y_i)^{k-1} y_i^t P_i B_i z_{i1} \\
& + \sum_{i=1}^{N}\Bigg[-\sum_{k=1}^{p}\{k\lambda_{\min}^{k-1}(P_i)\lambda_{\min}(Q_i) - \varsigma_i^{-1} d_{ik0}\}\|y_i\|^{2k} \\
& + 2\beta_i^* \|B_i^t P_i y_i\|^2 [1 + \|y_i\|^{2(p-1)}] \sum_{k=1}^{p} k\lambda_{\min}^{k-1}(P_i)\|y_i\|^{2(k-1)} \\
& + \varsigma^* d_2 \|B_i^t P_i y_i\|^2 \sum_{k=1}^{p} \lambda_{\max}^{2(k-1)}(P_i)\|y_i\|^{4(k-1)} \Bigg] \\
= & \sum_{i=1}^{N}\Bigg[\sum_{k=1}^{p} 2k(y_i^t \; P_i y_i)^{k-1} y_i^t P_i B_i z_{i1} \\
& - \{k\lambda_{\min}^{k-1}(P_i)\lambda_{\min}(Q_i) - \varsigma_i^{-1} d_{ik0}\}\|y_i\|^{2k} + \dot{V}_{i0}(y_i) \Bigg]. \qquad (8.46)
\end{aligned}$$

The actual gains β_i^* and ς_i are obtained in the final step. The state equations for y_i and z_{i1} are given as

$$\begin{aligned}
\dot{y}_i &= A_i y_i + B_i\{z_{i1} + r_{i1}(y_i, \hat{\beta}_i) + \theta_i^t \gamma_{i0}(y_1, \ldots, y_N)\}, \\
\dot{z}_{i1} &= x_{i2} - \frac{\partial r_{i1}}{\partial y_i}\dot{y}_i - \frac{\partial r_{i1}}{\partial \hat{\beta}_i}\tau_{i0}(y_i) + \theta_i^t \gamma_{i0}(y_1, \ldots, y_N, x_{i1}), \qquad (8.47) \\
&= x_{i2} + \nu_{i1}(y_i, z_{i1}, \hat{\beta}_i) + \sum_{\iota=0}^{1} \varphi_{i1}^{\iota}(y_i, \hat{\beta}_i)\theta_i^t \gamma_{i\iota}(y_1, \ldots, y_N, x_{i1}).
\end{aligned}$$

Step 1: Consider x_{i2} as the virtual control for the (y_i, z_{i1}) subsystem. Let $\hat{\theta}_i$ be the estimate of θ_i. Since ς^* is a function of the bounds on the interconnections, and hence unknown, it needs to be estimated. Likewise, define $\hat{\varsigma}^i$ as an estimate for ς^*. Consider the following composite Lyapunov function:

$$V_c = V_0 + \sum_{i=1}^{N}\{z_{i1}^2 + (\hat{\theta}_i - \theta_i)^t(\hat{\theta}_i - \theta_i) + (\hat{\varsigma}^i - \varsigma^*)^2\}. \qquad (8.48)$$

In the sequel, with a slight abuse of notation, we refer to $\gamma_{i\iota}$ as

$$\gamma_{i\iota}(y_1, \ldots, y_N, x_{i1}, \ldots, x_{i\iota})$$

with the understanding that γ_{i0} is $\gamma_{i0}(y_1, \ldots, y_N)$.

By differentiating $\dot{V}_c$ along the trajectories of (8.47) and combining the term

$$\sum_{k=1}^{p} k(y_i^t P_i y_i)^{(k-1)} y_i^t P_i B_i z_{i1}$$

with coefficients of z_{i1}, we obtain

$$\dot{V}_c \leq \sum_{i=1}^{N} \Bigg[\dot{\nu}_{i0}(y_i) - \sum_{k=1}^{p} \Big(k\lambda_{\min}^{k-1}(P_i)\lambda_{\min}(Q_i) - \varsigma_i^{-1} d_{ik0} \Big) \|y_i\|^{2k}$$
$$+ 2z_{i1} \Bigg(x_{i2} + \sum_{k=1}^{p} k(y_i^t P_i y_i)^{(k-1)} y_i^t P_i B_i + \nu_{i1}(y_i, z_{i1}, \hat{\beta}_i)$$
$$+ \sum_{\iota=0}^{1} \varphi_{i1}^{\iota}(y_i, \hat{\beta}_i)\theta_i^t \gamma_{i\iota}(y_1, \ldots, y_N, x_{i1}) \Bigg)$$
$$+ 2(\hat{\theta}_i - \theta_i)^t \dot{\hat{\theta}}_i + 2(\hat{\varsigma}^i - \varsigma^*)\dot{\hat{\varsigma}}^i \Bigg]$$
$$\leq \sum_{i=1}^{N} \Bigg[\dot{\nu}_{i0}(y_i) - \sum_{k=1}^{p} \Big(k\lambda_{\min}^{k-1}(P_i)\lambda_{\min}(Q_i) - \varsigma_i^{-1} d_{ik0} \Big) \|y_i\|^{2k}$$
$$+ 2z_{i1} \Bigg(x_{i2} + \sum_{k=1}^{p} k(y_i^t P_i y_i)^{(k-1)} y_i^t P_i B_i + \nu_{i1}(y_i, z_{i1}, \hat{\beta}_i)$$
$$+ \sum_{\iota=0}^{1} \varphi_{i1}^{\iota}(y_i, \hat{\beta}_i)\theta_i^t \gamma_{i\iota}(0, \ldots, 0, x_{i1}) \Bigg) + 2(\hat{\theta}_i - \theta_i)^t \dot{\hat{\theta}}_i + 2(\hat{\varsigma}^i - \varsigma^*)\dot{\hat{\varsigma}}^i \Bigg]$$
$$+ \sum_{i=1}^{N} \sum_{\iota=0}^{1} 2\|z_{i1}\varphi_{i1}^{\iota}\| \Big(\|\theta_i^t \gamma_{i\iota}(y_1, \ldots, y_N, x_{i1}) - \theta_i^t \gamma_{i\iota}(0, \ldots, 0, x_{i1})\| \Big). \quad (8.49)$$

Using bounds from (8.32) and (8.39), the last term in (8.49) can be written as

$$+ \sum_{i=1}^{N} \sum_{\iota=0}^{1} 2\|z_{i1}\varphi_{i1}^{\iota}\| \|\theta_i^t \gamma_{i\iota}(y_1, \ldots, y_N, x_{i1}) - \theta_i^t \gamma_{i\iota}(0, \ldots, 0, x_{i1})\|$$
$$\leq \sum_{i=1}^{N} \sum_{l=0}^{N} \sum_{\iota=0}^{1} \sum_{k=0}^{p_{i\iota}} 2\|z_{i1}\varphi_{i1}^{\iota}\eta_{i\iota}^{k}\| \varsigma_l^{1/2} \varsigma_l^{-(1/2)} \zeta_{il\iota}^{k} \|y_l\|^k$$
$$\leq \varsigma^* \sum_{i=1}^{N} \sum_{\iota=0}^{1} \varpi_{i\iota} z_{i1}^2 \|\varphi_{i1}^{\iota}\|^2 + \sum_{i=1}^{N} \sum_{k=1}^{p} \sum_{\iota=0}^{1} \varsigma_i^{-1} d_{ik\iota} \|y_i\|^{2k}, \quad (8.50)$$

where

$$d_{ik\iota} = \sum_{l=1}^{N} (\zeta_{li\iota}^{k})^2, \quad (8.51)$$

$$\varpi_{i\iota}(x_{i1}, \ldots, x_{i\iota}) = \sum_{k=1}^{p_{i\iota}} (\eta_{i\iota}^{k})^2 \quad (8.52)$$

and ς^* is given by (8.43) . Using (8.50), $\dot{V}_c$ is written as

$$\dot{V}_c \leq \sum_{i=1}^{N}\Bigg[\dot{\nu}_{i0}(y_i) - \sum_{k=1}^{p}\{k\lambda_{\min}^{k-1}(P_i)\lambda_{\min}(Q_i) - (2d_{ik0} + d_{ik1})\varsigma_i^{-1}\}\|y_i\|^{2k}$$
$$+ 2z_{i1}\Bigg[x_{i2} + \sum_{k=1}^{p} k(y_i^t P_i y_i)^{(k-1)} y_i^t P_i B_i + \nu_{i1}(y_i, z_{i1}, \hat{\beta}_i)$$
$$+ \frac{\varsigma^*}{2} z_{i1} \sum_{\iota=0}^{1} \varpi_{i\iota}\|\varphi_{i1}^{\iota}\|^2 + \sum_{\iota=0}^{1} \varphi_{i1}^{\iota}(y_i, \hat{\beta}_i)\theta_i^t \gamma_{i\iota}(0, \ldots, 0, x_{i1})\Bigg]$$
$$+ 2(\hat{\theta}_i - \theta_i)^t \dot{\hat{\theta}}_i + 2(\hat{\varsigma}^i - \varsigma^*)\dot{\hat{\varsigma}}^i\Bigg]. \tag{8.53}$$

With x_{i2} as the virtual control, we choose

$$c_{i1} = r_{i2}(y_i, z_{i1}, \hat{\beta}_i, \hat{\theta}_i, \dot{\hat{\varsigma}}^i), \tag{8.54}$$

where $c_{i1} > 0$. Define

$$z_{i2} = x_{i2} - r_{i2}(y_i, z_{i1}, \hat{\beta}_i, \hat{\theta}_i, \dot{\hat{\varsigma}}^i).$$

In terms of (y_i, z_{i1}, z_{i2}), $\dot{V}_1$ is given as

$$\dot{V}_1 \leq \sum_{i=1}^{N}\Bigg[\dot{\nu}_{i0}(y_i) - \sum_{k=1}^{p}\{k\lambda_{\min}^{k-1}(P_i)\lambda_{\min}(Q_i) - (2d_{ik0} + d_{ik1})\varsigma_i^{-1}\}\|y_i\|^{2k}$$
$$- 2c_{i1}z_{i1}^2 + -2z_{i1}z_{i2} + 2(\hat{\theta}_i - \theta_i)^t\Bigg\{\dot{\hat{\theta}}_i - z_{i1}\sum_{\iota=0}^{1}\varpi_{i1}^{\iota}(y_i, \hat{\beta}_i)\gamma_{i\iota}(0, \ldots, 0, x_{i1})\Bigg\}$$
$$+ 2(\hat{\varsigma}^i - \varsigma^*)\Bigg\{\dot{\hat{\varsigma}}^i - \frac{z_{i1}^2}{2}\varpi_{i1}\|\varphi_{i1}^{\iota}(y_i, \hat{\beta}_i)\|^2\Bigg\}\Bigg]. \tag{8.55}$$

Proceeding to avoid overparameterization, the adaptation laws for $\hat{\theta}_i$ and $\hat{\varsigma}^i$ are obtained in the final step. Inspired by [21], we define the following tuning functions:

$$\tau_{i2}(y_i, z_{i1}, \hat{\beta}_i) = z_{i1}\sum_{\iota=0}^{1}\varpi_{i1}^{\iota}(y_i, \hat{\beta}_i)\gamma_{i\iota}(0, \ldots, 0, x_{i1}), \tag{8.56}$$

$$\epsilon_{i1}(y_i, z_{i1}, \hat{\beta}_i) = \frac{z_{i1}^2}{2}\varpi_{i1}\|\varphi_{i1}^{\iota}(y_i, \hat{\beta}_i)\|^2. \tag{8.57}$$

Retaining the relevant terms, the (z_{i1}, z_{i2}) sub-subsystem can then be written as

$$\dot{z}_{i1} = z_{i2}\nu_{i1}(y_i, z_{i1}, \hat{\beta}_i) + r_{i2}(y_i, z_{i1}, \hat{\beta}_i, \hat{\theta}_i, \hat{\varsigma}^i)$$
$$+ \sum_{\iota=0}^{1}\varpi_{i1}^{\iota}(y_i, \hat{\beta}_i)\theta_i^t\gamma_{i\iota}(y_1, \ldots, y_N, x_{i1}),$$
$$\dot{z}_{i2} = x_{i3} - \frac{\partial r_{i2}}{\partial y_i}\dot{y}_i - \frac{\partial r_{i2}}{\partial z_{i1}}\dot{z}_{i1} - \frac{\partial r_{i2}}{\partial \hat{\beta}_i}\dot{\hat{\beta}}_i - \frac{\partial r_{i2}}{\partial \hat{\theta}_i}\dot{\hat{\theta}}_i$$

$$
\begin{aligned}
&- \frac{\partial r_{i2}}{\partial \hat{\varsigma}^i}\dot{\hat{\varsigma}}^i + \theta_i^t \gamma_{i2}(y_i, \ldots, y_N, x_{i1}, x_{i2}) \qquad (8.58)\\
&= x_{i3} + \nu_{i2}(y_i, z_{i1}, z_{i2}, \hat{\beta}_i, \hat{\theta}_i, \hat{\varsigma}^i)\\
&\quad + \dot{\hat{\theta}}_i^t \psi_{i2}(y_i, z_{i1}, \hat{\beta}_i, \hat{\varsigma}^i) + \dot{\hat{\varsigma}}^i \xi_{i2}(y_i, z_{i1}, \hat{\beta}_i, \hat{\theta}^i)\\
&\quad + \sum_{\iota=0}^{2} \varphi_{i2}^{\iota}(y_i, z_{i1}, \hat{\beta}_i, \hat{\theta}_i, \hat{\varsigma}^i)\theta_i^t \gamma_{i\iota}(y_1, \ldots, y_N, x_{i1}, x_{i2}).
\end{aligned}
$$

Step 2: From now on, the following notation is used unless specified explicitly:

$$
\begin{aligned}
&\nu_{ik}(y_i, z_{i1}, \ldots, z_{ik}, \hat{\beta}_i, \hat{\theta}_i, \hat{\varsigma}^i) = \nu_{ik},\\
&\gamma_{ik}(y_i, \ldots, y_N, x_{i1}, \ldots, x_{ik}) = \gamma_{ik},\\
&\varphi_{ik}(y_i, z_{i1}, \ldots, z_{i,k-1}, \hat{\beta}_i, \hat{\theta}_i, \hat{\varsigma}^i) = \varphi_{ik},\\
&\psi_{ik}(y_i, z_{i1}, \ldots, z_{i,k-1}, \hat{\beta}_i, \hat{\varsigma}^i) = \psi_{ik}, \qquad (8.59)\\
&\xi_{ik}(y_i, z_{i1}, \ldots, z_{i,k-1}, \hat{\beta}_i, \hat{\theta}_i) = \xi_{ik},\\
&\tau_{ik}(y_i, z_{i1}, \ldots, z_{ik}, \hat{\beta}_i, \hat{\theta}_i, \hat{\varsigma}^i) = \tau_{ik},\\
&\epsilon_{ik}(y_i, z_{i1}, \ldots, z_{ik}, \hat{\beta}_i, \hat{\theta}_i, \hat{\varsigma}^i) = \epsilon_{ik}.
\end{aligned}
$$

Consider as x_{i3} the virtual control for the $(y_i, z_{i1}, \ldots, z_{i2})$ subsystem. Consider the Lyapunov function:

$$
V_e = V_c + \sum_{i=1}^{N} z_{i2}^2. \qquad (8.60)
$$

On differentiating V_2 along the trajectories of (8.58), we obtain

$$
\begin{aligned}
\dot{V}_2 \le \sum_{i=1}^{N} \Bigg[&\dot{\nu}_{i0}(y_i) - \sum_{k=1}^{p} \{k\lambda_{\min}^{k-1}(P_i)\lambda_{\min}(Q_i) - (2d_{ik0} + d_{ik1})\varsigma_i^{-1}\}\|y_i\|^{2k}\\
&- 2c_{i1}z_{i1}^2 + 2z_{i2}\left(z_{i1} + x_{i3} + \nu_{i2} + \dot{\hat{\varsigma}}^i \xi_{i2} + \dot{\hat{\theta}}_i^t \psi_{i2} + \sum_{\iota=0}^{2} \varphi_{i2}^{\iota}\theta_i^t \gamma_{i\iota} \right)\\
&+ 2(\hat{\theta}_i - \theta_i)^t \{\dot{\hat{\theta}}_i - \tau_{i1}\} + 2(\hat{\varsigma}^i - \varsigma^*)\{\dot{\hat{\varsigma}}^i - \epsilon_{i1}\} \Bigg]. \qquad (8.61)
\end{aligned}
$$

By similarity to (8.50) , we obtain using (8.32) and (8.39), bounds for

$$
+ \sum_{i=1}^{N}\sum_{\iota=0}^{2} 2\|z_{i2}\varphi_{i2}^{\iota}\|\|\theta_i^t \gamma_{i\iota}(y_1, \ldots, y_N, x_{i1} \ldots x_{i\iota}) - \theta_i^t \gamma_{i\iota}(0, \ldots, 0, x_{i1} \ldots x_{i\iota})\|
$$

with $d_{ik\iota}$ and $\varpi_{i\iota}$ given by (8.51) and (8.52), respectively. On using these bounds, relation (8.61) can be expressed as

$$\dot{V}_e \leq \sum_{i=1}^{N} \Bigg[\dot{\nu}_{i0}(y_i) - \sum_{k=1}^{p} \{k\lambda_{\min}^{k-1}(P_i)\lambda_{\min}(Q_i) - (3d_{ik0} + 2d_{ik1} + d_{ik2})\varsigma_i^{-1}\}\|y_i\|^{2k}$$
$$- 2c_{i1}z_{i1}^2 + 2z_{i2}\Bigg\{ z_{i1} + x_{i3} + \nu_{i2} + \frac{\varsigma^*}{2} z_{i2} \sum_{\iota=0}^{2} \varpi_{i\iota}\|\varphi_{i2}^{\iota}\|^2$$
$$+ \theta_i^t \sum_{\iota=0}^{2} \varphi_{i2}^{\iota}\gamma_{i\iota}(0,\ldots,0,x_{i1},x_{i2}) + \dot{\hat{\theta}}_i^t \psi_{i2} + \dot{\hat{\varsigma}}^i \xi_{i2} \Bigg\}$$
$$+ 2(\hat{\theta}_i - \theta_i)^t\{\dot{\hat{\theta}}_i - \tau_{i1}\} + 2(\hat{\varsigma}^i - \varsigma^*)\{\dot{\hat{\varsigma}}^i - \epsilon_{i1}\} \Bigg]. \tag{8.62}$$

Selecting

$$x_{i3} = -\Bigg[z_{i1} + c_{i2}z_{i2} + \nu_{i2} + \tau_{i2}\psi_{i2} + \epsilon_{i2}\xi_{i2} + \frac{\hat{\varsigma}^i}{2} \sum_{\iota=0}^{2} \varpi_{i\iota}\|\varphi_{i2}^{\iota}\|^2$$
$$+ \theta_i^t \sum_{\iota=0}^{2} \varphi_{i2}^{\iota}\gamma_{i\iota}(0,\ldots,0,x_{i1},x_{i2}) \Bigg]$$
$$= r_{i3}(y_i, z_{i1}, z_{i2}, \hat{\beta}_i, \hat{\theta}_i, \hat{\varsigma}^i), \tag{8.63}$$

where τ_{i2} and ϵ_{i2} are tuning functions defined as

$$\tau_{i2} = \tau_{i1} + z_{i2} \sum_{\iota=0}^{2} \varphi_{i2}^{\iota}\gamma_{i\iota}(0,\ldots,0,x_{i1},x_{i2}),$$
$$\epsilon_{i2} = \epsilon_{i1} + \frac{1}{2} z_{i2}^2 \sum_{\iota=0}^{2} \varpi_{i\iota}\|\varphi_{i2}^{\iota}\|^2.$$

Recall that x_{i3} is not the control. Thus we define

$$z_{i3} = x_{i3} - r_{i3}(y_i, z_{i1}, z_{i2}, \hat{\beta}_i, \hat{\theta}_i, \hat{\varsigma}^i).$$

Then, $\dot{V}_e$ is given by

$$\dot{V}_e \leq \sum_{i=1}^{N} \Bigg[\dot{\nu}_{i0}(y_i) - \sum_{k=1}^{p} \{k\lambda_{\min}^{k-1}(P_i)\lambda_{\min}(Q_i) - (3d_{ik0} + 2d_{ik1} + d_{ik2})\varsigma_i^{-1}\}\|y_i\|^{2k}$$
$$- 2c_{i1}z_{i1}^2 - 2c_{i2}z_{i2}^2 + 2z_{i2}z_{i3} + 2z_{i2}\{\dot{\hat{\theta}}_i - \tau_{i1}\}^t \psi_{i2}$$
$$+ 2z_{i2}\{\dot{\hat{\varsigma}}^i - \epsilon_{i2}\}\xi_{i2} + 2(\hat{\theta}_i - \theta_i)^t\{\dot{\hat{\theta}}_i - \tau_{i2}\} + 2(\hat{\varsigma}^i - \varsigma^*)\{\dot{\hat{\varsigma}}^i - \epsilon_{i2}\} \Bigg]. \tag{8.64}$$

If the control u_i appears in the state equation for x_{i3}, the adaptation laws for $\hat{\theta}_i$ and $\hat{\varsigma}^i$ would be

$$\begin{aligned}\dot{\hat{\theta}}_i &= \tau_{i3}\\ &= \tau_{i2} + z_{i3}\sum_{\iota=0}^{3}\varphi_{i3}^{\iota}\gamma_{i\iota}(0,\ldots,0,x_{i1},x_{i2},x_{i3}),\end{aligned} \tag{8.65}$$

$$\begin{aligned}\dot{\hat{\varsigma}}^i &= \epsilon_{i3}\\ &= \epsilon_{i2} + \frac{1}{2}z_{i3}^2\sum_{\iota=0}^{3}\varpi_{i\iota}\|\varphi_{i3}^{\iota}\|^2.\end{aligned} \tag{8.66}$$

In the time derivative of the Lyapunov function $V_3 = V_2 + \sum_{i=1}^{N} z_{i2}^3$, the term

$$2z_{i2}\{\dot{\hat{\theta}}_i - \tau_{i2}\}^t\psi_{i2} + 2z_{i2}\{\dot{\hat{\varsigma}}^i - \epsilon_{i2}\}\xi_{i2} \tag{8.67}$$

can be expressed as $z_{i3}\lambda_{i3}(z_{i1}, z_{i2}, z_{i3}, \hat{\theta}_i, \hat{\varsigma}^i)$ and can be countered using the control $-\lambda_{i3}$. However, if u_i does not appear in $\dot{x}_{i3}$, (8.67) needs to be upgraded using tuning functions [21].

This is illustrated in Step m $(3 \le m \le n-\kappa-1)$, where the design procedure is now made recursive.

Step m $(3 \le m \le n-\kappa-1)$: Consider $x_{i,m+1}$ as the virtual control for the $(y_i, z_{i1}, \ldots, z_{im})$ subsystem, where z_{im} is given by

$$z_{im} = x_{im} - r_{im}(y_i, z_{i1}, \ldots, z_{i,m-1}, \hat{\beta}_i, \hat{\theta}_i, \hat{\varsigma}^i) \tag{8.68}$$

and

$$\dot{z}_{im} = x_{i,m+1} + \nu_{im} + \dot{\hat{\theta}}_i^t\psi_{im} + \dot{\hat{\varsigma}}^i\xi_{im} + \sum_{\iota=0}^{m-1}\varphi_{im}^{\iota}\theta_i^t\gamma_{i\iota}. \tag{8.69}$$

Consider the following Lyapunov function for the $(y_i, z_{i1}, \ldots, z_{im})$ subsystems:

$$V_m = V_{m-1} + \sum_{i=1}^{N} z_{im}^2. \tag{8.70}$$

By differentiating V_m along the trajectories of the $(y_i, z_{i1}, \ldots, z_{im})$ subsystem, we obtain

$$\begin{aligned}\dot{V}_m \le \sum_{i=1}^{N}\Bigg[&\dot{\nu}_{i0}(y_i) - \sum_{k=1}^{p}\Big\{k\lambda_{\min}^{k-1}(P_i)\lambda_{\min}(Q_i)\\ &- \big[md_{ik0} + (m-1)d_{ik1} + \cdots + d_{ik,m-1}\big]\varsigma_i^{-1}\Big\}\|y_i\|^{2k}\\ &- 2c_{i1}z_{i1}^2 - \cdots - 2c_{i,m-1}z_{i,m-1}^2\\ &+ 2\{\dot{\hat{\theta}}_i - \tau_{i,m-1}\}^t\sum_{j=2}^{m-1} z_{ij}\psi_{ij} + 2\{\dot{\hat{\varsigma}}^i - \epsilon_{i,m-1}\}\sum_{j=2}^{m-1} z_{ij}\xi_{ij}\end{aligned}$$

$$+ 2z_{im}\left(z_{i,m-1} + x_{i,m+1} + \nu_{im} + \dot{\hat{\theta}}_i^t \psi_{im} + \dot{\hat{\varsigma}}^i \xi_{im} + \sum_{\iota=0}^{m} \varphi_{im}^{\iota} \theta_i^t \gamma_{i\iota}\right)$$

$$+ 2(\hat{\theta}_i - \theta_i)^t \{\dot{\hat{\theta}}_i - \tau_{i,m-1}\} + 2(\hat{\varsigma}^i - \varsigma^*)\{\dot{\hat{\varsigma}}^i - \epsilon_{i,m-1}\}\Bigg]. \tag{8.71}$$

Using (8.32) and (8.39), we obtain the associated bounds

$$+ \sum_{i=1}^{N}\sum_{\iota=0}^{m} 2\|z_{im}\varphi_{im}^{\iota}\| \|\theta_i^t \gamma_{i\iota}(y_1, \ldots, y_N, x_{i1}, \ldots, x_{i\iota}) - \theta_i^t \gamma_{i\iota}(0, \ldots, 0, x_{i1}, \ldots, x_{i\iota})\|.$$

On applying these bounds to $\dot{V}_m$, we obtain

$$\begin{aligned}
\dot{V}_m \le \sum_{i=1}^{N} \Bigg[& \dot{\nu}_{i0}(y_i) - \sum_{k=1}^{p} \big\{ k\lambda_{\min}^{k-1}(P_i)\lambda_{\min}(Q_i) \\
& - \big[(m+1)d_{ik0} + m d_{ik1} + \cdots + d_{ikm}\big]\varsigma_i^{-1} \big\} \|y_i\|^{2k} \\
& - 2c_{i1}z_{i1}^2 - \cdots - 2c_{i,m-1}z_{i,m-1}^2 \\
& + 2\{\dot{\hat{\theta}}_i - \tau_{i,m-1}\}^t \sum_{j=2}^{m-1} z_{ij}\psi_{ij} + 2\{\dot{\hat{\varsigma}}^i - \epsilon_{i,m-1}\} \sum_{j=2}^{m-1} z_{ij}\xi_{ij} \\
& + 2z_{im}\Bigg(z_{i,m-1} + x_{i,m+1} + \nu_{im} + \frac{\varsigma^*}{2} z_{im} \sum_{\iota=0}^{2} \varpi_{i\iota} \|\varphi_{im}^{\iota}\|^2 \\
& + \dot{\hat{\theta}}_i^t \sum_{\iota=0}^{m} \varphi_{im}^{\iota} \gamma_{i\iota}(0, \ldots, 0, x_{i1}, \ldots, x_{im}) + \dot{\hat{\theta}}_i^t \psi_{im} + \dot{\hat{\varsigma}}^i \xi_{im}\Bigg) \\
& + 2(\hat{\theta}_i - \theta_i)^t \{\dot{\hat{\theta}}_i - \tau_{i,m-1} + 2(\hat{\varsigma}^i - \varsigma^*)\{\dot{\hat{\varsigma}}^i - \epsilon_{i,m-1}\} \Bigg].
\end{aligned} \tag{8.72}$$

Similarly define the tuning functions

$$\tau_{im} = \tau_{i,m-1} + z_{im} \sum_{\iota=0}^{m} \varphi_{im}^{\iota} \gamma_{i\iota}(0, \ldots, 0, x_{i1}, \ldots, x_{im}), \tag{8.73}$$

$$\epsilon_{im} = \epsilon_{i,m-1} + \frac{1}{2} z_{im}^2 \sum_{\iota=0}^{m} \varpi_{i\iota} \|\varphi_{im}^{\iota}\|^2. \tag{8.74}$$

We get

$$\begin{aligned}
& 2\{\dot{\hat{\theta}}_i - \tau_{i,m-1}\}^t \sum_{j=2}^{m-1} z_{ij}\psi_{ij} + 2\{\dot{\hat{\varsigma}}^i - \epsilon_{i,m-1}\} \sum_{j=2}^{m-1} z_{ij}\xi_{ij} \\
& \quad = 2\{\dot{\hat{\theta}}_i - \tau_{i,m}\}^t \sum_{j=2}^{m-1} z_{ij}\psi_{ij} + 2\{\dot{\hat{\varsigma}}^i - \epsilon_{i,m}\} \sum_{j=2}^{m-1} z_{ij}\xi_{ij}
\end{aligned}$$

$$+ 2z_{im}\left\{\sum_{\iota=0}^{m}\varphi_{im}^{\iota}\gamma_{i\iota}(0,\ldots,0,x_{i1},x_{i2},\ldots,x_{im})\sum_{j=2}^{m-1}z_{ij}\psi_{ij}\right.$$
$$\left. + \frac{1}{2}z_{im}\sum_{\iota=0}^{m}\varpi_{i\iota}\|\varphi_{im}^{\iota}\|^2\sum_{j=2}^{m-1}z_{ij}\xi_{ij}\right\}. \tag{8.75}$$

Applying (8.75) to (8.72), we have

$$\begin{aligned}\dot{V}_m \le \sum_{i=1}^{N}\Bigg[&\dot{\nu}_{i0}(y_i) - \sum_{k=1}^{p}\Big\{k\lambda_{\min}^{k-1}(P_i)\lambda_{\min}(Q_i)\\
&- \big[(m+1)d_{ik0} + md_{ik1} + \cdots + d_{ikm}\big]\varsigma_i^{-1}\Big\}\|y_i\|^{2k}\\
&- 2c_{i1}z_{i1}^2 - \cdots - 2c_{i,m-1}z_{i,m-1}^2\\
&+ 2\{\dot{\hat{\theta}}_i - \tau_{i,m}\}^t\sum_{j=2}^{m-1}z_{ij}\psi_{ij} + 2\{\dot{\hat{\varsigma}}^i - \epsilon_{i,m}\}\sum_{j=2}^{m-1}z_{ij}\xi_{ij}\\
&+ 2z_{im}\Bigg(z_{i,m-1} + x_{i,m+1} + \nu_{im}\\
&+ \sum_{\iota=0}^{m}\varphi_{im}^{\iota}\gamma_{i\iota}(0,\ldots,0,x_{i1},x_{i2},\ldots,x_{im})\sum_{j=2}^{m-1}z_{ij}\psi_{ij}\\
&+ \frac{1}{2}z_{im}\sum_{\iota=0}^{m}\varpi_{i\iota}\|\varphi_{im}^{\iota}\|^2\sum_{j=2}^{m-1}z_{ij}\xi_{ij} + \frac{\varsigma^*}{2}z_{im}\sum_{\iota=0}^{2}\varpi_{i\iota}\|\varphi_{im}^{\iota}\|^2\\
&+ \theta_i^t\sum_{\iota=0}^{m}\varphi_{im}^{\iota}\gamma_{i\iota}(0,\ldots,0,x_{i1},x_{i2},\ldots,x_{im}) + \dot{\hat{\theta}}_i^t\psi_{im} + \dot{\hat{\varsigma}}^i\xi_{im}\Bigg)\\
&+ 2(\hat{\theta}_i - \theta_i)^t\{\dot{\hat{\theta}}_i - \tau_{i,m-1}\} + 2(\hat{\varsigma}^i - \varsigma^*)\{\dot{\hat{\varsigma}}^i - \epsilon_{i,m-1}\}\Bigg].\end{aligned} \tag{8.76}$$

Choose the following virtual control for $x_{i,m+1}$:

$$\begin{aligned}x_{i,m+1} = -\Bigg[&z_{i,m-1} + c_{im}z_{im} + \nu_{im}\\
&+ \sum_{\iota=0}^{m}\varphi_{im}^{\iota}\gamma_{i\iota}(0,\ldots,0,x_{i1},x_{i2},\ldots,x_{im})\sum_{j=2}^{m-1}z_{ij}\psi_{ij}\\
&+ \frac{1}{2}z_{im}\sum_{\iota=0}^{m}\varpi_{i\iota}\|\varphi_{im}^{\iota}\|^2\sum_{j=2}^{m-1}z_{ij}\xi_{ij}\\
&+ \frac{\hat{\varsigma}^i}{2}z_{im}\sum_{\iota=0}^{m}\varpi_{i\iota}\|\varphi_{im}^{\iota}\|^2\end{aligned}$$

$$+\hat{\theta}_i^t \sum_{\iota=0}^{m} \varphi_{im}^{\iota}(y_i, z_{i1}, \ldots, z_{i,m-1}) \gamma_{i\iota}(0, \ldots, 0, x_{i1}, \ldots, x_{im})$$
$$\left. + \tau_{im}\psi_{im} + \epsilon_{im}\xi_{im} \right]$$
$$= r_{i,m+1}(y_i, z_{i1}, \ldots, z_{im}, \hat{\beta}_i, \hat{\theta}_i, \hat{\varsigma}^i) \tag{8.77}$$

and define

$$z_{i,m+1} = x_{i,m+1} - r_{i,m+1}(y_i, z_{i1}, \ldots, z_{im}, \hat{\beta}_i, \hat{\theta}_i, \hat{\varsigma}^i).$$

Then it is not difficult to see that $\dot{V}_{im}$ is given by

$$\begin{aligned} \dot{V}_m \le \sum_{i=1}^{N} \Bigg[& \dot{v}_{i0}(y_i) - \sum_{k=1}^{p} \Big\{ k\lambda_{\min}^{k-1}(P_i)\lambda_{\min}(Q_i) \\ & - \big[(m+1)d_{ik0} + md_{ik1} + \cdots + d_{ikm}\big]\varsigma_i^{-1}\Big\} \|y_i\|^{2k} \\ & - 2c_{i1}z_{i1}^2 - \cdots - 2c_{i,m}z_{i,m}^2 + 2z_{i,m+1}z_{im} \\ & + 2\{\dot{\hat{\theta}}_i - \tau_{i,m}\}^t \sum_{j=2}^{m} z_{ij}\psi_{ij} \\ & + 2\{\dot{\hat{\varsigma}}^i - \epsilon_{im}\} \sum_{j=2}^{m} z_{ij}\xi_{ij} + 2(\hat{\theta}_i - \theta_i)^t\{\dot{\hat{\theta}}_i - \tau_{im}\} \\ & + 2(\hat{\varsigma}^i - \varsigma^*)\{\dot{\hat{\varsigma}}^i - \epsilon_{im}\} \Bigg]. \end{aligned} \tag{8.78}$$

