

Introduction to Laplace Transforms for Engineers

C.T.J. Dodson, School of Mathematics, Manchester University

1 What are Laplace Transforms, and Why?

This is much easier to state than to motivate! We state the definition in two ways, first in words to explain it intuitively, then in symbols so that we can calculate transforms.

Definition 1

Given f , a function of time, with value $f(t)$ at time t , the Laplace transform of f is denoted \tilde{f} and it gives an average value of f taken over all positive values of t such that the value $\tilde{f}(s)$ represents an average of f taken over all possible time intervals of length s .

Definition 2

$$\mathcal{L}[f(t)] = \tilde{f}(s) = \int_0^{\infty} e^{-st} f(t) dt, \quad \text{for } s > 0. \quad (1.1)$$

A short table of commonly encountered Laplace Transforms is given in Section 7.5. Note that this definition involves integration of a product so it will involve frequent use of integration by parts—see Appendix Section 7.1 for a reminder of the formula and of the definition of an infinite integral like (1.1).

This immediately raises the question of why to use such a procedure. In fact the reason is strongly motivated by real engineering problems. There, typically we encounter models for the dynamics of phenomena which depend on rates of change of functions, eg velocities and accelerations of particles or points on rigid bodies, which prompts the use of ordinary differential equations (ODEs). We can use ordinary calculus to solve ODEs, provided that the functions are nicely behaved—which means continuous and with continuous derivatives. Unfortunately, there is much interest in engineering dynamical problems involving functions that input step change or spike impulses to systems—playing pool is one example. Now, there is an easy way to smooth out discontinuities in functions of time: simply take an average value over all time. But an ordinary average will replace the function by a constant, so we use a kind of moving average which takes continuous averages over all possible intervals of t . This very neatly deals with the discontinuities by encoding them as a smooth function of interval length s .

The amazing thing about using Laplace Transforms is that we can convert a whole ODE initial value problem into a Laplace transformed version as functions of s ,

simplify the algebra, find the transformed solution $\tilde{f}(s)$, then undo the transform to get back to the required solution f as a function of t .

Interestingly, it turns out that the transform of a derivative of a function is a simple combination of the transform of the function and its initial value. So a calculus problem is converted into an algebraic problem involving polynomial functions, which is easier.

There is one further point of great importance: calculus operations of differentiation and integration are linear. So the Laplace Transform of a sum of functions is the sum of their Laplace Transforms and multiplication of a function by a constant can be done before or after taking its transform.

In this course we find some Laplace Transforms from first principles, ie from the definition (1.1), describe some theorems that help finding more transforms, then use Laplace Transforms to solve problems involving ODEs.

2 Finding Laplace Transforms

We have three methods to find $\tilde{f}(s)$ for a given $f(t)$;

From the definition: Here we use (1.1) directly: eg

$$\text{For } f(t) = 1, \quad \mathcal{L}[1] = \int_0^{\infty} e^{-st} dt = \left[-\frac{1}{s} e^{-st} \right]_0^{\infty} = \frac{1}{s}.$$

$$\text{For } f(t) = t, \quad \mathcal{L}[t] = \int_0^{\infty} e^{-st} t dt = \left[-\frac{1}{s} e^{-st} t \right]_0^{\infty} + \int_0^{\infty} \frac{1}{s} e^{-st} = \frac{1}{s^2}.$$

$$\text{For } f(t) = \frac{dy}{dt}, \quad \mathcal{L}\left[\frac{dy}{dt}\right] = \int_0^{\infty} e^{-st} \frac{dy}{dt} dt = \left[e^{-st} y \right]_0^{\infty} + \int_0^{\infty} s e^{-st} y dt = -y(0) + s\tilde{y}(s).$$

For $f(t) = e^{at}$, a constant,

$$\mathcal{L}[e^{at}] = \int_0^{\infty} e^{-st} e^{at} dt = \int_0^{\infty} e^{-(s-a)t} dt = \left[-\frac{1}{s-a} e^{-(s-a)t} \right]_0^{\infty} = \frac{1}{s-a}, \quad s > a.$$

