Leonhard Euler:

Life, Work and Legacy

Robert E. Bradley
C. Edward Sandifer
Editors

STUDIES IN THE HISTORY AND PHILOSOPHY OF MATHEMATICS

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LEONHARD EULER: LIFE, WORK AND LEGACY

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Foreword

The articles in this volume were commissioned by the editors and have not appeared elsewhere. “The Truth About Königsberg,” by Brian Hopkins and Robin J. Wilson, is the only exception. It was published in May 2004 in *The College Mathematics Journal* and won The Mathematical Association of America’s George Pólya Award. It is included with the kind permission of The Mathematical Association of America.

Indirectly, this volume is a result of a contributed paper session on Euler organized by William Dunham and V. Frederick Rickey at the 2001 Joint Mathematics Meetings in New Orleans. This attracted Euler enthusiasts from all over North America. A few, Ronald Calinger, John Glaus and Edward Sandifer, met at a local restaurant to hatch the idea of The Euler Society. The Society had its first annual meeting in the summer of 2002.

Two years later, Arjen Sevenster, an editor at Elsevier, contacted us, saying, “I just came across the announcement of the Third Annual Meeting of the Euler Society: Euler 2004. I wonder, if you would be interested in editing a volume covering the same topics of the Meeting: Euler, his work and times, aiming to give a more or less complete picture.”

We soon found that The Euler Society alone could not provide the “complete picture” the project required. Here, our friend Rüdiger Thiele stepped in and helped introduce us to a number of his European colleagues. We owe the participation of Wolfgang Breidert, Peter Hoffmann, Teun Koetsier, Olaf Neumann, Karin Reich and Dieter Suisky to Thiele’s good efforts. He also invited Michael Raith to contribute, but sadly, Raith passed away before he could contribute. Rüdiger Thiele has dedicated his own contribution to the memory of Michael Raith. Peter Hoffmann wrote his chapter in German, and Rüdiger Thiele worked to render it into English, the language of the volume. Because of all of this, and because he is in all ways such a fine friend and colleague, we dedicate this volume to Rüdiger Thiele. Without his help, this volume would be much less than it is.

Others deserve our recognition. The staff at Elsevier, especially Andy Deelen and Simon Pepping, have been helpful whenever we have needed them, as has Henk Bos, the general editor of the series, *Studies in the History and Philosophy of Mathematics*. Our six-member Editorial Panel of Ronald Calinger, Lawrence D'Antonio, Stacy Langton, Rüdiger Thiele, Jeff Suzuki and Homer White, did yeoman work refereeing and editing the chapters. We also thank Pat Allaire, Ken Gittelson and Theresa Sandifer for their editorial assistance.

The authors of the chapters have been patient and professional with us, and have written some wonderful essays.

We also thank our wives and families.

Robert E. Bradley	C. Edward Sandifer
Garden City, NY	Newtown, CT

August 2006

Contents

Leonhard Euler: Life, Work and Legacy
Robert E. Bradley and C. Edward Sandifer (Editors)

Introduction

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The year 2007 marks 300 years since the birth of Leonhard Euler. This gives historians of mathematics their first opportunity since 1983, 200 years after his death, to celebrate an Eulerian anniversary. Academic celebrations have three traditional forms, books, meetings, and special issues of journals. This book, a collection of chapters written by outstanding Euler scholars from seven different countries, is one such celebration.

Chronology and tradition provide us four opportunities to celebrate each century, and this volume fits into a sequence of anniversary works that have come before, and, presumably, will come after. We should probably expect (and perhaps begin planning for) special Euler events in 2033, 250 years after his death.

The tradition of celebrating anniversaries is not a new one, but it is not as old as some might think. It is, of course, a social construct, and it seems to have arisen in the late 19th century, in the same historical context that gave us the rise of nationalism, organized sports, Manifest Destiny, leisure time and the Victorian era.

Thus, there were no Euler celebrations in 1807, in the midst of the Napoleanic wars. In 1830, just three years before the 50th anniversary of Euler's death, the St. Petersburg Academy published a kind of memorial issue of academy's *Mémoires*, to publish what the Academy thought were all the remaining unpublished Euler articles (E772 to E785). The timing of this event, though, seems more related to the Academy's desire to publish all of Euler's papers before everyone who knew him had died, and not correlated with any anniversary.

The first Euler anniversary event seems to have been a small seminar in Zürich on December 6, 1883, where Ferdinand Rudio delivered a short biographical talk on Euler. This seminar would probably be completely forgotten if Rudio had not published the text of his talk more than 25 years later in the wake of the 200th anniversary events of 1907.

The Americans were the first to move to initiate a major commemoration of Euler's 200th anniversary. The Carnegie Corporation, a philanthropic foundation endowed by the estate of American steel magnate Andrew Carnegie, solicited proposals for scientific projects. The American Mathematical Society proposed publishing the complete works of Leonhard Euler, but the Carnegie Corporation elected, instead, to fund astronomical observatories.

Just a few years later, a similar proposal was made to the Swiss Academy of Sciences and the Euler Commission was formed to collect and publish the *Leonhardi Euleri Opera omnia*. The *Opera omnia* project continues today and already extends to more than 70 volumes. It includes all of Euler's published papers and has started on his correspondence. The Editors are making plans to publish many of Euler's notebooks and other papers as well.

Various volumes of the *Opera omnia* were edited by some of the most outstanding mathematicians and Euler scholars of the 20th century: Constantin Carathéodory, Ferdinand Rudio, Clifford Truesdell, René Taton, Eduard Winter, Adolf Juškevič, Emil Fellmann and others too numerous to list. Their Editors' Introductions constitute some of the finest and most authoritative Euler scholarship of the 20th century, and students of Euler will benefit from their thoroughness and dedication for hundreds of years.

Much of the flurry of Euler scholarship shortly after 1907 was stimulated by the decision to begin the *Opera omnia*. The comprehensive *Verzeichnis der Schriften Leonhard Eulers* of Gustaf Eneström is a particularly important and useful example. Throughout this volume, authors consistently use Eneström's numbering system (e.g. E65, E101) to refer to Euler's books and papers.

In contrast to the marvelous scholarship of 1907, the 200th anniversary of Euler's birth, world events sometimes overshadow academic events, and we find no Eulerian observations in 1933, the 150th anniversary of his death.

The celebrations of 1957 were mostly confined to the German Democratic Republic and the Soviet Union. Their respective Academies of Science each published a *Sammelband*, the former edited by Kurt Schröder and the latter by M. A. Larent'ev, A. P. Juškevič and A. T. Grigor'jan, with contributions by important mathematicians including Gel'fond, Smirnov, Delaunay, Vinogradov and Erdös. The Academy of the German Democratic Republic also published four volumes of Euler's letters, including the important correspondence with Christian Goldbach, and one volume of his personal notes on the meetings of the Berlin Academy. Also, the German Democratic Republic, the Soviet Union and Switzerland all issued commemorative postage stamps.

The 200th anniversary of Euler's death, 1983, was highlighted by a marvelous volume edited by three members of the Editorial Committee of the *Opera omnia*. J. J. Burkhardt, E. A. Fellmann and W. Habicht. It contains articles by well-known mathematicians including André Weil, A. O. Gel'fond, Pierre Dugac and B. L. van der Waerden, and also comprehensive a 42-page bibliography of secondary literature about Euler, prepared by J. J. Burckhardt. This is an invaluable resource. Only the German Democratic Republic issued a commemorative stamp.

As we write this Introduction, so much has changed since 1983 that this volume could not have had its present form and content in earlier times. First and foremost, the Soviet Union and the German Democratic Republic are gone, and along with them much of the state sponsorship of such historical studies, especially in Russia. Hence, the language of the present volume is English, rather than German, Russian or a mixture of languages. Furthermore, there are more contributors from the 'West' than there were in 1957 or 1983, although thanks to the efforts of Ronald Calinger and especially of Rüdiger Thiele, scholars from Russia and the former German Democratic Republic are well represented. Additionally, the archives in Moscow, St. Petersburg and Berlin are far more open to western scholars than they were during the years of the Cold War. The implicit view that Euler studies were an essentially Eastern Bloc pursuit has evaporated.

The Internet has had three major influences on this project. First and most obvious, email has dramatically reduced the length of the editorial cycle. Hardworking authors and editors on different continents can progress from a first manuscript to page proofs in a matter of days, rather than waiting weeks for the postal system.

Second, a great many primary sources are now available on line. The complete works of Lagrange and Gauss, for example, are entirely on line. Euler's *Opera omnia* are not on line, but digital images of the original versions of over 95% of his published works, scanned from the original 18th century pages are available at The Euler Archive (www.eulerarchive.org). This amazing web site is the result of efforts of Lee Stemkoski and Dominic Klyve, then graduate students. These on-line versions lack the corrections and the Editors' Introductions of the *Opera omnia*, but they are available to everyone with an Internet connection and the editors of The Euler Archive are gradually adding links to commentaries and translations. The seventy-odd volumes of the *Opera omnia* are heavy, expensive and static, but they are also permanent, comprehensive and well-edited. By the time we celebrate the next Eulerian anniversary in 2033, perhaps the relative roles of paper editions like the *Opera omnia* and digital editions like The Euler Archive will be clearer.

The third major influence of the Internet is more subtle, and it affects both the style and the content of this volume. On the World Wide Web, the significance of an object depends both on its content and on the ways it is connected to other objects on the Web. The same is true of a mathematical text; its significance depends both on its content and on its context. Previous styles of discussing the history of mathematics tended to be taxonomical. We would describe and classify. Articles bore titles like "Euler's manuscript on number theory" or "Some facets of Euler's work on series," titles that promise to describe and classify. Now, though, we are more inclined to dwell on connections and give our articles titles that emphasize connections; "Euler and Lagrange on the foundations of analysis" or "Euler, D'Alembert and the logarithm function." The 1957 and 1983 volumes, in the style of their own times, had more articles with taxonomic and descriptive titles, while this 2007 volume has more connective and contextual titles.

The organization of this volume might not be obvious at first glance. We have put the chapters that seem more biographical and historical near the beginning of the book. This includes the articles by Calinger, Hoffmann, Polyakhova, Breidert and Fasanelli. We have put near the end those chapters that describe Euler's influence on the mathematicians that came after him. These are the articles by Grattan-Guinness, Caparrini, Reich and Suisky. In between are the more internalist articles dealing with Euler's mathematical and scientific work, grouped roughly by topic; astronomy, mechanics, analysis, geometry, number theory, probability and combinatorics. Within each group, articles tend to go from the more general to the more specific.

The reader is, of course, welcome to read the chapters in any order. However, those unfamiliar with details of Euler's life or with the astonishing scope of his accomplishments in mathematics and science will probably wish to begin with Ronald Calinger's biographical chapter, which follows immediately.

Robert E. Bradley and C. Edward Sandifer (Editors)

Leonhard Euler: Life and Thought

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As the European Enlightenment began in the 1720s, few new accomplishments in mathematics were expected. Although mathematics had not yet become a profession in the previous century, when most of its practitioners came from the aristocracy or positions in medicine or law, that period culminating in the inventions of differential calculus by Isaac Newton and Gottfried Leibniz was considered a great age in mathematics, leaving little to be developed. But some scholars anticipated a fecund era for the field.[1] Above all, the research of Leonhard Euler would prove them right. The Swiss-born Euler was to be one of the four preeminent mathematical scientists in history, the other three being Archimedes, Newton, and Carl Friedrich Gauss. Only for Newton and Euler did Gauss reserve the term *summus*.[2]

Driven by a passion for mathematics and natural science, a commitment to build a strong institutional base for them, and an insistent defense of reform Christianity, Euler made seminal contributions across the mathematical sciences and was arguably the most prolific mathematician in history. At

[1] Nicholas Fuss, "Lobrede auf Herrn Leonhard Euler," in *OO* I.1, pp. XLIII-XCV, trans. by John S. D. Glaus, 2005, p. 5, and Marc Parmentier, "L'optimisme mathématique," in G. W. Leibniz, *La naissance du calcul différentiel*, Paris: Librairie philosophique J. Vrin, 1989, pp. 41-51.

[2] Karin Reich, "Gauss' geistige Väter: nicht nur '*summus* Newton' sondern auch '*summus* Euler'," in *Wie der Blitz einschlägt, hat sich das Räthsel gelöst*, Göttingen: Niedersächsische Staats- und Universitätsbibliothek, 2005, pp. 105-115.

the core of his research were infinitary analysis, or differential calculus, and rational mechanics. Along with celestial mechanics, he made them the sciences *par excellence* of the eighteenth century. He was the principal creator of the calculus of variations and differential equations, and he pioneered the differential geometry of surfaces. In mechanics Euler, not Newton, formulated most of the fundamental differential equations before William Rowan Hamilton. Operating within Enlightenment rivalries, in his case with Jean d'Alembert, Alexis Clairaut, Daniel Bernoulli, and Colin Maclaurin, he led in transforming mechanics and astronomy into modern exact sciences based on calculus. Euler founded continuum mechanics and advanced the study of ballistics, cartography, dioptrics, the theory of elasticity, hydraulics, hydrodynamics, music theory, number theory, optics, and ship theory. Massive and fearless computations, an extraordinary application of analysis and analogies, an appeal to his near unerring instinct, and clarity in writing characterize his work. Not since Claudius Ptolemy had a single geometer so dominated all branches of the mathematical sciences. During the eighteenth century four royal science institutions, in Paris, London, St. Petersburg, and Berlin, eclipsed universities in scientific research. It was largely Euler's efforts that made the St. Petersburg and Berlin Academies of Science prominent European centers. The more than 810 of his articles and books, which fill seventy-four large volumes in the first three series of his *Opera omnia*, include approximately one-third of the entire corpus of research in mathematics, theoretical physics, and engineering mechanics published from 1726 to 1800, while the equivalent of research articles fill his extensive correspondence.[3]

[3] All but twenty of Euler's 868 books and memoirs are listed in the Eneström index. Of these, 819 fill seventy-four volumes of three hundred to six hundred pages each in the first three series of his *Opera omnia* (abbreviated *OO*). There is essentially no repetition in their materials. Series One on pure mathematics has twenty-nine volumes in thirty parts; Series Two on mechanics and astronomy thirty-one volumes in thirty-two parts. Series Three on physics and miscellany is comprised of twelve volumes. The Swiss Society of Sciences began the *Opera omnia* in 1907. Euler suffered one misfortune: six boxes filled with his papers were lost in 1766 during shipment across the Baltic Sea to St. Petersburg. To 1802, 707 of Euler's writings were published. In 1975 work commenced on a two-part fourth series of the *Opera*, which is projected to consist of at least ten volumes. Section A has begun to make accessible the over 2,840 surviving letters to and from him in French, Latin, German, Russian, and English. Section B will examine his twelve extant notebooks. Counting fragments discovered over the last four decades, this makes 886 works for him.

See also Rüdiger Thiele, "The Mathematics and Science of Leonhard Euler (1707 - 1783)" in Glen van Brummelen and Michael Kinyon, eds., *Mathematics and the Historian Craft*, Berlin: Springer, 2005, pp. 81-140 that includes an extensive bibliography, and Clifford Truesdell, "Leonhard Euler, Supreme Geometer (1707 - 1783)," in Harold E. Pagliaro, ed., *Irrationalism in the Eighteenth Century*, Cleveland: Case Western Reserve

1. Lineage, Youth, and Formal Education

Leonhard Euler was born on Friday April 15, 1707 (n. s.), in Basel, Switzerland. While most of Protestant and Orthodox Europe followed the Julian calendar or old style, the city had adopted the current Gregorian style in 1701. Euler's birth house was probably located in the neighborhood around St. Martin's Church near the center of the city close to the market quarter and ship landing on the Rhine River. He was the first child of Paul Euler, an Evangelical-Reformed minister, and Margaretha née Brucker. While "reformed" generally refers to the Protestantism of the Calvinists and Lutherans, Basel's variety was of a pietism stressing love and the inner life. Leonhard's mother Margaretha, the daughter of a hospital minister, was from a distinguished line of artists and humanistic scholars. Their son was baptized two days after his birth in the same St. Martin's church as his father had been.

The Euler (Äuler, Ewler, Öwler) family came from the town of Lindau on Lake Constance (the Bodensee) in the German Swiss Canton. *Au* is the diminutive of *Äule*, which refers to a small, wet field or meadow. Au appears in the names of many small German towns, such as Dessau and Nassau. The owner of an *Äule* was an *Äuler* (oyler). The Eulers were variously called Euler-Schoelpin, signifying squint-eyed, which suggests that they were susceptible to an eye malady. The first written record of an Euler appeared in 1287, but a documented continuous line did not commence until 1458. Lindau, though on the far side of the Canton from Basel, had many close economic, political, and religious ties with the town. Hans Georg Euler, the great-great grandfather of Leonhard and the grandson of the German-speaking patriarch Hans Euler, moved to Basel in 1594. Hans Georg obtained citizenship in Basel, became a comb- and brushmaker, fathered fifteen children in two marriages, and lived to be ninety. [4] Apparently the next three generations were artisans, most of them combmakers or tradesmen belonging to hospitality guilds. They built the family's modest financial base. In the fourth generation, four of the fourteen male cousins were able to become Basler Evangelical-Reformed ministers. These included Paul Euler, who matriculated at the University of Basel in 1685 at the age of fifteen. While at the university, Paul resided at the home of Jacob Bernoulli, under whose direction he wrote his senior thesis on ratios and proportions. He shared rooms with the young Johann I Bernoulli. Paul Euler completed his theological studies in 1693.

U. Press, 1972, pp. 53-54.

[4] Fritz Burckhardt, "Zur Genealogie der Familie Euler in Basel," in *Basler Jahrbuch 1908*, Basel: Heibing and Lichtenhahn, 1908, pp. 83-89.

Leonhard did not spend his early youth in Basel. In June 1708 his father was named pastor-designate to St. Martin's church in nearby Riehen-Bettingen. In November he was installed and the family moved to Riehen, about five kilometers northeast of Basel. It and Bettingen combined had a population of fourteen hundred. Supported by a sub-Mediterranean climate, these small villages were known for their rich vegetation, especially the white blossoms of the cherry trees in the late spring and the gold and red leaves on the grapes in the vineyards. The Eulers lived in a two-room parsonage until it was enlarged in 1712. One room was a study and the other living quarters. Of Leonhard's two younger sisters, Anna Marie was born in 1708 and Marie Magdalena in 1711. His paternal grandmother lived with the family to her death in 1712. Johann Heinrich, the fourth child of the Eulers, was not born until 1719, after his brother departed for studies at the Basel *Gymnasium*. Leonhard was a talented child, apparently cheerful and sociable. The simplicity of rural life together with the model of his parents has its reflection in the forthright nature and even disposition of the adult Euler.

Leonhard's parents were his first teachers. Familiar with the humanistic tradition, his mother Margaretha introduced him to Greek and Roman classics. The elementary instruction that his father Paul offered included mathematics, conceived as a subject underlying all natural knowledge. Paul began not with a geometry text but with Christoff Rudolff's *Coss*, or algebra, the German equivalent of the Italian *cosa* or unknown, a two-part work that Michael Stifel had expanded from 208 to 484 pages. Paul possibly employed a reprint from 1615 of the first edition published in 1553. After explaining place-value notation and the four basic arithmetical operations, it examines in verbal form first, second, and third degree equations. Euler's unfinished autobiography notes that he diligently studied the text for several years and made progress in solving its 434 problems, almost all of them first- or second-degree equations.[5] He did this before moving to lodge with his maternal grandmother in Basel and enroll in the city's *Gymnasium*, probably at age eight. Only an exceptional child of this age could have advanced previously through the difficult *Coss*.

The Basel *Gymnasium*, a Latin school, was in a pitiful state. Students were taught the Latin language and selections from ancient classics. Greek was optional. Teachers did not spare the rod, and fistfights broke out in the classroom. Like most parents, Euler's hired a tutor, in their case a young theologian named Johann Burckhardt, who sided with Johann I Bernoulli in disputes with British geometers and natural philosophers, especially Brook

[5] Leonhard Euler, "Autobiography," in Emil Fellmann, *Leonhard Euler*, Reinbeck bei Hamburg: Rowohlt Taschenbuch Verlag, 1995, pp. 11-13.

Taylor. Burckhardt taught Euler the humanities and mathematics, a subject earlier struck from the curriculum by a vote of the townspeople.

In 1720 Euler matriculated at the University of Basel into the Philosophical Faculty, essentially the school of arts and sciences. He was thirteen, at the time roughly the normal age for entering a university. The university was in decline. Its enrollment had fallen from over a thousand students a century earlier to just above a hundred. It had only nineteen professors, underpaid and most of them mediocre. The exception was Johann I Bernoulli. The Philosophical Faculty provided the general education preparatory to choosing a specialty for a higher degree. Through industry and a powerful memory, Euler mastered all his subjects. Apparently he skipped the dry introductory mathematics lectures of Bernoulli, as Charles Darwin and Albert Einstein would later avoid sessions of tedious college courses. At fourteen Euler gave a speech titled "Declamatio: De Arithmetica et Geometria" commending the superiority of geometry. After giving a speech in Latin praising temperance, he received in 1722 his *prima laurea*, roughly the Bachelor's degree. In the autumn of 1723 he completed his examinations for Master of Arts. In June 1724 the seventeen-year-old Euler officially received the degree upon giving a public lecture in Latin on his master's thesis, a comparison of the natural philosophy of René Descartes with that of Isaac Newton, along with the consequences of each.

In October 1723 Leonhard's father had required him to register for theology to prepare for becoming a rural pastor. He had mainly to study Greek, Hebrew, Protestant theology, and classical humanities. About this time he began to display his photographic memory by reciting long passages from Virgil's *Aeneid* by heart, if not the complete text. He could cite the first and last line on each page of his copy of the book. To the age of seventy, he could remember the *Aeneid* entirely. The theology curriculum allowed him to study mathematics. He had already begun to meet with Bernoulli in a tutorial. Spending most of his time on mathematics, he made little progress in his other subjects. At the university, Euler had become friends with Johann II Bernoulli, who probably helped his request for private lessons. The elder Bernoulli offered these to other students but refused to do so for Euler. Instead he advised the young scholar to read diligently some difficult books on mathematics, astronomy, and physics until he encountered obstacles. The two were to meet on Saturday afternoons, when Bernoulli would show Euler how to overcome the impediments and avoid unpromising routes to solutions. Euler devoted his full energy to reducing his questions to a very small number. When Bernoulli showed him how to conquer one difficulty, he was delighted that ten others disappeared. Bernoulli was discovering his student's genius. Euler's autobiography declares that reading masterworks in a tutorial with a skillful teacher "is the best method to succeed in [learning] mathematical subjects."[6] At least it was for so talented a student. It was probably in 1725 that the elder Bernoulli, now nearly sixty, traveled to Riehen to persuade his former roommate Paul to allow his son to transfer to mathematics.

In 1725 young Euler was seeking employment. Producing more graduates than needed for their own country, the Swiss had to export them. In 1725 Euler's friends Daniel and Nicholas II Bernoulli accepted positions at the new St. Petersburg Academy of Sciences. After the unexpected death of Nicholas II, Euler was invited in the fall of 1726 to join the academy in medicine with a 200-ruble pension, which he thought too small. Still he agreed to come as soon as the weather cleared. Meanwhile he enrolled in courses in anatomy and physiology. When the professor of physics in Basel died, the elder Bernoulli recommended that Euler apply to fill the vacancy. The specimen essay that he had to submit was a sixteen-page paper on acoustics, titled "De sono," which historians describe as his doctoral dissertation. It became a classic. But the university selected faculty by lottery, and young Euler was not a finalist. On April 5, 1727, three days after Benedict Stähelin became professor of physics, Euler left Basel forever. He was already acquiring a modest reputation. His essay on the masting of ships submitted to the Paris Academy annual prize competition that year won the *accessit* or honorable mention, though he had only boats on the Rhine and not ocean ships to observe.

2. Into the Colossus of the North: The Groundwork of Euler's Research

After a journey down the Rhine River to Frankfurt-am-Main and overland to the north through Hamburg to Lübeck, and a rough voyage along the coast of the Lübeck Bay and the Baltic Sea, Euler arrived in St. Petersburg in May 1727, two months after the death of Isaac Newton, to begin his illustrious career. Probably through the intervention of Jakob Hermann, Daniel Bernoulli, and Christian Goldbach, Euler was placed in the mathematics rather than medical division of the dynamic new science academy. It was in a state of consternation. Its benefactress, Empress Catherine I, the widow and successor of Peter I, had died a week earlier. The nobility in the new government saw the academy as a foreign intrusion in Russia and froze funds for it; Euler had to accept a post of medic in the Russian navy that was becoming the chief power bordering the Baltic. He served under Admi-

[6] As reproduced in Emil Fellman, *Leonhard Euler*, pp. 11-13.

ral de Sievers. Upon the accession of the empress Anne of Courland in 1730, the academy's fortunes improved. When Hermann and Georg Bilfinger departed, Daniel Bernoulli succeeded Hermann in the prestigious position of professor of mathematics. Euler declined the offer by the Russian navy for a promotion and returned full time to the academy, replacing Bilfinger as professor of physics. To 1733 Euler resided at Bernoulli's home, and they often dined together, occasionally arguing briefly over the sciences. Among the subjects they collaborated on was Bernoulli's forthcoming *Hydrodynamica*, published in 1738.

