Chapter 6 The Laplace transform

6.1 Introduction

The Laplace transform is closely related to the Fourier transform and we shall be considering this relationship in some detail. This will allow many of the properties of the Laplace transform to be deduced from those of the Fourier transform. You should already be familiar with the use of Laplace transforms for solving initial value differential equations and for carrying out transient analysis of electrical circuits and so these applications will not be covered in detail.

6.2 The one-sided Laplace transform

Given a function $f(t)$ which is assumed to be **causal** so that $f(t) = 0$ for $t < 0$, the (one-sided) Laplace transform of $f(t)$ is

$$F_L(s) = \int_0^\infty f(t) \exp(-st)\,\mathrm{d}t \tag{6.1}$$

In this definition s is assumed to be a **complex** variable $s = \sigma + \mathrm{j}\omega$. The set of values of s for which the integral converges **absolutely** is called the **region of absolute convergence** of the Laplace transform.

Theorem If the one-sided Laplace transform of the locally integrable function $f(t)$ converges absolutely at some complex number s_0, then it converges at every point s satisfying $\Re s \geq \Re s_0$.

Proof: We make use of the fact that the improper integral of a locally integrable function $\phi(t)$,

$$\int_0^\infty \phi(t)\,\mathrm{d}t \tag{6.2}$$

converges if and only if for any given $\epsilon > 0$ there exists $M > 0$ with the property that whenever t_1 and t_2 are chosen so that $t_2 > t_1 > M$,

$$\left| \int_{t_1}^{t_2} \phi(t)\,\mathrm{d}t \right| < \epsilon \tag{6.3}$$

(This is called the Cauchy condition for the existence of the improper integral.)

Let us now consider the Laplace transform integral at s for arbitrary limits $t_2 > t_1 > 0$

$$\left| \int_{t_1}^{t_2} f(t) \exp(-st)\,\mathrm{d}t \right| \leq \int_{t_1}^{t_2} |f(t) \exp(-s_0 t) \exp[-(s - s_0)t]|\,\mathrm{d}t \tag{6.4}$$

$$= \int_{t_1}^{t_2} |f(t) \exp(-s_0 t)| \exp[-\Re(s - s_0)t]\,\mathrm{d}t \tag{6.5}$$

$$\leq \int_{t_1}^{t_2} |f(t) \exp(-s_0 t)|\mathrm{d}t \tag{6.6}$$

where the last inequality follows from $\Re s \geq \Re s_0$ and $t_2 > t_1 > 0$. Since the Laplace transform converges absolutely at s_0, the integral

$$\int_0^\infty |f(t) \exp(-s_0 t)|\mathrm{d}t \tag{6.7}$$

exists and satisfies the Cauchy condition. Thus for a given $\epsilon > 0$, there exists $M > 0$ such that choosing $t_2 > t_1 > M$ will ensure that (6.6) is less than ϵ. This shows that the Laplace transform integral at s satisfies the Cauchy condition and thus must exist. Indeed the above also shows that the convergence of the Laplace transform integral at s is absolute.

This result shows that the region of absolute convergence of a one-sided Laplace transform is either

1. An open right half-plane $\Re s > \sigma_a$
2. A closed right half-plane $\Re s \geq \sigma_a$

where σ_a is a real number (which may be $\pm\infty$) called the **abscissa of absolute convergence** of the Laplace transform.

Exercise: Fill in the details required to establish the above claim. In particular consider why it is that the Laplace transform cannot be absolutely convergent at **any** point to the left of $s = \sigma_a$ and why it is that the line $s = \sigma_a$ itself must either be entirely inside or entirely outside the region of absolute convergence.

(*Technical note:* It is possible to work with the region of simple convergence of Laplace transforms where we only require convergence rather than absolute convergence of the integral. However this complicates the theory somewhat. The region of simple convergence need not coincide with the region of absolute convergence. In fact there are functions for which the Laplace transform is convergent everywhere but absolutely convergent nowhere. For more details see for example *Introduction to the Theory and Application of the Laplace Transformation* by Gustav Doetsch, Springer-Verlag 1974.)