From a property: There are a number of powerful theorems about the properties of transforms: eg $\mathcal{L}[af + bg] = a\mathcal{L}[f] + b\mathcal{L}[g]$

$$\mathcal{L}[3t + 4] = 3\frac{1}{s^2} + 4\frac{1}{s}.$$

$$\mathcal{L}[\cos at + i \sin at] = \mathcal{L}[e^{iat}] \quad \text{by DeMoivre.}$$

$$\mathcal{L}[e^{iat}] = \frac{1}{s - ia} = \frac{s}{s^2 + a^2} + \frac{ia}{s^2 + a^2}.$$

Hence, equating real and imaginary parts and using linearity

$$\mathcal{L}[\cos at] = \frac{s}{s^2 + a^2}$$

$$\mathcal{L}[\sin at] = \frac{a}{s^2 + a^2}.$$

We can apply the convolution property from the table to find

$$\mathcal{L}^{-1} \left[\frac{f(s)}{s} \right].$$

$$\mathcal{L}^{-1}[f(s)] = f(t), \quad \text{and } \mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1 = g(t),$$

so

$$\mathcal{L}^{-1} \left[\frac{f(s)}{s} \right] = \int_0^t f(\theta) d\theta.$$

From a list: Computer algebra packages like Mathematica, Matlab and Maple know Laplace Transforms of all the functions you are likely to encounter, so you have access to these online, and the packages have also an inversion routine to find a function f from a given \tilde{f} . There are books with long lists of transforms of known functions and compositions of functions; we give some in Section 7.5, which you should read through, eg some that are harder to calculate:

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}, \quad n = 0, 1, 2, \dots, \quad \mathcal{L}[t^{1/2}] = \frac{1}{2} \left(\frac{\pi}{s^3} \right)^{1/2}, \quad \mathcal{L}[t^{-1/2}] = \left(\frac{\pi}{s} \right)^{1/2}.$$

2.1 Exercises

1. Use the definition to prove the Shift Theorem:

$$\mathcal{L}[e^{at} f(t)] = \tilde{f}(s - a).$$

2. Deduce that

$$\mathcal{L}[e^{2t} \cos 4t] = \frac{s - 2}{s^2 - 4s + 20}.$$

3. Check that

$$\mathcal{L}[e^{-3t} t^3] = \frac{6}{(s + 8)^4}.$$

3 Finding inverse transforms using partial fractions

Given a function f , of t , we denote its Laplace Transform by $\mathcal{L}[f] = \tilde{f}$; the inverse process is written:

$$\mathcal{L}^{-1}[\tilde{f}] = f.$$

A common situation is when $\tilde{f}(s)$ is a polynomial in s , or more generally, a ratio of polynomials; then we use partial fractions to simplify the expressions. Given an expression for a Laplace transform of the form N/D where numerator N and denominator D are both polynomials of s , possibly in the form of factors, and N may be constant; use partial fractions:

- (i) if N has degree equal to or higher than D , divide N by D until the remainder is of lower degree than D
- (ii) For every linear factor like $(as + b)$ in D , write a partial fraction of the form $A/(as + b)$
- (iii) For every repeated factor like $(as + b)^2$ in D write two partial fractions of the form $A/(as + b)$ and $B/(as + b)^2$. Similarly for every repeated factor like $(as + b)^3$ in D write three partial fractions of the form $A/(as + b)$, $B/(as + b)^2$ and $C/(as + b)^3$; and so on.
- (iv) For quadratic factor $(as^2 + bs + c)$ write a partial fraction $(As + B)/(as^2 + bs + c)$.

For repeated quadratic factors write a series of partial fractions as in (iii), but with numerators of the form $(As + B)$ and successive powers of the quadratic factor as the denominators.

With a little more algebra you should in this way be able to write the original expression as a sum of simpler transforms, which are found in your table. You then add their inverse transforms together, to get the inverse of the original transform.

