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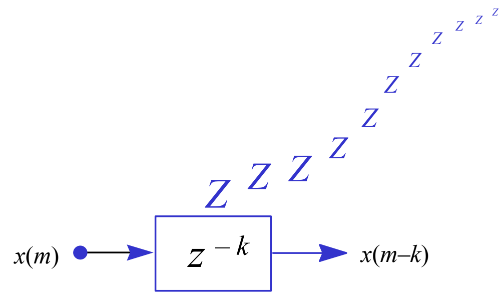


z-Transform

Z-transform, like the Laplace transform, is an indispensable mathematical tool for the design, analysis and monitoring of systems. The z-transform is the discrete-time counter-part of the Laplace transform and a generalisation of the Fourier transform of a sampled signal. Like Laplace transform the z-transform allows insight into the transient behaviour, the steady state behaviour, and the stability of discrete-time systems. A working knowledge of the z-transform is essential to the study of digital filters and systems. This chapter begins with the definition of the Laplace transform and the derivation of the z-transform from the Laplace transform of a discrete-time signal. A useful aspect of the Laplace and the z-transforms are the representation of a system in terms of the locations of the poles and the zeros of the system transfer function in a complex plane. In this chapter we derive the so-called z-plane, and its associated unit circle, from sampling the s-plane of the Laplace transform. We study the description of a system in terms the system transfer function. The roots of the transfer function, the so-called poles and zeros of transfer function, provide useful insight into the behaviour of a system. Several examples illustrating the physical significance of poles and zeros and their effect on the impulse and frequency response of a system are considered.

4.1 Introduction

The Laplace transform and its discrete-time counterpart the z-transform are essential mathematical tools for system design and analysis, and for monitoring the stability of a system. A working knowledge of the z-transform is essential to the study of discrete-time filters and systems. It is through the use of these transforms that we formulate a closed-form mathematical description of a system in the frequency domain, design the system, and then analyse the stability, the transient response and the steady state characteristics of the system.
A mathematical description of the input-output relation of a system can be formulated either in the time domain or in the frequency domain. Time-domain and frequency domain representation methods offer alternative insights into a system, and depending on the application it may be more convenient to use one method in preference to the other. Time domain system analysis methods are based on differential equations which describe the system output as a weighted combination of the differentials (i.e. the rates of change) of the system input and output signals. Frequency domain methods, mainly the Laplace transform, the Fourier transform, and the z-transform, describe a system in terms of its response to the individual frequency constituents of the input signal. In section 4.?? we explore the close relationship between the Laplace, the Fourier and the z-transforms, and the we observe that all these transforms employ various forms of complex exponential as their basis functions. The description of a system in the frequency domain can reveal valuable insight into the system behaviour and stability. System analysis in frequency domain can also be more convenient as differentiation and integration operations are performed through multiplication and division by the frequency variable respectively. Furthermore the transient and the steady state characteristics of a system can be predicted by analysing the roots of the Laplace transform or the z-transform, the so-called poles and zeros of a system.

4.2 Derivation of the z-Transform

The z-transform is the discrete-time counterpart of the Laplace transform. In this section we derive the z-transform from the Laplace transform a discrete-time signal. The Laplace transform $X(s)$, of a continuous-time signal $x(t)$, is given by the integral

$$X(s)=\int_{0-}^{\infty} x(t)e^{-st}\,dt \tag{4.1}$$

where the complex variable $s=\sigma+j\omega$, and the lower limit of $t=0^-$ allows the possibility that the signal $x(t)$ may include an impulse. The inverse Laplace transform is defined by

$$x(t)=\int_{\sigma_1-j\infty}^{\sigma_1+j\infty} X(s)e^{st}\,ds \tag{4.2}$$

where σ_1 is selected so that $X(s)$ is analytic (no singularities) for $s>\sigma_1$. The z-transform can be derived from Eq. (4.1) by sampling the continuous-time input signal $x(t)$. For a sampled signal $x(mT_s)$, normally denoted as $x(m)$ assuming the sampling period $T_s=1$, the Laplace transform Eq. (4.1) becomes

$$X(e^s)=\sum_{m=0}^{\infty} x(m)e^{-sm} \tag{4.3}$$

Substituting the variable e^s in Eq. (4.3) with the variable z we obtain the one-sided z-transform equation

$$X(z)=\sum_{m=0}^{\infty} x(m)z^{-m} \tag{4.4}$$

The two-sided z-transform is defined as

$$X(z)=\sum_{m=-\infty}^{\infty} x(m)z^{-m} \tag{4.5}$$

Note that for a one-sided signal, $x(m)=0$ for $m<0$, Eqs. (4.4) and (4.5) are equivalent.

4.2.1 The Relationship Between the Laplace, the Fourier, and the z-Transforms

The Laplace transform, the Fourier transform and the z-transform are closely related in that they all employ complex exponential as their basis function. For right-sided signals (zero-valued for negative time index) the Laplace transform is a generalisation of the Fourier transform of a continuous-time signal, and the z-transform is a generalisation of the Fourier transform of a discrete-time signal. In the previous section we have shown that the z-transform can be derived as the Laplace transform of a discrete-time signal. In the following we explore the relation between the z-transform and the Fourier transform. Using the relationship

$$z = e^{s} = e^{\sigma} e^{j\omega} = re^{j2\pi f} \tag{4.6}$$

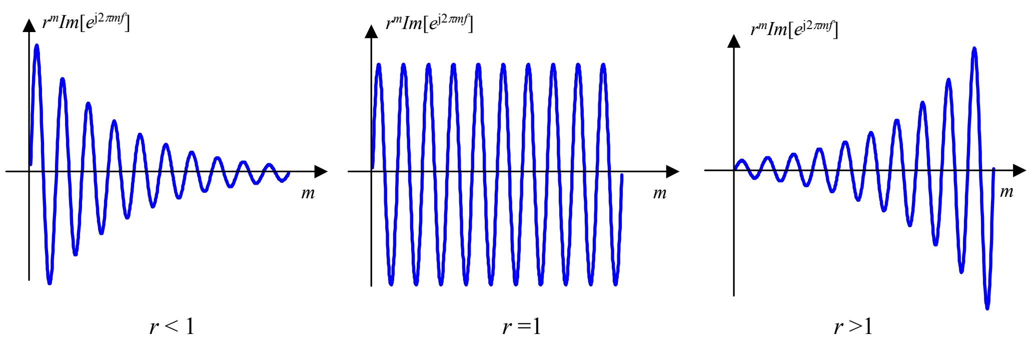


Figure 4.1 – The z-transform basis functions.

where $s=\sigma+j\omega$ and $\omega=2\pi f$, we can rewrite the z-transform Eq. (4.4) in the following form

$$X(z) = \sum_{m=0}^{\infty} x(m) r^{-m} e^{-j2\pi mf} \tag{4.7}$$

Note that when $r=e^{\sigma}=1$ the z-transform becomes the Fourier transform of a sampled signal given by

$$X(z = e^{-j2\pi f}) = \sum_{m=-\infty}^{\infty} x(m) e^{-j2\pi fm} \tag{4.8}$$

Therefore the z-transform is a generalisation of the Fourier transform of a sampled signal derived in sec. 3.xx. Like the Laplace transform, the basis functions for the z-transform are damped or growing sinusoids of the form $z^{-m} = r^{m} e^{-j2\pi fm}$ as shown in Fig. 4.1. These signals are particularly suitable for transient signal analysis. The Fourier basis functions are steady complex exponential, $e^{-j2\pi fm}$, of time-invariant amplitudes and phase, suitable for steady state or time-invariant signal analysis.