Step $n-\kappa$: In the $(n-\kappa-1)$th step, $x_{i,n-\kappa}$ is obtained as a virtual control input [$= r_{i,n-\kappa}(y_i, z_{i1}, \ldots, z_{i,n-\kappa-1}, \hat{\theta}_i, \hat{\varsigma}_i)$] for the $(y_i, z_{i1}, \ldots, z_{n-\kappa-1})$ subsystem. Define $z_{i,n-\kappa} = x_{i,n-\kappa} - r_{i,n-\kappa}(y_i, z_{i1}, \ldots, z_{i,n-\kappa-1}, \hat{\theta}_i, \hat{\varsigma}_i)$. Denoting $n-\kappa = \rho$ as the degree of mismatch, we obtain

$$\dot{z}_{i\rho} = v_i(x_i) + v_{i,\rho} + \dot{\hat{\theta}}_i^t \psi_{i,\rho} + \dot{\hat{\varsigma}}^i \xi_{i,\rho} + \sum_{\iota=0}^{\rho} \varphi_{i,\rho}\theta_i^t \gamma_{i,\rho} + \delta_i(y_i, x_i)u_i. \tag{8.79}$$

Consider the following Lyapunov function:

$$V_\rho = V_{\rho-1} + z_{i\rho}^2 \tag{8.80}$$

and differentiating (8.80) along the trajectories of the overall system, $\dot{V}_\rho$ is given by

$$\begin{aligned} \dot{V}_\rho \le \sum_{i=1}^{N} \Bigg[& \dot{v}_{i0}(y_i) - \sum_{k=1}^{p} \Big\{ k\lambda_{\min}^{k-1}(P_i)\lambda_{\min}(Q_i) \\ & - \big[\rho d_{ik0} + (\rho-1)d_{ik1} + \cdots + d_{ik,\rho-1}\big]\varsigma_i^{-1}\Big\} \|y_i\|^{2k} \\ & - 2c_{i1}z_{i1}^2 - \cdots - 2c_{i,\rho-1}z_{i,\rho-1}^2 \end{aligned}$$

$$
\begin{aligned}
&+2\{\dot{\hat{\theta}}_i-\tau_{i,\rho-1}\}^t\sum_{j=2}^{\rho-1}z_{ij}\psi_{ij}+2\{\dot{\hat{\varsigma}}^i-\epsilon_{i,\rho-1}\}\sum_{j=2}^{\rho-1}z_{ij}\xi_{ij}\\
&+2z_{i\rho}\left(z_{i,\rho-1}+\delta_i(y_i,x_i)u_i+\upsilon_i(x_i)+\nu_{i,\rho}\dot{\hat{\theta}}_i^t\psi_{i\rho}+\dot{\hat{\varsigma}}^i\xi_{i\rho}+\sum_{\iota=0}^{\rho}\delta_{i\rho}^{\iota}\theta_i^t\gamma_{i\iota}\right)\\
&+2(\hat{\theta}_i-\theta_i)^t\{\dot{\hat{\theta}}_i-\tau_{i,\rho-1}\}+2(\hat{\varsigma}^i-\varsigma^*)\{\dot{\hat{\varsigma}}^i-\epsilon_{i,\rho-1}\}\Bigg]. \qquad (8.81)
\end{aligned}
$$

As in (8.49), (8.53), we apply the associated bounds for

$$
\begin{aligned}
+\sum_{i=1}^{N}\sum_{\iota=0}^{\rho}2\|z_{i1}\varphi_{i\rho}^{\iota}\|\,\|&\theta_i^t\gamma_{i\iota}(y_1,\dots,y_N,x_{i1},\dots,x_{i\iota})\\
&-\theta_i^t\gamma_{i\iota}(0,\dots,0,x_{i1},\dots,x_{i\iota})\|
\end{aligned}
$$

to $\dot{V}_\rho$ to obtain

$$
\begin{aligned}
\dot{V}_\rho\le\sum_{i=1}^{N}\Bigg[&\dot{\nu}_{i0}(y_i)-\sum_{k=1}^{p}\Big\{k\lambda_{\min}^{k-1}(P_i)\lambda_{\min}(Q_i)\\
&-\big[(\rho+1)d_{ik0}+\rho d_{ik1}+\cdots+d_{ik,\rho}\big]\varsigma_i^{-1}\Big\}\|y_i\|^{2k}\\
&-2c_{i1}z_{i1}^2-\cdots-2c_{i,\rho-1}z_{i,\rho-1}^2\\
&+2\{\dot{\hat{\theta}}_i-\tau_{i,\rho-1}\}^t\sum_{j=2}^{\rho-1}z_{ij}\psi_{ij}+2\{\dot{\hat{\varsigma}}^i-\epsilon_{i,\rho-1}\}\sum_{j=2}^{\rho-1}z_{ij}\xi_{ij}\\
&+2z_{i\rho}\Bigg(z_{i,\rho-1}+\delta_i(y_i,x_i)u_i+\upsilon_i(x_i)+\nu_{i\rho}+\theta_i^t\sum_{\iota=0}^{\rho}\varphi_{i\rho}^{\iota}\gamma_{i\iota}(0,\dots,0,x_i)\\
&+\frac{\varsigma^*}{2}z_{i\rho}\sum_{\iota=0}^{\rho}\varpi_{i\iota}\|\varphi_{i\rho}^{\iota}\|^2+\dot{\hat{\theta}}_i^t\psi_{i\rho}+\dot{\hat{\varsigma}}^i\xi_{i\rho}\Bigg)\\
&+2(\hat{\theta}_i-\theta_i)^t\{\dot{\hat{\theta}}_i-\tau_{i,\rho-1}\}+2(\hat{\varsigma}^i-\varsigma^*)\{\dot{\hat{\varsigma}}^i-\epsilon_{i,\rho-1}\}\Bigg]. \qquad (8.82)
\end{aligned}
$$

The following decentralized control input u_i is applied for the ith subsystem:

$$
\begin{aligned}
u_i=\frac{1}{\delta_i(y_i,x_i)}\Bigg(&-\upsilon_i(x_i)-z_{i,\rho-1}-c_{i\rho}z_{i\rho}-\nu_{i,\rho}\\
&-\frac{\varsigma^*}{2}z_{i\rho}\sum_{\iota=0}^{\rho}\varpi_{i\iota}\|\varphi_{i\rho}^{\iota}\|^2-\hat{\theta}_i^t\sum_{\iota=0}^{\rho}\varphi_{i\rho}^{\iota}\gamma_{i\iota}(0,\dots,0,x_{i1})\\
&-\tau_{i\rho}\psi_{i\rho}-\epsilon_{i\rho}\xi_{i\rho}-\sum_{\iota=0}^{\rho}\varphi_{i\rho}^{\iota}\gamma_{i\iota}(0,\dots,0,x_{i1})\sum_{j=2}^{\rho-1}z_{ij}\psi_{ij}\\
&-\frac{1}{2}z_{i\rho}\sum_{\iota=0}^{\rho}\varpi_{i\iota}\|\varphi_{i\rho}^{\iota}\|^2\sum_{j=2}^{\rho-1}z_{ij}\xi_{ij}\Bigg) \qquad (8.83)
\end{aligned}
$$

with the adaptation laws

$$\begin{aligned}
\dot{\hat{\theta}}^t_i &= \tau_{i\rho} \\
&= \tau_{i,\rho-1} + z_{i\rho} \sum_{\iota=0}^{\rho} \varphi^\iota_{i\rho} \gamma_{i\iota}(0,\dots,0,x_i), \\
\tau_{ij} &= \tau_{i,j-1} + z_{ij} \sum_{\iota=0}^{j} \varphi^\iota_{ij} \gamma_{i\iota}(0,\dots,0,x_{i1},\dots,x_{ij}), \quad 2 \le j \le \rho, \\
\tau_{i1} &= z_{i1} \sum_{\iota=0}^{1} \varphi^\iota_{i1}(y_i) \gamma_{i\iota}(0,\dots,0,x_{i1}).
\end{aligned} \tag{8.84}$$

$$\begin{aligned}
\dot{\hat{\varsigma}}^i &= \epsilon_{i\rho} \\
&= \epsilon_{i,\rho-1} + \frac{1}{2} z^2_{i\rho} \sum_{\iota=0}^{\rho} \varpi_{i\iota} \|\varphi^\iota_{i\rho}\|^2, \\
\epsilon_{ij} &= \epsilon_{i,j-1} + \frac{1}{2} z^2_{ij} \sum_{\iota=0}^{j} \varpi_{i\iota} \|\varphi^\iota_{ij}\|^2, \quad 2 \le j \le \rho, \\
\epsilon_{i1}(y_i, z_i) &= \frac{z^2_{i1}}{2} \sum_{\iota=0}^{1} \varpi_{i\iota} \|\varphi^\iota_{i1}(y_i)\|^2.
\end{aligned} \tag{8.85}$$

The stability properties of the above designed decentralized adaptive control are stated in the following theorem.

Theorem 8.3 *Suppose that* (8.24), *with* $n_i = n$ *and uniform degree of mismatch for each subsystem, satisfies the conditions of Proposition* 8.1. *The control input* (8.83) *along with the adaptation laws* (8.45), (8.84), *and* (8.85), *obtained from the above systematic design procedure, results in the global uniform stability of the equilibrium*

$$z_i = \underline{0}, \quad \hat{\theta}_i = \theta_i, \quad \hat{\beta}_i = \beta^*_i, \quad \hat{\varsigma}^i = \varsigma^*, \quad 1 \le i \le N.$$

Furthermore, regulation of the state $\zeta(t)$ *is achieved,*

$$\lim_{t \leftrightarrow \infty} \zeta(t) = 0$$

for all initial conditions in $\Omega = \Re^{n_1+\dots+n_N}$.

Proof Applying (8.83) with adaptation laws (8.84) and (8.85), (8.82) becomes

$$\begin{aligned}
\dot{V}_\rho \le \sum_{i=1}^{N} \Bigg[& -2\beta^*_i \|B^t_i P_i y_i\|^2 [1 + \|y_i\|^{2(p-1)}] \sum_{k=1}^{p} k \lambda^{k-1}_{\min}(P_i) \|y_i\|^{2(k-1)} \\
& + \varsigma^* d_2 \|B^t_i P_i y_i\|^2 \sum_{k=1}^{p} \lambda^{2(k-1)}_{\max}(P_i) \|y_i\|^{4(k-1)}
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{k=1}^{p} \big\{ k\lambda_{\min}^{k-1}(P_i)\lambda_{\min}(Q_i) - \big[(\rho+1)d_{ik0} + \rho d_{ik1} \\
&+ (\rho-1)d_{ik2} + \cdots + 2d_{ik,\rho-1} + d_{ik,\rho}\big]\varsigma_i^{-1}\big\} \|y_i\|^{2k} \\
&- 2c_{i1}z_{i1}^2 - \cdots - 2c_{i,\rho}z_{i\rho}^2 \Bigg]. \qquad (8.86)
\end{aligned}
$$

The degree of freedom ς_i is chosen as

$$
\varsigma_i \geq \max_{k\in 1,\ldots i,\rho} \left\{ \frac{(\rho+1)d_{ik0} + \rho d_{ik1} + \cdots + 2d_{ik,\rho-1} + d_{ik,\rho}}{k\lambda_{\min}(Q_i)\lambda_{\min}^{k-1}(P_i)} \right\}. \qquad (8.87)
$$

We need to establish the existence of the gain β_i^* as a function of the interconnection measure ς^*, such that $\dot{V}_\rho$ is negative. The first two terms of $\dot{V}_\rho$ can be made negative by an appropriate choice of β^*. This can be seen as follows. The coefficients of terms $\|y_i\|^{2(k-1)}$ for $k = 2, 4, 6, \ldots \leq p$ are negative and are given by

$$
-2k\beta_i^*\lambda_{\min}^{k-1}(P_i)\|B_i^t P_i y_i\|^2.
$$

The coefficients of terms $\|y_i\|^{2(k-1)}$ for $k = 1, 3, 5, \ldots \leq p$ are

$$
\Delta_{i1} = -[2k\beta_i^*\lambda_{\min}^{k-1}(P_i) - \varsigma^* d_2\lambda_{\max}^{k-1}(P_i)]\|B_i^t P_i y_i\|^2
$$

and the coefficients of terms $\|y_i\|^{4(k-1)}$ for $k = 1 + \lfloor p/2 \rfloor, 2 + \lfloor p/2 \rfloor, \ldots, p$ are given by

$$
\Delta_{i2} = -[2\beta_i^*(2k-p)\lambda_{\min}^{2k-p-1}(P_i) - \varsigma^* d_2\lambda_{\max}^{2(k-1)}(P_i)]\|B_i^t P_i y_i\|^2.
$$

Therefore, choosing

$$
\beta_i^* = (\max\{\Lambda_{i1}, \Lambda_{i2}\})\varsigma^*, \qquad (8.88)
$$

where

$$
\begin{aligned}
\Lambda_{i1} &= \max_{k=1,3,5,\ldots\leq p} \left[\frac{d_2\lambda_{\max}^{k-1}(P_i)}{2k\lambda_{\min}^{k-1}(P_i)} \right], \\
\Lambda_{i2} &= \max_{k=1+\lfloor p/2 \rfloor,\ldots,p} \left[\frac{d_2\lambda_{\max}^{2(k-1)}(P_i)}{2(2k-p)\lambda_{\min}^{2k-p-1}(P_i)} \right]
\end{aligned}
$$

we have

$$
\begin{aligned}
\dot{V}_\rho &\leq -\sum_{i=1}^{N} \left\{ c_{i0}\sum_{k=1}^{p} \|y_i\|^{2k} + \sum_{j=1}^{\rho} c_{ij}z_{ij}^2 \right\}, \\
&c_{ij} > 0,\ 0 \leq j \leq \rho;\ 1 \leq i \leq N. \qquad (8.89)
\end{aligned}
$$

Thus, the solutions $y_i, z_{i1}, \ldots, z_{i,\rho}, \hat{\beta}_i, \hat{\varsigma}^i, \hat{\theta}_i, \forall 1 \leq i \leq N$ are bounded for all initial conditions and for all t. Thus, $\dot{y}_i, \dot{z}_{ij}$ are bounded, $\forall 0 \leq j \leq \rho; \leq i \leq N$, implying uniform continuity of y_i and z_{ij}. Moreover, since $V_\rho(t)$ is a positive, monotonically decreasing function, its limit $V_\rho(\infty)$ is well defined and

$$
\begin{aligned}
-\int_0^\infty \dot{V}_\rho dt &= \int_0^\infty \sum_{i=1}^{N} \left\{ c_{i0} \sum_{k=1}^{p} \|y_i(\tau)\|^{2k} + \sum_{j=1}^{\rho} c_{ij} z_{ij}^2(\tau) \right\} d\tau \\
&= V_\rho(0) - V_\rho(\infty) < \infty.
\end{aligned}
$$

Thus, $y_i, z_{ij} \in L_2$. Invoking Barbalat's lemma [35], we have

$$
\begin{aligned}
&\lim_{t\to\infty} y_i(t) = 0, \\
&\lim_{t\to\infty} z_{ij}(t) = 0; \quad 0 \le j \le \rho,\ 1 \le i \le N.
\end{aligned}
$$

Furthermore $\dot{y}_i, \dot{z}_{ij}, \dot{\hat{\theta}}_i, \dot{\hat{\varsigma}}^i$ and $\dot{\hat{\beta}}_i$ are uniformly continuous. Also

$$
\lim_{t\to\infty} \int_0^t \dot{(.)}(\tau) d\tau < \infty,
$$

where $(.) \in \{y_i, z_{ij}, \hat{\theta}_i, \hat{\varsigma}^i, \hat{\beta}_i\}$. Therefore

$$
\begin{aligned}
&\lim_{t\to\infty} \dot{y}_i(t) = 0, \quad \lim_{t\to\infty} z_{ij}(t) = 0, \quad \lim_{t\to\infty} \dot{\hat{\theta}}_i(t) = 0, \\
&\lim_{t\to\infty} \dot{\hat{\varsigma}}^i(t) = 0, \quad \lim_{t\to\infty} \dot{\hat{\beta}}_i = 0, \quad \forall 0 \le j \le \rho,\ 1 \le i \le N.
\end{aligned}
$$

Since $x_{i1}, \dots, x_{i\rho}$ can be expressed as smooth functions of y_i, $z_{i1}, \dots, z_{i,\rho}$, $\hat{\theta}_i$, $\hat{\varsigma}^i$, $\hat{\beta}_i$, using (8.68), and from the uniqueness of $\underline{0}$ as an equilibrium of (8.30), and (8.31), we obtain

$$
\begin{aligned}
&\lim_{t\to\infty} y_i(t) = 0, \\
&\lim_{t\to\infty} \dot{x}_{ij}(t) = 0, \quad \forall 0 \le j \le \rho;\ 1 \le i \le N.
\end{aligned}
$$

It now follows from (8.30) and (8.31) that $\lim_{t\to\infty} x_{ij}(t) = 0$. Finally, since

$$
(y_i^t, x_i^t)^t = \phi_i(\varsigma_i), \quad i = 1, \dots, N
$$

is a diffeomorphism with $\phi_i(0) = 0$, regulation is achieved in the original coordinates, that is,

$$
\lim_{t\to\infty} \varsigma_i(t) = 0, \quad i \in \{1, \dots, N\}. \qquad \square
$$

The above design procedure proceeds independently for each subsystem after Step 0, where the virtual control x_{i1} for the y_i subsystem is obtained. Therefore, as alluded to in Remark 8.8, the degree of mismatch and the number of states for each subsystem need not be the same. Thus, the results of the above theorem are true for the general case of different degrees of mismatch and number of states among the subsystems. Furthermore, for each subsystem, three parameters $(\hat{\beta}_i, \hat{\theta}_i, \hat{\varsigma}^i)$ need to be adapted. However, if $\gamma_{ij}(0, \dots, 0, x_{i1}, \dots, x_{ij}) = 0, \forall 1 \le j \le \rho$, only the parameters $\hat{\beta}_i$ and $\hat{\varsigma}^i$ need to be updated for the ith subsystem. These points are illustrated in the example below.

Remark 8.9 The class of large-scale nonlinear systems can be further extended to systems transformable to the following *decentralized pure feedback form:*

$$
\begin{aligned}
\dot{w}_{i1} &= w_{i2},\\
&\vdots\\
\dot{w}_{i,\kappa_i-1} &= w_{i,\kappa_i},\\
\dot{w}_{i,\kappa_i} &= w_{i,\kappa_i+1} + \theta_i^t \gamma_{i0}(w_{j1}, \ldots, w_{j,\kappa_j}, w_{i,\kappa_i+1} | 1 \le j \le N),\\
\dot{w}_{i,\kappa_i+1} &= w_{i,\kappa_i+2} + \theta_i^t \gamma_{i1}(w_{j1}, \ldots, w_{j,\kappa_j}, w_{i,\kappa_i+1}, w_{i,\kappa_i+2} | 1 \le j \le N),\\
&\vdots\\
\dot{w}_{i,n_i-1} &= w_{in} + \theta_i^t \gamma_{i,n_i-\kappa_i-1}(w_{j1}, \ldots, w_{j,\kappa_j}, w_{i,\kappa_i+1}, \ldots, w_{i,n_i} | 1 \le j \le N)\\
&\quad + \{\delta_{i0}(w_i) + \theta_i^t \delta_i(w_{j1}, \ldots, w_{j,\kappa_j}, w_{i,\kappa_i+1}, \ldots, w_{i,n_i} | 1 \le j \le N)\} u_i,\\
&\quad 1 \le i \le N.
\end{aligned}
\tag{8.90}
$$

The decentralized design can be carried out in similar steps as above. However, as discussed in [20], the region of attraction in this case is not global, but asymptotic regulation and tracking are guaranteed in regions for which a priori estimates are given.

8.4.3 Simulation Example 8.3

We consider a system comprised of two subsystems with degrees of mismatch $\rho_i = 1, i = 1, 2$.

$$
\begin{aligned}
\textit{Subsystem } 1:\quad \dot{y}_{11} &= y_{12},\\
\dot{y}_{12} &= x_{11} + \theta_1(\zeta_{11} y_{11} y_{12} + \zeta_{12} y_{21}^2), \qquad (8.91)\\
\dot{x}_{11} &= u_1 + \theta_1 x_{11} y_{21}^2.\\
\textit{Subsystem } 2:\quad \dot{y}_{21} &= x_{21} + \theta_2(\zeta_{21} y_{11}^2 + \zeta_{22} y_{12} y_{21}), \qquad (8.92)\\
\dot{x}_{21} &= u_2 + \theta_2 x_{21} y_{11}^2.
\end{aligned}
$$

The system is already in the decentralized strict feedback form and hence satisfies the conditions of Theorem 8.3. Identifying with (8.30) and (8.31)

$$
\begin{aligned}
\gamma_{10} &= \zeta_{11} y_{11} y_{21} + \zeta_{12} y_{21}^2,\\
\gamma_{11} &= x_{11} y_{21}^2,\\
\gamma_{20} &= \zeta_{21} y_{11}^2 + \zeta_{22} y_{12} y_{21},\\
\gamma_{21} &= x_{21} y_{11}^2.
\end{aligned}
$$

We design a decentralized adaptive control law for the above system following the steps outlined earlier.

Step 0: Since the interconnections are bounded by quadratic polynomials in the states, from (8.33) we have $p = 2$.

We choose

$$Q_1 = \text{diag}(10, 10), \qquad Q_2 = 10, \qquad \alpha_1 = \alpha_2 = 10.$$

Solving the ARE (8.34), we obtain

$$P_1 = \begin{bmatrix} 10.9545 & 1 \\ 1 & 1.0954 \end{bmatrix}, \quad P_2 = 1.$$

The *virtual* controls obtained from (8.45) are given by

$$\begin{aligned} r_{11} &= -(y_{11} + 1.0954 y_{12})\{10 + \hat{\beta}_1(1 + y_{11}^2 + y_{12}^2)\}, \\ r_{21} &= -y_{21}\{10 + \hat{\beta}_2(1 + y_{21}^2)\} \end{aligned}$$

with adaptation laws obtained from (8.45) with $\Gamma_1 = \Gamma_2 = 1$

$$\begin{aligned} \dot{\hat{\beta}}_1 &= (y_{11} + 1.0954 y_{12})^2(1 + y_{11}^2 + y_{12}^2) \\ &\quad \times \{1 + 2(10.9545 y_{11}^2 + 1.0954 y_{12}^2 + 2 y_{11} y_{12})\} \\ &= \tau_{10}(y_{11}, y_{12}), \\ \dot{\hat{\beta}}_2 &= y_{21}^2(1 + y_{21}^2)(1 + 2 y_{21}^2) \\ &= \tau_{20}(y_{21}). \end{aligned} \tag{8.93}$$

Define $z_{i1} = x_{i1} - r_{i1}$. The terms $\nu_{i1}(y_i, z_{i1}, \hat{\beta}_i)$ and $\varphi_{i1}^{\iota}(y_i, \hat{\beta}_i)$ in (8.47) are given by

$$\begin{aligned} \nu_{11}(y_1, z_{11}, \hat{\beta}_1) &= -\frac{\partial r_{11}}{\partial y_{11}} y_{12} - \frac{\partial r_{11}}{\partial y_{12}}(z_{11} + r_{11}) - \frac{\partial r_{11}}{\partial \hat{\beta}_1} r_{10}(y_{11}, y_{12}), \\ \nu_{21}(y_2, z_{21}, \hat{\beta}_2) &= -\frac{\partial r_{21}}{\partial y_{21}}(z_{21} + r_{21}) - \frac{\partial r_{21}}{\partial \hat{\beta}_2} r_{20}(y_{21}), \\ \varphi_{11}^0(y_{11}, y_{12}, \hat{\beta}_1) &= \frac{\partial r_{11}}{\partial y_{12}}, \\ \varphi_{21}^0(y_{21} \hat{\beta}_2) &= \frac{\partial r_{21}}{\partial y_{21}}, \\ \varphi_{11}^1 &= \varphi_{21}^1 = 1. \end{aligned}$$

Since $\gamma_{i1}(0, 0, x_{i1}) = 0$, $i = 1, 2$, the unknown parameter θ_i can be lumped with the coefficients of the polynomial interconnections, which are also unknown. Thus, only adaptation in $\hat{\varsigma}^i$ is required.

Step 1: From the bounds in (8.50), we obtain

$$\begin{aligned} \varpi_{10} &= \varpi_{20} = 2, \\ \varpi_{11} &= x_{11}^2, \\ \varpi_{21} &= x_{21}^2. \end{aligned}$$

Since the degree of mismatch is one, the input appears in this step and is obtained from (8.54), that is,

$$u_1 = -\frac{z_{11}}{2} - (y_{11} + 1.0954 y_{12})\{1 + 2(10.9545 y_{11}^2 + 1.0954 y_{12}^2 + 2 y_{11} y_{12})\}$$
$$+ y_{12}\frac{\partial r_{11}}{\partial y_{11}} + (z_{11} + r_{11})\frac{\partial r_{11}}{\partial y_{12}} + r_{10}\frac{\partial r_{11}}{\partial \hat{\beta}_1} - \hat{\varsigma}^1 z_{11}\left\{\left(\frac{\partial r_{11}}{\partial y_{12}}\right)^2 + \frac{x_{11}^2}{2}\right\}, \quad (8.94)$$

$$u_2 = -\frac{z_{21}}{2} - y_{21}(1 + 2y_{21}^2) + (z_{21} + r_{21})\frac{\partial r_{21}}{\partial y_{21}} + r_{20}\frac{\partial r_{21}}{\partial \hat{\beta}_2}$$
$$- \hat{\varsigma}^2 z_{21}\left\{\left(\frac{\partial r_{21}}{\partial y_{21}}\right)^2 + \frac{x_{21}^2}{2}\right\} \quad (8.95)$$

with adaptation for $\hat{\varsigma}^i$ given by (8.57) as

$$\dot{\hat{\varsigma}}^1 = z_{11}^2\left\{\left(\frac{\partial r_{11}}{\partial y_{12}}\right)^2 + \frac{1}{2}x_{11}^2\right\},$$
$$\dot{\hat{\varsigma}}^2 = z_{21}^2\left\{\left(\frac{\partial r_{21}}{\partial y_{21}}\right)^2 + \frac{1}{2}x_{21}^2\right\}. \quad (8.96)$$

The closed-loop system was simulated for the following initial conditions:

$$y_{11}(0) = 2.0, \quad y_{12}(0) = -2.0, \quad y_{21}(0) = 2.0,$$
$$x_{11}(0) = 1.0, \quad x_{21}(0) = -2.0, \quad (8.97)$$

$$\hat{\beta}_1(0) = \hat{\beta}_2(0) = \hat{\varsigma}^1(0) = \hat{\varsigma}^2(0) = 0. \quad (8.98)$$

The nominal values for θ_i, ς_{ij}, $1 \le i, j \le 2$, were chosen as 1.0. The closed-loop responses for the two subsystems are plotted in Figs. 8.7 and 8.8, respectively. The adaptation of parameters $\hat{\beta}_i$, $\hat{\varsigma}^i$ for the two subsystems is shown in Figs. 8.9 and 8.10, respectively.

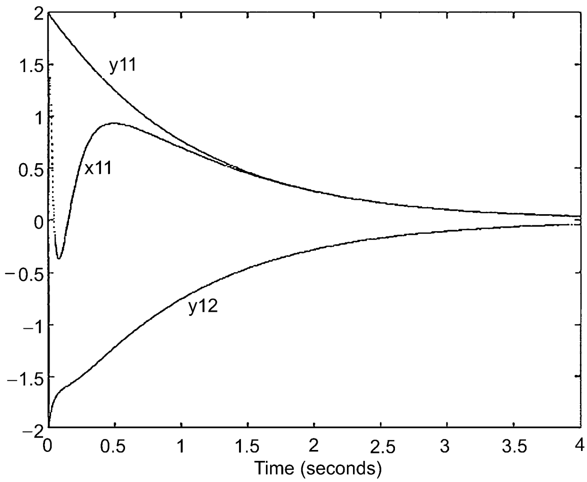


Fig. 8.7 Subsystem 1 closed-loop response

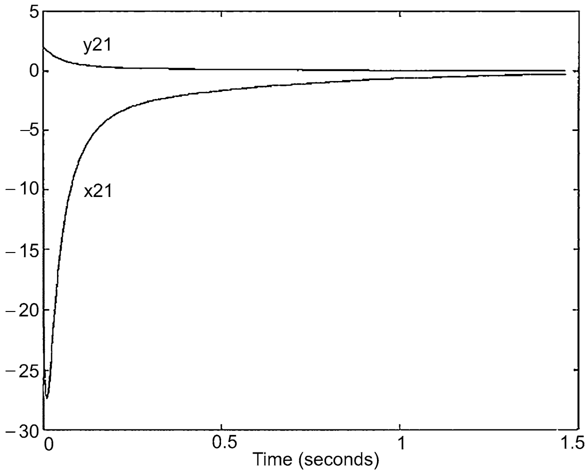


Fig. 8.8 Subsystem 2 closed-loop response

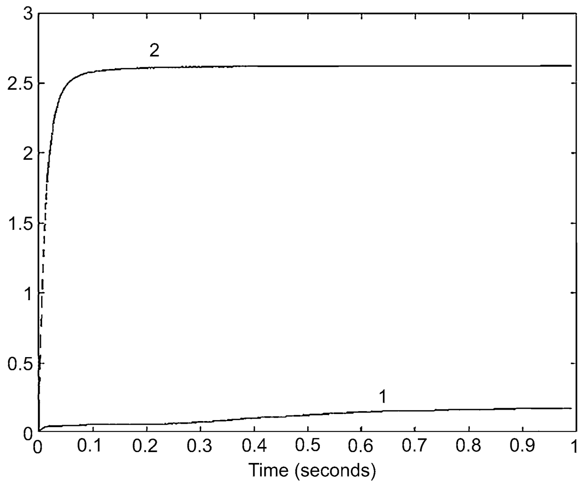


Fig. 8.9 Adaptation of $\hat{\beta}_1$ and $\hat{\beta}_2$

Remark 8.10 It is interesting to note that (8.94) and (8.95) maintain a robust performance to a wide class of perturbations in the system dynamics, as long as the interconnections are bounded by a quadratic polynomial in y_i. In this sense, the controller guarantees robustness to inaccurately modeled dynamics.

8.4.4 Simulation Example 8.4

Consider the when the system of (8.91) and (8.92) is modified to the following:

$$\begin{aligned} \textit{Subsystem } 1: \quad \dot{y}_{11} &= y_{12}, \\ \dot{y}_{12} &= x_{11} + \theta_1[(\zeta_{11} y_{11} y_{12} + \zeta_{12} y_{21}^2 + \zeta_{13} y_{12} y_{21}) \\ &\qquad + \zeta_{14} y_{12} + \zeta_{15} y_{11} \sin(y_{21})], \\ \dot{x}_{11} &= u_1 + \theta_1 x_{11}(y_{21}^2 + y_{21} y_{11}). \end{aligned} \tag{8.99}$$

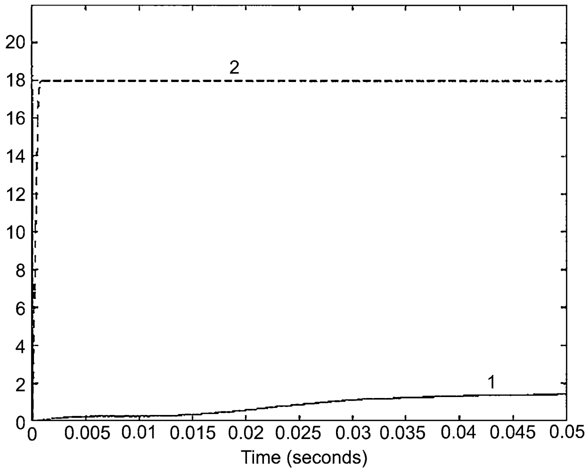


Fig. 8.10 Adaptation of $\hat{\zeta}_1$ and $\hat{\zeta}_2$

$$\begin{aligned} \textit{Subsystem 2:}\quad \dot{y}_{21} &= x_{21} + \theta_2[\zeta_{21} y_{11}^2 + \zeta_{22} y_{12} y_{21} + \zeta_{23} y_{11} y_{12} \\ &\quad + \zeta_{24} y_{12}^2 + \zeta_{25} \cos(y_{11}) \sin(y_{12})], \\ \dot{x}_{21} &= u_2 + \theta_2 x_{21} (y_{11}^2 + y_{12}^2). \end{aligned} \tag{8.100}$$

It is a straightforward task to show that the controller developed above will still maintain a robust performance. The adaptation parameters self-adjust to new (possibly higher) values to incorporate the additional perturbations.

Remark 8.11 An important issue in decentralized control design is redesigning decentralized controllers for the original subsystems if more subsystems are appended to the large-scale system. The design methodology proposed here obviates any need for controller redesign for the original subsystems if the order of the nonlinearities in the interconnections, due to the appended subsystem, is less than or equal to that of original system. This is generally true for most practical applications, where the interconnected subsystems are dynamically similar. In any case, defining p in (8.33) as the maximum possible order of all current and future interconnections will ensure that the same decentralized controller works for the modified subsystems.