3.1 Exercises

1. Show that

$$\mathcal{L}^{-1} \left[\frac{1}{2s + 3} \right] = \frac{1}{2} e^{-3t/2}.$$

2. Given

$$\tilde{f}(s) = \frac{1 + s}{(s + 3)(s - 2)}$$

show that

$$\tilde{f}(s) = \frac{\frac{2}{5}}{s + 3} + \frac{\frac{3}{5}}{s - 2}.$$

Deduce that

$$f(t) = \frac{2}{5} e^{-3t} + \frac{3}{5} e^{2t}.$$

3. Given

$$\tilde{f}(s) = \frac{1 + s}{s^2 + s + 1}$$

complete the square in the denominator to obtain

$$\frac{1 + s}{s^2 + s + 1} = \frac{1 + s}{(s + \frac{1}{2})^2 + \frac{3}{4}} = \frac{(s + \frac{1}{2}) + \frac{1}{2}}{(s + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}.$$

Use the Shift Theorem and the table of transforms to deduce

$$\mathcal{L}^{-1} \left[\frac{(s + \frac{1}{2}) + \frac{1}{2}}{(s + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} \right] = e^{-t/2} \cos \frac{\sqrt{3}}{2} t + \frac{1}{\sqrt{3}} e^{-t/2} \sin \frac{\sqrt{3}}{2} t.$$

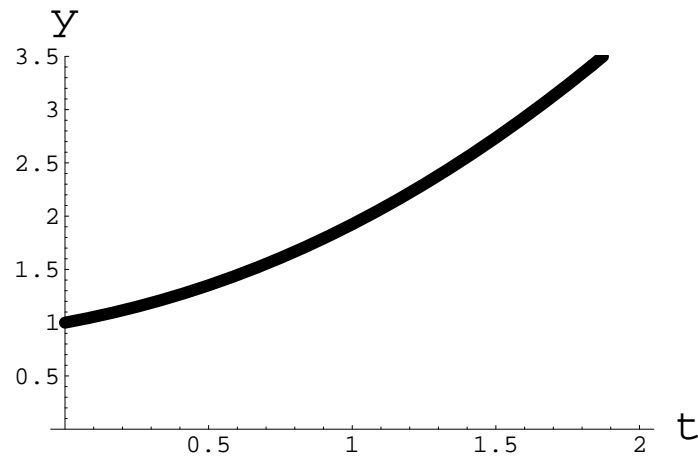


Figure 1: Solution of the ODE problem $2y' - y = \sin t$, $y(0) = 1$ in 4.2.

Note that this last expression can be simplified to

$$\frac{2\sqrt{3}}{2}e^{-t/2} \cos\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{6}\right)$$

by using the phase relationship between the sine and cosine functions.

4 Solving ODEs and ODE Systems

The application of Laplace Transform methods is particularly effective for *linear ODEs with constant coefficients*, and for *systems* of such ODEs. To transform an ODE, we need the appropriate initial values of the function involved and initial values of its derivatives. We illustrate the methods with the following programmed Exercises.

4.1 Exercises

1. For the ODE problem

$$2\frac{dy}{dt} - y = \sin t, \quad y(0) = 1. \quad (4.2)$$

- (a) obtain the transformed version as

$$2(s\tilde{y} - 1) - \tilde{y} = \frac{1}{s^2 + 1}.$$

- (b) Rearrange to get

$$\tilde{y}(s) = \frac{2s^2 + 3}{(2s - 1)(s^2 + 1)} = \frac{A}{2s - 1} + \frac{Bs + C}{s^2 + 1}$$

- (c) Show that $A = \frac{14}{5}$, $B = \frac{-2}{5}$, $C = \frac{-1}{5}$, and take the inverse transform to obtain the final solution to (4.2) as

$$y(t) = \frac{7}{5}e^{t/2} - \frac{2}{5}\cos t - \frac{1}{5}\sin t.$$

2. For the system of ODEs

$$\frac{dy}{dt} - \frac{dx}{dt} + y + 2x = e^t \quad (4.3)$$

$$\frac{dy}{dt} + \frac{dx}{dt} - x = e^{2t} \quad (4.4)$$

$$\text{Initial data: } x(0), y(0) = 1, \quad (4.5)$$

- (a) transform to obtain

$$(s\tilde{y} - y_0) - (s\tilde{x} - x_0) + \tilde{y} + 2\tilde{x} = \frac{1}{s-1} \quad (4.6)$$

$$(s\tilde{y} - y_0) + (s\tilde{x} - x_0) - \tilde{x} = \frac{1}{s-2}. \quad (4.7)$$

