The Fourier Transform

The Fourier transform is crucial to any discussion of time series analysis, and this chapter discusses the definition of the transform and begins introducing some of the ways it is useful.

We will use a *Mathematica*-esque notation. This includes using the symbol **I** for the square root of minus one. Also, what is conventionally written as **sin(t)** in *Mathematica* is **Sin[t]**; similarly the cosine is **Cos[t]**. Finally, the irrational number 2.71828... is represented by the symbol **E.**

The contents of this chapter are:

■ Fourier Series

Recall the Fourier series, in which a function **f[t]** is written as a sum of sine and cosine terms:

$$f[t] = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \,\mathrm{Cos}[nt] + \sum_{n=1}^{\infty} b_n \,\mathrm{Sin}[nt]$$

or equivalently:

$$f[t] = \sum_{n=-\infty}^{\infty} c_n \,E^{-Int} = \sum_{n=-\infty}^{\infty} c_n \,(\mathrm{Cos}[nt] - I\,\mathrm{Sin}[nt])$$

The coefficients are found from the fact that the sine and cosine terms are orthogonal, from which:

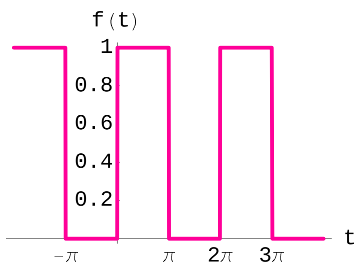
$$a_n = \frac{1}{\pi} \int_{t=0}^{2\pi} f[t] \, \text{Cos}[nt] \, dt$$

$$b_n = \frac{1}{\pi} \int_{t=0}^{2\pi} f[t] \, \text{Sin}[nt] \, dt$$

Fourier series are used, for example, to discuss the harmonic structure of the tonic and overtones of a vibrating string.

Note that the series represents either **f[t]** over a limited range of $0 < \mathbf{t} < 2\pi$, or we assume that the function is periodic with a period equal to 2π.

Also note that, as opposed to the Taylor series, the Fourier series can represent a discontinuous function:



$$f[t] = \frac{1}{2} + \frac{2}{\pi} \left(\frac{\text{Sin}[t]}{1} + \frac{\text{Sin}[3t]}{3} + \frac{\text{Sin}[5t]}{5} + \ldots \right)$$

■ Fourier Transform

In the previous section we defined the series over the interval $(0, 2\pi)$. Say instead we are interested in the interval (-L, L). Then the coefficients in the Fourier series are:

$$a_n = \frac{1}{L} \int_{-L}^{L} f[t] \, \text{Cos}\left[\frac{n\pi t}{L}\right] dt$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f[t] \, \text{Sin}\left[\frac{n\pi t}{L}\right] dt$$

Thus we write the series of **f** as a function of a dummy variable **x** as:

$$f[x] =$$
$$\frac{1}{2L}\int_{-L}^{L} f[t]\,dt + \frac{1}{L}\sum_{n=1}^{\infty} \mathrm{Cos}\left[\frac{n\pi x}{L}\right]\int_{-L}^{L} f[t]\,\mathrm{Cos}\left[\frac{n\pi t}{L}\right]dt +$$
$$\frac{1}{L}\sum_{n=1}^{\infty} \mathrm{Sin}\left[\frac{n\pi x}{L}\right]\int_{-L}^{L} f[t]\,\mathrm{Sin}\left[\frac{n\pi t}{L}\right]dt$$

The trigonometry relation:

$$\mathrm{Cos}[\theta_1 - \theta_2] =$$
$$\mathrm{Cos}[\theta_2 - \theta_1] = \mathrm{Cos}[\theta_1]\,\mathrm{Cos}[\theta_2] + \mathrm{Sin}[\theta_1]\,\mathrm{Sin}[\theta_2]$$

allows us to rewrite the expansion as:

$$f[x] = \frac{1}{2L}\int_{-L}^{L} f[t]\,dt + \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f[t]\,\mathrm{Cos}\left[\frac{n\pi}{L}(t-x)\right]dt$$