8.4.5 Simulation Example 8.5

For the simulation examples considered above, we append a third subsystem to the original system given by (8.91) and (8.92). The new system is given by

$$\begin{aligned} \textit{Subsystem 1:}\quad \dot{y}_{11} &= y_{12}, \\ \dot{y}_{12} &= x_{11} + \theta_1(\zeta_{11} y_{11} y_{21} + \zeta_{12} y_{21}^2 + \zeta_{13} y_{31}^2 + \zeta_{14} y_{21} y_{31}), \\ \dot{x}_{11} &= u_1 + \theta_1 x_{11} (y_{21}^2 + y_{31}^2). \end{aligned} \tag{8.101}$$

$$\begin{aligned} \textit{Subsystem 2:} \quad & \dot{y}_{21} = x_{21} + \theta_2(\zeta_{21} y_{11}^2 + \zeta_{22} y_{12} y_{21} + \zeta_{23} y_{11} y_{31}), \\ & \dot{x}_{21} = u_2 + \theta_2 x_{21}(y_{11}^2 + y_{11} y_{31}). \end{aligned} \tag{8.102}$$

$$\begin{aligned} \textit{Subsystem 3:} \quad & \dot{y}_{31} = x_{31} + \theta_3(\zeta_{31} y_{11}^2 + \zeta_{32} y_{21} y_{31}), \\ & \dot{x}_{31} = u_3 + \theta_3 x_{31} y_{21}^2. \end{aligned} \tag{8.103}$$

The decentralized control laws for the first and second subsystems are given by (8.94), (8.95) with adaptation laws (8.93) and (8.96). For the third subsystem

$$r_{31} = -y_{31}\{10 + \hat{\beta}_3(1 + y_{31}^2)\}.$$

Defining $z_{31} = x_{31} - r_{31}$, the control law for u_3 is given by

$$\begin{aligned} u_3 = & -\frac{z_{31}}{2} - y_{31}(1 + 2y_{31}^2) + (z_{31} + r_{31})\frac{\partial r_{31}}{\partial y_{31}} \\ & + r_{30}\frac{\partial r_{31}}{\partial \hat{\beta}_3} - \hat{\varsigma}^3 z_{31}\left\{\left(\frac{\partial r_{31}}{\partial y_{31}}\right)^2 + \frac{x_{31}^2}{2}\right\} \end{aligned} \tag{8.104}$$

with adaptation laws

$$\begin{aligned} \dot{\hat{\beta}}_3 &= y_{31}^2(1 + y_{31}^2)(1 + 2y_{31}^2) = r_{30}(y_{31}), \\ \dot{\hat{\varsigma}}^3 &= z_{31}^2\left\{\left(\frac{\partial r_{31}}{\partial y_{31}}\right)^2 + \frac{1}{2}x_{31}^2\right\}. \end{aligned} \tag{8.105}$$

The initial conditions used for simulations are given by (8.97) along with

$$y_{31}(0) = -2.0, \quad x_{31}(0) = 1.0.$$

The closed-loop responses are plotted in Figs. 8.11, 8.12 and 8.13. The plots validate the fact that the same control laws maintain a robust performance even when more subsystems are appended to the original system.

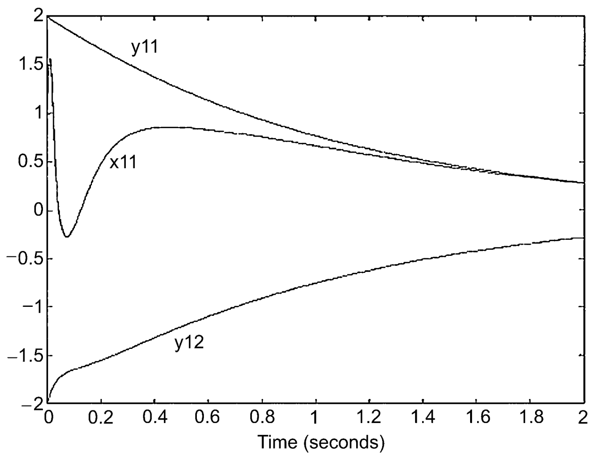


Fig. 8.11 Subsystem 1 closed-loop response

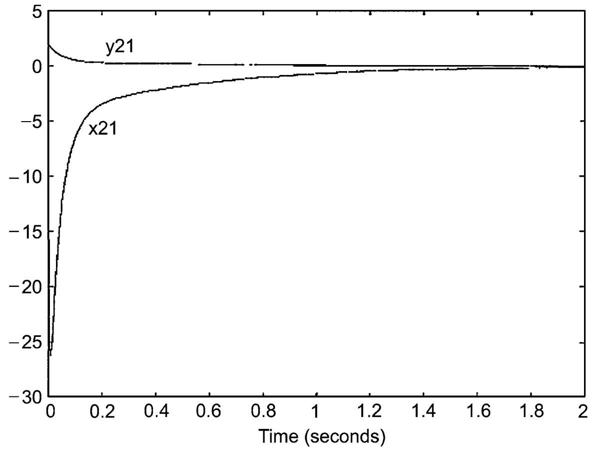


Fig. 8.12 Subsystem 2 closed-loop response

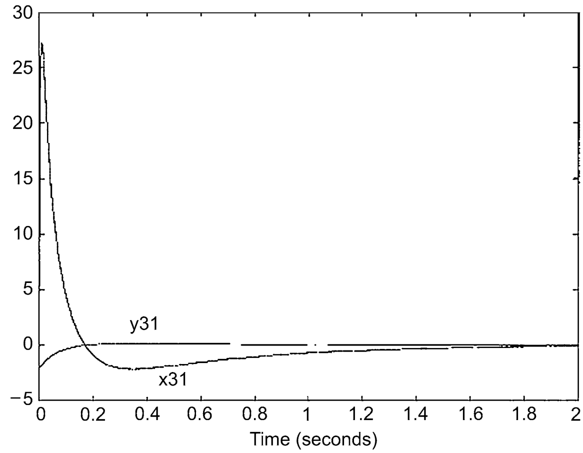


Fig. 8.13 Subsystem 3 closed-loop response

8.4.6 Tracking Behavior

The objective of the decentralized state tracking problem is to track a given reference model. To avoid notational complexity, we consider the case of uniform degrees of mismatch and number of states among the subsystems. Since the isolated subsystems are in the Brunovsky form, we choose a similar model for each subsystem of the interconnected system given by (8.30) and (8.31), that is,

$$\begin{aligned}
\dot{y}_{mi} &= A_i y_{mi} + B_i x_{mi1}, \\
\dot{x}_{mi1} &= x_{mi2}, \\
&\vdots \\
\dot{x}_{mi,n-\kappa-1} &= x_{mi,n-\kappa}, \\
\dot{x}_{mi,n-\kappa} &= K_i [y_{mi}^t x_{mi}^t]^t + b_{mi} r_i, \quad 1 \le i \le N,
\end{aligned} \tag{8.106}$$

where r_i is an external (reference) input to the ith subsystem model. We first define the state tracking errors $\tilde{y}_i = y_i - y_{mi}$ and $\tilde{x}_i = x_i - \tilde{x}_{mi}$. The design procedure in the previous section is applied to tracking error system

$$\begin{aligned}
\dot{\tilde{y}}_i &= A_i \tilde{y}_i + B_i \tilde{x}_{i1} + B_i \theta_i^t \gamma_{i0}(y_1, \ldots, y_N), \\
\dot{\tilde{x}}_{i1} &= \tilde{x}_{i2} + \theta_i^t \gamma_{i1}(y_1, \ldots, y_N, x_{i1}), \\
&\vdots \\
\dot{\tilde{x}}_{i,n-\kappa-1} &= \tilde{x}_{i,n-\kappa} + \theta_i^t \gamma_{i,n-\kappa-1}(y_1, \ldots, y_N, x_{i1}, \ldots, x_{i,n-\kappa-1}), \\
\dot{\tilde{x}}_{i,n-\kappa} &= v_i(y_i, x_i) + \theta_i^t \gamma_{i,n-\kappa}(y_1, \ldots, y_N, x_{i1}, \ldots, x_{i,n-\kappa}) \\
&\quad + \delta_i(y_i, x_i) u_i - K_i [y_{mi}^t x_{mi}^t]^t + b_{mi} r_i, \quad 1 \le i \le N
\end{aligned} \tag{8.107}$$

with certain distinctions which are outlined below.

Step 0: View $\tilde{x}_{i1}$ as the virtual control for the $\tilde{y}_i$ subsystem in (8.108). Therefore, we have

$$\begin{aligned}
\tilde{x}_{i1} &= -\alpha_i B_i^t P_i \tilde{y}_i - \hat{\beta}_i B_i^t P_i \tilde{y}_i \{1 + (\tilde{y}_i^t \tilde{y}_i)^{(p-1)}\} \\
&= r_{i1}(\tilde{y}_i, \hat{\beta}_i)
\end{aligned} \tag{8.108}$$

with adaptation law for $\hat{\beta}_i$ given by

$$\begin{aligned}
\dot{\hat{\beta}}_i &= \Gamma_i \| B_i^t P_i \tilde{y}_i \|^2 [1 + \|\tilde{y}_i\|^{2(p-1)}] \sum_{k=1}^{p} k (\tilde{y}_i^t P_i \tilde{y}_i)^{(k-1)} - \Gamma_{i1} \sigma_{i1} \hat{\beta}_i \\
&= \tau_{i0}(\tilde{y}_i),
\end{aligned} \tag{8.109}$$

where $\sigma_{i1} > 0$ incorporates the "σ*-modification*" to avoid parameter drift as originally proposed in [14]. The properties of the virtual control law (8.108) can be analyzed using the following Lyapunov function:

$$V_0 = \sum_{i=1}^{N} \left\{ \sum_{k=1}^{p} (\tilde{y}_i^t P_i \tilde{y}_i)^k + \Gamma_{i1}^{-1} (\hat{\beta}_i - \beta_i^*)^2 \right\} \tag{8.110}$$

with Γ_i a positive constant. In this case, the interconnection terms can be bounded using the inequality [36]

$$(|a_1| + |a_2|)^k \le 2^{k-1} (|a_1|^k + |a_2|^k) \tag{8.111}$$

as follows:

$$\begin{aligned}
&\| \theta_i^t \gamma_{i\iota}(y_1, \ldots, y_N, x_{i1} \ldots x_{i\iota}) - \theta_i^t \gamma_{i\iota}(0, \ldots, 0, x_{i1} \ldots x_{i\iota}) \| \\
&\quad \le \sum_{k=1}^{p_{i\iota}} \sum_{l=1}^{N} \eta_{i\iota}^k \zeta_{il\iota}^k (\|y_{ml}\| + \|\tilde{y}_l\|)^k \\
&\quad \le \sum_{k=1}^{p_{i\iota}} \sum_{l=1}^{N} 2^{p_{i\iota}-1} \eta_{i\iota}^k \zeta_{il\iota}^k \|y_{ml}\|^k + \sum_{l=1}^{N} \sum_{k=1}^{p_{i\iota}} 2^{p_{i\iota}-1} \eta_{i\iota}^k \zeta_{il\iota}^k \|\tilde{y}_l\|^k
\end{aligned}$$

$$\le N\zeta_{\max}^{\iota}\overline{y}_m\hat{\eta}_{i\iota}(x_{i1}\dots x_{i\iota})$$
$$\le \sum_{l=1}^{N}\sum_{k=1}^{p_{i\iota}} 2^{p_{i\iota}-1}\eta_{i\iota}^{k}\zeta_{il\iota}^{k}\|\tilde{y}_l\|^k, \tag{8.112}$$

where

$$\zeta_{\max}^{\iota} = \max_{i,j\in\{1,\dots,N\};k\in\{1,\dots,p\}}\{\zeta_{ij\iota}^{k}\}2^{p_{i\iota}-1},$$
$$\overline{y}_m = \max_{k\in\{1,\dots,p\},l\in\{1,\dots,N\},t}\|y_{ml}(t)\|^k,$$
$$\hat{\eta}_{i\iota}(x_{i1}\dots x_{i\iota}) = \sum_{j=1}^{p_{i\iota}}\eta_{i\iota}^{j},$$
$$\hat{\eta}_{i0} = p_{i0}.$$

$\dot{V}_0$ is now obtained following manipulations parallel to Step 0 in the regulation case with an additional term due to $\zeta_{\max}^{\iota}$.

Since, $\tilde{x}_{i1}$ is only the virtual control, define the error $\tilde{z}_{i1} = \tilde{x}_{i1} - \tilde{r}_{i1}(\tilde{y}_i, \hat{\beta}_i)$. In Step 1, the dynamics for the error term $\tilde{z}_{i1}$ are formulated, and the virtual control for $\tilde{x}_{i2}$ is obtained. In this case, due to the additional first term in (8.112), we also need

$$\max_{\iota\in\{1\dots n-\kappa\}}(N\zeta_{\max}^{\iota}\overline{y}_m)^2 = \zeta_{\max}.$$

We denote this estimate by $\hat{\zeta}_{\max}$.

The following Lyapunov function is used to analyze the properties of the virtual control designed in Step 1:

$$V_c = V_0 + \sum_{i=1}^{N}\{\tilde{z}_{i1}^2 + (\hat{\theta}_i - \theta_i)^t\Gamma_{i2}^{-1}(\hat{\theta}_i - \theta_i)$$
$$+ \Gamma_{i3}^{-1}(\hat{\varsigma}_i - \varsigma^*)^2 + \Gamma_{i4}^{-1}(\hat{\zeta}_{\max} - \zeta_{\max})^2\} \tag{8.113}$$

with $\Gamma_{ik} > 0$, $k = 2, 3, 4$. At each step m $(2 \le m \le n - \kappa - 1)$ of the design procedure, virtual control laws for $\tilde{x}_{i,m+1}$ are designed, and the error $\tilde{z}_{i,m+1}$ is defined as the difference between $\tilde{x}_{i,m+1}$ and the virtual control. The Lyapunov function at the mth step is given by $V_m = V_{m-1} + \sum_{i=1}^{N}\tilde{z}_{im}^2$. As with $\hat{\theta}_i$ and $\hat{\varsigma}^i$, tuning functions are designed at each step to avoid overparameterization in designing the adaptation law for $\hat{\zeta}_{\max}$. The tuning functions at step m $(2 \le m \le n - \kappa)$ are given by

$$\tau_{im} = \tau_{i,m-1} + \tilde{z}_{im}\sum_{\iota=0}^{m}\varphi_{im}^{\iota}\gamma_{i\iota}(0,\dots,0,x_{i1},\dots,x_{im}), \tag{8.114}$$

$$\epsilon_{im} = \epsilon_{i,m-1} + \frac{1}{2}\tilde{z}_{im}^2\sum_{\iota=0}^{m}\varpi_{i\iota}\|\varphi_{im}^{\iota}\|^2, \tag{8.115}$$

$$\omega_{im} = \omega_{i,m-1} + \frac{\tilde{z}_{im}^2}{2\varrho m}\left(\sum_{\iota=0}^{m}\|\hat{\eta}_{i\iota}\varphi_{im}^{\iota}\|\right)^2 \tag{8.116}$$

with

$$\tau_{i1}(\tilde{y}_i, \tilde{z}_{i1}, \hat{\beta}_i) = \tilde{z}_{i1} \sum_{\iota=0}^{1} \varphi_{i1}^{\iota}(\tilde{y}_i, \hat{\beta}_i) \cdot \gamma_{i\iota}(0, \dots, 0, x_{i1}), \tag{8.117}$$

$$\epsilon_{i1}(\tilde{y}_i, \tilde{z}_{i1}, \hat{\beta}_i) = \frac{\tilde{z}_{i1}^2}{2} \sum_{\iota=0}^{1} \varpi_{i\iota} \|\varphi_{i1}^{\iota}(\tilde{y}_i, \hat{\beta}_i)\|^2, \tag{8.118}$$

$$\omega_{i1}(\tilde{y}_i, \tilde{z}_{i1}, \hat{\beta}_i) = \frac{\tilde{z}_{im}^2}{2\varrho_1} \left(\sum_{\iota=0}^{1} \|\hat{\eta}_{i\iota} \varphi_{i1}^{\iota}(\tilde{y}_i, \hat{\beta}_i)\| \right)^2. \tag{8.119}$$

The virtual control for $\tilde{x}_{i,m+1}$ at the mth step is given by

$$\begin{aligned}
\tilde{x}_{i,m+1} = -\Bigg[& \tilde{z}_{i,m-1} + c_{im}\tilde{z}_{im} + \nu_{im} \\
& + \sum_{\iota=0}^{m} \varphi_{im}^{\iota} \gamma_{i\iota}(0, \dots, 0, x_{i1}, x_{i2}, \dots, x_{im}) \sum_{j=2}^{m-1} \tilde{z}_{ij} \psi_{ij} \\
& + \frac{1}{2} \tilde{z}_{im} \sum_{\iota=0}^{m} \varpi_{i\iota} \|\varphi_{im}^{\iota}\|^2 \sum_{j=2}^{m-1} \tilde{z}_{ij} \xi_{ij} \\
& + \frac{\tilde{z}_{im}}{2\varrho_m} \left(\sum_{\iota=0}^{m} \|\hat{\eta}_{i\iota} \varphi_{im}^{\iota}\| \right)^2 \sum_{j=2}^{m-1} \tilde{z}_{ij} \vartheta_{ij} + \frac{\hat{\varsigma}^i}{2} \tilde{z}_{im} \sum_{\iota=0}^{m} \varpi_{i\iota} \|\varphi_{im}^{\iota}\|^2 \\
& + \hat{\theta}_i^t \sum_{\iota=0}^{m} \varphi_{im}^{\iota}(\tilde{y}_i, \tilde{z}_{i1}, \dots, \tilde{z}_{i,m-1}) \gamma_{i\iota}(0, \dots, 0, x_{i1}, \dots, x_{im}) \\
& + \hat{\zeta}_{\max} \frac{\tilde{z}_{im}}{2\varrho_m} \left(\sum_{\iota=0}^{m} \hat{\eta}_{i\iota} \|\varphi_{im}^{\iota}\| \right)^2 + \tau_{im}\psi_{im} + \epsilon_{im}\xi_{im} + \omega_{im}\vartheta_{im} \Bigg] \\
= {} & r_{i,m+1}(\tilde{y}_i, \tilde{z}_{i1}, \dots, \tilde{z}_{im}, \hat{\beta}_i, \hat{\theta}_i, \hat{\varsigma}^i, \hat{\zeta}_{\max}),
\end{aligned} \tag{8.120}$$

where the notation used is defined analogously to the regulation case, see (8.59) for example, with the arguments replaced by the error terms.

In (8.120), ϑ_{im} is the coefficient of $\dot{\hat{\zeta}}_{\max}$ in the dynamics of $\tilde{z}_{im}$, and c_{im} and ϱ are positive design parameters effecting the magnitude of the tracking error.

The actual control input u_i appears in the $n - \kappa$ $(= \rho)$th step. The decentralized tracking control law is given by

$$\begin{aligned}
u_i = \frac{1}{\delta_i(y_i, x_i)} \Bigg[& -v_i(x_i) + K_i[y_{mi}^t x_{mi}^t]^t + b_{mi} r_i - \tilde{z}_{i,\rho-1} - c_{i\rho}\tilde{z}_{i\rho} - \nu_{i,\rho} \\
& - \frac{\hat{\varsigma}^i}{2} \tilde{z}_{i\rho} \sum_{\iota=0}^{\rho} \varpi_{i\iota} \|\varphi_{i\rho}^{\iota}\|^2 - \hat{\theta}_i^t \sum_{\iota=0}^{\rho} \varphi_{i\rho}^{\iota} \gamma_{i\iota}(0, \dots, 0, x_i)
\end{aligned}$$

$$
\begin{aligned}
&- \hat{\zeta}_{\max} \frac{\tilde{z}_{i\rho}}{2\varrho_\rho} \left(\sum_{\iota=0}^{\rho} \|\hat{\eta}_{i\iota} \varphi_{i\rho}^{\iota}\| \right)^2 - \tau_{i\rho}\psi_{i\rho} - \epsilon_{i\rho}\xi_{i\rho} - \omega_{i\rho}\vartheta_{i\rho} \\
&- \sum_{\iota=0}^{\rho} \varphi_{i\rho}^{\iota} \gamma_{i\iota}(0, \ldots, 0, x_i) \sum_{j=2}^{\rho-1} \tilde{z}_{ij}\psi_{ij} - \frac{1}{2}\tilde{z}_{i\rho} \sum_{\iota=0}^{\rho} \varpi_{i\iota} \|\varphi_{i\rho}^{\iota}\|^2 \sum_{j=2}^{\rho-1} \tilde{z}_{ij}\xi_{ij} \\
&- \frac{\tilde{z}_{i\rho}}{2\varrho_\rho} \left(\sum_{\iota=0}^{\rho} \|\hat{\eta}_{i\iota} \varphi_{im}^{\iota}\| \right)^2 \sum_{j=2}^{\rho-1} \tilde{z}_{ij}\vartheta_{ij} \Bigg]
\end{aligned} \tag{8.121}
$$

with the following adaptation laws:

$$
\begin{aligned}
\dot{\hat{\theta}}_i &= \Gamma_{i2}\tau_{i\rho} - \sigma_{i2}\Gamma_{i2}\hat{\theta}_i \\
&= \Gamma_{i2}\tau_{i,\rho-1} + \Gamma_{i2}\tilde{z}_{i\rho} \sum_{\iota=0}^{\rho} \varphi_{i\rho}^{\iota}\gamma_{i\iota}(0, \ldots, 0, x_i) - \sigma_{i2}\Gamma_{i2}\hat{\theta}_i,
\end{aligned} \tag{8.122}
$$

$$
\begin{aligned}
\dot{\hat{\varsigma}}^i &= \Gamma_{i3}\epsilon_{i\rho} - \sigma_{i3}\Gamma_{i3}\hat{\varsigma}^i \\
&= \Gamma_{i3}\epsilon_{i,\rho-1} + \frac{1}{2}\Gamma_{i3}\tilde{z}_{i\rho}^2 \sum_{\iota=0}^{\rho} \varpi_{i\iota}\|\varphi_{i\rho}^{\iota}\|^2 - \sigma_{i3}\Gamma_{i3}\hat{\varsigma}^i,
\end{aligned} \tag{8.123}
$$

$$
\begin{aligned}
\dot{\hat{\zeta}}_{\max} &= \Gamma_{i4}\omega_{i\rho} - \sigma_{i4}\Gamma_{i4}\hat{\zeta}_i \\
&= \Gamma_{i4}\omega_{i,\rho-1} + \frac{\Gamma_{i4}\tilde{z}_{i\rho}}{2\sigma_\rho} \left(\sum_{\iota=0}^{\rho} \|\hat{\eta}_{i\iota}\varphi_{i\rho}^{\iota}\| \right)^2 - \sigma_{i4}\Gamma_{i4}\hat{\zeta}_i,
\end{aligned} \tag{8.124}
$$

where the tuning functions τ_{ij}, ϵ_{ij}, and ω_{ij}, $2 \le j \le \rho$ are given by (8.114), (8.115), and (8.116), respectively. In (8.122)–(8.124), σ_{ik}'s are the σ-modification parameters.

The following theorem states the stability and tracking properties of the proposed decentralized tracking control, the proof of which is left as an exercise.

Theorem 8.4 *The control input* (8.121) *along with the adaptation laws* (8.109) *and* (8.122)–(8.124) *results in the global uniform boundedness of the error system* $[\tilde{y}_i, \tilde{x}_i, \hat{\beta}_i, \hat{\theta}_i, \hat{\varsigma}^i, \hat{\zeta}_{\max}]$, $1 \le i \le N$, *with respect to a compact set around the origin. Furthermore, the error* $\tilde{y}_i$ *can be made arbitrarily small by choosing the design parameters appropriately.*

Remark 8.12 The control effort at each step (k) is a function of $1/\sigma_k$ and c_{ik}. Thus, the choice of σ_k, c_{ik} illustrates a tradeoff between the magnitude of the tracking error and the control effort applied. It holds that the conclusions of the above theorem are true for the case of nonuniform degrees of mismatch and number of states among the subsystems. In the sequel, this will be apparent.

8.4.7 Simulation Example 8.6

We consider the example system given by (8.91) and (8.92). The parameters of the reference model (8.106) are given as

$$K_1 = [-27\ -27\ -9], \quad K_2 = [-9\ -6],$$
$$b_{m1} = [0\ 0\ 20]^t, \quad b_{m2} = [0\ 20]^t.$$

The reference signals to be tracked are $r_1(t) = r_2(t) = \sin(3t)$.

Step 0: Using the same α_i, P_i, and Q_i as in simulation example 8.3, the virtual controls in this step are designed using (8.108) as

$$\begin{aligned} r_{11} &= -(\tilde{y}_{11} + 1.0954\tilde{y}_{12})\{10 + \hat{\beta}_1(1 + \tilde{y}_{11}^2 + \tilde{y}_{12}^2)\}, \\ r_{21} &= -\tilde{y}_{21}\{10 + \hat{\beta}_2(1 + \tilde{y}_{21}^2)\}. \end{aligned} \tag{8.125}$$

For adaptation, we choose the adaptation gains $\Gamma_{ij} = 2$, $\sigma_{ij} = 0.5$. The adaptation for $\hat{\beta}_i$ given by (8.109), i.e.,

$$\begin{aligned} \dot{\hat{\beta}}_1 &= 2(\tilde{y}_{11} + 1.0954\tilde{y}_{12})^2(1 + \tilde{y}_{11}^2 + \tilde{y}_{12}^2) \\ &\quad \times \{1 + 2(10.9545\tilde{y}_{11}^2 + 1.0954\tilde{y}_{12}^2 + \tilde{2}y_{11}\tilde{y}_{12})\} - \hat{\beta}_1 \\ &= \tau_{10}(\tilde{y}_{11}, \tilde{y}_{12}), \\ \dot{\hat{\beta}}_2 &= 2\tilde{y}_{21}^2(1 + \tilde{y}_{21}^2)(1 + 2\tilde{y}_{21}^2) - \hat{\beta}_2 \\ &= \tau_{20}(\tilde{y}_{21}). \end{aligned} \tag{8.126}$$

Step 1: Define $\tilde{z}_{i1} = \tilde{x}_{i1} - r_{i1}$; ν_{i1}, φ_{i1}^{ι}, and $\varpi_{i\iota}$ are the same as in (8.91) and (8.92) with y_i and z_i replaced by $\tilde{y}_i$ and $\tilde{z}_i$, respectively. Since the degree of mismatch is one, the control input for each subsystem appears in this step and is obtained from (8.121) (using $\rho_j = 1.0$), i.e.,

$$\begin{aligned} u_2 &= -\frac{\tilde{z}_{11}}{2} - (\tilde{y}_{11} + 1.0954\tilde{y}_{12}) \\ &\quad \times \{1 + 2(10.9545\tilde{y}_{11}^2 + 1.0954\tilde{y}_{12}^2 + \tilde{2}y_{11}\tilde{y}_{12})\} \\ &\quad + \tilde{y}_{12}\frac{\partial r_{11}}{\partial \tilde{y}_{11}} + (z_{11} + r_{11})\frac{\partial r_{11}}{\partial \tilde{y}_{12}} \\ &\quad + r_{10}\frac{\partial r_{11}}{\partial \hat{\beta}_1} - \hat{\varsigma}^1\tilde{z}_{11}\left\{\left(\frac{\partial r_{11}}{\partial \tilde{y}_{12}}\right)^2 + \frac{x_{11}^2}{2}\right\} \\ &\quad - \hat{\zeta}_{\max}^1\frac{\tilde{z}_{11}}{2}\left(x_{11} - \frac{\partial r_{11}}{\partial \tilde{y}_{12}}\right)^2 + 27y_{m11} \\ &\quad + 27y_{m12} + 9x_{m11} - 20.0\sin(3t), \\ u_2 &= -\frac{\tilde{z}_{21}}{2} - \tilde{y}_{21}(1 + 2\tilde{y}_{21}^2) + (\tilde{z}_{21} + r_{21})\frac{\partial r_{21}}{\partial \tilde{y}_{21}} \end{aligned} \tag{8.127}$$

$$+ r_{20}\frac{\partial r_{21}}{\partial \hat{\beta}_2} - \hat{\varsigma}^2 \tilde{z}_{21}\left\{\left(\frac{\partial r_{21}}{\partial \tilde{y}_{21}}\right)^2 + \frac{x_{21}^2}{2}\right\}$$
$$- \hat{\zeta}_{\max}^2 \frac{\tilde{z}_{21}}{2}\left(x_{21} - \frac{\partial r_{21}}{\partial \tilde{y}_{21}}\right)^2 + 9y_{m21}$$
$$+ 6x_{m21} - 20.0\sin(3t)$$

with adaptation given by (8.123) and (8.124), i.e.,

$$\begin{aligned}
\dot{\hat{\varsigma}}^1 &= 2\tilde{z}_{11}^2\left\{\left(\frac{\partial r_{11}}{\partial \tilde{y}_{12}}\right)^2 + \frac{1}{2}x_{11}^2\right\} - \hat{\varsigma}^1, \\
\dot{\hat{\varsigma}}^2 &= 2\tilde{z}_{21}^2\left\{\left(\frac{\partial r_{21}}{\partial \tilde{y}_{21}}\right)^2 + \frac{1}{2}x_{21}^2\right\} - \hat{\varsigma}^2, \\
\dot{\hat{\zeta}}_{\max}^1 &= \tilde{z}_{11}^2\left(x_{11} - \frac{\partial r_{11}}{\partial \tilde{y}_{12}}\right)^2 - \hat{\zeta}_{\max}^1, \\
\dot{\hat{\zeta}}_{\max}^2 &= \tilde{z}_{21}^2\left(x_{21} - \frac{\partial r_{21}}{\partial \tilde{y}_{21}}\right)^2 - \hat{\zeta}_{\max}^2.
\end{aligned} \tag{8.128}$$

The initial conditions used are given by (8.97), along with $\hat{\zeta}_{\max}^1(0) = \hat{\zeta}_{\max}^2(0) = 0$. The closed-loop responses along with the reference model states are plotted in Figs. 8.14–8.18. From the plots, we see that the tracking error stays bounded.

The adaptation of parameters $\hat{\beta}_i$, $\hat{\varsigma}^i$, and $\zeta_{\max}^i$ are shown in Figs. 8.19, 8.20, and 8.21, respectively.

Remark 8.13 As before, it can be shown that the decentralized tracking control law maintains robustness to perturbations in the system dynamics as long as the order of the interconnections remains the same. Also, the same control law can be used if additional subsystems are appended to the original system.

Fig. 8.14 States y_{11} and y_{m11}

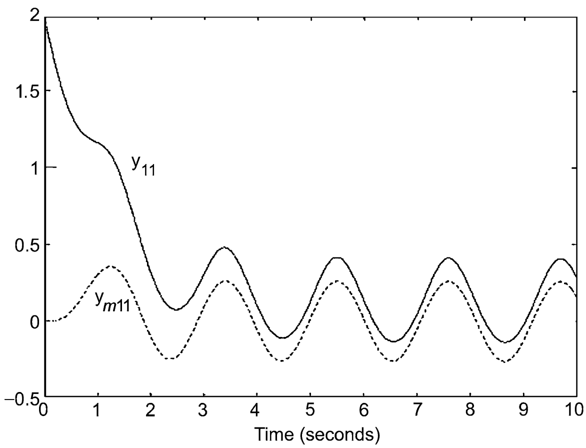


Fig. 8.15 States y_{12} and y_{m12}

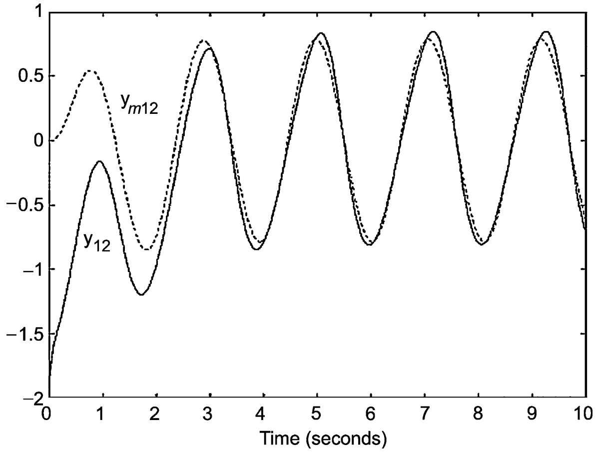


Fig. 8.16 States x_{11} and x_{m11}

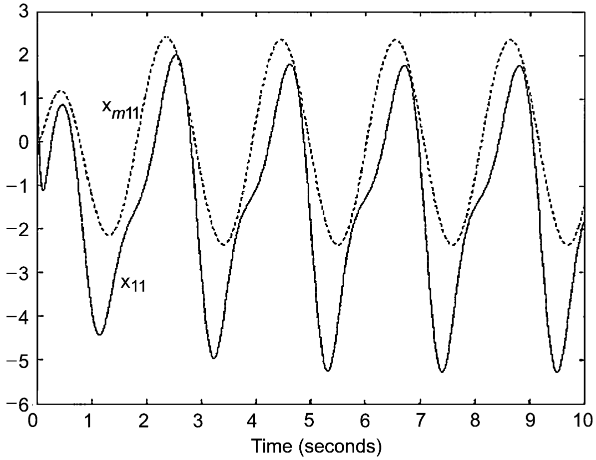
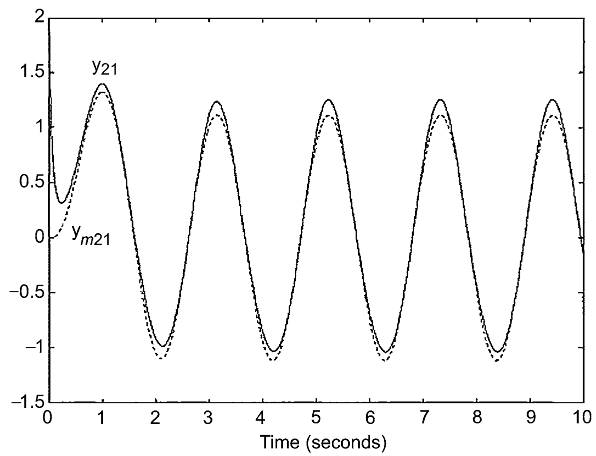


