Laplace Transformation

Laplace Transformation

- Definition:

$$f(t) \quad \rightarrow \quad \mathcal{L}\{f(t)\} = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt$$

time domain frequency domain

- Usefulness:

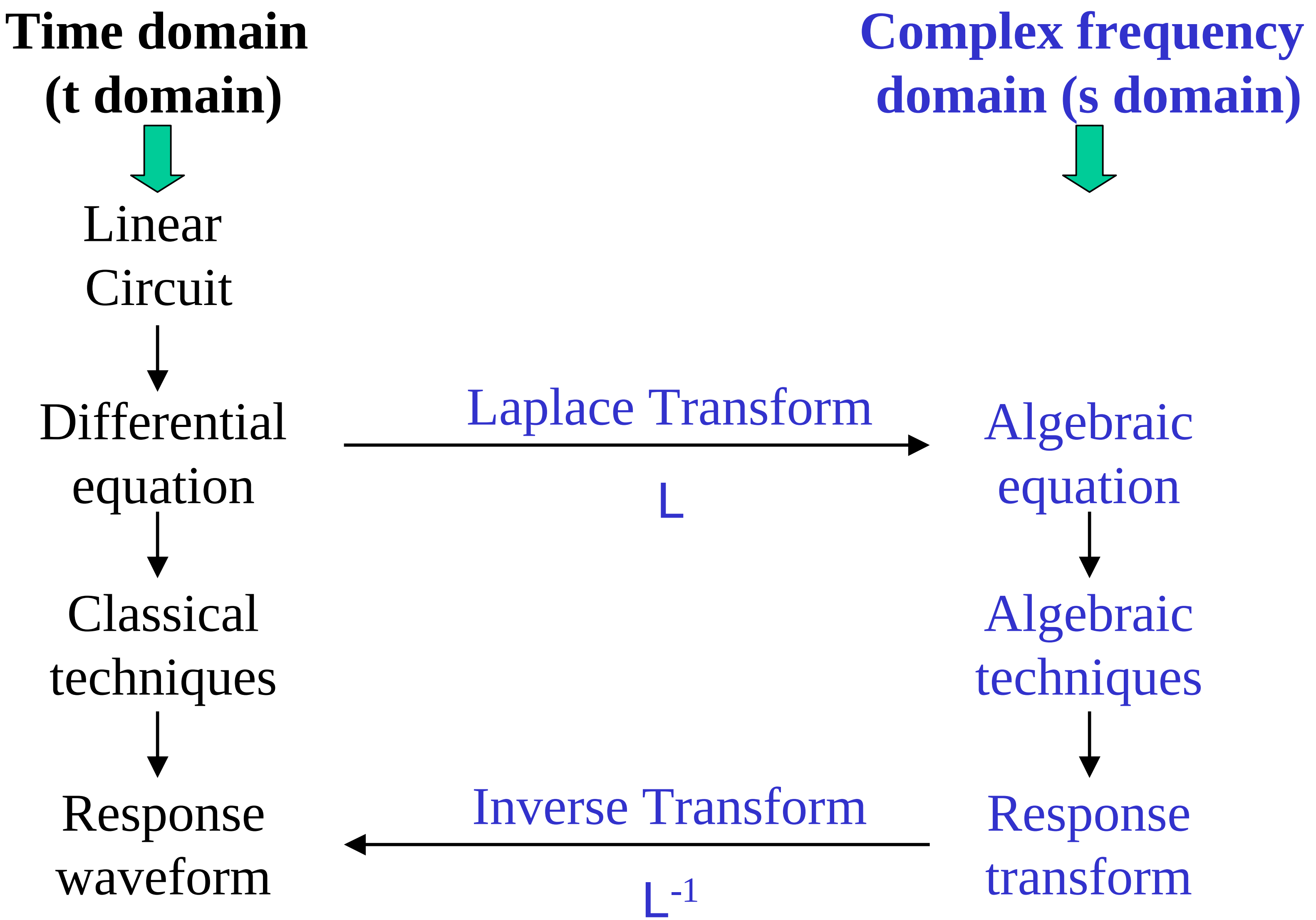
differential equations ⟶ algebraic equations

