2

The Laplace Transform

2–1 INTRODUCTION

The Laplace transform is one of the most important mathematical tools available for modeling and analyzing linear systems. Since the Laplace transform method must be studied in any system dynamics course, we present the subject at the beginning of this text so that the student can use the method throughout his or her study of system dynamics.

The remaining sections of this chapter are outlined as follows: Section 2–2 reviews complex numbers, complex variables, and complex functions. Section 2–3 defines the Laplace transformation and gives Laplace transforms of several common functions of time. Also examined are some of the most important Laplace transform theorems that apply to linear systems analysis. Section 2–4 deals with the inverse Laplace transformation. Finally, Section 2–5 presents the Laplace transform approach to the solution of the linear, time-invariant differential equation.

2–2 COMPLEX NUMBERS, COMPLEX VARIABLES, AND COMPLEX FUNCTIONS

This section reviews complex numbers, complex algebra, complex variables, and complex functions. Since most of the material covered is generally included in the basic mathematics courses required of engineering students, the section can be omitted entirely or used simply for personal reference.

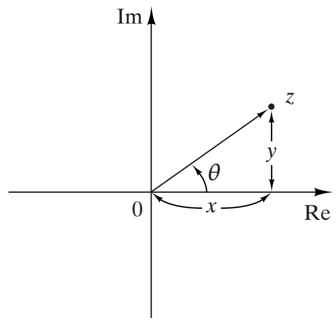

Figure 2–1 Complex plane representation of a complex number z.

Complex numbers. Using the notation $j = \sqrt{-1}$, we can express all numbers in engineering calculations as

$$z = x + jy$$

where z is called a *complex number* and x and jy are its *real* and *imaginary parts*, respectively. Note that both x and y are real and that j is the only imaginary quantity in the expression. The complex plane representation of z is shown in Figure 2–1. (Note also that the real axis and the imaginary axis define the complex plane and that the combination of a real number and an imaginary number defines a point in that plane.) A complex number z can be considered a point in the complex plane or a directed line segment to the point; both interpretations are useful.

The magnitude, or absolute value, of z is defined as the length of the directed line segment shown in Figure 2–1. The angle of z is the angle that the directed line segment makes with the positive real axis. A counterclockwise rotation is defined as the positive direction for the measurement of angles. Mathematically,

$$\text{magnitude of } z = |z| = \sqrt{x^2 + y^2}, \qquad \text{angle of } z = \theta = \tan^{-1}\frac{y}{x}$$

A complex number can be written in rectangular form or in polar form as follows:

$$\left.\begin{aligned} z &= x + jy \\ z &= |z|(\cos\theta + j\sin\theta) \end{aligned}\right\}\text{rectangular forms}$$

$$\left.\begin{aligned} z &= |z|\,\underline{/\theta} \\ z &= |z|\,e^{j\theta} \end{aligned}\right\}\text{polar forms}$$

In converting complex numbers to polar form from rectangular, we use

$$|z| = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}\frac{y}{x}$$

To convert complex numbers to rectangular form from polar, we employ

$$x = |z|\cos\theta, \qquad y = |z|\sin\theta$$

Complex conjugate. The *complex conjugate* of $z = x + jy$ is defined as

$$\bar{z} = x - jy$$

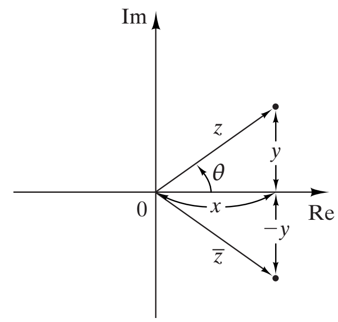

Figure 2–2 Complex number z and its complex conjugate $\bar{z}$.

The complex conjugate of z thus has the same real part as z and an imaginary part that is the negative of the imaginary part of z. Figure 2–2 shows both z and $\bar{z}$. Note that

$$z = x + jy = |z| \underline{/\theta} = |z| (\cos\theta + j\sin\theta)$$
$$\bar{z} = x - jy = |z| \underline{/-\theta} = |z| (\cos\theta - j\sin\theta)$$

Euler's theorem. The power series expansions of $\cos\theta$ and $\sin\theta$ are, respectively,

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

and

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

Thus,

$$\cos\theta + j\sin\theta = 1 + (j\theta) + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \cdots$$

Since

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

it follows that

$$\cos\theta + j\sin\theta = e^{j\theta}$$

This is known as *Euler's theorem*.

Using Euler's theorem, we can express the sine and cosine in complex form. Noting that $e^{-j\theta}$ is the complex conjugate of $e^{j\theta}$ and that

$$e^{j\theta} = \cos\theta + j\sin\theta$$
$$e^{-j\theta} = \cos\theta - j\sin\theta$$

we find that

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$
$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Complex algebra. If the complex numbers are written in a suitable form, operations like addition, subtraction, multiplication, and division can be performed easily.

Equality of complex numbers. Two complex numbers z and w are said to be equal if and only if their real parts are equal and their imaginary parts are equal. So if two complex numbers are written

$$z = x + jy, \qquad w = u + jv$$

then $z = w$ if and only if $x = u$ and $y = v$.

Addition. Two complex numbers in rectangular form can be added by adding the real parts and the imaginary parts separately:

$$z + w = (x + jy) + (u + jv) = (x + u) + j(y + v)$$

Subtraction. Subtracting one complex number from another can be considered as adding the negative of the former:

$$z - w = (x + jy) - (u + jv) = (x - u) + j(y - v)$$

Note that addition and subtraction can be done easily on the rectangular plane.

Multiplication. If a complex number is multiplied by a real number, the result is a complex number whose real and imaginary parts are multiplied by that real number:

$$az = a(x + jy) = ax + jay \qquad (a = \text{real number})$$

If two complex numbers appear in rectangular form and we want the product in rectangular form, multiplication is accomplished by using the fact that $j^2 = -1$. Thus, if two complex numbers are written

$$z = x + jy, \qquad w = u + jv$$

then

$$zw = (x + jy)(u + jv) = xu + jyu + jxv + j^2yv$$
$$= (xu - yv) + j(xv + yu)$$

In polar form, multiplication of two complex numbers can be done easily. The magnitude of the product is the product of the two magnitudes, and the angle of the product is the sum of the two angles. So if two complex numbers are written

$$z = |z| \underline{/\theta}, \qquad w = |w| \underline{/\phi}$$

then

$$zw = |z||w| \underline{/\theta + \phi}$$

Multiplication by j. It is important to note that multiplication by j is equivalent to counterclockwise rotation by 90°. For example, if

$$z = x + jy$$

then

$$jz = j(x + jy) = jx + j^2y = -y + jx$$

or, noting that $j = 1\,\underline{/90^\circ}$, if

$$z = |z|\,\underline{/\theta}$$

then

$$jz = 1\,\underline{/90^\circ}\,|z|\,\underline{/\theta} = |z|\,\underline{/\theta + 90^\circ}$$

Figure 2–3 illustrates the multiplication of a complex number z by j.

Division. If a complex number $z = |z|\,\underline{/\theta}$ is divided by another complex number $w = |w|\,\underline{/\phi}$, then

$$\frac{z}{w} = \frac{|z|\,\underline{/\theta}}{|w|\,\underline{/\phi}} = \frac{|z|}{|w|}\,\underline{/\theta - \phi}$$

That is, the result consists of the quotient of the magnitudes and the difference of the angles.

Division in rectangular form is inconvenient, but can be done by multiplying the denominator and numerator by the complex conjugate of the denominator. This procedure converts the denominator to a real number and thus simplifies division. For instance,

$$\frac{z}{w} = \frac{x + jy}{u + jv} = \frac{(x + jy)(u - jv)}{(u + jv)(u - jv)} = \frac{(xu + yv) + j(yu - xv)}{u^2 + v^2}$$
$$= \frac{xu + yv}{u^2 + v^2} + j\frac{yu - xv}{u^2 + v^2}$$

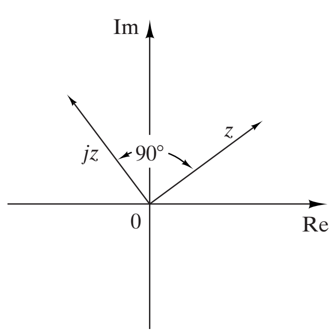

Figure 2–3 Multiplication of a complex number z by j.

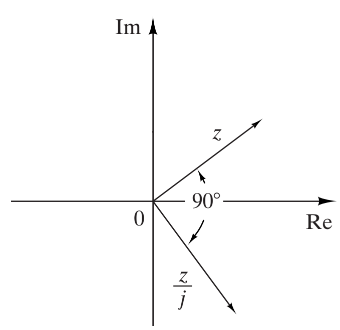

Figure 2–4 Division of a complex number z by j.

Division by j. Note that division by j is equivalent to clockwise rotation by 90°. For example, if $z = x + jy$, then

$$\frac{z}{j} = \frac{x + jy}{j} = \frac{(x + jy)j}{jj} = \frac{jx - y}{-1} = y - jx$$

or

$$\frac{z}{j} = \frac{|z| \angle \theta}{1 \angle 90^\circ} = |z| \angle \theta - 90^\circ$$

Figure 2–4 illustrates the division of a complex number z by j.

Powers and roots. Multiplying z by itself n times, we obtain

$$z^n = (|z| \angle \theta)^n = |z|^n \angle n\theta$$

Extracting the nth root of a complex number is equivalent to raising the number to the $1/n$th power:

$$z^{1/n} = (|z| \angle \theta)^{1/n} = |z|^{1/n} \angle \frac{\theta}{n}$$

For instance,

$$(8.66 - j5)^3 = (10 \angle -30^\circ)^3 = 1000 \angle -90^\circ = 0 - j\,1000 = -j\,1000$$

$$(2.12 - j2.12)^{1/2} = (9 \angle -45^\circ)^{1/2} = 3 \angle -22.5^\circ$$

Comments. It is important to note that

$$|zw| = |z||w|$$

and

$$|z + w| \neq |z| + |w|$$

Complex variable. A complex number has a real part and an imaginary part, both of which are constant. If the real part or the imaginary part (or both) are variables, the complex number is called a *complex variable*. In the Laplace transformation, we use the notation s to denote a complex variable; that is,

$$s = \sigma + j\omega$$

where σ is the real part and $j\omega$ is the imaginary part. (Note that both σ and ω are real.)

Complex function. A complex function $F(s)$, a function of s, has a real part and an imaginary part, or

$$F(s) = F_x + jF_y$$

where F_x and F_y are real quantities. The magnitude of $F(s)$ is $\sqrt{F_x^2 + F_y^2}$, and the angle θ of $F(s)$ is $\tan^{-1}(F_y/F_x)$. The angle is measured counterclockwise from the positive real axis. The complex conjugate of $F(s)$ is $\overline{F}(s) = F_x - jF_y$.

Complex functions commonly encountered in linear systems analysis are single-valued functions of s and are uniquely determined for a given value of s. Typically, such functions have the form

$$F(s) = \frac{K(s + z_1)(s + z_2)\cdots(s + z_m)}{(s + p_1)(s + p_2)\cdots(s + p_n)}$$

Points at which $F(s)$ equals zero are called *zeros*. That is, $s = -z_1, s = -z_2, \ldots, s = -z_m$ are zeros of $F(s)$. [Note that $F(s)$ may have additional zeros at infinity; see the illustrative example that follows.] Points at which $F(s)$ equals infinity are called *poles*. That is, $s = -p_1, s = -p_2, \ldots, s = -p_n$ are poles of $F(s)$. If the denominator of $F(s)$ involves k-multiple factors $(s + p)^k$, then $s = -p$ is called a *multiple pole* of order k or *repeated pole* of order k. If $k = 1$, the pole is called a *simple pole*.

As an illustrative example, consider the complex function

$$G(s) = \frac{K(s + 2)(s + 10)}{s(s + 1)(s + 5)(s + 15)^2}$$

$G(s)$ has zeros at $s = -2$ and $s = -10$, simple poles at $s = 0, s = -1$, and $s = -5$, and a double pole (multiple pole of order 2) at $s = -15$. Note that $G(s)$ becomes zero at $s = \infty$. Since, for large values of s,

$$G(s) \doteq \frac{K}{s^3}$$

it follows that $G(s)$ possesses a triple zero (multiple zero of order 3) at $s = \infty$. If points at infinity are included, $G(s)$ has the same number of poles as zeros. To summarize, $G(s)$ has five zeros $(s = -2, s = -10, s = \infty, s = \infty, s = \infty)$ and five poles $(s = 0, s = -1, s = -5, s = -15, s = -15)$.

