7. The Z-transform

7.1 Definition of the Z-transform

We saw earlier that complex exponential of the from $\{e^{jwn}\}$ is an eigen function of for a LTI System. We can generalize this for signals of the form $\{z^n\}$ where, z is a complex number.

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= \sum_{k=-0}^{\infty} h[k]z^{n-k} \\ &= \left(\sum_{k=-\infty}^{\infty} h[k]z^{-k}\right) z^n \\ &= H(z)z^n \end{aligned}$$

where

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-h} \tag{7.1}$$

Thus if the input signal is $\{z^n\}$ then output signal is $H(z)\{z^n\}$. For $z = e^{jw}$ w real (i.e for $|z| = 1$), equation (7.1) is same as the discrete-time fourier transform. The $H(z)$ in equation (7.1) is known as the bilateral z-transform of the sequence $\{h[n]\}$. We define for any sequence of a sequence $\{x[n]\}$ as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \tag{7.2}$$

where z is a complex variable. Writing z in polar form we get $z = re^{jw}$, where r is magnitude and ω is angle of z.

$$\begin{aligned} X(re^{jw}) &= \sum_{n=-\infty}^{\infty} x[n](re^{jw})^{-n} \\ &= \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-jwn} \qquad (7.3) \end{aligned}$$

This shows that $X(re^{jw})$ is Fourier transform of the sequence $\{r^{-n}x[n]\}$. When $r = 1$ the z-transform reduces to the Fourier transform of $\{x[n]\}$. From equation (7.3) we see that for convergence of z-transform that Fourier transform of the sequence $\{r^{-n}x[n]\}$ converges. This will happen for some r and not for others. The values of z - for which $\sum_{n=-\infty}^{\infty} r^{-n}|x[n]| < \infty$ is called the region of convergence(ROC). If the ROC contains unit circle (i.e. $r = 1$ or equivalently $|z| = 1$ then the Fourier transform also converges. Following examples show that we must specify ROC to completely specify the z-transform.

Example 1: Let $\{x[n]\} = \{a^n u[n]\}$, then

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ &= \sum_{n=0}^{\infty}(az^{-1})^n \end{aligned}$$

This is a geometric series and converges if $|az^{-1}| < 1$ or $|z| > |a|$. Then

$$X[z] = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, |z| > |a| \tag{7.4}$$

We see that $X(z) = 0$ at $z = 0$, and $1/X((z) = 0$ at $z = a$. Values of z where $X(z)$ is zero is called zero of $X(z)$ and value of z where $1/X(z)$ is zero is called a pole of $X(z)$. Here we see that ROC consists of a region in Z-plane which lies outside the circle centered at origin and passing through the pole. FIGURE

Example 2: Let, $\{y[n]\} = \{-a^n u[-n-1]\}$, then

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{-1} -a^n z^{-n} \\ &= \sum_{m=1}^{\infty} -a^{-m} z^m \end{aligned}$$

This is a geometric series which converges when $|a^{-1}z| < 1$, that is $|z| < |a|$ Then

$$\begin{aligned} Y(z) &= \frac{-a^{-1}z}{1-a^{-1}z} = -\frac{z}{a-z} \\ &= \frac{z}{z-a}, \qquad |z| < |a| \end{aligned} \tag{7.5}$$

FIGURE

Here the ROC is inside the circle of radius $|a|$. Comparing equation (7.4) and (7.5) we see that algebraic form of $X(z)$ and $Y(z)$ are same, but ROC are different and they correspond to two different sequences. Thus in specifying z-transform, we have to give functional form $X(z)$ and the region of convergence.
Now we state some properties of the region of convergence

7.2 Properties of the ROC

1. The ROC of $X(z)$ consists of an annular region in the z-plane, centered about the origin. This property follows from equation (7.3), where we see that convergence depends on r only.

2. The ROC does not certain any poles. Since at poles $X(z)$ does not converge.

3. The ROC is a connected region in z-plane. This property is proved in complex analysis.

4. If $\{x[n]\}$ is a right sided sequence, i.e. $x[n] = 0$, for $n < n_0$, and if the circle $|Z| = r_0$ is in the ROC, then all finite values of z, for which $|z| > r_0$ will also be in the ROC.
   For a right sided sequence