- (b) Rearranging,

$$(s+1)\tilde{y} - (s-2)\tilde{x} = \frac{1}{s-1} + 1 - 1 = \frac{1}{s-1} \quad (4.8)$$

$$s\tilde{y} + (s-1)\tilde{x} = \frac{1}{s-2} + 1 + 1 = \frac{2s-3}{s-2}. \quad (4.9)$$

- (c) To eliminate \tilde{y} , multiply (4.8) by s and (4.9) by $(s+1)$ then subtract, and deduce as follows

$$((s-1)(s+1) + s(s-2))\tilde{x} = \frac{(2s-3)(s+1)}{s-2} - \frac{s}{s-1}, \quad (4.10)$$

$$\tilde{x}(s) = \frac{2s^3 - 4s^2 + 3}{(s-1)(s-2)(2s^2 - 2s - 1)} \quad (4.11)$$

Then, by partial fractions,

$$\tilde{x}(s) = \frac{1}{s-1} + \frac{1}{s-2} - \frac{s-\frac{1}{2}}{(s-\frac{1}{2})^2 - (\frac{\sqrt{3}}{2})^2} - \frac{1}{\sqrt{3}} \frac{\frac{\sqrt{3}}{2}}{(s-\frac{1}{2})^2 - (\frac{\sqrt{3}}{2})^2}. \quad (4.12)$$

- (d) From the table of transforms, we can find $x(t)$ as

$$x(t) = e^t + e^{2t} - e^{t/2} \cosh\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}}e^{t/2} \sinh\left(\frac{\sqrt{3}}{2}t\right).$$

- (e) You can find $y(t)$ by differentiating and substituting $\frac{dx}{dt}$ in either of the system equations. Quicker here is to subtract the second equation from the first to obtain

$$-2\frac{dx}{dt} + y + x + 2x = e^t - e^{2t}$$

so

$$y(t) = 2\frac{dx}{dt} - 3x + e^t - e^{2t}.$$

The trajectory curve of the ODE system 4.3-4.5 in x, y -space, with time increasing along the curve from left to right, is shown in Figure 2.

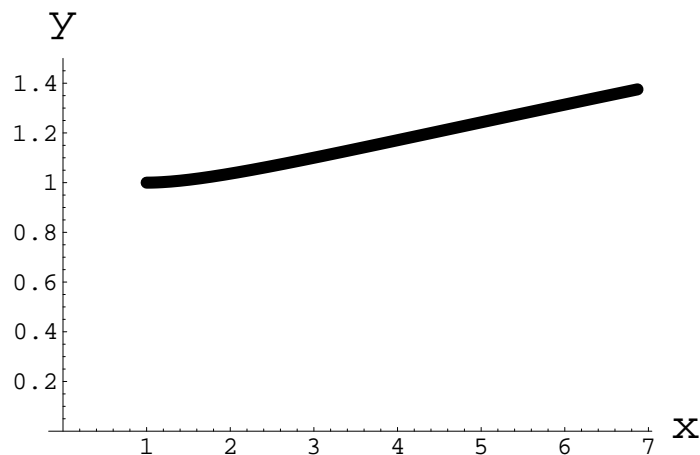


Figure 2: *Solution of the ODE system 4.3 to 4.5; here we plot the trajectory of the system in x, y -space, with parameter t representing time increasing along the curve from left to right.*

5 Impulse problems

Laplace transform methods are particularly valuable in handling differential equations involving impulse and step functions. The problem in the Exercise below represents the dynamics of a point, initially at rest, moving away from the origin along the y -axis under a constant acceleration of value 10 for $0 \leq t < 1$ and an extra impulse acceleration of size 10 is applied at $t = 1$. This is like a simple rocket boost, but can you solve it any other way? We use the Dirac impulse function $\delta(t - a)$ which is nonzero at $t = a$, but zero elsewhere while having unit total area under it:

$$\delta(t - a) = 0 \text{ if } (t \neq a) \text{ and } \int_{-\infty}^{\infty} \delta(t - a) dt = 1. \quad (5.13)$$