Let L $\to\infty$, ie. let the interval (-L, L) go to (-∞, ∞). We write:

$$\omega = \frac{n\pi}{L}$$
$$\Delta\omega = \frac{\pi}{L}$$

The meaning of $\Delta\omega$ is that it is the amount that ω changes for each time in the sum that n goes to n + 1. The first term in the series is the DC component of **f[t]**. In the limit as L $\to\infty$, the integral must be either infinity or zero; the latter is the only reasonable choice. So we now can write the expansion as:

$$f[x] = \frac{1}{\pi}\sum_{n=1}^{\infty}\Delta\omega\int_{-\infty}^{\infty} f[t]\,\mathrm{Cos}[\omega\,(t-x)]\,dt$$

Recall the notation, useful for multiple integrals that:

$$\int\left(\int f[x, y]\,dx\right)dy = \int dy\int dx\, f[x, y]$$

In the expression for **f[x]**, we replace the sum with an integral:

$$f[x] = \frac{1}{\pi} \int_0^{\infty} d\omega \int_{-\infty}^{\infty} dt\, f[t]\, \mathrm{Cos}[\omega\, (t - x)] =$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dt\, f[t]\, \mathrm{Cos}[\omega\, (t - x)]$$

Since the sine is odd:

$$\mathrm{Sin}[\theta] = -\mathrm{Sin}[-\theta]$$

we can write:

$$0 = -I \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dt\, f[t]\, \mathrm{Sin}[\omega\, (t - x)] \right)$$

Adding this to the expression for **f[x]** gives:

$$f[x] = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{+I\omega x} \int_{-\infty}^{\infty} dt\, f[t]\, e^{-I\omega t}$$

The Fourier transform $F_1[\omega]$ of **f[t]** is:

$$F_1[\omega] = \int_{-\infty}^{\infty} f[t]\, e^{-I\omega t}\, dt$$

Note that it is a function of ω. If we interpret **t** as the time, then ω is the angular frequency. Thus we have replaced a function of time with a spectrum in frequency.

The inverse Fourier transform takes **F[**ω**]** and, as we have just proved, reproduces **f[t]:**

$$f[t] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_1[\omega]\, e^{I\omega t}\, d\omega$$

You should be aware that there are other common conventions for the Fourier transform (which is why we labelled the above transforms with a subscript). For example, some texts use a different normalisation:

$$F_2[\omega] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f[t]\, e^{-I\omega t}\, dt$$

$$f[t] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_2[\omega]\, e^{I\omega t}\, dt$$

Still others reverse the transform and its inverse:

$$F_3[\omega] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f[t]\, e^{I\omega t}\, dt$$

$$f[t] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_3[\omega]\, e^{-I\omega t}\, dt$$

The only difference between the "type-2" definition and the "type-3" one is the relative signs of the real and imaginary parts of the transforms.

By default, *Mathematica* uses this "type-3" definition of the Fourier transform. In this class we will almost always be using the "type-1" convention.

Say we have a function of the position **x**: **g[x]**. Then the type-1 Fourier transform and inverse transform are:

$$G_1[k] = \int_{-\infty}^{\infty} g[x]\, e^{-Ikx}\, dx$$

and:

$$g[x] = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_1[k]\, e^{Ikx}\, dk$$

In this case the transform is a function of the wavenumber $\mathbf{k = 2\pi/\lambda}$.

■ Example and Interpretation

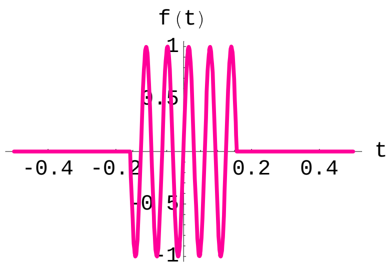
Say we have a function:

$$f[t\_] := \mathrm{Sin}[\omega_0\, t]\ /;\ \mathrm{Abs}[t] < \frac{n\pi}{\omega_0}$$

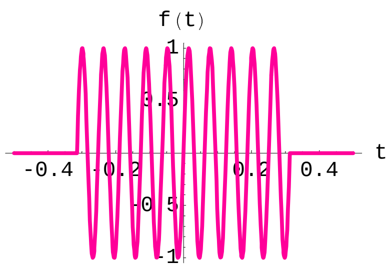
$$f[t\_] := 0\ /;\ \mathrm{Abs}[t] \geq \frac{n\pi}{\omega_0}$$

where **Abs** is *Mathematica*'s function for an absolute value. Also, the **t_** means any variable, and **:=** is the form of equal sign used in function definitions. The **/;** imposes the conditions under which the definition applies.