2–3 LAPLACE TRANSFORMATION

The Laplace transform method is an operational method that can be used advantageously in solving linear, time-invariant differential equations. Its main advantage is that differentiation of the time function corresponds to multiplication of the transform by a complex variable s, and thus the differential equations in time become algebraic equations in s. The solution of the differential equation can then be found by using a Laplace transform table or the partial-fraction expansion technique. Another advantage of the Laplace transform method is that, in solving the differential equation, the initial conditions are automatically taken care of, and both the particular solution and the complementary solution can be obtained simultaneously.

Laplace transformation. Let us define

$f(t)$ = a time function such that $f(t) = 0$ for $t < 0$

s = a complex variable

$\mathcal{L}$ = an operational symbol indicating that the quantity upon which it operates is to be transformed by the Laplace integral $\int_0^\infty e^{-st}\,dt$

$F(s)$ = Laplace transform of $f(t)$

Then the Laplace transform of $f(t)$ is given by

$$\mathscr{L}[f(t)] = F(s) = \int_0^\infty e^{-st}\,dt[f(t)] = \int_0^\infty f(t)e^{-st}\,dt$$

The reverse process of finding the time function $f(t)$ from the Laplace transform $F(s)$ is called *inverse Laplace transformation*. The notation for inverse Laplace transformation is $\mathscr{L}^{-1}$. Thus,

$$\mathscr{L}^{-1}[F(s)] = f(t)$$

Existence of Laplace transform. The Laplace transform of a function $f(t)$ exists if the Laplace integral converges. The integral will converge if $f(t)$ is piecewise continuous in every finite interval in the range $t > 0$ and if $f(t)$ is of exponential order as t approaches infinity. A function $f(t)$ is said to be of exponential order if a real, positive constant σ exists such that the function

$$e^{-\sigma t}|f(t)|$$

approaches zero as t approaches infinity. If the limit of the function $e^{-\sigma t}|f(t)|$ approaches zero for σ greater than σ_c and the limit approaches infinity for σ less than σ_c, the value σ_c is called the *abscissa of convergence*.

It can be seen that, for such functions as t, $\sin \omega t$, and $t \sin \omega t$, the abscissa of convergence is equal to zero. For functions like e^{-ct}, te^{-ct}, and $e^{-ct}\sin \omega t$, the abscissa of convergence is equal to $-c$. In the case of functions that increase faster than the exponential function, it is impossible to find suitable values of the abscissa of convergence. Consequently, such functions as e^{t^2} and te^{t^2} do not possess Laplace transforms.

Nevertheless, it should be noted that, although e^{t^2} for $0 \le t \le \infty$ does not possess a Laplace transform, the time function defined by

$$\begin{aligned} f(t) &= e^{t^2} && \text{for } 0 \le t \le T < \infty \\ &= 0 && \text{for } t < 0, T < t \end{aligned}$$

does, since $f(t) = e^{t^2}$ for only a limited time interval $0 \le t \le T$ and not for $0 \le t \le \infty$. Such a signal can be physically generated. Note that the signals that can be physically generated always have corresponding Laplace transforms.

If functions $f_1(t)$ and $f_2(t)$ are both Laplace transformable, then the Laplace transform of $f_1(t) + f_2(t)$ is given by

$$\mathscr{L}[f_1(t) + f_2(t)] = \mathscr{L}[f_1(t)] + \mathscr{L}[f_2(t)]$$

Exponential function. Consider the exponential function

$$\begin{aligned} f(t) &= 0 && \text{for } t < 0 \\ &= Ae^{-\alpha t} && \text{for } t \ge 0 \end{aligned}$$

where A and α are constants. The Laplace transform of this exponential function can be obtained as follows:

$$\mathscr{L}[Ae^{-\alpha t}] = \int_0^\infty Ae^{-\alpha t}e^{-st}\,dt = A\int_0^\infty e^{-(\alpha+s)t}\,dt = \frac{A}{s+\alpha}$$

In performing this integration, we assume that the real part of s is greater than $-\alpha$ (the abscissa of convergence), so that the integral converges. The Laplace transform $F(s)$ of any Laplace transformable function $f(t)$ obtained in this way is valid throughout the entire s plane, except at the poles of $F(s)$. (Although we do not present a proof of this statement, it can be proved by use of the theory of complex variables.)

Step function. Consider the step function

$$\begin{aligned} f(t) &= 0 \qquad \text{for } t < 0 \\ &= A \qquad \text{for } t > 0 \end{aligned}$$

where A is a constant. Note that this is a special case of the exponential function $Ae^{-\alpha t}$, where $\alpha = 0$. The step function is undefined at $t = 0$. Its Laplace transform is given by

$$\mathscr{L}[A] = \int_0^\infty Ae^{-st}\,dt = \frac{A}{s}$$

The step function whose height is unity is called a *unit-step function*. The unit-step function that occurs at $t = t_0$ is frequently written $1(t - t_0)$, a notation that will be used in this book. The preceding step function whose height is A can thus be written $A1(t)$.

The Laplace transform of the unit-step function that is defined by

$$\begin{aligned} 1(t) &= 0 \qquad \text{for } t < 0 \\ &= 1 \qquad \text{for } t > 0 \end{aligned}$$

is $1/s$, or

$$\mathscr{L}[1(t)] = \frac{1}{s}$$

Physically, a step function occurring at $t = t_0$ corresponds to a constant signal suddenly applied to the system at time t equals t_0.

Ramp function. Consider the ramp function

$$\begin{aligned} f(t) &= 0 \qquad \text{for } t < 0 \\ &= At \qquad \text{for } t \geq 0 \end{aligned}$$

where A is a constant. The Laplace transform of this ramp function is

$$\mathscr{L}[At] = A\int_0^\infty te^{-st}\,dt$$

To evaluate the integral, we use the formula for integration by parts:

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

In this case, $u = t$ and $dv = e^{-st}\,dt$. [Note that $v = e^{-st}/(-s)$.] Hence,

$$\mathscr{L}[At] = A\int_0^\infty te^{-st}\,dt = A\left(t\,\frac{e^{-st}}{-s}\bigg|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\,dt\right)$$
$$= \frac{A}{s}\int_0^\infty e^{-st}\,dt = \frac{A}{s^2}$$

Sinusoidal function. The Laplace transform of the sinusoidal function

$$\begin{aligned} f(t) &= 0 && \text{for } t < 0 \\ &= A\sin\omega t && \text{for } t \geq 0 \end{aligned}$$

where A and ω are constants, is obtained as follows: Noting that

$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

and

$$e^{-j\omega t} = \cos\omega t - j\sin\omega t$$

we can write

$$\sin\omega t = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$$

Hence,

$$\mathscr{L}[A\sin\omega t] = \frac{A}{2j}\int_0^\infty \left(e^{j\omega t} - e^{-j\omega t}\right)e^{-st}\,dt$$
$$= \frac{A}{2j}\,\frac{1}{s - j\omega} - \frac{A}{2j}\,\frac{1}{s + j\omega} = \frac{A\omega}{s^2 + \omega^2}$$

Similarly, the Laplace transform of $A\cos\omega t$ can be derived as follows:

$$\mathscr{L}[A\cos\omega t] = \frac{As}{s^2 + \omega^2}$$

Comments. The Laplace transform of any Laplace transformable function $f(t)$ can be found by multiplying $f(t)$ by e^{-st} and then integrating the product from $t = 0$ to $t = \infty$. Once we know the method of obtaining the Laplace transform, however, it is not necessary to derive the Laplace transform of $f(t)$ each time. Laplace transform tables can conveniently be used to find the transform of a given function $f(t)$. Table 2–1 shows Laplace transforms of time functions that will frequently appear in linear systems analysis. In Table 2–2, the properties of Laplace transforms are given.

Translated function. Let us obtain the Laplace transform of the translated function $f(t - \alpha)1(t - \alpha)$, where $\alpha \geq 0$. This function is zero for $t < \alpha$. The functions $f(t)1(t)$ and $f(t - \alpha)1(t - \alpha)$ are shown in Figure 2–5.

By definition, the Laplace transform of $f(t - \alpha)1(t - \alpha)$ is

$$\mathscr{L}[f(t - \alpha)1(t - \alpha)] = \int_0^\infty f(t - \alpha)1(t - \alpha)e^{-st}\,dt$$

TABLE 2–1 Laplace Transform Pairs

	$f(t)$	$F(s)$
1	Unit impulse $\delta(t)$	1
2	Unit step $1(t)$	$\frac{1}{s}$
3	t	$\frac{1}{s^2}$
4	$\frac{t^{n-1}}{(n-1)!} \quad (n = 1, 2, 3, \ldots)$	$\frac{1}{s^n}$
5	$t^n \quad (n = 1, 2, 3, \ldots)$	$\frac{n!}{s^{n+1}}$
6	e^{-at}	$\frac{1}{s+a}$
7	te^{-at}	$\frac{1}{(s+a)^2}$
8	$\frac{1}{(n-1)!}t^{n-1}e^{-at} \quad (n = 1, 2, 3, \ldots)$	$\frac{1}{(s+a)^n}$
9	$t^n e^{-at} \quad (n = 1, 2, 3, \ldots)$	$\frac{n!}{(s+a)^{n+1}}$
10	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
11	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
12	$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}$
13	$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$
14	$\frac{1}{a}(1-e^{-at})$	$\frac{1}{s(s+a)}$
15	$\frac{1}{b-a}(e^{-at}-e^{-bt})$	$\frac{1}{(s+a)(s+b)}$
16	$\frac{1}{b-a}(be^{-bt}-ae^{-at})$	$\frac{s}{(s+a)(s+b)}$
17	$\frac{1}{ab}\left[1+\frac{1}{a-b}(be^{-at}-ae^{-bt})\right]$	$\frac{1}{s(s+a)(s+b)}$

TABLE 2–1 (*continued*)

	$f(t)$	$F(s)$
18	$\frac{1}{a^2}(1 - e^{-at} - ate^{-at})$	$\frac{1}{s(s+a)^2}$
19	$\frac{1}{a^2}(at - 1 + e^{-at})$	$\frac{1}{s^2(s+a)}$
20	$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$
21	$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2 + \omega^2}$
22	$\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\omega_n\sqrt{1-\zeta^2}\,t$	$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
23	$-\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}\,t - \phi)$ $\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$	$\frac{s}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
24	$1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}\,t + \phi)$ $\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$	$\frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$
25	$1 - \cos\omega t$	$\frac{\omega^2}{s(s^2+\omega^2)}$
26	$\omega t - \sin\omega t$	$\frac{\omega^3}{s^2(s^2+\omega^2)}$
27	$\sin\omega t - \omega t\cos\omega t$	$\frac{2\omega^3}{(s^2+\omega^2)^2}$
28	$\frac{1}{2\omega}t\sin\omega t$	$\frac{s}{(s^2+\omega^2)^2}$
29	$t\cos\omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
30	$\frac{1}{\omega_2^2 - \omega_1^2}(\cos\omega_1 t - \cos\omega_2 t) \quad (\omega_1^2 \neq \omega_2^2)$	$\frac{s}{(s^2+\omega_1^2)(s^2+\omega_2^2)}$
31	$\frac{1}{2\omega}(\sin\omega t + \omega t\cos\omega t)$	$\frac{s^2}{(s^2+\omega^2)^2}$

TABLE 2–2 Properties of Laplace Transforms

1	$\mathscr{L}[Af(t)] = AF(s)$
2	$\mathscr{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$
3	$\mathscr{L}_\pm\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0\pm)$
4	$\mathscr{L}_\pm\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s) - sf(0\pm) - \dot{f}(0\pm)$
5	$\mathscr{L}_\pm\left[\frac{d^n}{dt^n}f(t)\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k} \overset{(k-1)}{f}(0\pm)$ where $\overset{(k-1)}{f}(t) = \frac{d^{k-1}}{dt^{k-1}}f(t)$
6	$\mathscr{L}_\pm\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \frac{\left[\int f(t)\,dt\right]_{t=0\pm}}{s}$
7	$\mathscr{L}_\pm\left[\iint f(t)\,dt\,dt\right] = \frac{F(s)}{s^2} + \frac{\left[\int f(t)\,dt\right]_{t=0\pm}}{s^2} + \frac{\left[\iint f(t)\,dt\,dt\right]_{t=0\pm}}{s}$
8	$\mathscr{L}_\pm\left[\int \cdots \int f(t)(dt)^n\right] = \frac{F(s)}{s^n} + \sum_{k=1}^{n} \frac{1}{s^{n-k+1}}\left[\int \cdots \int f(t)(dt)^k\right]_{t=0\pm}$
9	$\mathscr{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$
10	$\int_0^\infty f(t)\,dt = \lim_{s\to 0} F(s) \qquad \text{if } \int_0^\infty f(t)\,dt \text{ exists}$
11	$\mathscr{L}[e^{-at}f(t)] = F(s+a)$
12	$\mathscr{L}[f(t-\alpha)1(t-\alpha)] = e^{-\alpha s}F(s) \qquad \alpha \geq 0$
13	$\mathscr{L}[tf(t)] = -\frac{dF(s)}{ds}$
14	$\mathscr{L}[t^2f(t)] = \frac{d^2}{ds^2}F(s)$
15	$\mathscr{L}[t^nf(t)] = (-1)^n\frac{d^n}{ds^n}F(s) \qquad n = 1, 2, 3, \ldots$
16	$\mathscr{L}\left[\frac{1}{t}f(t)\right] = \int_s^\infty F(s)\,ds \qquad \text{if } \lim_{t\to 0}\frac{1}{t}f(t) \text{ exists}$
17	$\mathscr{L}\left[f\left(\frac{t}{a}\right)\right] = aF(as)$

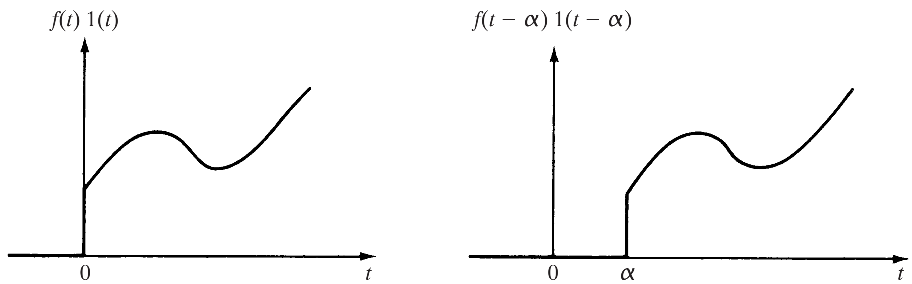

Figure 2–5 Function $f(t)1(t)$ and translated function $f(t-\alpha)1(t-\alpha)$.