- Analogy:

$$a \quad \rightarrow \quad \log a$$

$$a \cdot b \quad \rightarrow \quad \log a + \log b$$

Circuit Analysis Using Laplace Transforms



Basic Laplace Transform Pairs

Signal	Waveform $f(t)$	Transform F(s)
Impulse	$\delta(t)$	1
Step function	$u(t)$	$\frac{1}{s}$
Ramp	$tu(t)$	$\frac{1}{s^2}$
Exponential	$[e^{-\alpha t}]u(t)$	$\frac{1}{s+\alpha}$
Damped ramp	$[te^{-\alpha t}]u(t)$	$\frac{1}{(s+\alpha)^2}$
Sine	$[\sin \beta t]u(t)$	$\frac{\beta}{s^2+\beta^2}$
Cosine	$[\cos \beta t]u(t)$	$\frac{s}{s^2+\beta^2}$
Damped sine	$[e^{-\alpha t}\sin \beta t]u(t)$	$\frac{\beta}{(s+\alpha)^2+\beta^2}$
Damped cosine	$[e^{-\alpha t}\cos \beta t]u(t)$	$\frac{(s+\alpha)}{(s+\alpha)^2+\beta^2}$

Laplace Transform of Some Basic Functions

$$\mathcal{L}\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t) e^{-st} dt$$

$$= \int_{0^-}^{0^+} \delta(t) e^{-st} dt$$

$$= \int_{0^-}^{0^+} \delta(t) dt$$

$$= 1$$

Laplace Transform of Some Basic Functions

$$\mathcal{L}\{u(t)\} = \int_{0^-}^{\infty} u(t) e^{-st} dt$$

$$= \int_{0}^{\infty} e^{-st} dt$$

$$= \left[-\frac{1}{s} e^{-st} \right]_0^{\infty}$$

$$= \frac{1}{s}$$

Laplace Transform of Some Basic Functions

$$\mathcal{L}\{t \cdot u(t)\} = \int_{0^-}^{\infty} t \cdot u(t) e^{-st} dt$$

$$= \int_{0}^{\infty} t e^{-st} dt$$

$$= \left[t(-\frac{1}{s} e^{-st}) \right]_0^{\infty} + \frac{1}{s} \int_0^{\infty} e^{-st} dt$$

$$= \frac{1}{s^2}$$

Laplace Transform of Some Basic Functions

$$\mathcal{L}\{e^{-\alpha t}\} = \int_{0^-}^{\infty} e^{-\alpha t} e^{-st} dt$$

$$= \int_{0^-}^{\infty} e^{-(s+\alpha)t} dt$$

$$= \left[-\frac{1}{s+\alpha} e^{-(s+\alpha)t} \right]_0^{\infty}$$

$$= \frac{1}{s+\alpha}$$

Basic Laplace Transformation Properties

Properties	Time Domain	Frequency Domain
Independent variable	t	s
Signal representation	$f(t)$	$F(s)$
Uniqueness	$\mathscr{L}^{-1}\{F(s)\} (=) [f(t)]u(t)$	$\mathscr{L}\{f(t)\} = F(s)$
Linearity	$Af_1(t) + Bf_2(t)$	$AF_1(s) + BF_2(s)$
Integration	$\int_0^t f(\tau)d\tau$	$\frac{F(s)}{s}$
Differentiation	$\frac{df(t)}{dt}$	$sF(s) - f(0-)$
	$\frac{d^2 f(t)}{dt^2}$	$s^2 F(s) - sf(0-) - f'(0-)$
	$\frac{d^3 f(t)}{dt^3}$	$s^3 F(s) - s^2 f(0-) - sf'(0-) - f''(0-)$

Proofs of Basic Laplace Transformation Properties

$$\mathcal{L}\{\int_0^t f(\tau)d\tau\} = \frac{F(s)}{s}$$

Proof: $$\mathcal{L}\{\int_0^t f(\tau)d\tau\} = \int_0^\infty \left[\int_0^t f(\tau)d\tau\right] e^{-st}dt$$

$$= \left[-\frac{e^{-st}}{s}\int_0^t f(\tau)d\tau\right]_0^\infty - \int_0^\infty f(t)\left(-\frac{e^{-st}}{s}\right)dt$$

$$= \frac{1}{s}\int_0^\infty f(t)e^{-st}dt = \frac{F(s)}{s}$$

Proofs of Basic Laplace Transformation Properties

$$\mathcal{L}\{\frac{df(t)}{dt}\} = sF(s) - f(0^-)$$

Proof: $$\mathcal{L}\{\frac{df(t)}{dt}\} = \int_{0^-}^{\infty} \frac{df(t)}{dt} e^{-st} dt$$

$$= \left[f(t)e^{-st}\right]_{0^-}^{\infty} - \int_{0}^{\infty} f(t)\left(-se^{-st}\right)dt$$

$$= -f(0^-) + s\int_{0^-}^{\infty} f(t)e^{-st} dt = sF(s) - f(0^-)$$

Poles and Zeros of F(s)

$$F(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$$

$$F(s) = K \frac{(s - z_1)(s - z_2)\ldots(s - z_m)}{(s - p_1)(s - p_2)\ldots(s - p_n)}$$