For n = 5 and ω_0 = 100, this looks like:



Increasing n to 10 increases the number of cycles:



The Fourier transform is:

$$F[\omega] = \int_{-\infty}^{\infty} f[t]\, e^{-I\omega t}\, dt = \int_{-n\pi/\omega_0}^{n\pi/\omega_0} \mathrm{Sin}[\omega_0\, t]\, e^{-I\omega t}\, dt$$

This evaluates to:

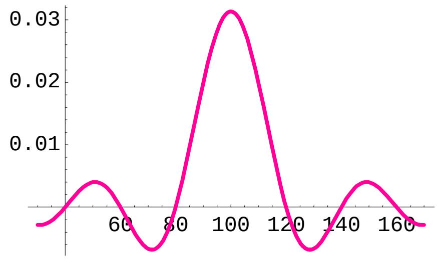
$$-I\sqrt{\frac{2}{\pi}}\left[\frac{\mathrm{Sin}[(\omega_0 - \omega)\, n\pi/\omega_0]}{2\,(\omega_0 - \omega)} - \frac{\mathrm{Sin}[(\omega_0 + \omega)\, n\pi/\omega_0]}{2\,(\omega_0 + \omega)}\right]$$

where of course I used *Mathematica* to actually do the integration.

For $\omega \approx \omega_0$ and ω_0 large:

$$IF[\omega] = \sqrt{\frac{1}{2\pi}}\ \frac{\mathrm{Sin}[(\omega_0 - \omega)\, n\pi / \omega_0]}{2\,(\omega_0 - \omega)}$$

For n = 5 and ω_0 = 100, the right hand side of the above looks like:



The zeroes in the above occur at:

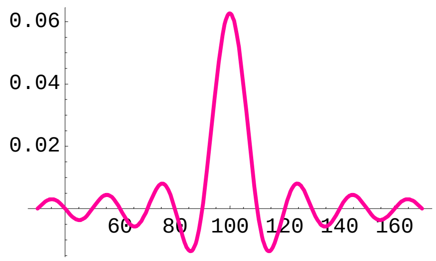
$$\frac{\omega_0 - \omega}{\omega_0}\, n = \frac{\delta\omega}{\omega_0}\, n = \pm 1,\ \pm 2,\ \pm 3,\ \ldots$$

Since the contributions outside the central maximum are small, we may take:

$$\delta\omega = \frac{\omega_0}{n}$$

to be a measure of the width of the peak.

For n = 10 and the same value of ω_0 the plot looks like:



Clearly, the width of the curve is now decreased.

Curves such as the above will occur sufficiently often that we will give the function that generates them a name: the *sinc*:

$$\mathrm{Sinc}[x] \equiv \frac{\mathrm{Sin}[\pi x]}{\pi x}$$

One interpretation of the above Fourier transform is that **F[ω]** is the frequency spectrum of a sine wave signal **f[t]** which is varying in time; thus ω is the angular frequency. The main frequency component occurs at the frequency of the sine wave, ω_0, but there are other frequency components that cancel out the signal for values of the time whose absolute value is greater than $\mathbf{n}\pi/\omega_0$. If we think about letting **n** go to infinity, then the sine wave is non-zero for all values of the time from $-\infty$ to ∞; in this case the width of the Fourier transform goes to zero and become a Dirac delta function centered at ω_0.

Thus if we have an infinite sine wave but only measure it for a finite period of time, the measurement will introduce "sidebands" in the frequency spectrum.

Another interpretation of the transform is that the symbol **t** is the finite width of a slit; the Fourier transform of **f[t]** is then the amplitude of the diffraction pattern of the slit. The fact that a wider slit produces a narrower transform means that to get, say, good dispersion of the high tones from a loudspeaker requires that the speaker be small.