By changing the independent variable from t to τ, where $\tau = t - \alpha$, we obtain

$$\int_0^\infty f(t-\alpha)1(t-\alpha)e^{-st}\,dt = \int_{-\alpha}^\infty f(\tau)1(\tau)e^{-s(\tau+\alpha)}\,d\tau$$

Noting that $f(\tau)1(\tau) = 0$ for $\tau < 0$, we can change the lower limit of integration from $-\alpha$ to 0. Thus,

$$\begin{aligned}\int_{-\alpha}^\infty f(\tau)1(\tau)e^{-s(\tau+\alpha)}\,d\tau &= \int_0^\infty f(\tau)1(\tau)e^{-s(\tau+\alpha)}\,d\tau \\ &= \int_0^\infty f(\tau)e^{-s\tau}e^{-\alpha s}\,d\tau \\ &= e^{-\alpha s}\int_0^\infty f(\tau)e^{-s\tau}\,d\tau = e^{-\alpha s}F(s)\end{aligned}$$

where

$$F(s) = \mathscr{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt$$

Hence,

$$\mathscr{L}[f(t-\alpha)1(t-\alpha)] = e^{-\alpha s}F(s) \qquad \alpha \geq 0$$

This last equation states that the translation of the time function $f(t)1(t)$ by α (where $\alpha \geq 0$) corresponds to the multiplication of the transform $F(s)$ by $e^{-\alpha s}$.

Pulse function. Consider the pulse function shown in Figure 2–6, namely,

$$\begin{aligned} f(t) &= \frac{A}{t_0} \qquad \text{for } 0 < t < t_0 \\ &= 0 \qquad \text{for } t < 0,\ t_0 < t \end{aligned}$$

where A and t_0 are constants.

The pulse function here may be considered a step function of height A/t_0 that begins at $t = 0$ and that is superimposed by a negative step function of height A/t_0

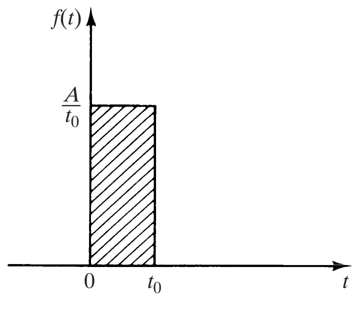

Figure 2–6 Pulse function.

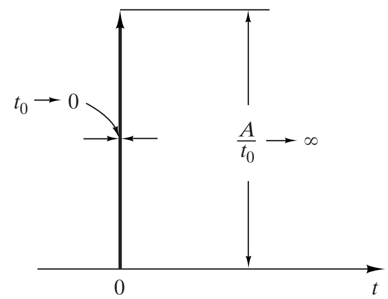

Figure 2–7 Impulse function.

beginning at $t = t_0$; that is,

$$f(t) = \frac{A}{t_0}1(t) - \frac{A}{t_0}1(t - t_0)$$

Then the Laplace transform of $f(t)$ is obtained as

$$\begin{aligned}\mathscr{L}[f(t)] &= \mathscr{L}\left[\frac{A}{t_0}1(t)\right] - \mathscr{L}\left[\frac{A}{t_0}1(t - t_0)\right] \\ &= \frac{A}{t_0 s} - \frac{A}{t_0 s}e^{-st_0} \\ &= \frac{A}{t_0 s}(1 - e^{-st_0}) \qquad (2\text{–}1)\end{aligned}$$

Impulse function. The impulse function is a special limiting case of the pulse function. Consider the impulse function

$$\begin{aligned}f(t) &= \lim_{t_0 \to 0} \frac{A}{t_0} &&\text{for } 0 < t < t_0 \\ &= 0 &&\text{for } t < 0, t_0 < t\end{aligned}$$

Figure 2–7 depicts the impulse function defined here. It is a limiting case of the pulse function shown in Figure 2–6 as t_0 approaches zero. Since the height of the impulse function is A/t_0 and the duration is t_0, the area under the impulse is equal to A. As the duration t_0 approaches zero, the height A/t_0 approaches infinity, but the area under the impulse remains equal to A. Note that the magnitude of the impulse is measured by its area.

From Equation (2–1), the Laplace transform of this impulse function is shown to be

$$\begin{aligned}\mathscr{L}[f(t)] &= \lim_{t_0 \to 0}\left[\frac{A}{t_0 s}(1 - e^{-st_0})\right] \\ &= \lim_{t_0 \to 0} \frac{\dfrac{d}{dt_0}[A(1 - e^{-st_0})]}{\dfrac{d}{dt_0}(t_0 s)} = \frac{As}{s} = A\end{aligned}$$

Thus, the Laplace transform of the impulse function is equal to the area under the impulse.

The impulse function whose area is equal to unity is called the *unit-impulse function* or the *Dirac delta function*. The unit-impulse function occurring at $t = t_0$ is usually denoted by $\delta(t - t_0)$, which satisfies the following conditions:

$$\delta(t - t_0) = 0 \qquad \text{for } t \neq t_0$$
$$\delta(t - t_0) = \infty \qquad \text{for } t = t_0$$
$$\int_{-\infty}^{\infty} \delta(t - t_0)\, dt = 1$$

An impulse that has an infinite magnitude and zero duration is mathematical fiction and does not occur in physical systems. If, however, the magnitude of a pulse input to a system is very large and its duration very short compared with the system time constants, then we can approximate the pulse input by an impulse function. For instance, if a force or torque input $f(t)$ is applied to a system for a very short time duration $0 < t < t_0$, where the magnitude of $f(t)$ is sufficiently large so that $\int_0^{t_0} f(t)\, dt$ is not negligible, then this input can be considered an impulse input. (Note that, when we describe the impulse input, the area or magnitude of the impulse is most important, but the exact shape of the impulse is usually immaterial.) The impulse input supplies energy to the system in an infinitesimal time.

The concept of the impulse function is highly useful in differentiating discontinuous-time functions. The unit-impulse function $\delta(t - t_0)$ can be considered the derivative of the unit-step function $1(t - t_0)$ at the point of discontinuity $t = t_0$, or

$$\delta(t - t_0) = \frac{d}{dt} 1(t - t_0)$$

Conversely, if the unit-impulse function $\delta(t - t_0)$ is integrated, the result is the unit-step function $1(t - t_0)$. With the concept of the impulse function, we can differentiate a function containing discontinuities, giving impulses, the magnitudes of which are equal to the magnitude of each corresponding discontinuity.

Multiplication of $f(t)$ by $e^{-\alpha t}$. If $f(t)$ is Laplace transformable and its Laplace transform is $F(s)$, then the Laplace transform of $e^{-\alpha t} f(t)$ is obtained as

$$\mathscr{L}[e^{-\alpha t} f(t)] = \int_0^{\infty} e^{-\alpha t} f(t) e^{-st}\, dt = F(s + \alpha) \tag{2–2}$$

We see that the multiplication of $f(t)$ by $e^{-\alpha t}$ has the effect of replacing s by $(s + \alpha)$ in the Laplace transform. Conversely, changing s to $(s + \alpha)$ is equivalent to multiplying $f(t)$ by $e^{-\alpha t}$. (Note that α may be real or complex.)

The relationship given by Equation (2–2) is useful in finding the Laplace transforms of such functions as $e^{-\alpha t} \sin \omega t$ and $e^{-\alpha t} \cos \omega t$. For instance, since

$$\mathscr{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2} = F(s) \qquad \text{and} \qquad \mathscr{L}[\cos \omega t] = \frac{s}{s^2 + \omega^2} = G(s)$$

it follows from Equation (2–2) that the Laplace transforms of $e^{-\alpha t} \sin \omega t$ and $e^{-\alpha t} \cos \omega t$ are given, respectively, by

$$\mathscr{L}[e^{-\alpha t} \sin \omega t] = F(s + \alpha) = \frac{\omega}{(s + \alpha)^2 + \omega^2}$$

and

$$\mathscr{L}[e^{-\alpha t} \cos \omega t] = G(s + \alpha) = \frac{s + \alpha}{(s + \alpha)^2 + \omega^2}$$

Comments on the lower limit of the Laplace integral. In some cases, $f(t)$ possesses an impulse function at $t = 0$. Then the lower limit of the Laplace integral must be clearly specified as to whether it is $0-$ or $0+$, since the Laplace transforms of $f(t)$ differ for these two lower limits. If such a distinction of the lower limit of the Laplace integral is necessary, we use the notations

$$\mathscr{L}_+[f(t)] = \int_{0+}^{\infty} f(t)e^{-st}\,dt$$

and

$$\mathscr{L}_-[f(t)] = \int_{0-}^{\infty} f(t)e^{-st}\,dt = \mathscr{L}_+[f(t)] + \int_{0-}^{0+} f(t)e^{-st}\,dt$$

If $f(t)$ involves an impulse function at $t = 0$, then

$$\mathscr{L}_+[f(t)] \neq \mathscr{L}_-[f(t)]$$

since

$$\int_{0-}^{0+} f(t)e^{-st}\,dt \neq 0$$

for such a case. Obviously, if $f(t)$ does not possess an impulse function at $t = 0$ (i.e., if the function to be transformed is finite between $t = 0-$ and $t = 0+$), then

$$\mathscr{L}_+[f(t)] = \mathscr{L}_-[f(t)]$$

Differentiation theorem. The Laplace transform of the derivative of a function $f(t)$ is given by

$$\mathscr{L}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0) \tag{2–3}$$

where $f(0)$ is the initial value of $f(t)$, evaluated at $t = 0$. Equation (2–3) is called the differentiation theorem.

