Digital Signal Processing

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1 Review of our Transforms

Definition 1. Discrete-Time Fourier Transform (DTFT):

$$\begin{aligned} S\left(e^{j2\pi f}\right) &= \sum_{n=-\infty}^{\infty} s(n)e^{-j2\pi fn} \\ s(n) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} S\left(e^{j2\pi f}\right)e^{j2\pi fn}df \end{aligned}$$

The DTFT is periodic with period 1; the limits on the integral for the inverse DTFT may be anything you choose as long as their difference is one, the length of a period. The DTFT is continuous in frequency, even though it is a transform of a discrete-time signal. Just as we needed to consider discrete-time representations of signals, we also need a transform that is dicrete in frequency. The Discrete Fourier Transform (DFT) is discrete in frequency and essentially samples the spectrum of a finite-length (or periodic) discrete-time signal.

Definition 2. Discrete Fourier Transform (DFT): For $s(n)$ non-zero for $n = 0, ..., N-1$, the DFT is

$$S(k) = \sum_{n=0}^{N_1} s(n)e^{-\frac{j2\pi kn}{N}} \text{ for } k = 0, ..., K-1.$$

The above is a DFT of length K. If $K = N$, we can also write a formula for the inverse DFT,

$$s(n) = \frac{1}{N}\sum_{k=0}^{N-1} S(k)e^{\frac{j2\pi kn}{N}}.$$

The complexity of computing the DFT is $O(N^2)$. This means that if our signal doubles in length, our computation is quadrupled. Since signal processing generally needs to be done in *real time*, computational complexity is a concern. The Fast Fourier Transform (FFT) is algorithm for computing the DFT (it is not another transform, just a method!) which requires only $O(Nlog(N))$ operations.

Example 1. FFT for DFT of length 4. For a length $N = 4$ discrete-time signal, the DFT, for $k = 0, .., 3$, is

$$\begin{aligned} S(k) &= s(0) + s(1)e^{-\frac{j2\pi k}{4}} + s(2)e^{-\frac{j4\pi k}{4}} + s(3)e^{-\frac{j6\pi k}{4}}, \\ &= s(0) + s(2)e^{-j\pi k} + s(1)e^{-\frac{j\pi k}{2}} + s(3)e^{-\frac{j3\pi k}{2}}, \\ &= s(0) + s(2)e^{-j\pi k} + e^{-\frac{j\pi k}{2}}\left(s(1) + s(3)e^{-\frac{j\pi k}{2}}\right). \end{aligned}$$

The form of the final line looks like a scaled sum of two length-2 DFTs. This method of re-writing the DFT actually reduces the number of operations needed.

Example 2. Compute the DFT of

$$s(n) = e^{\frac{j2\pi n}{T}}.$$

This signal is periodic with period T, so we find a length T DFT.

$$\begin{aligned} S(k) &= \sum_{n=0}^{T-1} e^{\frac{j2\pi n}{T}} e^{-\frac{j2\pi kn}{T}}, \\ &= \sum_{n=0}^{T-1} e^{\frac{j2\pi n(k-1)}{T}}. \end{aligned}$$

By orthogonality of the complex exponential signal, we have

$$S(k) = \begin{cases} \sum_{n=0}^{T-1} 1 = T & \text{for } k = 1, 1+T, 1+2T, ... \\ 0 & \text{otherwise} \end{cases}.$$

Example 3. Compute the DFT of

$$s(n) = \cos\left(\frac{2\pi n}{T}\right).$$

By Euler's formula, we have

$$s(n) = \frac{1}{2} e^{\frac{2\pi n}{T}} + \frac{1}{2} e^{-\frac{2\pi n}{T}}.$$

Let $s_1(n) = \frac{1}{2} e^{\frac{2\pi n}{T}}$ and $s_2(n) = \frac{1}{2} e^{-\frac{2\pi n}{T}}$. Then by the above example,

$$S_1(k) = \begin{cases} \frac{T}{2} & k = 1, 1+T, 1+2T, ... \\ 0 & \text{otherwise} \end{cases},$$

$$S_2(k) = \begin{cases} \frac{T}{2} & k = -1, -1+T, -1+2T, ... \\ 0 & \text{otherwise} \end{cases}.$$

Since the DFT is a linear function,

$$S(k) = S_1(k) + S_2(k) = \begin{cases} \frac{T}{2} & k = -1, 1, -1+T, 1+T, -1+2T, 1+2T, ... \\ 0 & \text{otherwise} \end{cases},$$

for $T \neq 1, 2$.