Yet another interpretation is that **f[t]** is the amplitude of an electromagnetic wave that is passing by us. The period of the sine wave itself is

$$T = \frac{2\pi}{\omega_0}$$

and there are **n** cycles of the sine wave in **f[t]**, so it takes a time:

$$\Delta t = n\,\frac{2\pi}{\omega_0}$$

for the wave to pass us.

The width in the peak of the Fourier transform is a way of saying there is an uncertainty in the "true" value of the frequency. But we know from Planck that the frequency is related to the energy **E** according to:

$$E = \frac{h}{2\pi}\,\omega$$

Thus, the uncertainty in the frequency corresponds to an uncertainty in the energy:

$$\Delta E = \frac{h}{2\pi}\,\delta\omega$$

Above we estimated $\delta\omega$ to be $\omega_0/\mathbf{n}$ so:

$$\Delta E = \frac{h\,\omega_0}{2\pi n}$$

Thus:

$$\Delta E\, \Delta t = \frac{\hbar h\, \omega_0}{2\,\pi\, n}\, 2\, \frac{n\pi}{\omega_0} = h$$

This is just a form of the Heisenberg uncertainty principle!

■ Oddness and Evenness

Symmetry arguments in Fourier theory often allows us to show directly that certain integrals vanish without needing to evaluate them. Also, often symmetry considerations allows us the reduce the limits of integration, which again can symplify calculations.

An even function **e[t]** is one such that **e[-t] = e[t]**; an example is the cosine. An odd function **o[t]** is one such that **o[-t] = -o[t]**; an example is the sine. Given an arbitrary function **f[t]**, we can extract the even and odd parts of it:

$$e[t] = 1/2\,(f[t] + f[-t])$$
$$o[t] = 1/2\,(f[t] - f[-t])$$

and:

$$f[t] = e[t] + o[t]$$

In general **e** and **o** are complex.

The Fourier transform of **f[t]** is:

$$F[\omega] = \int_{-\infty}^{\infty} f[t]\, e^{-I\omega t}\, dt =$$
$$\int_{-\infty}^{\infty} (e[t] + o[t])\,(\mathrm{Cos}[\omega t] - I\,\mathrm{Sin}[\omega t])\, dt =$$
$$\int_{-\infty}^{\infty} e[t]\,\mathrm{Cos}[\omega t]\, dt - I\int_{-\infty}^{\infty} o[t]\,\mathrm{Sin}[\omega t]\, dt =$$
$$2\int_{0}^{\infty} e[t]\,\mathrm{Cos}[\omega t]\, dt - 2\,I\int_{0}^{\infty} o[t]\,\mathrm{Sin}[\omega t]\, dt$$

Thus, for example, if **f[t]** has an even part that is real and an odd part that is imaginary, its Fourier transform is real.

■ The Convolution Theorem

I hope that after going through some of the interpretations of the Fourier transform above, you are already convinced that it is one of the "keys to the universe." Here we present one of the most important keys in the context of time series analysis.

Imagine we have a function **f[t]** whose Fourier transform is **F[ω]**, and another function **g[t]** whose transform is **G[ω]**. Then the convolution is:

$$f[t] * g[t] = \int_{-\infty}^{\infty} f[u]\, g[t-u]\, du$$

We write **g[t - u]** in terms of the inverse Fourier transform:

$$g[t-u] = \frac{1}{2\pi} \int_{-\infty}^{\infty} G[\omega]\, E^{I\omega(t-u)}\, d\omega$$

Thus:

$$f[t] * g[t] = \int_{-\infty}^{\infty} f[u] \frac{1}{2\pi} \int_{-\infty}^{\infty} G[\omega]\, E^{I\omega(t-u)}\, d\omega\, du = \frac{1}{2\pi} \int_{-\infty}^{\infty} G[\omega]\, E^{I\omega t} \int_{-\infty}^{\infty} f[u]\, E^{-I\omega u}\, du\, d\omega$$

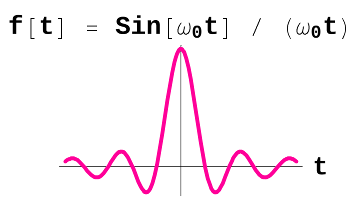
But the right hand integral above is just the Fourier transform of **f[u]**, so:

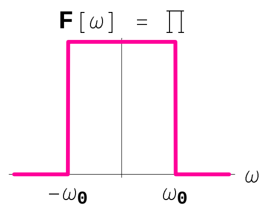
$$f[t] * g[t] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F[\omega]\, G[\omega]\, E^{I\omega t}\, d\omega$$

In words:

The inverse Fourier transform of a product of Fourier transforms is the convolution of the original functions.