5.1 Exercises

Consider the ODE initial value problem given by

$$y'' = 10 + 10\delta(t - 1), \quad y(0) = y'(0) = 0. \quad (5.14)$$

1. Begin by sketching the graph of the acceleration, y'' , to show the step increase.
2. Transforming according to the table, to get

$$s^2 \tilde{y} - sy(0) - y'(0) = \frac{10}{s} + 10e^{-s}$$

so, rearranging $\tilde{y}(s) = \frac{10}{s^3} + \frac{10e^{-s}}{s^2} = 5\frac{2}{s^3} + 10e^{-s}\frac{1}{s^2}$

3. From the table use the Delay property to deduce that

$$y(t) = 5t^2 + 10(t - 1)H(t - 1)$$

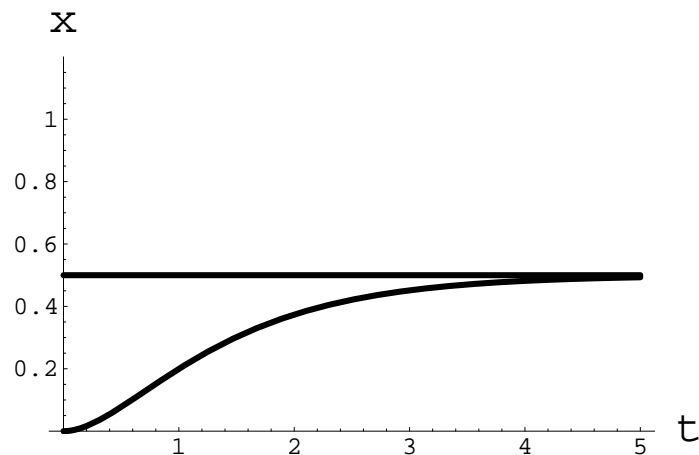


Figure 3: Solution of $\ddot{x} + 3\dot{x} + 2x = H(t)$, $x(0) = \dot{x}(0) = 0$ in (6.15); $x(t)$ is asymptotic to $x = \frac{1}{2}$, why is that obvious?

4. By interpreting the step function $H(t-1)$ up to and after $t = 1$, show that the impulse at $t = 1$ produces what you would expect: a discontinuity in **velocity** at $t = 1$. Sketch the full solution:

$$y(t) = 5t^2 \quad \text{for } t \leq 1 \quad (\text{so here } y'(t) = 10t)$$

$$y(t) = 5t^2 + 10(t-1) \quad \text{for } t > 1 \quad (\text{so here } y'(t) = 10t + 10).$$

6 Step Input problems

Here we consider the following initial value problem which involves a step input function—typical of many control-type problems:

$$\ddot{x} + 3\dot{x} + 2x = H(t), \quad x(0) = \dot{x}(0) = 0. \quad (6.15)$$

Here we have a second order ODE representing a system that is at rest until time $t = 0$, when a unit step input $H(t)$ is applied; we seek the output $x(t)$.

6.1 Exercises

1. Transform (6.15) to get

$$(s^2 + 3s + 2)\tilde{x}(s) = \frac{1}{s},$$

$$\tilde{x}(s) = \frac{1}{s} \left(\frac{1}{(s+2)(s+1)} \right) = \frac{1}{s} \left(\frac{-1}{(s+2)} + \frac{1}{(s+1)} \right).$$

2. Apply the Integration property to obtain

$$x(t) = \int_0^t (-e^{-2\theta} + e^{-\theta}) d\theta \quad (6.16)$$

$$= \left[\frac{1}{2}e^{-2\theta} - e^{-\theta} \right]_0^t \quad (6.17)$$

$$= \frac{1}{2}e^{-2t} - e^{-t} - \frac{1}{2} + 1 \quad (6.18)$$

$$= \frac{1}{2}e^{-2t} - e^{-t} + \frac{1}{2}. \quad (6.19)$$

3. Show that the solution $x(t)$ is asymptotic to $x = \frac{1}{2}$, cf Figure 3; why is this obvious as a steady state solution?