Fig. 8.17 States y_{21} and y_{m21}



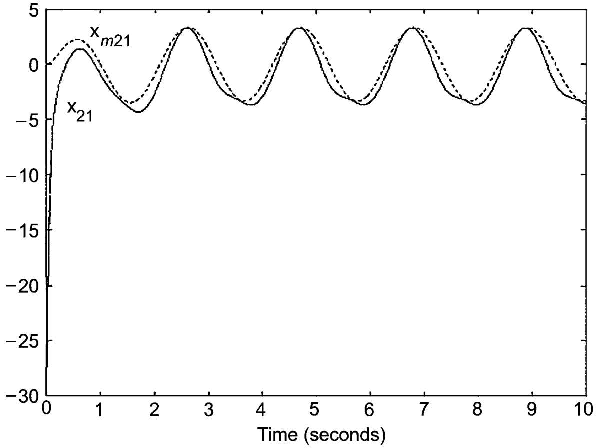


Fig. 8.18 States x_{21} and x_{m21}

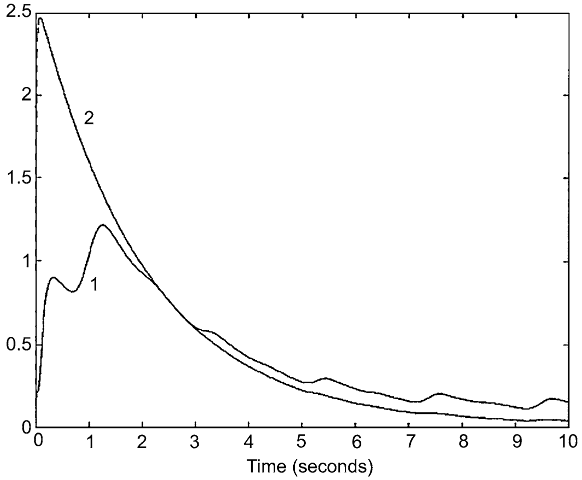


Fig. 8.19 Adaptation of $\hat{\beta}_1$ and $\hat{\beta}_2$

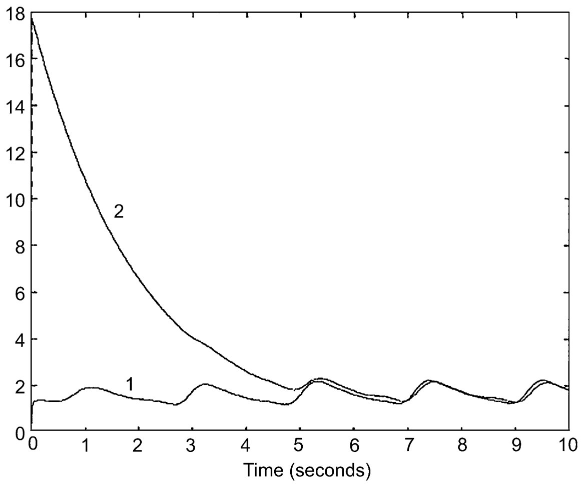


Fig. 8.20 Adaptation of ζ^1 and ζ^2

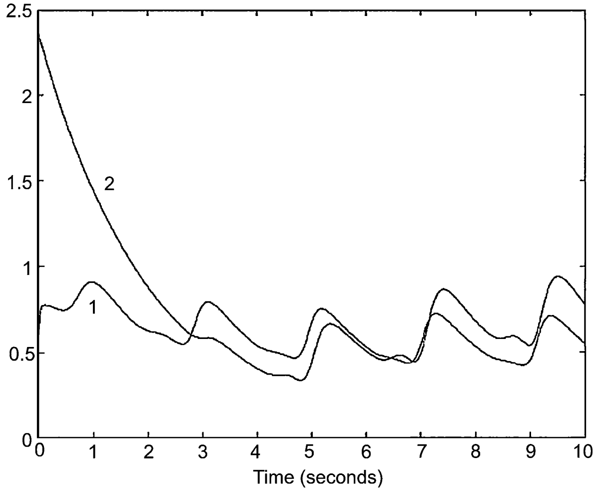


Fig. 8.21 Adaptation of $\hat{\zeta}^1$ and $\hat{\zeta}^2$

8.5 Proofs

To prove Theorem 2.1, we recall first the following result:

Lemma 8.1 *If $\beta(x)$ is a real valued function, $f(x)$ is a vector field, and $\omega(x)$ is a vector field, then*

$$L_f^r \beta\omega = \sum_{i=0}^{r} \binom{r}{i} (L_f^i \beta)(L_f^{r-i}\omega), \quad \textit{for } r = 1, 2, 3 \dots . \tag{8.129}$$

8.5.1 Proof of Theorem 2.1

Proof Let $\varphi_i(\zeta_i)$ and $\psi_i(\zeta_i)$ be two outputs for which the isolated subsystem has relative degree n_i. Let the degrees of mismatch with respect to the coordinate transformation corresponding to these two outputs be different. More precisely, $\exists\kappa_i, \mu_i$ (without loss of generality $\kappa_i < \mu_i$, so that the degrees of mismatch as defined in Definition 8.1 are different, i.e., $n_i - \kappa_i > n_i - \mu_i$) such that

$$\begin{aligned} &L_{f_{il}}^j \varphi_i \equiv 0, \quad 0 \le j \le \kappa_i - 1, \quad \forall \ell \in [1..p_i], \\ &L_{f_{il}}^{\kappa_i} \varphi_i \neq 0, \quad \text{for at least one } \ell \in [1..p_i], \end{aligned} \tag{8.130}$$

$$\begin{aligned} &L_{f_{im}}^j \psi_i \equiv 0, \quad j \in \{0, \dots, \kappa_i, \dots, \mu_i - 1\}, \quad \forall m \in [1..p_i], \\ &L_{f_{im}}^{\mu_i} \psi_i \neq 0, \quad \text{for at least one } m \in [1..p_i]. \end{aligned} \tag{8.131}$$

By construction, $\varphi_i(\zeta_i)$ and $\psi_i(\zeta_i)$ are obtained such that the codistributions generated by their gradients are annihilators of the following distribution:

$$\text{span}\{g_{i0}, ad_{f_{i0}}, g_{i0}, \ldots, ad_{f_{i0}}^{n_i-2} g_{i0}\}.$$

By virtue of feedback linearizability of the isolated subsystem, the above distribution is involutive and has dimension $n_i - 1$. Correspondingly, its annihilating codistribution is of dimension one. Thus $\exists \pi_i(\zeta_i) \neq 0$ such that

$$d\psi_i + \pi_i(\zeta_i) d\varphi_i = 0. \tag{8.132}$$

Let $\langle .,. \rangle$ denote the inner product. Consider $L_{f_{il}}^{\kappa_i} \psi_i + \pi_i(\zeta_i) L_{f_{il}}^{\kappa_i} \varphi_i$, where l is the index for all the vector fields for which (8.130) is satisfied. Then

$$\begin{aligned}
L_{f_{il}}^{\kappa_i} \psi_i + \pi_i(\zeta_i) L_{f_{il}}^{\kappa_i} \varphi_i &= \langle L_{f_{il}}^{\kappa_i - 1} d\psi_i, f_{il} \rangle + \langle \pi_i L_{F_{il}}^{\kappa_i - 1} d\varphi_i, f_{il} \rangle \\
&= \langle L_{f_{il}}^{\kappa_i - 1} d\psi_i, f_{il} \rangle + \langle L_{f_{il}}^{\kappa_i - 1} (\pi_i d\varphi_i), f_{il} \rangle \\
&\quad - \sum_{j=1}^{\kappa_i - 1} \binom{\kappa_i - 1}{\kappa_i - 1 - j} (L_{f_{il}}^{j} \pi_i) \langle L_{f_{il}}^{\kappa_i - j - 1} d\varphi_i, f_{il} \rangle
\end{aligned}$$

[using (8.129)]

$$\begin{aligned}
&= \langle L_{f_{il}}^{\kappa_i - 1} (d\psi_i + \pi_i d\varphi_i), f_{il} \rangle \\
&\quad - \sum_{j=1}^{\kappa_i - 1} \binom{\kappa_i - 1}{\kappa_i - 1 - j} (L_{f_{il}}^{j} \pi_i)(L_{f_{il}}^{\kappa_i - j} \varphi_i).
\end{aligned} \tag{8.133}$$

The last term in (8.133) is zero, since $L_{F_{il}}^{\kappa_i - j} \varphi_i = 0$, $1 \leq j \leq \kappa_i - 1$. Also from (8.132), the first term is zero. Therefore

$$L_{f_{il}}^{\kappa_i} \psi_i + \pi_i(\zeta_i) L_{F_{il}}^{\kappa_i} \varphi_i = 0. \tag{8.134}$$

But, $L_{f_{il}}^{\kappa_i} \psi_i = 0$, and since $\pi_i(\zeta_i) \neq 0$, (8.134) implies that $L_{f_{il}}^{\kappa_i} \varphi_i = 0$, which contradicts (8.130). Thus, necessarily

$$\kappa_i = \mu_i \quad \Longleftrightarrow \quad n_i - \kappa_i = n_i - \mu_i.$$

Equivalently, the degree of mismatch is the same for both transformations. □

8.5.2 *Proof of Proposition 8.1*

Proof Sufficiency. From Assumption 8.4, there exists a global diffeomorphism $w_i = \phi_i(\zeta_i)$, $\phi_i(0) = 0$, transforming the ith isolated subsystem (8.25) into

$$\begin{aligned}
\dot{w}_{ij} &= w_{i,j+1}, \quad 1 \leq j \leq n_i - 1, \\
\dot{w}_{i,n_i} &= v_i(w_i) + \delta_i(w_i) u_i
\end{aligned} \tag{8.135}$$

with $v_i(0) = 0$, $\delta_i(w_i) \neq 0$ $\forall w_i \in \Re^{n_i}$. For the ith subsystem, the representation of f_{i0}, g_{i0}, and g^{ik} in the transformed coordinates is

$$
\begin{aligned}
f_{i0} &= w_{i2}\frac{\partial}{\partial w_{i1}} + \cdots + w_{i,n_i}\frac{\partial}{\partial w_{i,n_i-1}} + v_i(w_i)\frac{\partial}{\partial w_{i,n_i}}, \\
g_{i0} &= \delta_i(w_i)\frac{\partial}{\partial w_{i,n_i}}, \\
g^{ik} &= \operatorname{span}\left\{\frac{\partial}{\partial w_{i,n_i}}, \ldots, \frac{\partial}{\partial w_{i,n_i-k}}\right\}, \quad 0 \le k \le n_i - 1,
\end{aligned} \tag{8.136}
$$

where $\partial/\partial w_{i1}, \ldots, \partial/\partial w_{in_i}$ are the coordinate vector fields associated with w_i the coordinates. Thus, condition (i) can be written as

$$
\begin{aligned}
&\left[\frac{\partial}{\partial w_{ij}}, f_{il}\right] \in \operatorname{span}\left\{\frac{\partial}{\partial w_{i,n_i}}, \ldots, \frac{\partial}{\partial w_{ij}}\right\}, \\
&1 \le i \le N,\ 1 \le l \le p_i,\ n_i - \rho_i \le j \le n_i.
\end{aligned} \tag{8.137}
$$

Furthermore, in the transformed coordinates

$$
\Omega^{ij} = \operatorname{span}\{dw_{i,n_i}, \ldots, dw_{i,n_i-j+1}\}, \quad 1 \le j \le n_i.
$$

Therefore, in transformed coordinates

$$
W^{ij} = \operatorname{span}\left\{\frac{\partial}{\partial w_{i1}}, \ldots, \frac{\partial}{\partial w_{i,n_i-j}}\right\},
$$

and condition (ii) can be written as

$$
\begin{aligned}
&\left[\frac{\partial}{\partial w_{kj}}, f_{il}\right] \in \operatorname{span}\left\{\frac{\partial}{\partial w_{i,n_i}}, \ldots, \frac{\partial}{\partial w_{i,\kappa_i}}\right\}, \\
&1 \le j \le \kappa_k,\ 1 \le l \le p_i,\ 1 \le i, k \le N, \\
&\kappa_i = n_i - \rho_i,
\end{aligned} \tag{8.138}
$$

where ρ_i is the degree of mismatch for the ith subsystem.

Hence, the representation of f_{il} in the transformed coordinates is given by

$$
\begin{aligned}
f_{il} &= \gamma_{i0l}(w_{j1}, \ldots, w_{j,\kappa_j} | 1 \le j \le N)\frac{\partial}{\partial w_{i,\kappa_i}} \\
&\quad + \gamma_{i1l}(w_{j1}, \ldots, w_{j,\kappa_j}, w_{i,k_i+1} | 1 \le j \le N)\frac{\partial}{\partial w_{i,\kappa_i+1}} + \cdots \\
&\quad + \gamma_{i,n_i-\kappa_i-1,l}(w_{j1}, \ldots, w_{j,\kappa_j}, w_{i,\kappa_i+1}, \ldots, w_{i,n_i-1} | 1 \le j \le N)\frac{\partial}{\partial w_{i,n_i-1}} \\
&\quad + \gamma_{i,n_i-\kappa_i,l}(w_{j1}, \ldots, w_{j,\kappa_j}, w_{i,\kappa_i+1}, \ldots, w_{i,n_i} | 1 \le j \le N)\frac{\partial}{\partial w_{i,n_i}}, \\
&\quad 1 \le i \le N,\ 1 \le l \le p_i
\end{aligned} \tag{8.139}
$$

which is the same as given in (8.26). This completes the proof of sufficiency.

Necessity. Given a diffeomorphism $w_i = \phi_i(\zeta_i)$ that transforms the original system (8.24)–(8.26), it is easy to verify that the coordinate-free conditions (i) and (ii) are satisfied for (8.26) and hence for (8.24). □

8.6 Decentralized Adaptive Tracker

8.6.1 Introduction

In recent years, the decentralized control of interconnected systems has been a popular research topic in control theory. Large-scale systems, such as transportation systems, power systems, communications systems, to name a few, are the essential features of our modern life. One of the early decentralized adaptive control methods was proposed in [17], which focused on the vital role of interconnections. It showed that interconnections, even if they are weak, can make a decentralized adaptive controller unstable. Decentralized adaptive controllers were essentially developed to guarantee boundedness and exponential convergence of the tracking and parameter errors to bounded residual sets. Several decentralized adaptive techniques have been developed in [2–4, 7, 12, 13, 38] and [30–41]. Mos of these works are confined to strictly decentralized adaptive control systems where no explicit communication exists between subsystems.

A particular class of these techniques is the model reference adaptive control (MRAC), where the objective of the theory is to give the performance specifications in terms of a model. The model represents the ideal response of the process to a command signal. MRAC has been extensively developed for continuous time systems [17] and discrete-time systems [8].

One of the main disadvantages of the known model reference decentralized adaptive control laws is that the convergence of local tracking errors only to a bounded residual set. Besides, the bounds of this set are unknown a priori and the size depends upon the bound for the strength of the unmodelled interconnections, so such adaptive schemes may be unsuitable for some applications. The need to develop new methods which would allow one to avoid this basic disadvantage is therefore apparent. In this regard, a modified local adaptive control scheme was proposed in [28] which improves the transient performance by utilizing an appropriate time-delay action in the centralized adaptive control. A further improvement was reported in [45] based on backstepping adaptive tracking. A more general class of interconnected systems with unmodelled dynamics was considered in [42, 44].

With the rapid advances in digital technology and computers, more and more control engineers would like to replace analogue controllers with digital controllers for the purpose of better reliability, lower cost, and more flexibility [9]. In the sequel, we focus attention on the digital redesign approach [1, 9] to construct the digital tracker for the sampled-data system. Based on the ideas contained in [28, 31] and the digital redesign method, a novel sampled-data decentralized adaptive approach is provided to the solution of the decentralized adaptive tracking problem in this article, so that the system output will follow any trajectory specified at sampling instant which may not be presented by the analytic reference model initially.

Based on the given plant, a well-designed reference model is proposed to fit the desired trajectory at discrete-time sampling instant first. Then, by invoking the digital redesign technique, we develop an acceptable digital tracker for the sampled-data decentralized adaptive system which closely matches the response of the continuous-time well-designed system with the same inputs and initial conditions, rather than designing a new controller using the digital control theory.

8.6.2 The Decentralized Adaptive Control Design

The decentralized adaptive control for linear time-invariant system has been studied since the 1980s. In [31], the author introduced a new and numerically efficient approach to the design of the decentralized adaptive controller for linear large-scale systems. The digital model-reference-based decentralized adaptive controller for the decentralized adaptive control system proposed in this chapter is mainly derived from the analogue adaptive controller proposed for the analogue system in [28, 31].

8.6.3 The Decentralized Adaptive Problem

Most available techniques focus on the design of centralized controllers, in which every input affects all controller outputs. The design method has received a great deal of attention in the control literature. We used the decentralized controller to realize the complicated centralized controller for the desired performance. The structure of the decentralized control system is shown in Fig. 8.22. Although this constraint on the controller structure may lead to performance deterioration when compared to a system with a single centralized controller, the decentralized control is used in most industrial control system designs. The advantages of decentralized control are summarized below:

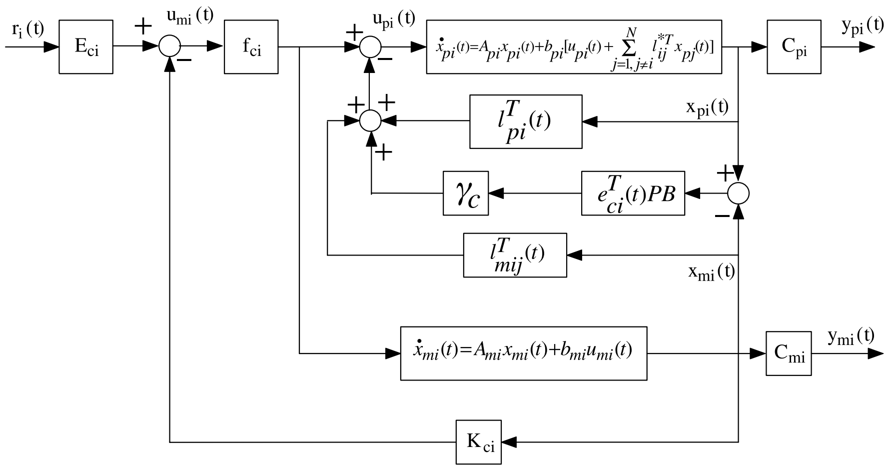


Fig. 8.22 The structure of decentralized control system

1. *Control hardware simplicity: the cost of implementation of a decentralized control system is clearly less than that of a centralized controller.*
2. *Control design and tuning simplicity: the decentralized controllers include far fewer parameters to result in great reduction in the time and cost of tuning.*

Consider a linear time-invariant system Σ_p consisting of N interconnected subsystems $\Sigma_{p1}, \Sigma_{p2}, \Sigma_{p3}, \ldots, \Sigma_{pN}$, which is described by [31] as follows:

$$\Sigma_{pi}\colon \quad \dot{x}_{pi}(t) = A_{pi}x_{pi}(t) + b_{pi}\left[u_{pi}(t) + \sum_{j=1, j\neq 1}^{N} l_{ij}^{*t} x_{pj}(t)\right], \tag{8.140}$$

$$y_{pi}(t) = c_{pi}x_{pi}(t), \tag{8.141}$$

where $i = 1, 2, \ldots, N$, $u_{pi}(t) \in \Re$ is the input, $y_{pi}(t) \in \Re$ is the output, $x_{pj}(t) \in \Re^{n_j \times 1}$ is the state vector of the jth subsystem and $x_{pi} \in \Re^{n_i \times 1}$ is the state vector of the ith subsystem at time t. The matrices $A_{pi} \in \Re^{n_i \times n_i}$, and the vectors $b_{pi} \in \Re^{n_i \times 1}$, $c_{pi} \in \Re^{1 \times n_i}$ are assumed to be known. Assume that all subsystems in (8.140) are completely controllable and the overall system is decentrally stabilized.

The corresponding N designed inner-loop reference models Σ_{mi}, not arbitrary assigned, are described as

$$\Sigma_{mi}\colon \quad \dot{x}_{mi}(t) = A_{mi}x_{mi}(t)b_{mi}u_{mi}(t), \tag{8.142}$$

$$y_{mi}(t) = c_{mi}x_{mi}(t), \tag{8.143}$$

where $i = 1, 2, \ldots, N$, $u_{mi}(t) \in \Re$ is the bounded control input, $y_{mi}(t) \in \Re$ is the bounded output, and $x_{mi}(t) \in \Re^{n_i \times 1}$, which is the tracking target of the state $x_{pi}(t)$ of the subsystem Σ_{pi}, is the corresponding state of the ith model Σ_{mi} at time t. The matrices $A_{mi} \in \Re^{n_i \times n_i}$ are asymptotically stable constant matrices of appropriate dimensions, and the constant vectors $b_{mi} \in \Re^{n_i \times 1}$, $c_{mi} \in \Re^{1 \times n_i}$ are identical to b_{pi}, c_{pi} in Σ_{pi} and Σ_{mi}, respectively, i.e. $b_{mi} = b_{pi}$ and $c_{mi} = c_{pi}$.

The terms $l_{ij}^{*t} x_{pj}(t)$ $(j \neq i)$, as shown in (8.140), corresponding to the perturbations on the subsystem Σ_{pi} due to subsystems $\Sigma_{pj, j\neq i}$, $j = 1, 2, \ldots, N$. To compensate for all the interconnections for achieving the decoupling close-loop system, eliminating $l_{ij}^{*t} x_{pj}(t)$ is the first control objective, and the second control objective is to estimate the parameters $l_{pi}(t)$ in (8.140), so that $x_{pi}(t)$ can asymptotically track $x_{mi}(t)$ with zero error.

As long as the given (A_{pi}, b_{pi}) pairs are controllable, one can have the inner-loop feedback gain K_{mi} in (8.144)

$$A_{mi} = A_{pi} - b_{pi}K_{mi}, \tag{8.144}$$

based on the linear quadratic regulator (LQR) design, without any restriction, to form the desired A_{mi} shown in (8.142). The optimal state-feedback control law is to minimize the following performance index:

$$J_i = \int_0^{\infty} \{x_{mi}^t(t) Q_i x_{mi}(t) u_{mi}^t(t) R_i u_{mi}(t)\} dt, \tag{8.145}$$

with $Q_i \geq 0$ and $R_i > 0$ for the plant subsystem Σ_{pi}. This inner-loop optimal control is given by

$$u_{mi}(t) = -K_{mi}x_{mi}(t), \tag{8.146}$$

where $K_{mi} = R_i^{-1} b_{pi}^t O_i$ [31], and O_i is the positive definite and symmetric solution of the following Riccati equation:

$$A_{pi}^t O_i + O_i A_{pi} - O_i b_{pi} R_i^{-1} b_{pi}^t O_i + Q_i = 0.$$

Then, the resulting system (8.142) becomes

$$\dot{x}_{mi}(t) = (A_{pi} - b_{pi}K_{mi})x_{mi}(t) + b_{pi}u_{mi}(t), \tag{8.147}$$

where the outer-loop control input $u_{mi}(t)$ is to be further designed in Sect. 8.7.1 so that $y_{mi}(t)$ will track the reference input $r_i(t)$ well.

For N subsystems Σ_{pi} and N reference submodels Σ_{mi}, there exists N controllers Σ_{ci} to compensate Σ_{pi}. At every instant t, the controller Σ_{ci} accesses only the state $x_{pi}(t)$ of the subsystem $\sum_{pi}$ and the complete knowledge of the desired states $x_{mi}(t)$ of all the reference models Σ_{mi}. It is desired to determine controllers Σ_{ci} to generate bounded inputs $u_{pi}(t)$ such that $x_{pi}(t)$ are bounded, and

$$\lim_{t\to\infty} \|e_{ci}(t)\| = \lim_{t\to\infty} \|x_{pi}(t) - x_{mi}(t)\| = 0.$$

Remark 8.14 The structure of the interconnections shown in (8.140) is identical to that of [7]. The structure assures the existence of a bounded control input $u_{pi}(t)$ which can compensate for all the interconnections, provided the vectors l_{ij} and the states $x_{pj}(t)$ are known to controller Σ_{ci}. It is assumed that each controller Σ_{ci} is aware only of the input $u_{pi}(t)$ and the states $x_{pj}(t)$ of the subsystem Σ_{pi} at every time instant t. It can also be restated that the adaptive control has to be carried out using only inputs and outputs (rather than the state vectors) of the subsystems, and the interconnections between them assume special forms [29].

8.6.3.1 A Model-Reference Adaptive Controller

The digital decentralized adaptive controller for the sampled-data large-scale interconnected system proposed in this chapter is mainly derived from the analogue decentralized adaptive controller proposed for the analogue system in [31]. Here, we briefly introduce the derivation of analogue decentralized adaptive controller in this section, and the complete proof is given in [31].

Consider the model-reference-based decentralized adaptive control (MRDAC) problem, as shown in Fig. 8.23, where all controllers access only to the input $u_{pc}(t)$ and the state $x_{pc}(t)$ of the subsystem. The linear controllable continue-time system and the reference model, described by (8.140)–(8.143), are restated globally, respectively, as

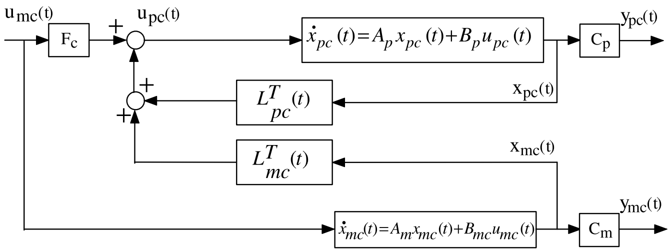


Fig. 8.23 The decentralized adaptive control system with the MRDAC

$$\dot{x}_{pc}(t) = A_p x_{pc}(t) B_p u_{pc}(t), \tag{8.148}$$

$$y_{pc}(t) = C_p x_{pc}(t), \tag{8.149}$$

$$\dot{x}_{mc}(t) = A_m x_{mc}(t) B_m u_{mc}(t), \tag{8.150}$$

$$y_{mc}(t) = C_m x_{mc}(t), \tag{8.151}$$

$$u_{pc}(t) = F_c u_{mc}(t) - L^t_{pc} x_{pc}(t) - L^t_c x_{mc}(t), \tag{8.152}$$

where $x_{pc}(t) \in \Re^{n\times 1}$, $n = n_1 + n_2 + \cdots + n_N$, $x_{mc}(t) \in \Re^{n\times 1}$, $u_{pc}(t) \in \Re^{N\times 1}$, $u_{mc}(t) \in \Re^{N\times 1}$, $y_{pc}(t) \in \Re^{N\times 1}$, $y_{mc}(t) \in \Re^{N\times 1}$, $A_p \in \Re^{n\times n}$, $A_m \in \Re^{n\times n}$, $B_p \in \Re^{n\times N}$, $B_m \in \Re^{n\times N}$, $C_p \in \Re^{N\times n}$, $C_m \in \Re^{N\times n}$ and

$$x_{pc}(t) = \begin{bmatrix} x_{p1}(t) \\ \vdots \\ x_{pN}(t) \end{bmatrix}, \quad u_{pc}(t) = \begin{bmatrix} u_{p1}(t) \\ \vdots \\ u_{pN}(t) \end{bmatrix},$$

$$y_{pc}(t) = \begin{bmatrix} y_{p1}(t) \\ \vdots \\ y_{pN}(t) \end{bmatrix}, \tag{8.153}$$

$$A_p = \begin{bmatrix} A_{p1} & b_{p1} l^{*t}_{12} & \dots & b_{p1} l^{*t}_{1N} \\ b_{p2} l^{*t}_{21} & A_{p2} & & b_{p2} l^{*t}_{2N} \\ \vdots & & \ddots & \vdots \\ b_{pN} l^{*t}_{N1} & b_{pN} l^{*t}_{N2} & \dots & A_{pN} \end{bmatrix},$$

$$B_p = \text{blockdiag}[b_{p1}\ b_{p2}\ \dots\ b_{pN}],$$

$$C_p = \text{blockdiag}[c_{p1}\ c_{p2}\ \dots\ c_{pN}],$$

$$x_{mc}(t) = \begin{bmatrix} x_{m1}(t) \\ \vdots \\ x_{mN}(t) \end{bmatrix}, \quad \begin{bmatrix} u_{m1}(t) \\ \vdots \\ u_{mN}(t) \end{bmatrix}, \tag{8.154}$$

$$y_{mc}(t) = \begin{bmatrix} y_{m1}(t) \\ \vdots \\ y_{mN}(t) \end{bmatrix},$$

$$\begin{aligned}
A_m &= \text{blockdiag}[A_{m1}\ A_{m2}\ \dots\ A_{mN}],\\
F_c &= \text{blockdiag}[f_{c1}\ f_{c2}\ \dots\ f_{cN}],\\
B_m &= \text{blockdiag}[b_{m1}\ b_{m2}\ \dots\ b_{mN}],\\
C_m &= \text{blockdiag}[c_{m1}\ c_{m2}\ \dots\ c_{mN}],\\
L_{pc}(t) &= \text{blockdiag}[l_{p1}\ l_{p2}\ \dots\ l_{pN}],\\
L_c(t) &= \begin{bmatrix} 0 & l_{12}(t) & \dots & l_{1N}(t)\\ l_{21}(t) & 0 & & l_{2N}(t)\\ \vdots & & \ddots & \vdots\\ l_{N1}(t) & l_{N2}(t) & \dots & 0 \end{bmatrix}.
\end{aligned} \tag{8.155}$$

The decentralized adaptive control problem is that while the controller Σ_{ci} is aware of the form of the interconnections (that is, $l_{ij}^{*t} x_{pj}(t)$), it has no knowledge of either l_{ij}^{*t} or $x_{pj}(t)$. As stated previously, the desired state $x_{mj}(t)$, in place of the state $x_{pj}(t)$, of the other subsystem Σ_{pj} is used as a part of its control input by the controller Σ_{ci} [31].

Proceeding further, we let the control input $u_{pi}(t)$ of the subsystem Σ_{pi} be

$$\begin{aligned}
u_{pi}(t) &= f_{ci} u_{mi}(t) - l_{pi}^t(t) x_{pi}(t) - \gamma_{ci} e_{ci}^t(t) P_i b_{pi}\\
&\quad - \sum_{j=1, j\neq i}^{N} l_{ij}^t(t) x_{mj}(t) \quad \text{for } i = 1, 2, \dots, N,
\end{aligned} \tag{8.156}$$

where the second term is introduced to match the reference model, the third term attempts to help stabilize the overall system and the last term is used to cancel the effect of perturbations due to the other subsystems Σ_{pj} and γ_{ci} is the extra compensating gain. The state-feedback gain $(l_{pi}(t), l_{ij}(t))$ are adjusted using the following adaptive laws from [31] as follows:

$$\frac{dl_{pi}(t)}{dt} = e_{ci}^t(t) P_i b_{pi} x_{pi}(t), \quad \text{for } i = 1, 2, \dots, N, \tag{8.157}$$

$$\frac{dl_{ij}(t)}{dt} = e_{ci}^t(t) P_i b_{mi} x_{mj}(t), \quad \text{for } i, j = 1, 2, \dots, N \text{ and } j \neq i. \tag{8.158}$$

The controlled subsystem is described by

$$\begin{aligned}
\Sigma_{pi}:\quad \dot{x}_{pi}(t) &= A_{pi} x_{pi}(t) + B_{pi} u_{pi}(t) + b_{pi} \sum_{j=1, j\neq i}^{N} l_{ij}^{*t} x_{pj}(t)\\
&= \big[A_{pi} - b_{pi} l_{pi}^t(t)\big] x_{pi}(t) + b_{pi} f_{ci} u_{mi}(t) - \gamma_{ci} b_{pi} e_{ci}^t(t) P_i b_{pi}\\
&\quad - b_{pi} \sum_{j=1, j\neq i}^{N} \big[l_{ij}^t(t) x_{mj}(t) - l_{ij}^{*t} x_{pj}(t)\big],
\end{aligned} \tag{8.159}$$

and the tracking error $e_{ci}(t)$ of Σ_{pi} is described by the differential equation

$$\begin{aligned}\dot{e}_{ci}(t) &= \dot{x}_{pi}(t) - \dot{x}_{mi}(t)\\ &= A_{mi}e_{ci}(t) - b_{pi}\tilde{l}^t_{pi}(t)x_{pi}(t) - \gamma_{ci}b_{pi}e^t_{ci}(t)P_i b_{pi}\\ &\quad - b_{pi}\sum_{j=1,j\neq i}^{N}\big[\tilde{l}^t_{ij}(t)x_{mj}(t) - \tilde{l}^{*t}_{ij}e_{cj}(t)\big],\end{aligned} \tag{8.160}$$

where $e_{ci}(t) = x_{pi}(t) - x_{mi}(t)$, $P_i = P_i^t > 0$ is the solution of the Lyapunov equation

$$\begin{aligned}A^t_{mi}P_i + P_i A_{mi} &= -Q_i,\\ Q_i = Q_i^t > 0, &\quad Q_i \in \Re^{n_i\times n_i}.\end{aligned} \tag{8.161}$$

Choosing the same Lyapunov function candidate as before

$$\begin{aligned}&V(e_{ci}(t), \tilde{l}_{pi}(t), \tilde{l}_{ij}(t))\\ &= \sum_{i=1}^{N}\big[e^t_{ci}(t)P_i e_{ci}(t) + \tilde{l}^t_{pi}(t)\tilde{l}_{pi}(t)\big] + \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\tilde{l}^t_{ij}(t)\tilde{l}_{ij}(t),\end{aligned} \tag{8.162}$$

the time derivative along any trajectory is given by

$$\begin{aligned}\dot{V}\big(e_{ci}(t), \tilde{l}_{pi}(t), \tilde{l}_{ij}(t)\big) &= \sum_{i=1}^{N} e^t_{ci}(t)\big[A^t_{mi}P_i + P_i A_{mi}\big]e_{ci}(t)\\ &\quad - \sum_{i=1}^{N} 2\gamma_{ci}[e^t_{ci}(t)P_i b_{pi}e^t_{ci}(t)P_i b_{pi}]\\ &\quad + \sum_{i=1}^{N} 2\tilde{l}^t_{pi}(t)\big[\dot{\tilde{l}}_{pi}(t) - e^t_{ci}(t)P_i b_{pi}x_{pi}(t)\big]\\ &\quad + \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} 2\tilde{l}^t_{ij}(t)\big[\dot{\tilde{l}}_{ij}(t) - e^t_{ci}(t)P_i b_{pi}x_{mj}(t)\big]\\ &\quad + \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} 2e^t_{ci}(t)P_i b_{pi}\tilde{l}^{*t}_{ij}e_{cj}(t)\\ &\quad + \sum_{i=1}^{N}\big[2\tilde{l}^t_{pi}(t)\dot{\tilde{l}}_{pi}(t)\big] + \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} 2\tilde{l}^t_{ij}(t)\dot{\tilde{l}}_{ij}(t).\end{aligned} \tag{8.163}$$

Substituting (8.161) and the adaptive laws (8.157) and (8.158) into (8.163), it yields that

$$\begin{aligned}\dot{V}\big(e_{ci}(t), \tilde{l}_{pi}(t), \tilde{l}_{ij}(t)\big) &= \sum_{i=1}^{N}\big[-e^t_{ci}(t)Q_i e_{ci}(t) - 2\gamma_{ci}(e^t_{ci}(t)P_i b_{pi})^2\big]\\ &\quad + \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} 2e^t_{ci}(t)P_i b_{pi}\tilde{l}^{*t}_{ij}e_{cj}(t).\end{aligned} \tag{8.164}$$

It follows that

$$\dot{V}\big(e_{ci}(t), \tilde{l}_{pi}(t), \tilde{l}_{ij}(t)\big) \le \sum_{i=1}^{N}\big[-\lambda_{\min}(Q_i)\|e_{ci}(t)\|^2 - 2\gamma_{ci}(e_{ci}^t(t)P_i b_{pi})^2\big] + \sum_{i=1}^{N}\sum_{j=1, j\ne i}^{N} 2e_{ci}^t(t)P_i b_{pi}\tilde{l}_{ij}^{*t} e_{cj}(t), \tag{8.165}$$

where $\lambda_{\min}(Q_i)$ is the smallest eigenvalue of Q_i.