   $$X(z) = \sum_{n=n_0}^{\infty} x[n]z^{-n}$$

   If n_0 is negative then we can write

   $$X(z) = \sum_{n=n_0}^{0} x[n]z^{-n} + \sum_{n=1}^{\infty} x[n]z^{-n}$$

   Let $Z = re^{jw}$, with $r > r_0$, then, $X(z)$ exists if
   $\sum_{n=n_0}^{-1} |x[n]|r^{-n} + \sum_{n=0}^{\infty} |x[n]|r^{-n}$ is finite.
   The first summation is finite as it consists of a finite number of terms. In the second summation note that each term is less than $|x[n]|r_0^{-n}$ as $r > r_0$. Since $\sum_{n=1}^{\infty} |x[n]|r_0^{-n}$ is finite by our assumption that circle with radius r_0 lies in ROC, the second sum is also finite. Hence values of z such that $|z| > r_0$ lies in ROC, except when $z = \infty$. At $z = \infty$, the

first summation will became infinite. So if $n_0 \geq 0$, i.e. the sequence $x[n]$ is causal, the value $z = \infty$ will lie in the ROC.

5. If $\{x[n]\}$ is left sided sequence, i.e. $x[n] = 0,\ n > n_0$ and $|z| = r_0$ lies in the ROC, the values of z function $0 < |z| < r_0$ also lie in the ROC. The proof is similar to the property 4. The point $z = 0$, will lie in the ROC if the sequence is purely anticausal ($x[n] = 0,\ n > 0$)

6. If $\{x[n]\}$ is non zero for, $n_1 \leq n \leq n_2$, then ROC is entire z-plane except possibly $z = 0$, and/or $z = \infty$
   In this case the $X(z)$ consists of finite number of terms and therefore it converges if each term infinite which is the case when z is different from 0 or ∞. $z = 0$ lies in ROC, if $n_2 \leq 0$, and $z = \infty$ lies in the ROC if $n_1 \geq 0$.

7. If $\{x[n]\}$ is two-sided sequence and if circle $|z| = r_0$ is in ROC, then ROC will consist of annular region in z-plane, which includes $|z| = r_0$
   We can express a two sided sequence as sum of a right sided sequence and a left sided sequence. Then using property 4 and 5 we get this property. Using property 2 and 3 we see what ROC will be banded by circles passing through the poles.

7.3 The inverse z-transform

The inverse z-transform is given by

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz \tag{7.6}$$

the symbol $\oint$ indicates contour integration, over a counter clockwise contour in the ROC of $X(z)$. If $X(z)$ consists of ratio of polynomials one can use Cauchy integral theorem to calculate the contour integral. There are some other alternative procedures also, which will be considered after discussing the properties of z-transform.

7.4 Properties of the z-transform:

We use the notation

$$\{x[n]\} \leftrightarrow X(z), \qquad ROC = R_x$$

to denote z-transform of the sequence $\{x[n]\}$.

1. Linearity:

The z-transform of a linear combination of two sequence is given by

$$a\{x[n]\} + b\{y[n]\} \leftrightarrow aX(z) + bY(z),\ \text{ROC contains}(R_x \cap Ry)$$

The algebraic form follows directly from the definition, equation (7.2). The linear combination is such that some zero's can cancel the poles, then the region of convergence may be larger. For example if the linear combination $\{a^n u[n] - a^n u[n-N]\}$ is a finite-length sequence, the ROC is entire z-plane except at $a = 0$, like individual ROCs are $|z| > |a|$. If the intersection of R_x and R_y is null set, the z-transform of the linear combination will not exist.

2. Time shifting

If we shift the time sequence, we get
$\{x[n - n_0]\} \leftrightarrow z^{-n_0}X(z)$, $ROC = R_x$ except for possible addition or deletion of $z = 0$ and/or $z = \infty$
We have

$$Y(z) = \sum_{n=-\infty}^{\infty} x[n - n_0]z^{-n}$$

changing variable, $m = n - n_0$

$$\begin{aligned} Y(z) &= \sum_{m=-0}^{\infty} x[m]z^{-(m+n_0)} \\ &= z^{-n_0} \sum_{m=-\infty}^{\infty} x[m]z^{-m} \\ &= z^{-n_0}X(z) \end{aligned}$$

The factor z^{-n_0} can affect the poles and zeros at $z = 0$, $z = \infty$

3. Multiplication by a exponential sequence:

$$\{z_0^n x[n]\} \leftrightarrow X(z/z_0), \quad ROC = \{z : \ z/z_0 \,\epsilon\, R_x\}$$

This follows directly from defining equation (7.2).

4. Differentiation of $X(z)$:

If we differentiate $X(z)$ term by term we get

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} x[n](-n)z^{-n-1}$$

Thus

$$-z\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} nx[n]z^{-n}$$

$\{nx[n]\} \leftrightarrow -z\frac{dX(z)}{dz},\ ROC = R_x,$ except possibly $z = 0,\ z = \infty$
The ROC does not change (except $z = 0,\ z = \infty$). This follows from the property that $X(z)$ is an analytic function.