Example 4. Find and plot the magnitude of the DFT for the discrete-time square wave.

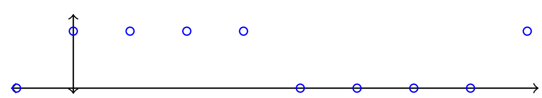

Figure 1: Discrete-time square wave with period $T = 8$.

We compute the DFT using our formula.

$$\begin{aligned}
S(k) &= \sum_{n=0}^{7} s(n) e^{-\frac{j2\pi kn}{8}}, \\
&= \sum_{n=0}^{3} e^{-\frac{j\pi kn}{4}}, \\
&= \left(\sum_{n=0}^{3} e^{-\frac{j\pi k}{4}} \right)^{n}, \\
&= \frac{1 - e^{-\frac{j\pi 4k}{4}}}{1 - e^{-\frac{j\pi k}{4}}}, \\
&= \frac{1 - e^{-\frac{j\pi k}{4}}}{1 - e^{-\frac{j\pi k}{4}}}.
\end{aligned}$$

We can simplify the above expression.

$$S(k) = \begin{cases} 4 & k = 0 \\ \frac{2}{1 - e^{-\frac{j\pi k}{4}}} & k \text{ odd} \\ 0 & k \text{ even} \end{cases} .$$

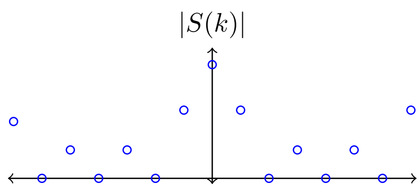

Figure 2: DFT for the discrete-time square wave.

This is a LPF. Note that for discrete-time filters, we characterize the filter by what happens within one period.

2 Filtering

We can use linear difference equations to describe LTI discrete-time filters.

$$y(n) = \sum_{i=1}^{K} a_i y(n-i) + \sum_{i=0}^{M} b_i x(n-i)$$

To find the transfer function for the associated filters, we assume that the input to the system has the form

$$x(n) = X e^{j2\pi fn},$$

and the output has the form

$$y(n) = Y e^{j2\pi fn}.$$

Then, we can find the general form for the transfer function by plugging this input-output pair into the difference equation.

$$\begin{aligned}
Ye^{j2\pi fn} &= \sum_{i=1}^{K} a_i Y e^{j2\pi f(n-i)} + \sum_{i=0}^{M} b_i X e^{j2\pi f(n-i)}, \\
&= Ye^{j2\pi fn} \sum_{i=1}^{K} a_i e^{-j2\pi i} + Xe^{j2\pi fn} \sum_{i=0}^{M} b_i e^{-j2\pi fi}, \\
\Rightarrow Y &= Y \sum_{i=1}^{K} a_i e^{-j2\pi i} + X \sum_{i=0}^{M} b_i e^{-j2\pi i}, \\
\Rightarrow Y\left(1 - \sum_{i=1}^{K} a_i e^{-j2\pi i}\right) &= X \sum_{i=0}^{M} b_i e^{-j2\pi i}, \\
\Rightarrow \frac{Y}{X} &= \frac{\sum_{i=0}^{M} b_i e^{-j2\pi i}}{1 - \sum_{i=1}^{K} a_i e^{-j2\pi i}}, \\
&= H\left(e^{j2\pi f}\right).
\end{aligned}$$