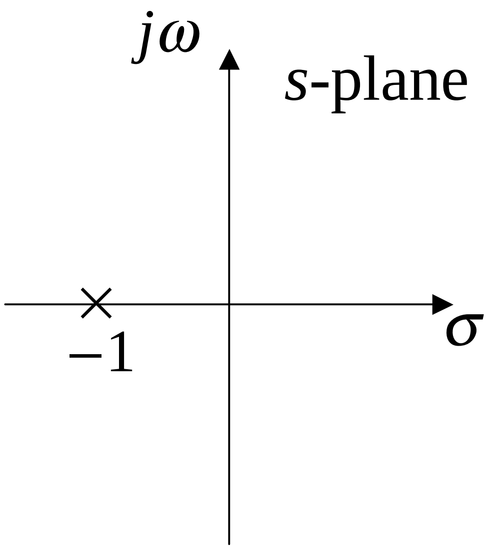
- Scale factor: $K = b_m/a_n$
- Poles: $s = p_k$ (k = 1, 2, ..., n)
- Zeros: $s = z_k$ (k = 1, 2, ..., m)

} Critical frequencies

Pole-Zero Diagrams

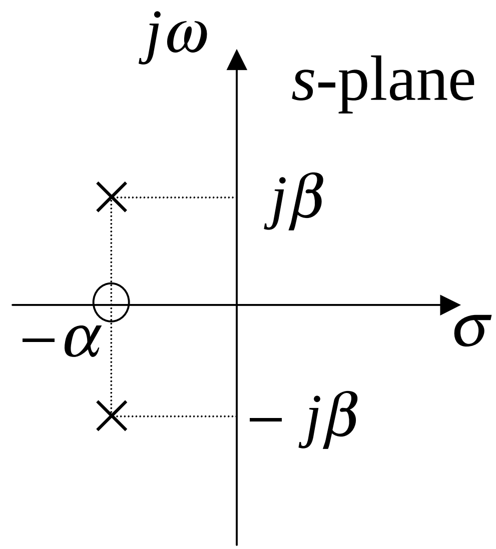
× pole location

○ zero location



$$F(s)=\frac{1}{s+1}$$

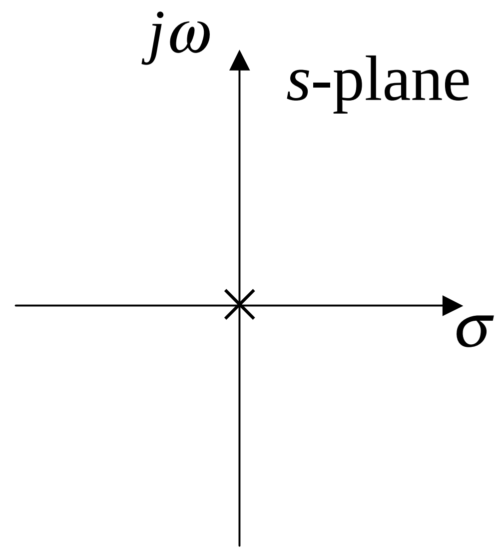
$\text{pole}: s=-1$



$$F(s)=\frac{A(s+\alpha)}{(s+\alpha)^2+\beta^2}$$

$\text{zero}: s=-\alpha$

$\text{poles}: s=-\alpha \pm j\beta$

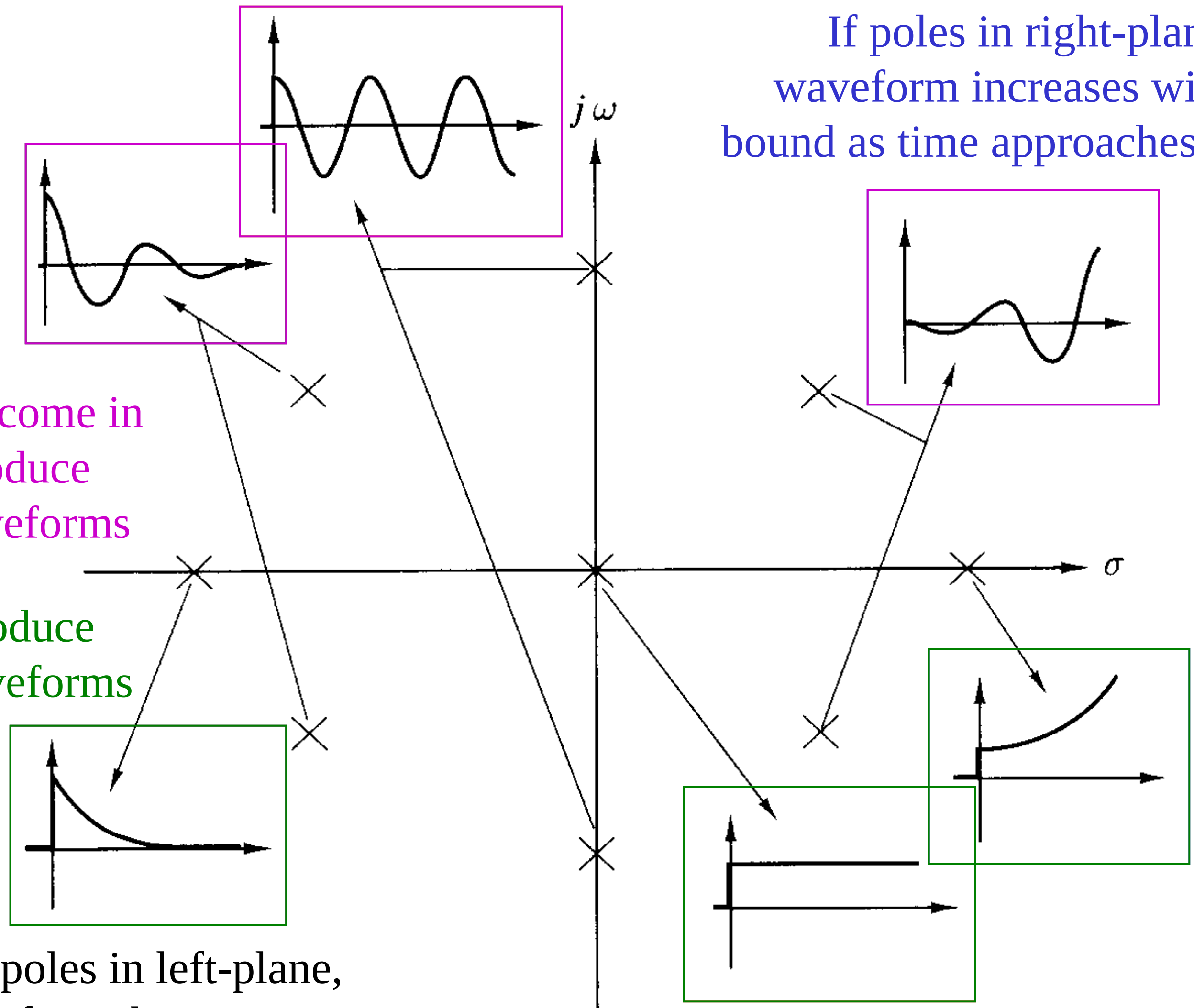


$$F(s)=\frac{1}{s}$$

$\text{pole}: s=0+j0$

Poles and Waveforms

If poles in right-plane,
waveform increases without
bound as time approaches infinity



Complex poles come in
pairs that produce
oscillatory waveforms

Real poles produce
exponential waveforms

If poles in left-plane,
waveform decays to zero
as time approaches infinity

If poles on j-axis,
waveform neither decays nor grows

Inverse Laplace Transforms

$$F(s) = K\frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}$$

- m < n, i.e., F(s) is a proper rational function

$$F(s) = \frac{k_1}{(s-p_1)} + \frac{k_2}{(s-p_2)} + \ldots + \frac{k_n}{(s-p_n)} \qquad k_i : \text{residues}$$

$$f(t) = \left(k_1 e^{p_1 t} + k_2 e^{p_2 t} + \ldots + k_n e^{p_n t}\right) u(t)$$