When Daniel Bernoulli returned to Basel in 1733, Euler succeeded him. Having an improved financial situation, in January 1734 he married Katharina Gsell, the daughter of artist Georg Gsell, originally from St. Gall in Switzerland. They purchased a wooden house on the tenth line of Vasilyevski Island. In November 27 their first child, Johann Albrecht, was born. Euler named him after the president of the academy, Baron Johann Albrecht de Korff, and had Christian Goldbach as the godfather. These were to be important allies. The Eulers would have a total of thirteen births, only three sons and two daughters surviving early childhood. Katharina organized and ran the entire household, leaving her husband more time for his research and writing. At the academy the genial Euler had some strained relations with its despotic and crude administrator, Johann Schumacher, especially over salary. But he avoided public disputes with Schumacher, who was a friend of his father-in-law. He attended parties at the Schumachers' house.

In St. Petersburg Euler pursued groundwork research that ranged across a broad spectrum in the sciences from number theory and music theory to astronomy and ballistics. The emphasis in his *Commentarii* memoirs is on infinitary analysis and rational mechanics.

In number theory Goldbach was the major source of encouragement, and he supplied problems demanding difficult solutions that resulted in early achievements for Euler. In a letter of December 1729, Goldbach had inquired whether Fermat's conjecture is true that all integers of the form $2^{2^n} + 1$ are prime. Euler confirmed it for $n = 1$, 2, 3, and 4, but by 1732 proved that $n = 5$ is a counterexample, that is, $2^{32} + 1 = 4,294,967,297 = 641 \times 6,700,417$. He had found that composite Fermat numbers must have divisors of the form $2^{n+1}k + 1$. Once he knew this, it would have been far easier to discover the divisor 641, the case when $k = 10$ gives $2^6(10) + 1$. In 1734 Euler discovered a fascinating interrelationship between natural logarithms and the harmonic series. That made it possible to compute Euler's constant or gamma, which is

$$\lim_{n\to\infty}\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n\right)=0.577215664\ldots$$

Gamma is one of the important real constants that appear in analysis, alongside the transcendental numbers π and e. To the present it is not known whether gamma is rational, algebraic, or transcendental. In the twentieth century G. H. Hardy promised to surrender his Savilian Professorship at Oxford to any scholar able to prove that gamma is algebraic.

During the 1730s Euler proceeded through a first stage preparatory for his *Introductio in analysin infinitorum* [E101,E102] of 1748 by completing exhaustive calculations, perfecting computational methods, and developing the three elementary classes of transcendental functions of infinitary analysis — the exponential, logarithmic, and trigonometric. Following Leibniz and Bernoulli he divided functions into two classes: algebraic and transcendental. It was likely Johann I Bernoulli who introduced him to the exponential function e^x, in which the exponent is the variable. Euler first employed the symbol in a posthumously published paper on gunnery written in 1728 or late 1727 [E853]. Newton and Bernoulli had independently defined it by the limit of the binomial expansion $(1+1/n)^n$ as n approaches infinity. Employing the series

$$e=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots,$$

given here in modern factorial notation, Euler later computed its value as 2.718281828, which is accurate to nine decimal places. In 1737 he computed e as a continued fraction [7],

$$2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+\cdots}}}}$$

In trigonometry Euler transformed Ptolemaic chords and half chords into numerical ratios, making the trigonometric lines into functions. Probably after reading the pioneering work of Roger Cotes and Abraham de Moivre's *Miscellanea analytica* of 1730, he discovered in 1737 that $(\cos .z+\sqrt{-1}\sin .z)n=\cos .nz+\sqrt{-1}sin.nz$ or in his later notation $e^{ix}=\cos x+i\sin x$, the cardinal formula of analytical trigonometry. His massive computations began to give logarithms a greater place in infinitary analysis, but he had not yet established e as the natural base for them or uniformly employed the letter i to represent $\sqrt{-1}$.

[7] Eli Maor, *e: The Story of a Number*, Princeton: Princeton U. Press, 1994, p. 151.

In 1736 the mayor of Danzig asked Euler to solve a recreational puzzle. The center of the city of Königsberg in East Prussia is an island surrounded by the River Pregel. Seven bridges spanned the river. The question was whether a traveler could cross over the bridges in a connected walk, going over each bridge only once, and return to the same spot. Euler considered the puzzle simple and solved it by reason alone. By connecting the number of bridges with the number of times the traveler entered each region, he showed that the transit under the given conditions is impossible. His memoir "Solutio problematis ad geometriam situs pertinentis" [E53], containing his results, was an early contribution to the evolving field of *geometria situs* or *analysis situs*, that is, topology. Although no graphs appear in the memoir, it is considered the initial contribution to graph theory.

A principal reason for Euler's becoming recognized by the late 1730s as the leading mathematician in Europe is his exact summation in 1734/35 in the solution of the Basel problem, the infinite series of reciprocals of square integers, whose value is denoted today as $\zeta(2)$, where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. For more than seventy-five years, geometers had attempted to sum precisely this slowly converging series. James Stirling's *Methodus differentialis* of 1730 gives what at the time was the closest approximation, accurate to eight decimal places, 1.644034066. While computing it in four different ways, Euler unexpectedly arrived at the exact sum of $\pi^2/6$. In December 1734 he reported his finding to the St. Petersburg Academy. Through much labor, he exactly summed $\zeta(n)$ for even values of n up to 12, but he failed to do so for odd values. That Euler could not solve this part of the problem suggests that no simple solution exists. Tome 7 of the academy's *Commentarii* for 1734/35 did not appear in print until 1740; meanwhile Euler's summation circulated through the mail. Because his finding lacked a proof and thus rigor, he was criticized even after he supplied one in 1743. Euler connected the zeta function with the distribution of prime numbers and in a memoir of 1737 introduced his famous product decomposition formula: for P the set of primes,

$$\prod_{p\in P} \left(1 - p^{-s}\right)^{-1} = \prod_{p\in P} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots\right).$$

Multiplying out the right side gives $\zeta(s)$. The case $s = 1$ is the harmonic series, which diverges. Since the corresponding product must have infinitely many factors, Euler had indirectly proved the infinitude of primes.

Even more important for Euler's growing reputation in the late 1730s is the publication of his 980-page *Mechanica* [E15,E16] in two volumes as a supplement to the *Commentarii* in 1736.[8] It broke decisively with the

[8] See also Clifford Truesdell, "The Rational Mechanics of Flexible or Elastic Bodies,

geometric format for mechanics and introduced the differential equations of what are now called mass points. Except for Benjamin Robins in England, who questioned its systematic use of differential equations, geometers and natural philosophers across Europe including Johann I Bernoulli hailed it as a landmark in physics. Its review in the Parisian *Mémoires de Trevoux* in 1740 concludes that Euler had created modern mechanics. In his *Mécanique analytique* of 1788, Joseph Louis Lagrange called the *Mechanica* "the first large publication where analysis has been applied to the science of motion." But it is erroneous to portray the *Mechanica* as simply translating into infinitary analysis much of Newton's *Principia mathematica* of 1687. The *Mechanica* is Euler's first major work in his highly ambitious program to resolve computationally the motion of bodies that are elastic, fluid, flexible, and rigid — a task, which alone would have exhausted most scientific geniuses, that he successfully completed as a component in his research program over the next twenty years.

Among Euler's writings in St. Petersburg to 1741 are four books. With the *Mechanica*, the *Scientia navalis* [E110,E111] on shipbuilding and navigation and the *Tentamen novae theoriae musicae* (Introduction to a New Theory of Music) [E33] comprise his first St. Petersburg period trilogy. Although completed in 1738, the *Scientia navalis* was not published until Euler was in Berlin in 1749. A treatise largely finished in 1731, the *Tentamen novae theoriae musicae* was not printed until 1739. It expands upon Ptolemy's music theory and stresses numerical proportions of harmony. Euler was a talented teacher. For the academic *Gymnasium*, of which he was a member of the examination board, he wrote a two-part text *Einleitung zur Rechenkunst* (An Introduction to Arithmetic) [E17,E35] that came out in print in two parts in 1738 and 1740.

The academy charged Euler with many duties. He served on the weights and measures commission and cooperated in the testing of fire pumps, saws, and scales. Spurred by Johannes Andreas Segner's water wheel, he began to develop hydraulic machines. He sent articles to a journal introducing the general public to the sciences and reviewed papers by others, including an essay on the quadrature of the circle. His activities extended even to ordering ink and paper for the academic printing press. He crucially assisted the chief state project of the early academy, the Second Kamchatka (or Great Northern) Expedition that lasted from 1734 to 1743. It was to prepare the first accurate general map of the Russian empire and its regions by determining latitude and longitude astronomically and making better geodetic measurements. It was the most heavily funded project at the early academy. Euler continued to aid the French astronomer Joseph-Nicholas Delisle, and

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1638 - 1788" in *OO* II.11, part 2, esp. pp. 15-141.

from 1735 he directed the geography section of the academy. He computed ephemerides and in 1737 provided uniform instructions to the expedition's geodesists, especially for the Russian scholar Vasilii Tatischev, who was studying the economic possibilities of the high Urals. Euler encouraged Gerhard Müller to collect archives on the expedition as primary sources for a future history of Siberia.

Amid the controversies and rivalries that abounded in the Enlightenment, the initially dominant Cartesians argued to the 1740s with the Newtonians at the Paris Academy of Sciences over which science was superior, while in German universities, Berlin, and St. Petersburg to the 1750s, Leibnizians and Wolffians opposed Newtonian science. Euler selectively synthesized consistent elements from the first three, added his original thoughts, and rejected Wolffian metaphysics. Cartesians and Newtonians disagreed in one instance over the shape of Earth. Its determination would be a crucial element in indicating which science was superior. The Cartesian theory of vortices or whirlpools of ether in the heavens predicted an elongated spindle, while Book III of Newton's *Principia mathematica* surmised an oblate spheroid flattened at the poles.

This scientific dispute, along with an upsurge in French cartography and a desire to test improved surveying instruments, prompted the French monarchy and the Paris Academy to send geodetic expeditions to Peru from 1735 to 1744 and to Lapland from 1736 to 1737 to measure an arc of meridian and thereby demonstrate the true shape of the planet. (Pierre) Louis Moreau de Maupertuis (1698 - 1759), who led the Lapland expedition, claimed upon his return that his findings showed Newton to be correct. Euler praised Maupertuis' data but did not find them alone definitive. He awaited the data from the Peru expedition. In 1738 by assuming that the internal density of Earth is variable, not homogeneous, Euler correctly computed the shape as less flattened at the poles than had Newton. Earth is more an orange than a melon. Euler asserted that he still accepted Newton's inverse-square law of attraction, but thought it possibly in need of a slight modification.

Euler's reputation was sharply increasing with his winning of the prestigious *Prix de Paris*, the annual prize of the Paris Academy, three years in a row beginning in 1738. To the competitions for these foremost scientific awards of the eighteenth century, he was to submit eighteen essays. His reception of the prize twelve times added to one under the name Johann Albrecht, his eldest son, is as yet unmatched. His closest competitor, Daniel Bernoulli, won ten times. The Paris prizes were offered in theoretical and applied subjects in alternate years. Euler gained them in both categories, five of them for practical papers on navigation and shipbuilding.

In 1738 Euler shared the *Prix de Paris* for an essay on the nature and properties of fire and in 1740 on the tides. At mid-century natural philosophers were examining the most volatile of the four Aristotelian elements – earth, air, fire, and water – and the theory of combustion. The dominant phlogiston theory of combustion held that substances rich in phlogiston burn readily. But the inability to weigh phlogiston exactly was raising doubts. Euler proposed instead that fire results from the bursting of glass-like balls of compressed air in the pores of bodies. Two anonymous competitors in 1738 who lost were Emilie du Châtelet and Voltaire, whose *Elements of Newton's Philosophy* appeared that year. Voltaire blamed his loss on the Cartesian dominance at the academy, but his paper was unoriginal. The judges recognized that his evidence on the heating of iron derived from the research of Petrus van Musschenbroek of Utrecht. A principal test in confirming Newtonian dynamics on the continent was the tides. Euler and Daniel Bernoulli were two of the winners of the 1740 prize for papers on their ebb and flow. While both must have started with the same results, they differed in the method of reaching them. A strict Newtonian, Bernoulli based his work on the inverse-square law of gravitational attraction and disagreed with Euler, who accepted Newtonian attraction but believed the inverse-square law alone insufficient to describe all celestial motion and instead began with a theory of vortices. Lunar motion and the nature and orbits of comets were two additional crucial tests for Newtonian dynamics. The available data were contradictory, and Euler advocated more exact observations with new telescopes, along with creating differential equations based on either Newtonian dynamics or a slight revision to give each point in the course of comets.

Possessed of a diffident disposition, the genial Euler rarely became upset and tended to avoid disputes in St. Petersburg, working within the shelter of the academy. Throughout the 1730s, the academy had powerful antagonists in the censors of the Russian Orthodox Church. They prohibited publication of books or articles supporting Copernican heliocentric astronomy, particularly the Russian *philosophe* Antiokh Cantemir's translation in 1730 of Bernard Fontenelle's *Conversation on the Plurality of Worlds*, which had appeared in 1686. It discusses Copernicus, Galileo, Kepler, and Descartes on astronomy. Despite protests from the senior astronomer Joseph N. Delisle, the censors suppressed its printing by the academic press until 1740. Euler quietly supported this project. In rejecting Newton's corpuscular theory of light Euler did take a strong public stance. In their correspondence in 1737, Johann I Bernoulli stated that he was attempting to synthesize the ideas of Newton and the wave theory of Christian Huygens. Holding to a strict analogy between the propagation of sound in air and that of light in the ether, young Euler initially proposed a theory close to the Malebranchean notion that in the medium of an elastic ether pressure vibrations or waves produce light. Euler also began critiques of Wolffian philosophy and science, which were prominent in German-speaking Europe. Euler rejected them, finding mathematical errors in Wolff's books including *Cosmologia generalis* of 1731, and questioned the existence of animate monads as the elemental substance of the physical world. For the moment, he kept his emerging criticisms within a small circle. He wanted to refine his ideas through further research before presenting them to a wider public.

Except for two serious health problems, the deaths of three daughters in infancy, and the quartering of soldiers in his house, to which he strenuously objected, little had disturbed the quiet life of Euler during the late 1730s. Occasionally he was to suffer from dangerous fevers. The first occurred in early 1735, when the academy had Euler calculate tables confirming the midday correction for determining the latitude of St. Petersburg. In his eulogy of Euler, Nicholas Fuss attributed the fever to exhaustion from indefatigable labor on this project. But Euler had derived a formula that allowed him to complete the task quickly. In 1735 stress, a fever, headaches, and harsh weather weakened his health. He did not yet suffer a problem with his eyesight. That came in the summer of 1738, when a near fatal fever and an infection produced an abscess in his right eye. According to Fuss, whose information comes from an elderly Euler, he suffered the complete loss of sight in the right eye and was set on the path to total blindness. But portraits of Euler and correspondence from the time suggest a gradual weakening of vision in the right eye with occasional partial remissions.

Through the summer of 1740 the Euler family seemed settled in St. Petersburg. Euler's sixteen-year-old brother Heinrich, who had arrived in 1735 to live with the family, was studying art, and in July 1740 a second son, Karl Johann, was born. But when Delisle joined the Second Kamchatka Expedition in Siberia, he left Euler and Gottfried Heinsius with the entire work of the geography section at the academy. This alarmed Euler, who believed that more than reading and writing his examination of geographical charts and land maps was overstraining his eyes. His right eye was already weak, and meticulously studying charts might lead to the deterioration of vision in his left eye, if not total blindness.

In the midst of Euler's health problems Frederick II, who at the age of twenty-eight had ascended the Prussian throne at the end of May, invited him to join a renovated Royal Brandenburg Society of Sciences. The first offer Euler found too small. He asked for the same salary that he was receiving in St. Petersburg. But getting out of imperial Russia was an appealing prospect. After the death of Empress Anne in late 1740, life became dangerous in the Russian capital, especially for foreigners. Euler was asked to cast a horoscope for the two-month old tsar Ivan VI, but passed the honor to Georg Krafft, who was known as the court astrologer. The interregnum until Elizabeth gained the throne in a coup a year later was a time of turbulence. In February 1741 Euler went to the new Prussian ambassador, Baron Axel von Mardefeld, and learned that Frederick had agreed to match his salary. He accepted the offer but could not leave St. Petersburg until June. He had first to negotiate his release from the academy, which Schumacher opposed, holding that he could not be dismissed from his contract for a year; and he suffered from another fever. After his seventy-three-year old teacher Johann I Bernoulli learned of Euler's move to Berlin, he sent Euler a letter in September hoping that he would travel as far as his home city to visit his parents. Bernoulli's most ardent desire was to have the opportunity to see Euler once more before he died.[9]

Euler was among the eminent scholars Frederick was attempting to recruit. The Prussian monarch, later celebrated as a hero of German nationalism, was actually Francophile, speaking German only with difficulty and relishing French refinement and wit above the more stolid character of his German compatriots, and his relations with the bourgeois Basler republican Euler were to be strained; but his interest in enhancing the reputation of the academy was paramount here. For co-presidents, he wanted Wolff and Maupertuis.[10] But Wolff returned from exile in Marburg to the University of Halle, while in April 1741 the Austrians captured Maupertuis, who was in Frederick's entourage, after the battle of Mollwitz during the First Silesian War. Recognized as a leading French scholar by an officer, he was not killed but sent to Vienna, where Maria Theresa feted him. He returned to Paris until 1745. Euler was to aspire to the presidency, but never achieved it.

3. In Frederician Berlin: At the Apex of His Career

On July 25, 1741, after a rough four-week journey on the Baltic Sea with a brief stop in Stettin, the Euler family arrived in Berlin, moving into the Barboness house at the Potsdam Bridge near Unter den Linden Street. Euler's brother Heinrich left for Paris and Italian cities to continue his study of art. From March Euler had awaited a letter from the monarch. Occupied with the First Silesian War, Frederick did not write until September 1741

[9] Paul Heinrich Fuss, ed., *Correspondance mathématique et physique de quelques célèbres du XVIIIéme siècle*, New York: Johnson Reprint Corporation, 1968, 1st ed. 1843, vol. II, p. 58.
[10] Adolf Harnack, *Geschichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin*, Berlin: Reichsdruckeri, 1900, 2nd ed., Olms, 1970, vol. I.1, pp. 254-259.

from his camp in Reichenbach welcoming him. He was pleased to learn that Euler was satisfied to be in Berlin and gave orders for the General Directory to pay his annual salary of 1,600 Reichsthaler. Frederick declared that should Euler still need anything, he had only to await the return of the king. Even so, no salary payment, official appointment, or meeting was forthcoming. Through the end of 1741, Euler and his family lived on credit. This did not dampen his relief to be in Berlin.

In October 1742 Euler wrote to Goldbach that he had purchased for 2,000 Reichsthaler a lovely house with a large garden. The land, which was near to where the king planned to build the new science academy, included the current 20 and 21 Behrenstrasse. Today it is across from the Comic Opera. Greatly modified except for the front façade, it is currently the residence of the Bavarian Representative. Since Euler first had to have repairs made, the family could not move into the house until Michaelmas, September 29, 1743. In the spring the Eulers planted a garden. The common vegetables in such gardens at the time were potatoes and tomatoes. The Euler family shared vegetables with neighbors and sold any surplus.

For the next twenty-three years Euler resided at 21 Behrenstrasse. By 1746 his family had grown to seven. Joining Johann Albrecht and Karl Johann were Katharina Helene, born in 1741, Christoph in 1743, and Charlotte the next year. Each of these survived to adulthood. Yet the family suffered from the high infant mortality of the time. A son born in 1747, twin daughters two years later, and another son in 1750 died before their first birthday. Euler enjoyed taking his children to the zoo and watching the bear cubs play. Before bed, he often read Scripture to them. Euler was now addicted to pipe tobacco. His letters to the Basel-born theologian Johann Kaspar Wettstein-Sarasin, chaplain to the duchess of Wales, have many requests for his preferred "good tobacco" from England, which was sent to him through Amsterdam. Euler had few relaxations. Playing the clavier was the foremost, and he invited composers to his house to give recitals of their new works. In Berlin he was becoming a talented chess player. At Euler's house a small circle of colleagues gathered around him. To 1756 Russian students sent by the St. Petersburg Academy for Euler to tutor boarded there. On weekdays from 10:00 to 11:00 he taught them and a few other noble children mathematics, astronomy, and physics.

In September 1742 the Royal Brandenburg Society followed the protocol of welcoming "the famous professor of mathematics Mr. Euler." He was now pressing for the founding of the Berlin or Prussian Academy of Sciences. In wartime Prussia funding from the sole source of the sale of almanacs was insufficient. Euler improved the computation of ephemerides, streamlined production, and urged sales in Silesia that increased income from 10,000 to 13,000 Reichsthaler but not his projected 16,000. The sale of almanacs was a royal monopoly. In a letter of June 1743, the king scornfully rebuked Euler for his "pretended funding." The monarch believed that his "abstract calculations from the grandeur of algebra" went against the basic rules of computation and would yield debits rather than great revenue.[11] But Euler did not relent. Displeased with the slow progress, the king's favorite Count Samuel von Schmettau and other court nobility founded the *Nouvelle Societé littéraire* in 1743 to serve as a basis for the academy. Euler attended its sessions but refused to be the director of its mathematical section until it and the old Brandenburg Society were joined to form the planned science academy.

In January 1744 Frederick, as part of a spectacular ritual occasion preceding his thirty-second birthday, formally founded the Berlin Academy, consisting of four small classes with their directors elected for life. The classes were experimental philosophy, the term roughly equivalent to physics; mathematics; speculative philosophy; and literature along with philology. Four curators from the nobility closely associated with the monarch joined the directors in managing the academy. To Delisle in St. Petersburg Euler wrote with some continuing dissatisfaction, "I was very much mistaken when I thought they would put the new Academy on the same footing as that of Paris. The thing is done. We have joined into one body the old and the new society under the name of an Academy of Sciences."[12] The king, whose displeasure with the frank, independent, and bourgeois Euler was to persist, appointed another director of the mathematics class. What most interested Frederick at the academy was not Euler's research in transforming the mathematical sciences but his translation of Robins' book on ballistics.

The year 1746 was important in the institutional history of the academy. Frederick agreed to have Maupertuis draw up a new constitution based on that of the Paris Academy. The new statutes lacked the element of democracy that the *Nouvelle Societé littéraire* had enjoyed. Upon their completion, the king in June confirmed Maupertuis as perpetual president with autocratic powers, and the members nominally elected Euler director of the mathematics class for life. It had taken Euler five years to obtain the position projected in his invitation. The academy had an annual prize. The topic for 1746 was the cause of winds. Euler headed the committee that selected the winner, the respected Jean d'Alembert, for an abstract paper containing new differential equations. Daniel Bernoulli, another contestant, found the paper of d'Alembert weak; between Euler and Bernoulli, his best friend, a small strain was developing. Though the king proclaimed

[11] *OO*, IVA.6, p. 303.

[12] Mary Terrall, *The Man Who Flattened the Earth: Maupertuis and the Sciences in the Enlightenment*, Chicago: U. of Chicago Press, 2002, p. 239.

himself the protector of the academy, it was to continue to experience financial difficulties, for few members had pensions and revenue from the sale of almanacs remained inadequate to equip the observatory and run the academy.

Euler had many tasks at the academy. His immediate assignment was to supervise the construction of the observatory and then its operation. The new academy building was not completed until 1752. As a member of the academy's directorate, Euler headed the editorial committee selecting papers for the *Mémoires* section of its journal, and he managed its library. He and the academy secretary Samuel Formey served as a buffer between the Frenchman Maupertuis and the German-speaking members. Maupertuis suffered from poor health. By the 1750s he had to be away more often and for longer periods. During those absences, Euler was the acting president of the academy. All of this did not overtax him. He also prepared almanacs and various geographical maps to be sold to finance the academy, and he oversaw its botanical garden, personally ordering plants and trees.