This method for finding the transfer function is the same method we used for circuits. Alternatively, we could also used the unit sample function (impulse), $\delta(n)$, as the input and find the corresponding output, $h(n)$, which we call the *impulse response*.

$$\begin{aligned}
x(n) = e^{j2\pi fn} &\quad\rightarrow\quad y(n) = H(e^{j2\pi f})e^{j2\pi fn} \\
x(n) = \delta(n) &\quad\rightarrow\quad y(n) = h(n) \\
DTFT\{h(n)\} &\quad=\quad H(e^{j2\pi fn})
\end{aligned}$$

Example 5. Find the impulse response and transfer funcion for the system defined by the following linear difference equation:

$$y(n) = \frac{1}{2}y(n-2) + \frac{1}{3}x(n-1).$$

We assume zero initial condition ($y(-2) = 0$). With an input of $x(n) = \delta(n)$, we fill out a table using the formula for the difference equation.

n	$\delta(n)$	$h(n)$
-2	0	0
-1	0	0
0	1	0
1	0	$\frac{1}{3}$
2	0	0
3	0	$\frac{1}{2}\frac{1}{3}$
4	0	0
5	0	$\left(\frac{1}{2}\right)^2 \frac{1}{3}$

If we continued the table we would see that

$$h(n) = \begin{cases} \left(\frac{1}{2}\right)^{\frac{n-1}{2}} \frac{1}{3} & n = 1, 3, 5, ... \\ 0 & \text{otherwise} \end{cases}.$$

This is an IIR filter since $h(n)$ has an infinite number of non-zero values. To find the transfer function, we take the DTFT of the impulse response.

$$
\begin{aligned}
H(e^{j2\pi f}) &= \sum_{n=-\infty}^{\infty} h(n)e^{-j2\pi fn}, \\
&= \sum_{n=1,3,\ldots} \left(\frac{1}{2}\right)^{\frac{n-1}{2}} \frac{1}{3}e^{-j2\pi fn}, \\
&= \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{k} \frac{1}{3}e^{-j2\pi f(2k+1)}, \\
&= \frac{1}{3}e^{-j2\pi f} \sum_{k=0}^{\infty} \left(\frac{1}{2}e^{-j4\pi f}\right)^{k}, \\
&= \frac{1}{3}e^{-j2\pi f} \frac{1}{1-\frac{1}{2}e^{-j4\pi f}}.
\end{aligned}
$$

If we plotted the magnitude of the DTFT, we would see that it is a LPF.

In general is an IIR filter also an LPF? No. In class, we saw the example where

$$y(n) = ay(n-1) + bx(n),$$

which was an IIR filter. The filter was a LPF for $0 < a < 1$ and a HPF for $-1 < a < 0$.

Example 6. Find the impulse response and transfer function for the system defined by the following linear difference equation:

$$y(n) = 5x(n-1) + 6x(n-7).$$

Again, we let $x(n) = \delta(n)$ and make a table to keep track of the outputs at each time step.

n	$\delta(n)$	$h(n)$
-1	0	0
0	1	0
1	0	5
2	0	0
3	0	0
4	0	0
5	0	0
6	0	0
7	0	6
8	0	0

The impulse response, $h(n)$, has only two non-zero values; the filter is a FIR filter. We can write the impulse response as

$$h(n) = 5\delta(n-1) + 6\delta(n-7),$$

which is just the impulse function plugging into the linear difference equation (in this case, there is no simpler way to write it than it is already written). The DTFT is

$$
\begin{aligned}
H(e^{j2\pi f}) &= \sum_{n=-\infty}^{\infty} h(n)e^{-j2\pi fn}, \\
&= 5e^{-j2\pi f} + 6e^{-j2\pi f(7)}.
\end{aligned}
$$

This is a LPF.

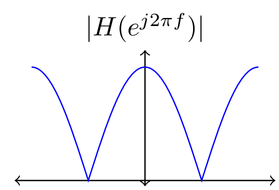

Figure 3: DTFT for FIR filter.