For a given function $f(t)$, the values of $f(0+)$ and $f(0-)$ may be the same or different, as illustrated in Figure 2–8. The distinction between $f(0+)$ and $f(0-)$ is important when $f(t)$ has a discontinuity at $t = 0$, because, in such a case, $df(t)/dt$ will

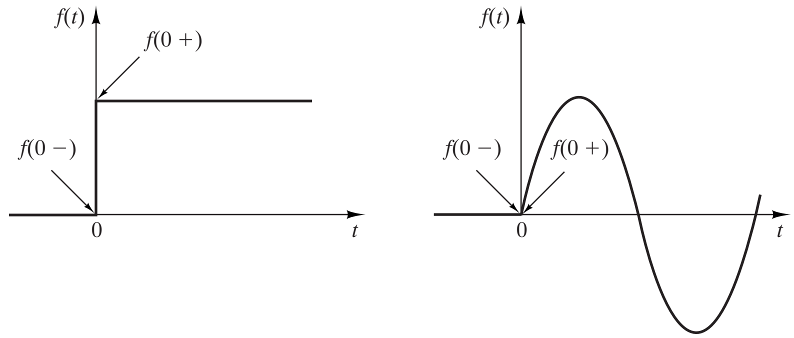

Figure 2–8 Step function and sine function indicating initial values at $t = 0-$ and $t = 0+$.

involve an impulse function at $t = 0$. If $f(0+) \neq f(0-)$, Equation (2–3) must be modified to

$$\mathcal{L}_+\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0+)$$

$$\mathcal{L}_-\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0-)$$

To prove the differentiation theorem, we proceed as follows: Integrating the Laplace integral by parts gives

$$\int_0^\infty f(t)e^{-st}\,dt = f(t)\frac{e^{-st}}{-s}\bigg|_0^\infty - \int_0^\infty \left[\frac{d}{dt}f(t)\right]\frac{e^{-st}}{-s}\,dt$$

Hence,

$$F(s) = \frac{f(0)}{s} + \frac{1}{s}\mathcal{L}\left[\frac{d}{dt}f(t)\right]$$

It follows that

$$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0)$$

Similarly, for the second derivative of $f(t)$, we obtain the relationship

$$\mathcal{L}\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s) - sf(0) - \dot{f}(0)$$

where $\dot{f}(0)$ is the value of $df(t)/dt$ evaluated at $t = 0$. To derive this equation, define

$$\frac{d}{dt}f(t) = g(t)$$

Then

$$\mathcal{L}\left[\frac{d^2}{dt^2}f(t)\right] = \mathcal{L}\left[\frac{d}{dt}g(t)\right] = s\mathcal{L}[g(t)] - g(0)$$
$$= s\mathcal{L}\left[\frac{d}{dt}f(t)\right] - \dot{f}(0)$$
$$= s^2F(s) - sf(0) - \dot{f}(0)$$

Similarly, for the nth derivative of $f(t)$, we obtain

$$\mathcal{L}\left[\frac{d^n}{dt^n}f(t)\right] = s^nF(s) - s^{n-1}f(0) - s^{n-2}\dot{f}(0) - \cdots - \overset{(n-1)}{f}(0)$$

where $f(0), \dot{f}(0), \ldots, \overset{(n-1)}{f}(0)$ represent the values of $f(t), df(t)/dt, \ldots, d^{n-1}f(t)/dt^{n-1}$, respectively, evaluated at $t = 0$. If the distinction between $\mathcal{L}_+$ and $\mathcal{L}_-$ is necessary, we substitute $t = 0+$ or $t = 0-$ into $f(t), df(t)/dt, \ldots, d^{n-1}f(t)/dt^{n-1}$, depending on whether we take $\mathcal{L}_+$ or $\mathcal{L}_-$.

Note that, for Laplace transforms of derivatives of $f(t)$ to exist, $d^nf(t)/dt^n$ ($n = 1, 2, 3, \ldots$) must be Laplace transformable.

Note also that, if all the initial values of $f(t)$ and its derivatives are equal to zero, then the Laplace transform of the nth derivative of $f(t)$ is given by $s^nF(s)$.

Final-value theorem. The final-value theorem relates the steady-state behavior of $f(t)$ to the behavior of $sF(s)$ in the neighborhood of $s = 0$. The theorem, however, applies if and only if $\lim_{t\to\infty} f(t)$ exists [which means that $f(t)$ settles down to a definite value as $t \to \infty$]. If all poles of $sF(s)$ lie in the left half s plane, then $\lim_{t\to\infty} f(t)$ exists, but if $sF(s)$ has poles on the imaginary axis or in the right half s plane, $f(t)$ will contain oscillating or exponentially increasing time functions, respectively, and $\lim_{t\to\infty} f(t)$ will not exist. The final-value theorem does not apply to such cases. For instance, if $f(t)$ is a sinusoidal function $\sin \omega t$, then $sF(s)$ has poles at $s = \pm j\omega$, and $\lim_{t\to\infty} f(t)$ does not exist. Therefore, the theorem is not applicable to such a function.

The final-value theorem may be stated as follows: If $f(t)$ and $df(t)/dt$ are Laplace transformable, if $F(s)$ is the Laplace transform of $f(t)$, and if $\lim_{t\to\infty} f(t)$ exists, then

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

To prove the theorem, we let s approach zero in the equation for the Laplace transform of the derivative of $f(t)$, or

$$\lim_{s\to 0}\int_0^\infty \left[\frac{d}{dt}f(t)\right]e^{-st}\,dt = \lim_{s\to 0}\,[sF(s) - f(0)]$$

Since $\lim_{s\to 0} e^{-st} = 1$, if $\lim_{t\to\infty} f(t)$ exists, then we obtain

$$\int_0^\infty \left[\frac{d}{dt}f(t)\right]dt = f(t)\Big|_0^\infty = f(\infty) - f(0)$$
$$= \lim_{s\to 0} sF(s) - f(0)$$

from which it follows that

$$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Initial-value theorem. The initial-value theorem is the counterpart of the final-value theorem. Using the initial-value theorem, we are able to find the value of $f(t)$ at $t = 0+$ directly from the Laplace transform of $f(t)$. The theorem does not give the value of $f(t)$ at exactly $t = 0$, but rather gives it at a time slightly greater than zero.

The initial-value theorem may be stated as follows: If $f(t)$ and $df(t)/dt$ are both Laplace transformable and if $\lim_{s\to\infty} sF(s)$ exists, then

$$f(0+) = \lim_{s\to\infty} sF(s)$$

To prove this theorem, we use the equation for the $\mathscr{L}_+$ transform of $df(t)/dt$:

$$\mathscr{L}_+\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0+)$$

For the time interval $0+ \leq t \leq \infty$, as s approaches infinity, e^{-st} approaches zero. (Note that we must use $\mathscr{L}_+$ rather than $\mathscr{L}_-$ for this condition.) Hence,

$$\lim_{s\to\infty} \int_{0+}^{\infty} \left[\frac{d}{dt}f(t)\right] e^{-st}\, dt = \lim_{s\to\infty} [sF(s) - f(0+)] = 0$$

or

$$f(0+) = \lim_{s\to\infty} sF(s)$$

In applying the initial-value theorem, we are not limited as to the locations of the poles of $sF(s)$. Thus, the theorem is valid for the sinusoidal function.

Note that the initial-value theorem and the final-value theorem provide a convenient check on the solution, since they enable us to predict the system behavior in the time domain without actually transforming functions in s back to time functions.

Integration theorem. If $f(t)$ is of exponential order, then the Laplace transform of $\int f(t)\, dt$ exists and is given by

$$\mathscr{L}\left[\int f(t)\, dt\right] = \frac{F(s)}{s} + \frac{f^{-1}(0)}{s} \tag{2–4}$$

where $F(s) = \mathscr{L}[f(t)]$ and $f^{-1}(0) = \int f(t)\, dt$, evaluated at $t = 0$. Equation (2–4) is called the integration theorem.

The integration theorem can be proven as follows: Integration by parts yields

$$\mathscr{L}\left[\int f(t)\, dt\right] = \int_0^{\infty} \left[\int f(t)\, dt\right] e^{-st}\, dt$$

$$= \left[\int f(t)\, dt\right] \frac{e^{-st}}{-s}\Bigg|_0^{\infty} - \int_0^{\infty} f(t) \frac{e^{-st}}{-s}\, dt$$

$$= \frac{1}{s}\int f(t)\,dt\bigg|_{t=0} + \frac{1}{s}\int_0^\infty f(t)e^{-st}\,dt$$
$$= \frac{f^{-1}(0)}{s} + \frac{F(s)}{s}$$

and the theorem is proven.

Note that, if $f(t)$ involves an impulse function at $t = 0$, then $f^{-1}(0+) \neq f^{-1}(0-)$. So if $f(t)$ involves an impulse function at $t = 0$, we must modify Equation (2–4) as follows:

$$\mathcal{L}_+\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \frac{f^{-1}(0+)}{s}$$
$$\mathcal{L}_-\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \frac{f^{-1}(0-)}{s}$$

We see that integration in the time domain is converted into division in the s domain. If the initial value of the integral is zero, the Laplace transform of the integral of $f(t)$ is given by $F(s)/s$.

The integration theorem can be modified slightly to deal with the definite integral of $f(t)$. If $f(t)$ is of exponential order, the Laplace transform of the definite integral $\int_0^t f(t)\,dt$ can be given by

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s} \tag{2–5}$$

To prove Equation (2–5), first note that

$$\int_0^t f(t)\,dt = \int f(t)\,dt - f^{-1}(0)$$

where $f^{-1}(0)$ is equal to $\int f(t)\,dt$, evaluated at $t = 0$, and is a constant. Hence,

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \mathcal{L}\left[\int f(t)\,dt - f^{-1}(0)\right]$$
$$= \mathcal{L}\left[\int f(t)\,dt\right] - \mathcal{L}[f^{-1}(0)]$$

Referring to Equation (2–4) and noting that $f^{-1}(0)$ is a constant, so that

$$\mathcal{L}[f^{-1}(0)] = \frac{f^{-1}(0)}{s}$$

we obtain

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s} + \frac{f^{-1}(0)}{s} - \frac{f^{-1}(0)}{s} = \frac{F(s)}{s}$$

Note that, if $f(t)$ involves an impulse function at $t = 0$, then $\int_{0+}^{t} f(t)\,dt \neq \int_{0-}^{t} f(t)\,dt$, and the following distinction must be observed:

$$\mathscr{L}_+\left[\int_{0+}^{t} f(t)\,dt\right] = \frac{\mathscr{L}_+[f(t)]}{s}$$
$$\mathscr{L}_-\left[\int_{0-}^{t} f(t)\,dt\right] = \frac{\mathscr{L}_-[f(t)]}{s}$$

2–4 INVERSE LAPLACE TRANSFORMATION

The inverse Laplace transformation refers to the process of finding the time function $f(t)$ from the corresponding Laplace transform $F(s)$. Several methods are available for finding inverse Laplace transforms. The simplest of these methods are (1) to use tables of Laplace transforms to find the time function $f(t)$ corresponding to a given Laplace transform $F(s)$ and (2) to use the partial-fraction expansion method. In this section, we present the latter technique. [Note that MATLAB is quite useful in obtaining the partial-fraction expansion of the ratio of two polynomials, $B(s)/A(s)$. We shall discuss the MATLAB approach to the partial-fraction expansion in Chapter 4.]

Partial-fraction expansion method for finding inverse Laplace transforms. If $F(s)$, the Laplace transform of $f(t)$, is broken up into components, or

$$F(s) = F_1(s) + F_2(s) + \cdots + F_n(s)$$

and if the inverse Laplace transforms of $F_1(s), F_2(s), \ldots, F_n(s)$ are readily available, then

$$\begin{aligned}\mathscr{L}^{-1}[F(s)] &= \mathscr{L}^{-1}[F_1(s)] + \mathscr{L}^{-1}[F_2(s)] + \cdots + \mathscr{L}^{-1}[F_n(s)] \\ &= f_1(t) + f_2(t) + \cdots + f_n(t)\end{aligned}$$

where $f_1(t), f_2(t), \ldots, f_n(t)$ are the inverse Laplace transforms of $F_1(s), F_2(s), \cdots, F_n(s)$, respectively. The inverse Laplace transform of $F(s)$ thus obtained is unique, except possibly at points where the time function is discontinuous. Whenever the time function is continuous, the time function $f(t)$ and its Laplace transform $F(s)$ have a one-to-one correspondence.

For problems in systems analysis, $F(s)$ frequently occurs in the form

$$F(s) = \frac{B(s)}{A(s)}$$

where $A(s)$ and $B(s)$ are polynomials in s and the degree of $B(s)$ is not higher than that of $A(s)$.

The advantage of the partial-fraction expansion approach is that the individual terms of $F(s)$ resulting from the expansion into partial-fraction form are very simple functions of s; consequently, it is not necessary to refer to a Laplace transform table if we memorize several simple Laplace transform pairs. Note, however, that in applying the partial-fraction expansion technique in the search for the inverse Laplace transform of $F(s) = B(s)/A(s)$, the roots of the denominator polynomial $A(s)$ must be known in advance. That is, this method does not apply until the denominator polynomial has been factored.

Consider $F(s)$ written in the factored form

$$F(s) = \frac{B(s)}{A(s)} = \frac{K(s + z_1)(s + z_2)\cdots(s + z_m)}{(s + p_1)(s + p_2)\cdots(s + p_n)}$$

where $p_1, p_2, \ldots, p_n$ and $z_1, z_2, \ldots, z_m$ are either real or complex quantities, but for each complex p_i or z_i, there will occur the complex conjugate of p_i or z_i, respectively. Here, the highest power of s in $A(s)$ is assumed to be higher than that in $B(s)$.

In the expansion of $B(s)/A(s)$ into partial-fraction form, it is important that the highest power of s in $A(s)$ be greater than the highest power of s in $B(s)$ because if that is not the case, then the numerator $B(s)$ must be divided by the denominator $A(s)$ in order to produce a polynomial in s plus a remainder (a ratio of polynomials in s whose numerator is of lower degree than the denominator). (For details, see **Example 2–2**.)

