Inverse Laplace Transform

So far, we have dealt with the problem of finding the Laplace transform for a given function f(t), t > 0,

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt$$

Now, we want to consider the inverse problem, given a function F(s), we want to find the function f(t) whose Laplace transform is F(s).
We introduce the notation

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

to denote such a function f(t), and it is called the **inverse Laplace transform** of F.

Remark: The inverse Laplace transform is not unique:
If

$$g(t) = \begin{cases} 1 \text{ if } 0 < t < 3 \\ -8 \text{ if } t = 3 \\ 1 \text{ if } t > 3 \end{cases}$$

then $\mathcal{L}\{g(t)\} = 1/s$
and $\mathcal{L}\{1\} = 1/s$
So, both functions have the same Lapalce transform, therefore 1/s has two inverse transforms.
But, the only continuous function with Laplace transform 1/s is f(t) =1.

A crude, but sometimes effective method for finding inverse Laplace transform is to construct the table of Laplace transforms and then use it in reverse to find the inverse transform.

Example:
1) Since $\mathcal{L}\{1\} = 1/s$, then $\mathcal{L}^{-1}\{1/s\} = 1$
2) Since $\mathcal{L}\{t\} = 1/s^2$, then $\mathcal{L}^{-1}\{1/s^2\} = t$
3) Since $\mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}$, then $\mathcal{L}^{-1}\{\frac{s}{s^2 + a^2}\} = \cos at$

The following properties will simplify our calculations:
1) **Linear Property**
If c_1 and c_2 are constants,

$$\mathcal{L}^{-1}\{c_1F_1(s) + c_2F_2(s)\} = c_1\mathcal{L}^{-1}\{F_1(s)\} + c_2\mathcal{L}^{-1}\{F_2(s)\}$$

Example:
$\mathcal{L}^{-1}\{\frac{4}{s} - \frac{3}{s^2}\} = 4\ \mathcal{L}^{-1}\{1/s\} - 3\ \mathcal{L}^{-1}\{1/s^2\} = 4 - 3t.$

2) **Inverse Translation Property**

Given F(s) = $\mathcal{L}$ {f(t)}, since $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$,

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t) = e^{at}\,\mathcal{L}^{-1}\{F(s)\}$$

Example:

1) Find $\mathcal{L}^{-1}\{\frac{15}{s^2+4s+13}\}$

$$\mathcal{L}^{-1}\{\frac{15}{s^2+4s+13}\} = \mathcal{L}^{-1}\{\frac{15}{s^2+4s+4+9}\} = \mathcal{L}^{-1}\{\frac{15}{\left(s^2+4s+4\right)+9}\} = \mathcal{L}^{-1}\{\frac{3(5)}{(s+2)^2+3^3}\}$$

$$= 5\ \mathcal{L}^{-1}\{\frac{3}{(s+2)^2+3^3}\} = 5\ \mathcal{L}^{-1}\{\frac{3}{(s-(-2))^2+3^3}\} = 5\ e^{-2t}\,\mathcal{L}^{-1}\{\frac{3}{s^2+3^2}\} = 5\ e^{-2t}\sin 3t$$

2) Find $\mathcal{L}^{-1}\{\frac{s+1}{s^2+6s+25}\}$

$$\mathcal{L}^{-1}\{\frac{s+1}{s^2+6s+25}\} = \mathcal{L}^{-1}\{\frac{s+1}{s^2+6s+9+16}\} = \mathcal{L}^{-1}\{\frac{s+1}{\left(s^2+6s+9\right)+16}\} =$$

$$\mathcal{L}^{-1}\{\frac{(s+3)-2}{(s+3)^2+4^2}\} = \mathcal{L}^{-1}\{\frac{s+3}{(s+3)^2+4^2}\} - \mathcal{L}^{-1}\{\frac{2}{(s+3)^2+4^2}\} = e^{-3t}\,\mathcal{L}^{-1}\{\frac{s}{s^2+4^2}\} -$$

$$e^{-3t}\,\tfrac{1}{2}\,\mathcal{L}^{-1}\{\frac{2(2)}{s^2+4^2}\} = e^{-3t}[\cos 4t - \tfrac{1}{2}\sin 4t]$$

3) Find $\mathcal{L}^{-1}\{\frac{1}{s^3+s}\}$.

$$\mathcal{L}^{-1}\{\frac{1}{s^3+s}\} = \mathcal{L}^{-1}\{\frac{1}{s\left(s^2+1\right)}\}$$

Let's use partial fractions

$$\frac{1}{s\left(s^2+1\right)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} = \frac{As^2+A+Bs^2+Cs}{s\left(s^2+1\right)}$$

then, $1 = (A+B)s^2 + Cs + A$

A = 1, C = 0 , A + B = 0, B = -1

$$\mathcal{L}^{-1}\{\frac{1}{s\left(s^2+1\right)}\} = \mathcal{L}^{-1}\{\frac{1}{s} - \frac{s}{s^2+1}\} = \mathcal{L}^{-1}\{\frac{1}{s}\} - \mathcal{L}^{-1}\{\frac{s}{s^2+1}\} = 1 - \cos t$$

3) Inverse Shifting Property

Since $\mathcal{L}\{u_a(t)\, f(t-a)\} = e^{-as}\, \mathcal{L}\{f(t)\} = e^{-as}\, F(s)$,

then $\mathcal{L}^{-1}\{ e^{-as}\, F(s)\} = u_a(t)\, f(t-a)$

Example:

1) Find $\mathcal{L}^{-1}\{ \frac{4}{s} - \frac{3e^{-3s}}{s} - 2\frac{e^{-7s}}{s} \}$