Here is an example. The Fourier transform of a Sinc function is just the rectangle function that in the Convolution chapter we gave the symbol $\prod$:





Thus **F[ω]** only passes frequencies with an absolute value less than ω_0. So if we convolve a Sinc function with some signal **g[t]**, the Sinc is performing as an ideal "low pass filter."

The example shows that we can consider the design of a filter in two separate domains:

1. In the frequency domain we can just multiply the Fourier transform of the original time series by some desired filter, a low pass filter in the above example.

2. In the time domain, we can convolve the time series with the inverse Fourier transform of the desired filter.

The fact that the amplitude of the Sinc function approaches zero asymptotically as **t** → ±∞ means that doing the full convolution would require measuring **f[t]** from **t** = -∞ to **t** = +∞. Since this is physically impossible, we have proved that the ideal low pass filter can not be built.

■ Discrete Fourier Transforms

So far in this chapter we have only considered continuous functions **f[t]**. Here we extend to a time series that is a sample of **f[t]**. This section is divided into three subsections: **Definitions, Examples**, and **Implementation.**

■ Definitions

If **f[t]** is a times series of length **n**, then we replace the continous Fourier transform:

$$F[\omega] \;=\; \int_{-\infty}^{\infty} f[t]\, e^{-I\omega t}\, dt$$

with a sum:

$$F[\omega_j] \;=\; \left(\sum_{k=0}^{n-1} f[t_k]\, E^{-I\,\omega_j\, t_k}\right) \Delta$$

We wish to evaluate t_k in the above. Recall that Δ is the sampling interval so the time is:

$$t_k = k\,\Delta$$

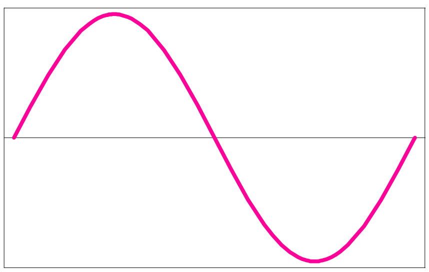
Imagine that **f[t]** is periodic and we have sampled over one complete period **T** so $f_0 = f_n$. Then:

$$T = n\,\Delta$$

In order to evaluate ω_j we will think for a moment about Fourier *series*. The term ω_0 is zero and is the DC component, corresponding to the a_0 term in the series. The next term:

$$\omega = \frac{2\pi}{T}$$

corresponds to the first allowed vibration, which for a vibrating string is the *tonic* and determines the note that is being played:



The next term is:

$$\omega = 2\left(\frac{2\pi}{T}\right)$$

and represents the first *harmonic* of a vibrating string:

So in general the frequencies are:

$$\omega_j = j\left(\frac{2\pi}{T}\right)$$

So now we can write the discrete Fourier transform as:

$$F[\omega_j] = \left(\sum_{k=0}^{n-1} f_k\, E^{-I\, j\, \left(\frac{2\pi}{T}\right) k\Delta}\right) \Delta = \left(\sum_{k=0}^{n-1} f_k\, E^{-I\, 2\pi\, j\, k/n}\right) \Delta$$

To get the inverse Fourier transform:

$$f[t_k] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F[\omega]\, E^{I\omega t}\, dw = \left(\frac{1}{2\pi} \sum_{j=0}^{n-1} F_j\, E^{I\,\omega_j\, t_k}\right) \delta\omega$$

The value of $\delta\omega$ is just how much ω_j changes with each change from **j** to **j + 1**. We just saw that it is:

$$\delta\omega = \frac{2\pi}{T} = \frac{2\pi}{n\Delta}$$

So the discrete inverse Fourier transform is:

$$f[t_k] = \left(\frac{1}{n} \sum_{j==}^{n-1} F_j\, E^{I\, 2\pi\, j\, k/n}\right) \frac{1}{\Delta}$$