From 1746 to 1748, Euler received offers of positions from familiar quarters outside Berlin and a major scientific honor. Having lost many members, including Delisle, the St. Petersburg Academy invited Euler to return in 1746, but he declined in June. The next year Schumacher asked him to review a paper on heat and cold by the Russian chemist, physicist, and poet Mikhail Lomonosov. Since Lomonosov had studied under Wolff and translated into Russian his text on physics, Schumacher expected an unfavorable review from Euler, now an outspoken opponent of Wolff. To his surprise, Euler praised the research of Lomonosov. His support for the talented Russian would continue. In June 1746 Euler wrote to Wettstein that he wished to be made a fellow of the Royal Society of London. Its *Philosophical Transactions* continued to be essential to his work, and he was closely following the observations of English astronomers, particularly James Bradley, who had discovered an aberration of stellar light in 1729 and was near to establishing the nutation or slight wobbling of Earth's axis. Euler's rivals Daniel Bernoulli and Alexis Clairaut had already been chosen fellows. But at that time a prospective candidate had to apply for membership, which Euler refused to do. Wettstein proceeded to nominate him, and he was elected in 1747. Fellows had to submit a manuscript. Euler initially offered his unpublished *Scientia navalis*. When word reached St. Petersburg, the Russian admiralty finally agreed after a near decade's delay to finance its publication. In January 1748 Johann I Bernoulli died, and the University of Basel offered his professorial chair to Euler, who never responded. He chose to remain on a larger stage and believed that the post should go to Daniel Bernoulli.

His increasing accolades notwithstanding, the republican commoner Euler was not socially popular at the Prussian royal court. Eighteenth-century Prussian society was aristocratic. Dictionaries defined courtly social graces. Euler was likely put off by both the privileges and the manners of the nobility. Episodes from Berlin allow a glimpse into his conduct in public. Condorcet's *éloge* relates that at a social gathering at the salon of Queen Mother Sophie Dorothea, probably shortly after New Year's Day 1742, she was puzzled that Euler kept responding to questions only in monosyllables. "Why do you not wish to speak with me?" she chided him. "Madame," he replied, "because I have just come from a country [Russia] where if you speak, you are done for."[13] He had reason to say so. The St. Petersburg academician Christian Gross had been sentenced to hang for speaking out on the wrong side during the November palace revolution there.

More significant were Euler's complex relations with Frederick II. Though impressed with his accomplishments, the king and the court nobility considered him lacking in courtly manners – the same charge that had been leveled against Galileo.[14] In a letter of October 1746 to his brother August Wilhelm, Frederick disparaged Euler's epigrams with new curves and described him as a "powerful calculator . . . useful to the republic of science" and a "Doric column" essential to the foundation of science, "though anything but elegant."[15] The king doubted that conversing with Euler would be edifying. Referring to Euler's eyesight problem, Frederick in November 1748 wrote to Voltaire with disdain, "We have here a great Cyclops of geometry."[16] Yet Euler seemed to enjoy conversations, put people at ease in them, had a sense of humor, and was a good story teller. A prodigious reader, he also spoke knowledgably on the classics and humanities.

In Berlin from 1741 to 1766, Euler wrote or completed more than 380 memoirs and books. Their combination of depth, originality, range, and sheer number is unmatched in the history of mathematics. While he continued to stress infinitary analysis and rational mechanics, he expanded the core of his research to astronomy, optics, and ballistics. He began to study electricity and magnetism as well. Relatively free of state projects during

[13] Marie-Jean-Antoine-Nicolas Caritat, Marquis de Condorcet, "Éloge de M. Euler," *Histoire de l'Académie royale des Sciences*, Paris, 1783 (printed 1786), pp. 37-68 and Nicholas Fuss, "Lobrede auf Herrn von Leonhard Euler" (1783) in *OO*, I.1, pp. XLII-XCV.

[14] Mario Biagioli, *Galileo Courtier: The Practice of Science in the Culture of Absolutism*, Chicago: U. Chicago Press, 1993, chapter 1.

[15] A. P. Juškevič and E. Winter, eds., *Die Berliner und die Petersburger Akademie der Wissenschaften im Briefwechsel Leonhard Eulers*, Berlin: Akademie-Verlag, 1959, part I, p. 3.

[16] As quoted in Rüdiger Thiele, *Leonhard Euler*, Leipzig: B. G. Teubner Verlagsgesellschaft, 1982, p. 106.

his first years in Berlin, Euler immersed himself in extensive and profound research. This short biography has space to cover only a portion of the highlights of his work in Berlin.

The first of Euler's landmark books published during the 1740s was his *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes ...* [E65] (The Method of Finding Curves That Show Some Property of Maximum and Minimum), completed in St. Petersburg in 1741. Its main body was sent to the publisher, Marcus-Michael Bousquet, in May 1743 and its appendices in December. It appeared in print the next year. The *Methodus inveniendi* presents Euler's invention of the initial stage of the classical calculus of variations, which sought to determine maximal and minimal lengths of plane curves in the course of movements, if any exist, and extrema among values of integrals, often called functionals.[17] While Euler replaced the previous *ad hoc* cases of problems with general solutions and arranged a hundred problems in eleven categories, making the calculus of variations an independent branch of calculus, his attention to curves kept for it a largely geometric format. He gave his differential equation or first necessary condition for supplying extrema for a class of functionals, and chapter 3 contains the most elegant solution of the time to the brachistochrone problem, the curve of quickest descent. At the urging of Daniel Bernoulli, Euler added two appendices. The first, the initial general tract on the mathematical theory of elasticity, includes the vibrating membrane problem and Euler's buckling formula for determining critical load and deciding the strength of columns. The second appendix contains a general form of the principle of least action. According to its twentieth-century editor, Constantin Carathéodory, the *Methodus inveniendi* is "one of the most beautiful mathematical works ever written."[18]

Even as he completed *Methodus inveniendi*, Euler remained active in number theory, achieving in 1741 the first of three proofs of Fermat's little theorem. In modern symbols, $a^{p-1} \equiv 1 \pmod{p}$, with p a prime and a relatively prime to p. Euler opened a seven years' campaign to prove all of Fermat's conjectures on the sums of squares. In April 1742 he wrote to Clairaut regretting the loss of the proofs of conjectures made by Fermat, noting that he had been able to prove only a few during the previous fourteen years. Euler asked whether some unpublished papers of the great man with proofs existed. He was attempting, unsuccessfully, to draw Clairaut into number theory. Clairaut replied that he had "never heard of Fermat's

[17] Jacques Hadamard, "Le calcul fonctionelles," *L'enseignement mathématiques*, 1912, pp. 1-18, in his *Oeuvres*, vol. 4. See also Rüdiger Thiele, "Euler and the Calculus of Variations," in this volume.

[18] Constantin Carathéodory, preface to *OO*, I.24, p. XI.

theories, nor do I know what happened to his papers." He held that the study of number theory was "not fashionable and is said to be dry."[19] Clairaut described as extremely subtle but unnecessary Euler's discovery of a method to determine whether a large number is prime. At the time the Berlin Academy was also attempting to develop a *perpetuum mobile.* Clairaut declared that to be impossible.

In 1742 and 1743 Euler also pursued a second phase of exhaustive calculations in infinitary analysis preparatory to his *Introductio.* Apparently a breakthrough in his perfecting of computational methods prompted this. He devised better techniques to sum the zeta functions for even integers less than 26. Employing Taylor expansions, deft substitutions, and other techniques, Euler summed infinite series, especially for π and e, for the most part more precisely than any predecessor. Continued fractions, for example, gave $e = 2.718281845904$. Euler had earlier studied Moivre's formula $(\cos x + i \sin x)^n = \cos nx + i \sin nx$, though not yet with the i notation, and he now returned to it, strengthening the connection between the ordinary trigonometric functions and both the exponential and the logarithmic ones, although extending their domains to the complex numbers. By 1744 he obtained the equation $e^{ix} = \cos x + i \sin x$, from which it follows that $e^{i\pi} = -1$.[20] Since Euler defined the natural logarithm as the inverse of the exponential function, this meant that $\ln(-1) = i\pi$. Thus, the logarithm of -1 is an imaginary number. Euler and d'Alembert soon debated the nature of logarithms of negative numbers.[21] The study that begins with Moivre's formula is believed to be Euler's concluding work on the *Introductio.* In May 1743 Euler had signed a contract with Bousquet to publish it, but he did not submit the entire manuscript for a year.

The comet of February and March 1742 must have heightened Euler's interest in its orbit and in celestial mechanics in general. He wrote to Clairaut, Delisle, and Heinsius for their observations of the comet, so that by com-

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[19] *OO*, IVA.5, p. 5

[20] Although he apparently never exactly wrote it in the modern form, this relation is now called Euler's identity. Today it is usually written as $e^{i\pi}+1 = 0$, which Richard Feynman has called "the most remarkable formula in math," because it connects these five most important mathematical constants and has significant applications in mathematics, for example, providing additional rigorous proofs of the infinity of primes. (Paul J. Nahin, *An Imaginary Tale: The Story of* $\sqrt{-1}$, Princeton: Princeton U. Press, 1998, p. 67.) This name for the relation probably arose in an early nineteenth century reference.

[21] Nearly a century and a quarter later, the Euler identity was essential to the stringent recognition that not every real number is the root of an algebraic equation: that transcendental numbers must exist. Charles Hermite proved in 1873 that e is transcendental, and afterward attention turned to π, which was long suspected to be the same. In a proof based on Euler's formula, Ferdinand Lindemann was the first to demonstrate the transcendancy of π, and David Hilbert improved upon his proof.

parison of sightings from separate places he could map its position more precisely. After adding observations of the comet of 1743, Euler found the orbits of both to be nearly parabolic ellipses. He continued his research on computing more accurately the irregularities and apsides of planetary and lunar orbits. In competition with Clairaut and d'Alembert, he addressed the thorny three-body problem. His textbook *Theoria motuum planetarum et cometarum* [E66] of 1744 contains the first differential equations for computing each point in the orbits of Earth and Mars.

Apparently Euler rather than Frederick and his courtiers in 1742 proposed translating into German Robins' *Principles of gunnery*, for long a subject in applied mathematics that had strong support of the court.[22] Euler had studied artillery fire for over a decade and admired this work of Robins. His *Methodus* includes an investigation of curves that apply to ballistics. The problem of the curved paths described by mortar projectiles in the air, or any other fluid, remained a challenge, though Euler credited Newton with solving it. Johann Bernoulli attempted to determine whether air resistance is proportional to v or v^2, and Brooke Taylor and others underestimated it for projectiles at higher speeds. Euler, like Newton, investigated resistance in both discontinued, rarefied fluids and air. (That the rarefied does not apply in our world did not stop Euler from studying possible computations for it. That kind of venture into purely theoretical situations is not uncommon among mathematicians.) Concentrating on internal ballistics, his *Neue Grundsätze der Artillerie*, [E77], published in 1745, also depended on chemical science. It begins with an examination of the nature of air and fire, citing Daniel Bernoulli's "incomparable *Hydrodynamica*," besides considering the influence of different temperatures on the compression of air. Essentially Euler devised thermal equations of state. He lacked a general formula for air resistance, instead computing its effect through a variety of rules, but found that at higher velocity only there is no back pressure on the projectile. When applied for lesser speeds, his techniques underestimate the resistance. In the combined *Neue Grundsätze* and a series of articles to 1753, Euler devised the first accurate differential equations for ballistic motion in the atmosphere.

To the *Neue Grundsätze*, Euler added annotations and appendices five times longer than the original book. He covered explosives and artillery topics that Robins had not. Proceeding as a physicist, Euler emphasized practical details and numerical computations rather than guiding principles. Although imperfect, the *Neue Grundsätze* began to transform a collection of separate rules into the first scientific work on gunnery. Publishing

[22] See Brett D. Steele, "Muskets and Pendulums: Benjamin Robins, Leonhard Euler, and the Ballistics Revolution," *Technology and Culture*, **35** (1994), pp. 348-382.

this book in German indicates that it was intended to improve the competency of lower military officers. Euler's later tables for mortar fire, included in a French translation, were studied by Napoleon Bonaparte and used to World War II.

From its start in 1746, the new academy had a major fissure over its prize topic for 1747, the monadic doctrine. The issue placed Euler against the Wolffian philosophers, who held sway in German universities. The monadic doctrine was central to the metaphysics of Leibniz and that of Wolff. Leibniz had proposed monads – animate, elastic, immaterial, geometric points of energy – as constituting the smallest components of matter. Wolff redefined these as souls and generally indivisible atoms. Academicians were not supposed to participate in the annual competitions. Euler broke that rule when he anonymously attacked the doctrine in the pamphlet *Gedancken von den Elementen der Cörper ...* (Thoughts on the Elements of Bodies ...) [E81]. Of molecules, as he termed them, Euler ascribed as their most basic properties impenetrability, drawing on Newton, and infinite divisibility, a concept probably taken from Euclid and Leibniz. He maintained that the laws of physics cannot empirically verify or quantify monads.

To this point Wolff, now in the autumn of his career, had expressed admiration for Euler's research in the sciences. But Euler's treatise incensed him, and he wrote in November 1746 to complain to the academy president Maupertuis, who had not initially followed the quarrel since he could not read German. Wolff dismissed Euler's affecting "a certain supremacy in the Republic of Letters" and declared his preference for "the thought of Leibniz in metaphysics and philosophy to the profundity of Euler."[23] Wolffian philosophers joined the attack on Euler's position. This controversy revealed an institutional split between the academy and universities in north German states. At Maupertuis' wish, Samuel Formey, the academy historiographer and future secretary, translated the articles from German into French. Although Formey claimed to be neutral, he became Euler's leading opponent. Even after Daniel Bernoulli urged him not to engage in this metaphysical debate, Euler in 1747 in the tract *Rettung der göttlichen Offenbarung* (The Rescue of Divine Revelation) [E92] extended his argument to an exercise in physicotheology. He claimed that Leibniz's monadic doctrine and principle of the pre-established harmony between mind and body contradicted the traditional Christian concept of original sin. That year Johann Justi won the Berlin Academy prize for a modest paper criticizing the concept of monads. Euler's *Anleitung zur Naturlehre* [E842], his

[23] A letter of 15 November 1746 as cited in William Clark, Jan Golinski, and Simon Schaffer, eds., *The Sciences in Enlightened Europe*, Chicago: U. Chicago Press, 1999, p. 442.

principal work on the theory of matter, was now completed, but it was lost, not rediscovered until 1844, and published in 1862.

Euler's battle with the Wolffian freethinkers was only beginning. In his strictly rational method, Wolff gave the principle of sufficient reason a lesser status than the principle of contradiction, which went against Leibniz and Johann I Bernoulli. Euler decried the Wolffian use of the principle of sufficient reason. The two sides also differed over mathematics. As he developed infinitary analysis, Euler far surpassed Wolff, who retained the primacy of geometry. Euler's *Introductio in analysin infinitorum*, probably the most influential textbook in modern mathematics, was in press, and would appear soon in 1748.

Meanwhile, Euler was making important advances in optics, mechanics, and the treatment of space and time. These appeared separately in three of Euler's papers written or published from 1746 to 1750. After rejecting Newton's corpuscular theory of light a decade earlier, Euler in his "Nova theoria lucis et colorum" (New Theory of Light and Colors) [E88] of 1746 presented the most comprehensive medium theory of light during the Enlightenment. Only for light as a pulse motion in an elastic ether did he accept Huygens' analogy between light and sound. Synthetic to a degree in optics, he began to transform into algebraic language Newton's geometric wave equations for vibratory motion. East of the Rhine, Euler's optics prevailed. From 1747 he intensely pursued a general method applying to all types of mechanical systems, whether they be continuous or discrete. That year he could treat discrete systems, but not fluid and solid media. In 1750 he completed his "Discovery of a New Principle of Mechanics" [E177], which did not appear in the academy's *Mémoires* until 1752. In it he finally recognized that the principle of linear momentum, Newton's second law, addresses all these systems. He was the first mathematician to express that law by a set of differential equations of motion. He called them "the first principles of mechanics." Today they are known as "the Newtonian equations." In "Réflexions sur l'espace et le tem[p]s" (Reflections on Space and Time) [E149], written in 1748 and published in the *Mémoires* in 1750, Euler argued that space and time must be absolute, for they envelope the principles of mechanics whose truths are incontestable. In effect appealing to mechanics, he rejected metaphysical first principles to determine their nature. He dismissed the claim by metaphysicians that space and time are relative, that is, imaginary and destitute of all reality. Inertia and the motion of solid and fluid bodies, he insisted, contradict their view of time as simply a succession of events.

Exploration and cartography continued to draw Euler's close attention. In a letter to Wettstein-Sarasin in England that appeared in the *Philosophical Transactions* of the Royal Society for 1747 [E107], he commented on the Russian search for a strait or northwest passage across North America, which provides a navigable water connection between Hudson's Bay and the Pacific Ocean. Euler called the ongoing search a "glorious undertaking." Captain Vitus Bering, who had led Russia's Second Kamchatka Expedition into the northeast of Siberia and died in 1741, suspected that the northern lands across from Siberia were connected to California and thus there was no water passage. Euler worried that Bering was correct but awaited proof from explorers.

In 1748 only seventy-five years beyond its birth, calculus had gone barely beyond its infancy, lacking a framework that identified and systematically arranged fundamental concepts along with a program for the development of the field. In a letter of July 1744 to Goldbach Euler described the two-volume *Introductio* "as a prodromus to *analysin infinitorum*," that is, a precalculus text. He was beginning to provide its necessary structure.

Since Leibniz, continental geometers had developed a theory of functions. Presenting functions as autonomous objects and making them central to calculus, the first volume of the *Introductio* builds a mostly methodical and comprehensive theory for algebraic and transcendental functions. "A function of a variable quantity [is] an analytical expression composed in any way whatsoever of the [changing] variable quantity and numbers or [fixed] constant quantities."[24] Chapters two and three give rules for formally combining and manipulating them and transmuting them "into other forms." The definition of function was evolving. Euler gave the value of e to twenty-three decimal places in Book II of the *Introductio*, and chapter eight of Book I takes π to 127 decimal places. After intensely approximating π, Euler reported this value from another geometer.[25] "For the sake of brevity," Euler wrote, "we will use the symbol π for the number."[26] Beyond adopting the symbol π, this master notation builder introduced in the *Introductio* the trigonometric functions as cos., sin., tang., cot., sec., and cosec.. But he had xx instead of x^2, as did almost everyone else for more than a century. This notation was also a convenience for typesetters. For natural logarithms base e he had lx for $\ln x$. He employed $f(x)$ from his earlier work but on this occasion used i and j for infinitely large numbers. He had not yet adopted the 'lazy eight' symbol ∞ for infinity.[27] Book Two of the *Introductio* unified Cartesian or analytic and for the first time put it into its modern form. Euler also completed the manuscript for the

[24] Leonhard Euler, *Introduction to Analysis of the Infinite, Book I*, trans. by John D. Blanton, New York: Springer-Verlag, 1988. pp. 1-5.
[25] Not until 1794 was the figure found wrong in the 113th place.
[26] Leonhard Euler, *Introduction to Analysis of the Infinite*, p. 101.
[27] The English mathematician John Wallis had introduced this symbol in the second half of the seventeenth century.

Institutiones calculi differentialis [E212] in 1748, but it took seven years to publish.

In 1749 Euler's *Scientia navalis* on the construction of ships and ship propulsion was published. It provides optimal ship designs, taking into consideration stability, handling, and speed, which oppose one another. Outstanding in both theoretical and applied mathematics, the *Scientia navalis* continued Euler's program founding continuum mechanics. After its release, he was concerned that the text was too difficult for navigators and began a revision to simplify it. The timing of publication posed a problem. Three years earlier Euler's old competitor Pierre Bouguer, victorious over him in the Paris prize competition in 1727, had published his *Traité du navire*, which allowed Bouguer to claim priority for naval science. But Euler was the first to give the principles of hydrostatics with variational solutions using differential equations.

Among the hundreds of discoveries Euler now made in mathematics was an algorithm in number theory to generate pairs of amicable or friendly numbers, which he published in the brief memoir "De numeris amicabilis" [E100] in *Nova acta eruditorum* in 1747. Two numbers are amicable pairs if each is the sum of the proper divisors of the other. In antiquity 220 and 284 had been known to be amicable, while it is probable that Thabit ibn Qurra in the ninth century found 17,296 and 18,416. In the seventeenth century, Pierre Fermat added a third pair, 9,363,548 and 9,457,506, and he gave a rule for computing amicable numbers if their factors are primes. The liklihood is that throughout history prior to Euler, only two other pairs may have been discovered. His "De numeris amicabilis" gives twenty-seven new pairs, a nine-fold increase over those previously well known. Euler's "Theoremata circa divisores numerorum," [E134] written in 1747 and published in volume 1 of the *Novi commentarii* three years later, returns to the subject with a potent new method for generating pairs. Introducing the concept of number theoretical functions, he invented the sigma function, which gives the sum of the divisors of a given number n. Building upon methods of Fermat with prime numbers, Euler obtained sixty-one more amicable pairs.[28]

As early as 1750 Euler, who had a habit of introducing subjects that he later developed, addressed what became in the late nineteenth century combinatorial topology. He supplied in solid geometry an elaborate but flawed proof that $v + f = e + 2$, interrelating in a convex polyhedron the number of vertices (v), edges (e), and faces (f). The modern form, $v - e + f = 2$, defines the Euler characteristic.

[28] For an explanation of his method, see Ed Sandifer, "How Euler Did It: Amicable Numbers," *MAA Online*, www.maa.org, November 2005, 6 pages.

In the late 1740s, the rivalries between d'Alembert, Daniel Bernoulli, Clairaut, and Euler intensified. The competition was not necessarily one against all others. Bernoulli, for instance, more strongly opposed d'Alembert than did Euler, and he, Clairaut, and Euler occasionally had d'Alembert as a common opponent.

A problem that challenged Bernoulli and Euler from the 1730s was the vibrating string problem, of interest in calculating musical frequencies. By 1743 they had formulated linear differential equations for loaded strings with clamped ends. In 1746 d'Alembert derived the linear wave equation for small vibrations of a string with such fixed ends. Although not the first partial differential equation and not in close accord with observations, it was the first to gain wide attention. But d'Alembert added to it many unnecessary conditions. He maintained that the string at time = 0 was in the equilibrium position, while Euler imagined the string at rest having an initial position differing from equilibrium, which would set the string in motion when it was let go. Euler, whose results did not appear in print until 1749, produced an alternate solution, allowing in modern terminology any piecewise continuous initial shape for the vibrating string. This was contrary to Alembert's solution, which allowed only one initial shape of the string. It had to be continuous or what is today called smooth, with continuously differentiable functions given by a single equation. A verbose controversy thus arose in 1751 over not the physical problem but the initial conditions. Bernoulli, who never agreed to start resolving the oscillating systems with differential equations, introduced in 1753 the concept of a composition of simple nodes and trigonometric series. Although the polemic added little to mechanics, Euler found nearly everything known today in that field concerning the vibrating string, and his abandonment of Leibniz's law of continuity opened the path leading to a more general conception of the function.

In pure mathematics Euler and d'Alembert corresponded between 1746 and 1751 mainly on the logarithms of negative numbers, which both found filled with paradoxes,[29] and imaginary roots of algebraic equations. Maupertuis and one of Euler's Basel cousins, the physicist Reinhard Battier, delivered some of the messages. The tone of d'Alembert's letters suggests an attempt to stump Euler and an effort of self-promotion.

After Euler in a letter of December 1746 tactfully rejected d'Alembert's idea that $\log(-x) = \log(x)$ for positive numbers x, d'Alembert responded in January 1747 that the new information disturbed him. He asked Euler to delete a portion of his treatise, titled "Recherches sur le calcul integral,"

[29] In chapter 21 of [E102], for example, Euler cites the paradoxes that make difficult the extension of the domain of the logarithm function to the set of all complex numbers.

sent the previous year for the Berlin Academy *Mémoires*, which discusses the $\log(-x)$. The logarithm controversy followed.[30] In March d'Alembert defined logarithms as the inverse of the exponential and challenged, using the curve of the exponential, Euler's position that the logarithm of a negative number is imaginary. The last argument in his letter reasons that $-1 = 1/-1$, so $\log(-1) = 0$. d'Alembert's contention "that logarithms of negative numbers are real," Euler responded in April, was "not fully correct," and he offered counterexamples. His letter has the symbol π to represent what is now 2π. After informing d'Alembert in August that he had removed the perplexing portion of his *Mémoires* paper, the next month Euler read to the academy his memoir, "Sur les logarithmes des nombres négatifs et imaginaires" (On the Logarithms of Negative and Imaginary Numbers) [E807], but he withheld it from publication. It appeared posthumously in 1862. The debate continued to December 1747, when Maupertuis informed Euler that d'Alembert wished to suspend "his work in mathematics for a little while to reestablish [his] health."[31]

Euler wanted to drop the subject, but d'Alembert persisted with letters in 1748 that in Euler's judgment offered little new about logarithms of negative numbers and were mainly argumentative. The debate diminished and ended for Euler in September, when he commented that "the matter of imaginary logarithms is no longer so familiar to me that I may rigorously respond to your latest remarks"[32] and by October d'Alembert had abandoned the argument. Still Euler submitted for the *Mémoires* in 1749 a lengthy memoir, "De la controverse entre Mssrs. Leibnitz et Bernoulli sur les logarithmes des nombres négatifs et imaginaires" [E168], defending his position. It appeared two years later, deepening discord between the two men.