Partial-fraction expansion when $F(s)$ involves distinct poles only. In this case, $F(s)$ can always be expanded into a sum of simple partial fractions; that is,

$$F(s) = \frac{B(s)}{A(s)} = \frac{a_1}{s + p_1} + \frac{a_2}{s + p_2} + \cdots + \frac{a_n}{s + p_n} \tag{2–6}$$

where $a_k (k = 1, 2, \ldots, n)$ are constants. The coefficient a_k is called the *residue* at the pole at $s = -p_k$. The value of a_k can be found by multiplying both sides of Equation (2–6) by $(s + p_k)$ and letting $s = -p_k$, giving

$$\begin{aligned}\left[(s + p_k)\frac{B(s)}{A(s)}\right]_{s=-p_k} &= \left[\frac{a_1}{s + p_1}(s + p_k) + \frac{a_2}{s + p_2}(s + p_k) + \cdots \right.\\ &\qquad \left. + \frac{a_k}{s + p_k}(s + p_k) + \cdots + \frac{a_n}{s + p_n}(s + p_k)\right]_{s=-p_k}\\ &= a_k\end{aligned}$$

We see that all the expanded terms drop out, with the exception of a_k. Thus, the residue a_k is found from

$$a_k = \left[(s + p_k)\frac{B(s)}{A(s)}\right]_{s=-p_k} \tag{2–7}$$

Note that since $f(t)$ is a real function of time, if p_1 and p_2 are complex conjugates, then the residues a_1 and a_2 are also complex conjugates. Only one of the conjugates, a_1 or a_2, need be evaluated, because the other is known automatically.

Since

$$\mathscr{L}^{-1}\left[\frac{a_k}{s + p_k}\right] = a_k e^{-p_k t}$$

$f(t)$ is obtained as

$$f(t) = \mathscr{L}^{-1}[F(s)] = a_1 e^{-p_1 t} + a_2 e^{-p_2 t} + \cdots + a_n e^{-p_n t} \qquad t \geq 0$$

Example 2–1

Find the inverse Laplace transform of

$$F(s) = \frac{s + 3}{(s + 1)(s + 2)}$$

The partial-fraction expansion of $F(s)$ is

$$F(s) = \frac{s + 3}{(s + 1)(s + 2)} = \frac{a_1}{s + 1} + \frac{a_2}{s + 2}$$

where a_1 and a_2 are found by using Equation (2–7):

$$a_1 = \left[(s + 1)\frac{s + 3}{(s + 1)(s + 2)}\right]_{s=-1} = \left[\frac{s + 3}{s + 2}\right]_{s=-1} = 2$$

$$a_2 = \left[(s + 2)\frac{s + 3}{(s + 1)(s + 2)}\right]_{s=-2} = \left[\frac{s + 3}{s + 1}\right]_{s=-2} = -1$$

Thus,

$$\begin{aligned} f(t) &= \mathscr{L}^{-1}[F(s)] \\ &= \mathscr{L}^{-1}\left[\frac{2}{s + 1}\right] + \mathscr{L}^{-1}\left[\frac{-1}{s + 2}\right] \\ &= 2e^{-t} - e^{-2t} \qquad t \geq 0 \end{aligned}$$

Example 2–2

Obtain the inverse Laplace transform of

$$G(s) = \frac{s^3 + 5s^2 + 9s + 7}{(s + 1)(s + 2)}$$

Here, since the degree of the numerator polynomial is higher than that of the denominator polynomial, we must divide the numerator by the denominator:

$$G(s) = s + 2 + \frac{s + 3}{(s + 1)(s + 2)}$$

Note that the Laplace transform of the unit-impulse function $\delta(t)$ is unity and that the Laplace transform of $d\delta(t)/dt$ is s. The third term on the right-hand side of this last equation is $F(s)$ in **Example 2–1**. So the inverse Laplace transform of $G(s)$ is given as

$$g(t) = \frac{d}{dt}\delta(t) + 2\delta(t) + 2e^{-t} - e^{-2t} \qquad t \geq 0-$$

Comment. Consider a function $F(s)$ that involves a quadratic factor $s^2 + as + b$ in the denominator. If this quadratic expression has a pair of complex-conjugate roots, then it is better not to factor the quadratic, in order to avoid complex numbers. For example, if $F(s)$ is given as

$$F(s) = \frac{p(s)}{s(s^2 + as + b)}$$

where $a \geq 0$ and $b > 0$, and if $s^2 + as + b = 0$ has a pair of complex-conjugate roots, then expand $F(s)$ into the following partial-fraction expansion form:

$$F(s) = \frac{c}{s} + \frac{ds + e}{s^2 + as + b}$$

(See **Example 2–3** and **Problems A–2–15, A–2–16**, and **A–2–19**.)

Example 2–3

Find the inverse Laplace transform of

$$F(s) = \frac{2s + 12}{s^2 + 2s + 5}$$

Notice that the denominator polynomial can be factored as

$$s^2 + 2s + 5 = (s + 1 + j2)(s + 1 - j2)$$

The two roots of the denominator are complex conjugates. Hence, we expand $F(s)$ into the sum of a damped sine and a damped cosine function.

Noting that $s^2 + 2s + 5 = (s + 1)^2 + 2^2$ and referring to the Laplace transforms of $e^{-\alpha t} \sin \omega t$ and $e^{-\alpha t} \cos \omega t$, rewritten as

$$\mathcal{L}[e^{-\alpha t} \sin \omega t] = \frac{\omega}{(s + \alpha)^2 + \omega^2}$$

and

$$\mathcal{L}[e^{-\alpha t} \cos \omega t] = \frac{s + \alpha}{(s + \alpha)^2 + \omega^2}$$

we can write the given $F(s)$ as a sum of a damped sine and a damped cosine function:

$$F(s) = \frac{2s + 12}{s^2 + 2s + 5} = \frac{10 + 2(s + 1)}{(s + 1)^2 + 2^2}$$
$$= 5\frac{2}{(s + 1)^2 + 2^2} + 2\frac{s + 1}{(s + 1)^2 + 2^2}$$

It follows that

$$f(t) = \mathcal{L}^{-1}[F(s)]$$
$$= 5\mathcal{L}^{-1}\left[\frac{2}{(s + 1)^2 + 2^2}\right] + 2\mathcal{L}^{-1}\left[\frac{s + 1}{(s + 1)^2 + 2^2}\right]$$
$$= 5e^{-t} \sin 2t + 2e^{-t} \cos 2t \qquad t \geq 0$$

Partial-fraction expansion when $F(s)$ involves multiple poles. Instead of discussing the general case, we shall use an example to show how to obtain the partial-fraction expansion of $F(s)$. (See also **Problems A–2–17** and **A–2–19**.)

Consider

$$F(s) = \frac{s^2 + 2s + 3}{(s + 1)^3}$$

The partial-fraction expansion of this $F(s)$ involves three terms:

$$F(s) = \frac{B(s)}{A(s)} = \frac{b_3}{(s+1)^3} + \frac{b_2}{(s+1)^2} + \frac{b_1}{s+1}$$

where b_3, b_2, and b_1 are determined as follows: Multiplying both sides of this last equation by $(s+1)^3$, we have

$$(s+1)^3 \frac{B(s)}{A(s)} = b_3 + b_2(s+1) + b_1(s+1)^2 \tag{2–8}$$

Then, letting $s = -1$, we find that Equation (2–8) gives

$$\left[(s+1)^3 \frac{B(s)}{A(s)}\right]_{s=-1} = b_3$$

Also, differentiating both sides of Equation (2–8) with respect to s yields

$$\frac{d}{ds}\left[(s+1)^3 \frac{B(s)}{A(s)}\right] = b_2 + 2b_1(s+1) \tag{2–9}$$

If we let $s = -1$ in Equation (2–9), then

$$\frac{d}{ds}\left[(s+1)^3 \frac{B(s)}{A(s)}\right]_{s=-1} = b_2$$

Differentiating both sides of Equation (2–9) with respect to s, we obtain

$$\frac{d^2}{ds^2}\left[(s+1)^3 \frac{B(s)}{A(s)}\right] = 2b_1$$

From the preceding analysis, it can be seen that the values of b_3, b_2, and b_1 are found systematically as follows:

$$\begin{aligned}
b_3 &= \left[(s+1)^3 \frac{B(s)}{A(s)}\right]_{s=-1} \\
&= (s^2 + 2s + 3)_{s=-1} \\
&= 2 \\
b_2 &= \left\{\frac{d}{ds}\left[(s+1)^3 \frac{B(s)}{A(s)}\right]\right\}_{s=-1} \\
&= \left[\frac{d}{ds}(s^2 + 2s + 3)\right]_{s=-1} \\
&= (2s + 2)_{s=-1} \\
&= 0 \\
b_1 &= \frac{1}{2!}\left\{\frac{d^2}{ds^2}\left[(s+1)^3 \frac{B(s)}{A(s)}\right]\right\}_{s=-1} \\
&= \frac{1}{2!}\left[\frac{d^2}{ds^2}(s^2 + 2s + 3)\right]_{s=-1} \\
&= \frac{1}{2}(2) = 1
\end{aligned}$$

We thus obtain

$$
\begin{aligned}
f(t) &= \mathcal{L}^{-1}[F(s)] \\
&= \mathcal{L}^{-1}\left[\frac{2}{(s+1)^3}\right] + \mathcal{L}^{-1}\left[\frac{0}{(s+1)^2}\right] + \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] \\
&= t^2 e^{-t} + 0 + e^{-t} \\
&= (t^2 + 1)e^{-t} \qquad t \geq 0
\end{aligned}
$$

2–5 SOLVING LINEAR, TIME-INVARIANT DIFFERENTIAL EQUATIONS

In this section, we are concerned with the use of the Laplace transform method in solving linear, time-invariant differential equations.

The Laplace transform method yields the complete solution (complementary solution and particular solution) of linear, time-invariant differential equations. Classical methods for finding the complete solution of a differential equation require the evaluation of the integration constants from the initial conditions. In the case of the Laplace transform method, however, this requirement is unnecessary because the initial conditions are automatically included in the Laplace transform of the differential equation.

If all initial conditions are zero, then the Laplace transform of the differential equation is obtained simply by replacing d/dt with s, d^2/dt^2 with s^2, and so on.

In solving linear, time-invariant differential equations by the Laplace transform method, two steps are followed:

1. By taking the Laplace transform of each term in the given differential equation, convert the differential equation into an algebraic equation in s and obtain the expression for the Laplace transform of the dependent variable by rearranging the algebraic equation.
2. The time solution of the differential equation is obtained by finding the inverse Laplace transform of the dependent variable.

In the discussion that follows, two examples are used to demonstrate the solution of linear, time-invariant differential equations by the Laplace transform method.

Example 2–4

Find the solution $x(t)$ of the differential equation

$$\ddot{x} + 3\dot{x} + 2x = 0, \qquad x(0) = a, \qquad \dot{x}(0) = b$$

where a and b are constants.

Writing the Laplace transform of $x(t)$ as $X(s)$, or

$$\mathcal{L}[x(t)] = X(s)$$

we obtain

$$
\begin{aligned}
\mathcal{L}[\dot{x}] &= sX(s) - x(0) \\
\mathcal{L}[\ddot{x}] &= s^2X(s) - sx(0) - \dot{x}(0)
\end{aligned}
$$

The Laplace transform of the given differential equation becomes

$$[s^2X(s) - sx(0) - \dot{x}(0)] + 3[sX(s) - x(0)] + 2X(s) = 0$$

Substituting the given initial conditions into the preceding equation yields

$$[s^2X(s) - as - b] + 3[sX(s) - a] + 2X(s) = 0$$

or

$$(s^2 + 3s + 2)X(s) = as + b + 3a$$

Solving this last equation for $X(s)$, we have

$$X(s) = \frac{as + b + 3a}{s^2 + 3s + 2} = \frac{as + b + 3a}{(s+1)(s+2)} = \frac{2a + b}{s + 1} - \frac{a + b}{s + 2}$$

The inverse Laplace transform of $X(s)$ produces

$$\begin{aligned} x(t) &= \mathscr{L}^{-1}[X(s)] = \mathscr{L}^{-1}\left[\frac{2a + b}{s + 1}\right] - \mathscr{L}^{-1}\left[\frac{a + b}{s + 2}\right] \\ &= (2a + b)e^{-t} - (a + b)e^{-2t} \qquad t \geq 0 \end{aligned}$$

which is the solution of the given differential equation. Notice that the initial conditions a and b appear in the solution. Thus, $x(t)$ has no undetermined constants.