$\mathcal{L}^{-1}\{ \frac{4}{s} - \frac{3e^{-3s}}{s} - 2\frac{e^{-7s}}{s} \} = 4\, \mathcal{L}^{-1}\{\frac{1}{s}\} - 3\, \mathcal{L}^{-1}\{e^{-3s}\frac{1}{s}\} - 2\, \mathcal{L}^{-1}\{e^{-7s}\frac{1}{s}\} = 4 - 3u_3(t)\, f(t-3) - 2u_7(t)\, f(t-7)$ where $f(t) = \mathcal{L}^{-1}\{\frac{1}{s}\} = 1$.

$$\mathcal{L}^{-1}\{ \frac{4}{s} - \frac{3e^{-3s}}{s} - 2\frac{e^{-7s}}{s} \} = 4 - 3u_3(t) - 2u_7(t) = \begin{cases} 4-0-0 \text{ if } 0<t<3 \\ 4-3-0 \text{ if } 3<t<7 \\ 4-3-2 \text{ if } t>7 \end{cases} = \begin{cases} 4 \text{ if } 0<t<3 \\ 1 \text{ if } 3<t<7 \\ -1 \text{ if } t>7 \end{cases}$$

2) Find $\mathcal{L}^{-1}\{ e^{-4s}\left(\frac{2}{s^2} - \frac{5}{s}\right)\}$

Since $F(s) = \frac{2}{s^2} - \frac{5}{s}$

$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{\frac{2}{s^2} - \frac{5}{s}\} = 2\, \mathcal{L}^{-1}\{\frac{1}{s^2}\} - 5\, \mathcal{L}^{-1}\{\frac{1}{s}\} = 2t - 5 = f(t)$

Now,

$\mathcal{L}^{-1}\{ e^{-4s}\left(\frac{2}{s^2} - \frac{5}{s}\right)\} = \mathcal{L}^{-1}\{e^{-4s}\, F(s)\} = u_4(t)\, f(t-4) = u_4(t)\, [2(t-4) - 5] = u_4(t)\, [2t-13] =$

$$\begin{cases} 0 \text{ if } 0<t<4 \\ 2t-13 \text{ if } t>4 \end{cases}$$

4) The Convolution Integral

Sometimes it is possible to identify a Laplace transform H(s) as the product of two other transforms F(s) and G(s) that are the transforms of two known functions f(t) and g(t). We will introduce a "generalize product" called convolution and H(s) would be the transform of the convolution of f and g.

Definition: Let f(t) and g(t) be two piecewise continuous functions on [0, b] and of exponential order with constant a. We call **convolution** of the functions f and g and denote it by $f * g$ to the integral

$$f * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

the integral is known as the **convolution integral**.

Properties:
1) **Commutative**:
$f * g = g * f$
<u>Proof</u>:
Since $f * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau$,
Make the substitution u = t - τ, then τ = t – u.
Since 0 < τ < t, then t < u < 0 and dτ = -du,
so

$$f * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau = \int_t^0 f(t-u)g(u)(-du) = -\int_t^0 f(t-u)g(u)du = \int_0^t f(t-u)g(u)du =$$
$$g * f(t)$$

2) **Distributive**:
$f * (g + k) = f*g + f*k$

3) **Associative**:
$(f * g)*k = f*(g*k)$

$f * 0 = 0*f = 0$

Remark : $f * 1 \neq f$
If f(t) = cos t, then

$$f * 1(t) = \int_0^t \cos(t-\tau)\ 1\ d\tau = -\sin(t-\tau)\Big|_0^t = -\sin 0 + \sin t = \sin t \neq f(t)$$

Theorem: In the above conditions, we have
$\mathcal{L}\{f*g\} = \mathcal{L}\{f\}\cdot\mathcal{L}\{g\} = F(s)\cdot G(s)$
then

$$\mathcal{L}^{-1}\{F(s)\cdot G(s)\} = f * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t g(\tau)f(t-\tau)d\tau$$

Example:
1)

$$\mathcal{L}^{-1}\{\frac{1}{s(s^2+1)}\} = \mathcal{L}^{-1}\{\frac{1}{s}\cdot\frac{s}{s^2+1}\} = f*g = g*f$$

Since $\mathcal{L}^{-1}\{\frac{1}{s}\} = 1$ and $\mathcal{L}^{-1}\{\frac{1}{s^2+1}\} = \sin t$

$$f * g(t) = \int_0^1 \sin\ \tau\ d\tau = -\cos\tau\Big|_0^t = 1 - \cos t$$

2) $\mathcal{L}^{-1}\{\frac{1}{s^2 - s - 12}\} = \mathcal{L}^{-1}\{\frac{1}{s-4} \cdot \frac{1}{s+3}\} = f*g = g*f$

Since $\mathcal{L}^{-1}\{\frac{1}{s-4}\} = e^{4t}\, \mathcal{L}^{-1}\{\frac{1}{s}\} = e^{4t}1$ and $\mathcal{L}^{-1}\{\frac{1}{s+3}\} = e^{-3t}\, \mathcal{L}^{-1}\{\frac{1}{s}\} = e^{-3t}1$

$$f * g(t) = \int_0^t e^{4(t-\tau)} e^{-3\tau} d\tau = \int_0^t e^{4t} e^{-4\tau} e^{-3\tau} d\tau = e^{4t} \int_0^t e^{-7\tau} d\tau = e^{4t} \frac{e^{-7\tau}}{-7}\Big|_0^t = e^{4t}\left[\frac{e^{-7t}}{-7} + \frac{1}{7}\right] =$$

$$\frac{e^{4t}}{7} - \frac{e^{-3t}}{7}$$