Leading most to the break between the two men was competition not in pure mathematics but in fluid mechanics. In 1749 d'Alembert submitted an awkward paper, *Essai d'une nouvelle théorie de la résistance des fluids*, for the Berlin Academy prize competition. He introduced the concept of fluid pressure and derived several correct partial differential equations on plane and rotationally symmetric flow, but knitted these together painfully into a fabric of conjecture and error. Although the three man review jury chaired by Euler found d'Alembert's paper the best entry, Euler believed that it did not warrant the prize. The academy's rejection of d'Alembert's paper together with all others on fluid mechanics for the prize in 1750 came

[30] See Robert E. Bradley, "Euler, d'Alembert and the Logarithm Function," in this volume.

[31] *OO*, IVA.5, p. 273. Euler wished him success in his recovery.

[32] *Ibid.*, p. 294.

as a blow to him. He thought Euler responsible, and the junior astronomer Augustin Nathaniel Grischow, a member of the review jury and an acquaintance of d'Alembert, confirmed this. The academy remanded the prize to the 1752 competition, but d'Alembert did not reenter.[33] In 1750 and 1751 Euler prepared papers on fluid mechanics simplifying and generalizing the hydraulic theories of Johann I and Daniel Bernoulli, bringing the subject into its final form. Without citing his source, he built upon d'Alembert's concept of pressure and correct partial differential equations. d'Alembert sent a bellicose letter to Euler in September 1751, breaking with him, and criticized him in the entry "Hydrodynamique" in volume 8 of the *Encyclopédie.* In that article, which corrected Daniel Bernoulli's hydrodynamics, d'Alembert held that Euler "should have given my work greater substance on this subject and agree on the utility that he was able to obtain from it."[34] The immediate response of Euler to his letter is not known. Since he did not acknowledge the controversy subsequently, the breach did not soon heal. Expanding upon the awkward pioneering labors of d'Alembert, Euler in three classic papers from 1753 to 1755 developed for all of hydrodynamics an elegant mathematical foundation utilizing analysis and algebra. These papers, which were not published until 1757, praise the researches on fluids of Bernoulli, Clairaut, and d'Alembert, while establishing the importance of Euler's contribution. It is "impossible" he wrote," not to admire the agreement between their profound meditations and the simplicity of the principles from which I have drawn my two equations, and to which I was led immediately by the first axioms of mechanics."[35]

At the turn into the 1750s Clairaut, d'Alembert, and Euler vied for supremacy in attempts to resolve the three-body or perturbational problem. In celestial mechanics the great unanswered question was whether Newtonian attraction alone can describe all celestial motions.

By 1742 Euler, who was preparing lunar tables, had urged the compilation of more telescopic observations. Although he accepted the inverse-square law, he wanted to explore whether it gives only close approximations for some celestial motions and thus requires a small correction for precision for objects at short distances from each other within our solar system, such as Earth and the moon or Saturn and Jupiter in conjunction, but negligible at greater distances. After pointing out in 1743 that Newton had not fully explained the motion of the moon's apogee, the farthest distance

[33] E. Winter, ed., *Die Registres der Berliner Akademie der Wissenschaften*, 1746-1766, Berlin: Akademie-Verlag, 1957, p. 150.
[34] *OO*, IVA.1, p. 9 and Jean d'Alembert, "Hydrodynamique," in *Encyclopédie*, vol. 8, p. 371 ff. , trans. by John S. D. Glaus
[35] Thomas Hankins, *Jean d'Alembert: Science and the Enlightenment*, Oxford: Clarendon Press, 1970, p. 50.

from Earth, Clairaut began to reexamine it. At the Paris Academy in 1747, he announced that his differential equations and first-order approximations based solely on attraction confirmed only roughly half of the observed motion of the lunar apogee, that is, 20° rather than the observed 40° of the annual precession. Even though his result was not new, for Newton had found the same without differential equations in his *Principia*, the calculation gave heart to critics of Newton.

In April 1748 Clairaut, who was a member of the prize commission, told Euler of his capture of the Paris Academy prize, attempting to ascertain solely from Newton's law of attraction the perturbations in the orbit of Saturn caused by Jupiter. Utilizing observations made by Jacques Cassini at the Paris observatory, Euler improved procedures for computing these secular inequalities. He invented new trigonometric series, moving the study of perturbations beyond tiresome numerical integrations. Drawing on the latest, modified observations made with most accurate telescopes, he computed coefficients for successive terms of these series. For varying motion his new differential equations employed arbitrary constants, which perturbations actually cause to vary extremely slowly. His own erroneous figures he attributed to imprecise observations and mostly to a slight inaccuracy in the inverse-square law at large, interplanetary distances. Pleased that Euler's memoir agreed with him, in November Clairaut recommended correcting Newton's inverse-square law of attraction by adding the small inverse of the fourth power of distance.

But upon realizing that second-order approximations are crucial for determining the motion of lunar apogee, Clairaut found his first computations to be at fault. The second-order calculation produced a precession of the apogee $3°2'6''$ per lunar cycle, or about 40° per solar year. The actual value is $3°4'11''$. Clairaut retracted his statement that the lunar motion was contrary to Newtonian attraction and announced his new results at the Paris Academy in 1749. When Euler learned of this, he redid his derivations in July but found no error. His mistaken treatment of the lunar orbit as a rotating ellipse suggested that he needed only first-order approximations with differential equations. He was convinced that his calculations were beyond doubt and thus that Clairaut was wrong. While Bernoulli said that he had pointed Clairaut in the right direction, neither Euler nor d'Alembert knew the new procedure. Clairaut informed his Swiss colleague Gabriel Cramer that Euler had written twice that year describing "his fruitless efforts to find the same theory as I, and he begged me to tell him how I arrived at them."[36] He wanted the new method released so that it could be

[36] As quoted in Jean Itard, "Alexis-Claude Clairaut," in Charles C. Gillispie, ed. in chief, *Dictionary of Scientific Biography*, New York: Scribner, 1971, vol. 3, p. 283.

mathematically proven correct or at fault, as he thought it. Whatever the outcome, he believed its resolution would considerably benefit astronomy, physics, and infinitary analysis.

In 1749 when the St. Petersburg Academy asked for the topic for its competition to begin in 1751, Euler sent a list of four. Its members selected the first, whether lunar inequalities occur in accordance with Newtonian theory. In January 1750 Clairaut wrote to Euler that he found the prize question "very interesting," but he worried that no St. Petersburg academicians were competent to judge it and that it was simply Euler and not his colleagues in Russia who had defined the topic.[37] After Euler in a letter in June expressed impatience awaiting the publication of his complete method, Clairaut submitted for the competition the paper "Theory of the Moon Deduced Only from the Principle of attraction." Euler, who was the chief judge, read it with "infinite satisfaction" and learned its method. After revising his researches on lunar theory and carrying his complicated approximations of apsidal motion far enough, he discovered the source of his own errors. For his past hardheadedness, Euler asked for Clairaut's pardon. He lavished praise on him, asserting that his work was exemplary and gravitational attraction "is entirely sufficient to explain the motion of" lunar apogee. These lunar studies and the prize-winning paper of Clairaut gave "quite a new luster to the [gravitational attraction] theory of the great Newton."[38] But Euler still harbored a belief, partly from his study of magnetism, that a small corrective factor, in this case negligible, was needed for Newton's inverse-square law. Clairaut's computations on lunar motion he considered to be of the highest degree of difficulty, and he noted that prior efforts of other astronomers to show accord between lunar apogee and Newtonian attraction had been insufficient.

Another significant question for astronomers was whether it is possible to deduce quantitatively the precession of the celestial equinoxes, the points when the celestial equator crosses the ecliptic, and the nutation, wobbling, of Earth's axis. Since antiquity, astronomers had known the equinoctial points are advancing through the zodiac. Put another way, they were witnessing the proceeding of Earth's axis through the fixed stars. These two phenomena posed a further test of Newtonian attraction. In 1748 the British royal astronomer James Bradley announced his discovery of nutation. In attracting Earth's equatorial bulge, the sun and moon produce precession. Euler's prize-winning paper in 1748 considered precession in a perturbed planet's orbit, and in correspondence with d'Alembert he enthusiastically broached the subject. d'Alembert's *Recherches sur la precession*

[37] *OO*, IVA.1, pp. 83-88.
[38] *OO*, II.24, p. 1

des equinoxes of 1749 was the first study to deduce both correctly. Although unable to follow his clumsy and difficult differential equations, Euler returned to the subject in a paper the next year [E171] without mentioning him. He devised a different method. d'Alembert claimed that both methods were the same and was again angered that he was not cited. Euler's article appeared, moreover, in the volume of the *Mémoires* dated 1749, so a reader could not tell which was first. Euler quickly surpassed d'Alembert in perfecting the differential equations. But in volume six of the *Histoire* of the Berlin Academy for 1750, he gave d'Alembert his deserved credit for priority. That volume appeared in print two years later. By 1751 d'Alembert was addressing more general French Enlightenment thought with his masterful *Preliminary Discourse to the Encyclopedia of Diderot.*[39]

While Euler was not generally popular at court, Frederick II valued his applied mathematics and the prizes he won abroad. In a *History of My Times*, the king was to praise him as "an ornament of the court." After the War of the Austrian Succession ended in 1748, Frederick II assigned Euler more state projects, including three in 1749. He was charged to find ways of leveling the 70-kilometer Finow canal joining the Oder River with the Havel. That canal made Stettin a maritime port for Prussia. Euler took his fifteen-year old son Johann Albrecht to assist him on this project. His report concentrated on regulating pressure at the many locks. In September he was asked to increase the hydraulic pressure in pumps and pipes of the fountains at the royal residence of Sans Souci in Potsdam. Frederick wanted the water jets from the fountains to reach a hundred feet in height and thus compare with the jets at Versailles. Euler found that before forcing water to those levels the wooden pipes even with metal binding would explode, but in computing hydraulic pressure he omitted the effects of friction with embarrassing results. Frederick later blamed the failure on the vanity of mathematics. On September 15, the king assigned Euler a problem in what is today recreational mathematics, designing a lottery for Prussia. In the wake of two recent wars, the kingdom needed funds for pensions to widows. Lottery income would help. Euler briefly set aside hydraulics, and in two days had a version of a lottery offering a chance of drawing five numbers from a group of ninety. For this lottery, he computed the fair price of tickets and the increase in profit margins. But over the course of his career he gave little attention to probability, later writing only eight papers on the subject. In April 1749, fourteen years before the official Prussian lottery began, Euler had written to Goldbach about his success in a different lottery: "I

[39] See Jean Le Rond d'Alembert, *Preliminary Discourse to the Encyclopedia of Diderot* (1751), trans. and with an intro. by Richard N. Schwab, Chicago: U. Chicago Press, 1995.

have won this day in a lottery 600 Reichsthaler, which was just as good as if I had won the Paris prize this year."[40]

Euler's strife with the Wolffians continued in the Maupertuis-König affair, the leading scientific controversy of the century. The central issue was over who should be granted priority for the principle of least action, which now has the title Euler-Maupertuis: did it belong to the great Leibniz or to Maupertuis? In his *Essay on Cosmology* of 1750, Maupertuis restated the law: "Whenever any change occurs in nature, the amount of action is always the smallest." He argued that it was the final law of mechanics, which Descartes and Newton had sought, and he declared that it had universal application and had originated in three of his papers in the Paris Academy *Mémoires*. Maupertuis began to formulate it in "Loix du repos des corps" in 1740, published it in "Accord du différentes loix de nature ..." in 1744, and two years later in "Les loix du mouvement et du repos ..." called it the "general principle" of mechanics. In the article "Action" in volume one of the *Encyclopédie* in 1751, d'Alembert endorsed its claim to universality. But in "De universali principio aequilibri et motus" in Leipzig's *Nova acta eruditorum*, also in 1751, the Swiss mathematician Johann Samuel König made counterclaims, which questioned the extent of the application of the principle and thus Maupertuis' scientific achievement, along with his integrity.

König's article surprised Maupertuis, and with Euler and the young Swiss Johann Bernard Merian, a tireless ally in metaphysics, he planned a rebuttal. Euler actually deserved priority for the principle, for it had appeared in the appendix on elastic curves to his *Methodus inveniendi*, where he wrote "nothing at all takes place in the universe in which some rule of maximum or minimum does not appear." Yet he steadfastly defended Maupertuis. Maupertuis lacked the mathematical skill to devise differential equations for the principle, so Euler was providing these in a series of articles. Journalists in the German press and university faculty, including Johann Christof Gottsched and his Leipzig circle, generally treated Maupertuis with hostility, especially for his autocratic handling of the Berlin Academy. In the growing debate, Euler appealed to scientific authority, contrasting the accomplishments of the "illustrious president" with those of "the professor," whose grasp of mechanics "was worthy only of contempt."[41] He dismissed König's journalistic allies as "public quibblers," who lacked adequate knowledge to make judgments in the sciences.[42] König's response

[40] Paul Heinrich Fuss, ed., *Correspondance mathématique et physique du quelques célèbres géomètres du XVIIIéme Siècle*, vol. 1, p. 497.
[41] Mary Terrall, *The Man Who Flattened the Earth: Maupertuis*, p. 300.
[42] *Ibid.*, pp. 300-301.

was unlikely enough to cast doubt upon his : an injustice that he helped perpetuate himself, for the letter was found more than a century later.

König based his claim on a copy of a portion of an unpublished letter of Leibniz to Jakob Hermann in Basel in October 1707 defining action and maintained that the origins of the principle of least action had appeared in Aristotle. Maupertuis demanded that he produce the original letter and requested the assistance of Frederick II, for the quarrel was an attack on the king's academy. The monarch wrote to magistrates in Bern and Basel asking them to check among Hermann's papers for the missing letter, and the Bernoullis helped in Basel. Both searches, completed by February 23, 1752, proved fruitless. No letter was found. Maupertuis then asked Abraham Kästner to look among Leibniz's papers in the Leipzig Library. On April 5 Kästner reported that he had failed to find even one letter from Leibniz to Hermann. On completion of the research into the matter, Maupertuis called for an extraordinary meeting of the academy on April 13, 1752, and invited two noble curators to lead it. He intended to be absent. To be sure that the German-speaking members not fluent in French could understand it, Euler read in Latin the report on König [E186]. It was published in French, "Exposé concernant l'examen de la lettre de Leibniz." [E176] Euler found König's position untenable and asserted that the letter fragment was "a forgery, either to malign Maupertuis or falsely to give exaggerated praise for the great Leibniz, who required no such help."[43] For that conclusion, resting on circumstantial evidence, Euler has drawn reproach. Only half the academicians attended the meeting, but those present unanimously endorsed Euler's ruling on the letter. The letter in question, discovered more than a century later, does not present a complete version of the principle of least action.

In a series of articles Euler and Merian defended the academy's judgment, while many German and Dutch journalists and university faculty opposed it. In two replies König rejected what he defined as the tyranny of the academy and considered the public alone "his natural judge." In 1752 the debate turned from a scientific quarrel into a literary affair when Voltaire, who was in Potsdam visiting Frederick, issued his caustic satire *Diatribe du Docteur Akakia, medicin du Pape, et du natif St-Malo*, a compilation of brief pamphlets ridiculing Maupertuis. Docteur Akakia (or Guiless), who denotes Voltaire, pilloried the hapless native from St. Malo, who is Maupertuis. Article fifteen of its first edition, which describes Professor Euler as "our lieutenant" and "lieutenant general" to Maupertuis, still praises him as "a very great geometer" belonging to a line from Copernicus, Kepler,

[43] Louis Gustave Du Pasquier, *Leonhard Euler et ses amis*, Paris: J. Hermann, 1927, p. 105.

and Leibniz through Johann Bernoulli, recognizes that he put the principle of least action into mathematical formulas, and declares that scholars who can understand his work find it to be that of complete genius.[44] But article nineteen holds that Euler never learned philosophy, that the formulas of this "phoenix of algebraists" led in one instance to the notion that a body dropped through a hole to the center of Earth would return to the surface, and questioned his trust in resolving paradoxes by his calculations more than logical analysis. In a play on the principle under discussion, Voltaire proclaimed that among geometers Euler sought to produce the maximum of calculations in published works. Later editions of *Docteur Akakia* drop reference to Euler.

Even after the king issued an anonymous pamphlet titled "Lettre d'un académician de Berlin à un académicien de Paris," defending the academy decision and eulogizing Maupertuis, Voltaire continued his attack. Outraged at Voltaire's effrontery, the king in December had *Docteur Akakia* burned by the public executioner at several places in Berlin. Whether the booklet damaged the health of Maupertuis, who had earlier suffered a series of illnesses, is arguable. Having won his case at the Berlin Academy, Maupertuis in the stubbornness of his criticism of König lost it before Europe's republic of letters. Growing pressure for freedom of the press against absolutist controls seems to have influenced the intensity of the press accounts. But in this dispute Maupertuis and Euler were not isolated, as is sometimes believed. In 1753 Georg Wolfgang Krafft at the University of Tübingen held that Maupertuis was the sole author of the principle, and the next year d'Alembert in the *Encyclopédie* article "Cosmologie" praised Maupertuis for fashioning into a single law the impacts of hard and elastic bodies, rejected the claims of König, and cited the elegance and directness of Euler's applications.

In 1753 Euler's mother, who was used to living in the country, had him purchase for her for 6,000 Reichsthaler a pleasant estate in Charlottenburg, an area then outside Berlin with royal and noble residences. After his brother Heinrich died in 1750, he had persuaded her to come live with him. That year Euler, Katherina, and Albrecht traveled to Frankfurt to Katherina's Dutch cousin Johannes Michael van Loen, a historian and theologian, to meet his mother, who had arrived there from Basel. While a devoted father, Euler sent the younger children and a tutor to live with his mother. A decade later Berlin academicians began the story that Euler had small children playing about him as he wrote. The relocation of the younger children, along with Albrecht's failure to remember his father's doing any household chores while he was young, casts doubt on it. But Euler appears

[44] Otto Spiess, *Leonhard Euler*, Leipzig: Verlag von Huber & Co., 1929, p. 135.

to have kept up close contacts with his children. For relaxation, exercise, and thinking in solitude, he liked to take frequent walks to Charlottenburg, which was about a mile from his Berlin house.

From May 1753 to July 1754, Euler was acting president of the academy. Earlier in April Maupertuis had asked Frederick that he be allowed to "relinquish the administrative details of the academy during my absence to Professor Euler,"[45] whose integrity, brilliance, and zeal for the institution justified the appointment. The king approved the request. In his attempts to recuperate from a lung disease, Maupertuis was staying away longer from Berlin in the healthy "native air" of France.

Euler, who was uncomfortable with management, had to follow limits set by Maupertuis. He showed himself to be conscientious, closefisted with finances, and sometimes stubborn, but always equitable. The king's desire to add luster to his court made urgent the recruitment and retention of distinguished members. After the astronomer Johann Kies left for Tübingen, Euler from June 1753 secretively pursued Tobias Mayer at Georg-August University in Göttingen. The academy treasury had for the post only 550 Reichsthaler, an amount little more than Mayer was receiving, but he could earn another 150 Reichsthaler for preparing a calendar. Frederick was expected to increase academy funding and pensions, the latter by reducing the number of academicians. Upon his return Maupertuis would certainly provide a higher salary, Euler assured Mayer. Meanwhile the monarch was upset in February 1754 when chemists and physicians fought noisily in public over their candidates for the same position. "A certain degree of anarchy reigns in your academy," he wrote to Maupertuis urging his return as soon as possible.[46] Looking forward to working with Euler, Mayer in July requested a minimum annual pension of 650 Reichsthaler, another 100 for moving expenses, and free housing.[47] That month the Russian academy sent a letter asking Euler to extend to Mayer an offer bringing higher pay, but he withheld that information. In August Maupertuis authorized a pension of 700 Reichsthaler, 100 for moving costs, and free accommodations. On the twenty-seventh Euler extended these terms. On the thirty-first he informed Mayer of the Russian offer, though recommending that he reject it. Mayer did so, but when he requested that his resignation from the Georg-August University be granted, it was not. Instead Hanoverian officials asked him to state conditions that would persuade him to remain. To

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[45] Letter of 24 April 1753 to David Köhler, as cited in Du Pasquier, *Leonhard Euler et ses amis*, p. 110.
[46] As cited in Mary Terrall, *The Man Who Flattened the Earth: Maupertuis*, p. 349.
[47] Eric G. Forbes, ed., *The Euler-Mayer Correspondence (1751 - 1755)*, New York: American Elsevier Inc., 1971, p. 89.

his surprise they met them, giving a salary above the Berlin amount and sole directorship of the nearly completed Göttingen Observatory. Stating to Euler that the Hanoverians thought more highly of him than he deserved, Mayer declined the Berlin Academy post.[48] He was grieved not to be able to work directly with Euler, he wrote in October, and hoped that this decision would not harm their written association.

From 1754 Euler's influence was expanding beyond the academy to German universities. Confident in Euler's identification of talent in the sciences, Frederick appointed him to choose a successor to Wolff as professor and departmental chair of mathematics at the University of Halle. Although Euler knew that Daniel Bernoulli would decline, he offered his friend the post. Next Euler nominated the mathematician Johannes Andreas von Segner from Göttingen, a severe critic of the mathematical and scientific foundations of Wolffian philosophy. Before the appointment in 1755, Euler persuaded the monarch to purchase all of Wolff's equipment for Segner's research. In May Frederick had him invite the Swiss anatomist Albrecht von Haller, who had recently retired from Göttingen, to be chancellor at Halle. Euler was not enthusiastic. The king wanted Halle to attain eminence among German universities, and Euler believed that Segner would contribute more to that than von Haller. Having to comply with the king's wishes, he proposed a salary of 2,000 Reichsthaler. In response von Haller requested 3,000 Reichsthaler in August and did not wish to undertake the post for another decade. Frederick found von Haller's terms excessive and left further negotiations to Euler, who offered an unacceptable 2,400 Reichsthaler. In December von Haller declined to accept the chancellorship. But the relations between the monarch and Euler were for the moment good. Frederick was especially pleased that Euler had him elected one of the ten foreign members of the Paris Academy of Sciences in 1756.

In astronomy, which remained a primary area of Euler's research from 1751 to 1756, a critical question was whether the secular change in the mean motion of planets is cyclical or linear. For confirming that Newtonian attraction alone explains the mechanical operations in the solar system, resolving it for the large planets Jupiter and Saturn was essential. In 1752 Euler won another Paris Academy prize for a memoir using flawed algebra to conclude wrongly that Jupiter and Saturn were both accelerating. Clairaut and d'Alembert competed independently to develop a more accurate lunar theory than that given in Euler's *Theoria motus lunae* [E187], his first lunar theory, in 1753. d'Alembert's two-volume *Recherches sur differens points importans du systme du monde*, which followed the next year, had been completed in 1751. d'Alembert's text, which was the first to

[48] Eric G. Forbes, ed., *The Euler-Mayer Correspondence (1751 - 1755)*, pp. 91 - 92.

propose an ordered sequence of increasingly refined, successive approximations, depended more on mathematical refinements than on observations. Clairaut was d'Alembert's principal opponent, criticizing his "long and tedious calculations' and finding his work careless and his tables inferior.[49] Prior lunar tables based on Newtonian dynamics had erred by more than $5'$ of longitude, while none of these three geometers had errors of less than $3'$, which was insufficient for determining longitude at sea within a degree.

Refining equations from Euler's 1748 prize-winning paper and employing superior astronomical equipment and observations involving the distance of the moon from a fixed star, Mayer was the first to produce accurate enough lunar tables to find longitude within a half degree. To d'Alembert's attack on his and Mayer's work, Euler responded to Mayer in June calling d'Alembert a "braggard" and promised to refute his "unfounded and jealous allegations."[50] In May 1755 Euler urged Mayer to enter the competition for the 20,000 sterling prize the British parliament promised in 1714, for he had perfected the method to determine the location of the moon to within $1'$, which allowed for finding longitude at sea to a half degree. Euler's relations with d'Alembert were broken. For a decade after 1752, he also stopped corresponding with Clairaut, but there is no evidence of a falling out. More likely their interests were diverging. After Euler failed to be elected a foreign member of the Paris Academy of Sciences in 1753, a vacancy that occurred in 1755 when Abraham de Moivre died was reserved for the president of the Royal Society of London. But Clairaut probably helped arrange to have Euler elected simultaneously in June as an associate member.