A similar relationship exists between the Laplace transform and the Fourier transform of a continuous time signal. The Laplace transform is a one-sided transform with the lower limit of integration at $t = 0^-$, whereas the Fourier transform Eq. (3.21) is a two-sided transform with the lower limit of integration at $t = -\infty$. However for a one-sided signal, which is zero-valued for $t < 0^-$, the limits of integration for the Laplace and the Fourier transforms are identical. In that case if the variable *s* in the Laplace transform is replaced with the frequency variable j2π*f* then the Laplace integral becomes the Fourier integral. Hence for a one-sided signal, the Fourier transform is a special case of the Laplace transform corresponding to s=j$2\pi f$ and σ=0.

Example 4.1 Show that the Laplace transform of a sampled signal is periodic with respect to the frequency axis jω of the complex frequency variable $s=\sigma+\mathrm{j}\omega$.

Solution: In Eq. (4.3) substitute $s+\mathrm{j}k2\pi$, where k is an integer variable, for the frequency variable s to obtain

$$
\begin{aligned}
X(e^{s+\mathrm{j}k2\pi}) &= \sum_{m=-\infty}^{\infty} x(m)e^{-(s+\mathrm{j}k2\pi)m} = \sum_{m=-\infty}^{\infty} x(m)e^{-sm}\underbrace{e^{-\mathrm{j}k2\pi m}}_{=1} \\
&= \sum_{m=-\infty}^{\infty} x(m)e^{-sm} = X(e^{s})
\end{aligned}
\tag{4.9}
$$

Hence the Laplace transform of a sample signal is periodic with a period of 2π as shown in Fig. 4.2.a

4.3 The z-Plane and The Unit Circle

The frequency variables of the Laplace transform $s=\sigma+\mathrm{j}\omega$, and the z-tranform $z=re^{\mathrm{j}\omega}$ are complex variables with real and imaginary parts and can be visualised in a two dimensional plane. Figs. 4.2.a and 4.2.b shows the *s*-plane of the Laplace transform and the *z*-plane of z-transform. In the *s*-plane the vertical jω–axis is the frequency axis, and the horizontal σ-axis gives the exponential rate of decay, or the rate of growth, of the amplitude of the complex sinusoid as also shown in Fig. 4.1. As shown

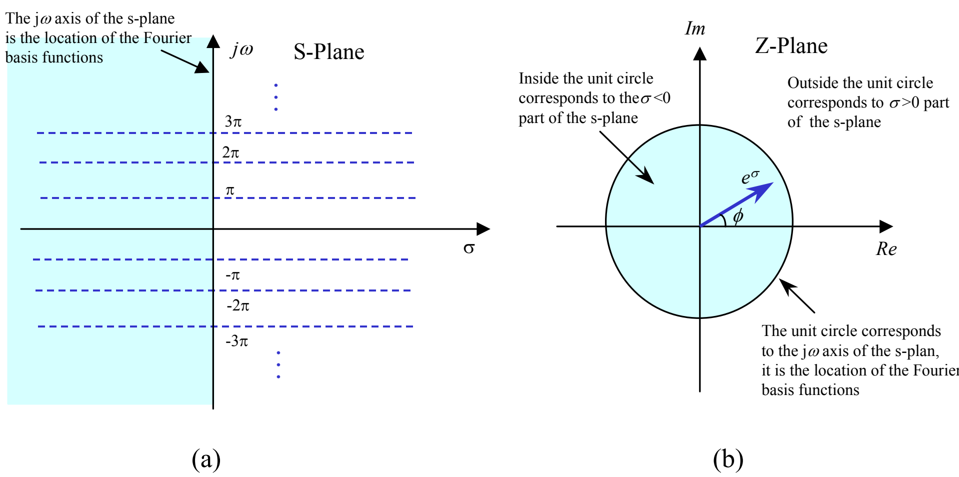


Figure 4.2 - Illustration of (a) the S-plane and (b) the Z-plane.

in example 4.1 when a signal is sampled in the time domain its Laplace transform, and hence the s-plane, becomes periodic with respect to the $j\omega$–axis. This is illustrated by the periodic horizontal dashed lines in Fig 4.2.a. Periodic processes can be conveniently represented using a circular polar diagram such as the z-plane and its assciated unit circle. Now imagine bending the $j\omega$–axis of the s-plane of the sampled signal of Fig. 4.2.a in the direction of the left hand side half of the s-plane to form a circle such that the points π and $-\pi$ meet. The resulting circle is called the *unit circle*, and the resulting diagram is called the z-plane. The area to the left of the s-plane, i.e. for $\sigma < 0$ or $r = e^{\sigma} < 1$, is mapped into the area inside the unit circle, this is the region of stable causal signals and systems. The area to the right of the s-plane, $\sigma > 0$ or $r = e^{\sigma} > 1$, is mapped onto the outside of the unit circle this is the region of unstable signals and systems. The $j\omega$–axis, with $\sigma = 0$ or $r = e^{\sigma} = 1$, is itself mapped onto the unit circle line. Hence the cartesian co-ordinates used in s-plane for continuous time signals Fig. 4.2.a, is mapped into a polar representation in the z-plane for discrete-time signals Fig 4.2.b.

Fig. 4.3 illustrates that an angle of 2π, i.e. once round the unit circle, corresponds to a frequency of F_s Hz where F_s is the sampling frequency. Hence a frequency of f Hz corresponds to an angle ϕ given by

$$\phi = \frac{2\pi}{F_s} f \quad \text{radians} \tag{4.10}$$

For example at a sampling rate of F_s=40 kHz, a frequency of 5 kHz corresponds to an angle of $2\pi \times 5/40 = 0.05\pi$ radians or 45 degrees.

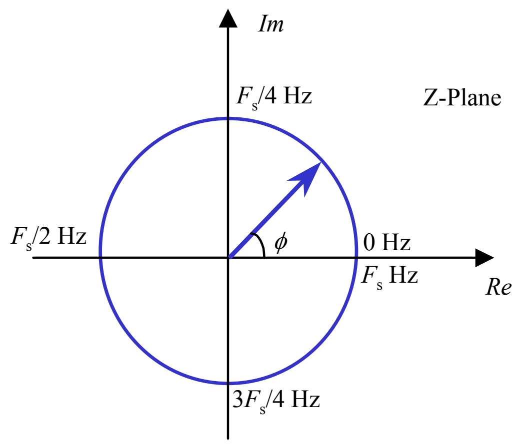
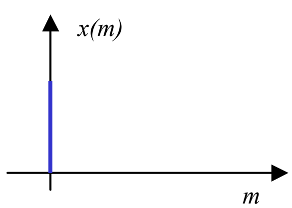


Figure 4.3 - Illustration of mapping a frequency of *f* Hz to an angle of ϕ radians.

4.3.1 The Region of Convergence (ROC)

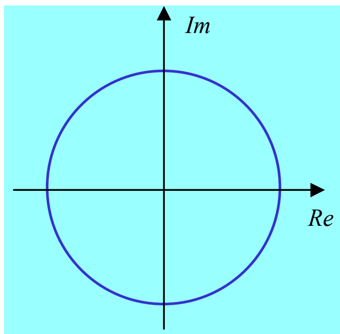
Since the z-transform is an infinite power series, it exists only for those values of the variable *z* for which the series converges to a finite sum. The region of convergence (ROC) of $X(z)$ is the set of all the values of *z* for which $X(z)$ attains a finite computable value.