If we now write $s = \sigma + \mathrm{j}\omega$ in the Laplace transform integral, we see that

$$F_L(\sigma + \mathrm{j}\omega) = \int_0^\infty f(t)\exp(-\sigma t)\ \exp(-\mathrm{j}\omega t)\,\mathrm{d}t \tag{6.8}$$

If we compare this with the Fourier transform for a function $g(t)$ (using the ω form rather than the ν form)

$$G(\omega) = \int_{-\infty}^\infty g(t)\exp(-\mathrm{j}\omega t)\,\mathrm{d}t \tag{6.9}$$

it is clear that we can consider the Laplace transform of f on the vertical line $\sigma + \mathrm{j}\omega$ as a Fourier transform of the function

$$g(t) = f(t)\exp(-\sigma t) \tag{6.10}$$

With this definition,

$$G(\omega) = F_L(\sigma + \mathrm{j}\omega) \tag{6.11}$$

In the region of absolute convergence of the Laplace transform, we see that $f(t)\exp(-\sigma t)$ is absolutely integrable and so its Fourier transform is well-defined in the classical sense. The inverse Fourier transform allows us to recover $g(t)$

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^\infty G(\omega)\exp(\mathrm{j}\omega t)\,\mathrm{d}\omega \tag{6.12}$$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty F_L(\sigma + \mathrm{j}\omega)\exp(\mathrm{j}\omega t)\,\mathrm{d}\omega \tag{6.13}$$

Using (6.10) this tells us how to recover $f(t)$ from F_L since

$$f(t) = \exp(\sigma t) g(t) \tag{6.14}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F_L(\sigma + \mathrm{j}\omega) \exp[(\sigma + \mathrm{j}\omega)t] \, \mathrm{d}\omega \tag{6.15}$$

This is the inverse Laplace transform relationship. It is more commonly written as a contour integral over $s = \sigma + \mathrm{j}\omega$

$$f(t) = \frac{1}{\mathrm{j}2\pi} \int_{\sigma - \mathrm{j}\infty}^{\sigma + \mathrm{j}\infty} F_L(s) \exp(st) \, \mathrm{d}s \tag{6.16}$$

where this is simply a different notation for exactly the same integral as in (6.15). The value of σ must be chosen to lie within the region of absolute convergence of F_L. Notice that the integral is carried out on a **line** parallel to the imaginary axis in the complex plane. There are usually many such lines within the region of absolute convergence. Performing the integral along **any** of these lines must give the same result. In this sense, we may regard the Laplace transform as a family of Fourier transforms of $f(t)\exp(-\sigma t)$ for many different values of σ. The effect of the damped exponential is to control the function at large values of t so that the result is absolutely integrable.

Example: Calculate the inverse Laplace transform of

$$F_L(s) = \frac{\exp(-s)}{s^2 + a^2} \tag{6.17}$$

whose region of absolute convergence is $\Re s > 0$.

Substituting into the inverse Laplace transform relationship, we see that

$$f(t) = \frac{1}{\mathrm{j}2\pi} \int_{\sigma - \mathrm{j}\infty}^{\sigma + \mathrm{j}\infty} \frac{e^{s(t-1)}}{s^2 + a^2} \, \mathrm{d}s \tag{6.18}$$

The integral is over a line L in the right-hand half-plane parallel to the imaginary axis starting at $\sigma - \mathrm{j}\infty$ and ending at $\sigma + \mathrm{j}\infty$. We wish to use complex integration methods to evaluate this integral. The function $F_L(s)$ and hence the integrand is only defined on the region of absolute convergence but we can easily extend the integrand to the entire complex plane by defining

$$I(s) = \frac{e^{s(t-1)}}{s^2 + a^2} \tag{6.19}$$

This function is meromorphic, which means that it is analytic everywhere on the complex plane except at a set of isolated singularities at which there are poles. Since this function agrees with the required integrand on the line L, it will have the same integral over L. By extending the function in this way, however, it is possible to use the techniques of complex variable theory to evaluate the integral by means of residues and contour integration.

(*Informal note:* If you like, think of the contour integration as a mathematical trick used to evaluate the integral over the line L. As part of this trick, we need to extend the integrand to $I(s)$ over the whole complex plane so that we can draw a contour through a region outside the region of convergence of the Laplace transform. The extended integrand $I(s)$ needs to agree with the true integrand only along the line L so that it will give the correct integral which is required to recover $f(t)$.)

The extended integrand $I(s)$ has simple poles at $s = \mathrm{j}a$ and $s = -\mathrm{j}a$. We need to consider two cases:

- If $t < 1$, $I(s)$ becomes large in the left-hand half-plane (as $\Re s \to -\infty$) and small in the right-hand half-plane (as $\Re s \to \infty$). Let us consider completing the contour in the right-hand half plane with a large semicircle C_R whose radius we will ultimately take to infinity.