Example 2–5

Find the solution $x(t)$ of the differential equation

$$\ddot{x} + 2\dot{x} + 5x = 3, \qquad x(0) = 0, \qquad \dot{x}(0) = 0$$

Noting that $\mathscr{L}[3] = 3/s$, $x(0) = 0$, and $\dot{x}(0) = 0$, we see that the Laplace transform of the differential equation becomes

$$s^2X(s) + 2sX(s) + 5X(s) = \frac{3}{s}$$

Solving this equation for $X(s)$, we obtain

$$\begin{aligned} X(s) &= \frac{3}{s(s^2 + 2s + 5)} \\ &= \frac{3}{5}\frac{1}{s} - \frac{3}{5}\frac{s + 2}{s^2 + 2s + 5} \\ &= \frac{3}{5}\frac{1}{s} - \frac{3}{10}\frac{2}{(s+1)^2 + 2^2} - \frac{3}{5}\frac{s + 1}{(s+1)^2 + 2^2} \end{aligned}$$

Hence, the inverse Laplace transform becomes

$$\begin{aligned} x(t) &= \mathscr{L}^{-1}[X(s)] \\ &= \frac{3}{5}\mathscr{L}^{-1}\left[\frac{1}{s}\right] - \frac{3}{10}\mathscr{L}^{-1}\left[\frac{2}{(s+1)^2 + 2^2}\right] - \frac{3}{5}\mathscr{L}^{-1}\left[\frac{s + 1}{(s+1)^2 + 2^2}\right] \\ &= \frac{3}{5} - \frac{3}{10}e^{-t}\sin 2t - \frac{3}{5}e^{-t}\cos 2t \qquad t \geq 0 \end{aligned}$$

which is the solution of the given differential equation.

EXAMPLE PROBLEMS AND SOLUTIONS

Problem A–2–1

Obtain the real and imaginary parts of

$$\frac{2 + j1}{3 + j4}$$

Also, obtain the magnitude and angle of this complex quantity.

Solution

$$\frac{2 + j1}{3 + j4} = \frac{(2 + j1)(3 - j4)}{(3 + j4)(3 - j4)} = \frac{6 + j3 - j8 + 4}{9 + 16} = \frac{10 - j5}{25}$$
$$= \frac{2}{5} - j\frac{1}{5}$$

Hence,

$$\text{real part} = \frac{2}{5}, \qquad \text{imaginary part} = -j\frac{1}{5}$$

The magnitude and angle of this complex quantity are obtained as follows:

$$\text{magnitude} = \sqrt{\left(\frac{2}{5}\right)^2 + \left(\frac{-1}{5}\right)^2} = \sqrt{\frac{5}{25}} = \frac{1}{\sqrt{5}} = 0.447$$

$$\text{angle} = \tan^{-1}\frac{-1/5}{2/5} = \tan^{-1}\frac{-1}{2} = -26.565°$$

Problem A–2–2

Find the Laplace transform of

$$\begin{aligned} f(t) &= 0 && t < 0 \\ &= te^{-3t} && t \geq 0 \end{aligned}$$

Solution Since

$$\mathscr{L}[t] = G(s) = \frac{1}{s^2}$$

referring to Equation (2–2), we obtain

$$F(s) = \mathscr{L}[te^{-3t}] = G(s + 3) = \frac{1}{(s + 3)^2}$$

Problem A–2–3

What is the Laplace transform of

$$\begin{aligned} f(t) &= 0 && t < 0 \\ &= \sin(\omega t + \theta) && t \geq 0 \end{aligned}$$

where θ is a constant?

Solution Noting that

$$\sin(\omega t + \theta) = \sin \omega t \cos \theta + \cos \omega t \sin \theta$$

we have

$$\mathcal{L}[\sin(\omega t + \theta)] = \cos\theta\,\mathcal{L}[\sin\omega t] + \sin\theta\,\mathcal{L}[\cos\omega t]$$
$$= \cos\theta\frac{\omega}{s^2+\omega^2} + \sin\theta\frac{s}{s^2+\omega^2}$$
$$= \frac{\omega\cos\theta + s\sin\theta}{s^2+\omega^2}$$

Problem A–2–4

Find the Laplace transform $F(s)$ of the function $f(t)$ shown in Figure 2–9. Also, find the limiting value of $F(s)$ as a approaches zero.

Solution The function $f(t)$ can be written

$$f(t) = \frac{1}{a^2}1(t) - \frac{2}{a^2}1(t-a) + \frac{1}{a^2}1(t-2a)$$

Then

$$F(s) = \mathcal{L}[f(t)]$$
$$= \frac{1}{a^2}\mathcal{L}[1(t)] - \frac{2}{a^2}\mathcal{L}[1(t-a)] + \frac{1}{a^2}\mathcal{L}[1(t-2a)]$$
$$= \frac{1}{a^2}\frac{1}{s} - \frac{2}{a^2}\frac{1}{s}e^{-as} + \frac{1}{a^2}\frac{1}{s}e^{-2as}$$
$$= \frac{1}{a^2 s}(1 - 2e^{-as} + e^{-2as})$$

As a approaches zero, we have

$$\lim_{a\to 0} F(s) = \lim_{a\to 0}\frac{1 - 2e^{-as} + e^{-2as}}{a^2 s} = \lim_{a\to 0}\frac{\frac{d}{da}(1 - 2e^{-as} + e^{-2as})}{\frac{d}{da}(a^2 s)}$$
$$= \lim_{a\to 0}\frac{2se^{-as} - 2se^{-2as}}{2as} = \lim_{a\to 0}\frac{e^{-as} - e^{-2as}}{a}$$

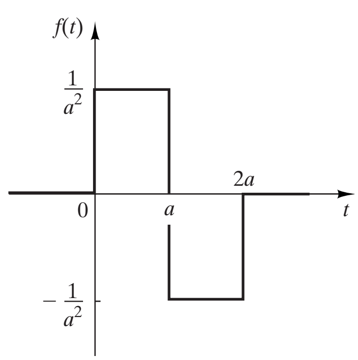

Figure 2–9 Function $f(t)$.

$$= \lim_{a \to 0} \frac{\frac{d}{da}(e^{-as} - e^{-2as})}{\frac{d}{da}(a)} = \lim_{a \to 0} \frac{-se^{-as} + 2se^{-2as}}{1}$$
$$= -s + 2s = s$$

Problem A–2–5

Obtain the Laplace transform of the function *f*(*t*) shown in Figure 2–10.

Solution The given function *f*(*t*) can be defined as follows:

$$\begin{aligned} f(t) &= 0 && t \leq 0 \\ &= \frac{b}{a}t && 0 < t \leq a \\ &= 0 && a < t \end{aligned}$$

Notice that *f*(*t*) can be considered a sum of the three functions $f_1(t)$, $f_2(t)$, and $f_3(t)$ shown in Figure 2–11. Hence, *f*(*t*) can be written as

$$\begin{aligned} f(t) &= f_1(t) + f_2(t) + f_3(t) \\ &= \frac{b}{a}t \cdot 1(t) - \frac{b}{a}(t - a) \cdot 1(t - a) - b \cdot 1(t - a) \end{aligned}$$

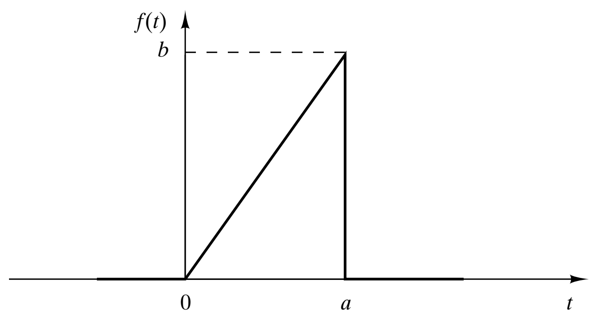

Figure 2–10 Function *f*(*t*).

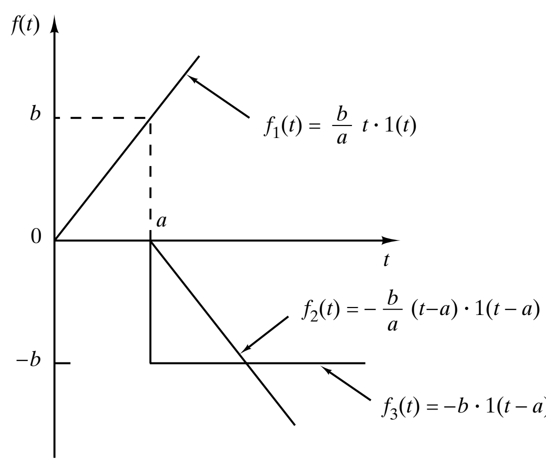

Figure 2–11 Functions $f_1(t)$, $f_2(t)$, and $f_3(t)$.

Then the Laplace transform of $f(t)$ becomes

$$F(s) = \frac{b}{a}\frac{1}{s^2} - \frac{b}{a}\frac{1}{s^2}e^{-as} - b\frac{1}{s}e^{-as}$$
$$= \frac{b}{as^2}(1 - e^{-as}) - \frac{b}{s}e^{-as}$$

The same $F(s)$ can, of course, be obtained by performing the following Laplace integration:

$$\mathscr{L}[f(t)] = \int_0^a \frac{b}{a}te^{-st}\,dt + \int_a^\infty 0\,e^{-st}\,dt$$
$$= \frac{b}{a}t\frac{e^{-st}}{-s}\Big|_0^a - \int_0^a \frac{b}{a}\frac{e^{-st}}{-s}\,dt$$
$$= b\frac{e^{-as}}{-s} + \frac{b}{as}\frac{e^{-st}}{-s}\Big|_0^a$$
$$= b\frac{e^{-as}}{-s} - \frac{b}{as^2}(e^{-as} - 1)$$
$$= \frac{b}{as^2}(1 - e^{-as}) - \frac{b}{s}e^{-as}$$

Problem A–2–6

Prove that if the Laplace transform of $f(t)$ is $F(s)$, then, except at poles of $F(s)$,

$$\mathscr{L}[tf(t)] = -\frac{d}{ds}F(s)$$
$$\mathscr{L}[t^2f(t)] = \frac{d^2}{ds^2}F(s)$$

and in general,

$$\mathscr{L}[t^nf(t)] = (-1)^n\frac{d^n}{ds^n}F(s) \qquad n = 1, 2, 3, \ldots$$

Solution

$$\mathscr{L}[tf(t)] = \int_0^\infty tf(t)e^{-st}\,dt = -\int_0^\infty f(t)\frac{d}{ds}(e^{-st})\,dt$$
$$= -\frac{d}{ds}\int_0^\infty f(t)e^{-st}\,dt = -\frac{d}{ds}F(s)$$

Similarly, by defining $tf(t) = g(t)$, the result is

$$\mathscr{L}[t^2f(t)] = \mathscr{L}[tg(t)] = -\frac{d}{ds}G(s) = -\frac{d}{ds}\left[-\frac{d}{ds}F(s)\right]$$
$$= (-1)^2\frac{d^2}{ds^2}F(s) = \frac{d^2}{ds^2}F(s)$$

Repeating the same process, we obtain

$$\mathscr{L}[t^nf(t)] = (-1)^n\frac{d^n}{ds^n}F(s) \qquad n = 1, 2, 3, \ldots$$

Problem A–2–7

Find the Laplace transform of

$$\begin{aligned} f(t) &= 0 & t < 0 \\ &= t^2 \sin \omega t & t \geq 0 \end{aligned}$$

Solution Since

$$\mathscr{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2}$$

referring to **Problem A–2–6**, we have

$$\mathscr{L}[f(t)] = \mathscr{L}[t^2 \sin \omega t] = \frac{d^2}{ds^2}\left[\frac{\omega}{s^2 + \omega^2}\right] = \frac{-2\omega^3 + 6\omega s^2}{(s^2 + \omega^2)^3}$$

Problem A–2–8

Prove that if the Laplace transform of $f(t)$ is $F(s)$, then

$$\mathscr{L}\left[f\left(\frac{t}{a}\right)\right] = aF(as) \qquad a > 0$$

Solution If we define $t/a = \tau$ and $as = s_1$, then

$$\begin{aligned} \mathscr{L}\left[f\left(\frac{t}{a}\right)\right] &= \int_0^\infty f\left(\frac{t}{a}\right)e^{-st}\,dt = \int_0^\infty f(\tau)e^{-as\tau}a\,d\tau \\ &= a\int_0^\infty f(\tau)e^{-s_1\tau}\,d\tau = aF(s_1) = aF(as) \end{aligned}$$

Problem A–2–9

Prove that if $f(t)$ is of exponential order and if $\int_0^\infty f(t)\,dt$ exists [which means that $\int_0^\infty f(t)\,dt$ assumes a definite value], then

$$\int_0^\infty f(t)\,dt = \lim_{s\to 0} F(s)$$

where $F(s) = \mathscr{L}[f(t)]$.