Example 4.2 Determine the z-transform, the region of convergence, and the Fourier transform of the following signal



$$x(m) = \delta(m) = \begin{cases} 1 & m = 0 \\ 0 & m \neq 0 \end{cases} \tag{4.11}$$

Solution: Substituting for $x(m)$ in the z-transform Eq. (4.4) we obtain

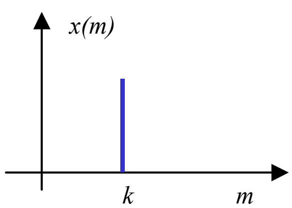


$$X(z) = \sum_{m=-\infty}^{\infty} x(m) z^{-m} = \delta(0) z^0 = 1 \qquad (4.12)$$

For all values of the variable z we have $X(z)$=1, hence as shown in the shaded area of the left hand side figure the region of convergence is the entire z-plane.
The Fourier transform of $x(m)$ may be obtained by evaluating $X(z)$ in Eq. (4.12) at $z = e^{j\omega}$ as

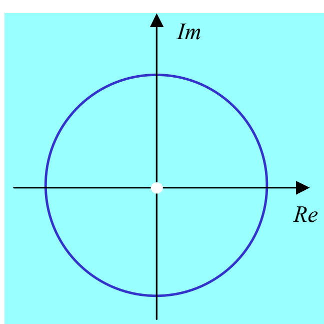
$$X(e^{j\omega}) = 1 \qquad (4.13)$$

Example 4.3 Determine the z-transform, the region of convergence, and the Fourier transform of the following signal



$$x(m) = \delta(m-k) = \begin{cases} 1 & m = k \\ 0 & m \neq k \end{cases} \qquad (4.14)$$

Solution: Substituting for $x(m)$ in the z-transform Eq. (4.4) we obtain

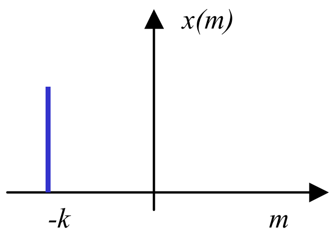


$$X(z) = \sum_{m=-\infty}^{\infty} \delta(m-k) z^{-m} = z^{-k} \qquad (4.15)$$

The z-transform is $X(z) = z^{-k} = 1/z^k$. Hence $X(z)$ is finite-valued for all the values of z except for z=0. As shown by the shaded area of the left hand side figure, the region of convergence is the entire z-plane except the point z=0. The Fourier transform is obtained by evaluating $X(z)$ in Eq. (4.15) at $z = e^{j\omega}$ as

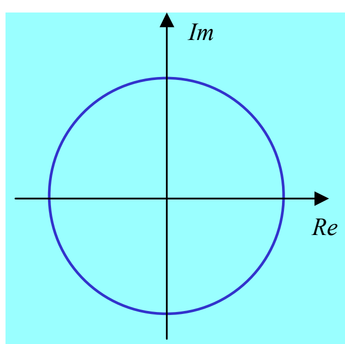
$$X(e^{j\omega}) = e^{-j\omega k} \qquad (4.16)$$

Example 4.4 Determine the z-transform, the region of convergence, and the Fourier transform of the following signal



$$x(m) = \delta(m+k) = \begin{cases} 1 & m = -k \\ 0 & m \neq -k \end{cases} \tag{4.17}$$

Solution: Substituting for $x(m)$ in the z-transform Eq. (4.4) we obtain

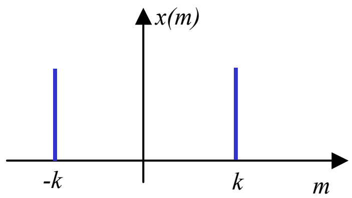


$$X(z) = \sum_{m=-\infty}^{\infty} x(m) z^{-m} = z^{k} \tag{4.18}$$

The z-transform is $X(z) = z^{k}$. Hence $X(z)$ is finite-valued for all the values of z except for $z = \infty$. As shown by the shaded area of the left hand side figure, the region of convergence is the entire z-plane except the point $z = \infty$ which is not shown. The Fourier transform is obtained by evaluating $X(z)$ in Eq. (4.18) at $z = e^{j\omega}$ as

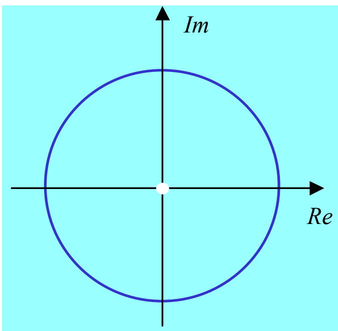
$$X(e^{j\omega}) = e^{j\omega k} \tag{4.19}$$

Example 4.5 Determine the z-transform, the region of convergence, and the Fourier transform of the following signal



$$x(m) = \delta(m+k) + \delta(m-k) = \begin{cases} 1 & m = \pm k \\ 0 & m \neq \pm k \end{cases} \tag{4.20}$$

Solution: Substituting for $x(m)$ in the z-transform Eq. (4.4) we obtain

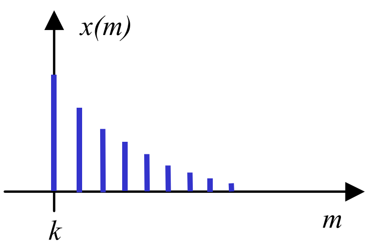


$$X(z) = \sum_{m=-\infty}^{\infty} x(m) z^{-m} = z^{k} + z^{-k} \tag{4.21}$$

The z-transform is Hence $X(z)$ is finite-valued for all the values of z except for z=0 and $z=\infty$. As shown by the shaded area of the left hand side figure, the region of convergence is the entire z-plane except the points z=0 and $z=\infty$ not shown. The Fourier transform is obtained by evaluating Eq. (4.21) at $z=e^{j\omega}$ as

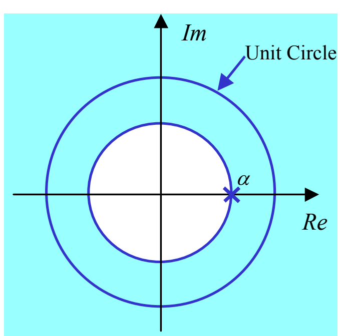
$$X(e^{j\omega}) = e^{-j\omega k} + e^{+j\omega k} = 2\cos(\omega k) \tag{4.22}$$

Example 4.6 Determine the z-transform and region of convergence of



$$x(m) = \begin{cases} \alpha^{m} & m \ge 0 \\ 0 & m < 0 \end{cases} \tag{4.23}$$

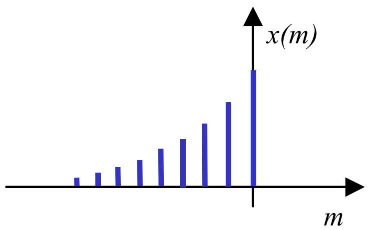
Solution: Substituting for $x(m)$ in the z-transform Eq. (4.4) we obtain



$$X(z) = \sum_{m=-\infty}^{\infty} x(m) z^{-m} = \sum_{m=0}^{\infty} \alpha^{m} z^{-m} = \sum_{m=0}^{\infty} \left(\alpha z^{-1}\right)^{m} \tag{4.24}$$

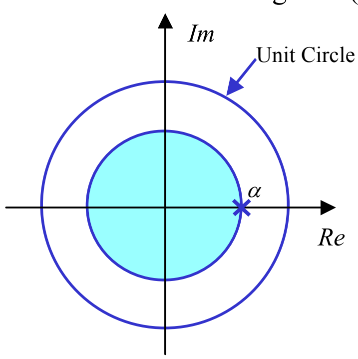
This infinite power series converges only if $|\alpha z^{-1}| < 1$. Therefore the ROC is $|z|>|\alpha|$. As shown by the shaded area of the left hand side figure, the region of convergence excludes a disc of radius α.