- How to determine k's?
  - Distinct poles: $k_i = (s-p_i)F(s)\big|_{s=p_i}$
  - Repeated poles of multiplicity r: r associated residues

$$k_i^0 = (s-p_i)^r F(s)\big|_{s=p_i}$$

$$k_i^q = \frac{d^q}{ds^q}(s-p_i)^r F(s)\big|_{s=p_i} \qquad q = 1\cdots r-1$$

Example of Inverse Laplace Transforms

$$F(s) = K\frac{(s-z_1)}{(s-p_1)(s-p_2)(s-p_3)} = \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \frac{k_3}{s-p_3}$$

- Multiply with $(s\text{-}p_1)$

$$(s-p_1)F(s) = K\frac{(s-z_1)}{(s-p_2)(s-p_3)} = k_1 + \frac{k_2(s-p_1)}{s-p_2} + \frac{k_3(s-p_1)}{s-p_3}$$

- Set $s = p_1$

$$(s-p_1)F(s)\Big|_{s=p_1} = K\frac{(p_1-z_1)}{(p_1-p_2)(p_1-p_3)} = k_1$$

- Similarly,

$$k_2 = (s-p_2)F(s)\Big|_{s=p_2}$$

$$k_3 = (s-p_3)F(s)\Big|_{s=p_3}$$

Example of Inverse Laplace Transforms

$$F(s) = K\frac{(s-z_1)}{(s-p_1)(s-p_2)^2} = \frac{k_1}{s-p_1} + \frac{k_2^0}{(s-p_2)^2} + \frac{k_2^1}{s-p_2}$$

- Multiply with $(s-p_2)^2$

$$(s-p_2)^2 F(s) = K\frac{(s-z_1)}{(s-p_1)} = \frac{k_1(s-p_2)^2}{s-p_1} + k_2^0 + k_2^1(s-p_2)$$

- Set $s = p_2$

$$(s-p_2)^2 F(s)\Big|_{s=p_2} = K\frac{(p_2-z_1)}{(p_2-p_1)} = k_2^0$$

- Differentiate w.r.t. s

$$\frac{d}{ds}(s-p_2)^2 F(s)\Big|_{s=p_2} = k_2^1$$

Inverse Laplace Transforms

- m ≥ n, i.e., F(s) is an improper rational function
  - For example: $$F(s) = \frac{s^3 + 6s^2 + 12s + 8}{s^2 + 4s + 3}$$
    $$= s + 2 + \frac{s+2}{s^2 + 4s + 3} = s + 2 + \frac{1/2}{s+1} + \frac{1/2}{s+3}$$
  - Recall: $\mathcal{L}\{\delta(t)\} = 1$
    $$\mathcal{L}\left\{\frac{d\delta(t)}{dt}\right\} = s\mathcal{L}\{\delta(t)\} - \delta(0^-) = s$$
  - Inverse transform: $$f(t) = \underbrace{\frac{d\delta(t)}{dt} + 2\delta(t)}_{\text{Pathologic waveforms}} + \left[\tfrac{1}{2}e^{-t} + \tfrac{1}{2}e^{-3t}\right]u(t)$$

- Pathological waveforms usually do not occur in real circuits
- Improper rational functions may occur during manipulation

Additional Laplace Transform Properties and Pairs

Feature	Time Domain	Frequency Domain
Simple complex poles	$\left[2\lvert k\rvert e^{-\alpha t}\cos(\beta t + \angle k)\right]u(t)$	$\dfrac{k}{s+\alpha-j\beta} + \dfrac{k^*}{s+\alpha+j\beta}$
Double complex poles	$\left[2\lvert k\rvert t e^{-\alpha t}\cos(\beta t + \angle k)\right]u(t)$	$\dfrac{k}{(s+\alpha-j\beta)^2} + \dfrac{k^*}{(s+\alpha+j\beta)^2}$
t translation	$[f(t-a)]u(t-a)$	$e^{-as}F(s)$
s translation	$e^{-at}f(t)$	$F(s+a)$
Scaling	$f(at)$	$\dfrac{1}{a}F\left(\dfrac{s}{a}\right)$
Initial value	$\lim_{t\to 0+} f(t)$	$\lim_{s\to\infty} sF(s)$
Final value	$\lim_{t\to\infty} f(t)$	$\lim_{s\to 0} sF(s)$
Convolution	$\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau$	$F_1(s)F_2(s)$

Proofs of Some Additional Properties

t translation: If $\mathcal{L}\{f(t)\} = F(s)$,

then $\mathcal{L}\{f(t-\alpha)u(t-\alpha)\} = e^{-\alpha s}F(s)$ for $\alpha > 0$