Note that in both the transform and its inverse, by making the usual choice that the sampling interval Δ is one simplifies the definition. We shall make that choice for the remainder of this chapter. Thus:

$$F_j = \sum_{k=0}^{n-1} f_k\, E^{-I\, 2\pi\, j\, k/n}$$

$$f_k = \frac{1}{n} \sum_{j=0}^{n-1} F_j\, E^{I\, 2\pi\, j\, k/n}$$

We recall that F_j is a shorthand for **F[**ω_j], where:

$$\omega_j = j \left(\frac{2\pi}{n\Delta}\right) = j \left(\frac{2\pi}{n}\right)$$

and that f_k is a shorthand for **f[**t_k**]**, where:

$$t_k = k\Delta = k$$

■ **Example**

As already mentioned, the built in discrete Fourier transform routines in *Mathematica* use a different convention than we use by default in these notes. Earlier, we called *Mathematica*'s choice a "type-3" definition, so we write:

$$\text{F}_3[[\text{j}]] = \frac{1}{\sqrt{\text{n}}} \sum_{\text{k}=1}^{\text{n}} \text{f}[[\text{k}]]\, \text{E}^{\text{I}\, 2\, \pi\, (\text{j}-1)\, (\text{k}-1)\, /\, \text{n}}$$

Here the sign of the exponential is different than the type-1 definition. Also note that **k** goes from 1 through **n**, because that is the way the *Mathematica* accesses members of a list. Recall that we usually write the elements of a time series as:

$$\text{timeSeries} = \{f_0,\ f_1,\ f_2,\ \ldots,\ f_{n-1}\}$$

Then *Mathematica*'s access of elements of the list is:

$$\text{timeSeries}[[\text{k}]] = f_{k-1}$$

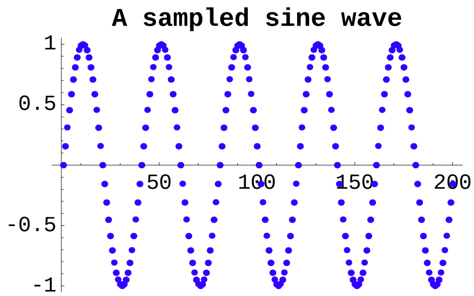
The *Mathematica* function implementing this definition is called **Fourier**.

The inverse Fourier transform in *Mathematica*, **InverseFourier**, is:

$$\text{f}_3[[\text{k}]] = \frac{1}{\sqrt{\text{n}}} \sum_{\text{j}=1}^{\text{n}} \text{F}_3[[\text{j}]]\, \text{E}^{-\text{I}\, 2\, \pi\, (\text{j}-1)\, (\text{k}-1)\, /\, \text{n}}$$

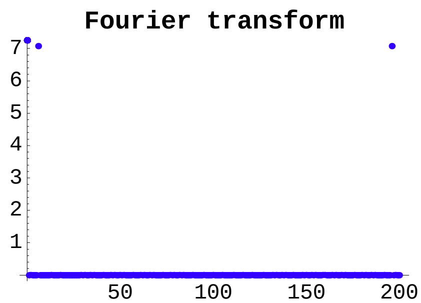
It is fairly simple to use *Mathematica*'s functions to implement the "type-1" conventions that we have been using, but in this subsection we will not bother. The only real difference between the two conventions is the signs of the real and imaginary parts and the normalisation.

Say we have a n = 200 element time series of five cycles of a sine wave:



The period is 40, so the angular frequency is $2\pi/40$.

Then using **Fourier** on the time series gives us the transform, whose absolute values are:



Note: there is a bug somewhere in the *Mathematica* → PostScript → PDF sequence that puts a superfluous dot on the vertical axis of the above plot. It should not be there. I am working with the vendor of *Mathematica* to try to find and fix the bug.