Example 4.7 Determine the *z*-transform and region of convergence of the left-sided sequence



$$x(m) = \begin{cases} 0 & m \geq 0 \\ \alpha^m & m < 0 \end{cases} \qquad (4.25)$$

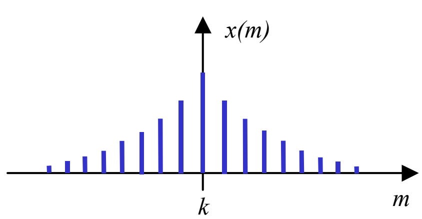
Solution: Substituting for $x(m)$ in the z-transform Eq. (4.4) we obtain



$$X(z) = \sum_{m=-\infty}^{\infty} x(m) z^{-m} = \sum_{m=-\infty}^{0} \alpha^m z^{-m} = \sum_{m=0}^{\infty} \left(\alpha^{-1} z\right)^m \qquad (4.26)$$

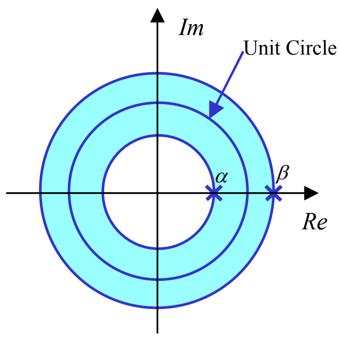
This infinite power series converges only if $|\alpha^{-1} z| < 1$. Therefore the ROC is $|z| < |\alpha|$. As shown by the shaded area of the left hand side figure, the region of convergence is confined to a disc of radius α.

Example 4.7 Determine the *z*-transform and region of convergence of the left-sided sequence



$$x(m) = \begin{cases} \alpha^m & m \geq 0 \\ \beta^m & m < 0 \end{cases} \qquad (4.27)$$

Solution: Substituting for $x(m)$ in the z-transform Eq. (4.4) we obtain



$$X(z)=\sum_{m=-\infty}^{\infty} x(m)z^{-m} = \sum_{m=0}^{\infty} \alpha^{m} z^{-m} + \sum_{m=-\infty}^{1} \beta^{m} z^{-m} \quad (4.28)$$

$$X(z)=\sum_{m=0}^{\infty} \left(\alpha z^{-1}\right)^{m} + \sum_{m=1}^{\infty} \left(\beta^{-1} z\right)^{m} \quad (4.29)$$

This infinite power series converges only if $|\alpha z^{-1}|<1$ and $|\beta^{-1}z|<1$. As shown by the shaded area of the left hand side figure, the region of convergence corresponds to the area of $\alpha < z < \beta$.

4.4 Properties of the z-Transform

As *z*-transform is a generalisation of the Fourier transform of a sampled signal it has similar properties to the Fourier Transform as described in the following.

Linearity Given two signals

$$x_1(m) \overset{z}{\Leftrightarrow} X_1(z) \quad (4.30)$$

and

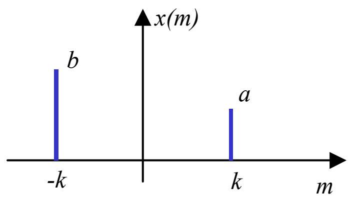
$$x_2(m) \overset{z}{\Leftrightarrow} X_2(z) \quad (4.31)$$

then the linearity implies that for any linear combination of $x_1(m)$ and $x_2(m)$ we have

$$a_1 x_1(m) + a_2 x_2(m) \overset{z}{\Leftrightarrow} a_1 X_1(z) + a_2 X_2(z) \quad (4.32)$$

Eq. (4.32) is known as the *superposition principle*.

Example 4.7 Given the following two signals



$$x_1(m) = \delta(m-k) \tag{4.33}$$

$$x_2(m) = \delta(m+k) \tag{4.34}$$

Determine the z-transform of

$$x(m) = a x_1(m-k) + b x_2(m+k) \tag{4.35}$$

Solution: Substituting for $x(m)$ in the z-transform Eq. (4.4) we obtain

$$\begin{aligned} X(z) &= \sum_{m=-\infty}^{\infty} \big(a\delta(m-k) + b\delta(m+k)\big) z^{-m} \\ &= \sum_{m=-\infty}^{\infty} a\delta(m-k) z^{-m} + \sum_{m=-\infty}^{\infty} b\delta(m+k) z^{-m} = az^{-k} + bz^{+k} \end{aligned} \tag{4.36}$$

It is clear in the second line of the above solution that the z-transform of the combination of two time domain signals $x(m)=x_1(m)+x_2(m)$ can be written as the sum of the z-transforms of the individual signals $x_1(m)$ and $x_2(m)$.

Time Shifting

The variable z has a useful interpretation in terms of time delay. If

$$x(m) \overset{z}{\Leftrightarrow} X(z)$$

then

$$x(m-k) \overset{z}{\Leftrightarrow} z^{-k} X(z) \tag{4.37}$$

This property can be proved by taking the z-transform of $x(m-k)$

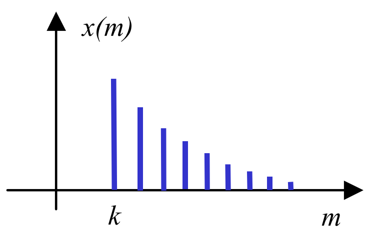
$$X(z) = \sum_{m=-\infty}^{\infty} x(m-k) z^{-m} = \sum_{n=-\infty}^{\infty} x(n) z^{-(n+k)} = z^{-k} \sum_{n=-\infty}^{\infty} x(n) z^{-n} = z^{-k} X(z) \tag{4.38}$$

where we have made a variable substitution $n=m-k$. That is the effect of a time shift by k sampling-interval-time units is equivalent to multiplication of the z-transform by z^{-k}. *Note that* z^{-1} *delays the signal by 1 unit and* z^{-k} *by k units, and* z^{+1} *is a noncausal unit time advance, and* z^{+k} *advances a signal in time by k units.*

(a) Causal k-unit delay (b) Noncausal k-unit advance

Figure 4.4 – Illustration of the variable z^{-k} as k-unit delay operator.

Example 4.8 Determine the z-transform and region of convergence of a time-delayed version of example 4.6 given as



$$x(m)=\begin{cases}\alpha^{m-k} & m \ge k \\ 0 & m<0\end{cases} \tag{4.39}$$

Solution: Substituting for $x(m)$ in the z-transform Eq. (4.4) we obtain

$$\begin{aligned} X(z) &= \sum_{m=-\infty}^{\infty} x(m) z^{-m} = \sum_{m=k}^{\infty} \alpha^{m-k} z^{-m} = z^{-k} \sum_{m=k}^{\infty} \alpha^{m-k} z^{-(m-k)} \\ &= z^{-k} \sum_{m=0}^{\infty} \alpha^{m} z^{-m} = z^{-k} \sum_{m=0}^{\infty} \left(\alpha z^{-1}\right)^{m} \end{aligned} \tag{4.40}$$

This infinite power series converges only if $|\alpha z^{-1}|<1$ and $z \neq 0$. Therefore the ROC is $|z|>|\alpha|$ which excludes z=0 as required..

Multiplication by an Exponential Sequence (Frequency Modulation)

The z-transform relation for the product of a signal $x(m)$ and the exponential sequence z_0^m is

$$z_0^m x(m) \overset{z}{\Leftrightarrow} X(z / z_0) \tag{4.41}$$

this property can be shown by substituting $z_0^m x(m) \overset{z}{\Leftrightarrow} X(z/z_0)$ in the z-transform equation

$$\sum_{m=-\infty}^{\infty} x(m) z_0^m z^{-m} = \sum_{m=k}^{\infty} x(m)(z/z_0)^{-m} = X(z/z_0) \tag{4.42}$$

Note that for the case when $z = e^{j\omega}$ and $z_0 = e^{j\omega_0}$ then we have the frequency modulation equation

$$e^{j\omega_0 m} x(m) \overset{F}{\Leftrightarrow} X(e^{j(\omega-\omega_0)}) \tag{4.43}$$

Convolution

For two signals $x_1(m)$ and $x_2(m)$

$$x_1(m) \overset{z}{\Leftrightarrow} X_1(z)$$
$$x_2(m) \overset{z}{\Leftrightarrow} X_2(z)$$

the convolutional property states that

$$x_1(m) * x_2(m) \overset{z}{\Leftrightarrow} X_1(z) X_2(z) \tag{4.44}$$

where the asterisk sign * denotes the convolution operation. That is the convolution of two signals in the time domain is equivalent to multiplication of their z-transforms and vice versa.