5. Conjugation of a complex sequence

$$\{x^*[n]\} \leftrightarrow X^*(z^*),\ ROC = R_x$$

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} x^*[n]z^{-n} \\ &= \left(\sum_{n=-\infty}^{\infty} x[n](z^*)^{-n}\right)^* \\ &= X^*(z^*) \end{aligned}$$

Since ROC depends only an magnitude $|z|$ it does not change.

6. Time Reversal:

$$\{x[-n]\} \leftrightarrow X(1/z)$$

$$ROC = \{z : \frac{1}{z}\ \epsilon R_X\}$$

We have

$$Y(z) = \sum_{n=-\infty}^{\infty} x[-n]z^{-n}$$

putting $m = -n$

$$\begin{aligned} y(z) &= \sum_{m=-\infty}^{\infty} x[m]z^{m} \\ &= X(1/z) \end{aligned}$$

If we combine it with the previous property, we get

$$\{x^*[-n]\} \leftrightarrow X^*(1/z^*),\ ROC = \{z : \frac{1}{z}\epsilon R_x\}$$

7. Convolution of sequence

$$\{x[n]\} * \{y[n]\} \leftrightarrow X(z)Y(z), \text{ ROC contains } R_x \cap R_y$$

The z-transform of the convolution is

$$\sum_{n=-\infty}^{\infty} (\sum_{k=-\infty}^{\infty} x[k]y[n-k])z^{-n}$$

Interchanging the order of summation

$$= \sum_{k=-\infty}^{\infty} x[k] \sum_{n=-\infty}^{\infty} y[n-k]z^{-n}$$

using time shifting property (or changing index of summation)

$$\begin{aligned} &= \sum_{k=-\infty}^{\infty} x[k]z^{-k}Y(z) \\ &= X(z)Y(z) \end{aligned}$$

If there is pole-zero cancelation, the ROC will be larger than the common ROC of two sequence.
Convolution property plays an important role in analysis of LTI system. An LTI system, which produces a delay of n_0, has the transfer function z^{-n_0}, therefore delay of n_o units is often depicted by z^{-n_0}
FIGURE

8. Complex convolution theorem:

If we multiply two sequences then

$$\{x[n]y[n]\} \leftrightarrow \frac{1}{2\pi j} \oint X(v)Y(z/v)v^{-1}dv, \text{ ROC contains } \{zw, z\epsilon R_x, w\epsilon R_y\}$$

This can be proved using inverse z-transform definition.

9. Initial value Theorem:

If $x[n]$ is zero for $n < 0$, i.e. $x[n]$ is causal, then

$$x[0] = \lim_{z \to \infty} X(z)$$

Taking limit term by term in $X(z)$, we get the above result.

10. Parseval's relation:

$$\sum_{n=-\infty}^{\infty} x[n]y^*[n] \leftrightarrow \frac{1}{2\pi j}\oint X(v)Y^*(1/v^*)v^{-1}dv$$

These properties are summarized in take 7.1

Table 7.1 z-transform properties

Sequence	Transform	ROC
1. $\{x[n]\}$	$X(z)$	R_x
2. $a\{x[n]\}+b\{y[n]\}$	$aX(z)+bY(z)$	contains $R_x \cap R_y$
3. $\{x[n-n_0]\}$	$z^{-n_o}X(z)$	R_x, except change at $z=o, z=\infty$
4. $\{z_0^n x[n]\}$	$X(z/z_0)$	$\{z/z_0 \epsilon R_x\}$
5. $\{nx[n]\}$	$-z\frac{dX(z)}{dz}$	R_x, except change at $-z=0, z=\infty$
6. $\{x^*[n]\}$	$X^*(z^*)$	R_x
7. $\{x[-n]\}$	$X(1/z)$	$\{1/Z\epsilon R_x\}$
8. $\{x[n]\}*\{y[n]\}$	$X(z)Y(z)$	Contains $R_x \cap R_y$
9. $\{x[n]y[n]\}$	$\frac{1}{2\pi j}\oint X(v)Y(z/v)u_{dv}^{-1}$	Contains R_xR_y

1. Methods of inverse z-transform

We can use the contour integration and the equation (7.6) to calculate inverse z-transform. This equation has to be evaluated for all values of n, which can be quite complicated in many cases. Here we give two simple methods for the inverse transform computation.