It is therefore concluded that a sufficient condition for $\dot{V}$ to be negative-semidefinite along any trajectory is that

$$\gamma_{ci} > \frac{1}{2}(N-1)\max_j\left(\frac{\|l_{ij}^*\|^2}{\lambda_{\min}(Q_j)}\right), \tag{8.166}$$

and some $\bar{\gamma}_{ci}$ exists such that whenever $\gamma_{ci} \ge \bar{\gamma}_{ci}$, $\dot{V}$ is negative-semi definite, thus V is a Lyapunov function for all systems. Choosing such a γ_{ci} for each subsystem, it is followed that $e_{ci}(t), \tilde{l}_{pi}(t)$ and $\tilde{l}_{ij}(t)$ are bounded for all i, which implies that $\dot{e}_{ci}(t)$ is bounded and $\lim_{t\to\infty} e_c(t) = 0$.

8.7 The Digital Redesign of the Decentralized Adaptive Control System

The digital redesign is desired to find the digital controller from the available analogue controller Σ_{ci}, so that the digitally redesigned sampled-data states are able to closely match those of the original analogously controlled system. The aforementioned prediction based digital redesign method [9], developed for the digital redesign of a state-feedback system, is utilized to find the state-matching digital controller for the analogue control system. The detailed derivation and properties of the prediction based digital controller $u_d(kT)$ can be found in [9].

Here, the digital redesign approach is introduced which leads to the novel digitally redesigned model reference- based adaptive controller for the sampled data decentralized adaptive control system by digitizing the analogue decentralized adaptive controller.

8.7.1 The Digital Redesign Methodology

For $y_{mi}(t)$ to track the reference input $r_i(t)$ well, consider the linear controllable continuous-time reference model described previously in (8.142) and (8.143), as shown in Fig. 8.24, the tracker design of linear continuous-time reference model is given by

$$u_{mc}(t) = -K_c x_{mc}(t) + E_c r(t), \tag{8.167}$$

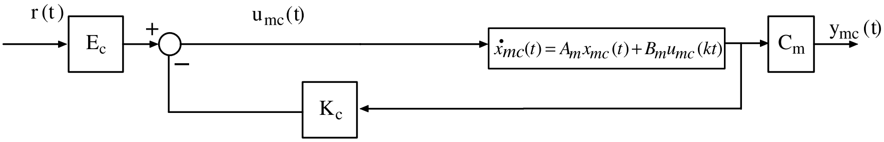


Fig. 8.24 The continuous-time control of a reference model

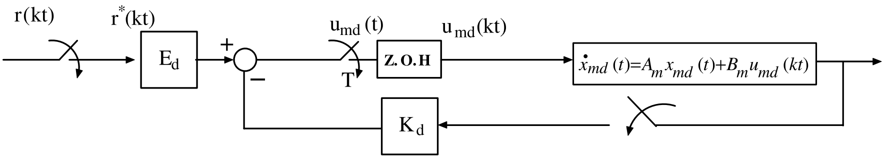


Fig. 8.25 The digital redesign sampled-data control of a reference model

where $K_c \in \Re^{N\times n}$ is the state-feedback gain, $E_c \in \Re^{N\times N}$ is the feed-forward gain, and $r(t) = [r_1(t)\ r_2(t)\ \ldots\ r_N(t)]^t \in \Re^N$ is the piecewise-constant reference input vector $r(t) = r(kT)$ for $kT \le t \le (k+1)T$. The controlled closed-loop reference model becomes

$$\dot{x}_{mc}(t) = (A_m - B_m K_c)x_{mc}(t) + B_m E_c r(t). \tag{8.168}$$

By linear quadratic optimal tracker algorithm [1], let the performance index be

$$J = \int_0^\infty \{(C_m x_{mc}(t) - r(t))^t Q_c (C_m x_{mc}(t) - r(t)) + u_{mc}^t(t) R_c u_{mc}(t)\}dt, \tag{8.169}$$

with $Q_c \ge 0$ and $R_c > 0$, which yields $K_c = R_c^{-1} B_m Z$ and $E_c = -R_c^{-1} B_m^t [(A_m - B_m K_c)^{-1}]^t C_m Z$. Here Z is the positive definite and symmetric solution of the following Ricatti equation

$$A_m^t Z + Z A_m - Z B_m R_c^{-1} B_m^t Z + C_m^t Q_c C_m = 0. \tag{8.170}$$

The prediction-based digital redesign method is utilized to realize a digitally redesigned controller. Thus, the digitally redesigned sampled-data state in Fig. 8.25 is able to closely match that of the original analogously controlled system in Fig. 8.24. The digitally redesigned controller $u_{md}(kT)$ achieved from the analogue controller $u_{mc}(t)$ in (8.167) can be described as

$$u_{md}(kT) = -K_d x_{md}(kT) + E_d r^*(kT), \tag{8.171}$$

where $K_d \in \Re^{m\times n}$ is the digital state feedback gain, $E_d \in \Re^{m\times m}$ is the digital feed-forward gain and $T > 0$ is the sampling period. The discrete-time state $x_{md}(kT)$ can be decided from the discrete-time model of the analogue model (8.150) as

$$x_{md}(kT + T) = G_m x_{md}(kT) + H_m u_{md}(kT), \tag{8.172}$$

where

$$G_m = e^{AmT},$$
$$H_m = \int_0^t e^{Am\tau} d\tau B_m = \begin{cases} \left[I_n T + A_m \frac{T^2}{2!} + A_m^2 \frac{T^3}{3!} + \cdots\right] B_m, \\ [G_m - I_n] A_m^{-1} B_m, \quad \text{if } A_m \text{ is invertible.} \end{cases} \tag{8.173}$$

The digital gains (K_d, E_d) in (8.171) can be determined from the analogue gains (K_c, E_c) in (8.167) as

$$K_d = (I_m + K_c H_m)^{-1} K_c G_m, \tag{8.174}$$
$$E_d = (I_m + K_c H_m)^{-1} E_c, \tag{8.175}$$

and $r^*(kT)$ in (8.171), the alternative form of the original reference input $r(t)$ at time step $t = kT$ with one-step ahead amplitude $r(kT + T)$ for tracking purpose is set to be

$$r^*(kT) = r(kT + T).$$

A zero-order holder (Z.O.H.) is utilized here.

8.7.2 An Improved Redesign Adaptive Controller

Consider the linear controllable continuous-time system, shown in Fig. 8.23 and described in (8.148) and (8.149), controlled adaptively by the continuous-time state-feedback controller which is described as

$$\begin{aligned} u_{pc} &= F_c u_{mc}(t) - L_{pc}^t x_{pc}(t) - \gamma_c e_c^t P B_p - L_c^t(t) x_{mc}(t) \\ &= F_c u_{mc}(t) - L_{pc}^t(t) x_{pc}(t) - \gamma_c B_p^t(t) P e_c(t) - L_c^t(t) x_{mc}(t), \end{aligned} \tag{8.176}$$

where the second term is introduced to match the reference model, the third term attempts to help stabilize the overall system and the last term is used to cancel the effect of the interconnections.

The state feed-forward gain $F_c \in \Re^{N\times N}$ is set to be unity and $(L_{pc}(t), L_c(t))$ is the state feedback gain, the extra compensating gain γ_c can be optimized by **EP** (evolution programming) or **GA** (genetic algorithm) method under performance consideration, $u_{mc}(t) = u_{mc}(kT)$ for $kT \le t < (k+1)^t$ is the piecewise-constant reference input vector, and the controlled closed-loop system thus becomes

$$\dot{x}_{pc}(t) = A_{pc}(t) x_{pc}(t) - B_p \gamma_c e_c^t(t) P B_p - B_p L_c^t(t) x_{mc}(t) + B_p F_c u_{mc}(t), \tag{8.177}$$

where $A_{pc}(t) = A_p - B_p L_{pc}^t(t)$.

The extra compensating input signal gains (F_c, γ_c) are compounded as

$$\gamma_c = \text{blockdiag}[\gamma_{c1}\ \gamma_{c2}\ \ldots\ \gamma_{cN}],$$
$$F_c = \text{blockdiag}[f_{c1}\ f_{c2}\ \ldots\ f_{cN}],$$

where the components of $u_{pc}(t)$ in (8.154) and (8.155) are adjusted adaptively by (8.157) and (8.158), respectively.

Let the corresponding hybrid system be described by the state equation

$$\dot{x}_{pd}(t) = A_p x_{pd}(t) B_p u_{pd}(t), \tag{8.178}$$

where $u_{pd}(t) \in \Re^{N\times 1}$ is the piecewise-constant input vector, satisfying

$$u_{pd}(t) = u_{pd}(kT) \quad \text{for } kT \le t \le (k+1)^t \tag{8.179}$$

and the sampling period $T > 0$. The discrete-time state feedback controller shown in Fig. 8.26 is assumed to be

$$\begin{aligned} u_{pd}(kT) = F_d(kT)u_{md}(kT) - L^t_{pd}(kT)x_{pd}(kT) \\ - \gamma_d(kT)e^t_d(kT)PB_p - L^t_d(kT)x_{md}(kT), \end{aligned} \tag{8.180}$$

where $F_d(kT) \in \Re^{N\times N}$ is the digital state feed-forward gain, $L_{pd}(kT), L_{md}(kT) \in \Re^{N\times n}$ are the digital state feedback gains, and $u^*_{md}(kT) \in \Re^{N\times 1}$ is the piecewise constant reference input vector which is determined in terms of $u_{md}(t)$ for tracking purpose. The digitally controlled closed-loop system is then

$$\begin{aligned} \dot{x}_{pd}(t) = A_p x_{pd}(t) + B_p[F_d(kT)u_{md}(kT) - L^t_{pd}(kT)x_{pd}(kT) \\ - \gamma_d(kT)e^t_d(kT)PB_p - L^t_{md}(kT)x_{md}(kT)] \\ \text{for } kT \le t \le (k+1)T. \end{aligned} \tag{8.181}$$

Notice that the notation A in Fig. 8.26 is defined as $A \equiv \text{diag}[A_{p1}\ A_{p2}\ \dots\ A_{pN}]$.

The digital redesign is desired to find the digital gains ($F_d(kT)$, $L_{pd}(kT)$, $L_{md}(kT)$) in (8.180) from the analogue gains ($F_c, L_{pc}(t), L_c(t)$) in (8.176), with the zero-order-hold device utilized for (8.176), so that the digital closed-loop state in (8.181) is able to closely match the analogue one in (8.177) at all the sampling instants, for the given $u_{mc}(t) = u_{mc}(kT)$ for $kT \le t \le (k+1)^t$.

The continuous-time state stated in (8.148), at $t = t_v = kT + vT$ for $0 \le v \le 1$ where v is the tuning parameter, is obtained as

$$\begin{aligned} x_{pc}(t_v) &= e^{A_p(t_v - kT)}x_{pc}(kT) + \int_{kT}^{kT+vT} e^{Ap(t_v-\tau)}B_p u_{pc}(\tau)d\tau \\ &\approx e^{A_p(t_v - kT)}x_{pc}(kT) + \int_{kT}^{kT+vT} e^{A_p(t_v-\tau)}B_p d\tau u_{pc}(t_v) \\ &= G^{(v)}_p x_{pc}(kT) + H^{(v)}_p u_{pc}(tv), \end{aligned} \tag{8.182}$$

where $u_{pc}(t_v)$ is assumed to be the piecewise-continuous input, and

$$\begin{aligned} G^{(v)}_p &= e^{A_p(t_v-kT)} = e^{A_p vT} = (e^{A_p T})^v = G^v_p, \\ H^{(v)}_p &= \int_{kT}^{kT+vT} e^{A_p(t_v-\tau)}B_p d\tau = \int_0^t e^{A_p\tau}B_p d\tau \\ &= \begin{cases} \left[I_n T + A_p \frac{T^2}{2!} + A^2_p \frac{T_3}{3!} + \cdots\right]B_p, \\ [G^{(v)}_p - I_n]A^{-1}_p B_p, \quad \text{if } A_p \text{ is invertible.} \end{cases} \end{aligned}$$

It must be noted that $(G_p^{(v)} - I_n)A_p^{-1}$ is a short-hand notation, which is well defined and can be verified by the cancellation of A_p^{-1} in the series expansion of the term $(G_p^{(v)} - I_n)$. Therefore, the invertibility of matrix A_p is no longer required.

Likewise, the discrete-time state stated in (8.178), at $t = t_v = kT + vT$ for $0 \leq v \leq 1$, is found to be

$$\begin{aligned} x_{pd}(t_v) &= e^{A_p(t_v - kT)} x_{pd}(kT) + \int_{kT}^{kT+vT} e^{A_p(t_v-\tau)} B_p u_{pd}(\tau) d\tau \\ &\approx e^{A_p(t_v - kT)} x_{pd}(kT) + \int_{kT}^{kT+vT} e^{A_p(t_v-\tau)} B_p d\tau u_{pd}(kT) \\ &= G_p^{(v)} x_{pd}(kT) + H_p^{(v)} u_{pd}(kT). \end{aligned} \tag{8.183}$$

Thus, under the assumption of $x_{pc}(kT) = x_{pd}(kT)$, it results from (8.182) and (8.183) that to obtain the state $x_{pc}(t_v) = x_{pd}(t_v)$, it is necessary to have $u_{pc}(t_v) = u_{pd}(KT)$ which leads to the digital prediction-based controller

$$\begin{aligned} u_{pd}(kT) &= u_{pc}(t_v) \\ &= F_c u_{mc}(t_v) - L_{pc}^t(t_v) x_{pc}(t_v) - \gamma_c e_c^t(t_v) P B_p - L_c^t(t_v) x_{mc}(t_v) \\ &= F_c u_{mc}(t_v) - L_{pc}^t(t_v) x_{pd}(t_v) - \gamma_c e_c^t(t_v) P B_p - L_c^t(t_v) x_{md}(t_v) \\ &= F_c u_{mc}(t_v) - L_{pc}^t(t_v)[G_p^{(v)} x_{pd}(kT) + H_p^{(v)} u_{pd}(kT)] \\ &\quad - \gamma_c \big(G_p^{(v)} x_{pd}(kT) + H_p^{(v)} u_{pd}(kT) \\ &\quad - G_m^{(v)} x_{mc}(kT) + H_m^{(v)} u_{mc}(t_v)\big)^t P B_p \\ &\quad - L_c^t(t_v)(G_m^{(v)} x_{mc}(kT) + H_m^{(v)} u_{mc}(t_v)), \end{aligned} \tag{8.184}$$

where the future state $x_{pd}(t_v)$, with the substitution described in (8.177), needs to be predicted by basing on the available causal signals, $x_{pd}(kT)$ and $u_{pd}(kT)$. For practical applications, v can be considered as a tuning parameter for the desired closeness between the predicted digital and analogue states. If $v = 1$, then the pre-requisite $x_{pc}(kT + T) = x_{pd}(kT + T)$ is ensured. Thus, for $k = 0, 1, 2, \ldots$, solving for $u_{pd}(kT)$ from (8.178), the desired prediction based digital controller results in

$$\begin{aligned} u_{pd}(kT) &= F_d u_{md}(kT) - L_{pd}(kT) x_{pd}(kT) - L_d(kT) x_{md}(kT) \\ &\quad - \gamma_d e_d(kT)^t P B_p \\ &= F_d u_{md}(kT) - L_{pd}(kT) x_{pd}(kT) - L_d(kT) x_{md}(kT) \\ &\quad - \gamma_d B_p^t P e_d(kT), \end{aligned} \tag{8.185}$$

where

$$\begin{aligned} u_{md}(kT) &= u_{mc}(kT + T) = -K_d x_{md}(kT) + E_d r^*(kT), \\ e_d(kT) &= G_p x_{pd}(kT) - G_m x_{md}(kT), \end{aligned}$$

and

$$F_d(kT) = \big(I + L_{pc}(kT+T)H_p + \gamma_c B_p^t P H_p\big)^{-1} \times \big(F_c - L_c(kT+T)H_m + \gamma_c B_p^t P H_m\big), \tag{8.186}$$

$$L_{pd}(kT) = \big(I + L_{pc}(kT+T)H_p + \gamma_c B_p^t P H_p\big)^{-1} L_{pc}(kT+T)G_p, \tag{8.187}$$

$$L_d(kT) = \big(I + L_{pc}(kT+T)H_p + \gamma_c B_p^t P H_p\big)^{-1} L_c(kT+T)G_m, \tag{8.188}$$

$$\gamma_d = \big(I + L_{pc}(kT+T)H_p + \gamma_c B_p^t P H_p\big)^{-1} \gamma_c, \tag{8.189}$$

in which

$$\begin{aligned}
L_{pc}(kT+T) &\cong \frac{T}{2} B_p^t P[(x_{pc}(kT+T) - x_{mc}(kT+T))x_{pc}(kT+T)(x_{pc}(kT) \\
&\quad - x_{mc}(kT))x_{pc}^t(kT)] \\
&\cong \frac{T}{2} B_p^t P\{[G_p x_{pd}(kT) + H_p u_{pd}(kT) \\
&\quad - G_m x_{md}(kT) - H_m u_{md}(kT)][x_{pd}^t(kT)G_p^t + u_{pd}^t(kT)H_p^t] \\
&\quad + x_{pd}(kT)x_{pd}^t(kT) - x_{md}(kT)x_{pd}^t(kT)\}, \\
L_c(kT+T) &\cong \frac{T}{2} B_p^t P[(x_{pc}(kT+T) - x_{mc}(kT+T))x_{md}^t(kT+T) \\
&\quad + (x_{pc}(kT) - x_{mc}(kT))x_{md}^t(kT)] \\
&\cong \frac{T}{2} B_p^t P\big\{[G_p x_{pd}(kT) + H_p u_{pd}(kT) \\
&\quad - G_m x_{md}(kT) - H_m u_{md}(kT)][x_{md}^t(kT)G_m^t + u_{md}^t(kT)H_m^t] \\
&\quad + x_{pd}(kT)x_{md}^t(kT) - x_{md}(kT)x_{md}^t(kT)\big\}, \\
G_p &= e^{A_p T}, \quad H_p = (G_p - I)A_p^{-1}B_p, \\
P &= \mathrm{diag}[P_1\ P_2\ \ldots\ P_N].
\end{aligned}$$

8.7.3 Incorporating Optimal Tracker

As shown in Fig. 8.26, combining the interconnected system in Fig. 8.22 and the reference model with optimal tracker in Fig. 8.23 results in the **MRDAC** with optimal tracker, where the **MRDAC** ensures that the state of the system exactly tracks the one of reference model, while the latter is regulated by the tracker to follow the reference input as close as possible.

The main advantage of this structure is that the design procedure of overall control can be separated into two parts, therefore, the controller design of the two parts can be considered individually so that the complexity of the overall controller design is simplified considerably. Here, we would like to point out the proposed methodology can be easily extended from **SISO** subsystem to the **MIMO** subsystems.

The overall control methodology is summarized in the sequence of analogue and digital control block diagrams as Figs. 8.27 and 8.28.

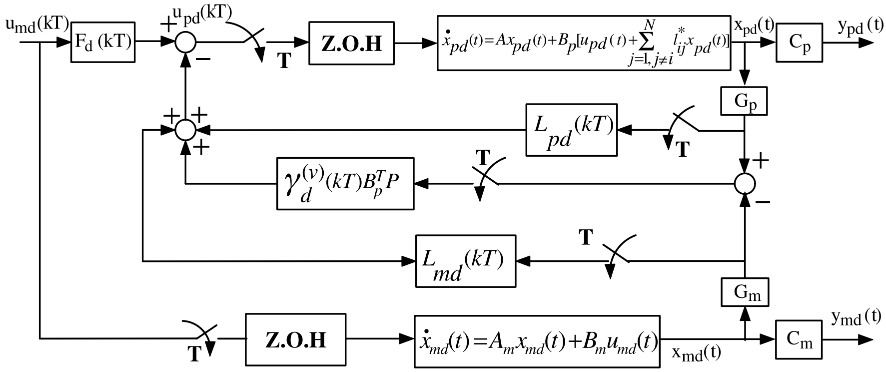


Fig. 8.26 The digitally redesigned sampled-data control

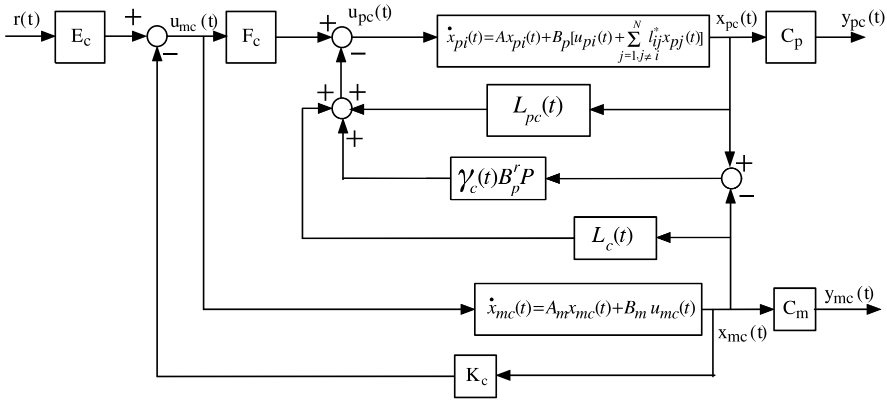


Fig. 8.27 The optimal tracker for MRDAC adaptive control system

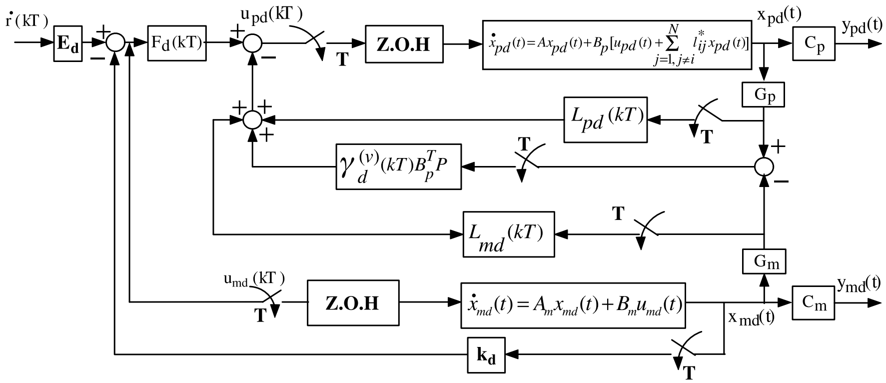


Fig. 8.28 The digitally redesigned tracker for MRDAC adaptive control system

8.7.4 Simulation Example 8.7

Consider a two-input two-output system Σ_p consisting of two subsystems $(\Sigma_{p1}, \Sigma_{p2})$ described by

$$\begin{aligned}\Sigma_{p1}:\quad \dot{x}_{p1}(t) &= A_{p1}x_{p1}(t) + b_{p1}[u_{p1}(t) + l_{12}^{*t}x_{p2}(t)],\\ \Sigma_{p2}:\quad \dot{x}_{p2}(t) &= A_{p2}x_{p2}(t) + b_{p2}[u_{p2}(t) + l_{21}^{*t}x_{p1}(t)],\end{aligned}$$

and stable reference model described by

$$\begin{aligned}\Sigma_{m1}:\quad \dot{x}_{m1}(t) &= A_{m1}x_{m1}(t) + B_{m1}u_{m1}(t),\\ \Sigma_{m2}:\quad \dot{x}_{m2}(t) &= A_{m2}x_{m2}(t) + B_{m2}u_{m2}(t),\end{aligned}$$

where

$$\begin{aligned}
x_{p1}(t) &= \begin{bmatrix} x_{p11}(t)\\ x_{p12}(t)\end{bmatrix}, \quad x_{p2}(t)\begin{bmatrix} x_{p21}(t)\\ x_{p22}(t)\end{bmatrix} \in \Re^2,\\
x_{m1}(t) &= \begin{bmatrix} x_{m11}(t)\\ x_{m12}(t)\end{bmatrix}, \quad x_{m2}(t)\begin{bmatrix} x_{m21}(t)\\ x_{m22}(t)\end{bmatrix} \in \Re^2,\\
A_{p1} &= \begin{bmatrix} 2 & -2\\ -7 & -2\end{bmatrix}, \quad A_{p2} = \begin{bmatrix} 0 & 4\\ 13 & 2\end{bmatrix}, \quad b_{p1} = \begin{bmatrix} 1\\ -1\end{bmatrix},\\
b_{p2} &= \begin{bmatrix} 1\\ 3\end{bmatrix}, \quad c_{p1} = \begin{bmatrix} 1\ 2\end{bmatrix}, \quad c_{p2} = \begin{bmatrix} 2\ 1\end{bmatrix},\\
A_{m1} &= \begin{bmatrix} -6 & 2\\ 1 & -6\end{bmatrix}, \quad A_{m2} = \begin{bmatrix} -4 & 2\\ 1 & -4\end{bmatrix},\\
b_{m1} &= b_{p1}, \quad b_{m2} = b_{p2}, \quad c_{m1} = c_{p1}, \quad c_{m2} = c_{p2},\\
&u_{p1}(t),\ u_{p2}(t),\ u_{m1}(t),\ u_{m2}(t),\ y_{p1}(t),\ y_{p2}(t), y_{m1}(t),\ y_{m2}(t) \in R.
\end{aligned}$$

The interconnections between the two subsystems, are represented by

$$l_{12}^{*t} = [1\ 1], \quad l_{21}^{*t}[1\ 1]$$

and the desired linear feedback gains, as defined by (8.144), are given as $K_{m1} = [-8\ 4]$, $K_{m2}[-4\ -2]$. Both the controllers Σ_{c1}, Σ_{c2} are aware of the desired outputs $(x_{m1}(t), x_{m2}(t))$.

The symmetric positive-definite matrices P_1, P_2, solutions of the Lyapunov equations described in (8.158), are

$$P_1 = \begin{bmatrix} 0.087 & 0.0221\\ 0.0221 & 0.0907\end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} 0.1384 & 0.0536\\ 0.0536 & 0.1518\end{bmatrix}$$

respectively. The continuous-time decentralized adaptive controller for the interconnected system is designed as

$$\begin{aligned}u_{p1}(t) &= F_1u_{m1}(t) - l_{p1}^t(t)x_{p1}(t) - \gamma_1 e_1^t(t)P_1b_{p1} - l_{12}^t(t)x_{m2}(t),\\ u_{p2}(t) &= F_2u_{m2}(t) - l_{p2}^t(t)x_{p2}(t) - \gamma_2 e_2^t(t)P_2b_{p2} - l_{21}^t(t)x_{m1}(t),\end{aligned}$$

where the parameters are adjusted adaptively according to the control laws stated in (8.157) and (8.158). And the reference model is regulated by the tracker described in (8.167) where

$$k_c = \begin{bmatrix} 988.95 & 497.88 & 0 & 0 \\ 0 & 0 & 499.71 & 999.14 \end{bmatrix},$$

$$E_c = \begin{bmatrix} 999.74 & 0 \\ 0 & 1000 \end{bmatrix}.$$

Applying the digital redesign method proposed in Sect. 8.7, the interconnected system and the reference model are controlled by (8.171) and (8.180), respectively, where the sampling period is $T = 0.1$ (s) and

$$G_p = \begin{bmatrix} 1.3234 & -0.1792 & 0.2168 & 0.1720 \\ -0.7479 & 0.8577 & -0.2245 & -0.1784 \\ 0.1431 & 0.1505 & 1.2904 & 0.4819 \\ 0.3482 & 0.3676 & 1.5662 & 1.5310 \end{bmatrix},$$

$$H_p = \begin{bmatrix} 0.1233 & 0.0302 \\ -0.1284 & -0.0310 \\ -0.0002 & 0.1764 \\ -0.0006 & 0.4342 \end{bmatrix},$$

$$G_m = \begin{bmatrix} 0.5543 & 0.1101 & 0 & 0 \\ 0.0551 & 0.5543 & 0 & 0 \\ 0 & 0 & 0.6770 & 0.1345 \\ 0 & 0 & 0.0673 & 0.6770 \end{bmatrix},$$

$$H_m = \begin{bmatrix} 0.0686 & 0 \\ -0.0720 & 0 \\ 0 & 0.1058 \\ 0 & 0.2519 \end{bmatrix}.$$

The digital gains (K_d, E_d) in (8.171), determined from the analogue gains (K_c, E_c) in (8.167), are thus

$$k_d = \begin{bmatrix} 17.4354 & 11.6587 & 0 & 0 \\ 0 & 0 & 1.3274 & 2.4342 \end{bmatrix} \quad \text{and}$$

$$E_d = \begin{bmatrix} 30.2831 & 0 \\ 0 & 3.2733 \end{bmatrix}.$$

Finally, the trajectories generated by the aforementioned digital controller are as close as possible to the original continuous-time state trajectories. The trajectories of the controller inputs with signal responses and states are shown in Figs. 8.29–8.38. To show the robustness of the proposed, let the tracker have good performance in the beginning, but the first subsystem input is artificially reduced to 5 of the determined input by external factor in 13 sec. Figs. 8.39 and 8.40 show that the proposed decentralized controller induces a good robustness on the decoupling of the closed-loop controlled system. When the inputs of parts of the system are broken, the others

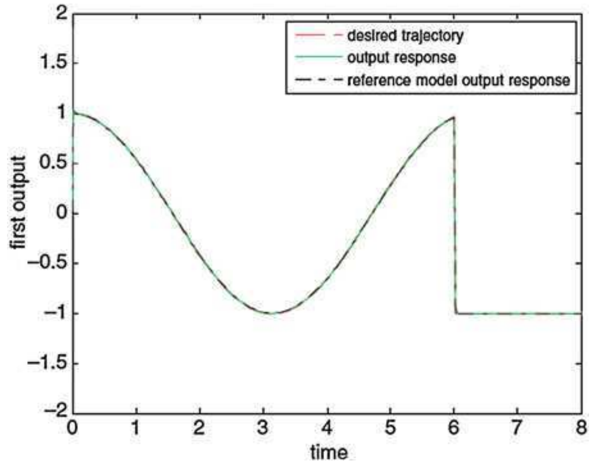


Fig. 8.29 First output responses using analogue controller

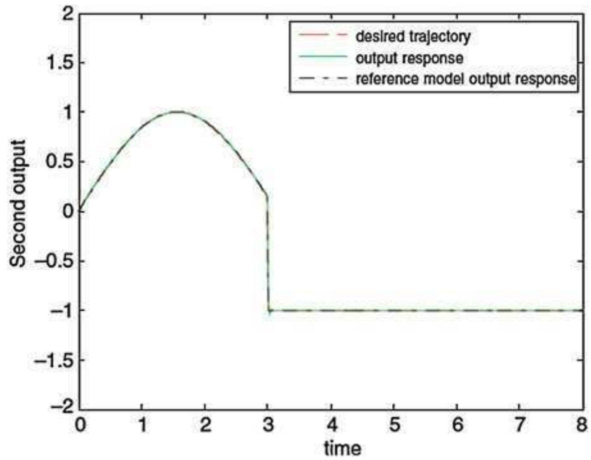


Fig. 8.30 Second output responses using analogue controller

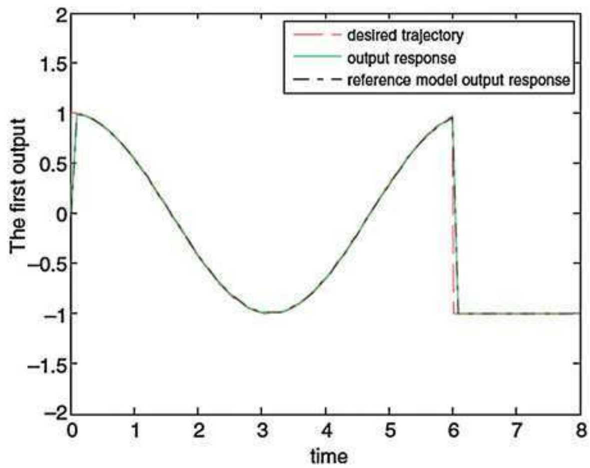


Fig. 8.31 Output responses for the first output by digital controller ($T = 0.1$ s)

Fig. 8.32 Output responses for the second output by digital controller ($T = 0.1$ s)

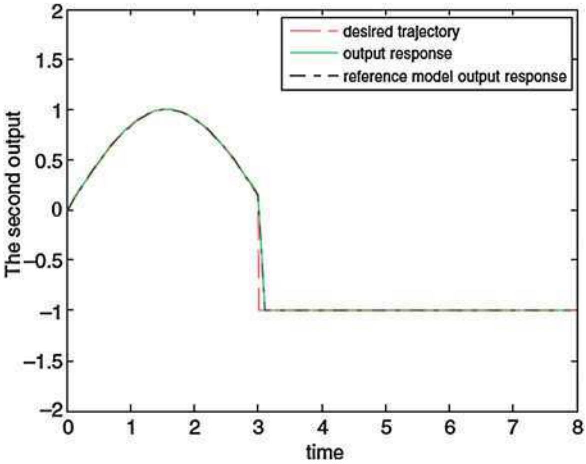


Fig. 8.33 The first-output comparisons of various output responses for $t = 0 \sim 8$ s ($T = 0.1$ s)

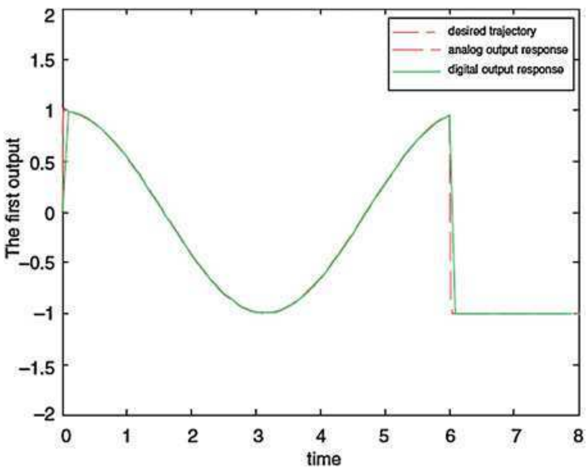
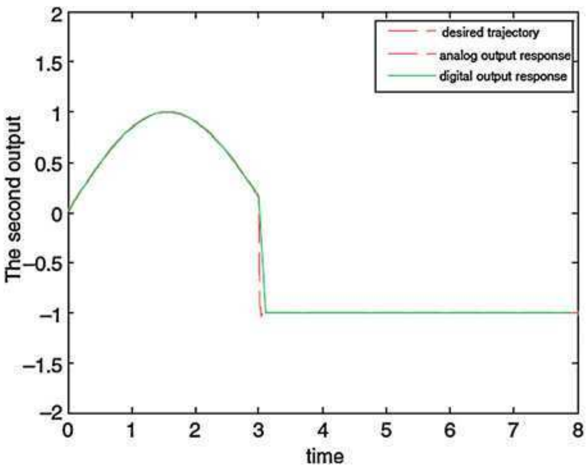


Fig. 8.34 The second-output comparisons of various output responses for $t = 0 \sim 8$ s ($T = 0.1$ s)



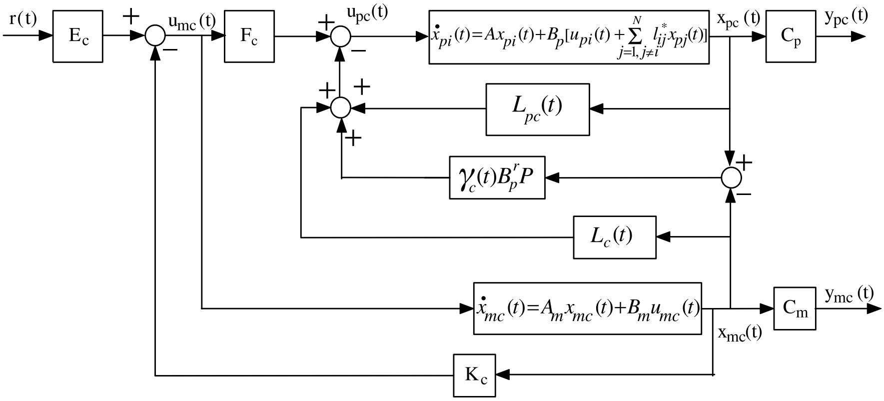


Fig. 8.35 Comparisons of various state x_{11} between plant and reference model ($T = 0.1$ s)

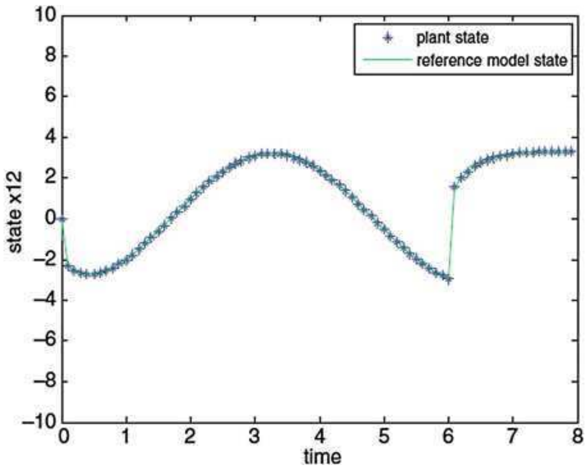


Fig. 8.36 Comparisons of various state x_{12} between plant and reference model ($T = 0.1$ s)

are not influenced entirely, so that the other digitally controlled systems still follow the reference inputs rapidly.