  This contour encloses no singularities of $I(s)$ and so the integral vanishes, i.e.,

$$\left(\int_L + \int_{C_R}\right) I\left(s\right)\,\mathrm{d}s = 0 \tag{6.20}$$

  It is easy to see that as the radius of C_R increases, the second integral tends to zero since the integrand $I(s)$ becomes small more rapidly than the arc length. Hence

$$f(t) = \frac{1}{\mathrm{j}2\pi}\int_L I(s)\,\mathrm{d}s = 0 \quad \text{for } t < 1. \tag{6.21}$$

- If $t > 1$, $I(s)$ becomes small in the left-hand half-plane (as $\Re s \to -\infty$) and large in the right-hand half-plane (as $\Re s \to \infty$). Thus, let us consider complete the contour in the left-hand half plane with a large semicircle C_L whose radius we will ultimately take to infinity.

  This contour encloses two singularities of $I(s)$. By the residue theorem

$$\left(\int_L + \int_{C_L}\right) I\left(s\right)\,\mathrm{d}s = \mathrm{j}2\pi\left[\mathrm{Res}\left(I\left(s\right),\mathrm{j}a\right) + \mathrm{Res}\left(I\left(s\right),-\mathrm{j}a\right)\right] \tag{6.22}$$

  Recall that the residue at a pole $s = p$ is the coefficient of the term $1/(s-p)$ in the Laurent expansion of the function. Since

$$I(s) = \frac{1}{\mathrm{j}2a}\left(\frac{1}{s-\mathrm{j}a} + \frac{1}{s+\mathrm{j}a}\right)e^{s(t-1)} \tag{6.23}$$

  we see that

$$\mathrm{Res}(I(s),\mathrm{j}a) = \frac{1}{\mathrm{j}2a}e^{\mathrm{j}a(t-1)}, \qquad \mathrm{Res}(I(s),-\mathrm{j}a) = -\frac{1}{\mathrm{j}2a}e^{-\mathrm{j}a(t-1)} \tag{6.24}$$

  Hence

$$\left(\int_L + \int_{C_L}\right) I\left(s\right)\,\mathrm{d}s = \frac{\pi}{a}\left[e^{\mathrm{j}a(t-1)} - e^{-\mathrm{j}a(t-1)}\right] = \frac{\mathrm{j}2\pi}{a}\sin\left[a\left(t-1\right)\right] \tag{6.25}$$

Combining the results yields

$$f(t) = \frac{1}{a}u(t-1)\sin[a(t-1)] \tag{6.26}$$

Exercise: Ensure that you can recover $u(t)t^{n-1}/(n-1)!$ by explicitly calculating the inverse Laplace transform of s^{-n}.

In practice, we usually do not use the explicit inverse transform relationship. Instead, we reduce the expression into a sum of standard forms (e.g. using partial fractions) and then use a table of Laplace transforms to work backwards. Under some conditions, it may be possible to write an expression as a product of two familiar transforms and to use the convolution theorem to evaluate the desired inverse transform.

6.3 Properties of the one-sided Laplace transform

The following properties of the one-sided Laplace transform may readily be proven. We use the notation $f(t) \leftrightarrow F_L(s)$ to indicate a Laplace transform pair.

1. Linearity

$$f(t) + g(t) \leftrightarrow F_L(s) + G_L(s) \tag{6.27}$$

2. Differentiation

$$f'(t) \leftrightarrow sF_L(s) - f(0) \tag{6.28}$$

$$f^{(n)}(t) \leftrightarrow s^n F_L(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0) \tag{6.29}$$

$$-tf(t) \leftrightarrow F_L'(s) \tag{6.30}$$

$$(-1)^n t^n f(t) \leftrightarrow F_L^{(n)}(s) \tag{6.31}$$

3. Integration

$$\int_0^t f(\tau)\,\mathrm{d}\tau \leftrightarrow \frac{1}{s} F_L(s) \tag{6.32}$$

4. Convolution

$$\int_0^t f(\tau) g(t-\tau)\,\mathrm{d}\tau \leftrightarrow F_L(s) G_L(s) \tag{6.33}$$

5. Scaling and translation

$$e^{at} f(t) \leftrightarrow F_L(s-a) \tag{6.34}$$

$$\frac{1}{c} f(t/c) \leftrightarrow F_L(cs) \qquad (c > 0) \tag{6.35}$$

$$f(t-b)u(t-b) \leftrightarrow e^{-bs} F_L(s) \tag{6.36}$$

6. Initial value theorem

$$\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF_L(s) \tag{6.37}$$

7. Final value theorem

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF_L(s) \tag{6.38}$$

6.4 The two-sided Laplace transform

This is simply the Laplace transform defined for functions which need not be zero for negative arguments. The transform pair is

$$F_L(s) = \int_{-\infty}^{\infty} f(t) \exp(-st)\,\mathrm{d}t \tag{6.39}$$

$$f(t) = \frac{1}{\mathrm{j}2\pi} \int_{\sigma - \mathrm{j}\infty}^{\sigma + \mathrm{j}\infty} F_L(s) \exp(st)\,\mathrm{d}s \tag{6.40}$$