Almost all the values are close to with zero. The sixth element has a value of 7.07. In the previous subsection we showed that the frequency values in the Fourier transform are:

$$\omega_j = j\left(\frac{2\pi}{T}\right) = j\left(\frac{2\pi}{n\Delta}\right)$$

which here is:

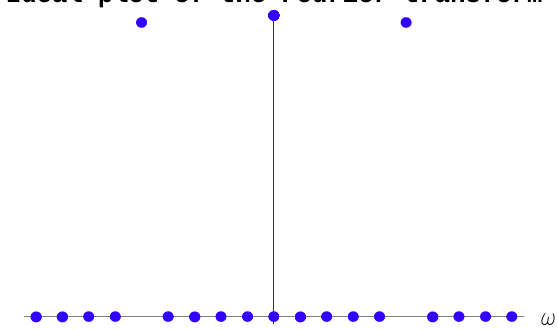
$$\omega_j = j\left(\frac{\pi}{100}\right)$$

The maximum in the transform occurs at the sixth element, which corresponds to j = 5. Thus:

$$\omega_j = 5\left(\frac{\pi}{100}\right) = \frac{\pi}{20}$$

This is precisely the frequency of the original sine wave.

Note that element 196 of the Fourier transform also has a value of 7.07. This is called an "alias" of the peak occuring in the sixth element. If you go through the Fourier transform definitions above you can convince yourself that the frequency can just as well be a negative number as a positive one. The alias is a representation of the negative frequency component of the Fourier transform. Ideally we would like it to look something like:



Note: the above plot also shows the same bug as the previous one. There is an extra dot high on the vertical axis. Ignore it.

But since the Fourier transform is just a list of numbers, not {frequency, number} pairs, *Mathematica* couldn't do that. Instead it joined the negative frequency values to the end of the positive frequency ones.

We will have a lot more to say about aliasing in the next chapter.

■ Implementation

One "brute force" way of calculating the sums of a Fourier transform is to define:

$$W \equiv E^{-I2\pi/n}$$

Then:

$$F_j = \sum_{k=0}^{n-1} f_k E^{-I\,2\,\pi\,j\,k\,/\,n} = \sum_{k=0}^{n-1} f_k W^{jk}$$

We can think of **f** as a vector of length n, and **W** as a matrix of dimension **n × k**. This multiplication requires $\boldsymbol{n}^{2}$ calculations, and evaluating the sum requires a smaller number of operations to generate the powers of **W**. Thus calculating the Fourier transform this way is a O(n^2) process. Doubling the number of points in a time sample quadruples the time necessary to calculate the transform; tripling the number of points requires nine times as much time.

However, there is a *Fast Fourier Transform* algorithm that can give an immense improvement. The basic idea is that we split the sum into two parts:

$$F_j = \sum_{k=0}^{n/2-1} f_{2k}\, E^{-I\,2\,\pi\,j\,2\,k/n} + \sum_{k=0}^{n/2-1} f_{2k+1}\, E^{-I\,2\,\pi\,j\,(2\,k+1)/n}$$

The first sum involves the even terms in **f**, and the second sum the odd ones.

Using the same **W** as before, we can write:

$$F_j = \sum_{k=0}^{n/2-1} f_{2k}\, E^{-I\,2\,\pi\,j\,k/(n/2)} +$$

$$W^k \sum_{k=0}^{n/2-1} f_{2k+1}\, E^{-I\,2\,\pi\,j\,k/(n/2)} = F_k^e + W^k\, F_k^o$$

We can apply the same procedure recursively to the sums represented by F_k^e and F_k^o. Eventually, if **n** is a power of two, we end up with no summations at all, just a product of terms.

It turns out that this procedure is O(n $\log_2$n). So, doubling the number of points in the time series only doubles the time necessary to calculate the transform using this algorithm, and tripling the number of points increases the time by about 4.75.

Many people treat this Fast Fourier Transform as "magic", and it is does seem magical in its properties. But notice that it all hinges on **n** being a power of two. For example, if one constructs a time series:

$$\text{timeSeries} = \sum_{t=1}^{32700} E^{-t/(32700/6)}$$

the values range from 0.999817 to 0.00247. *Mathematica*'s **Fourier** in one computing environment took 13.12 seconds to transform this series. Since the last terms in the series are so close to zero, it is quite reasonable to add zeroes to the end of the time series. We added 68 zeroes, so the total length of the series becomes 32,768 = 2^{15}. Now it took **Fourier** only 5.16 seconds to do the transform.

The lesson, then, is that if one is taking data in the lab and later the data will be Fourier transformed, if there is a lot of data the total number of data points should always be a factor of two.

■ Author