In 1755 Euler's *Institutiones calculi differentialis* (Foundations of Differential Calculus) [E212] appeared in print. This second part of his trilogy on calculus he had completed in 1748, when he was forty one. The *Differential Calculus* identifies the elementary principles of the field and was the first text generally to organize the field systematically. In revising Leibniz's calculus, Euler introduced and made standard the "differential coefficient," which eliminated indeterminacy over higher differentials. Chapter III contends that the foundations of the calculus of the infinitely small are not so mysterious as was thought. Euler's calculus of zeroes distinguishes between arithmetical and geometrical proportions. The differential, he asserted, is zero. Euler argued that analytic expressions can be given as power series and treated functions as formal representations for computing rather than as the modern mappings. The *Differential Calculus* sets out the first extensive research program for differential calculus and related topics. In

[49] Thomas Hankins, *Jean d'Alembert: Science and the Enlightenment*, p. 37.
[50] Eric G. Forbes, ed., *The Euler-Mayer Correspondence (1751 - 1755)*, p. 88.

assessing its initial impact, it needs to be noted that four-hundred-six of the five-hundred copies of the first edition remained unsold after six years.[51]

Characteristically in transforming infinitary analysis, applying it to mechanics, and contributing to analytic number theory, Euler was bold, nearly tireless, agile, and occasionally bizarre, devising fertile analogies, algorithms, and formulas and making massive computations. He was unmatched in inventing unorthodox methods for summations of infinite series with deft interpolations, approximations, and substitutions. He followed his peerless analytical intuition. When blocked from making further advance in some field, he awaited a breakthrough, meanwhile pursuing other interests. When the breakthrough occurred, he returned to an area to perfect methods and add extensive computations. A formalist, Euler made great demands on himself for calculations and was almost always correct. And he required the most accurate observations and experiments to confirm his equations in mechanics. He made a few errors. These and his scarce lapses in rigor have gotten his work portrayed as "happy go lucky analysis" or as reckless.[52] This judgment seems to stress his infrequent errors and might apply to another with a lesser intuition but seems mistaken for him. For Euler, who had hundreds of discoveries to announce, in a few cases without proofs, the later motto of Carl Gauss, "*pauca sed matura*" (few but ripe), does not hold. For proofs, tests for convergence of infinite series were only beginning to evolve. Euler supplied one, the integral test. A general theory of convergence and satisfactory foundations for calculus would not come until the early nineteenth century.

A frequent topic in Euler's correspondence to 1756 was electricity. He sought to know its physical causes and examined reports on its medical applications. In 1752 he obtained and studied a copy of Benjamin Franklin's *Experiments and Observations on Electricity* in French translation. Euler was later saddened to learn that in July 1753 Georg Richmann, making measurements in St. Petersburg related to Franklin's kite experiment, had not taken necessary precautions and was killed. He surely helped his son Johann Albrecht with the paper on electricity that won the prize of the St. Petersburg Academy in 1755. At the moment that institution was at ebb in its research. According to its *Protocols*, only three to five members attended most meetings.

Euler's mastery of the mathematical sciences was attended by a strong concern for practical scientific technology, especially optical equipment. Eu-

[51] Clifford Truesdell, *An Idiot's Fugitive Essays on Science*, Berlin: Springer-Verlag, 1984, p. 296.
[52] H. Weyl, "David Hilbert, 1862 - 1943," no. 131 in *Gesammelte Abhandlungen* Berlin: Springer, 1968, p. 124.

ler encouraged the construction of a two-foot telescope with lenses he designed in order to be able to detect Jupiter's satellites distinctly. One result from his study of pumps and the motion of fluids was his effort to improve the theory of turbines, which he presented in detail in his memoir "Complete Theory of Machines Which are Placed into Motion by Their Reaction to Water" [E222] in the academy *Mémoires* for 1754. Another difficult scientific challenge for Euler was developing a compound achromatic refractor to improve telescopes. Chromatic dispersion limited them. In London John Dollond, the optician to the king, who corrected errors in Newton's optical experiments, maintained that he had achieved largely achromatic lenses by building upon Euler's related mathematics, and for a long time he held a royal monopoly on their manufacture. But Euler rejected his claim to have constructed them. Dollond did not release his optical measurements that indicated that no refractive indices exist for different colored rays in different media and that instead each must be checked individually. Perhaps thinking as a theoretical physicist, Euler wanted an analogy to the eye or a mathematical law covering them all. The Royal Society of London, to which Dollond belonged, declared that he was criticizing Newton. To the contrary, Euler believed that his research on lenses was aligned with Newton's and that he was simply adding computations from infinitary analysis.

In August 1755 Euler received a letter from his foremost new colleague, the nineteen-year-old Ludovico de la Grange Tournier of Turin, better known as Joseph Louis Lagrange. The letter was revolutionary in its proposal for a way of eliminating the tedium of Euler's geometric considerations in *Methodus inveniendi* by reducing it entirely to analytic techniques. Lagrange began to provide the delta algorithm of analytic variations, which produced the Euler differential equation or first necessary condition for maxima or minima and more. His letter started an epic in what Euler renamed the calculus of variations. Lagrange perfected his ideas and published them in the 1760/61 issue of the *Miscellanea Taurinensia.* Euler withheld articles on the subject until afterward in order to give Lagrange full credit for his discovery.

Through the Seven Years' War from 1756 to 1763, Euler remained in Berlin. The war pitted Prussia and Britain against Imperial Austria together with France and Russia over Silesia. Acting as a Prussian patriot, Euler volunteered to translate messages in Russian intercepted by the Prussian military. In the fall of 1760 he took his son Karl to study for his medical degree at Halle, which was outside the battle zones. For his sons he would write papers for competitions. Karl's memoir on the average variable motion of planets won the annual prize of the Paris Academy of Sciences for 1760. Before October, when the Russian army entered Berlin, many fled the city, but Euler stayed. Although the Russian general Count Gottlob

Heinrich Totleben promised that his Charlottenburg estate would be safe, Russian soldiers pillaged it. When Totleben learned of this, he exclaimed that he had not come to make war on the sciences and reimbursed Euler for all losses, giving sums totaling more than their value. When another general informed Empress Elizabeth of the incident, she sent an additional 4,000 rubles.

Since the death of Maupertuis in 1759, Euler had been acting president of the Berlin Academy in close alliance with Merian. He hoped to be named president, but his relations were deteriorating with Frederick. The king did not seem to comprehend that Euler's work in the mathematical sciences was of greater significance than the writings of the Paris savants. He had only one candidate in mind, d'Alembert, who was a "*philosophe*, critic, science editor of the famous *Encyclopédie*, a man of noble ancestry, and French."[53] The last two were doubtless of more than small importance to Frederick. Above the sciences, the monarch preferred witticisms, freethinking, and poetry, which was not a favorite subject of Euler. In a letter to d'Alembert on the musing of mathematicians regarding poetry, Frederick wrote: "A certain geometer, who lost an eye while calculating, decided to compose a minuet with *a* times *b*. If it had been played in front of Apollo, the poor soul would have been skinned alive, as was Marsyas."[54] Probably from an actual episode, the monarch wrote disapprovingly of "a certain son of Euclid" who was distracted with his computations while at the theater, even during the most dramatic scenes.

In June 1763, Euler wrote to his friend Gerhard Müller, the secretary of the St. Petersburg Academy, about d'Alembert, who had just rejected a lucrative position there. Along with other French savants, he chose to continue with a small pension in Paris rather than accept a greater one in St. Petersburg. In reference to d'Alembert's "unbearable arrogance," Euler declared that he should "understand that he is not at all suited for [the Berlin presidency]."[55] Euler rejected as cavalier and counter to abundant experience d'Alembert's efforts to contradict Bernoulli's hydrodynamic theory and defined his debates with Clairaut in astronomy as disgraceful. Euler was troubled by d'Alembert's possible recommendation of the Chevalier Louis de Jaucourt, author of the article "Monarchie" in the *Encyclopédie* describing limits of royal power, as a substitute for the presidency, which would open the way for several radical French thinkers to be admitted to

[53] Adolf Harnack, *Geschichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin*, vol. 1.1, p. 355.
[54] As cited in Du Pasquier, *Leonhard Euler et ses amis*, p. 116.
[55] *OO*, IVA.1, R1885, p. 314.

the academy.[56]

Later in June d'Alembert, who had first met with Frederick in Wesel in 1755, had a second encounter in Berlin. For over a decade the monarch had entreated him to come to the Berlin Academy, providing him with monetary gifts. There was unease at court over receiving such a distinguished French thinker. Euler feared the worst, that either d'Alembert or another Frenchman would be named president. Eager to pay his respects, d'Alembert met with Euler and was astounded by his colossal memory, his knowledge by heart of the growing number of computational formulas in analysis, and the clarity of his logic. He had no intention of accepting the presidency. Euler was slightly embarrassed when d'Alembert recommended him for it and got his pension raised. From his salary, prize monies, and investments, Euler was now wealthy. He was enchanted with the meeting with d'Alembert. In October he wrote to Goldbach: "Our friendship is perfect, and one cannot tell me enough times of the pleasant things that M. d'Alembert has said on my behalf to the king."[57] While he would not move to Berlin permanently, d'Alembert through correspondence with Frederick became "the secret president" of the academy.

Euler's growing correspondence with Müller shows that from at least 1761 he was seriously contemplating going back to Russia. His letter of May 17 states that he had sold his Charlottenburg estate for 8,500 Reichsthaler, freeing himself in general to depart Berlin. But before the end of the war, he would not be in a position to accept an offer from St. Petersburg.

From the close of the war in 1763 to 1766, Euler remained in Berlin. In 1763 he became the head of the powerful French Calvinist Consistory. To raise funds for the Prussian economy devastated by the Seven Years War, Euler recommended a new lottery and acted as a business broker to increase porcelain manufacturing. In July Catherine II had the academy assessor Grigoriz N. Teplov propose terms for his return to Russia. He recommended that Euler be conference secretary and director of the mathematics class of the academy with an annual pension of 1,800 rubles, that Johann Albrecht be an ordinary professor with a pension of 600 rubles, that Euler's wife receive a widow's pension, and that there be 500 rubles to fund the family's trip to St. Petersburg.[58] Apparently desiring a higher pension, Euler declined this offer in July but expressed continuing interest in a possible return. He now wished only to pursue his research. Because of a teaching requirement, he had not considered taking a university position

[56] *OO*, IVA.1, p. 313.

[57] Paul Heinrich Fuss, ed., *Correspondance mathématique et physique, du quelques célèbres géomètres du XVIIIéme Siècle*, p. 668

[58] *OO*, IVA.1, p. 438.

in Holland. Euler wrote that he had been working continuously on his *Integral Calculus* and hoped that the St. Petersburg Academy would publish it as soon as possible. He also wanted the St. Petersburg Academy reorganized and reformed, largely through a study of other academies, in order to recapture its early dynamism. It began to implement these changes before his arrival and to fill vacant positions with his candidates. From 1763 Euler reminded Müller of his small annual pension from the St. Petersburg Academy, purchased books for it, and continued to send articles for its *Novi commentarii.* In 1765 the two men discussed the possible publication of Goldbach's correspondence.

From the winter of 1763/64 Euler's position in Berlin had worsened. Reasserting absolutism in Prussia, Frederick assumed the academy presidency. Selection of members and finances became critical issues. After Maupertuis' death, Euler had managed both. The nine corresponding members selected in 1760 included Lagrange and three Germans, among them Gotthold Ephraim Lessing. Partly from his unhappiness with Lessing, Frederick inserted in 1764 a reservation clause that allowed academicians no voice in choosing new members. From 1764 to 1766 Frederick named seven Frenchmen and two Swiss as new members. This looked to be a vindication of the fear that Euler fear had prognosticated in a letter of October 1763 to Goldbach that the Berlin Academy was being transformed "into a French academy," meaning an academy open to radical French ideas, seemed accurate. Euler could not block the king's selection of French Encyclopedists and, to Euler's moral distaste, Claude Adrien Helvetius, the author of the hedonistic *De l'esprit.* In the winter of 1763/64 Frederick had ordered a record made of the academy's finances during the war. He was dissatisfied with them and held Euler responsible. The total amount spent was 25,000 Reichsthaler. A nasty quarrel erupted over the annual sales of almanacs. The king lacked confidence in its commissioner, David Köhler, whom Euler defended. Köhler in fact pocketed some calendar funds. In 1765 Frederick appointed a five-man commission led by Euler to review sales. The majority concluded that a better monitoring of sales should increase revenues from 13,000 to 16,000 Reichsthaler. Without informing them, Euler wrote directly to the king in June with a proposal to keep Köhler in charge with additional supervision. His letter met a rebuke from Frederick, who responded "To be sure I cannot reckon curves, but I do know that 16,000 thaler are more than 13,000." [59] Although the king approved one of two pensions of 400 Reichsthaler for Johann Albrecht, Euler now lost control of finances. His position in Berlin was untenable. His struggle for the autonomy of science in Prussia had failed.

[59] J. D. E. Preuss, ed., *Oeuvres de Frederic le Grand*, Berlin, 1852, vol. xx, p. 209.

As he lost his capacity to lead the Berlin Academy in 1765, Euler began to separate from its activities and made no secret of his desire to relocate. In July, when Euler was excluded from public celebrations, the Prussian state minister J. L. Dorville asked him to nominate candidates to be royal librarian. One the three names that Euler submitted was a fictitious Wegelin of St. Gall. The selection committee's Heinrich de Catt and Formey soon realized what had happened. To avoid embarrassment, Euler later indicated that Wegelin had withdrawn his name. In December he wrote to the Russian High Chancellor Count Michail Voroncov describing his situation in Berlin and giving terms for him and his sons to return to Russia.

Empress Catherine II was an enlightened ruler who cultivated the arts, showered gifts upon her science academy, and offered its members protection. To bring immediate prestige to her academy, she especially wanted to acquire Euler. She directed her ambassador in Berlin, Prince Vladimir Dolgorukij, to negotiate with him and grant any request that he made. Euler posed comparatively stiff financial terms for a man of science, asking double Catherine's previous offer. He requested the position of vice president of the academy with an annual salary of 3,000 rubles for him, a widow's pension of 1,000 rubles for his wife, an ordinary professorship with a salary of 1,000 rubles for Johann Albrecht, and guaranteed positions in medicine for Karl and the military for Christoph. He wanted a staff of young mathematicians assembled around him, free housing and heating, and exemption of his house from military quartering. For a scholar of Euler's stature, the empress thought this an inexpensive bargain. After Voroncov indicated her positive response in a letter in January 1766, Dolgorukij officially confirmed it.

Euler had already written to the secretary of the St. Petersburg Academy, Jakob Stählin, that his ties with Berlin were completely severed. A personal element was added to his unhappiness with Frederick. The king had unsuccessfully forbad the marriage of one of Euler's daughters to a nobleman and would not guarantee an officer's position in the army for his son Christoph. After the Seven Years' War, Frederick chose only aristocrats as officers. If he was not invited to St. Petersburg, Euler claimed that he and his family might relocate in Switzerland, Ukraine, or some Russian province.

But Euler had first to obtain permission from Frederick to leave. His first two requests in February 1766 were met with silence, a known strategy of the monarch. Euler's extended family of fourteen, except for his youngest son Christoph who was in the Prussian army, was readied to leave that month. He knew that Frederick would refuse to release Christoph from his military service. In March, Euler and his eldest son stopped attending meetings of the Berlin Academy. With Swiss independence and tenacity, Euler asked twice more for permission to depart. After the third letter, Frederick ordered him on March 17 to desist in his repeated request. Euler did not. Instead his letter of April 30 asks that he and his sons Johann Albrecht and Karl be allowed to return to Russia. With no word of thanks, Frederick wrote tersely in two lines in May, "I permit you to quit and depart for Russia."[60] The king probably did not want needlessly to antagonize his "dear sister" Catherine II. On May 29 Euler and Johann Albrecht said their farewells at the Berlin Academy, and on June 9 the Euler party, consisting of the fourteen family members and four servants, left Berlin. Only Christoph had to remain in Prussia. The leave-taking was emotional; Prussians of royal blood, notably the Margrave of Brandenburg-Schwedt and his daughters, joined with Euler's other students in expressing their regrets.

The Euler party traveled overland in a small wagon caravan. At the invitation of Prince Adam Czartoryski, Euler visited the Polish capital of Warsaw, where he met with the new king Stanislaus August Poniatowski and was feted for ten days. Upset with the loss of an eminent scholar whose achievements he valued even as the Swiss commoner's ways offended his aristocratic preferences, Frederick wrote disparagingly of Euler and announced his pleasure that in September 1766 d'Alembert was able to have Lagrange succeed Euler.

Prominent among the profusion of scientific accomplishments that had occupied Euler during his Berlin years after 1760 is a collection of 234 letters, later entitled *Letters to a German Princess* [E343,E344,E417], which were sent from 1760 to 1762. When the correspondence began, Princess Charlotte was fifteen. Euler and her father, the future Margrave Friedrich Heinrich von Brandenburg-Schwedt, shared an interest in music. He occasionally visited their palatial residence and played music with the father. The letters written in French address natural philosophy, that is, the sciences, together with philosophy and religion. They have three natural divisions: general science, 1 - 79, which cover the topics of music, the air, optics, gravity, cosmology, the tides, and the theory of matter, especially the property of impenetrability and monads; philosophy, 80 - 133, including liberty, spirits, Christianity, language, syllogisms, evil, happiness, certainty of scientific, moral, and historical truths, foundations of knowledge, divisibility, and monads; and physical questions, 134 - 234, spanning electricity, magnetism, lenses, the telescope, the microscope, and stellar distances.

The *Letters* are a high popularization presenting Euler's synthesis in the sciences plus his original insights, along with an apologia. He examined the major natural philosophies, the Cartesian, Newtonian, Leibnizian, and Wolffian, in greater detail and with a better command than any other

[60] *OO*, IVA.6, p. 393.

popular rendering, for example the Newtonian texts of Pemberton and Voltaire. Euler strongly defended his religious views against freethinkers and French Encyclopedists.[61] Letter 85 describes Leibniz's pre-established harmony between mind and matter as "destructive to human liberty." In letter 115 Euler, like the Christian Platonists, divided proofs of truth into three classes: the sensible, intellectual, and historical and moral. For the first, he accepted Lockean sensationalism and held common ground with the Wolffian Alexander Baumgarten, whose *Metaphysica* of 1739 and the *Aesthetica* of 1750 hold that the senses have their own rules and perfection. Although Euler was comparatively weak in philosophy, Immanuel Kant read the *Letters* before criticizing the Wolffian rational methods and Goethe before his research on optics. The *Letters to a German Princess* were not put into print until 1768 and 1772 in St. Petersburg, probably because Frederick was critical of the margrave. Written in an absorbing manner and with great clarity, the work met with extraordinary success. It was translated by 1800 from the original French into eight other languages – Russian, German, Dutch, Swedish, Italian, English, Spanish, and Danish. By 1840 it ran to over forty editions.

In 1765 Euler published his *Theoria motus corporum solidorum* (Mechanics of Solid Bodies) [E289]. This landmark work, which is connected with his *Mechanica* on the mechanics of point masses, is often called his second mechanics. It is the final piece in his program on a subject that took him about thirty years to complete. He had previously provided differential equations for the motion of fluid, elastic, and flexible bodies. The *Theoria motus corporum solidorum* presents clearly and in detail his analytical revision of the entire theory of rigid bodies.

In 1765 the British parliament for the first time awarded prize monies for lunar tables for finding longitude aboard a ship when it is not in sight of land. Growing trade and commerce had brought state patronage to support the improvement of ocean navigation. After Euler urged him to apply for these prizes, Mayer in 1757 had sent the British board of longitude his lunar tables that applied equations from Euler and claimed the longitude prize. British participation in the Seven Years' War and the partiality of the board that did not wish to give prizes to foreigners delayed the decision. It urged John Harrison to construct a more practicable method for determining longitude at sea. Shortly after his death in February 1765, Mayer's wife and surviving children were awarded part of the smallest of

[61] See Ronald Calinger, "Euler's *Letters to a Princess of Germany* As an Expression of his Mature Scientific Outlook," *Archive Hist. Exact Sci.*, **15**, 1976, pp. 211-233 and Andreas Speiser, *Leonhard Euler und die deutsche Philosophie*, Zurich: Orell Füssli, 1939.

three bounties for the prize.[62] It was fixed at the sum of £3,000, and Euler received £300. John Harrison received half of the major prize of £20,000 for developing in England a marine chronometer giving longitude better than the lunar method that he had successfully tested the year before. Once the chronometer became affordable in the 1780s, it was widely adopted. But the Mayer lunar tables remained in navigation almanacs and aided sea travel for more than a century. Continuing his research in optics, Euler had completed in 1765 the manuscript for his *Théorie générale de la dioptrique* ("Dioptrics" or "General Theory of Lenses") [E363].

4. During the Reign of Catherine the Great: The Second St. Petersburg Years

On July 28, 1766, the Euler family arrived in St. Petersburg. He was fifty-nine. The flamboyant Catherine II received him in royal fashion, exceeding the terms of his contract. She sent him 10,800 rubles for the purchase of a large, two-story house complete with furniture on the banks of the Great Neva near the academy. She had set it aside for him and his family. The empress also provided one of her cooks to run their kitchen. Euler's return was triumphant. Catherine granted him and his two oldest sons the honor of a lengthy audience. She knew that some support existed to raise him to the nobility, but she declared that his fame was greater than any noble title.[63] In 1765 Russia had suffered a great loss in the sciences when Mikhail Lomonosov died. Euler and Albrecht successfully began to rehabilitate the academy. Catherine, who had already agreed to add eight academicians, pledged to Euler to have the institution reorganized in a way that would give more autonomy to scientific pursuits and reduce internal strife. As the eldest and most distinguished member, he was to head all conference meetings and was most responsible for selecting new members.

A significant development in the sciences after the first quarter of the century had been international cooperation on expeditions. Even during the Seven Years' War, the Paris and St. Petersburg academies and the Royal Society of London attempted to measure the transit of Venus as it passes between Earth and the disk of the sun.[64] The transits occur in pairs eight years apart and the pairings are separated by more than a

[62] Eric G. Forbes, ed., *The Euler-Mayer Correspondence (1751 - 1755)*, pp. 18-19.
[63] Otto Spiess, *Leonhard Euler*, p. 187.
[64] See Harry Woolf, *The Transits of Venus: A Study of Eighteenth-Century Science*, Princeton: Princeton U. Press, 1959.

century.[65] Its accurate measurement was crucial to determining the mean horizontal solar parallax and, from this, to computing the exact distance of Earth from the sun. In 1761 Stepan Rumovskij, Euler's foremost Russian student, led a team to Selenginsk, Siberia, to make astronomical and meteorological observations but bad weather precluded gathering satisfactory data. In 1763 and 1764 Euler corresponded with the French astronomer Joseph Lalande on different observations of the transit and lenses made in Paris and London.[66] In 1768 the academy ordered from Peter Dollond a range of telescopes and other precision optical instruments, many with achromatic lenses. Better equipped and better prepared, Rumovskij now as the academy's chief astronomer embarked in 1769 on a second trek that took him to Kola, Siberia, to measure the transit. His expedition with thirteen members included Albrecht Euler. It obtained better results, but environmental conditions still hampered it and led to some guesses in the data.

In May 1771 a great fire broke out in St. Petersburg, destroying about 550 houses, including Euler's. In the general confusion, the helpless, almost blind Euler, who was in his bedclothes, might have died had not his Basler handyman Peter Grimm rushed into the house to rescue him. The library and furniture were destroyed, but Count Vladimir Orlov saved the manuscripts. Catherine compensated Euler with 6,000 rubles to build a new house, which was quickly done. In September 1771 he had an eye operation to correct a cataract producing near blindness. Johann Albrecht and nine physicians gathered around him to observe the brief operation. The restoration of sight to his left eye brought a moment of rejoicing. But in October a complication, possibly an infection, left him legally blind and in occasional pain. Euler described his loss of sight as providing "one less distraction." He could still perform difficult computations in his head, such as summing infinite series to fifty decimal places. The academy held meetings three times a week. He was absent from them from September 1771 to May 1772, when he resumed attending.

In 1773 Euler asked Daniel Bernoulli to recommend an assistant from Basel. He was proud that he had never lost his Swiss accent and liked to employ the Basel dialect. In July Bernoulli sent Nicholas Fuss to live with him and be his personal secretary. Fuss was to be Euler's closest associate for the final decade of his life and would marry his granddaughter

[65] The pairings after 1600 are 1631 and 1639, 1761 and 1769, 1874 and 1882, and 2004 and 2012.

[66] They also discussed the path of the comet of 1759, and Lalande expressed his joy over the reconciliation between d'Alembert and Euler.

Albertina,[67] Albrecht's second daughter in 1784.

In September 1773 the French *philosophe* Denis Diderot, the editor of the *Encyclopédie*, arrived in St. Petersburg. He was quite ill. From 1759 Catherine had read the early volumes of his great work and was his patroness. Euler probably attended his induction into the St. Petersburg Academy. According to *A Budget of Paradoxes* by Augustus de Morgan in 1872, Euler posed to the atheist Diderot a meaningless algebraic equation claiming to prove the existence of God. Supposedly when the Frenchman did not recognize this sham, he was deeply embarrassed at court and left Russia. But Euler would not have approved of such demeaning behavior. And since Diderot had begun his career as a mathematics teacher, he would have known enough elementary mathematics to detect such a stratagem. He stayed only five months. Catherine found his mind exceptional and requested a memorandum on founding a university that he recommended be open and without social distinction among students,[68] but she thought impractical his recommendations on law and agriculture, and he had an annoying habit of grabbing her knee during conversations. To help relieve Diderot's strapped financial situation, she purchased his library, but she permitted him to keep it as long as he lived.