8.8 Notes and References

In this chapter we considered the control problem for a class of time varying nonlinear large-scale systems with time delays in the interconnections, and the interconnections can be nonlinear. An adaptive state feedback controller is proposed that is independent of time delays, and render the closed-loop system uniformly ultimately bounded stable. The result is also applied to control a class of interconnected systems whose nominal system is linear, and the corresponding state feedback controller and adaptive laws are obtained. Finally, numerical examples are given to

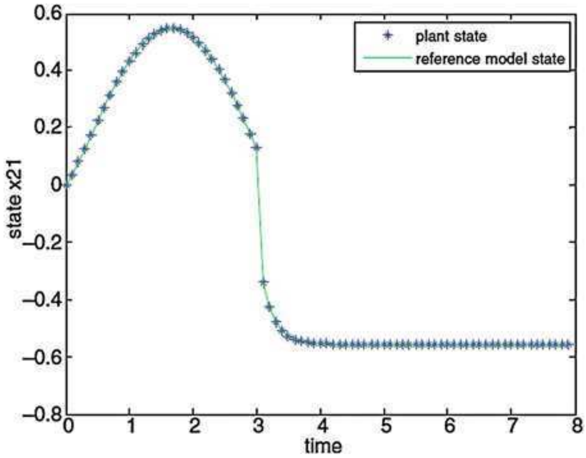


Fig. 8.37 Comparisons of various state x_{21} between plant and reference model ($T = 0.1$ s)

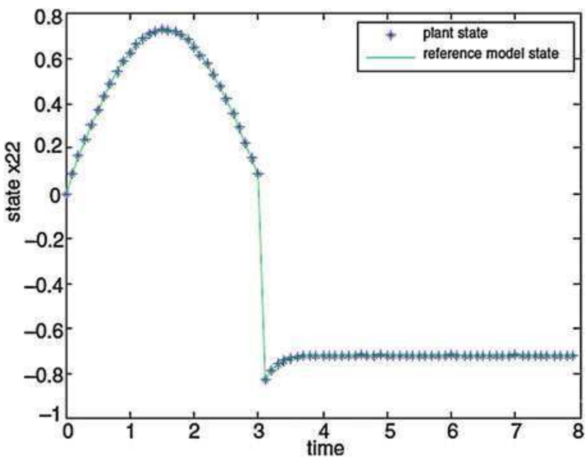


Fig. 8.38 Comparisons of various state x_{22} between plant and reference model ($T = 0.1$ s)

demonstrate the validity of the results developed in this chapter. It is shown from the example that the results obtained are effective and feasible. Therefore our results can be expected to have some applications to practical control problems of uncertain dynamic interconnected systems with time delay.

This chapter further extends the class of large-scale nonlinear systems for which decentralized controllers can be designed. This class is identified by systems transformable to a decentralized strict feedback form for which the matching condition is not satisfied. Geometric conditions for the existence of a parameter independent, decentralized diffeomorphism are presented. Higher order interconnections bounded by unknown pth order polynomials are considered. A constructive, stepwise procedure for decentralized control design is presented. Adaptation laws are obtained to update the control gains to counter uncertainties in the interconnections. Global regulation is proved, and its effectiveness is shown using a simulation example. Robustness of the developed control laws to perturbations in the system dynamics is

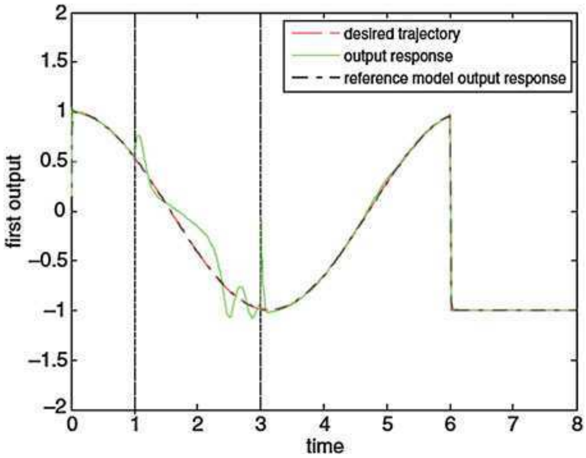


Fig. 8.39 Response of the first subsystem by analogue controller with 5% reduced input

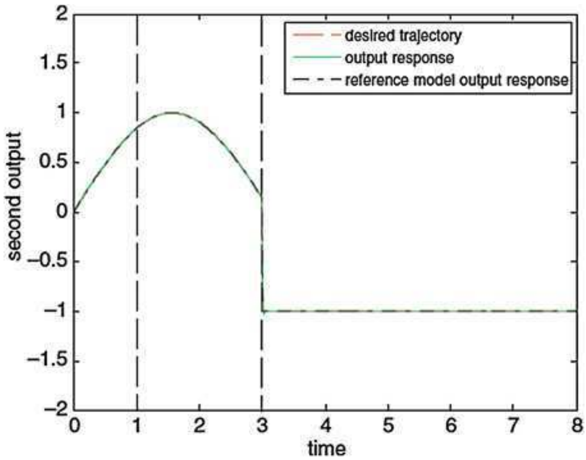


Fig. 8.40 Response of the second subsystem by analogue controller with 5% reduced input

established through simulations. Furthermore, it is shown that no redesign of decentralized controllers is required for the original subsystems if new subsystems are appended to the system. Finally, an adaptive model reference decentralized tracking control design is outlined for the same class of large-scale systems. Global uniform boundedness of the tracking error to a compact set is obtained in this case.

Finally, it is shown that the exact tracking of individual subsystem outputs is theoretically possible by the strictly decentralized control, where the controller of each subsystem has no knowledge of the inputs or states of the other subsystems. To achieve the exact tracking, the desired states or outputs of the other subsystems Σ_{mj} must be accessible to each controller Σ_{ci} of subsystem Σ_{pi} and be used in place of the corresponding inaccessible states or outputs to compensate for the perturbation signals from them.

The awareness of the desired states of the other subsystems implies that there is implicit cooperation among the subsystems, and communication between subsystems becomes necessary when some of the subsystems change their desired trajectories. From a theoretical point of view, the problem reduces to that of demonstrating overall system stability. By using the high gain controller, the individual subsystems are made sufficiently stable so that the overall stability is assured in spite of interconnections. This is accomplished by using appropriate feedback and proving the existence of a Lyapunov function for the overall system.

With the utilization of the reference model representing the desired states, the design procedure of the controller is separated into two parts, one ensures that the state of the reference model is exactly tracked by the one of the systems and the other makes the reference model follow the reference input as closely as possible. Thus, the controller design of the two parts can be considered individually so that the complexity of the overall controller design is considerably simplified. Since the perfect model following is possible in the ideal case, it is believable intuitively that the bounds on the errors obtained will also be smaller.

The control methodology is further applied to a class of large-scale sampled-data systems with interconnected strengths. It is shown that in the sampled data decentralized adaptive control, it is theoretically possible to asymptotically track desired outputs with a desired performance and shows that the proposed decentralized controller induces a good robustness on the decoupling of the closed-loop controlled system. The prediction-based digital redesign methodology is utilized to find the new digital controllers for effective digital control of the analogue plant. An illustrative example of interconnected linear system is presented to demonstrate the effectiveness of the proposed design methodology.

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Chapter 9
Mathematical Tools

9.1 Finite Dimensional Spaces

In what follows, we will introduce some of the fundamental notions in linear algebra. The treatment is essentially meant to be as assembly of analytical tools with references given at the end of the chapter. We provide few proofs so as to encourage the reader to gain practice with the machinery and results.

9.1.1 Vector Spaces

The structure introduced hereafter is the corner-stone of system theory, that of a *vector space*, also called a linear space. Let $x_j, y_j \in \Re$ (or $\mathcal{C}$), $j = 1, 2, \ldots, n$. Then the n-dimensional vectors $x,\ y$ are defined by $x = [x_1\ x_2\ \ldots\ x_n]^t$, $y = [y_1\ y_2\ \ldots\ y_n]^t$ $\in \Re^n$, respectively, where $\Re^n = \Re \times \cdots \times \Re$. A nonempty set $\mathcal{X}$ of elements $x, y, \ldots$ is called the *real* (*or complex*) *vector space* (*or real* (*complex*) *linear space*) by defining two algebraic operations, *vector additions*, *and scalar multiplication*, *in* $x = [x_1\ x_2\ \ldots\ x_n]^t$.

Given two vector spaces $\mathcal{X}_1$ and $\mathcal{X}_2$ with the same associated scalar field, we use $\mathcal{X}_1 \times \mathcal{X}_2$ to denote the vector space formed by their Cartesian product. Thus every element of $\mathcal{X}_1 \times \mathcal{X}_2$ is of the form

$$(x_1, x_2) \quad \text{where } x_1 \in \mathcal{X}_1 \text{ and } x_2 \in \mathcal{X}_2.$$

A nonempty subset $\mathcal{G} \subset \Re^n$ is called a *linear subspace* of $\Re^n$ if $x + y$ and αx are in $\mathcal{G}$ whenever x and y are in $\mathcal{G}$ for any scalar α. A set of elements $X = \{x_1,\ x_2,\ \ldots,\ x_n\}$ is said to be a *spanning set* for a linear subspace $\mathcal{G}$ of $\Re^n$ if every elements $g \in \mathcal{G}$ can be written as a linear combination of the $\{x_j\}$. That is, we have

$$\mathcal{G} = \{g \in \Re : g = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \text{ for some scalars } \alpha_1, \alpha_2,\ \ldots,\ \alpha_n\}.$$

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Sometimes the shorthand notation

$$\text{span}\{x_1, x_2, \ldots, x_n\}$$

is used. A spanning set X is said to be a *basis* for $\mathcal{G}$ if no element x_j of the spanning set X of $\mathcal{G}$ can written as a linear combination of the remaining elements $x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n$, that is, x_j, $1 \le j \le n$ form a linearly independent set. It is frequent to use $x_j = [0\ 0\ \ldots\ 0\ 1\ 0\ \ldots\ 0]^t$ the jth unit vector. The geometric ideas of linear vector spaces had led to the concepts of "*spanning a space*" and a "*basis for a space*".

The n-dimensional Euclidean space, denoted throughout this book by $\Re^n$, is a linear vector space equipped by the inner product

$$\langle x, y\rangle = x^t y = \sum_{j=1}^{n} x_j y_j.$$

Let $\mathcal{X}$ be a linear space over the *field* **F** (typically **F** is the field of real numbers $\Re$ or complex numbers $\mathcal{C}$). Then a function

$$\|.\| : \mathcal{X} \to \Re$$

that maps $\mathcal{X}$ into the real numbers $\Re$ is a norm on $\mathcal{X}$ iff

1. $\|x\| \ge 0$, $\forall x \in \mathcal{X}$ (non-negativity)
2. $\|x\| = 0 \Leftrightarrow x = 0$ (positive definiteness)
3. $\|\alpha x\| = |\alpha|\|x\|$ $\forall x \in \mathcal{X}$ (homogeneity with respect to $|\alpha|$)
4. $\|x + y\| \le \|x\| + \|y\|$, $\forall x, y \in \mathcal{X}$ (triangle inequality)

Given a linear space $\mathcal{X}$, there are many possible norms on it. For a given norm $\|.\|$ on $\mathcal{X}$, the pair $(\mathcal{X}, \|.\|)$ is used to indicate $\mathcal{X}$ endowed with the norm $\|.\|$.

9.1.2 Norms of Vectors

The class of L_p-norms is defined by

$$\|x\|_p = \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p}, \quad \text{for } 1 \le p < \infty,$$
$$\|x\|_\infty = \max_{1 \le j \le n} |x_j|.$$

The three most commonly used norms are $\|x\|_1$, $\|x\|_2$ and $\|x\|_\infty$. All p-norms are equivalent in the sense that if $\|x\|_{p1}$ and $\|x\|_{p2}$ are two different p-norms, then there exist positive constants c_1 and c_s such that

$$c_1 \|x\|_{p1} \le \|x\|_{p2} c_2 \ \|x\|_{p1}, \quad \forall x \in \Re^n.$$

Induced norms of matrices For a matrix $A \in \Re^{n\times n}$, the *induced p-norm* of A is defined by

$$\|A\|_p \triangleq \sup_{x\neq 0} \frac{\|Ax\|_p}{\|x\|_p} = \sup_{\|x\|_p=1} \|Ax\|_p.$$

Obviously, for matrices $A \in \Re^{m\times n}$ and $B \in \Re^{n\times r}$, we have the *triangle inequality*

$$\|A + B\|_p \leq \|A\||_p + \|B\|_p.$$

It is easy to show that the *induced norms* are also equivalent in the same sense as for the vector norms, and satisfying

$$\|AB\|_p \leq \|Ax\|_p \|B\|_p, \quad \forall A \in \Re^{n\times m}, \forall B \in \Re^{m\times r}$$

which is known as the *submultiplicative property*. For $p = 1, 2, \infty$, we have the corresponding induced norms as follows

$$\|A\|_1 = \max_j \sum_{s=1}^{n} |a_{sj}| \quad (\textit{column sum}),$$

$$\|A\|_2 = \max_j \sqrt{\lambda_j(A^t A)},$$

$$\|A\|_\infty = \max_s \sum_{j=1}^{n} |a_{sj}| \quad (\textit{row sum}).$$

9.1.3 Some Basic Topology

We start by defining the notion of *neighborhood* of a point in the vector space $\mathcal{V}$. To do this, we first define the *unit ball* with respect to a basis. Suppose that $\{u_1, u_2, \dots, u_n\}$ is a basis for the vector space $\mathcal{V}$. The open unit ball $\mathcal{B}$ in this basis is defined by

$$\mathcal{B}(u_1, \dots, u_n) = \{\alpha_1 u_1 + \cdots + \alpha_n u_n \in \mathcal{V} : \alpha_j \in \Re, \alpha_1^2 + \cdots + \alpha_n^2 < 1\}.$$

This set contains all the points that can be expressed, in the basis, with the vector of coefficients α inside the unit sphere of $\Re^n$ and clearly it is basis-dependent.

Next we define the notion of neighborhood of a point, which intuitively means any set that totally surrounds the given point in the vector space.

A subset $\mathsf{N}(0)$ *of the vector space* $\mathcal{V}$ i*s a neighborhood of the zero element if there exists a basis* $u_1, u_2, \dots, u_n$ *for* $\mathcal{V}$ *such that*

$$\mathcal{B}(u_1, \dots, u_n) \subset \mathsf{N}(0).$$

Further, a subset $\mathsf{N}(w) \subset \mathcal{V}$ *is a neighborhood of the point* $w \in \mathcal{V}$ *if the set*

$$\mathsf{N} = \{v \in \mathcal{V} : v + w\mathsf{N}(w)\}$$

is a neighborhood of the zero element. Alternatively, this implies that a set is a neighborhood of zero provided that one of its subsets is the unit ball in some basis element.

9.1.4 Convex Sets

A set $\mathbf{M} \subset \Re^n$ is said to be *open* if every vector $x \in \mathbf{M}$, there is an ε-neighborhood of x

$$\mathcal{N}(x, \varepsilon) = \{z \in \Re^n | \, \|z - x\| < \varepsilon\}$$

such that $\mathcal{N}(x, \varepsilon) \subset \mathbf{M}$.

A set is *closed* iff its complement in $\Re^n$ is open; *bounded* if there $r > 0$ such that $\|x\| < r$, $\forall x \in \mathbf{S}$; and *compact* if it is closed and bounded; *convex* if for every $x, y \in \mathbf{S}$, and every real number α, $0 < \alpha < 1$, the point $\alpha\, x + (1 - \alpha)\, x \in \mathbf{S}$.

Let us now define the line segment that joins two points in $\mathcal{V}$. Suppose that $v_1, v_2 \in \mathcal{V}$, then we define the line segment $\mathbf{L}(v_1, v_2)$ between them as the set of points

$$\mathbf{L}(v_1, v_2) = \{v \in \mathcal{V} : v = \mu v_1 + (1 - \mu) v_2, \text{ for some } \mu \in [0, 1]\}.$$

Clearly the end points of the line segment are v_1 and v_2, which occur in the parametrization when $\mu = 1$ and $\mu = 0$, respectively. We can now turn to the idea of *convexity*. Suppose that $\mathbf{K}$ is a nonempty subset of the vector space $\mathcal{V}$. Then $\mathbf{K}$ is defined to be *convex set* if *for any* $v_1, v_2 \in \mathsf{K}$, *the line segment* $\mathbf{L}(v_1, v_2)$ *is a subset of* $\mathbf{K}$, see Fig. 9.1. This simply means that given two points in a convex set, the line segment between them is also in the set. Note in particular that subspaces and linear varieties (a linear variety is a translation of linear subspaces) are convex. Also the empty set is considered convex. Clearly any vector space is convex, as is any subset $\{\nu\}$ of a vector space containing only a single element. Consider the expression

$$v = \mu_1 v_1 + \cdots + \mu_n v_n, \quad \mu_1 + \cdots + \mu_n = 1$$

which provides a clear generalization to an average of n points $v_1, \ldots, v_n$. Extending this further, the generalization of the line segment between two points to n points yields a point inside the perimeter defined by the points $v_1, \ldots, v_n$. This is illustrated in Fig. 9.2. Building on this intuition from $\Re^2$, we extend the idea to an arbitrary vector space $\mathcal{V}$. Given $v_1, \ldots, v_n$ we define the *convex hull* of these points by

$$\text{con}(\{v_1, \ldots, v_n\}) = \left\{ v \in \mathcal{V} : v = \sum_{k=1}^{n} \mu_k v_k, \; \mu_k \in [0, 1], \; \sum_{k=1}^{n} \mu_k = 1 \right\}.$$

With reference to Fig. 9.2 this set is made of the points inside the perimeter, that is, *the convex hull of the points* $v_1, \ldots, v_n$ *is simply the set composed of all weighted averages of these points*. In particular, we have that for two points

$$\mathbf{L}(v_1, v_2) = \text{con}(\{v_1, v_n\}).$$

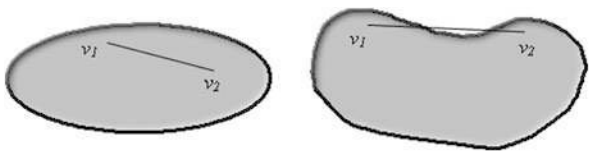

Fig. 9.1 Convex (*left*) and nonconvex (*right*) sets

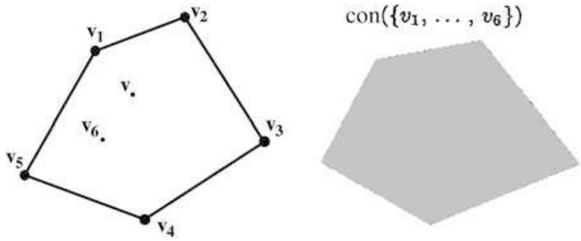

Fig. 9.2 Convex hull of finite number of points in $\Re^2$

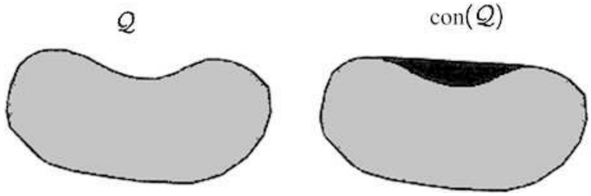

Fig. 9.3 Convex hull of a set $\mathcal{R}$

It is not difficult to show that if $\mathcal{R}$ is convex, then it necessarily contains any convex hull formed from a collection of its points. Generalizing this to an arbitrary set. Given a set $\mathcal{R}$, we define its convex hull con($\mathcal{R}$) by

$$\text{con}(\mathcal{R}) = \{v \in \mathcal{V} : \text{ there exist } n \text{ and } \{v_1, \ldots, v_n\} \in \mathcal{R}$$
$$\text{such that } v \in \text{con}(\{v_1, \ldots, v_n\})\}.$$

In brief, the convex hull of $\mathcal{R}$ is the collection of all possible weighted averages of points in $\mathcal{R}$.

The following facts provide important properties for convex sets and convex hull.

1. Let $\mathcal{C}_j$, $j = 1, \ldots, m$ be a family of m convex sets in $\Re^n$. Then the intersection $\mathcal{C}_1 \cap \mathcal{C}_2 \cap \cdots \cap \mathcal{C}_m$ is convex.
2. Let $\mathcal{C}$ be a convex set in $\Re^n$ and $x_0 \in \Re^n$. Then the set $\{x_0 + x : x \in \mathcal{C}\}$ is convex.
3. A set $\mathbf{K} \subset \Re^n$ is said to be *convex cone* with vertex x_0 if $\mathbf{K}$ is convex and $x \in \mathbf{K}$ implies that $x_0 + \lambda x \in \mathbf{K}$ for any $\lambda \geq 0$.
4. The subset condition $\mathcal{R} \subset \text{con}(\mathcal{R})$ is satisfied.
5. The convex hull con($\mathcal{R}$) is convex.
6. The relationship $\text{con}(\mathcal{R}) = \text{con}(\text{con}(\mathcal{R}))$ holds.
7. A set $\mathcal{R}$ is convex if and only if $\text{con}(\mathcal{R}) = \mathcal{R}$ is satisfied, see Fig. 9.3.

An important class of convex cones is the one defined by the positive semidefinite ordering of matrices, that is, $A_1 \geq A_2 \geq A_3$. Let $P \in \Re^{n \times n}$ be a positive semidefinite matrix. The set of matrices $X \in \Re^{n \times n}$ such that $X \geq P$ is a convex cone in $\Re^{n \times n}$.

9.1.5 Continuous Functions

A function $f : \Re^n \to \Re^m$ is said to be *continuous* at a point x if $f(x+\delta x) \to f(x)$ whenever $\delta x \to 0$. Equivalently, f is continuous at x if, given $\varepsilon > 0$, there is $\delta > 0$ such that

$$\|x - y\| < \varepsilon \quad \Longrightarrow \quad \|f() - f(y)\| < \varepsilon.$$

A function f is continuous on a set of $\mathbf{S}$ if it is a continuous at every point of $\mathbf{S}$, and it is uniformly continuous on $\mathbf{S}$ if given $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ (dependent only on ε), such that the inequality holds for all $x, y \in \mathbf{S}$

A function $f : \Re \to \Re$ is said to be *differentiable* at a point x if the limit

$$\dot{f}(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

exists. A function $f : \Re^n \to \Re^m$ is *continuously differentiable* at a point x (a set $\mathbf{S}$) if the partial derivatives $\partial f_j / \partial x_s$ exist and continuous at x (at every point of $\mathbf{S}$) for $1 \le j \le m$, $1 \le s \le n$ and the *Jacobian matrix* is defined as

$$\mathbf{J} = \left[\frac{\partial f}{\partial x}\right] = \begin{bmatrix} \partial f_1/\partial x_1 & \dots & \partial f_1/\partial x_n \\ \vdots & \ddots & \vdots \\ \partial f_m/\partial x_1 & \dots & \partial f_m/\partial x_n \end{bmatrix} \in \Re^{m \times n}.$$

9.1.6 Function Norms

Let $f(t) : \Re_+ \to \Re$ be a continuous function or piecewise continuous function. The p-norm of f is defined by

$$\|f\|_p = \left(\int_0^\infty |f(t)|^p dt\right)^{1/p}, \quad \text{for } p \in [1, \infty),$$
$$\|f\|_\infty = \sup_{t \in [0,\infty)} |f(t)|, \quad \text{for } p = \infty.$$

By letting $p = 1, 2, \infty$, the corresponding normed spaces are called $\mathbf{L}_1$, $\mathbf{L}_2$, $\mathbf{L}_\infty$, respectively. More precisely, let $f(t)$ be a function on $[0, \infty)$ of the signal spaces, they are defined as

$$\mathbf{L}_1 \triangleq \left\{ f(t) : \Re_+ \longrightarrow \Re \;\middle|\; \|f\|_1 = \int_0^\infty |f(t)| dt < \infty, \text{ convolution kernel} \right\},$$
$$\mathbf{L}_2 \triangleq \left\{ f(t) : \Re_+ \longrightarrow \Re \;\middle|\; \|f\|_2 = \int_0^\infty |f(t)|^2 dt < \infty, \text{ finite energy} \right\},$$
$$\mathbf{L}_\infty \triangleq \left\{ f(t) : \Re_+ \longrightarrow \Re \;\middle|\; \|f\|_\infty = \sup_{t \in [0,\infty)} |f(t)| < \infty, \text{ bounded signal} \right\}.$$

From a signal point of view, the 1-*norm*, $\|x\|_1$ of the signal $x(t)$ is the integral of its absolute value, the square $\|x\|_2^2$ of the 2-*norm* is often called the energy of the signal $x(t)$, and the ∞-*norm* is its absolute maximum amplitude or peak value. It must be emphasized that the definitions of the norms for vector functions are not unique.

In the case of $f(t) : \Re_+ \to \Re^n$, $f(t) = [f_1(t)\ f_2(t)\ \dots\ f_n(t)]^t$ which denote a continuous function or piecewise continuous vector function, the corresponding p-norm spaces are defined as

$$\mathbf{L}_p^n \triangleq \left\{ f(t) : \Re_+ \longrightarrow \Re^n \mid \|f\|_p = \int_0^\infty \|f(t)\|^p dt < \infty, \text{ for } p \in [1, \infty) \right\},$$
$$\mathbf{L}_\infty^n \triangleq \left\{ f(t) : \Re_+ \longrightarrow \Re^n \mid \|f\|_\infty = \sup_{t \in [0,\infty)} \|f(t)\| < \infty \right\}.$$

9.1.7 Mean Value Theorem

Assume that $f : \Re^n \to \Re$ is continuously differentiable at each point x of an open set $\mathbf{S} \subset \Re^n$. Let x and y be two points of $\mathbf{S}$ such that

$$L(x, y) = \{z \mid z = \theta x + (1 - \theta)y, 0 < \theta < 1\} \subset \mathbf{S},$$

where $L(x, y)$ is a line segment connecting x and y. Then there exists a point z of $L(x, y)$ such that

$$f(y) - f(x) = \left.\frac{\partial f}{\partial x}\right|_{x=z} (y - x).$$

9.1.8 Implicit Function Theorem

Assume that $f : \Re^n \times \Re^m \to \Re^n$ is continuously differentiable at each point (x, y) of an open set $\mathbf{S} \subset \Re^n \times \Re^m$. Let (x_0, y_0) be a point in $\mathbf{S}$ for which $f(x_0, y_0) = 0$ and for which the Jacobian matrix $[\partial f/\partial x](x_0, y_0)$ is nonsingular. Then there exist neighborhoods $U \subset \Re^n$ of x_0 and $V \subset \Re^m$ of y_0 such that for each $y \in V$ the equation $f(x, y) = 0$ has a unique solution $x \in U$. Moreover, this solution can be given as $x = g(y)$, where g is continuously differentiable at $y = y_0$.

For a detailed account of the foregoing two theorems, the reader is referred to [1].

9.2 Matrix Theory

In this section, we focus on matrix theory and solicit some basic facts and useful relations from linear algebra and calculus of matrices. The material are stated along with some hints whenever needed but without proofs unless we see the benefit of providing a proof. We start by introducing the concept of a *linear mapping* between vector spaces. The mapping $M : \mathcal{V} \to \mathcal{W}$ is linear if

$$M(\alpha v_1 + \beta v_2) = \alpha M v_1 + \beta M v_2$$

for all $v_1, v_2 \in \mathcal{V}$ and all scalars α and β. Here $\mathcal{V}$ and $\mathcal{W}$ are vector spaces with the same associated filed $\mathbf{F}$. The space $\mathcal{V}$ is called the *domain* of the mapping, and $\mathcal{W}$ its *codomain.*

Given bases $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_n\}$ for $\mathcal{V}$ and $\mathcal{W}$, respectively, we associate scalars m_{jk} with the mapping M, defining them such that they satisfy

$$M v_k = m_{1k} w_1 + m_{2k} w_2 + \cdots + m_{mk} w_m$$

for each $1 \le k \le n$. Namely, given any basis vector v_k, the coefficients are the coordinates of $M v_k$ in the selected basis of $\mathcal{W}$. It turns out that these mn numbers m_{jk} completely specify the linear mapping M. To see this is true, consider any vector $v \in \mathcal{V}$, and let $w = Mv$. We can express both vectors in their respective bases as

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n, \quad w = \beta_1 w_1 + \beta_2 w_2 + \cdots + \beta_m w_m.$$

Now we have

$$\begin{aligned} w = Mv = M(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) \\ = \alpha_1 M v_1 + \alpha_2 M v_2 + \cdots + \alpha_n M v_n \\ = \sum_{k=1}^{n} \sum_{j=1}^{m} \alpha_k m_{jk} w_j \\ = \sum_{j=1}^{m} \left(\sum_{k=1}^{n} \alpha_k m_{jk} \right) w_j \end{aligned}$$

and therefore by uniqueness of the coordinates we must have

$$\beta_j = \sum_{k=1}^{n} \alpha_k m_{jk}, \quad j = 1, \ldots, m.$$

To express this relationship in a more convenient form, can write the set of numbers m_{jk} as the $m \times n$ matrix

$$[M] = \begin{bmatrix} m_{11} & \ldots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{m1} & \ldots & m_{mn} \end{bmatrix}.$$

Then via the standard matrix product, we have

$$\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix} = \begin{bmatrix} m_{11} & \ldots & m_{1n} \\ \vdots & \ddots & \ldots \\ m_{m1} & \ldots & m_{mn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}.$$

In summary any linear mapping M between vector spaces can be regarded as a matrix M mapping $\Re^n$ to $\Re^m$ via matrix multiplication. It should be noted that the numbers m_{jk} depend intimately on the bases $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_n\}$. Frequently we use only one basis for $\mathcal{V}$ and one for $\mathcal{W}$ and thus there is need to distinguish between the map M and the basis dependent matrix $[A]$. We will henceforth write M to denote either the map or the matrix, making which is meant context dependent.

When reference is made to matrix function $M(t)$, we have the form:

$$M(t) = \begin{bmatrix} m_{11}(t) & \ldots & m_{1n}(t) \\ \vdots & \ddots & \vdots \\ m_{m1}(t) & \ldots & m_{mn}(t) \end{bmatrix}.$$

9.2.1 Fundamental Subspaces

Building upon the foregoing section, the idea now is to introduce four important subspaces which are useful. The entire linear vector space of a specific problem can be decomposed into the sum of these subspaces.

The *column space* of a matrix $A \in \Re^{n \times m}$ is the space spanned by the columns of A, is also called the *range space* of A, denoted by $\mathcal{R}[A]$. Similarly, the *row space* of A is the space spanned by the rows of A. Since the column rank of a matrix is the dimension of the space spanned by the columns and the row rank is the dimension of the space spanned by the rows, it is clear that the spaces $\mathcal{R}[A]$ and $\mathcal{R}[A^t]$ have the same dimension $r = \text{rank}(A)$.

The *right null space* of $A \in \Re^{n \times m}$ is the space spanned by all vectors x that satisfy $Ax = 0$, and is denoted $\mathcal{N}[A]$. The right null space of A is also called the *kernel* of A. The *left null space* of A is the space spanned by all vectors y that satisfy $y^t A = 0$. This space is denoted $\mathcal{N}[A^t]$, since it is also characterized by all vectors y such that $A^t y = 0$.