The region of absolute convergence of a two-sided Laplace transform consists of points in a strip bounded by two lines parallel to the imaginary axis. There are thus two abscissae of absolute convergence $-\infty \leq \sigma_1 < \sigma_2 \leq \infty$ such that the integral converges absolutely within $\sigma_1 < \Re s < \sigma_2$. It is easy to see that this must be the case since we can write $f(t)$ as a sum of a causal part and an anticausal part. The region of absolute convergence of f is the intersection of the regions of absolute convergence of the two parts.

For the two-sided Laplace transform, it is **essential** to specify the region of absolute convergence together with the function $F_L(s)$ since different $f(t)$ can have the same $F_L(s)$ but with different regions of absolute convergence.

Example: What are the two-sided Laplace transforms of

- $u(t)\exp(-at)$
- $-u(-t)\exp(-at)$

Substituting these into the definition, we see that the Laplace transform of $u(t)\exp(-at)$ is

$$\int_0^{\infty} \exp[-(a+s)t]\,\mathrm{d}t = \frac{1}{s+a} \quad \text{provided that } \sigma > -a \tag{6.41}$$

The Laplace transform of $-u(-t)\exp(-at)$ is

$$\int_{-\infty}^{0} \exp[-(a+s)t]\,\mathrm{d}t = \frac{1}{s+a} \quad \text{provided that } \sigma < -a \tag{6.42}$$

These two different functions have the same expression for the Laplace transform but different regions of absolute convergence.

Exercises:

1. Use the inverse Laplace transform relationship to recover each of the above functions from $1/(s+a)$ using the appropriate region of absolute convergence. Ensure that you understand how to get the both forms in the regions $t < 0$ and $t > 0$.

2. Consider the function

$$F_L(s) = \frac{\exp(2s)}{(s+2)(s+3)} \tag{6.43}$$

Calculate the inverse Laplace transform given that the region of absolute convergence is

(a) $\Re s < -3$

(b) $-3 < \Re s < -2$

(c) $\Re s > -2$

6.5 Example: Solution of the diffusion equation

Consider the differential equation for heat diffusion in a one-dimension

$$\kappa \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial \theta}{\partial t}$$

together with the initial and boundary conditions

$$\theta(x,0) \qquad \text{is specified,} \tag{6.44}$$

$$\theta(-\infty,t) = \theta(\infty,t) = 0 \qquad \text{for all } t \tag{6.45}$$

To solve this problem, we use a Fourier transform in x and a (one-sided) Laplace transform in t. In general, a Fourier transform is appropriate when the boundary conditions at $\pm\infty$ are homogeneous and a Laplace transform is useful for initial-value problems. We introduce

$$\Theta_F(u,t) = \int_{-\infty}^{\infty} \theta(x,t)\exp(-\mathrm{j}2\pi ux)\,\mathrm{d}x \tag{6.46}$$

and

$$\Theta_{FL}(u,s) = \int_0^{\infty} \Theta_F(u,t)\exp(-st)\,\mathrm{d}t \tag{6.47}$$

Taking the Fourier transform (in x) of the differential equation yields

$$\kappa(\mathrm{j}2\pi u)^2\Theta_F(u,t) = \frac{\partial\Theta_F}{\partial t} \tag{6.48}$$

Taking the Laplace transform (in s) of this yields

$$-4\pi^2\kappa u^2\Theta_{FL}(u,s) = s\Theta_{FL}(u,s) - \Theta_F(u,0) \tag{6.49}$$

Solving this yields

$$\Theta_{FL}(u,s) = \frac{\Theta_F(u,0)}{s + 4\pi^2\kappa u^2} \tag{6.50}$$

The inverse Laplace transform of this is

$$\Theta_F(u,t) = \Theta_F(u,0)\exp(-4\pi^2\kappa u^2 t) \tag{6.51}$$

The desired solution is found by taking an inverse Fourier transform. The right-hand side becomes a convolution of $\theta(x,0)$ with the inverse Fourier transform of $\exp(-4\pi^2\kappa u^2 t)$ which is

$$\frac{1}{\sqrt{4\pi\kappa t}}\exp\left(-\frac{x^2}{4\kappa t}\right) \tag{6.52}$$

Thus,

$$\theta(x,t) = \int_{-\infty}^{\infty} \frac{\theta(\xi,0)}{\sqrt{4\pi\kappa t}}\exp\left(-\frac{(x-\xi)^2}{4\kappa t}\right)\mathrm{d}\xi \tag{6.53}$$

If the initial temperature distribution is a delta function, this spreads out in time into a Gaussian of width proportional to $\sqrt{t}$. This scaling of the width is characteristic of a diffusion process.