In November 1773 Euler's wife Katharina died at the age of sixty-seven. The loss enormously complicated domestic life, for Katharina had managed everything and he had done nothing. Euler was determined to remain independent and not rely on his sons, this despite the custom at the time for an elderly parent to reside with the children and be under their care. While knowing that his sons would oppose it, at Christmas time in 1775 Euler broached the subject of a second marriage. Arguing that St. Petersburg society would not understand his choice, they blocked it. Euler now suffered another high fever and was under the care of his son Karl for two weeks. In July 1776 Euler simply announced without consultation his impending marriage to Katharina's half sister, Salome Abigail, who was fifty-three. In St. Petersburg, the Euler family attended the Reform church congregation, essentially Calvinist, that was probably near the academy and his home.[69] Its pastor performed the wedding ceremony at Euler's house.

A day before he became Comptroller General of France in August 1774, Baron Anne-Robert Turgot wrote to Louis XVI proposing the publication of two works by "the famous Leonard Euler" for textbooks in the naval

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[67] (1766 - 1829). Their eldest son, Paul Fuss (1798 - 1855), was the permanent secretary of the academy from 1800 to 1826.

[68] Alexander Vucinich, *Science in Russian Culture: A History to 1860*, Stanford: Stanford U. Press, 1963, pp. 192-193.

[69] Euler, Johann Albrecht, and Fuss tried unsuccessfully to end a rift among the French, Swiss, and German members in the church.

and artillery academies. One of them was the *Treatise on the Construction and Maneuver of Vessels* [E426], Euler's second ship theory, available in French since the preceding year. Euler had been unhappy that the *Scientia navalis* was too difficult for use among seamen. After returning to St. Petersburg, he met often with Admiral Knowles, who pointed to the difficulties the work presented to common users. Euler removed nonessential complications, and the *Treatise on the Construction and Maneuver of Vessels* presented a complete ship theory comprehensible to navigators. The other work that interested Turgot, *The New Principles of Gunnery* [E77] by Robins with commentary by Euler, had first to be translated into French. In October 1775 Turgot informed Euler that the king had agreed to fund these publications and to send Euler 1,000 rubles "as a token of the esteem he has for you." Not to be outdone, Catherine gave him 2,000 rubles. Euler was delighted. Turgot invited him to correct any errors in the originals. The new French edition of the *Treatise on the Construction and Maneuver of Vessels* appeared in 1776 and that of *The New Principles of Gunnery* in 1783.

In the summer of 1778 Johann III Bernoulli visited Euler. In his diary Bernoulli noted that Euler's general health remained good, but his vision was so poor that he could not recognize faces. Even so, he could write clearly with chalk in large symbols on a blackboard better than many people with sight. On a secret mission to St. Petersburg in 1780 concerning problems from the first partition of Poland, the Prussian margrave Friedrich Heinrich visited Euler, who was confined to bed, and spent several hours holding his hand and conversing with him about history and law.

During his second St. Petersburg period, a still energetic Euler became even more prolific in articles. After 1765, he completed four hundred fifteen or over fifty percent of his total memoirs. Even his near blindness did not slow him. At his service were his genius, his disciplinary intuition, and his phenomenal memory and ability with mental calculations, together with a small research team. It consisted of his sons Albrecht and Christoph, together with Anders Johann Lexell, Wolfgang Ludwig Krafft, Semjon Kirillovič Kotel'nikov, and Rumovskij. They were joined by Michail Evseevič Golovin and principally Fuss, who in 1774 Euler recommended be made adjuncts at the academy. By 1783 they comprised half the regular members of the academy. Krafft's father Georg had collaborated with Euler during his first St. Petersburg period, and Golovin was a relative of Lomonosov. Euler set the topics and initial problems for each of his theoretical papers, most of them brief. His research team worked at his residence around a large table with a chalk board in the middle. Although Euler could write distinctly on it in large symbols even after 1771, apparently his assistants used it most. When alone, the nearly blind Euler would walk around it for exercise. A sheen could be discerned from where he had run his hand along its edges. During working sessions participants made computations and Euler reviewed them to eliminate errors. Afterward all papers on the table were put into a large portfolio. Fuss and Golovin principally recorded Euler's dictation. For Euler's papers published after 1766, Fuss also made the computations of more than 160 and Golovin 70. In addition, some papers have the handwriting of his sons. Three hundred of his second St. Petersburg papers appeared posthumously.

To 1773 Euler wrote books, though his correspondence fell precipitously. He had completed in Berlin three of the four books that began to be published in1768. The first volume of his *Institutionum calculi integralis*, printed in 1768 [E342], and the second in 1769 [E366], together comprises the last part of his great trilogy on the calculus. It has hundreds of his discoveries regarding ordinary and partial differential equations. Volume three [E385] on the calculus of variations, which follows Lagrange's analytic methods, was published in 1770. Euler's *Letters to a German Princess* appeared in French in three volumes from 1768 to 1772. between 1768 and 1774 Rumovskij translated it from French into Russian.[70] Lexell helped edit the *Dioptrics* [E367,E386,E404], which was published in three volumes from 1769 to 1771. Euler's influential *Complete Introduction to Algebra* [E387,E388] appeared in Russian in 1768 and 1769, and the next year in German. It also would have editions in English, Dutch, Italian, and French. From 1770 Euler prepared his 775-page *Theoria motuum lunaea* [E418], his second lunar theory published in 1772. Among other topics it addresses the three (mutually-gravitating) bodies' problem. Three of his assistants, his son Albrecht, Krafft, and Lexell, who significantly assisted with computations, are named on the title page of the *Theoria motuum lunaea*. Euler's last book, the *Treatise on the Construction and Maneuver of Vessels*, was published in French the next year. By then his correspondence had declined to less than twenty letters a year. His letters with Lagrange, an important source of his research on number theory, calculus, and mechanics, ended in March 1775 on the topics of elliptic integrals, paradoxes in integration, and lost proofs of Fermat. After 1777 he sent less than five letters a year.

From 1775 to 1782 General Count Vladimir Orlov, the academy director from 1766 to 1775, was succeeded by a follower of his, the minor poet Sergej Domashnev. The Orlov brothers had been central in Catherine's coup against her husband in 1762, and she rewarded them with positions. Neither man was supportive of the institution. Orlov believed that the

[70] This was the first of five editions. The second appeared in 1785, the third in 1790-91, the fourth in 1796, and the fifth with only the first two books in 1808. All editions were published in St. Petersburg.

academy was useless and that the sciences were making the world more evil. Domashnev was similar to Schumacher: arrogant, abusive and a squanderer of funds. Disgusted, Euler began withdrawing from the business of the academy under Orlov leaving the post of head of the academic commission in 1774. As the *Protocols* show, under Domashnev he stopped attending sessions altogether in January 1777. Even so, he continued to submit a stream of articles read by Krafft, Fuss, and Golovin on such topics as the lunar orbit, the integration of irrational formulas, continued fractions, and paradoxes in the calculus of variations. After Domashnev did not respond to a letter from the academicians protesting his breach of the rules of academic protocols, in December 1782 they sent a letter objecting to his actions to the academic commission and . Euler had his name added to the signatures. After a two months' inquest, Domashnev was dismissed. The academicians could no longer be neglected politically.

In January 1783 when Princess Catherine Romanova Dashkova, another favorite of Catherine II, was made the director of the academy, she asked that along with Albrecht and Fuss, the man she called "Euler the Great" should ride in her carriage to her first session. She begged Euler, who felt honored, to be able to enter the academy on his arm and to have him introduce her. Fuss guided the steps of Euler. His presence moved the academicians to tears. When another professor, Jakob Stählin, took the seat of honor next to the director, Princess Dashkova turned to Euler and to the delight of his son and Fuss proclaimed, "Sit where you want and the place you choose will naturally be the first among all."[71] That was the last session he attended. With support from Catherine the Great, the St. Petersburg Academy began to pay homage to Euler during his life by commissioning for the assembly hall an allegorical mural of the wisdom of geometry, which included a board filled with formulas and calculations from Euler's second lunar theory – an extraordinary distinction for a man of science for the time.

To his death on September 18, 1783, Euler remained enthusiastic and perspicacious in research and teaching. He was teaching four of his grandchildren elementary mathematics, and on his last morning he instructed a grandson, who was gifted in the sciences, and made mental calculations that were put on the slate in his study about how high hot-air balloons could rise. News of the success that June by the Montgolfier brothers in launching balloon ascents in Paris, which became the topic of the moment within the republic of letters, had just arrived in St. Petersburg. Euler took lunch with his assistants Fuss, Lexell, and Lexell's family. Lexell was

[71] Princess Ekaterina Dashkova, *The Memoirs of Princess Dashkova*, trans. and ed. by Kyril Fitzlyon, Durham: Duke U. Press, 1995, p. 334.

to be his successor. They discussed the orbit of the planet Uranus, which William Herschel had just discovered in March of 1781, another topic that engrossed the European reading public. Euler enjoyed dictating computations on aerodynamics and the orbit of Herschel's planet for his assistants to put on the blackboard and record. At tea time about five, he had been playing with his grandson a little, and he was drinking some tea and smoking a pipe. Suddenly the pipe fell from his hand. "My pipe," he exclaimed, and he bent over to reclaim it but was unsuccessful and stood up. For a year Euler had endured vertigo and weakening health. Now he suffered a stroke. Clasping both hands to his chest, he said, "I am dying" and lost consciousness, which he never regained. He died about eleven that night. He was seventy-six years of age. Twenty years earlier in reflecting on the state of the soul after death, Euler had presumed that there will be a suspension of the union between the body and the soul, in which the senses influence even our dreams. In death, he believed, "we will find ourselves in a more perfect state of dreaming." [72]

Euler was buried on Vasilyevsky Island in the Lutheran section of the Smolensk Cemetery, which was mainly for members of the Russian Orthodox Church. The four major royal science academies in Berlin, London, Paris, and St. Petersburg as well as societies in Lisbon and Turin, to all of which he belonged, announced their profound loss. Antoine-Nicolas de Condorcet, the secretary of the Paris Academy of Sciences, Fuss in St. Petersburg, and Formey in Berlin delivered the principal eulogies. On October 23, 1783, the Imperial Academy of Sciences in St. Petersburg held its memorial meeting for Euler. Princess Dashkova presided, and an archbishop along with many noble dignitaries were among the attendants. In the eulogy Fuss depicted the life of Euler as a triumph of the human spirit and the man an exemplar in the effort of his century "in enlightening the world." [73] At the demand of the princess, the officers of the Imperial Academy honored Euler's memory materially in 1785 by installing a half-length marble bust by Jean-Dominique Rachette in the library hall of the Kunstkammer. Euler had promised Count Orlov to prepare enough articles to appear in the academy's *Acta* and *Nova acta* for the next twenty years but left enough for over forty, transcribed by his assistants. By the first decades of the nineteenth century the modest gravestone of Euler could not be located in the Smolensk Cemetery, even by Fuss, a participant in the funeral. Euler's grave marker was not rediscovered until 1830 at the

[72] Leonhard Euler, *Letters . . . Addressed to a German Princess*, New York:Harper, 1840, vol. I, p. 310.

[73] Nicholas Fuss, "Lobrede auf Herrn Leonhard Euler," trans. by John S. D. Glaus, pp. 1-2.

burial of one of his daughters-in-law. In 1837 the academy replaced it with a lasting monument constructed out of pink Finnish marble with the simple inscription, *Leonhardo Eulero, Academia Petropolitanae*, for a scholar who according to Turgot had "honor[ed] humanity with his genius and science with his style." [74]

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Leonhard Euler: Life, Work and Legacy
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Leonhard Euler and Russia

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1. Introduction

Every biography of Leonhard Euler, the most distinguished mathematician of the 18th century, attaches special significance to Euler's relations to Russia and the St. Petersburg Academy.

Leonhard Euler was born in Basel in 1707. His father was a vicar who was interested in mathematics and who had attended lectures of Jakob Bernoulli (1654-1705) an important mathematician in his time.

Leonhard showed mathematical ability an early age. His father expected him to study theology. However during his study at the University of Basel he followed his father in a different way and was attracted by the lectures delivered by the famous mathematician Johann Bernoulli (1667-1748) the brother of Jakob Bernoulli. Despite the outstanding performances that Euler gave in various dissertations, his applications for an academic career in his native Switzerland failed. But in 1725 his friends Nikolaus (1695-1728) and Daniel Bernoulli (1700-1782), both sons of his teacher Johann Bernoulli, got positions as professors at the newly founded Academy of Sciences in St. Petersburg, and later both invited Euler to follow them. In 1727 Euler accepted an appointment as adjunct of the Petersburg Academy. In his biography of Euler, Rüdiger Thiele (born in 1943) remarked that in Euler's homeland, where the few available academic positions were occupied in any case, he would never have received such a generous opportunity for development as he did in Petersburg. So it was a decisive factor for Euler that he had no option to stay in Switzerland, but instead had to leave the country for Petersburg, the center of the Russian Enlightenment, where he found an appropriate sphere of action.[1] After Euler had accepted his appointment in St. Petersburg, Christian Wolff (1679-1754), a leading figure of the German Enlightenment, wrote to him from Marburg on April 20, 1727 that Euler would travel "into the paradise of the scholars."[2]

2. The First Petersburg Period

Indeed, for Euler this characterization of life in St. Petersburg proved to be true in an exceptional way. Above all, the Petersburg Academy offered chairs and appointments to young scientists. The staff of the Petersburg Academy was generously taken care of. The scientists received a fixed salary, free accommodation and an additional remuneration for firewood and such, which would have been unusual at German universities. Professors received a salary of at least of 600 rubles a year, adjuncts 300. Among the five adjuncts affiliated with the various branches of the Petersburg Academy, Euler was the one who most frequently lectured on new scientific discoveries in the academic conferences. In 1730 he and all of the other adjuncts were nominated to the rank of professor.

Circumstances in Petersburg gave the academic staff complete freedom to choose research projects. This proved to be extraordinarily advantageous for Euler. Stubborn scientific debates on Leibniz' and Newton's views of the world divided the Academy. Euler joined the side of the Newtonian view. That is why he sought mathematical proofs for various statements of Newton, especially in mechanics. As a consequence, a two-volume introduction to mechanics, *Mechanica sive motus scientia analytice exposita*, appeared in 1736.

In his first Petersburg period Euler systematically extended his field of research; in addition to mathematical and physical questions he considered problems of astronomy, geography (temporarily he even was director of the Geographical Department of the Academy), theory of music and many other fields. Euler was one of the most active members of the Academy. In the academic conferences he regularly lectured on his new scientific re-

[1] Thiele, R: *Leonhard Euler*. Leipzig: Teubner 1982, p. 29.
[2] Quoted according to A.P. Juškevič and E. Winter, *Die Berliner und die Petersburger Akademie der Wissenschaften im Briefwechsel Leonhard Eulers*, part 1. Berlin: Akademie Verlag 1959, p. 41.

sults. In 1736 alone Euler published eleven mathematical contributions in that year's volume of the Petersburg Academy journal, the *Commentarii Academiae scientiarum Imperialis Petropolitanae.* At the same time he competed in several prize problems of the Paris Academy.

Shortly after his arrival Euler learned Russian and he spoke it so well that he was able to express himself by speaking and writing, which was exceptional among the foreign members.[3] Admittedly, he did not speak without an accent, but he did so in German too, where he maintained his Swiss accent. On his way to Russia Büsching (1724-1793) met Euler in Berlin on December 17, 1749 and reported that Euler "spoke his mother tongue in such a strong dialect he was hardly understandable."[4] Furthermore, Büsching added these comments describing Euler: "On his right eye he has a fistula robbing him of half of his sight and which looks rather nauseating. In general the celebrated algebraists are in the habit of being scowling and tiring minds in social intercourse, but Euler is a very lively and alert person, especially in the company of the friends."[5]

After the death of Tsarina Anna (1693-1740), the political circumstances in St. Petersburg became unsafe. At the same time, Euler received an enticing offer from the Prussian King Frederick II (1712-1786) and decided to move to Berlin in 1741.

Euler had arrived in St. Petersburg in 1727 as an unknown but budding scientist. When he left the city on the River Neva in 1741, he had grown to be a researcher with an international reputation. Euler himself later felt that the Petersburg years had forged him into a strong scientist, as can be seen in various surviving letters that were written in his Berlin period.

3. The Berlin Period

In Berlin Euler lived rather independently. The Prussian Academy opened in 1746. It succeeded the *Societas Regia Scientiarum* that had been founded in 1700 and Euler became director of the Mathematics Section. Up to this

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[3] A.P. Juškevič, Leonard Ėjler. Žizn i tvorčestvo, in: *Razvitie idej Leonarda Ėjlera i sovremannaja nauka.* N.N. Bogoljubov et al., eds. Moscow: Nauka 1988, pp. 15-46, cit. p. 25.

[4] P. Hoffmann, *Anton Friedrich Büsching. Ein leben im Zeitalter der Aufklärung.* Berlin-Verlag Arno Spitz: Berlin: 2000. "... redet seine Muttersprache so grob, dass man ihn kaum verstehen kann.", p. 36.

[5] Ibid. "An dem rechten Auge hat er eine Fistel, die ihm die Hälfte seines Gesichts beraubt und ziemlich ekelhaft aussieht. Es pflegen zwar die grossen Algebraisten gemeinglich finstere Köpfe und im Umgang beschwerliche Köpfe zu sein, er aber ist sehr belebt und aufgeweckt, insonderheit wenn er sich unter Bekannten befindet."

time Euler had published five papers and submitted others to the journal of the Society, the *Miscellanea Berolinensis.* Because of his comprehensive knowledge and his broad scientific and administrative activities, Euler soon gained an extraordinary reputation in Berlin and the Berlin Academy.

During the Berlin period Euler's productivity was so great that in those days no one Academy was able to publish all of his papers. Euler published about half of his paper in Berlin and the other half in St. Petersburg. In addition Euler repeatedly submitted papers to the Paris Academy and competed in the Paris Prize competition. From time to time Euler sent long papers from Berlin to St. Petersburg for publication, and some long books were printed in Berlin on at the expense of the Petersburg Academy. Furthermore, important manuscripts were published in St. Petersburg including, among others, *Scientia navalis seu tractatus de constructione ac dirigendis navibus* (1749), *Theoria motus lunae* (1753), and *Institutiones calculi diffentialis* (1755). It is noteworthy that originally the *Scientia navalis* was intended to be printed in Berlin at the Petersburg Academy's expenses, but there was no printer in Berlin who could handle the complicated typesetting. So Euler sent the manuscript to the printer of the Petersburg Academy, though the proof-reading was done in Berlin.

These and other activities show that during the years he was in Berlin Euler remained *de facto* an active member of the Petersburg Academy. As a foreign member of the Academy he got an annual pension of 200 rubles. Characteristic of Euler's feelings for Russia is the fact in Berlin he frequently remembered his Petersburg years. There are many reports on this matter that have not yet been comprehensively analyzed. In the 25 years Euler lived in Berlin a total of about 800 letters were exchanged, on average three letters each month to or from St. Petersburg [6] – and this calculation includes the time of the Seven Years War, in which the correspondence was almost brought to a standstill and each letter was sent, despite enormous difficulties, through neutral regions.

Frederick II once asked Euler how he had gained his knowledge. Euler responded that he himself "and all the others who were so lucky as to belong to the Imperial Russian Academy eventually must confess that all of what we are we owe to the very advantageous circumstances we found ourselves in at this Academy. Furthermore, as far as I am concerned, without this magnificent opportunity I would have been forced to undertake some study in which I would probably have been remained but a bungler." [7]

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[6] A.P. Juškevič, *Leonard Ejler*, p. 33.

[7] Letter to J.D. Schumacher from 7./18. November 1749. In: *Die Berliner und die Petersburger Akademie der Wissenschaften im Briefwechsel Leonhard Eulers*, part 2. A.P. Juškevič and E. Winter, eds. Berlin: Akademie-Verlag 1961. "... und alle übrige,

Büsching met Euler in December 1749 and made this note: Euler "entertained us with an extensive narration on the nature of Russians whom he had came to know during his 14-year stay in St. Petersburg. He praised the extraordinary intellect and the skillfulness of the Russian farmers and assured us that in comparison with the Russian farmers, the rural inhabitants of the Marches [the area surrounding Berlin, the Marches or the Mark Brandenburg] are like clod-hoppers." [8]

Euler often mentioned the Russian gift of practical ingenuity. When the Petersburg Academy wanted a physicist who at the same time could work as a mechanic, Euler wrote to Johann Daniel Schumacher (1690-1761), the chief official of the academic chancery: "Obviously we are not far from the moment that if we need skillful people in Germany then we will have to take a Russian." [9] In any case Euler was not in a position to recommend a candidate suited for this job from Germany for the Petersburg Academy. Therefore he continued: "There were skillful mechanics but they did not study. In my opinion the Academy would most reliably get such a skillful man if a young person who has studied and who possessed a good basic knowledge of mathematics were urged to do all kinds of mechanical work like wood-turning, clock-making, and glass-grinding, for which the Academy itself would offer wonderful opportunities. After this, such a person could be sent here for a few years where he could increase his theoretical knowledge with me and with Herr Dr. Lieberkühn [1711-1756] who runs a workshop in his house for glass-grinding and uses various machines. He would have all advantages of practical instruction." [10] Euler's

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welche das Glück gehabt, einige Zeit bey der russisch-Kaiserlichen Academie zu stehen, müssen gestehen, dass wir alles, was wir sind, den vortheilhaften Umständen, worin wir uns daselbst befunden, schuldig sind. Dann was mich betrifft, so würde ich in Ermangelung dieser herrlichen Gelegenheit genöthiget gewesen seyn, mich auf ein ander Studium hauptsächlich zu legen, worinn ich allem ansehen nach doch nur ein Stümper würde geworden seyn", p. 182 (letter 106).

[8] Quoted according to P. Hoffmann, *Anton Friedrich Büsching*. Euler "unterhielt uns mit einer weitläufigen Erzählung von der Beschaffenheit der Russen, die er in der Zeit seines 14-jährigen Aufenthalts in Petersburg genauer kennengelernet. Den Verstand und die Geschicklichkeit der russischen Bauren rühmte er ungemein und versicherte, dass die märkischen in Vergleichung mit denselben wie Klötze wären", p. 36.

[9] Letter to J.D. Schumacher from 19./30. June 1753. In: *Die Berliner und die Petersburger Akademie der Wissenschaften im Briefwechsel Leonhard Eulers*, part 2. "Allem Ansehen nach sind wir nicht mehr weit von dem Zeitpunkt entfernt, dass, wenn man in Teutschland wird geschickte Leute nötig haben, man solche aus Russland wird verschreiben müssen", p. 312 (letter 231).

[10] "Geschickte Mechanicos gäbe es zwar noch, welche aber nicht studirt haben. Meiner Meynung nach würde die Academie am sichersten zu einem solchen geschickten Mann gelangen, wenn sie einen jungen Menschen, der studiert und in Mathematicis einen guten Grund gelegt hätte, zu aller Gattung mechanischen Arbeiten als Drechseln, Uhrmachen,

following conclusions are particularly expressive: "A Russian seems to be more skillful than a German because a German artist and craftsman is seldom able to make anything that he did not learn to make, whereas I have always seen with surprise that even the most common Russian people make attempts and they are often successful. Such behavior is necessary for such purposes." [11] Euler continued in case his proposal were accepted by the Academy "certainly the safest way to fill the vacancy of a mechanic would be to fill it soon and to fill the post with such a person so it could never be done by a foreigner." [12]

Again and again Euler made great efforts to promote young Russian scientists. From September 1743 to summer 1744 Krill Grigor'evič Razumoskij (1728-1803) the future President of the Petersburg Academy and his private tutor Grigorij Nikolaevič Teplov (1725-1771,) later Secretary and Councilor of the Academic Chancery, stayed in Berlin in Euler's house. [13] This was one of the educational journeys the nobility usually undertook – in this case a larger influence cannot be overlooked. Several other of Euler's disciples – Semen Kirillovič Kotel'nikov (1723-1806), Stepan Jakovlevič Rumovskij (1734-1812), and Michail Sofronov (1729-1760) - stayed in Euler's house in Berlin between 1751 and 1756. Later Kotel'nikov and Rumovskij became members of the Petersburg Academy. The highly gifted Sofronov, who had already shown a tendency towards alcoholism in Berlin, did not realize his full potential.

Euler is well known for the expert opinions he gave on papers written by young Russian scientists. Let us only recall his opinion concerning papers written by Michail Vasil'evič Lomonosov (1711-1765) and Nikita Ivanovič Popov (1720-1782). [14] And in a letter dated 27 January/7 February 1756 Euler reported on the good progress his disciples Kotel'nikov and Rumovskij had made in mathematics: "... and I hope they will soon be able

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Glassschleiffen anhalten wollte, wozu bey der Academie selbst die schönste Gelegenheit wäre. Hierauf könnte man einen solchen Menschen auf etliche Jahre hierher schicken, wo er sich bey mir in Theoreticis fester setzen, bey dem H. Dr. Lieberkühn aber, welcher die künstliche Werkstatt in seinem Hause von Glassschleiffen und anderen Maschinen unterhält, alle Vortheile in practis erlernen könnte", ibid. pp. 312f (letter 231).