Proof: $\mathcal{L}\{f(t-\alpha)u(t-\alpha)\} = \int_{0^-}^{\infty} f(t-\alpha)u(t-\alpha)e^{-st}dt$

$= \int_0^{\infty} f(t')e^{-s(t'+\alpha)}dt'$ Let $t' = t - \alpha$

$= e^{-\alpha s}\int_0^{\infty} f(t')e^{-st}dt' = e^{-\alpha s}F(s)$

Proofs of Some Additional Properties

Scaling: If $\mathcal{L}\{f(t)\} = F(s)$,

then $\mathcal{L}\{f(\alpha t)\} = \frac{1}{\alpha} F(\frac{s}{\alpha})$ for $\alpha > 0$

Proof: $\mathcal{L}\{f(\alpha t)\} = \int_{0^-}^{\infty} f(\alpha t) e^{-st} dt$

$$= \int_0^{\infty} f(t') e^{-(s/\alpha)t'} d(\frac{t'}{\alpha}) \qquad \text{Let } t' = \alpha t$$

$$= \frac{1}{\alpha} \int_0^{\infty} f(t') e^{-(s/\alpha)t'} dt' = \frac{1}{\alpha} F(\frac{s}{\alpha})$$

Proofs of Some Additional Properties

Initial/Final Values: $$\lim_{t\to 0^+} f(\mathrm{t}) = \lim_{s\to\infty} sF(s)$$

$$\lim_{t\to\infty} f(\mathrm{t}) = \lim_{s\to 0} sF(s)$$

Proof: $$sF(s) - f(0^-) = \int_{0^-}^{\infty} \frac{df(t)}{dt} e^{-st} dt$$

$$\lim_{s\to\infty}(sF(s) - f(0^-)) = \lim_{s\to\infty} \int_{0^-}^{0^+} \frac{df(t)}{dt} e^{-st} dt + \lim_{s\to\infty} \int_{0^+}^{\infty} \frac{df(t)}{dt} e^{-st} dt$$

$$= \lim_{s\to\infty} \int_{0^-}^{0^+} \frac{df(t)}{dt} e^{-0} dt + 0 = f(0^+) - f(0^-)$$

$$\Rightarrow \lim_{s\to\infty} sF(s) = f(0^+)$$

Proofs of Some Additional Properties

Proof of Initial/Final Values:

$$\lim_{s\to 0}(sF(s) - f(0^-)) = \lim_{s\to 0}\int_{0^-}^{\infty} \frac{df(t)}{dt} e^{-st} dt$$

$$= \lim_{s\to 0}\int_{0^-}^{\infty} \frac{df(t)}{dt} e^{-0} dt = f(\infty) - f(0^-)$$

$$\Rightarrow \lim_{s\to 0} sF(s) = \lim_{t\to\infty} f(t)$$

- Initial-value property holds when F(s) is a proper rational function ($f(0^+)$ is well defined)
- Final-value property holds when poles of sF(s) are in the left-half plane (except for a simple pole at the origin)

Proofs of Some Additional Properties

Convolution Property: $\mathcal{L}^{-1}\{F_1(s)F_2(s)\} = \int_0^t f_1(\tau) f_2(t-\tau)d\tau$

$$= \int_0^t f_2(\tau) f_1(t-\tau)d\tau$$

Proof:

$$F_1(s) = \int_{0^-}^{\infty} f_1(\tau)e^{-s\tau}d\tau$$

$$F_1(s)F_2(s) = \int_{0^-}^{\infty} f_1(\tau)\left(F_2(s)e^{-s\tau}\right)d\tau)$$

$$= \int_{0^-}^{\infty} f_1(\tau)\left(\int_{0^-}^{\infty} f_2(t-\tau)u(t-\tau)e^{-st}dt\right)d\tau)$$

$$= \int_{0^-}^{\infty}\left(\int_{0^-}^{t} f_1(\tau) f_2(t-\tau)d\tau\right)e^{-st}dt$$

Example of Convolution Property

Show: $\mathcal{L}^{-1}\left\{\frac{1}{(s+\alpha)^2}\right\} = te^{-\alpha t}$

Proof: $\mathcal{L}^{-1}\left\{\frac{1}{(s+\alpha)} \cdot \frac{1}{(s+\alpha)}\right\} = \int_0^t e^{-\alpha\tau} e^{-\alpha(t-\tau)} d\tau$

$$= \int_0^t e^{-\alpha t} d\tau = te^{-\alpha t}$$