1. Inverse transform by partial fraction expansion:

This is method is useful when z-transform is ratio of polynomials. A rational $X(z)$ can be expressed as

$$X(z) = \frac{N(z)}{D(z)}$$

where $N(z)$ and $D(z)$ are polynomials in z^{-1}. If degree M of the numerator polynomial $N(z)$ is greater than or equal to the degree N of the denominator polynomial $D(z)$, we can divide $N(z)$ by $D(z)$ and re-express $X(z)$ as

$$X(z) = \sum_{k=0}^{M-N} a[k]z^{-k} + \frac{N_1(z)}{D(z)}$$

where the degree of polynomial $N_1(z)$ is strictly less than that of $D(z)$. For simplicity let us assume that all poles are simple. Then

$$X(z) = \sum_{k=0}^{M-N} B_k z^{-k} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k Z^{-l}}$$

where $A_k = (1 - d_k Z^{-1})\frac{N_1(z)}{D(z)}|_{Z=d_k}$
Example: Let

$$X(z) = \frac{1 + 2z^{-1}}{(1 - \cdot 2\, z^{-1})(1 + \cdot 6z^{-1})}$$

The partial fraction expression is

$$\begin{aligned} X(z) &= \frac{A_1}{1 - \cdot 2\, z^{-1}} + \frac{A_2}{(1 + \cdot 6\, z^{-1})} \\ A_1 &= (1 - \cdot 2\, z^{-1})X(z)|_{Z=\cdot 2} = \frac{1 + 2z^{-1}}{1 + \cdot 6\, z^{-1}}|_{z=\cdot 2} = 2.75 \\ A_2 &= (1 - \cdot 6\, z^{-1})X(z)|_{Z=-\cdot 6} = \frac{1 + 2z^{-1}}{1 - \cdot 2z^{-1}}|_{z=\cdot -6} = -1.75 \\ X(z) &= \frac{2.75}{1 - \cdot 2z^{-1}} - \frac{1.75}{1 + \cdot 6z^{-1}} \end{aligned}$$

The inverse z-transform depends on the ROC. If ROC is $|z| > \cdot 6$, then ROCs associated with each term is outside a circle(so that common ROC is outside a circle), sequences are causal. Using linearity property and z-transform of $a^n u[n]$ we get

$$x[n] = 2.75(0.2)^n u[n] - 1.75(-.6)^n u[n]$$

If the ROC is $.2 < |z| < .6$, the ROC of the term $\frac{1}{1 - .2z^{-1}}$ should be outside the circle $|z| = .2$, and ROC for $\frac{1}{1 + .6z^{-1}}$ should be $|z| < .6$. Hence we get the sequence as

$$x[n] = 2.75(.2)^n u[n] + 1.75(-.6)^n u[-n-1]$$

Similarly if ROC is $|Z| < .2$ we get a noncausal sequence

$$x[n] = -2.75(.2)^n u[-n-1] + 1.75(-.6)^n u[-n-1]$$

If $X(z)$ has multiple poles, the partial fraction has slightly different form. If $X(z)$ has a pole of order s at $z = d_i$, and all other poles are simple Then

$$X(z) = \sum_{k=0}^{M-N} B_k z^{-k} + \sum_{k=1, k\neq i}^{N} \frac{A_k}{1 - d_k Z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1 - d_i z^{-1})^m}$$

where A_k and B_k are obtained as before, the coefficients C_m are given by

$$C_m = \frac{1}{(s-m)!(-d_i)^{s-m}} \left\{ \frac{d^{s-m}}{dw^{s-m}} (1-d_i w)^s X(w^{-1}) \right\}_{w=d_i^{-1}}$$

If there are more multiple poles, there will be more terms like the third term. Using linearity and differentiation properties we get some useful z-transform pairs given in Table 7.2

Table 7.2 Some useful z-transform pairs

Sequence	Transform	ROC
1. $\{\delta[n]\}$	1	All z
2. $\{\delta[n-m]\}$	z^{-m}	All z, except 0(if $m>0$) or ∞(if $m<0$)
3. $\{a^n u[n]\}$	$\frac{1}{1-az-1}$	$\|z\| > \|a\|$
4. $\{-a^n u[-n-1]\}$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
5. $\{na^n u[n]\}$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
6. $\{-na^n u[-n-1]\}$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
7. $\{r^n \cos w_0 n\, u[n]\}$	$\frac{1-r\cos w_0 z^{-1}}{1-2r\,\cos w_0 z^{-1}+r^2 z^{-2}}$	$\|z\| > r$
8. $\{r^n \sin w_0 n\, u[n]\}$	$\frac{\sin w_0\, z^{-1}}{1-2r\cos w_0 z^{-1}+r^2 z^{-2}}$	$\|z\| > r$
9. $\{a^n,\ 0 \le n \le N-1\}$	$\frac{1-a^N z^{-N}}{1-a\,z^{-1}}$	$\|z\| > 0$

2. Inverse Transform via long division:

For causal sequence the z-transform $X(z)$ can be exported into a pure series in z^{-1}. In the series expansion, the coefficient multiplying the term z^{-n} is $x[n]$. If $X(z)$ is anticausal then we expand in terms of poles of z.