[11] "Ein Russ scheint insonderheit dazu geschickter zu seyn als ein Teutscher, denn da ein teutscher Künstler und Handwerker selten etwas zu verfertigen imstande ist, was er nicht gelernt hat, so habe ich immer mit Verwunderung gesehen, dass auch die gemeinsten Russen alles unternehmen und mehrentheils glücklich ausführen. Solche Ingenia werden aber zu einer solchen Arbeit unumgänglich erfordert", ibid. p. 313 (letter 231).

[12] "... so würde dieses gewiss das sicherste Mittel seyn, die erledigte mechanische Stelle bald auf eine solche Art zu besetzen als durch einen Ausländer nimmer mehr geschehen würde", ibid. p. 313 (letter 231).

[13] See "Vorwort (Preface)", ibid. p. 6.

[14] Letter to Schumacher from 8/19 April 1749, ibid. p. 162 (letter 92).

to make applications with such a success that it may well happen that they surpass all of what has been done by foreigners in this science."[15]

Euler's letters to Petersburg frequently contain his memories of Petersburg. In this spirit he wrote a letter to the Secretary of the Academic Chancery Johann Daniel Schumacher about the observatory in Petersburg: "The observatory in Petersburg can rightly be praised because for many years now they have not economized at all in the purchase of the instruments they needed and moreover even the building was favorably designed for astronomical purposes so that we here [in Berlin] do not know a better model to propose."[16] This statement came to have special importance because after the death of the astronomer Christfried Kirch (1694-1740) the Berlin observatory fell into ruin. Euler reported to Joseph-Nicolas Delisle (1688-1768) in Paris on April 23, 1743: "Since the death of Mr. Kirch, the observatory of the Societé has been in a sad state, so that it no longer can be used for making observations. One could believe that the King has destined this place to be another fortress, and for this reason he has not made any efforts on behalf of the needs of astronomy."[17]

The Berlin Academy was designed to meet the needs of the Prussian king. After the death of the President Pierre-Louis Moreau de Maupertuis (1698-1759) the king himself directed the Academy. Frederick II and Euler were very different in origin, view of life, and character. The king appreciated brilliant and witty conversation, had a tendency to cynicism, and showed no deeper understanding of mathematics, although he saw its practical use. To Frederick, Euler was merely a member of his Academy and although the glory of Euler shined at the Academy, Frederick did not engage Euler in conversation. After the death of Maupertuis, the king did not recognize the excellent work Euler had done when Euler did his utmost and used his influence as Director of the Mathematical Section to advance the

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[15] "... et j'espère qu'ils seront bientôt en état d'être employés avec un tel succès qu'on pourra bien se passer tout à fait des étrangers dans cette science [mathématiques]", ("... and I hope that they will soon be in a position to be employed with such success that they may well, in fact, surpass the foreigners in this science [mathematics].") ibid. p. 414 (letter 321).

[16] Das Observatorium in Petersburg kann sich mit Recht rühmen, dass es von so vielen Jahren her an Anschaffung aller nöthigen Instrumenten nichts ist erspahret worden, und über diese ist auch das Gebäude so vorteilhaft zum Endzweck der Astronomie angelegt worden, dass wir allhier kein besseres Modell vorzuschlagen wissen", ibid. p.86 (letter 35).

[17] L. Ėjler i Ž.-N. Delil' v ich perepiske 1735-1765. In: *Russko-francuzskie naučnye svjazi.* Leningrad: Nauka 1968. "L'Observatoire de la Societé a été jusque à present depuis la mort de M-r Kirch dans un mauvais etat, de sort qu'on n'a presque rien pu observer. On croiroit que le roy avoit destiné cette place à un autre batiment, et par cette raison on n'a pas voulu faitre aucune depense pour les besoin de l'astronomie," p. 162.

development of the Academy. The king had never considered appointing Euler to the Presidency - in this the king certainly was negligent.

Euler stuck to his principles, the upright views that were forged by the middle-class of his Swiss homeland and at the same time he advocated a deeply religious world-view. He repeatedly attacked the free-thinkers in papers in which he provocatively criticized the opinions of the French philosophers of Enlightenment who dominated Frederick's court, and he also criticized the philosophies of Gottfried Wilhelm Leibniz (1645-1716) and Christian Wolff (1679-1754).

In addition to the instability and uncertainty at the Berlin Academy, there were the difficulties of the Seven Years War. In 1760, Berlin was occupied by Russian troops. This also affected Euler's estate at Charlottenburg outside of Berlin. Euler sent a letter to the Secretary of the Petersburg Academy dated October 7/18, 1760 that said, "We had a visitor here who on all other occasions would have been very welcome. It was always my wish that if Berlin were to be taken by foreign troops then it might be done by Russians. I had the delight to get to know so many brave Russian officers." [18] In the following letter, though, Euler reports on the devastation of his Charlottenburg estate - the order of safe keeping issued by the Russian command came too late to protect Euler's properties. In addition Euler made a report recommending that his case be referred to the Russian tsarina for the treatment he had received from the Russian officers, saying that as a foreign member of the Petersburg Academy he certainly should be compensated by the Russian court for the damage inflicted. In further letters to authorities of the Petersburg Academy this topic was repeatedly discussed. Euler did not rebuild his Charlottenburg estate but he sold it a few years later.

For Euler, his circumstances in Berlin, especially those caused by the war, became unbearable. The depreciation of the currency also had an impact on Euler's standard of living. Although Euler had taken on the many duties of the President of the Berlin Academy, he did not occupy the presidency and the King did not remunerate him accordingly. What is more, even after the end of the war he could not expect rapid improvement.

When he recognized that he could not realize his ambitions, Euler returned to St. Petersburg in 1766. At this time he was 59 years old. This

[18] Letter to Müller from 7/18 October 1760. In: *Die Berliner und die Petersburger Akademie der Wissenschaften im Briefwechsel Leonhard Eulers*, part 1. A.P. Juškevič and E. Winter, eds. Berlin: Akademie-Verlag 1959. "Wir haben hier einen Besuch gehabt, welcher mir bey allen anderen Gelegenheiten höchst angenehm gewesen wäre. Doch habe ich immer gewünscht, dass, wenn Berlin von fremden Truppen eingenommen werden sollte, solches von den russischen geschehen möchte. Ich habe also das Vergnügen gehabt, so viele wackere russische Herren Offiziere kennen zu lernen", p. 161 (letter 120).

is an age at which such a decision is made only after careful consideration. However despite all the quarrels of the first Petersburg period, Euler obviously had fond memories of his stay in Petersburg.

The Prussian king was very reluctant to let Euler leave and he needed to be asked several times before approving Euler's dismissal. Not until Euler resigned all of his academic posts did the King finally give in with a few laconic words and accept Euler's application for dismissal. It is characteristic of the King that he did not find any positive words as a reward for the enormous work Euler did during two and a half decades at the Academy.

4. The Second Petersburg Period

In 1766 Euler returned to St. Petersburg. There he was welcomed with honor. He received many gifts, which at once enabled him to buy a house located in a distinguished area near the Academy building. He was regularly seen having audiences with the Russian tsarina Catherine II (1729-1796), who talked to him and listened attentively.

Euler's relations with the Director of the Petersburg Academy once again became difficult, but this was ultimately not so important. The Director was Vladimir Grigor'evic Orlov (1743-1831), 23 years old and a younger brother of count Aleksej Grigor'evic Orlov (1737-1808), the favorite of Catherine II. At any rate, Euler soon withdrew from his official academic duties at the Petersburg Academy, but this did not hamper him at all in his concentrated scientific work.

Euler's second Petersburg period lasted 17 years, until Euler's death. It became a time of rich harvest for him. He published several general monographs during these years - among them the three volumes of the *Dioptrica* (1769-1771), probably prepared in Berlin, *Institutionum calculi integralis* (1768-1770), also in three volumes, and the book *Theoria motuum lunae* (1772) which was published by his son Johann Albrecht (1734-1800), Wolfgang Krafft (1743-1814), and Andreas Johann Lexell (1740-1784) under the direction of Leonhard Euler. It contained the lunar tables *Novae tabulae Lunares* as an appendix.

Although Euler went completely blind in 1771 he continued working. He wrote and published many general papers. Due to his comprehensive memory and his outstanding power of imagination Euler was able to dictate even complicated mathematical investigations to his young colleagues. He found able minds and these young men became distinguished mathematicians under his supervision. The most famous is Nikolaus Fuss (1755-1829), who was born in Basel and married Albertine Euler (1766-1829), a daughter of

Euler's oldest son Johann Albrecht, in 1784.

Even after Euler became blind, his productivity was so huge that the Petersburg Academy was unable to publish all of his papers. When he died, many papers were waiting to be published, some of which appeared soon after Euler's death but many not until decades later.

In 1783 the highly honored Leonhard Euler died. In Petersburg his work left a trail that can be followed through to our days. In 1769 Euler's oldest son Johann Albrecht had been appointed as Perpetual Secretary of the Conferences of the Petersburg Academy, a position in which he remained until his death in 1800. His successor was Nikolaus Fuss, also trained by Euler.

5. Euler's Legacy

During his lifetime Euler was always in close touch with Russia. Therefore, it is not surprising that after Euler's death the Petersburg Academy gave intense care to the publishing of his remaining papers. The Academy needed about fifty years in total to publish the many completed papers that were left by Euler.[19] Many of his sketches, notices, and calculations are still to be published in the collected works, the *Opera omnia Euleri*, in the not too distant future.

The publication of Euler's collected work was considered for the first time in the middle of the 19th century in Petersburg. They began with the publication of important letters.[20] However, this first attempt at a comprehensive edition did not get beyond two volumes published in 1849.[21] The task was too much for the Petersburg Academy alone and, despite repeated efforts, the Academy did not find anyone willing to take part in this project. They negotiated mostly with Carl Gustav Jacobi (1804-1851), a member of the Berlin Academy, but due to the financial limitations the project broke down.[22]

[19] See G.N. Matvievskaja, O rukopisnom nasledie i zapisnych knigach Ėjlera (On the handwritten estate and Euler's notebooks), in: *Razvitie idej Leonarda Ėjlera* (The development of Euler's ideas). Moscow:Nauka:1988, pp. 102-121, cit. p. 124.

[20] See G.N. Matvievskaja, O rukopisnom nasledie i zapisnych knigach Ėjlera (On the handwritten estate and Euler's notebooks), in: *Razvitie idej Leonarda Ėjlera* (The development of Euler's ideas). Moscow:Nauka:1988, pp. 102-121, cit. p. 124.

[21] L. Euler, *Commentationes arithmeticae collectae*, 2 vols. St. Petersburg 1849; L. Euler, *Opera posthuma mathematica et physica*. St. Petersburg 1862.

[22] See P. Stäckel and W. Ahrens, Briefwechsel zwischen C.G. Jacobi und P.H. von Fuss über die Herausgabe der Werke Leonhard Eulers, in: *Bibliotheca mathematica* (**3**) 8 (1907) 233-306; also an elarged separat edition Leipzig: Teuber 1908.

When the Schweizerische Naturforschende Gesellschaft (Swiss Society of Natural Sciences) was first considering the publication of a complete edition and offered the Petersburg Academy its ideas, the Petersburg Academy agreed at once.[23] An Euler Committee was founded in the Swiss Society and it soon presented a plan that was supported by the Petersburg Academy. In 1910 the Petersburg Academy placed the Euler materials that were in the possession of the Academy at the Euler Commission's disposal under the condition of a speedy return. The Euler Commission made photocopies but the return did not take place until in 1947 and 1948.[24] When the material was handed over to the Euler Commission, Boris L'vovič Modzalevskij (1874-1928) made a list of the Euler estate that was published as a preprint.[25] Russian scientists were involved in the editorial work that started in these years. Even before World War I the Russian mathematicians and members of the Petersburg Academy of Sciences, Aleksandr Michajlovič Ljapunov (1857-1918) and Andrej Andreevič Markov (1856-1922) had each undertaken the editing of two volumes of the series and sent the completed manuscript to the Euler Commission in Zurich. However, there were other volumes ready for printing. That is why Ljapunov's volumes were not published until 1920 and 1932, and Markov's volumes were not published until 1941 and 1944 only then in a revised edition.[26]

Due to political conditions in the late 1920's and the 1930's, the cooperation between Soviet scientists and the Euler Commission in Switzerland almost came to a standstill. Nevertheless, in the Soviet Union Euler's heritage continued to be investigated thoroughly. In 1935, on the occasion of the 150th anniversary of Euler's death, an omnibus volume was published, and many writings of Euler were translated into Russian and published.[27]

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[23] See K.-R. Biermann, Aus der Vorgeschichte der Euler-Ausgabe 1783-1907; J.J. Burckhardt, Die Eulerkommission der Schweizerischen Naturforschenden Gesellschaft, both in: *Leonhard Euler. Beiträge zu Leben und Werk.* J.J. Burckhardt et a. eds. Basel: Birkhäuser 1983, pp. 489-500, 501-510.

[24] G.N. Matvievskaja, *O rukopisnom nasledie* (On the handwritten estate), p. 125.

[25] B.L. Modzalevskij, *Perečen' rukopisej Ėjlera, chranjaščichsja v Archive Konferencii imp. Akademii nauk* (List of Euler's manuscripts stored in the Archive of the Conferences of the Imperial Acedemy of Sciences), Preprint 1910; G. Eneström, Bericht an die Eulerkommission der Schweizerischen Naturforschenden Gesellschaft über die Euler Manuskripte der Petersburger Akademie, in: *Jahresbericht der Deutschen Mathematiker-Vereinigung* **22**, 1-2 (1913) Abt. 2, pp. 191-205.

[26] See E.P. Ožigova, Ob učastie Peterburgskoj Akademii nauk (Akademii nauk SSSR) v izdanii trudov Ėjlera (On the role of the Petersburg Academy for publishing Euler's works), in: *Razvitie idej Leonarda Ėjlera* (The development of Euler's ideas), pp. 60-8, cit. pp. 73f.

[27] L. Ėiler, Metod nachoždenija krivich linij (...), Moscow-Leningrad, Gos. Techn.-

When the Euler estate came back to Russia to the Archive of the Leningrad Academy, Soviet scientists got new opportunities for extensive research and they made vigorous use of the opportunity. In 1958 Gleb K. Michailov (born in 1929) and Vladimir Ivanovič Smirnov (1887-1974) gave the first report on those activities.[28] Furthermore in 1962 and 1965 a very detailed but not annotated list of the Euler material preserved in the Archive of the Academy was published in two volumes.[29] Even without annotations, the first volume contains a list of 2,268 letters from and to Euler stored in the Petersburg Archive. Since the 1950's, the Soviet Academy and now the Russian Academy of Sciences have devoted particular attention to the opening and editing of Leonhard Euler's correspondence, which had not been included in the original plans Opera omnia Euleri. In cooperation with the Deutsche Akademie der Wissenschaften zu Berlin (German Academy of Sciences in Berlin), the general correspondence[30] appeared in three volumes and the correspondence between Euler and Christian Goldbach (1690-1764) was published.[31] In 1963 a volume of selected scientific

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teoretič. izd. "Obrazovanie," 1934; ders., Novaja teorija dviženija luny (...), Leningrad, Izd. AN SSSR, 1934; ders., Osnovy dinamiki toski (...) Moscow-Leningrad, ONTI / Glavn. red. techn.-teoretič. lit-ry, 1938; ders., Differencial'noe isčislenie (Institutiones calculi differentialis), Moscow, Gostechizdat 1949; ders., Integtral'noe osčislenie(Institutionum calculi inegralis), 3 vols. Moscow, Gos. Izd. techn.-teoret. lit., 1956-1958; ders., Izbrannye kartografičeskie stat'i (...), Moscow, Geodezizcat, 1959; ders., Vvedenie v analiz bezkonečnych (...), Moscow, Fizmatgiz, 1961; ders., Issledovanija po ballistike (...), Moscow, Fizmatgiz, 1962; further papers were published in the omnibus "Variacionnye principi mechaniki" (Variational principles of mechanics), Moscow, Fizmatgiz 1959.

[28] G.I. Michajlov and V.I. Smirnov, Neopublikovannye materialy Leonarda Ėjlera v archive Akademii nauk SSSR (Unpublished material of Euler in the archive of the Academy of Sciences of the Soviet Union), in: *Leonard Ėjler. Sbornik statej v cest' 250-letija so dnja roždenija*, predstavlennych Akademii nauk SSSR (Collection of papers in honor of the 250th birthday of Euler). Moscow: Izdatel'stvo Akademii nauk SSSR, cit. p. 47.

[29] *Rukopisnye materialy Leonarda Ėjlera v Archive Akademiiu nauk SSSR* (Handwritten material of Euler in the Archive of the Academy of Sciences of the Soviet Union), 2 vols.(Trudy Archiva 17 and 20). Moscow-Leningrad: Izdatel'stvo Akademii nauk SSSR 1962, Leningrad: Nauka1965.

[30] *Die Berliner und die Petersburger Akademie der Wissenschaften im Briefwechsel Leonhard Eulers*, A.P. Juškevič and E. Winter, eds., in cooperation with P. Hoffmann, T.N. Klado and Ju. Ch. Kopelevič, 3 vols. (=Quellen und Studien zur Geschichte Osteuropas, III/1-3). Berlin: Akademie-Verlag 1959, 1961, 1976.

[31] *Leonhard Euler und Christian Goldbach. Briefwechsel 1829-1764*. A.P. Juškevič and E. Winter, eds., in cooperation with P. Hoffmann, T.N. Klado and Ju. Ch. Kopelevič (Abhandlungen der Deutschen Akademie der Wissenschaften zu Berlin, Klasse für Philosophie, Geschichte, Staats-, Rechts- und Wirtschaftswissenschaften, Jahrgang 1965, No. 1). Berlin: Akademie-Verlag 1965.

letters written by Euler to 19 scientists was published, and all letters were translated into Russian.[32] A list of Euler's letters were edited in Russian by Adol'f Pavlovič Juškevič (1906-1993) and Vladimir Ivanovič Smirnov which contained all known letters in Russia and outside. In total the list contains 2,654 letters from and to Euler along with a short abstract.[33]

In the 1970's cooperation between the Euler Commission in Zurich and the Soviet Academy was intensified because of the extension of the Euler edition.[34] The correspondence and the scientific notes will be collected in a new fourth series of the *Opera omnia Euleri*. In 1975 the first volume of this series was published and contained a revised list enumerating 2,892 letters of the correspondence.[35]

6. Conclusion

The topic "Euler and Russia" can be extended by some further thoughts. During his lifetime Euler was concerned with preserving his status as a citizen of Basel. He remained a Swiss citizen all his life. However, at the same time he had built up a special relation with Russia and the Russians that cannot be understood by a rational explanation alone. Many of Euler's statements show an inward solidarity with his adopted home, Russia, and they are not just lip service. It is notable in the way Euler made remarks about his first Petersburg period that he never made about his Berlin period. Obviously in Berlin he did not feel at home, probably partly because of his Swiss dialect. In the urban St. Petersburg he felt more comfortable. On the other hand, Euler was a cosmopolitan in the spirit of the Enlightenment, and accordingly he did not break off his relations with the Berlin Academy after leaving Berlin, as one might have expected. On the contrary, he kept in touch with the Academy and in later years he even wrote some letters to Frederick II.

In Russia, the Russian scientific community copiously restored his confidence. Rightly, Euler entered Russian history of sciences as one of its leading figures.

[32] L. Ėjler,*Pis'ma k učenym* (Letters to scholars). Moscow-Leningrad: Izdatel'stvo Akademii nauk SSSR 1963.
[33] L. Ėjler. *Perepiska. Annotirovannyj ukazatel'* (Correspondence. Annotated catalog). Leningrad: Nauka 1967.
[34] J.J. Burkhardt, Die Euler-Kommission der Schweizerischen Naturforschenden Gesellschaft: Ein Beitrag zur Editionsgeschichte, in: *Leonhard Euler 1707-1783.* J.J. Burckhardt et al., eds. Basel: Birkhäuser 1983, pp. 501-510.
[35] Leonardi Euleri opera omnia, ser. 4. Leonardi Euleri commercium epsitolicum. Vol A1: *Descriptio commercii epistolici.* Birkhäuser: Basel 1975.

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Leonhard Euler: Life, Work and Legacy
Robert E. Bradley and C. Edward Sandifer (Editors)

Princess Dashkova, Euler, and the Russian Academy of Sciences

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In January 1783 Empress Catherine II, the Great, of Russia appointed Princess Ekaterina (Catherine) Dashkova director of the Imperial Academy of Sciences in St. Petersburg, a position that the princess would fill to 1794 and officially hold for two more years. This selection and the return of Leonhard Euler to Russia in 1766 were Catherine's most important efforts toward renovating her academy of sciences and restoring its European reputation in research. An enlightened despot, she recognized that the sciences were vital to the growth of Russia as a powerful, modern state within Europe rather than a backward realm at the fringe of the continent. While stressing the utility of the sciences, for example the contributions of geography to the preparation of reliable maps and atlases, and the practical value of mineralogy, metallurgy, natural history, and naval science, she substantially supported research in astronomy, mechanics, and higher pure mathematics. [1] After sketching to 1783 the life of Princess Dashkova, this

[1] A. Kahan, "Entrepreneurship in the Early Development of Iron Manufacturing in

paper will examine the efforts of Euler to improve the management of the Imperial Academy, the extraordinary deference that the Russian aristocrat and friend of the empress gave to the plain-spoken Swiss-born commoner, and her major achievements in directing the institution to 1790 and so bringing Russia more into the Enlightenment.

1. Princess Dashkova: Life in Brief to 1783

In March 1744 Princess Ekaterina Dashkova was born Countess Vorontsova in St. Petersburg. Her mother was Marfa and her father Count Roman I. Vorontsov,[2] who became a Russian senator in 1760. He was the eldest brother of Count Michael Vorontsov,[3] the Grand Chancellor of Russia from 1758 to 1762 under Tsarina (Empress) Elizabeth and later Tsar Peter III. Raised in the family of her uncle, the Grand Chancellor, the young countess was well educated, studying European literature, especially the writers of the French Enlightenment. She collected a private library of 900 volumes. Among the dignitaries she encountered in the Grand Chancellor's house was the future Catherine the Great,[4] whom she met in 1758. Countess Vorontsova quickly became an admirer of the grand duchess's polished shrewdness. Count Vorontsov was also a friend of the poet, historian, grammarian, and chemist Michael Lomonosov, who often visited. In 1759 the countess married Prince Michael Dashkov. They and their children would at first live principally in St. Petersburg but often visited Moscow and their nearby country estate.[5]

In 1761 Princess Dashkova resided in her uncle's dacha next to the grand duchess, who during her stay in Oranienbaum occasionally came to have supper with the uncle or invited the princess to visit her in the evenings. The grand duchess and the princess shared an interest in cosmopolitan culture and readings in history and French political theory. "Bayle, Montesquieu, Voltaire, and Boileau" she wrote, were her "favorite authors."[6]

Russia," *Economic Development and Cultural Change*, **10** (1961-62), pp. 395-422.

[2] (1707 - 1783). See the Russian Academy of Sciences: Institute for the History of Science and Technology, St. Petersburg Branch, the History of Ideas Section for information on the Vorontsovs.

[3] (1714 - 1767)

[4] (1729 - 1796)

[5] See Gunther Schlegelberger, *Die Fuerstin Daschkova - eine biographische Studie zu Geschichte Katharinas II*, Berlin: Jumker and Duennhaubt, 1935.

[6] *The Memoirs of Princess Dashkova*, trans. and ed. by Kyril Fitzlyon, Durham: Duke University Press, 1995, esp. p. 33. The first edition of this book appeared in London with John Calder Publishers Ltd. in 1958.

Two others whom she carefully read were Jean-Jacques Rousseau, whom she disdained, and Claude Adrien Helvetius, the author in 1758 of the irreligious *De l'esprit (Essay on the Mind).* Dashkova's love of books was to continue lifelong. She spent all her pocket money on obtaining them. The German-born Catherine had taken easily to Russia, adopting the Orthodox religion and learning the Russian language. Catherine's encouragement of Dashkova's readings and the support she gave to the progress of her young friend suggest that she was recruiting allies for coming political battles. Princess Ekaterina could also provide invaluable information, for her sister was the mistress of the Crown Prince. In December 1761 Empress Elizabeth died and was succeeded by Peter III, whose failure on his accession to mention his pregnant wife possibly indicated an intention to replace Catherine.