Differentiation in the z-Domain

Given

$$x(m) \overset{z}{\Leftrightarrow} X(z)$$

then

$$mx(m) \overset{z}{\Leftrightarrow} -z\, dX(z)/dz \tag{4.45}$$

This property can be proved by taking the derivative of the *z*-transform equation w.r.t. the variable *z* as

$$\begin{aligned}\frac{dX(z)}{dz} &= \frac{d}{dz}\sum_{m=-\infty}^{\infty} x(m)z^{-m} = -\sum_{m=\infty}^{\infty} mx(m)z^{-m-1} \\ &= -z^{-1}\sum_{m=\infty}^{\infty} m\,x(m)z^{-m}\end{aligned} \tag{4.46}$$

4.5 Transfer Function, Poles and Zeros

Consider the general linear time-invariant difference equation describing the input-output relationship of a discrete-time linear system

$$y(m) = \sum_{k=1}^{N} a_k y(m-k) + \sum_{k=0}^{M} b_k x(m-k) \tag{4.47}$$

In Eq. (4.47) the signal $x(m)$ is the system input, $y(m)$ is the system output, and a_k and b_k are the system coefficients. Taking the z-transform of Eq. (4.47) we obtain

$$Y(z) = \sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{k=0}^{M} b_k z^{-k} X(z) \tag{4.48}$$

Eq. (4.48) can be rearranged and expressed in terms of the ratio of a numerator polynomial $Y(z)$ and a denominator polynomial $X(z)$ as

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 - a_1 z^{-1} - \cdots - a_N z^{-N}} = \frac{\sum_{k=0}^{M} bz^{-k}}{1 - \sum_{k=1}^{N} az^{-k}} \tag{4.49}$$

$H(z)$ is known as the system transfer function. The frequency response of a system $H(\omega)$ may be obtained by substituting $z = e^{j\omega}$ in Eq. (4.49).

4.5.1 Poles and Zeros

One of the most useful aspects of the z-transform analysis is the description of a system in terms of the so-called poles and zeros of the system. The zeros of a transfer function *H*(*z*) are the values of the variable *z* for which the transfer function (or equivalently its numerator) is zero. Therefore the zeros are the roots of numerator polynomial in Eq. (4.49). The poles of *H*(*z*) are the values of the variable *z* for which *H*(*z*) is infinite. This happens when the denominator of *H*(*z*) is zero. Therefore the poles of *H*(*z*) are the roots of the denominator polynomial of Eq. (4.49).
To obtain the poles and zeros of *H*(*z*) rewrite the numerator and denominator polynomials to avoid negative powers of the variable *z* as

$$H(z) = \frac{b_0 z^{-M}}{z^{-N}} \times \frac{z^M + (b_1/b_0) z^{M-1} + \cdots + (b_M/b_0)}{z^N - a_1 z^{N-1} - \cdots - a_N} \tag{4.50}$$

Now the numerator and denominator polynomials of *H*(*z*) may be factorised and expressed as

$$H(z) = b_0\, z^{-M+N} \times \frac{(z-z_1)(z-z_2)\cdots(z-z_M)}{(z-p_1)(z-p_2)\cdots(z-p_N)} \tag{4.51}$$

or

$$H(z) = G\, z^{-M+N} \times \frac{\prod_{k=1}^{M}(z-z_k)}{\prod_{k=1}^{N}(z-p_k)} \tag{4.52}$$

where the gain $G=b_0$, and z_ks are the zeros and p_ks the poles of *H*(*z*) respectively. Thus the transfer function *H*(*z*) has *M* finite zeros i.e. the roots of the numerator polynomial at $z=z_1, z_2, \ldots, z_M$, *N* finite poles i.e. the roots of the denominator polynomial at $z=p_1, p_2, \ldots, p_N$, and $|N\text{-}M|$ zeros (if $N > M$), or poles (if $N < M$), at the origin $z=0$. Poles or zeros may also occur at $z=\infty$. A zero exists at infinity if $H(z=\infty)=0$ and a pole exists at infinity if $H(z=\infty)=\infty$. If we count the number of poles and zeros at zero and infinity we find that *H*(*z*) has exactly the same number of poles as zeros. A further important point to note is that for a system with real-valued coefficients a_k and b_k, complex-valued poles or zeros always occur in complex conjugate pairs.
The description of a system in terms of its poles and zeros is an extremely useful abstraction of a system. The poles represent the roots of the feedback part of the transfer function of a system. For a stable system the poles should have a magnitude of

less than one and lie inside the unit circle as shown in the following examples. The zeros represent the roots of the feed forward part of the transfer function of a system. There is no restriction on the values of zeros other than that required to obtain a desired frequency or impulse response.

A useful graphical abstraction of the transfer function of a discrete-time system $H(z)$ is the *pole-zero plot* in a complex polar plane. The location of the poles are shown by crosses (×) and the locations of the zeros by circles (**O**) as described in the following examples.

Example 4.9 Find the *z*-transform and plot the pole-zero diagram of the following right-sided discrete-time signal

$$x(m) = \begin{cases} \alpha^m & m \geq 0 \\ 0 & m < 0 \end{cases} \tag{4.53}$$

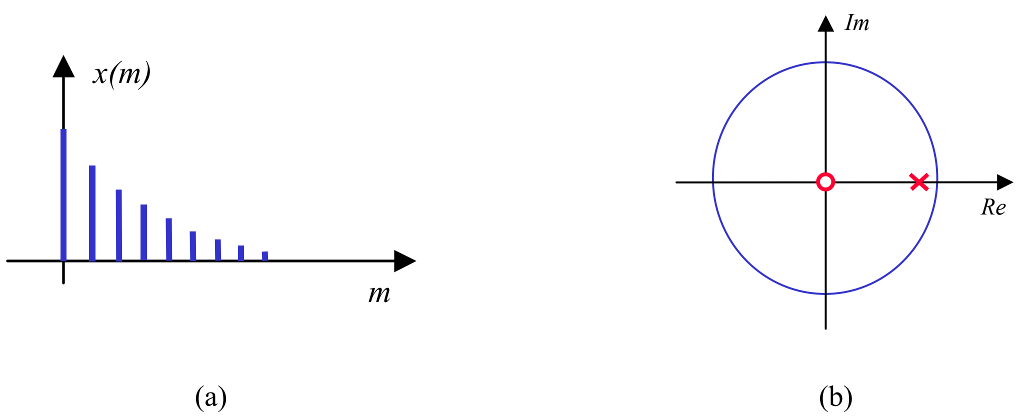


Figure 4.5 (a) An exponentially decaying signal, (b) The pole-zero representation of the signal in (a).

Solution: Substituting for $x(m)$ in the z-transform Eq. (4.4) we obtain

$$X(z) = \sum_{m=0}^{\infty} \alpha^m z^{-m} = \sum_{m=0}^{\infty} \left(\alpha z^{-1}\right)^m \tag{4.54}$$

For $|a|<1$, using the convergence formula for a geometic series in appendix A, this power series converges to

$$X(z) = \frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha} \tag{4.55}$$

Therefore $X(z)$ in this case has a single zero at z=0 and a single pole at z=α. The pole-zero plot is shown in Fig 4.5.b.

4.5.2 The Response of a Single (First-Order) Zero or Pole

The effect of a first order zero is to introduce a deep or trough in the frequency spectrum of the signal at the frequencies 0 or π radians, where π corresponds to half the sampling frequency. The effect of a first order pole is to introduce a peak in the frequency spectrum of the signal at frequencies 0 or π. These are illustrated in the following examples.