The dimensions of the four spaces $\mathcal{R}[A]$, $\mathcal{R}[A^t]$, $\mathcal{N}[A]$ and $\mathcal{N}[A^t]$ are to be determined in the sequel. Since $A \in \Re^{n \times m}$, we have the following

$$\begin{aligned} r &\triangleq \text{rank}(A) = \text{dimension of column space } \mathcal{R}[A], \\ \dim \mathcal{N}[A] &\triangleq \text{dimension of right null space } \mathcal{N}[A], \\ n &\triangleq \text{total number of columns of } A. \end{aligned}$$

Hence the dimension of the null space $\dim \mathcal{N}[A] = n - r$. Using the fact that $\text{rank}(A) = \text{rank}(A^t)$, we have

$$\begin{aligned} r &\triangleq \text{rank}(A^t) = \text{dimension of row space } \mathcal{R}[A^t], \\ \dim \mathcal{N}[A^t] &\triangleq \text{dimension of left null space } \mathcal{N}[A^t], \\ m &\triangleq \text{total number of rows of } A. \end{aligned}$$

Hence the dimension of the null space $\dim\mathcal{N}[A^t] = m - r$. These facts are summarized below

$$\mathcal{R}[A^t] \triangleq \text{row space of } A\text{: dimension } r,$$
$$\mathcal{N}[A] \triangleq \text{right null space of } A\text{: dimension } n - r,$$
$$\mathcal{R}[A] \triangleq \text{column space of } A\text{: dimension } r,$$
$$\mathcal{N}[A^t] \triangleq \text{left null space of } A\text{: dimension } n - r.$$

Note from these facts that the entire n-dimensional space can be decomposed into the sum of the two subspaces $\mathcal{R}[A^t]$ and $\mathcal{N}[A]$. Alternatively, the entire m-dimensional space can be decomposed into the sum of the two subspaces $\mathcal{R}[A]$ and $\mathcal{N}[A^t]$.

An important property is that $\mathcal{N}[A]$ and $\mathcal{R}[A^t]$ are *orthogonal subspaces*, that is, $\mathcal{R}[A^t]^\perp = \mathcal{N}[A]$. This has the meaning that every vector in $\mathcal{N}[A]$ is orthogonal to every vector in $\mathcal{R}[A^t]$. In the same manner, $\mathcal{R}[A]$ and $\mathcal{N}[A^t]$ are *orthogonal subspaces*, that is, $\mathcal{R}[A]^\perp = \mathcal{N}[A^t]$. The construction of the fundamental subspaces is appropriately attained by the singular value decomposition.

9.2.2 Change of Basis and Invariance

Suppose that $\{v_1, v_2, \ldots, v_n\}$ is chosen as a basis for $\mathcal{V}$. Then any vector $x \in \mathcal{V}$, there are unique scalars $x_v = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \in \Re^n$ such that $x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$. In turn, this raises the question: how can we effectively move between this basis and another basis $\{u_1, u_2, \ldots, u_n\}$ for $\mathcal{V}$? That is, given $x \in \mathcal{V}$, how are the coordinate vectors $x_v, x_u \in \Re^n$ related? To answer this question, suppose that each vector u_k is expressed by

$$u_k = t_{1k} v_1 + t_{2k} v_2 + \cdots + t_{mk} v_m$$

in the basis $\{v_1, v_2, \ldots, v_n\}$. Then the coefficients t_{jk} define the matrix

$$T = \begin{bmatrix} t_{11} & \ldots & t_{1n} \\ \vdots & \ddots & \vdots \\ t_{n1} & \ldots & t_{nn} \end{bmatrix}$$

which is obviously nonsingular since it represents the identity mapping $I_{\mathcal{V}}$ in the bases $\{v_1, v_2, \ldots, v_n\}$ and $\{u_1, u_2, \ldots, u_n\}$. Then the relationship between the two coordinate vectors is

$$T x_u = x_v.$$

Now suppose that $M : \mathcal{V} \to \mathcal{V}$ and that $M_v : \Re^n \to \Re^n$ is the representation of M on the basis $\{v_1, v_2, \ldots, v_n\}$ and M_u is the representation of M on the basis $\{u_1, u_2, \ldots, u_n\}$. How is M_u related to M_v?

To examine this, take any $x \in \mathcal{V}$ and let x_v, x_u be its coordinates in the respective bases, and z_v, z_u be the coordinates of Ax. Then we have

$$z_u = T^{-1} z_v = T^{-1} A_v x_v = T^{-1} A_v T x_u.$$

Since the above identity and

$$z_u = A_u x_u$$

both hold for every x_u, we conclude that

$$A_u = T^{-1} A_v T$$

which is frequently called a *similarity transformation.*

Now the notion of *invariance* of a subspace to a mapping is presented. We say that a subspace $\mathcal{S} \subset \mathcal{V}$ is M-invariant if $M : \mathcal{V} \to \mathcal{V}$ and

$$M\mathcal{S} \subset \mathcal{S}.$$

It is readily seen that every map has at least two invariant subspaces, the *zero subspace* and entire domain $\mathcal{V}$. For subspaces $\mathcal{S}$ of intermediate dimension, the *invariance property* is expressed most clearly by saying the associate matrix has the form

$$[M] = \begin{bmatrix} M_1 & M_2 \\ 0 & M_4 \end{bmatrix},$$

where we assumed that our basis for $\mathcal{V}$ is obtained by extending a basis for $\mathcal{S}$.

9.2.3 Calculus of Vector-Matrix Functions of a Scalar

The differentiation and integration of time functions involving vectors and matrices arises in solving state equations, optimal control and so on. This section summarizes the basic definitions of differentiation and integration on vectors and matrices. A number of formulas for the derivative of vector-matrix products are also included.

The derivative of a matrix function $M(t)$ of a scalar is the matrix of the derivatives of each element in the matrix

$$\frac{dM(t)}{dt} = \begin{bmatrix} \frac{dM_{11}(t)}{dt} & \cdots & \frac{dM_{1n}(t)}{dt} \\ \vdots & \ddots & \vdots \\ \frac{dM_{m1}(t)}{dt} & \cdots & \frac{dM_{mn}(t)}{dt} \end{bmatrix}.$$

The integral of a matrix function $M(t)$ of a scalar is the matrix of the integral of each element in the matrix

$$\int_a^b M(t)dt = \begin{bmatrix} \int_a^b M_{11}(t)dt & \ldots & \int_a^b M_{1n}(t)dt \\ \vdots & \ddots & \vdots \\ \int_a^b M_{m1}(t)dt & \ldots & \int_a^b M_{mn}(t)dt \end{bmatrix}.$$

The Laplace transform of a matrix function $M(t)$ of a scalar is the matrix of the Laplace transform of each element in the matrix

$$\int_a^b M(t)e^{-st}dt = \begin{bmatrix} \int_a^b M_{11}(t)e^{-st}dt & \dots & \int_a^b M_{1n}(t)e^{-st}dt \\ \vdots & \ddots & \vdots \\ \int_a^b M_{m1}(t)e^{-st}dt & \dots & \int_a^b M_{mn}(t)e^{-st}dt \end{bmatrix}.$$

The scalar derivative of the product of two matrix time-functions is

$$\frac{d(A(t)B(t))}{dt} = \frac{A(t)}{dt}B(t) + A(t)\frac{B(t)}{dt}.$$

This result is analogous to the derivative of a product of two scalar functions of a scalar, except caution must be used in reserving the order of the product. An important special case follows:

The scalar derivative of the inverse of a matrix time-function is

$$\frac{dA^{-1}(t)}{dt} = -A^{-1}\frac{A(t)}{dt}A(t).$$

9.2.4 Derivatives of Vector-Matrix Products

The derivative of a real scalar-valued function $f(x)$ of a real vector $x = [x_1, \dots, x_n]^t \in \Re^n$ is defined by

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{bmatrix},$$

where the partial derivative is defined by

$$\frac{\partial f(x)}{\partial x_j} \triangleq \lim_{\Delta x_j \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x_j}, \quad \Delta x = [0 \ \dots \ \Delta x_j \ \dots \ 0]^t.$$

An important application arises in the Taylor's series expansion of $f(x)$ about x_0 in terms of $\delta x \triangleq x - x_0$. The first three terms are

$$f(x) = f(x_0) + \left(\frac{\partial f(x)}{\partial x}\right)^t \delta x + \frac{1}{2}\delta x^t \left[\frac{\partial^2 f(x)}{\partial x^2}\right]\delta x,$$

where

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{bmatrix},$$

$$\frac{\partial^2 f(x)}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f(x)}{\partial x}\right)^t = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f(x)}{\partial x_n^2} \end{bmatrix}.$$

The derivative of a real scalar-valued function $f(A)$ with respect to a matrix

$$A = \begin{bmatrix} A_{11} & \dots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \dots & A_{nn} \end{bmatrix} \in \Re^{n\times n}$$

is given by

$$\frac{\partial f(A)}{\partial A} = \begin{bmatrix} \frac{\partial f(A)}{\partial A_{11}} & \cdots & \frac{\partial f(A)}{\partial A_{1n}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f(A)}{\partial A_{n1}} & \cdots & \frac{\partial f(A)}{\partial A_{nn}} \end{bmatrix}.$$

A vector function of a vector is given by

$$v(u) = \begin{bmatrix} v_1(u) \\ \vdots \\ v_n(u) \end{bmatrix},$$

where $v_j(u)$ is a function of the vector u. The derivative of a vector function of a vector (the *Jacobian*) is defined as follows

$$\frac{\partial v(u)}{\partial u} = \begin{bmatrix} \frac{\partial v_1(u)}{\partial u_1} & \cdots & \frac{\partial v_1(u)}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial v_n(u)}{\partial u_1} & \cdots & \frac{\partial v_n(u)}{\partial u_m} \end{bmatrix}.$$

Note that the *Jacobian* is sometimes defined as the transpose of the foregoing matrix. A special case is given by

$$\frac{\partial (Su)}{\partial u} = S, \qquad \frac{\partial (u^t Ru)}{\partial u} = 2u^t R$$

for arbitrary matrix S and symmetric matrix R.

The following section include useful relations and results from linear algebra.

9.2.5 Positive Definite and Positive Semidefinite Matrices

A matrix P is positive definite if P is real, symmetric and $x^t Px > 0, \forall x \neq 0$. Equivalently, if all the eigenvalues of P have positive real parts. A matrix S is positive semidefinite if S is real, symmetric and $x^t Px \geq 0, \forall x \neq 0$.

Since the definiteness of the scalar $x^t Px$ is a property only of the matrix P, we need a test for determining definiteness of a constant matrix P. Define a *principal* *submatrix* of a square matrix P as any square submatrix sharing some diagonal elements of P. Thus the constant, real, symmetric matrix $P \in \Re^{n\times n}$ is positive definite ($P > 0$) if either of these equivalent conditions holds:

- All eigenvalues of P are positive
- The determinant of P is positive
- All successive principal submatrices of P (minors of successively increasing size) have positive determinants

9.2.6 Matrix Ellipsoid

Given three matrices $X^t = X \in \Re^{m\times m}$, $Y \in \Re^{m\times p}$ and $0 < Z^t = Z \in \Re^{p\times p}$, consider the following set

$$\left\{K \in \Re^{m\times p} : [I\ K]\begin{bmatrix} X & Y \\ \bullet & Z \end{bmatrix}\begin{bmatrix} I \\ K \end{bmatrix} \le 0\right\}.$$

This set is called a *matrix ellipsoid*. Some of the relevant properties are

- *The matrix ellipsoid can be written as*

$$(K - K_o)Z(K - K_o)^t \le R,$$

  where R *is the radius and* $K_o = -YZ^{-1}$ *is the center of the ellipsoid.*
- *A matrix ellipsoid is nonempty if and only if the radius* $R = YZ^{-1}Y^t - X \ge 0$.
- *If* $X = YZ^{-1}Y^t$ *the matrix ellipsoid is a singleton.*
- *A matrix ellipsoid is a compact convex set.*

9.2.7 Power of a Square Matrix

For positive m, A^m for a square matrix A is defined as $AA\cdots A$, with m terms in the product. For negative m, let $m = -n$, where n is positive; $A^m = (A^{1-})^n$. It follows that $A^pA^q = A^{p+q}$, for any integers p and q, positive or negative, and likewise that $(A^p)^q = A^{pq}$.

A polynomial in A is a matrix $p(A) = \sum_{j=1}^{m} \alpha_j A^j$, where the α_j are scalars. Any two polynomials is the same matrix commute—that is,

$$p(A)q(A) = q(A)p(A),$$

where p and q are polynomials. It follows that

$$p(A)q^{-1}(A) = q^{-1}(A)p(A)$$

and that such *rational functions* of A also commute.

9.2.8 Exponential of a Square Matrix

Let A be a square matrix. Then it can shown that the series

$$I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

converges, in the sense that the $(j-k)$th entry of the partial sums of the series converges for all j and k. The sum is defined as e^A. It follows that

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots.$$

Moreover,

$$p(A)e^{At} = e^{At}p(A)$$

for any polynomial A, and $e^{-At} = [e^{At}]^{-1}$.

9.2.9 Eigenvalues and Eigenvectors of a Square Matrix

Let A be an $n \times n$ matrix. The polynomial $\det[sI - A]$ is termed the *characteristic polynomial* of A and the zeros of this polynomial are the *eigenvalues* of matrix A.

If λ_j is an eigenvalue of A, there always exists a least one vector x satisfying

$$Ax = \lambda_j x.$$

The vector x is termed an *eigenvector* of matrix A. If λ_j is not a repeated eigenvalue-that is, if it is a simple zero of the characteristic polynomial, to within a scalar multiple x is unique. If not, there may be more than one eigenvector associated with λ_j. If λ_j is real, the entries of x are real, whereas if λ_j is complex, the entries of x are complex.

If A has zero entries everywhere off the main diagonal—that is, if a_{jk} for all $j, k, j \neq k$, the A is termed *diagonal*. It follows trivially from the definition of an eigenvalue that the diagonal entries of the diagonal A are precisely the eigenvalues of A.

It is also true that for a general matrix A,

$$\det(A) = \prod_{j=1}^{n} \lambda_j.$$

If A is singular, A possesses at least one zero eigenvalue.

The eigenvalues of a rational function $r(A)$ of A are the numbers $r(\lambda_j)$, where λ_j are the eigenvalues of A. For example, the eigenvalues of e^{At} are $e^{\lambda_j t}$.

9.2.10 The Cayley-Hamiltonian Theorem

A formal definition of the Cayley-Hamiltonian theorem is that *Every square matrix satisfies its own characteristic equation*. Let A be a square matrix, and let

$$\det[sI - A] = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n$$

then

$$A^n + \alpha_1 A^{n-1} + \cdots + \alpha_n I = 0.$$

From the Cayley-Hamiltonian theorem, it follows that any analytic function $f(A)$ of A are expressible as a linear combination of $\{I, A, A^{n-1}\}$—that is, A^m for any $m \geq n$ and e^A.

9.2.11 Trace Properties

The trace of a square matrix P, trace(P), equals the sum of its diagonal elements or equivalently the sum of its eigenvalues. A basic property of the trace is invariant under cyclic perturbations, that is,

$$\text{trace}(AB) = \text{trace}(BA),$$

where AB is square. Successive applications of the above results yield

$$\text{trace}(ABC) = \text{trace}(BCA) = \text{trace}(CAB),$$

where ABC is square. In general,

$$\text{trace}(AB) = \text{trace}(B^t A^t).$$

Another result is that

$$\text{trace}(A^t BA) = \sum_{k=1}^{p} a_k^t B a_k,$$

where $A \in \Re^{n \times p}$, $B \in \Re^{n \times n}$ and $\{a_k\}$ are the columns of A. The following identities on trace derivatives are noted

$$\begin{aligned}
\frac{\partial(\text{trace}(AB))}{\partial A} &= \frac{\partial(\text{trace}(A^t B^t))}{\partial A} = \frac{\partial(\text{trace}(B^t A^t))}{\partial A}, \\
&= \frac{\partial(\text{trace}(BA))}{\partial A} = B^t, \\
\frac{\partial(\text{trace}(AB))}{\partial B} &= \frac{\partial(\text{trace}(A^t B^t))}{\partial B} = \frac{\partial(\text{trace}(B^t A^t))}{\partial B}, \\
&= \frac{\partial(\text{trace}(BA))}{\partial B} = A^t, \\
\frac{\partial(\text{trace}(BAC))}{\partial A} &= \frac{\partial(\text{trace}(B^t C^t A^t))}{\partial A} = \frac{\partial(\text{trace}(C^t A^t B^t))}{\partial A},
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\partial(\text{trace}(ACB))}{\partial A} = \frac{\partial(\text{trace}(CBA))}{\partial A}, \\
&= \frac{\partial(\text{trace}(A^t B^t C^t))}{\partial A} = B^t\ C^t, \\
\frac{\partial(\text{trace}(A^t BA))}{\partial A} &= \frac{\partial(\text{trace}(BAA^t))}{\partial A} = \frac{\partial(\text{trace}(AA^t B))}{\partial A}, \\
&= (B + B^t)A.
\end{aligned}
$$

Using these basic ideas, a list of matrix calculus results are given below

$$
\begin{aligned}
\frac{\partial(\text{trace}(AX^t))}{\partial X} &= A, \quad \frac{\partial(\text{trace}(AXB))}{\partial X} = A^t B^t, \\
\frac{\partial(\text{trace}(AX^t B))}{\partial X} &= BA, \quad \frac{\partial(\text{trace}(AX))}{\partial X^t} = A, \\
\frac{\partial(\text{trace}(AX^t))}{\partial X^t} &= A^t, \quad \frac{\partial(\text{trace}(AXB))}{\partial X^t} = BA, \\
\frac{\partial(\text{trace}(AX^t B))}{\partial X^t} &= A^t B^t, \quad \frac{\partial(\text{trace}(XX))}{\partial X} = 2X^t, \\
\frac{\partial(\text{trace}(XX^t))}{\partial X} &= 2X, \\
\frac{\partial(\text{trace}(AX^n))}{\partial X} &= \left(\sum_{j=0}^{n-1} X^j\ A\ X^{n-j-1}\right)^t, \\
\frac{\partial(\text{trace}(AXBX))}{\partial X} &= A^t X^t B^t + B^t X^t A^t, \\
\frac{\partial(\text{trace}(AXBX^t))}{\partial X} &= A^t XB^t + AXB, \\
\frac{\partial(\text{trace}(X^{-1}))}{\partial X} &= -\left(X^{-2}\right)^t, \\
\frac{\partial(\text{trace}(AX^{-1}B))}{\partial X} &= -\left(X^{-1}BAX^{-1}\right)^t, \\
\frac{\partial(\text{trace}(AB))}{\partial A} &= B^t + B - \text{diag}(B).
\end{aligned}
$$

9.2.12 Kronecker Product and vec

Let $A \in \Re^{m\times n}$, $B \in \Re^{p\times r}$. The product $C \in \Re^{mp\times nr}$ defined as

$$
C = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ a_{21}B & \cdots & a_{2n}B \\ \vdots & & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix}
$$

and written $C = A \otimes B$ is termed the *Kronecker product* of matrices A and B. In case A and B are square, the set of eigenvalues of C is given by $\lambda_j(A)\lambda_k(B), \forall j, k$. The Kronecker product is associative-that is,

$$(A \otimes B)(C \otimes D) = AC \otimes BD, \quad (A \otimes B)^t = A^t \otimes B^t.$$

Let $A \in \Re^{m \times n}$. The column mn-vector, obtained by stacking column 2 of A after column 1, column 3 of A after column 2, and so forth, is termed vec A.

If M, N are matrices for which the product MN can be formed, then

$$\begin{aligned} \text{vec}(MN) &= [I \otimes M]\,\text{vec}\,N \\ &= [N^t \otimes I]\,\text{vec}\,M. \end{aligned}$$

9.2.13 Partitioned Matrices

Given a partitioned matrix (matrix of matrices) of the form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A, B, C and D are of compatible dimensions. Then:

(1) if A^{-1} exists, a Schur complement of M is defined as $D - CA^{-1}B$, and
(2) if D^{-1} exists, a Schur complement of M is defined as $A - BD^{-1}C$.

When A, B, C and D are all $n \times n$ matrices, then:

$$\text{(a)} \quad \det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A)\det(D - CA^{-1}B), \quad \det(A) \neq 0;$$

$$\text{(b)} \quad \det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(D)\det(A - BD^{-1}C), \quad \det(D) \neq 0.$$

In the special case, we have

$$\det\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = \det(A)\det(C),$$

where A and C are square. Since the determinant is invariant under row, it follows

$$\begin{aligned} \det\begin{bmatrix} A & B \\ C & D \end{bmatrix} &= \det\begin{bmatrix} A & B \\ C - CA^{-1}A & D - CA^{-1}B \end{bmatrix} \\ &= \det\begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix} = \det(A)\det(D - CA^{-1}B) \end{aligned}$$

which justifies the forgoing result.

Given matrices $A \in \Re^{m \times n}$ and $B \in \Re^{n \times m}$, then

$$\det(I_m - AB) = \det(I_n - BA).$$

In case that A is invertible, then $\det(A^{-1}) = \det(A)^{-1}$.

9.2.14 The Matrix Inversion Lemma

Suppose that $A \in \Re^{n \times n}$, $B \in \Re^{n \times p}$, $C \in \Re^{p \times p}$, and $D \in \Re^{p \times n}$. Assume that A^{-1} and C^{-1} both exist. Then

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(DA^{-1}B + C^{-1})^{-1}DA^{-1}.$$

In the case of partitioned matrices, we have the following result

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B\Xi^{-1}CA^{-1} & -A^{-1}B\Xi^{-1} \\ -\Xi^{-1}CA^{-1} & \Xi^{-1} \end{bmatrix}$$
$$\Xi = (D - CA^{-1}B)$$

provided that A^{-1} exists. Alternatively,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} \Xi^{-1} & -\Xi^{-1}BD^{-1} \\ -D^{-1}C\Xi^{-1} & D^{-1} + D^{-1}C\Xi^{-1}BD^{-1} \end{bmatrix},$$
$$\Xi = (D - CA^{-1}B)$$

provided that D^{-1} exists.

For a square matrix Y, the matrices Y and $(I + Y)^{-1}$ commute, that is, given that the inverse exists

$$Y(I + Y)^{-1} = (I + Y)^{-1}Y.$$

Two additional inversion formulas are given below

$$Y(I + XY)^{-1} = (I + YX)^{-1}Y,$$
$$(I + YX)^{-1} = I - YX(I + YX)^{-1}.$$

The following result provides conditions for the positive definiteness of a partitioned matrix in terms of its submatrices. The following three statements are equivalent:

$$(1) \quad \begin{bmatrix} A_o & A_a \\ A_a^t & A_c \end{bmatrix} > 0,$$
$$(2) \quad A_c > 0, \quad A_o - A_a A_c^{-1} A_a^t > 0,$$
$$(3) \quad A_a > 0, \quad A_c - A_a^t A_o^{-1} A_a > 0.$$

9.2.15 Strengthened Version of Lemma of Lyapunov

The basic lemma of Lyapunov states that *for positive definite C, there exists a unique positive definite P such that*

$$PA + A^t P + C = 0$$

if and only if $\operatorname{Re}\lambda_j(A) < 0$. The first strengthening states that *if* $[A, D]$ *is completely observable*, *there exists a unique positive definite* P *such that*

$$PA + A^t P = -DD^t$$

if and only if $\operatorname{Re}\lambda_j(A) < 0$. The second strengthening states that *if* $[A, D]$ *is completely detectable, there exists a unique nonnegative definite* P *such that*

$$PA + A^t P = -DD^t$$

if and only if $\operatorname{Re}\lambda_j(A) < 0$.

In all cases where P exists,

$$P = \int_0^\infty e^{A^t t} DD^t e^{At} dt.$$

9.2.16 The Singular Value Decomposition

The singular value decomposition (SVD) is a matrix factorization that has found a number of applications to engineering problems. The SVD of a matrix $M \in \Re^{n\times m}$ is

$$M = USV^\dagger = \sum_{j=1}^{p} \sigma_j U_j V_j^\dagger,$$

where $U \in \Re^{\alpha\times\alpha}$ and $V \in \Re^{\beta\times\beta}$ are unitary matrices ($U^\dagger U = UU^\dagger = I$ and $V^\dagger V = VV^\dagger I$); $S \in \Re^{\alpha\times\beta}$ is a real, diagonal (but not necessarily square); and $p = \min(\alpha, \beta)$. *The singular values* $\{\sigma_1, \sigma_2, \ldots, \sigma_\beta\}$ *of* M *are defined as the positive square roots of the diagonal elements of* $S^t S$, and are ordered from largest to smallest.

To proceed further, we recall a result on unitary matrices. If U is a unitary matrix ($U^\dagger U = I$), then the transformation U preserves length, that is:

$$\begin{aligned}\|Ux\| &= \sqrt{(Ux)^\dagger(Ux)} = \sqrt{x^\dagger U^\dagger Ux},\\ &= \sqrt{x^\dagger x} = \|x\|.\end{aligned}$$

As a consequence, we have

$$\begin{aligned}\|Mx\| &= \sqrt{x^\dagger M^\dagger Mx} = \sqrt{x^\dagger VS^t U^\dagger USV^\dagger x},\\ &= \sqrt{x^\dagger VS^t SV^\dagger x}.\end{aligned}$$

To evaluate the maximum gain of matrix M, we calculate the maximum norm of the above equation to yield

$$\max_{\|x\|=1} \|Mx\| = \max_{\|x\|=1} \sqrt{x^\dagger VS^t SV^\dagger x} = \max_{\|\tilde{x}\|=1} \sqrt{\tilde{x}^\dagger VS^t S\tilde{x}}.$$

Note that maximization over $\tilde{x} = Vx$ is equivalent to maximizing over x since V is invertible and preserves the norm (equals 1 in this case). Expanding the norm yield

$$\max_{\|x\|=1} \|Mx\| = \max_{\|\tilde{x}\|=1} \sqrt{\tilde{x}^{\dagger} V S^t S \tilde{x}},$$
$$= \max_{\|\tilde{x}\|=1} \sqrt{\sigma_1^2 |\tilde{x}_1|^2 + \sigma_2^2 |\tilde{x}_2|^2 + \cdots + \sigma_\beta^2 |\tilde{x}_\beta|^2}.$$

The foregoing expression is maximized, given the constraint $\|\tilde{x}\| = 1$, when $\tilde{x}$ is concentrated at the largest singular value; that is $|\tilde{x}| = [1\ 0\ \ldots\ 0]^t$. The maximum gain is then

$$\max_{\|x\|=1} \|Mx\| = \sqrt{\sigma_1^2 |1|^2 + \sigma_2^2 |0|^2 + \cdots + \sigma_\beta^2 |0|^2} = \sigma_1 = \sigma_M.$$

In words, this reads *The maximum gain of a matrix is given be the maximum singular value* σ_M. Following similar lines of development, it is easy to show that

$$\min_{\|x\|=1} \|Mx\| = \sigma_\beta = \sigma_m,$$
$$= \begin{cases} \sigma_p, & \alpha \geq \beta, \\ 0, & \alpha < \beta. \end{cases}$$

A property of the singular values is expressed by

$$\sigma_M(M^{-1}) = \frac{1}{\sigma_m(M)}.$$

9.3 Some Bounding Inequalities

In the sequel, all mathematical inequalities are proved for completeness. They are termed facts afterwards due to their high frequency of usage in the analytical developments.

9.3.1 Bounding Inequality A

For any real matrices Σ_1, Σ_2 *and* Σ_3 *with appropriate dimensions and* $\Sigma_3^t\, \Sigma_3 \leq I$, *it follows that*

$$\Sigma_1 \Sigma_3 \Sigma_2 + \Sigma_2^t \Sigma_3^t \Sigma_1^t \leq \alpha \Sigma_1 \Sigma_1^t + \alpha^{-1} \Sigma_2^t \Sigma_2, \quad \forall \alpha > 0.$$

Proof This inequality can be proved as follows. Since $\Phi^t \Phi \geq 0$ holds for any matrix Φ, then take Φ as

$$\Phi = [\alpha^{1/2} \Sigma_1 - \alpha^{-1/2} \Sigma_2].$$

Expansion of $\Phi^t \Phi \geq 0$ gives $\forall \alpha > 0$

$$\alpha \Sigma_1 \Sigma_1^t + \alpha^{-1} \Sigma_2^t \Sigma_2 - \Sigma_1^t \Sigma_2 - \Sigma_2^t \Sigma_1 \geq 0$$

which by simple arrangement yields the desired result. □

9.3.2 Bounding Inequality B

Let Σ_1, Σ_2, Σ_3 and $0 < R = R^t$ *be real constant matrices of compatible dimensions and $H(t)$ be a real matrix function satisfying $H^t(t)H(t) \leq I$. Then for any $\rho > 0$ satisfying $\rho \Sigma_2^t \Sigma_2 < R$, the following matrix inequality holds*:

$$(\Sigma_3 + \Sigma_1 H(t) \Sigma_2) R^{-1} (\Sigma_3^t + \Sigma_2^t H^t(t) \Sigma_1^t) \leq \rho^{-1} \Sigma_1 \Sigma_1^t + \Sigma_3 \big(R - \rho \Sigma_2^t \Sigma_2\big)^{-1} \Sigma_3^t.$$

Proof The proof of this inequality proceeds like the previous one by considering that

$$\Phi = [(\rho^{-1} \Sigma_2 \Sigma_2^t)^{-1/2} \Sigma_2 R^{-1} \Sigma_3^t - (\rho^{-1} \Sigma_2 \Sigma_2^t)^{-1/2} H^t(t) \Sigma_1^t].$$

Recall the following results

$$\begin{aligned}
\rho \Sigma_2^t \Sigma_2 &< R, \\
[R - \rho \Sigma_2^t \Sigma_2]^{-1} &= [R^{-1} + R^{-1} \Sigma_2^t [\rho^{-1} I - \Sigma_2 R^{-1} \Sigma_2^t]^{-1} \Sigma_2 R^{-1} \Sigma_2
\end{aligned}$$

and

$$H^t(t) H(t) \leq I \quad \Longrightarrow \quad H(t) H^t(t) \leq I.$$

Expansion of $\Phi^t \Phi \geq 0$ under the condition $\rho \Sigma_2^t \Sigma_2 < R$ with standard matrix manipulations gives

$$\begin{aligned}
&\Sigma_3 R^{-1} \Sigma_2^t H^t(t) \Sigma_1^t + \Sigma_1 H(t) \Sigma_2 R^{-1} \Sigma_3^t + \Sigma_1 H(t) \Sigma_2 \Sigma_2^t H^t(t) \Sigma_1^t \\
&\quad \leq \rho^{-1} \Sigma_1 H(t) H^t(t) \Sigma_1^t + \Sigma_3^t R^{-1} \Sigma_2 [\rho^{-1} I \Sigma_2 \Sigma_2^t]^{-1} \Sigma_2 R^{-1} \Sigma_3^t \quad \Longrightarrow \\
&(\Sigma_3 + \Sigma_1 H(t) \Sigma_2) R^{-1} (\Sigma_3^t + \Sigma_2^t H^t(t) \Sigma_1^t) - \Sigma_3 R^{-1} \Sigma_3^t \\
&\quad \leq \rho^{-1} \Sigma_1 H(t) H^t(t) \Sigma_1^t + \Sigma_3^t R^{-1} \Sigma_2 [\rho^{-1} I - \Sigma_2 \Sigma_2^t]^{-1} \Sigma_2 R^{-1} \Sigma_3^t \quad \Longrightarrow \\
&(\Sigma_3 + \Sigma_1 H(t) \Sigma_2) R^{-1} (\Sigma_3^t + \Sigma_2^t H^t(t) \Sigma_1^t) \\
&\quad \leq \Sigma_3 [R^{-1} + \Sigma_2 [\rho^{-1} I - \Sigma_2 \Sigma_2^t]^{-1} \Sigma_2 R^{-1}] \Sigma_3^t + \rho^{-1} \Sigma_1 H(t) H^t(t) \Sigma_1^t \\
&\quad = \rho^{-1} \Sigma_1 H(t) H^t(t) \Sigma_1^t + \Sigma_3 \big(R - \rho \Sigma_2^t \Sigma_2\big)^{-1} \Sigma_3^t
\end{aligned}$$

which completes the proof. □

9.3.3 Bounding Inequality C

For any real vectors β, ρ and any matrix $Q^t = Q > 0$ with appropriate dimensions, it follows that

$$-2\rho^t\beta \leq \rho^t Q\rho + \beta^t Q^{-1}\beta.$$

Proof Starting from the fact that

$$[\rho + Q^{-1}\beta]^t Q[\rho + Q^{-1}\beta] \geq 0, \quad Q > 0$$

which when expanded and arranged yields the desired result. □

9.3.4 Bounding Inequality D

For any quantities u and v of equal dimensions and for all $\eta_t = i \in \mathcal{S}$, it follows that the following inequality holds

$$\|u + v\|^2 \leq [1 + \beta^{-1}]\|u\|^2 + [1 + \beta]\|v\|^2 \tag{9.1}$$

for any scalar $\beta > 0$, $i \in \mathcal{S}$.

Proof Since

$$[u + v]^t[u + v] = u^t u + v^t v + 2u^t v, \tag{9.2}$$

it follows by taking norm of both sides of (9.2) for all $i \in \mathcal{S}$ that

$$\|u + v\|^2 \leq \|u\|^2 + \|v\|^2 + 2\|u^t v\|. \tag{9.3}$$

We know from the triangle inequality that

$$2\|u^t v\| \leq \beta^{-1}\|u\|^2 + \beta\|v\|^2. \tag{9.4}$$

On substituting (9.4) into (9.3), it yields (9.1). □

9.3.5 Young's Inequality

For any scalars $\varepsilon > 0$, $p > 1$, $q = (1 - p^{-1})^{-1} > 1$ and vectors $a \in \Re^n$ and $b \in \Re^n$, it follows that

$$a^t b \leq \varepsilon|a|^p/p + |b|^q/(q^{\varepsilon^{q-1}}).$$

9.4 Gronwall-Bellman Inequality

Gronwall-Bellman Inequality Let $\sigma : [a, b] \to \Re$ be continuous and $\beta : [a, b] \to \Re$ be continuous and nonnegative. If a continuous function $z : [a, b] \to \Re$ satisfies

$$z(t) \le \sigma(t) + \int_a^b \beta(s)z(s)ds$$

for $a \le t \le b$, then on the same interval

$$z(t) \le \sigma(t) + \int_a^b \sigma(s)\beta(s)\exp\left[\int_s^t \beta(s)ds\right].$$

In particular, if $\sigma(t) \equiv \sigma$ is a constant, then

$$z(t) \le \sigma \exp\left[\int_s^t \beta(s)ds\right].$$

If, in addition, $\beta(t) \equiv \beta \ge 0$ is a constant, then

$$z(t) \le \sigma \exp[\beta(t-a)].$$

Proof Let $y(t) = \int_a^b \beta(s)z(s)ds$ and $w(t) = y(t) + \sigma(t) - z(t) \ge 0$. Then, z is differentiable and

$$\dot{(z)} = \beta(t)z(t) = \beta(t)y(t) + \beta(t)\sigma(t) - \beta(t)w(t)$$

which describes a linear state equation with an associated state transition function

$$\phi(t, s) = \exp\left[\int_s^t \beta(\tau)d\tau\right].$$

Since $y(a) = 0$, we have

$$y(t) = \int_a^t \phi(t, s)[\beta(s)\sigma(s) - \beta(s)w(s)]ds.$$

Observe that

$$\int_a^t \phi(t, s)\beta(s)w(s)ds \ge 0.$$

Therefore,

$$y(t) \le \int_a^t \exp\left[\int_s^t \beta(\tau)d\tau\right]\beta(s)\sigma(s)ds.$$

Since $z(t) \le \sigma(t) + y(t)$, the proof is completed in the general case.