7 Appendix

7.1 Integration by Parts and Infinite Integrals

Recall the product rule for differentiation, when $y = uv$:

$$\frac{d(uv)}{dt} = v \frac{du}{dt} + u \frac{dv}{dt}$$

Suppose we integrate both sides with respect to t :

$$\int \frac{d(uv)}{dt} dt = \int v \frac{du}{dt} dt + \int u \frac{dv}{dt} dt$$

On the left, we are integrating the derivative, so we get back to the original function, which is uv . Rearranging, this becomes:

$$\int u \frac{dv}{dt} dt = uv - \int v \frac{du}{dt} dt.$$

Now this gives us a useful formula for integrating products: we let one of the functions be u and let the other be $\frac{dv}{dt}$, next work out v and $\frac{du}{dt}$, then use the formula.

7.2 Infinite Integrals

The integral of a function over an infinite interval is the limit of the integral over a finite interval as the bound on the interval tends to infinity. In symbols:

$$\int_0^{\infty} f(t) dt = \lim_{k \rightarrow \infty} \int_0^k f(t) dt,$$

So, first do the finite form of the integral then find the limiting value as we let k tend to ∞ .

7.3 Scientific Wordprocessing with L^AT_EX

This pdf document with its hyperlinks was created using L^AT_EX which is the standard (free) mathematical wordprocessing package; more information can be found via the webpage:

<http://www.ma.umist.ac.uk/kd/latexut/pdfbyex.htm>

7.4 Computer Algebra Methods

The computer algebra package Mathematica can be used to find and invert Laplace Transforms, it was used to produce the graphics of functions in these notes. See:

<http://www.ma.umist.ac.uk/kd/mmaprogs/AREADMEFILE>

for beginning Mathematica. Similarly, Maple and Matlab also can be used for working with Laplace Transforms and for creating graphics.

7.5 Frequently used Laplace Transforms

Function $f(t)$	Transform $\tilde{f}(s) = \int_0^\infty e^{-st} f(t) dt$
1	$1/s$
t^n , for $n = 0, 1, 2, \dots$	$n!/s^{n+1}$
$t^{1/2}$	$\frac{1}{2}(\pi/s^3)^{1/2}$
$t^{-1/2}$	$(\frac{\pi}{s})^{1/2}$
e^{at}	$1/(s-a)$
$\sin \omega t$	$\omega/(s^2 + \omega^2)$
$\cos \omega t$	$s/(s^2 + \omega^2)$
$t \sin \omega t$	$2\omega s/(s^2 + \omega^2)^2$
$t \cos \omega t$	$(s^2 - \omega^2)/(s^2 + \omega^2)^2$
$e^{at} t^n$	$n!/(s-a)^{n+1}$
$e^{at} \sin \omega t$	$\omega/((s-a)^2 + \omega^2)$
$e^{at} \cos \omega t$	$(s-a)/((s-a)^2 + \omega^2)$
$\sinh \omega t$	$\omega/(s^2 - \omega^2)$
$\cosh \omega t$	$s/(s^2 - \omega^2)$
Impulse (Dirac δ): $\delta(t-a)$ ($\neq 0$ at $t=a$, else $= 0$)	e^{-as}
Step function: $H_a(t)$ ($= 0$ for $t < a$ and $= 1$, $t \geq a$)	e^{-as}/s
Delay of g : $H_a(t)g(t-a)$	$e^{-as}\tilde{g}(s)$
Shift of g : $e^{at}g(t)$	$\tilde{g}(s-a)$
Convolution : $f(t) * g(t) = \int_0^t f(t-\tau)g(\tau) d\tau$	$\tilde{g}(s)\tilde{f}(s)$
Integration : $1 * g(t) = \int_0^t g(\tau) d\tau$	$\frac{1}{s}\tilde{g}(s)$
Derivative : y'	$s\tilde{y}(s) - y(0)$
y''	$s^2\tilde{y}(s) - sy(0) - y'(0)$