Example 4.10 Consider the first order feed forward system of Fig 4.6 given by

$$y(m) = \alpha x(m-1) + x(m) \tag{4.56}$$

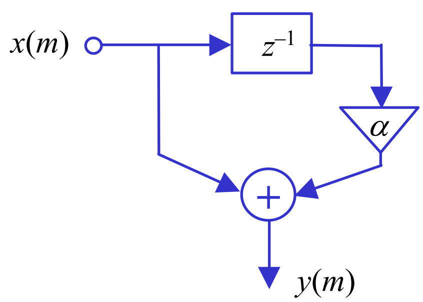


Figure 4.5 (a) A first-order feed forward discrete-time system.

Taking the z-transform of Eq. (4.56) yields

$$Y(z) = \alpha z^{-1} X(z) + X(z) = X(z)(1+\alpha z^{-1}) \tag{4.57}$$

From Eq. (4.57) the z-transfer function is given by

$$H(z) = \frac{Y(z)}{X(z)} = 1 + \alpha z^{-1} \tag{4.58}$$

Equating $H(z)$=0 yields a zero at $z = -\alpha$. Substituting $z = e^{-j\omega}$ in Eq. (4.58) yields the frequency response

$$H(e^{j\omega}) = 1 + \alpha e^{-j\omega} \tag{4.59}$$

Now at an angular frequency ω=0, $H(0)$=1+α and at ω=π; $H(\pi)$=1−α. Hence when α=1, $H(0)$=0. Fig. 4.6 shows the variation of the frequency response of the first order

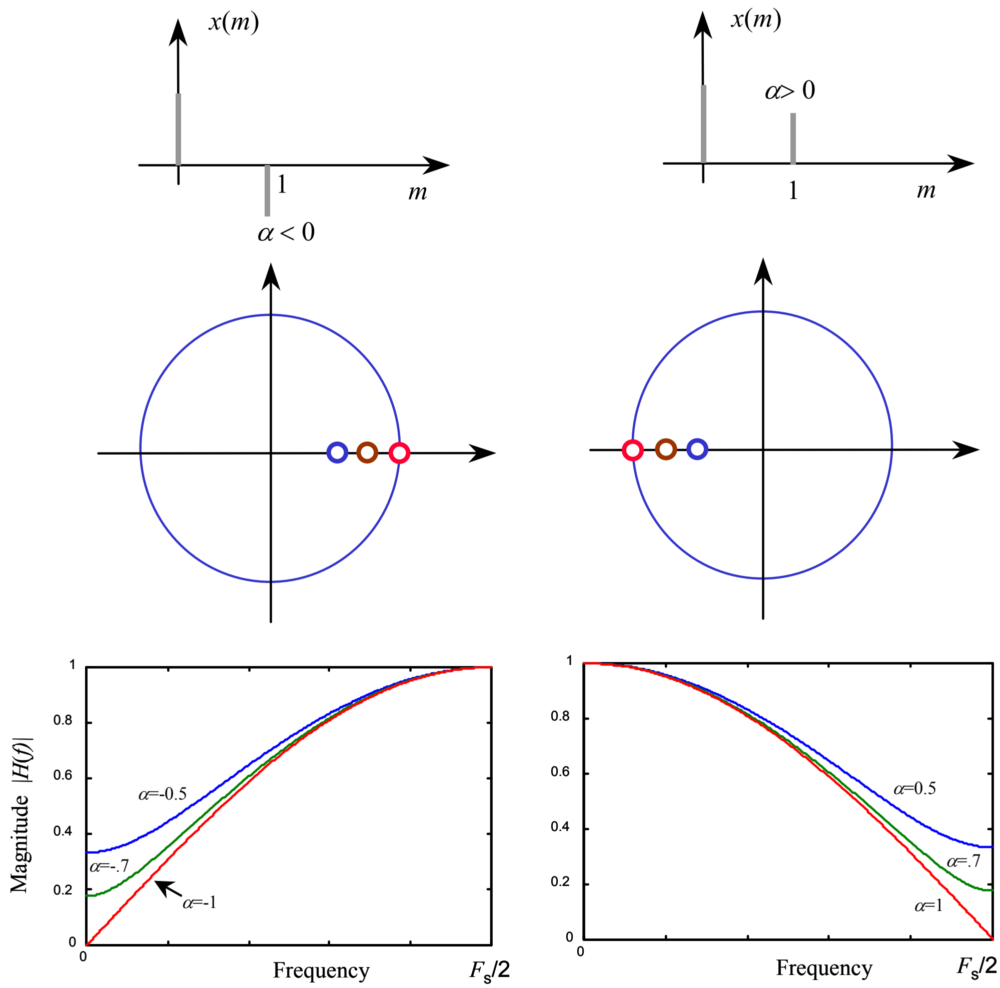


Figure 4.6 Illustration of the impulse response, the pole-zero diagram, and the frequency response of a first order system with a single zero, for the varying values of the zero α.

single-zero system with the radius of the zero α. On the frequency axis the angular frequency ω=π corresponds to a frequency of $F_s/2$ Hz where F_s is the sampling rate. Note that for the positive values of α the system has a high-pass frequency response, and conversely for the negative values of α the system has a low-pass frequency response.

Example 4.11 Consider the first order feedback system of Fig 4.7 given by

$$y(m) = \alpha\, y(m-1) + x(m) \tag{4.60}$$

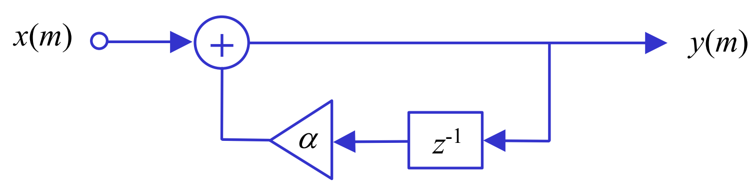


Figure 4.7 (a) A first order feedback discrete-time system.

Taking the z-transform of Eq. (4.60) yields

$$Y(z) = \alpha\, z^{-1} Y(z) + X(z) \tag{4.61}$$

From Eq. (4.61) the transfer function is given by

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-\alpha\, z^{-1}} = \frac{z}{z-\alpha} \tag{4.62}$$

The transfer function $H(z)$ has a pole at z=α and a zero at the origin. Substituting $z = e^{j\omega}$ in Eq. (4.58) gives the frequency response

$$H(e^{j\omega}) = \frac{1}{1-\alpha\, e^{-j\omega}} \tag{4.63}$$

Now at ω=0; $H(0)=1/(1-\alpha)$ and at ω=π; $H(\pi)=1/(1+\alpha)$. Fig. 4.8 shows the variation of the frequency response of the first order single-pole system with the pole radius α. On the frequency axis the angular frequency ω=π corresponds to a frequency of $F_s/2$

Hz where F_s is the sampling rate. Note that for the positive values α of the system has a low-pass frequency response, and conversely for the negative values of α the system has a high-pass frequency response.