Exercises:

1. Instead of a rod of infinite length, consider a rod of length L extending between $x = 0$ and $x = L$. The initial temperature $\theta(x,0)$ is again specified but the boundary conditions are now $\theta(0,t) = \theta(L,t) = 0$. Using the fact that $\theta(x,t)$ can be expanded as a sine series

$$\theta(x,t) = \sum_{k=1}^{\infty} c_k(t)\sin\left(\frac{k\pi x}{L}\right) \tag{6.54}$$

calculate an expression for $\theta(x,t)$ in terms of $\theta(x,0)$.

2. The density of neutrons n in a cubical block of uranium 235 of side a is governed by the equation
$$\kappa\nabla^2 n + \gamma n = \frac{\partial n}{\partial t} \tag{6.55}$$
where κ and γ are constants. At the surfaces of the block ($x = 0$, $x = a$, $y = 0$, $y = a$, $z = 0$, $z = a$) the neutron density is zero. Find the value of a such that the block becomes critical.

Hint: Write n as a three-dimensional sine series
$$n(x, y, z, t) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} c_{klm}(t)\sin\left(\frac{k\pi x}{a}\right)\sin\left(\frac{l\pi y}{a}\right)\sin\left(\frac{m\pi z}{a}\right) \tag{6.56}$$

6.6 The Laplace transform and the matrix exponential

Consider a system of coupled linear differential equations with constant coefficients which can be written in matrix form as
$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x} \qquad \text{where } \mathbf{x}(0) \text{ is given.} \tag{6.57}$$
In this equation, $\mathbf{x}$ is a column vector with components $x_1(t)$,..., $x_n(t)$ and $\mathbf{A}$ is an $n \times n$ matrix. By analogy with the case where $\mathbf{A}$ is a scalar, the solution of this system is written as
$$\mathbf{x}(t) = \exp(\mathbf{A}t)\mathbf{x}(0) \tag{6.58}$$
This matrix exponential is interpreted in terms of a power series, namely
$$\exp(\mathbf{A}t) = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} + ... \tag{6.59}$$
We can use Laplace transforms to find an explicit closed form for $\exp(\mathbf{A}t)$ when a specific matrix is involved. For example suppose that
$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \tag{6.60}$$
Taking the Laplace transform of the differential equation, we see that the Laplace transform $\mathbf{X}(s)$ of $\mathbf{x}(t)$ is given by $\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0)$. The Laplace transform of $\exp(\mathbf{A}t)$ is thus
$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{pmatrix} s & -1 \\ 2 & s+3 \end{pmatrix}^{-1} \tag{6.61}$$
$$= \frac{1}{s^2 + 3s + 2}\begin{pmatrix} s+3 & 1 \\ -2 & s \end{pmatrix} \tag{6.62}$$
Taking the inverse Laplace transform shows that
$$\exp(\mathbf{A}t) = \begin{pmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{pmatrix} \tag{6.63}$$
An alternative way of calculating the matrix exponential involves the use of the eigenvalues and eigenvectors of the matrix $\mathbf{A}$. It is easy to check that the eigenvalues of $\mathbf{A}$ are -1 and -2 with corresponding eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$. This means that we can write
$$\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \tag{6.64}$$

where $\mathbf{V}$ is the matrix whose columns are the eigenvectors of $\mathbf{A}$ and $\mathbf{D}$ is the diagonal matrix of eigenvalues. For any power of $\mathbf{A}$ it is clear that $\mathbf{A}^n = (\mathbf{V}\mathbf{D}\mathbf{V}^{-1})^n = \mathbf{V}\mathbf{D}^n\mathbf{V}^{-1}$ and $\mathbf{D}^n$ is simply found by raising the diagonal elements to the n'th power. From the power series definition we see that

$$\begin{aligned}\exp(\mathbf{A}t) = \mathbf{V}\exp(\mathbf{D}t)\mathbf{V}^{-1} &= \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{pmatrix} \end{aligned} \tag{6.65}$$

This coincides with the form found above. The second method fails if the matrix $\mathbf{A}$ is **defective**, i.e., if the eigenvectors of $\mathbf{A}$ do not span the entire space.