Example : Let

$$X(z) = \frac{1+2z^{-1}}{(1-.2z^{-1})(1+.6z^{-1})}, \quad ROC\ |z| > \ .6$$

This is a causal sequence, long division gives
LONG DIVISION EQUATION(to be done as image)

This gives $x[0]=1,\ x[1]=1.6,\ x[2]=-.52,\ x[3]=.4$,.....
We can see that it is not easy to write the n^{th} term.

Example 2:

$$X(z) = ln(1+az^{-1}),\ |z|\ >\ |a|$$

Using the pure series expansion for $ln(1+x)$ with $|x| < |$, we obtain

$$
\begin{aligned}
X(z) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n} \\
x[n] &= \begin{cases} (-1)^{n+1} \dfrac{a^n}{n}, & n \geq 1 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

Analysis of LTI system using z-transform:

From the convolution property we have

$$Y(z) = H(z)\,X(z)$$

where $X(z)$, $Y(z)$are $H(z)$ are z-transforms of input sequence $\{x[n]\}$, output sequence $\{y[n]\}$ and impulse response $\{h[n]\}$ respectively. The $H(z)$ is referred to as system function or transfer function of the system. For z on the unit circle ($z = e^{jw}$), $H(z)$ reduces to the frequency response of the system, provided that unit circle is in the ROC for $H(z)$.
A causal LTI system has impulse response $\{h[n]\}$ such that $h[n] = 0$, $n < 0$. Thus ROC of $H(z)$ is exterior of a circle in z-plane including $z = \infty$. Thus a discrete time LTI system is causal if and only if ROC is exterior of a circle which includes infinity.
An LTI system is stable if and only if impulse response $\{h[n]\}$ is absolutely summable. This is equivalent to saying that unit circle is in the ROC of $H(z)$.
For a causal and stable system ROC is outside a circle and ROC contains the unit circle. That means all the poles are inside the unit circle. Thus a causal LTI system is stable if and if only if all the poles inside unit circle.

LTI systems characterized by Linear constant coefficient difference equation:

For the system characterized by

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

We take the z-transform of both sides and use linearity and the time shift property to get

$$\begin{aligned}\sum_{k=0}^{N} a_k z^{-k} Y(z) &= \sum_{k=0}^{M} b_k z^{-k} X(z) \\ H(z) = \frac{Y(z)}{X(z)} &= \frac{\sum\limits_{k=0}^{M} b_k z^{-k}}{\sum\limits_{k=0}^{N} a_k z^{-k}}\end{aligned}$$

Thus the system function is always a rational function. We can write it by inspection. Numerator polynomial coefficients are the coefficients of $x[n-k]$ and denominator coefficients are coefficients of $y[n-k]$. The difference equation by itself does not provide information about the ROC, it can be determined by conditions like causality and stability

System Function and block diagram representation:

The use of z-transform allows us to replace time domain operation such as convolution time shifting etc with algebraic operations.
Consider the parallel interconnection if two system, as
FIGURE 7.1

shown in figure 7.1. The impulse response of the over all system is

$$\{h[n]\} = \{h_1[n]\} + \{h_2[n]\}$$

From linearity of the z-transform,

$$H(z) = H_1(z) + H_2(z)$$

Similarly, the impulse response of the series connection in figure 7.2 is

$$\{h[n]\} = \{h_1[n]\} * \{h_2[n]\}$$

FIGURE 7.2

From the convolution property.

$$H(z) = H_1(z)H_2(z)$$

The z-transform of the interconnection of linear system can be obtained by algebraic means. For example consider the feed back connection in figure 7.3

FIGURE 7.3
We have

$$\begin{aligned}
Y(z) &= H_1(z)E(z) \\
E(z) &= X(z) - Z(z) \\
&= X(z) - Y(z)H_2(z)
\end{aligned}$$

or

$$\begin{aligned}
Y(z) &= H_1(z)[X(z) - Y(z)H_2(z)] \\
\frac{Y(z)}{X(z)} &= H(z) = \frac{H_1(z)}{1 + H_1(z)H_2(z)}
\end{aligned}$$