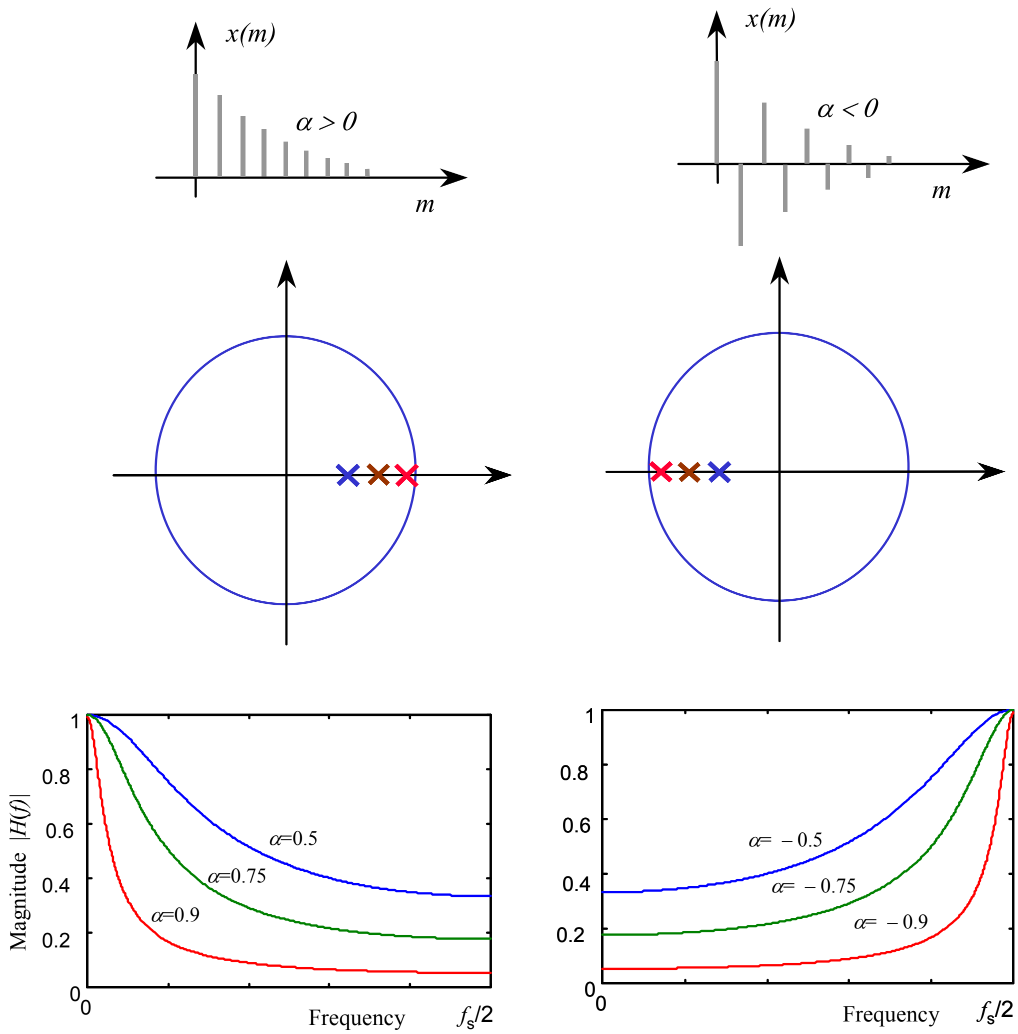


Figure 4.8 Illustration of the impulse response, the pole-zero diagram, and the frequency response of a first order system with a single pole, for the varying values of the pole radius α.

4.5.3 The Response of a Second Order Pair of Zeroes or Poles

The effect of a second order pair of zeros is to introduce a deep or trough in the frequency spectrum of the signal at a frequency that depends on the angular position of the zeros. The effect of a second order pair of poles is to introduce a peak in the frequency spectrum of the signal at a frequency that depends on the angular position of the poles in the unit circle.

Example 4.12 Consider the second order feed forward system of Fig 4.9 given by

$$y(m) = b_0 x(m) + b_1 x(m-1) + b_2 x(m-2) \tag{4.64}$$

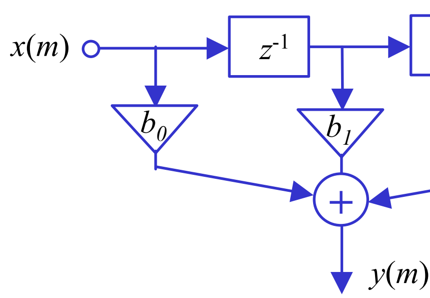


Figure 4.9 (a) A second-order feed forward discrete-time system.

Taking the z-transform of Eq. (4.64) yields

$$Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) \tag{4.65}$$

From Eq. (4.65) the transfer function is given by

$$H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} \tag{4.66}$$

Eq. (4.66) can be factorised and expressed in terms of the zeros of the transfer function as

$$H(z) = G\left(1 - z_1 z^{-1}\right)\left(1 - z_1^* z^{-1}\right) \tag{4.67}$$

where the gain factor $G=b_0$. Note that since the coefficients of the transfer function polynomial in Eq. (4.62) are real-valued, the roots of the polynomial have to be either complex conjugates or real-valued. For a pair of complex conjugate poles $z_1 = re^{j\phi}$ and $z_1^* = re^{-j\phi}$ Eq. (4.67) can be written in a polar form in terms of the angular frequency ϕ and the radius of the poles r as

$$\begin{aligned} H(z) &= G\left(1 - re^{j\phi} z^{-1}\right)\left(1 - re^{-j\phi} z^{-1}\right) \\ &= G\left(1 - 2r\cos(\phi) z^{-1} + r^2 z^{-2}\right) \end{aligned} \tag{4.68}$$

Comparing Eqs. (4.62) and (4.64) we have $b_0=G$, $b_1=2r\cos(\phi)$ and $b_2=r^2$.

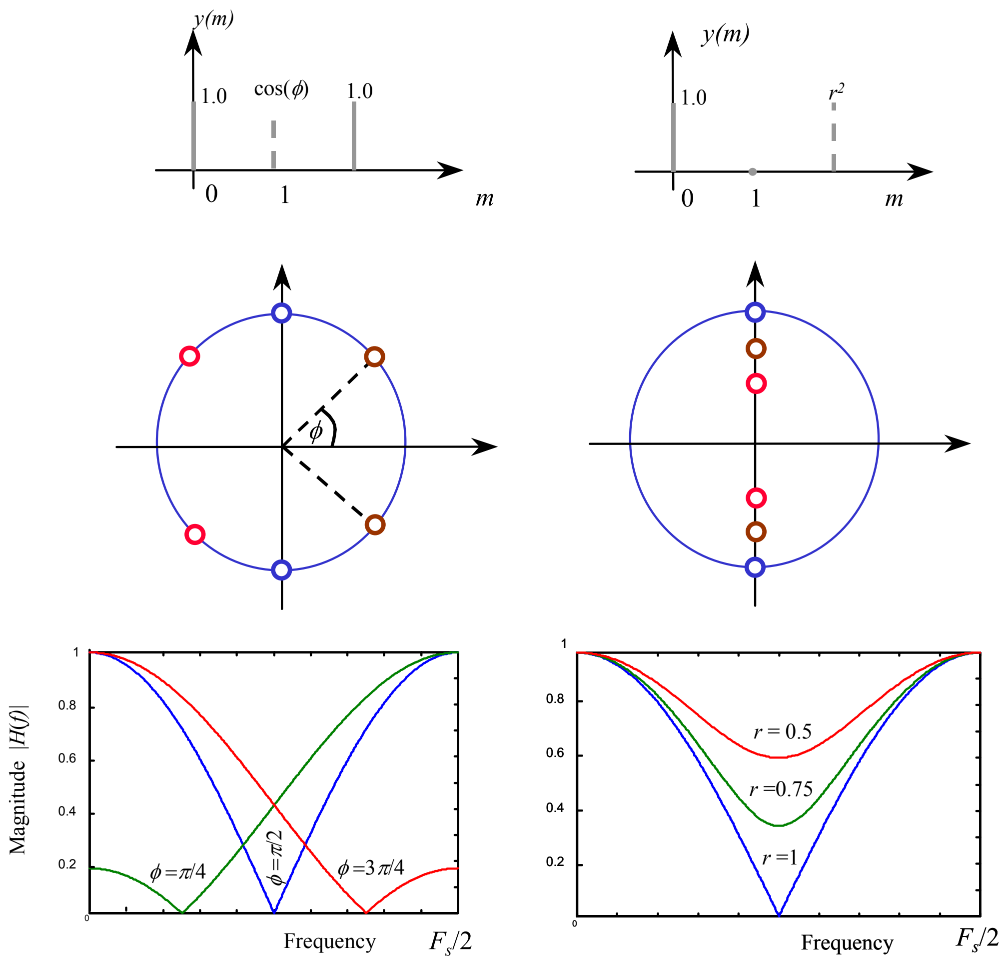


Figure 4.10 Illustration of the impulse response, the pole-zero diagram, and the frequency response of a second order system with a a pair of complex conjugate zeros, for the varying values of the angular position ϕ and the radius r of the zeros.