Solution Note that

$$\int_0^\infty f(t)\,dt = \lim_{t\to\infty}\int_0^t f(t)\,dt$$

Referring to Equation (2–5), we have

$$\mathscr{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$$

Since $\int_0^\infty f(t)\,dt$ exists, by applying the final-value theorem to this case, we obtain

$$\lim_{t\to\infty}\int_0^t f(t)\,dt = \lim_{s\to 0} s\frac{F(s)}{s}$$

or

$$\int_0^\infty f(t)\,dt = \lim_{s\to 0} F(s)$$

Problem A–2–10

The convolution of two time functions is defined by

$$\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau$$

A commonly used notation for the convolution is $f_1(t)*f_2(t)$, which is defined as

$$f_1(t)*f_2(t) = \int_0^t f_1(\tau)f_2(t-\tau)\,d\tau = \int_0^t f_1(t-\tau)f_2(\tau)\,d\tau$$

Show that if $f_1(t)$ and $f_2(t)$ are both Laplace transformable, then

$$\mathcal{L}\left[\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau\right] = F_1(s)F_2(s)$$

where $F_1(s) = \mathcal{L}[f_1(t)]$ and $F_2(s) = \mathcal{L}[f_2(t)]$.

Solution Noting that $1(t-\tau) = 0$ for $t < \tau$, we have

$$\begin{aligned}\mathcal{L}\left[\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau\right] &= \mathcal{L}\left[\int_0^\infty f_1(\tau)f_2(t-\tau)1(t-\tau)\,d\tau\right] \\ &= \int_0^\infty e^{-st}\left[\int_0^\infty f_1(\tau)f_2(t-\tau)1(t-\tau)\,d\tau\right]dt \\ &= \int_0^\infty f_1(\tau)\,d\tau \int_0^\infty f_2(t-\tau)1(t-\tau)e^{-st}\,dt\end{aligned}$$

Changing the order of integration is valid here, since $f_1(t)$ and $f_2(t)$ are both Laplace transformable, giving convergent integrals. If we substitute $\lambda = t - \tau$ into this last equation, the result is

$$\begin{aligned}\mathcal{L}\left[\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau\right] &= \int_0^\infty f_1(\tau)e^{-s\tau}\,d\tau \int_0^\infty f_2(\lambda)e^{-s\lambda}\,d\lambda \\ &= F_1(s)F_2(s)\end{aligned}$$

or

$$\mathcal{L}[f_1(t)*f_2(t)] = F_1(s)F_2(s)$$

Thus, the Laplace transform of the convolution of two time functions is the product of their Laplace transforms.

Problem A–2–11

Determine the Laplace transform of $f_1(t)*f_2(t)$, where

$$\begin{aligned}f_1(t) &= f_2(t) = 0 && \text{for } t < 0 \\ f_1(t) &= t && \text{for } t \geq 0 \\ f_2(t) &= 1 - e^{-t} && \text{for } t \geq 0\end{aligned}$$

Solution Note that

$$\begin{aligned}\mathcal{L}[t] &= F_1(s) = \frac{1}{s^2} \\ \mathcal{L}[1 - e^{-t}] &= F_2(s) = \frac{1}{s} - \frac{1}{s+1}\end{aligned}$$

The Laplace transform of the convolution integral is given by

$$\mathcal{L}[f_1(t)*f_2(t)] = F_1(s)F_2(s) = \frac{1}{s^2}\left(\frac{1}{s} - \frac{1}{s+1}\right)$$
$$= \frac{1}{s^3} - \frac{1}{s^2(s+1)} = \frac{1}{s^3} - \frac{1}{s^2} + \frac{1}{s} - \frac{1}{s+1}$$

To verify that the expression after the rightmost equal sign is indeed the Laplace transform of the convolution integral, let us first integrate the convolution integral and then take the Laplace transform of the result. We have

$$f_1(t)*f_2(t) = \int_0^t \tau[1 - e^{-(t-\tau)}]\,d\tau$$
$$= \int_0^t (t-\tau)(1 - e^{-\tau})\,d\tau$$
$$= \int_0^t (t - \tau - te^{-\tau} + \tau e^{-\tau})\,d\tau$$

Noting that

$$\int_0^t (t-\tau)\,d\tau = \frac{t^2}{2}$$
$$\int_0^t te^{-\tau}\,d\tau = -te^{-t} + t$$
$$\int_0^t \tau e^{-\tau}\,d\tau = -\tau e^{-\tau}\Big|_0^t + \int_0^t e^{-\tau}\,d\tau = -te^{-t} - e^{-t} + 1$$

we have

$$f_1(t)*f_2(t) = \frac{t^2}{2} - t + 1 - e^{-t}$$

Thus,

$$\mathcal{L}\left[\frac{t^2}{2} - t + 1 - e^{-t}\right] = \frac{1}{s^3} - \frac{1}{s^2} + \frac{1}{s} - \frac{1}{s+1}$$

Problem A–2–12

Prove that if $f(t)$ is a periodic function with period T, then

$$\mathcal{L}[f(t)] = \frac{\int_0^T f(t)e^{-st}\,dt}{1 - e^{-Ts}}$$

Solution

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt = \sum_{n=0}^{\infty}\int_{nT}^{(n+1)T} f(t)e^{-st}\,dt$$

By changing the independent variable from t to $\tau = t - nT$, we obtain

$$\mathcal{L}[f(t)] = \sum_{n=0}^{\infty} e^{-nTs}\int_0^T f(\tau + nT)e^{-s\tau}\,d\tau$$

Since $f(t)$ is a periodic function with period T, $f(\tau + nT) = f(\tau)$. Hence,

$$\mathcal{L}[f(t)] = \sum_{n=0}^{\infty} e^{-nTs} \int_0^T f(\tau) e^{-st}\, d\tau$$

Noting that

$$\begin{aligned}\sum_{n=0}^{\infty} e^{-nTs} &= 1 + e^{-Ts} + e^{-2Ts} + \cdots \\ &= 1 + e^{-Ts}(1 + e^{-Ts} + e^{-2Ts} + \cdots) \\ &= 1 + e^{-Ts}\left(\sum_{n=0}^{\infty} e^{-nTs}\right)\end{aligned}$$

we obtain

$$\sum_{n=0}^{\infty} e^{-nTs} = \frac{1}{1 - e^{-Ts}}$$

It follows that

$$\mathcal{L}[f(t)] = \frac{\displaystyle\int_0^T f(t) e^{-st}\, dt}{1 - e^{-Ts}}$$

Problem A–2–13

What is the Laplace transform of the periodic function shown in Figure 2–12?

Solution Note that

$$\begin{aligned}\int_0^T f(t) e^{-st}\, dt &= \int_0^{T/2} e^{-st}\, dt + \int_{T/2}^T (-1) e^{-st}\, dt \\ &= \frac{e^{-st}}{-s}\bigg|_0^{T/2} - \frac{e^{-st}}{-s}\bigg|_{T/2}^{T} \\ &= \frac{e^{-(1/2)Ts} - 1}{-s} + \frac{e^{-Ts} - e^{-(1/2)Ts}}{s} \\ &= \frac{1}{s}\left[e^{-Ts} - 2e^{-(1/2)Ts} + 1\right] \\ &= \frac{1}{s}\left[1 - e^{-(1/2)Ts}\right]^2\end{aligned}$$

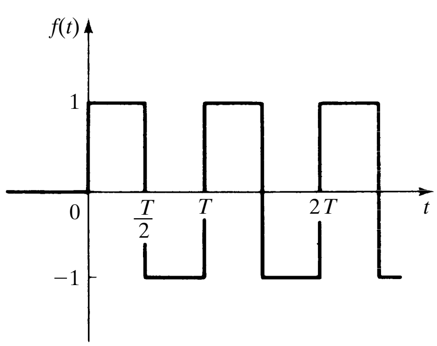

Figure 2–12 Periodic function (square wave).

Consequently,

$$F(s) = \frac{\int_0^T f(t)e^{-st}\,dt}{1 - e^{-Ts}} = \frac{(1/s)[1 - e^{-(1/2)Ts}]^2}{1 - e^{-Ts}}$$
$$= \frac{1 - e^{-(1/2)Ts}}{s[1 + e^{-(1/2)Ts}]} = \frac{1}{s}\tanh\frac{Ts}{4}$$

Problem A–2–14

Find the initial value of $df(t)/dt$, where the Laplace transform of $f(t)$ is given by

$$F(s) = \mathscr{L}[f(t)] = \frac{2s + 1}{s^2 + s + 1}$$

Solution Using the initial-value theorem, we obtain

$$\lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\frac{s(2s + 1)}{s^2 + s + 1} = 2$$

Since the $\mathscr{L}_+$ transform of $df(t)/dt = g(t)$ is given by

$$\mathscr{L}_+[g(t)] = sF(s) - f(0+)$$
$$= \frac{s(2s + 1)}{s^2 + s + 1} - 2 = \frac{-s - 2}{s^2 + s + 1}$$

the initial value of $df(t)/dt$ is obtained as

$$\lim_{t\to 0+}\frac{df(t)}{dt} = g(0+) = \lim_{s\to\infty} s[sF(s) - f(0+)]$$
$$= \lim_{s\to\infty}\frac{-s^2 - 2s}{s^2 + s + 1} = -1$$

To verify this result, notice that

$$F(s) = \frac{2(s + 0.5)}{(s + 0.5)^2 + (0.866)^2} = \mathscr{L}[2e^{-0.5t}\cos 0.866t]$$

Hence,

$$f(t) = 2e^{-0.5t}\cos 0.866t$$

and

$$\dot{f}(t) = -e^{-0.5t}\cos 0.866t + 2e^{-0.5t}0.866\sin 0.866t$$

Thus,

$$\dot{f}(0) = -1 + 0 = -1$$

Problem A–2–15

Obtain the inverse Laplace transform of

$$F(s) = \frac{cs + d}{(s^2 + 2as + a^2) + b^2}$$

where a, b, c, and d are real and a is positive.

Solution Since $F(s)$ can be written as

$$F(s) = \frac{c(s+a) + d - ca}{(s+a)^2 + b^2}$$
$$= \frac{c(s+a)}{(s+a)^2 + b^2} + \frac{d - ca}{b} \frac{b}{(s+a)^2 + b^2}$$

we obtain

$$f(t) = ce^{-at} \cos bt + \frac{d - ca}{b} e^{-at} \sin bt$$

Problem A–2–16

Find the inverse Laplace transform of

$$F(s) = \frac{1}{s(s^2 + 2s + 2)}$$

Solution Since

$$s^2 + 2s + 2 = (s + 1 + j1)(s + 1 - j1)$$

it follows that $F(s)$ involves a pair of complex-conjugate poles, so we expand $F(s)$ into the form

$$F(s) = \frac{1}{s(s^2 + 2s + 2)} = \frac{a_1}{s} + \frac{a_2 s + a_3}{s^2 + 2s + 2}$$

where a_1, a_2, and a_3 are determined from

$$1 = a_1(s^2 + 2s + 2) + (a_2 s + a_3)s$$

By comparing corresponding coefficients of the s^2, s, and s^0 terms on both sides of this last equation respectively, we obtain

$$a_1 + a_2 = 0, \qquad 2a_1 + a_3 = 0, \qquad 2a_1 = 1$$

from which it follows that

$$a_1 = \frac{1}{2}, \qquad a_2 = -\frac{1}{2}, \qquad a_3 = -1$$

Therefore,

$$F(s) = \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s+2}{s^2 + 2s + 2}$$
$$= \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{(s+1)^2 + 1^2} - \frac{1}{2}\frac{s+1}{(s+1)^2 + 1^2}$$

The inverse Laplace transform of $F(s)$ is

$$f(t) = \frac{1}{2} - \frac{1}{2}e^{-t}\sin t - \frac{1}{2}e^{-t}\cos t \qquad t \geq 0$$

Problem A–2–17

Derive the inverse Laplace transform of

$$F(s) = \frac{5(s+2)}{s^2(s+1)(s+3)}$$

Solution

$$F(s) = \frac{5(s+2)}{s^2(s+1)(s+3)} = \frac{b_2}{s^2} + \frac{b_1}{s} + \frac{a_1}{s+1} + \frac{a_2}{s+3}$$

where

$$a_1 = \frac{5(s+2)}{s^2(s+3)}\bigg|_{s=-1} = \frac{5}{2}$$

$$a_2 = \frac{5(s+2)}{s^2(s+1)}\bigg|_{s=-3} = \frac{5}{18}$$

$$b_2 = \frac{5(s+2)}{(s+1)(s+3)}\bigg|_{s=0} = \frac{10}{3}$$

$$b_1 = \frac{d}{ds}\left[\frac{5(s+2)}{(s+1)(s+3)}\right]_{s=0}$$

$$= \frac{5(s+1)(s+3) - 5(s+2)(2s+4)}{(s+1)^2(s+3)^2}\bigg|_{s=0} = -\frac{25}{9}$$