When $\sigma(t) \equiv \sigma$, we have

$$\begin{aligned}\int_a^t \exp\left[\int_s^t \beta(\tau)d\tau\right]ds &= -\int_a^t \frac{d}{ds}\left\{\exp\left[\int_s^t \beta(\tau)d\tau\right]\right\}ds \\ &= -\left\{\exp\left[\int_s^t \beta(\tau)d\tau\right]\right\}\Bigg|_{s=a}^{s=t} \\ &= -1 + \exp\left[\int_a^t \beta(\tau)d\tau\right]\end{aligned}$$

which establishes the part of the lemma when σ is constant. The remaining part when both σ and β are constants follows by integration. □

9.5 Schur Complements

Schur complements Given a matrix Ω composed of constant matrices $\Omega_1, \Omega_2, \Omega_3$, where $\Omega_1 = \Omega_1^t$ and $0 < \Omega_2 = \Omega_2^t$ as follows

$$\Omega = \begin{bmatrix} \Omega_1 & \Omega_3 \\ \Omega_3^t & \Omega_2 \end{bmatrix}.$$

We have the following results:

(A) $\Omega \geq 0$ if and only if either

$$\begin{cases} \Omega_2 \geq 0, \\ \Pi = \Upsilon \Omega_2, \\ \Omega_1 - \Upsilon \Omega_2 \Upsilon^t \geq 0 \end{cases} \tag{9.5}$$

or

$$\begin{cases} \Omega_1 \geq 0, \\ \Pi = \Omega_1 \Lambda, \\ \Omega_2 - \Lambda^t \Omega_1 \Lambda \geq 0 \end{cases} \tag{9.6}$$

hold where Λ, Υ are some matrices of compatible dimensions.

(B) $\Omega > 0$ if and only if either

$$\begin{cases} \Omega_2 > 0, \\ \Omega_1 - \Omega_3 \Omega_2^{-1} \Omega_3^t > 0 \end{cases}$$

or

$$\begin{cases} \Omega_1 \geq 0, \\ \Omega_2 - \Omega_3^t \Omega_1^{-1} \Omega_3 > 0 \end{cases}$$

hold where Λ, Υ are some matrices of compatible dimensions.

In this regard, matrix $\Omega_1 - \Omega_3 \Omega_2^{-1} \Omega_3^t$ ($\Omega_2 - \Omega_3^t \Omega_1^{-1} \Omega_3$) is often called the Schur complement of Ω_2 (of Ω_1) in Ω.

Proof (A) To prove (9.5), we first note that $\Omega_2 \geq 0$ is necessary. Let $z^t = [z_1^t \; z_2^t]$ be a vector partitioned in accordance with Ω. Thus we have

$$z^t \Omega z = z_1^t \Omega_1 z_1 + 2 z_1^t \Omega_3 z_2 + z_2^t \Omega_2 z_2. \tag{9.7}$$

Select z_2 such that $\Omega_2 z_2 = 0$. If $\Omega_3 z_2 \neq 0$, let $z_1 = -\pi \Omega_3 z_2$, $\pi > 0$. Then it follows that

$$z^t \Omega z = \pi^2 z_2^t \Omega_3^t \Omega_1 \Omega_3 z_2 - 2\pi z_2^t \Omega_3^t \Omega_3 z_2$$

which is negative for a sufficiently small $\pi > 0$. We thus conclude $\Omega_1 z_2 = 0$ which then leads to $\Omega_3 z_2 = 0$, $\forall z_2$ and consequently

$$\Omega_3 = \Upsilon \Omega_2 \tag{9.8}$$

for some Υ.

Since $\Omega \geq 0$, the quadratic term $z^t \Omega z$ possesses a minimum over z_2 for any z_1. By differentiating $z^t \Omega z$ from (9.7) wrt z_2^t, we get

$$\frac{\partial(z^t \Omega z)}{\partial z_2^t} = 2\Omega_3^t z_1 + 2\Omega_2 z_2 = 2\Omega_2 \Upsilon^t z_1 + 2\Omega_2 z_2.$$

Setting the derivative to zero yields

$$\Omega_2 \Upsilon z_1 = -\Omega_2 z_2 \tag{9.9}$$

Using (9.8) and (9.9) in (9.7), it follows that the minimum of $z^t \Omega z$ over z_2 for any z_1 is given by

$$\min_{z_2} z^t \Omega z = z_1^t [\Omega_1 - \Upsilon \Omega_2 \Upsilon^t] z_1$$

which proves the necessity of $\Omega_1 - \Upsilon \Omega_2 \Upsilon^t \geq 0$.

On the other hand, we note that the conditions (9.5) are necessary for $\Omega \geq 0$ and since together they imply that the minimum of $z^t \Omega z$ over z_2 for any z_1 is nonnegative, they are also sufficient.

Using similar argument, conditions (9.6) can be derived as those of (9.5) by starting with Ω_1.

The proof of (B) follows as direct corollary of (A). □

9.6 Lemmas

The basic tools and standard results that are utilized in robustness analysis and resilience design in the different chapters are collected hereafter.

Lemma 9.1 *The matrix inequality*

$$-\Lambda + S\Omega^{-1} S^t < 0 \tag{9.10}$$

holds for some $0 < \Omega = \Omega^t \in \Re^{n \times n}$, *if and only if*

$$\begin{bmatrix} -\Lambda & S\mathcal{X} \\ \bullet & -\mathcal{X} - \mathcal{X}^t + \mathcal{Z} \end{bmatrix} < 0 \tag{9.11}$$

holds for some matrices $\mathcal{X} \in \Re^{n \times n}$ *and* $\mathcal{Z} \in \Re^{n \times n}$.

Proof (⇒) By Schur complements, inequality (9.10) is equivalent to

$$\begin{bmatrix} -\Lambda & S\Omega^{-1} \\ \bullet & -\Omega^{-1} \end{bmatrix} < 0. \tag{9.12}$$

Setting $\mathcal{X} = \mathcal{X}^t = \mathcal{Z} = \Omega^{-1}$, we readily obtain inequality (9.11).

($\Leftarrow$) Since the matrix $[I \;\; S]$ is of full rank, we obtain

$$\begin{bmatrix} I \\ S^t \end{bmatrix}^t \begin{bmatrix} -\Lambda & S\mathcal{X} \\ \bullet & -\mathcal{X} - \mathcal{X}^t + \mathcal{Z} \end{bmatrix} \begin{bmatrix} I \\ S^t \end{bmatrix} < 0$$
$$\Longleftrightarrow \quad -\Lambda + S\mathcal{Z}S^t < 0$$
$$\Longleftrightarrow \quad -\Lambda + S\Omega^{-1}S^t < 0, \quad \mathcal{Z} = \Omega^{-1} \tag{9.13}$$

which completes the proof. □

Lemma 9.2 *The matrix inequality*

$$AP + PA^t + D^t R^{-1} D + M < 0 \tag{9.14}$$

holds for some $0 < P = P^t \in \Re^{n \times n}$, *if and only if*

$$\begin{bmatrix} A\mathcal{V} + \mathcal{V}^t A^t + M & P + A\mathcal{W} - \mathcal{V} & D^t R \\ \bullet & -\mathcal{W} - \mathcal{W}^t & 0 \\ \bullet & \bullet & -R \end{bmatrix} < 0 \tag{9.15}$$

holds for some $\mathcal{V} \in \Re^{n \times n}$ *and* $\mathcal{W} \in \Re^{n \times n}$.

Proof ($\Rightarrow$) By Schur complements, inequality (9.14) is equivalent to

$$\begin{bmatrix} AP + PA^t + M & D^t R \\ \bullet & -R \end{bmatrix} < 0. \tag{9.16}$$

Setting $\mathcal{V} = \mathcal{V}^t = P$, $\mathcal{W} = \mathcal{W}^t = R$, it follows from Lemma 9.1 with Schur complements that there exists $P > 0$, $\mathcal{V}$, $\mathcal{W}$ such that inequality (9.15) holds.

($\Leftarrow$) In a similar way, Schur complements to inequality (9.15) imply that:

$$\begin{bmatrix} A\mathcal{V} + \mathcal{V}^t A^t + M & P + A\mathcal{W} - \mathcal{V} & D^t R \\ \bullet & -\mathcal{W} - \mathcal{W}^t & 0 \\ \bullet & \bullet & -R \end{bmatrix} < 0$$
$$\Longleftrightarrow \quad \begin{bmatrix} I \\ A \end{bmatrix} \begin{bmatrix} A\mathcal{V} + \mathcal{V}^t A^t + M + D^t R^{-1} D & P + A\mathcal{W} - \mathcal{V} \\ \bullet & -\mathcal{W} - \mathcal{W}^t \end{bmatrix} \begin{bmatrix} I \\ A \end{bmatrix}^t < 0$$
$$\Longleftrightarrow \quad AP + PA^t + D^t R^{-1} D + M < 0, \quad \mathcal{V} = \mathcal{V}^t \tag{9.17}$$

which completes the proof. □

The following lemmas are found in [10].

Lemma 9.3 *Given any* $x \in \Re^n$:

$$\max\{[x^t RH\Delta Gx]^2 : \Delta \in \Re\} = x^t RHH^t Rxx^t G^t Gx.$$

Lemma 9.4 *Given matrices* $0 \le X = X^t \in \Re^{p \times p}$, $Y = Y^t < 0 \in \Re^{p \times p}$, $0 \le Z = Z^t \in \Re^{p \times p}$, *such that*

$$[\xi^t \; Y \; \xi]^2 - 4[\xi^t \; X \; \xi \; \xi^t \; Z \; \xi]^2 > 0$$

for all $0 \neq \xi \in \Re^p$ *is satisfied. Then there exists a constant* $\alpha > 0$ *such that*

$$\alpha^2 X + \alpha Y + Z < 0.$$

The following lemma can be found in [3].

Lemma 9.5 *For a given two vectors* $\alpha \in \Re^n$, $\beta \in \Re^m$ *and matrix* $N \in \Re^{n\times m}$ *defined over a prescribed interval* Ω, *it follows for any matrices* $X \in \Re^{n\times n}$, $Y \in \Re^{n\times m}$, *and* $Z \in \Re^{m\times m}$, *the following inequality holds*

$$-2\int_{\Omega} \alpha^t(s) N\beta(s) ds \leq \int_{\Omega} \begin{bmatrix} \alpha(s) \\ \beta(s) \end{bmatrix}^t \begin{bmatrix} X & Y - N \\ Y^t - N^t & Z \end{bmatrix} \begin{bmatrix} \alpha(s) \\ \beta(s) \end{bmatrix} ds,$$

where

$$\begin{bmatrix} X & Y \\ Y^t & Z \end{bmatrix} \geq 0.$$

An algebraic version of Lemma 9.5 is stated below.

Lemma 9.6 *For a given two vectors* $\alpha \in \Re^n$, $\beta \in \Re^m$ *and matrix* $N \in \Re^{n\times m}$ *defined over a prescribed interval* Ω, *it follows for any matrices* $X \in \Re^{n\times n}$, $Y \in \Re^{n\times m}$, *and* $Z \in \Re^{m\times m}$, *the following inequality holds*

$$\begin{aligned} -2\alpha^t N\beta &\leq \begin{bmatrix} \alpha \\ \beta \end{bmatrix}^t \begin{bmatrix} X & Y - N \\ Y^t - N^t & Z \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \\ &= \alpha^t X\alpha + \beta^t(Y^t - N^t)\alpha + \alpha^t(Y - N)\beta + \beta^t Z\beta \end{aligned}$$

subject to

$$\begin{bmatrix} X & Y \\ Y^t & Z \end{bmatrix} \geq 0.$$

The following lemma can be found in [9].

Lemma 9.7 *Let* $0 < Y = Y^t$ *and* M, N *be given matrices with appropriate dimensions. Then it follows that*

$$Y + M\Delta N + N^t\Delta^t M^t < 0, \quad \forall \Delta^t\Delta \leq I$$

holds if and only if there exists a scalar $\varepsilon > 0$ *such that*

$$Y + \varepsilon M M^t + \varepsilon^{-1} N^t N < 0.$$

In the following lemma, we let $X(z) \in \Re^{n\times p}$ be a matrix function of the variable z. A matrix $X_*(z)$ is called the orthogonal complement of $X(z)$ if $X^t(z)X_*(z) = 0$ and $X(z)X_*(z)$ is nonsingular (of maximum rank).

Lemma 9.8 *Let* $0 < L = L^t$ *and* X, Y *be given matrices with appropriate dimensions. Then it follows that the inequality*

$$L(z) + X(z)PY(z) + Y^t(z)P^tX^t(z) > 0 \tag{9.18}$$

holds for some P *and* $z = z_0$ *if and only if the following inequalities*

$$X_*^t(z)L(z)X_*(z) > 0, \qquad Y_*^t(z)L(z)Y_*(z) > 0 \tag{9.19}$$

hold with $z = z_0$.

It is significant to observe that feasibility of matrix inequality (9.18) with variables P and z is equivalent to the feasibility of (9.19) with variable z and thus the matrix variable P has been eliminated from (9.18) to form (9.19). Using Finsler's lemma [2], we can express (9.19) in the form

$$L(z) - \beta X(z)X^t(z) > 0, \qquad L(z) - \beta Y(z)Y^t(z) > 0 \tag{9.20}$$

for some $\beta \in \Re$.

Lemma 9.9 *For any constant matrix* $0 < \Sigma \in \Re^{n\times n}$, *scalar* $\sigma < \tau(t) < \varrho$ *and vector function* $\dot{x} : [-\varrho, -\sigma] \to \Re^n$ *such that the following integration is well-defined, then it holds that*

$$-(\varrho - \sigma)\int_{t-\varrho}^{t-\sigma} \dot{x}^t(s)\Sigma\dot{x}(s)ds \le -[x(t-\sigma) - x(t-\varrho)]^t\Sigma[x(t-\sigma) - x(t-\varrho)].$$

Lemma 9.10 *Given constant matrices* $\Omega_1, \Omega_2, \Omega_3$, *where* $\Omega_1 = \Omega_1^T$ *and* $\Omega_2 = \Omega_2^T$, *then* $\Omega_1 + \Omega_3^T\Omega_2^{-1}\Omega_3 < 0$ *if and only if*

$$\begin{bmatrix} \Omega_1 & \Omega_3^T \\ \Omega_3 & -\Omega_2 \end{bmatrix} < 0 \quad or \quad \begin{bmatrix} -\Omega_2 & \Omega_3^T \\ \Omega_3 & \Omega_1 \end{bmatrix} < 0.$$

The following is a statement of the reciprocal projection lemma.

Lemma 9.11 *Let* $P > 0$ *be a given matrix. The following statements are equivalent*:

(i) $M + Z + Z^t < 0$;
(ii) *the LMI problem*

$$\begin{bmatrix} M + P - (V + V^t) & V^t + Z^t \\ V + Z & -P \end{bmatrix} < 0$$

is feasible with respect to the general matrix V.

Lemma 9.12 *For any constant matrix* $0 < M^t = M \in \Re^{n\times n}$, *scalar* $\phi > 0$, *if there exists a vector function* $r(s) : [0, \phi] \to \Re^n$ *such that the following integrations are well-defined then*

$$-\phi\int_{\phi}^{0} r^t(s)Mr(s)ds \ge \left[\int_0^{\phi} r(s)ds\right]^t M\left[\int_0^{\phi} r(s)ds\right].$$

Lemma 9.13 (The **S** Procedure) *Denote the set* $\mathsf{Z} = \{z\}$ *and let* $\mathcal{F}(z)$, $\mathcal{Y}_1(z)$, $\mathcal{Y}_2(z), \ldots, \mathcal{Y}_k(z)$ *be some functionals or functions. Define domain* D *as*

$$\mathsf{D} = \{z \in \mathsf{Z} : \mathcal{Y}_1(z) \geq 0,\ \mathcal{Y}_2(z) \geq 0, \ldots,\ \mathcal{Y}_k(z) \geq 0\}$$

and the two following conditions:

(I) $\mathcal{F}(z) > 0, \forall z \in \mathsf{D}$,
(II) $\exists \varepsilon_1 \geq 0,\ \varepsilon_2 \geq 0, \ldots, \varepsilon_k \geq 0$ *such that* $\mathcal{S}(\varepsilon, z) = \mathcal{F}(z) - \sum_{j=1}^{k} \varepsilon_j \mathcal{Y}_j(z) > 0$ $\forall z \in \mathsf{Z}$. *Then* (II) *implies* (I).

9.7 Stability Theorems

9.7.1 Lyapunov-Razumikhin Theorem

Here the idea is based on the following argument: because the future states of the system depend on the current and past states' values the Lyapunov function should become functional—more details in Lyapunov Krasovskii method—which may complicate the condition formulation and the analysis. To avoid using functional; Razumakhin made his theorem which is based on formulating Lyapunov functions not functionals. First one should build a Lyapunov function $V(x(t))$ which is zero when $x(t) = 0$ and positive otherwise, then the theorem does not require $\dot{V} < 0$ always but only when the $V(x(t))$ for the current state becomes equals to $\bar{V}$ which is given by

$$\bar{V} = \max_{\theta \in [-\tau, 0]} V(x(t + \theta)) \tag{9.21}$$

the theorem statement is given by ([4]):

Lyapunov-Razumikhin Theorem *Suppose* f *is a functional that takes time* t *and initial values* x_t *and gives a vector of* n *states* $\dot{x}$ *and* u, v, w *are class* $\mathcal{K}$ *functions* $u(s)$ *and* $v(s)$ *are positive for* $s > 0$ *and* $u(0) = v(0) = 0$, v *is strictly increasing. If there exists a continuously differentiable function* $V : \Re \times \Re^n \to \Re$ *such that*

$$u(\|x\|) \leq V(t, x) \leq v(\|x\|) \tag{9.22}$$

and the time derivative of V *along the solution* $x(t)$ *satisfies* $\dot{V}(t, x) \leq -w(\|x\|)$ *whenever* $V(t + \theta, x(t + \theta)) \leq V(t, x(t))\theta \in [-\tau, 0]$, *then the system is uniformly stable.*

If in addition $w(s) > 0$ *for* $s > 0$ *and there exists a continuous non-decreasing function* $p(s) > s$ *for* $s > 0$ *such that* $\dot{V}(t, x) \leq -w(\|x\|)$ *whenever* $V(t + \theta, x(t + \theta)) \leq p(V(t, x(t)))$ *for* $\theta \in [-\tau, 0]$, *then the system is uniformly asymptotically stable. If in addition* $\lim u(s)_{s \to \infty} = \infty$ *then the system is globally asymptotically stable.*

The argument behind the theorem is like this: $\bar{V}$ is serving as a measure for the V in the interval $t-\tau$ to t then if $V(x(t))$ is less than $\bar{V}$ then it's not necessary that $\dot{V}<0$, but if $V(x(t))$ becomes equals to $\bar{V}$ then $\dot{V}$ should be <0 such that V will not grow.

The procedure can be explained more by the following discussion: consider system and a selected Lyapunov function $V(x)$ which is positive semi-definite. By taking the time derivative of this Lyapunov function we get $\dot{V}$. According to Razumikhin theorem this term does not always need to be negative, but if we added the following term $a(V(x)-V(x_t))$, $a>0$ to $\dot{V}$ then the term

$$\dot{V}+a(V(x)-V(x_t)) \tag{9.23}$$

should always be negative. Then by looking at this term we find that this condition is satisfied if $\dot{V}<0$ and $V(x)\leq V(x_t)$ meaning that the system states are not growing in magnitude and it is approaching the origin (stable system). Or whenever $a(V(x)<V(x_t))$ and $\dot{V}>0$ but $\dot{V}<|a(V(x)-V(x_t))|$ then although $\dot{V}$ is positive and states are increasing but the Lyapunov function is limited by an upper bound and it will not grow without limit. The third case is that both of them are negative and it's clear that it is stable. This condition insures uniformly stability meaning that the states may not reach the origin but it is contained is a domain say ε which obey the primary definition of the stability. To extends this theorem for asymptotic stability we can consider adding the term $p(V(x(t)))-V(x_t)$ where $p(.)$ is a function that has the following characteristics

$$p(s)>s$$

and then the condition becomes

$$\dot{V}+a(p(V(x(t)))-V(x_t))<0, \quad a>0. \tag{9.24}$$

By this when the system reaches some value which make $p(V(x(t)))=V(x_t)$ requires $\dot{V}$ to be negative but at this instant $V(x(t)<V(x_t)$ then in the coming τ interval the $V(x)$ will never reaches $V(x_t)$ and the maximum value in this interval is the new $V(x_t)$ which is less than the previous value and with the time the function keeps decreasing until the states reach the origin.

9.7.2 Lyapunov-Krasovskii Theorem

The Razumikhin theorem attempts to construct Lyapunov function while Lyapunov-Krasovskii uses functionals because V which can be considered as an indicator for the internal power in the system is function of x_t , then it's logically to consider V which is a function of function and hence a functional. The terms of $V(x_t)$ should contains terms for the x in the interval $(t-\tau)$ to t and $\dot{V}$ should be <0 to ensure asymptotic stability. This method will be covered in more detail in a next section.

In many cases, Lyapunov-Razumikhin can be found as a special case of Lyapunov-Krasovskii which make the former more conservative. Lyapunov-Krasovskii method tries to build a Lyapunov functional which is function in x_t and the time derivative of this Lyapunov function should be negative for the system to be stable. Previously there were criticism on Lyapunov-Krasovskii that it can be used for system with the third category of delay mentioned in Sect. 2.2.2 only when $\dot{\tau} \leq \mu \leq 1$, but the recent results resolve this problem as we see next chapter. Another criticism is that the Krasovskii methods can not deal with delay in the second category but also the recent results in this method succeed to include this case [3]. The remaining advantage of the Razumikhin method is its simplicity, but the Krasovskii method proved to give less conservative results, the thing that gives it the interest of most of the researchers in the recent years. Before going to the theorem we have to define the following notations

$$\begin{aligned} \phi &= x_t, \\ \|\phi\|_c &= \max_{\theta \in [-\tau, 0]} x(t+\theta). \end{aligned} \tag{9.25}$$

Lyapunov-Krasovskii Theorem ([4]) *Suppose f is a functional that takes time t and initial values x_t and gives a vector of n states $\dot{x}$ and u, v, w are class $\mathcal{K}$ functions $u(s)$ and $v(s)$ are positive for $s > 0$ and $u(0) = v(0) = 0$, v is strictly increasing. If there exists a continuously differentiable function $V : \Re \times \Re_n \rightarrow \Re$ such that*

$$u(\|\phi\|) \leq V(t, x) \leq v(\|\phi\|_c) \tag{9.26}$$

and the time derivative of V along the solution $x(t)$ satisfies

$$\dot{V}(t, x) \leq -w(\|\phi\|) \quad \textit{for } \theta \in [-\tau, 0]$$

then the system is uniformly stable. If in addition $w(s) > 0$ for $s > 0$ then the system is uniformly asymptotically stable. If in addition $\lim\ u(s)_{s\rightarrow\infty} = \infty$ *then the system is globally asymptotically stable.*

It's clear that V is a functional and $\dot{V}$ should always be negative.

When considering a special class of systems which consider the case of linear time invariant system with multiple discrete time delay which is given by [4]

$$\dot{x}(t) = A_o x(t) + \sum_{j=1}^{m} A_j x(t - h_j), \tag{9.27}$$

$h_j,\ j = 1, 2, \ldots, m$ are constants then this case is a simplified case and in spite of that the Lyapunov-Krasovskii functional that gives a necessary and sufficient condition for the system stability is given by

$$\begin{aligned} V(x_t) &= x'(t)U(0)x(t) \\ &+ \sum_{k=1}^{m}\sum_{k=1}^{m} x'(t+\theta_2)A'_k \int_0^{-h_k} U(\theta_1+\theta_2+h_k-h_j)A_j x(t+\theta_1)d\theta_1 d\theta_2 \\ &+ \sum_{m}^{k=1} \int_0^{-h_k} x'(t+\theta)[(h_k+\theta)R_k + W_k]x(t+\theta)d\theta, \end{aligned} \tag{9.28}$$

where $W_0, W_1, \ldots, W_m$, $R_1, R_2, \ldots, R_m$ are positive definite matrices and U is given by

$$\frac{d}{d\tau}U(\tau) = U(\tau)A_0 + \sum_{k=1}^{m} U(\tau - h_k)A_k, \quad \tau \in \left[0, \max_k(h_k)\right]. \tag{9.29}$$

This theorem were found by trying to imitate the situation of delay free systems by finding the state transition matrix and then use it to find P that make

$$x'(t)(PA + A'P)x(t) = -Q, \quad Q > 0,\ P > 0.$$

This Lyapunov functional gives a necessary and sufficient condition for the system stability; but finding the U for this equation is very difficult "and involves solving algebraic ordinary and partial differential equations with appropriate boundary conditions which is obviously unpromising" [4]. And even if we can find this U; the resulting functional leads to a complicated system of partial differential equations yielding infinite dimension LMI. That is why many authors considered special forms of it and thus derived simpler but more conservative, sufficient conditions which can be represented by appropriate set of LMIs.

This is the case for LTI system with fixed time delay then considering time varying delay or generally nonlinear system make it more difficult. But looking at these terms one can have idea about the possible terms which can be used in the simplified functional.

9.7.3 Halany Theorem

The following fundamental result plays an important role in the stability analysis of time-delay systems. Suppose that constant scalars k_1 and k_2 satisfy $k_1 > k_2 > 0$ and $y(t)$ is a non-negative continuous function on $[t_o - \tau, t_o]$ satisfying

$$\frac{dy(t)}{dt} \le -k_1 y(t) + k_2 \bar{y}(t) \tag{9.30}$$

for $t \ge t_0$, where $\tau \ge 0$ and

$$\bar{y}(t) = \sup_{t-\tau \le s \le t} \{y(s)\}.$$

Then, for $t \ge t_0$, we have

$$y(t) \le \bar{y}(t_0)\exp(-\sigma(t - t_0)),$$

where $\sigma > 0$ is the unique solution of the following equation

$$\sigma = k_1 - k_2 \exp(\sigma\tau).$$

It must be emphasized that Lyapunov-Krasovskii theorem, Lyapunov-Razumikhin theorem and Halanay theorem can be effectively used to derive stability conditions when the time-delay is time-varying, continuous but not necessarily differentiable. Experience and the available literature show that the Lyapunov-Krasovskii theorem is more usable particularly for obtaining delay-dependent stability and stabilization conditions.

9.7.4 Types of Continuous Lyapunov-Krasovskii Functionals

In this section, we provide some Lyapunov-Krasovskii functionals and their time-derivatives which are of common use in stability studies throughout the text.

$$V_1(x) = x^t P x + \int_{-\tau}^{0} x^t(t+\theta) Q x(t+\theta) d\theta, \tag{9.31}$$

$$V_2(x) = \int_{-\tau}^{0} \left[\int_{t+\theta}^{t} x^t(\alpha) R x(\alpha) d\alpha \right] d\theta, \tag{9.32}$$

$$V_3(x) = \int_{-\tau}^{0} \left[\int_{t+\theta}^{t} \dot{x}^t(\alpha) W \dot{x}(\alpha) d\alpha \right] d\theta, \tag{9.33}$$

where x is the state vector, τ is a constant delay factor and the matrices $0 < P^t = P$, $0 < Q^t = Q$, $0 < R^t = R$, $0 < W^t = W$ are appropriate weighting factors.

Standard matrix manipulations lead to

$$\dot{V}_1(x) = \dot{x}^t P x + x^t P \dot{x} + x^t(t) Q x(t) - x^t(t-\tau) Q x(t-\tau), \tag{9.34}$$

$$\begin{aligned} \dot{V}_2(x) &= \int_{-\tau}^{0} \left[x^t(t) R x(t) - x^t(t+\alpha) R x(t+\alpha) \right] d\theta \\ &= \tau x^t(t) R x(t) - \int_{-\tau}^{0} x^t(t+\theta) R x(t+\theta) d\theta, \end{aligned} \tag{9.35}$$

$$\dot{V}_3(x) = \tau \dot{x}^t(t) W x(t) - \int_{t-\tau}^{t} \dot{x}^t(\alpha) W \dot{x}(\alpha) d\alpha. \tag{9.36}$$

9.7.5 Some Discrete Lyapunov-Krasovskii Functionals

In this section, we provide some a general-form of discrete Lyapunov-Krasovskii functionals and their first-difference which can be used in stability studies of discrete-time throughout the text.

$$\begin{aligned} V(k) &= V_o(k) + V_a(k) + V_c(k) + V_m(k) + V_n(k), \\ V_o(k) &= x^t(k) \mathcal{P}_\sigma x(k),\; V_a(k) = \sum_{j=k-d(k)}^{k-1} x^t(j) \mathcal{Q}_\sigma x(j), \\ V_c(k) &= \sum_{j=k-d_m}^{k-1} x^t(j) \mathcal{Z}_\sigma x(j) + \sum_{j=k-d_M}^{k-1} x^t(j) \mathcal{S}_\sigma x(j), \\ V_m(k) &= \sum_{j=-d_M+1}^{-d_m} \sum_{m=k+j}^{k-1} x^t(m) \mathcal{Q}_\sigma x(m), \end{aligned}$$

$$V_n(k) = \sum_{j=-d_M}^{-d_m-1} \sum_{m=k+j}^{k-1} \delta x^t(m) \mathcal{R}_{a\sigma} \delta x(m) + \sum_{j=-d_M}^{-1} \sum_{m=k+j}^{k-1} \delta x^t(m) \mathcal{R}_{c\sigma} \delta x(m), \quad (9.37)$$

where

$$\begin{aligned} 0 < \mathcal{P}_\sigma &= \sum_{j=1}^{N} \lambda_j \mathcal{P}_j, \quad 0 < \mathcal{Q}_\sigma = \sum_{j=1}^{N} \lambda_j \mathcal{Q}_j, \quad 0 < \mathcal{S}_\sigma = \sum_{j=1}^{N} \lambda_j \mathcal{S}_j, \\ 0 < \mathcal{Z}_\sigma &= \sum_{j=1}^{N} \lambda_j \mathcal{Z}_j, \quad 0 < \mathcal{R}_{a\sigma} = \sum_{j=1}^{N} \lambda_j \mathcal{R}_{aj}, \quad 0 < \mathcal{R}_{c\sigma} = \sum_{j=1}^{N} \lambda_j \mathcal{R}_{cj} \end{aligned} \quad (9.38)$$

are weighting matrices of appropriate dimensions. Consider now a class of discrete-time systems with interval-like time-delays can be described by:

$$\begin{aligned} x(k+1) &= A_\sigma x(k) + D_\sigma x(k - d_k) + \Gamma_\sigma \omega(k), \\ z(k) &= C_\sigma x(k) + G_\sigma x(k - d_k) + \Sigma_\sigma \omega(k), \end{aligned} \quad (9.39)$$

where $x(k) \in \Re^n$ is the state, $z(k) \in \Re^q$ is the controlled output and $\omega(k) \in \Re^p$ is the external disturbance which is assumed to belong to $\ell_2[0, \infty)$. In the sequel, it is assumed that d_k is time-varying and satisfying

$$d_m \le d_k \le d_M, \quad (9.40)$$

where the bounds $d_m > 0$ and $d_M > 0$ are constant scalars. The system matrices containing uncertainties which belong to a real convex bounded polytopic model of the type

$$\begin{aligned} &[A_\sigma, D_\sigma, \dots, \Sigma_\sigma] \\ &\in \widehat{\Xi}_\lambda := \left\{ [A_\lambda, D_\lambda, \dots, \Sigma_\lambda] = \sum_{j=1}^{N} \lambda_j [A_j,\ D_j, \dots, \Sigma_j], \lambda \in \Lambda \right\}, \end{aligned} \quad (9.41)$$

where Λ is the unit simplex

$$\Lambda \triangleq \left\{ (\lambda_1, \dots, \lambda_N) : \sum_{j=1}^{N} \lambda_j = 1, \lambda_j \ge 0 \right\}. \quad (9.42)$$

Define the vertex set $\mathcal{N} = \{1, \dots, N\}$. We use $\{A, \dots, \Sigma\}$ to imply generic system matrices and $\{A_j, \dots, \Sigma_j, j \in \mathcal{N}\}$ to represent the respective values at the vertices. In what follows, we provide a definition of exponential stability of system (9.39):

A straightforward computation gives the first-difference of $\Delta V(k) = V(k+1) - V(k)$ along the solutions of (9.39) with $\omega(k) \equiv 0$ as:

$$
\begin{aligned}
\Delta V_o(k) &= x^t(k+1)\mathcal{P}_\sigma x(k+1) - x^t(k)\mathcal{P}_\sigma x(k) \\
&= [A_\sigma x(k) + D_\sigma x(k-d_k)]^t \mathcal{P}_\sigma [A_\sigma x(k) + D_\sigma x(k-d_k)] - x^t(k)\mathcal{P}_\sigma x(k), \\
\Delta V_a(k) &\leq x^t(k)\mathcal{Q}x(k) - x^t(k-d(k))\mathcal{Q}x(k-d(k)) + \sum_{j=k-d_M+1}^{k-d_m} x^t(j)\mathcal{Q}x(j), \\
\Delta V_c(k) &= x^t(k)\mathcal{Z}x(k) - x^t(k-d_m)\mathcal{Z}x(k-d_m) + x^t(k)\mathcal{S}x(k) \\
&\quad - x^t(k-d_M)\mathcal{S}x(k-d_M), \qquad (9.43) \\
\Delta V_m(k) &= (d_M - d_m)x^t(k)\mathcal{Q}x(k) - \sum_{j=k-d_M+1}^{k-d_m} x^t(k)\mathcal{Q}x(k), \\
\Delta V_n(k) &= (d_M - d_m)\delta x^t(k)\mathcal{R}_a \delta x(k) + d_M \delta x^t(k)\mathcal{R}_c \delta x(k) \\
&\quad - \sum_{j=k-d_M}^{k-d_m-1} \delta x^t(j)\mathcal{R}_a \delta x(j) - \sum_{j=k-d_M}^{k-1} \delta x^t(j)\mathcal{R}_c \delta x(j).
\end{aligned}
$$

9.8 Notes and References

The topics covered in this chapter is meant to provide the reader with general platform containing the basic mathematical information needed for further examination of switched time-delay systems. These topics are properly selected from standard books and monographs on mathematical analysis. For further details, the reader is referred to the standard texts [1, 5–8] where fundamentals are provided.

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