Fig. 4.10.a illustrates the variation of the zero-frequency with the angular position of the zeros for a complex conjugate pair of zeros. Fig. 4.10.b illustrates the variation of the depth and the bandwidth the trough introduced by a complex conjugate pair of zeros with the radius of the zeros.

Example 4.13 Consider the second order feedback system of Fig. 4.11 given by

$$y(m) = a_2 y(m-2) + a_1 y(m-1) + g x(m) \tag{4.65}$$

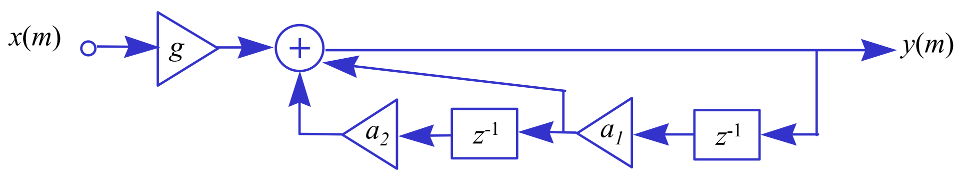


Figure 4.11 (a) A second-order feed back discrete-time system.

Taking the z-transform of Eq. (4.65) we have

$$Y(z) = a_2 z^{-2} Y(z) + a_1 z^{-1} Y(z) + gX(z) \tag{4.66}$$

Rearranging Eq. (4.66) we obtain the z-transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{g}{1 - a_1 z^{-1} - a_2 z^{-2}} \tag{4.67}$$

Eq. (4.67) can be expressed in terms of the poles of the z-transfer function as

$$H(z) = \frac{g}{\left(1 - z_1 z^{-1}\right)\left(1 - z_1^* z^{-1}\right)} \tag{4.68}$$

where $g=1/a_0$. Note that since the coefficients of the polynomial $H(z)$ are real-valued, the roots of this polynomial have to be either complex conjugate or real. For a pair of complex conjugate poles $z_1 = re^{j\phi}$ and $z_1^* = re^{-j\phi}$ Eq. (4.68) can be rewritten in a polar form in terms of the angular frequency and the radius of the poles as

$$\begin{aligned} H(z) &= \frac{g}{\left(1 - re^{j\phi} z^{-1}\right)\left(1 - re^{-j\phi} z^{-1}\right)} \\ &= \frac{g}{1 - 2r\cos(\phi) z^{-1} + r^2 z^{-2}} \end{aligned} \tag{4.69}$$

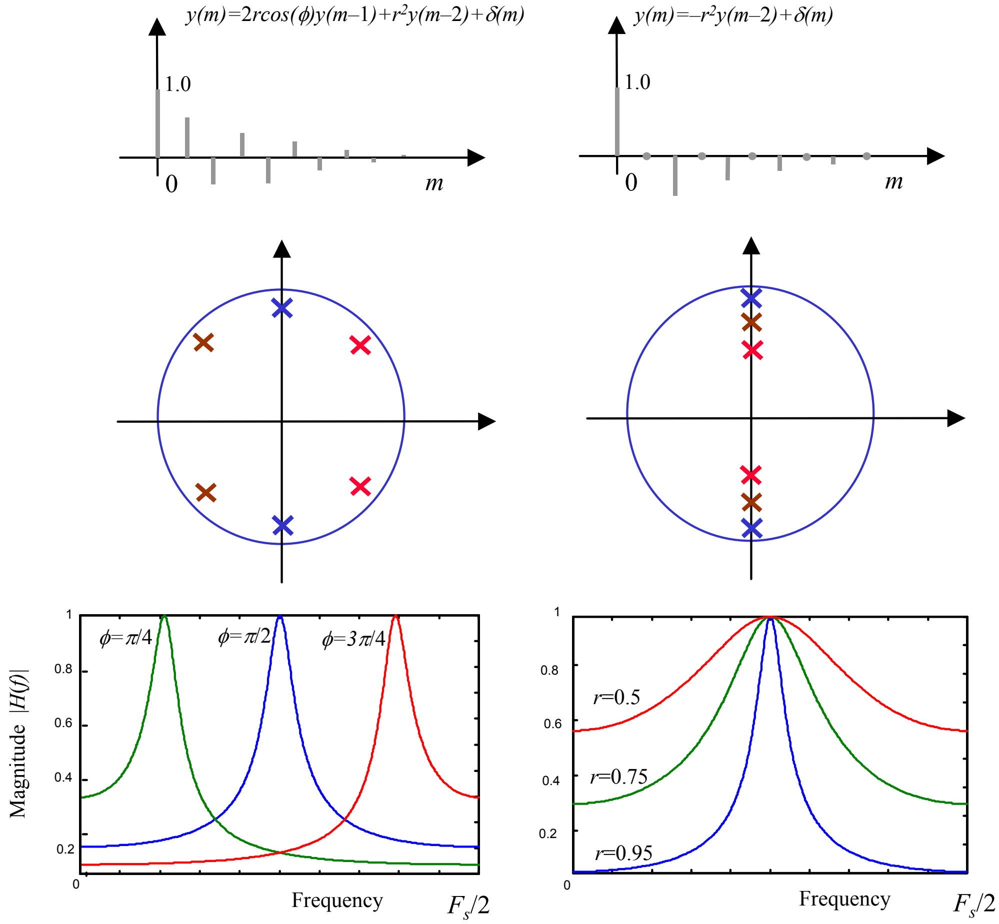


Figure 4.102 Illustration of the impulse response, the pole-zero diagram, and the frequency response of a second order system with a pair of complex conjugate zeros, for the varying values of the the angular position and the radius of the zeros.

Fig. 4.12.a illustrates the variation of the resonance frequency with the angular position of the poles for a complex conjugate pair of poles. Fig. 4.12.b illustrates the variation of the bandwidth of the resonance from a complex conjugate pair of poles with the radius of the poles.

4.6 Inverse z-Transform

The inverse z-transform can be obtained using one of two methods: (a) the inspection method, (b) the partial fraction method. In the inspection method each simple term of a polynomial in z, $H(z)$, is substituted by its time-domain equivalent. For the more complicated functions of z, the partial fraction methods is used to describe the polynomial in terms of simpler terms, and then each simple term is substituted by its time-domain equivalent term.

4.6.1 Inverse z-transform by Inspection

In this method the discrete-time equation for a signal or a system is obtained from its z-transform by recognising simple z-transform pairs and substituting the time domain terms for their corresponding z-domain terms.

Example 4.13 Find the inverse z-transform of

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-\alpha z^{-1}} \tag{4.70}$$

Solution: From Eq. (4.70) we have

$$Y(z) = \alpha z^{-1} Y(z) + X(z) \tag{4.71}$$

By inspection and through the substitution of $z^{-k}Y(z)$ for $y(m-k)$ and $z^{-k}X(z)$ for $x(m-k)$ we obtain the discrete-time equivalent of Eq. (4.71)

$$y(m) = \alpha\, y(m-k) + x(m) \tag{4.72}$$

Now if the input $x(m)$ is a discrete-time impulse $\delta(m)$ given by

$$\delta(m) = \begin{cases} 1 & m = 0 \\ 0 & m \neq 0 \end{cases} \tag{4.73}$$

then the output of the feedback system of Eq. () will be

$$y(m) = \begin{cases} \alpha^m & m \geq 0 \\ 0 & m < 0 \end{cases} \tag{4.74}$$