Thus,

$$F(s) = \frac{10}{3}\frac{1}{s^2} - \frac{25}{9}\frac{1}{s} + \frac{5}{2}\frac{1}{s+1} + \frac{5}{18}\frac{1}{s+3}$$

The inverse Laplace transform of $F(s)$ is

$$f(t) = \frac{10}{3}t - \frac{25}{9} + \frac{5}{2}e^{-t} + \frac{5}{18}e^{-3t} \qquad t \geq 0$$

Problem A–2–18

Find the inverse Laplace transform of

$$F(s) = \frac{s^4 + 2s^3 + 3s^2 + 4s + 5}{s(s+1)}$$

Solution Since the numerator polynomial is of higher degree than the denominator polynomial, by dividing the numerator by the denominator until the remainder is a fraction, we obtain

$$F(s) = s^2 + s + 2 + \frac{2s+5}{s(s+1)} = s^2 + s + 2 + \frac{a_1}{s} + \frac{a_2}{s+1}$$

where

$$a_1 = \frac{2s+5}{s+1}\bigg|_{s=0} = 5$$

$$a_2 = \frac{2s+5}{s}\bigg|_{s=-1} = -3$$

It follows that

$$F(s) = s^2 + s + 2 + \frac{5}{s} - \frac{3}{s+1}$$

The inverse Laplace transform of $F(s)$ is

$$f(t) = \mathscr{L}^{-1}[F(s)] = \frac{d^2}{dt^2}\delta(t) + \frac{d}{dt}\delta(t) + 2\delta(t) + 5 - 3e^{-t} \qquad t \geq 0-$$

Problem A–2–19

Obtain the inverse Laplace transform of

$$F(s) = \frac{2s^2 + 4s + 6}{s^2(s^2 + 2s + 10)} \tag{2–10}$$

Solution Since the quadratic term in the denominator involves a pair of complex-conjugate roots, we expand $F(s)$ into the following partial-fraction form:

$$F(s) = \frac{a_1}{s^2} + \frac{a_2}{s} + \frac{bs + c}{s^2 + 2s + 10}$$

The coefficient a_1 can be obtained as

$$a_1 = \left.\frac{2s^2 + 4s + 6}{s^2 + 2s + 10}\right|_{s=0} = 0.6$$

Hence, we obtain

$$\begin{aligned} F(s) &= \frac{0.6}{s^2} + \frac{a_2}{s} + \frac{bs + c}{s^2 + 2s + 10} \\ &= \frac{(a_2 + b)s^3 + (0.6 + 2a_2 + c)s^2 + (1.2 + 10a_2)s + 6}{s^2(s^2 + 2s + 10)} \end{aligned} \tag{2–11}$$

By equating corresponding coefficients in the numerators of Equations (2–10) and (2–11), respectively, we obtain

$$\begin{aligned} a_2 + b &= 0 \\ 0.6 + 2a_2 + c &= 2 \\ 1.2 + 10a_2 &= 4 \end{aligned}$$

from which we get

$$a_2 = 0.28, \qquad b = -0.28, \qquad c = 0.84$$

Hence,

$$\begin{aligned} F(s) &= \frac{0.6}{s^2} + \frac{0.28}{s} + \frac{-0.28s + 0.84}{s^2 + 2s + 10} \\ &= \frac{0.6}{s^2} + \frac{0.28}{s} + \frac{-0.28(s + 1) + (1.12/3) \times 3}{(s + 1)^2 + 3^2} \end{aligned}$$

The inverse Laplace transform of $F(s)$ gives

$$f(t) = 0.6t + 0.28 - 0.28e^{-t}\cos 3t + \frac{1.12}{3}e^{-t}\sin 3t$$

Problem A–2–20

Derive the inverse Laplace transform of

$$F(s) = \frac{1}{s(s^2 + \omega^2)}$$

Solution

$$\begin{aligned} F(s) &= \frac{1}{s(s^2 + \omega^2)} = \frac{1}{\omega^2}\left(\frac{1}{s} - \frac{s}{s^2 + \omega^2}\right) \\ &= \frac{1}{\omega^2}\frac{1}{s} - \frac{1}{\omega^2}\frac{s}{s^2 + \omega^2} \end{aligned}$$

Thus, the inverse Laplace transform of $F(s)$ is obtained as

$$f(t) = \mathscr{L}^{-1}[F(s)] = \frac{1}{\omega^2}(1 - \cos \omega t) \qquad t \geq 0$$

Problem A–2–21

Obtain the solution of the differential equation

$$\dot{x} + ax = A \sin \omega t, \qquad x(0) = b$$

Solution Laplace transforming both sides of this differential equation, we have

$$[sX(s) - x(0)] + aX(s) = A\frac{\omega}{s^2 + \omega^2}$$

or

$$(s + a)X(s) = \frac{A\omega}{s^2 + \omega^2} + b$$

Solving this last equation for $X(s)$, we obtain

$$\begin{aligned} X(s) &= \frac{A\omega}{(s + a)(s^2 + \omega^2)} + \frac{b}{s + a} \\ &= \frac{A\omega}{a^2 + \omega^2}\left(\frac{1}{s + a} - \frac{s - a}{s^2 + \omega^2}\right) + \frac{b}{s + a} \\ &= \left(b + \frac{A\omega}{a^2 + \omega^2}\right)\frac{1}{s + a} + \frac{Aa}{a^2 + \omega^2}\frac{\omega}{s^2 + \omega^2} - \frac{A\omega}{a^2 + \omega^2}\frac{s}{s^2 + \omega^2} \end{aligned}$$

The inverse Laplace transform of $X(s)$ then gives

$$\begin{aligned} x(t) &= \mathscr{L}^{-1}[X(s)] \\ &= \left(b + \frac{A\omega}{a^2 + \omega^2}\right)e^{-at} + \frac{Aa}{a^2 + \omega^2}\sin \omega t - \frac{A\omega}{a^2 + \omega^2}\cos \omega t \qquad t \geq 0 \end{aligned}$$

PROBLEMS

Problem B–2–1

Derive the Laplace transform of the function

$$f(t) = 0 \qquad t < 0$$
$$= te^{-2t} \qquad t \geq 0$$

Problem B–2–2

Find the Laplace transforms of the following functions:

(a) $f_1(t) = 0 \qquad t < 0$
$\qquad = 3\sin(5t + 45°) \qquad t \geq 0$

(b) $f_2(t) = 0 \qquad t < 0$
$\qquad = 0.03(1 - \cos 2t) \qquad t \geq 0$

Problem B–2–3

Obtain the Laplace transform of the function defined by

$$f(t) = 0 \qquad t < 0$$
$$= t^2 e^{-at} \qquad t \geq 0$$

Problem B–2–4

Obtain the Laplace transform of the function

$$f(t) = 0 \qquad t < 0$$
$$= \cos 2\omega t \cdot \cos 3\omega t \qquad t \geq 0$$

Problem B–2–5

What is the Laplace transform of the function $f(t)$ shown in Figure 2–13?

Problem B–2–6

Obtain the Laplace transform of the pulse function $f(t)$ shown in Figure 2–14.

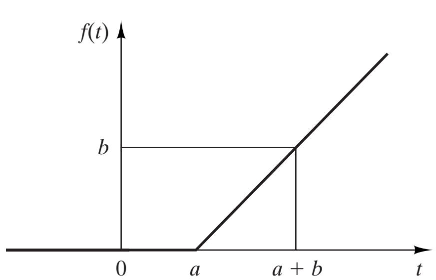

Figure 2–13 Function $f(t)$.

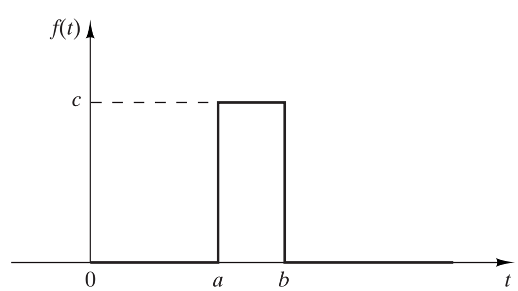

Figure 2–14 Pulse function.

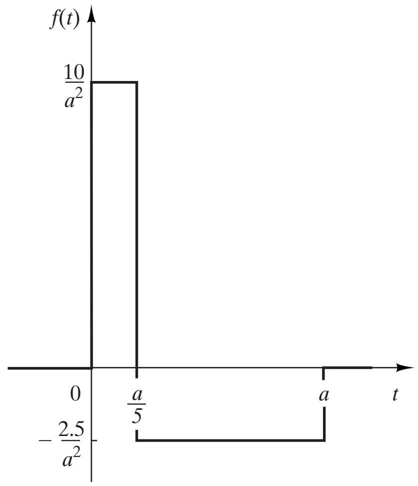

Figure 2–15 Function $f(t)$.

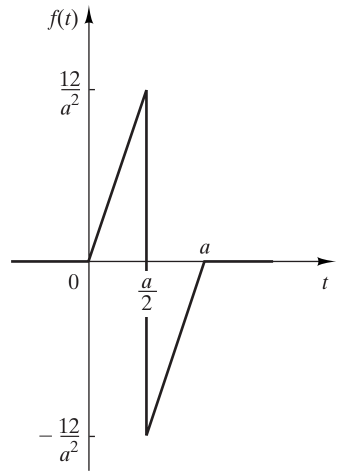

Figure 2–16 Function $f(t)$.

Problem B–2–7

What is the Laplace transform of the function $f(t)$ shown in Figure 2–15? Also, what is the limiting value of $\mathscr{L}[f(t)]$ as a approaches zero?

Problem B–2–8

Find the Laplace transform of the function $f(t)$ shown in Figure 2–16. Also, find the limiting value of $\mathscr{L}[f(t)]$ as a approaches zero.

Problem B–2–9

Given

$$F(s) = \frac{5(s+2)}{s(s+1)}$$

obtain $f(\infty)$. Use the final-value theorem.

Problem B–2–10

Given

$$F(s) = \frac{2(s+2)}{s(s+1)(s+3)}$$

obtain $f(0+)$. Use the initial-value theorem.

Problem B–2–11

Consider a function $x(t)$. Show that

$$\dot{x}(0+) = \lim_{s\to\infty}[s^2X(s) - sx(0+)]$$

Problem B–2–12

Derive the Laplace transform of the third derivative of $f(t)$.

Problem B–2–13

What are the inverse Laplace transforms of the following functions?

(a) $$F_1(s) = \frac{s+5}{(s+1)(s+3)}$$

(b) $$F_2(s) = \frac{3(s+4)}{s(s+1)(s+2)}$$

Problem B–2–14

Find the inverse Laplace transforms of the following functions:

(a) $$F_1(s) = \frac{6s+3}{s^2}$$

(b) $$F_2(s) = \frac{5s+2}{(s+1)(s+2)^2}$$

Problem B–2–15

Find the inverse Laplace transform of

$$F(s) = \frac{2s^2+4s+5}{s(s+1)}$$

Problem B–2–16

Obtain the inverse Laplace transform of

$$F(s) = \frac{s^2+2s+4}{s^2}$$

Problem B–2–17

Obtain the inverse Laplace transform of

$$F(s) = \frac{s}{s^2+2s+10}$$

Problem B–2–18

Obtain the inverse Laplace transform of

$$F(s) = \frac{s^2+2s+5}{s^2(s+1)}$$

Problem B–2–19

Obtain the inverse Laplace transform of

$$F(s) = \frac{2s+10}{(s+1)^2(s+4)}$$

Problem B–2–20

Derive the inverse Laplace transform of

$$F(s) = \frac{1}{s^2(s^2 + \omega^2)}$$

Problem B–2–21

Obtain the inverse Laplace transform of

$$F(s) = \frac{c}{s^2}(1 - e^{-as}) - \frac{b}{s}e^{-as}$$

where $a > 0$.

Problem B–2–22

Find the solution $x(t)$ of the differential equation

$$\ddot{x} + 4x = 0, \qquad x(0) = 5, \qquad \dot{x}(0) = 0$$

Problem B–2–23

Obtain the solution $x(t)$ of the differential equation

$$\ddot{x} + \omega_n^2 x = t, \qquad x(0) = 0, \qquad \dot{x}(0) = 0$$

Problem B–2–24

Determine the solution $x(t)$ of the differential equation

$$2\ddot{x} + 2\dot{x} + x = 1, \qquad x(0) = 0, \qquad \dot{x}(0) = 2$$

Problem B–2–25

Obtain the solution $x(t)$ of the differential equation

$$\ddot{x} + x = \sin 3t, \qquad x(0) = 0, \qquad \dot{x}(0) = 0$$