It was soon apparent that the deficiencies of Peter were putting Russia in crisis and great debt. In June 1762 Catherine organized and orchestrated a successful *coup d'état.* She had long planned it, but she was deeply in debt. She had first to amass funds and wait as opposition to the emperor increased. The Orlovs and the Dashkovs gave her crucial support. Three of the five Orlov brothers were officers who supplied military muscle for the *coup* from the Izmailovskii guard regiment. The dashing General Gregory Orlov had become Catherine's lover. At age nineteen, Princess Dashkova was not simply a friend of Catherine but a political figure with growing influence in the imperial court and within the aristocracy in St. Petersburg. During the *coup* on 27 June, she put on a military overcoat and urged the Orlovs and Catherine to act with greater dispatch lest it fail.[7] Perhaps for political reasons, the empress afterward declared that the princess had taken only a minimal role in it.

After the coronation of Catherine II in September 1762 in Moscow, the traditional location for the ceremony for all Russian tsars,[8] the Dashkovs resided with her in St. Petersburg in the Winter Palace and received strong financial support from her. One indicator of her advancing political strength is that as the empress became known as Catherine the Great, the princess would be recognized as Catherine the Little. After her husband Prince Michael died in 1764, she remained for five years on her country estate. From 1769 to 1771 she journeyed to the Rhine valley, Frankfurt, and Berlin as well as Geneva, where she spent time alone with Voltaire. Upon her

[7] *Ibid.*, pp. 74 - 77. See also E. R. Dashkova, "Der Staatsreich von 1762. Denkwürdigkeiten der Fürstin Dashkow" in *Denkwürdigkeiten der Kaiserin Katherina II von Russland*, Ebenhausen bei München, 1916, pp. 275-306.

[8] Moscow was considered the third Rome, the second being Constantinople. The Russian title tsar derives from Caesar.

return to St. Petersburg in 1771 she had to overcome a rash of malicious gossip. The story spread that she was boasting of being the daughter of a liaison between her mother and a Count Panin; another had it that she was the count's lover. Since she had almost no personal property, for her wedding dowry had largely been spent on paying her husband's debts and the education of her children, Catherine II generously gave her 60,000 rubles to purchase a landed estate in the countryside.[9]

From 1776 to 1782 during an estrangement from the empress, Princess Dashkova traveled to western and central Europe, visiting England, Scotland, Ireland, France, Holland, the German states, and Italian cities, including Pisa, Lucca, Rome, Naples, and Venice. Without stopping after Venice, she crossed the Tyrolese mountains to spend time in the high society of Vienna, where she dined with the wily chancellor, Prince Wenzel Anton von Kaunitz.[10] Princess Dashkova wrote regularly to Catherine II, reporting on her studies of schools, universities, economics, culture, landscape architecture, and gardening. She visited parliaments, hospitals, museums, parks, and archaeological excavations and collected herbs and mineral samples. She met political leaders, philosophers, scientists, writers, actors, artists, and painters. She even composed music, mainly church hymns admired by the English tragedian David Garrick[11] for their Rousseauan "pathetic simplicity." Princess Dashkova had an extensive correspondence with Enlightenment figures, exchanging letters with the Encyclopedists Diderot, Voltaire, and Jean d'Alembert, a member of Russia's Imperial Academy of Sciences since 1764.[12] From 1776 to 1779 she was in Edinburgh, where her younger son Pavel (Paul) was a university student. The city's leading scholars visited her salon. Among her guests were the historians William Robertson and Arthur Young, the chemist Joseph Black, and the economist Adam Smith. Out of gratitude, she was to present Edinburgh University with a magnificent cabinet of Russian medals.

In 1781 in Paris, Princess Dashkova met Benjamin Franklin. His reputation for being a distinguished man of science and advocate of human rights

[9] *The Memoirs of Princess Dashkova*, trans. and ed. by Kyril Fitzlyon, pp. 139-140.
[10] *Ibid.*, p. 179.
[11] (1717 - 1779)
[12] The Encyclopedists were the writers for Diderot's famous *Encyclopedia*, the greatest collaborative work of the Enlightenment. It appeared in twenty-eight volumes over twenty-one years, starting in 1751. Among its authors were representatives of the more radical segment of the French Enlightenment. Europe's republic of letters, its readership multiplying in London, Berlin, Munich, Paris, Turin, St. Petersburg, and other cities, pursued the challenge that Immanuel Kant would later pose: "Sapere aude! (Dare to know)." See Frank A. Kafker, *The Encyclopedists as a Group*, Oxford: Voltaire Foundation, 1996.

was well accepted. She found him "a very superior man who combined profound erudition with simplicity of dress and manner, whose modesty was unaffected, and who had great indulgence for other people." [13] She was now being placed among the Enlightenment luminaries. The French sculptor Jean-Antoine Houdon, who made a marble sculpture of Jean-Jacques Rousseau, two of Voltaire, both of them now in the Hermitage Museum, and a half-length terracotta of Franklin, at present in the Louvre, produced in 1780 a full-length sculpture of her in Paris. [14]

In July 1782 Princess Dashkova returned to Russia to a rapprochement with Catherine II, who was extremely cordial. The empress presented her with a landed estate in Belorussia with 2,500 peasant serfs, which was on a par with gifts to her court officials. [15] The princess's energy, patriotism, amiable disposition, understanding of the importance of the sciences to Russia, and friendships with many scientists and writers in Europe's republic of letters built the foundation for her coming appointment to the directorship of the Imperial Academy. [16] The several portraits painted by Russian artists in the 1780s suggest the princess's increasing importance. Dmitri Levitsky made the most prominent of these, a half length portrait, and copies in 1784. [17]

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[13] *The Memoirs of Princess Dashkova*, trans. and ed. by Kyril Fitzlyon, p. 228.

[14] St. Petersburg has the famous bronze figure of "Dashkova sitting with an open book." It is located on the socle pedestal of the fifteen-meter monument to Catherine the Great designed by the Russian painter M. Mikeshin and constructed in 1873.

[15] See *The Memoirs of Princess Dashkova*, trans. and ed. by Kyril Fitzlyon, pp. 194 and Sue Ann Prince, ed., *The Princess & the Patriot: Ekaterina Dashkova, Benjamin Franklin, and the Age of Enlightenment, Transactions of the American Philosophical Society*, vol. 96, pt. 1, Philadelphia: American Philosophical Society, 2005, 129 pp., esp. p. 25.

[16] See *Mon histoire: Mémoires d'une femme de lettres russe à l'époque des lumières*, ed. by Alexandre Woronzoff-Dachkoff *et al.*, Paris: L'Harmattan, 1999

[17] (1740 - 1822). The oil portrait was among the gifts that the princess gave her English guests the Wilmot sisters, who spent several years on her country estate. Catherine Wilmot (1773 - 1824) received a manuscript copy of her *Mon histoire* in French in 1807 and Martha (1775 - 1873) letters from Catherine II and the oil portrait the next year. Martha Wilmot, who married Reverend William Bradford, edited along with Henry Coburn *Memoirs of the Princess Dashkaw, Lady of Honour to Catherine II, Written by Herself Comprising Letters of the Empress and Other Correspondence*, which was published in London in 1840. See also The Marchioness of Londonderry and H. Montgomery Hyde, eds., *The Russian Journals of Martha and Catherine Wilmot, 1803 - 1808*, London: Macmillan and Co., 1934, reissued in New York by Arno Press in 1971. The oil portrait disappeared until identified in a Russian exhibition in London in 1935. It was item w127 of the catalogue. Afterwards Mrs. Marjorie Merriweather Post, who founded the Hillwood Museum, purchased it. The portrait is now above the central stairway of the Hillwood Museum in Washington, D.C. See A. Bird, *Eighteenth Century Russian Painters in Western Collections: the Connoisseur*, 1971, pp. 78 -83.

2. Academic Governance: Euler, Orlov, and Domashnev

Meanwhile, in 1765 Catherine II had begun her attempt to revitalize the Imperial Academy of Sciences.[18] A major component of her project was her invitation to Euler to return from Berlin. In November he wrote to his friend the astronomer Joseph Nicholas Delisle in Paris that after the death of Pierre-Louis de Maupertuis, the president of the Berlin Academy, the selection of new members no longer depended upon the academicians and that he had decided to return to Russia. Catherine II, he observed, promised a grand reform of her Imperial Academy of Sciences that would regain its previous luster. She intended to raise the regular pension of professors to a thousand rubles, provide them with housing near the academy, and make eight additional appointments.[19] Euler, who was assigned to find scholars to fill these positions, asked Delisle to indicate possible candidates. Following Euler's recommendation, Catherine II in October 1766 replaced with a control commission the chancery overseeing the academy and stripped the academy president Count Kirill Razumovskij[20] of his legal prerogatives but not the post. Razumovskij, the hetman of Ukraine,[21] whom Empress Elizabeth had appointed president of the institution at the age of eighteen in 1746, was a prominent courtier but while he had been a student of Euler in Berlin in 1743 and 1744, he was showing scant interest in the sciences. To improve the operation of the academy, the empress created as a supplement to him the auxiliary post of director, to which she transferred presidential powers besides adding a large enough staff to control its departments. The consulting commission was part of the directorship. It included five academicians: Euler, his son Johann Albrecht, Semyon Kotel'nikov, Johann Lehman, and Stepan Rumovskij along with Jakob Stählin, the conference secretary to 1769. The director, who was also to be a member of the committee, and the nobles associated with the academic directorship assured the empress's dominance over the academy.

[18] Catherine saw technology and science as fundamental to the growth of her state. Her manifesto of 1763 had already addressed drawing to Russia artisans with skills in handicrafts and the production of luxury items, along with manufacturers. Granting skilled people an additional round of privileges, including tax exemptions, religious toleration, freedom from military service, and funding for constructing factories, brought more than 30,000 immigrants to Russia, most of them from the German states.

[19] Leonhard Euler, *Opera omnia: commercium epistolicum*, IVA.1, ed. by A. Juškevič, V. Smirnov, W. Habicht, Basel: Birkhäuser Verlag, 1975, p. 105.

[20] (1728 - 1803)

[21] The hetman was the leader of the Don Cossacks and Ukraine on the southern border of the Russian empire.

The academic control commission was supposed to prepare detailed plans for a thorough reorganization of the academy and to administer its affairs until a new director was named and then to assist him. From his first academic meeting after his return to St. Petersburg in August 1766, Euler worked steadily for a decade to better the administration. At the end of 1766 he submitted to the commission a memorandum that recommended streamlining the editorial board, improving the management of publications in the academy, and increasing the financial terms for foreign scientists invited to Russia. In February 1767 he sought as additional academic privileges a raise in salaries and the elimination of censorship on imported books. Other proposals were for relief from teaching assignments, a reduction in internal strife, and the establishment of more adequate funding through an expanded sale of books, journals, and calendars. While in Berlin, Euler had effectively supported the pioneering work in St. Petersburg of Michael Lomonosov,[22] who campaigned for educating native Russian scientists. Lomonosov, Russia's first eminent man of science, helped found physical chemistry, challenged the dominant phlogiston theory of combustion on the grounds that phlogiston could have different weights or be weightless, and supported the wave theory of light. Throughout his career in Brandenburg-Prussia and Russia, Euler sought a measure of autonomy for the sciences. Following upon recent efforts of Lomonosov, who by order of President Razumovskij had headed the educational branch of the academy from 1760 to 1765,[23] he advocated giving academicians a greater voice in running their institution. He was less willing than Lomonosov to accept complete aristocratic control over the academy, but only very briefly obtained limited independence for it.[24] Opposition initially by the first of two power-seeking directors and courtiers associated with the academic commission thwarted all of Euler's administrative proposals. Eventually in a letter of

[22] (1711 - 1765)

[23] As there was disorder and disorganization in the academic gymnasium and university in St. Petersburg, Lomonosov developed a plan in the early 1750s to establish Moscow University. Founded by Empress Elizabeth in 1755, it quickly became the leading educational center in Russia, including the sciences among its strengths. Moscow University practically realized Peter the Great's idea of the unity of the sciences and education as well as helping fulfill in part Lomonosov's vision of creating distinguished Russian universities.

[24] Euler must have agreed with Lomonosov on the position of scientists in eighteenth-century Russia. They commanded no prestige . They were state employees with no posts in the state bureaucracy and no rights to rise to the nobility. This was opposite to the situation in western and central Europe. Lomonosov considered this social limitation damaging to the advance of the sciences in Russia but was unable to convince the imperial court of this. Not until 1790 did Catherine II permit scientists to become state consultants.

February 1774 to Orlov, he pointedly resigned from the academic commission for himself and his son Johann Albrecht.[25] His best way now to serve the academy, he wrote in September, was by educating and coaching new students. As the *doyen* and most distinguished member of the academy, though, Euler headed the twice-weekly general conferences in the absence of the president and together with his son, who was made conference secretary in 1769, had the chief responsibility for choosing successors to fill vacant positions.

Neither Euler's administrative efforts nor his blindness after his cataract operation in 1771 slowed his extraordinary productivity in research and publications. During his second St. Petersburg period, he wrote almost 415 memoirs and books, which amount to just over half his total publications.[26] These include the completion of two of the three volumes of his *Institutiones calculi integralis*, the first published in 1768 and the second in 1770, his three-volume *Dioptrica*, which appeared from 1769 to 1771, and his 775-page *Theoria motuum lunae*, published in 1772 and containing his second lunar theory. Most of his articles appeared posthumously. He set out most of the topics for the annual prize competitions of the academy and was always one of the judges. He helped to organize the famous expeditions to diverse regions of Russia to observe the transit of Venus before the disc of the sun in 1769 and several others involving solar eclipses. Data from the observations of the transit of Venus allowed him to determine the parallax angle of the disc of the sun, which was needed to compute exactly the distance between Earth and the sun. Although Euler's correspondence diminished after he left Berlin, he continued to use that means of spreading his ideas. Letters to D. Bernoulli, Joseph Louis Lagrange, Nicholas de Condorcet, and other geometers address integral calculus, while another to the Royal Society of London discusses dioptrics and lunar theory. In a letter to the Berlin Academy Euler reported on the research he had accomplished after essentially losing his sight in 1768. He exchanged letters with noble Russian officials and two monarchs, Frederick II of Prussia and Stanislaw August of Poland, for whom he improved geographical coordinates to improve Polish cartography.

General Count Vladimir Orlov,[27] a graduate of Leipzig University who was the first director of the Imperial Academy, served from 1766 to Decem-

[25] Euler, *Opera omnia*, IVA.1, p. 326. A day before, in a letter of February 2, 1774, he had asked Orlov to free him of work in the geography department because of his failing eyesight.

[26] Emil A. Fellmann, "Leonhard Euler: Ein Essay über Leben und Werk," in *Leonhard Euler: 1707 - 1783: Beiträge zu Leben und Werk*, Basel: Birkhäuser Verlag, 1983, pp. 31 - 32.

[27] (1743 - 1831)

ber 1774; his successor the minor poet Sergei G. Domashnev, [28] a follower of the Orlovs, held the post from 1775 to 1782. The appointments were mainly rewards for being among the leading supporters of Catherine's *coup*. Neither of the two understood scientific research or had much respect for it. They both rejected the academy's brief degree of self-government organized separately by Lomonosov and Euler. The court members of the commission and the first two directors, *primus inter pares* on it, wanted to reestablish a despotic bureaucracy. During a visit to St. Petersburg Count Sigismund Ehrenreich Redern of the Berlin Academy, whose atlas Euler had published in 1762, was horrified at Orlov's administration. "My God," he exclaimed to Euler, "what an extraordinary kind of person you have for the president [actually director] of the academy–who is against all scholars, regards the academy as useless, and believes with Rousseau that science would make the world only more evil." [29] It was initially thought that Orlov would provide leadership, but in his preoccupation with the great Pugachev rebellion he came to neglect his duties and retired. Domashnev, though a highly intelligent graduate of Moscow University, was a worse choice. He squandered the academy's finances, was rude and arbitrary toward its members, and flagrantly violated their institutional rights. He left vacancies unfilled, rarely attended meetings, embezzled funds from the academy treasury, and took for his own collection books ordered for the library. In April 1782 Domashnev arbitrarily removed Kotel'nikov from the academy and soon after attempted to transfer Kotel'nikov's cabinet of natural history to Peter Pallas.

In the ensuing controversy, the academicians insisted that this "authoritative act would overturn the entire system of academic obligation." [30] In August they wrote that the dismissal of Kotel'nikov violated the academy's 1747 charter, asked Domashnev to withdraw it, and expressed no confidence in him. In November 1782 they sent him a further letter of protest. The letter was met with silence. The academicians then openly forwarded the letter to the control commission with a call for Domashnev to be discharged, and Catherine was informed of the conflict. Euler was among the signatures. After a two month inquest, Catherine dismissed Domashnev. Under her the Russian government would no longer ignore the few rights of the academy's men of science.

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[28] (1743 - 1795)

[29] As quoted in Alexander Vucinich, *Science in Russian Culture: A History to 1860*, Stanford: Stanford University Press, 1963, p. 141.

[30] Michael D. Gordin, "Arduous and Delicate Task," in Sue Ann Prince, ed., *The Princess & the Patriot*, p. 10.

3. Princess Dashkova as Imperial Academy Director

On 24 January 1783 with the stroke of a pen, Catherine II by decree dissolved the academic control commission and appointed Princess Dashkova the director of the Imperial Academy of Sciences. Thus began an epoch in the history of the academy. The previous night when the empress offered her the position, the princess "had been first struck dumb with astonishment," then but quickly insisted that "I cannot accept any office which is beyond my capacities."[31] Her majesty rejected the objection and declared her abler than her predecessors. Princess Dashkova knew that the empress had the tact, shrewdness, and perseverance to get a positive answer, so resistance would be futile. At the staff meeting of the empress on the morning of the twenty-fourth, Domashnev was present. When he attempted to describe to the princess the duties she would carry as academy director, she politely rebuffed him, saying that she "would treat its members with perfect impartiality."[32]

At thirty-eight the princess, who was Russia's first stateswoman and prominent female aristocratic manager, faced a daunting challenge, beginning with ending the disarray in the operations of the academy and reestablishing stability. Salaries of members of the debt-mired academy were in arrears, buildings in disrepair or not begun, libraries and laboratories depleted, and many scientific expeditions not yet organized. Although well educated, Princess Ekaterina was not a scientist. Apparently she had conducted a sober, unbiased assessment of her possibilities for the directorship and concluded that strong administrative and organizational skills, in both of which she had talent, together with her intellectual ability, could produce beneficial results for years. And she had greater authority than her two predecessors. She reported directly to Catherine. But in recognition of the prerogatives of the academicians, she would not be installed in the position until the academy elected her. Its members were pleased, and on January 28 they respectfully and unanimously voted for her.

On Monday 30 January 1783, Princess Dashkova was to be formally presented and announced in the Imperial Academy Conference Hall. Relieved that the era of Domashnev had ended, all the academicians and professors of the academy gladly agreed to gather again in the conference hall in what is now referred to as the old academy building. The former palace of Princess Praskovea, the sister-in-law of Peter the Great, it had been since 1728 the main academy building. The *Mémoires* of Princess Ekaterina de-

[31] *The Memoirs of Princess Dashkova*, trans. and ed. by Kyril Fitzlyon, p. 200.
[32] *Ibid.*, p. 204.

scribe in detail her meeting on Sunday with members and on Monday with Euler and the events of that her first day in the academy.

> "Early the previous morning, which was a Sunday, I received the visit of all professors, officers, and servants of the academy. I told them that I should visit the academy the next day, and I begged them that if they had any business to discuss with me, they come to my room at any hour convenient to them without waiting to be announced.
>
> I spent the evening reading the reports given me and tried to make myself familiar with the labyrinth into which I was about to venture, for I was entirely convinced that my slightest mistake would become known and criticized. I also tried not to forget the names of the most important governmental inspectors and officers of the academy.
>
> The next day before going there I paid a visit to the great Euler. I say 'great' because he was, without any doubt, the greatest geometer and mathematician of his age, besides being familiar with every branch of science: his industry was such that even after he lost his sight he continued his researches and made discoveries, dictating his work to Mr. Fuss, who was married to his grand-daughter. He left behind him a great deal of material that went to enrich the publications of the academy for many years after his death.
>
> Disgusted, like everyone else, with Domashnev's behavior, he no longer attended the academy and took no interest in its proceedings, apart from adding his name to an occasional protest with the other members and even writing directly to her Majesty whenever Domashnev took it into his head to launch into some ruinous undertaking.
>
> I begged him to accompany me to the academy at least this once, adding that I did not claim he should bother to attend it in future, but that as this would be my first appearance at a sitting of the scientific body, I wanted to be introduced by him. He seemed flattered by my great consideration for him. We had known each other for a long time, and I may venture to say that he had a high regard for me ever since I was a very young woman, some fifteen years before I assumed the directorship of the academy.
>
> Euler came with me in my carriage, to which I also invited his son, the permanent secretary of the academy, as well as his grandson Mr. Fuss, who since the great man was blind had the task of guiding his steps. [33]
>
> As I entered the conference hall, I said to the professors and other members assembled that though I was an ignorant person myself, I wanted to mark my respect for science and could find no more solemn

[33] Perhaps the princess did not remember for her memoirs that Fuss had not yet married Euler's granddaughter.

and impressive way of doing it than by being introduced by Mr. Euler. I spoke these few words before sitting down and noticed that Professor Mr. Stählin had taken his place next to the Director's chair.... I therefore turned to Mr. Euler and told him to sit down where he thought fit, for any place he occupied would always be the first. His son and grandson were not alone in showing appreciation and pleasure at my remark, for the eyes of the professors, who all had the highest respect for the venerable old man, were filled with tears." [34]

The year 1783 brought increasing recognition for Princess Dashkova. Following the princess's initiative, Catherine II founded in September the Russian Academy of Language, Linguistics, Literature, and Dictionaries. Initially proposed by Lomonosov, it became an essential element in the fruitful development of Russian culture during the period, defined as the empress's Golden Age, in which she made St. Petersburg a major educational and cultural center. The princess convinced intellectuals of differing social status to produce over a five year period the first Russian dictionary, for which she and Catherine separately wrote some entries. The nobility, who spoke French, found it merely an inconvenience. The empress herself also wrote children's stories and short plays, which she helped stage. The theater became popular. On the advice of the princess, Catherine II instructed Russia's ambassadors to purchase paintings, sketches, carvings, medals, and books, including Voltaire's library. Initially many were placed in the Winter Palace. Following the empress's example, the wealthier members of the aristocracy decorated their homes with fine paintings and sculptures. Princess Dashkova was named president of the Russian Academy of Language in 1783 and generally received active support for it from the imperial court.For the next dozen years, she oversaw the two most important academies in Russia.

As the Imperial Academy director, Princess Ekaterina set out an ambitious program, in general tackling an array of problems with a striking success that she would report to the empress in 1786. Underlying her program was an attack on the dominant view in Russia that theoretical science was at opposites to practical applications and that a wall separated them. Instead Princess Dashkova sought to bring out their interconnections and the impact of progress in the sciences upon daily life. She recognized that heavy debt and the scantiness of funding were her greatest problem." [35] As a start to building a sound financial base for the academy, she prepared for the empress a detailed account of its holdings. She visited the chancellery that oversaw its administration and finances and informed its officials that

[34] *The Memoirs of Princess Dashkova*, trans. and ed. by Kyril Fitzlyon, pp. 205 - 206.
[35] *Ibid.*, p. 206.

henceforth it was "the common duty of us all [to] redress these [past financial] abuses, [such as embezzlement], the shortest and most efficient method of achieving this being to squander nothing and stop all misappropriations."[36] Under Domashnev embezzlement had been rife and publicly known. A traditional Enlightenment way to correct financial arrears was to ask the monarch for funds, and Catherine had pledged these. But the princess requested only a small amount for medals and earned almost the entire budget through honest, impersonal, and meticulous administration. She noted that debts to booksellers in Holland, Paris, and Russia were a principal drain on finances. By astutely increasing by thirty percent the price of academy publications, she soon reduced debts substantially.[37]

The maintenance, rehabilitation, and construction of academy buildings were fundamental to a healthy academy. Princess Dashkova solved many related troubles. Attentive to detail, she looked at even the smallest problems, such as heating fireplaces and renting out empty basements for storerooms. Two apartment houses for academy members and the restoration of Gottorp's Globe in the circular hall of the Kunstkammer tower were two of her largest construction projects. After the fire of 1747, the Kunstkammer had been rebuilt from 1754 to 1758, but in its tower the large Gottorp's Globe was only partly reconstructed. A rotating spherical globe-planetarium about four meters across, it had been presented to Peter the Great in 1713 by Duke Karl Friedrich of Holstein-Gottorp for his aid against Sweden in the Great Northern War. Twelve people can sit within the globe and observe the stars and the moon. The inner surface has more than a thousand golden points representing stars, reflected by the light of a central candle source. As a replacement for the burned design of the seventeenth century according to maps of the sixteenth century, Princess Ekaterina had the academician Friedrich Theodor von Schubert[38] supply a timely version of the design of the geographical map on the outer surface of the globe. The only detail that remains from the eighteenth century on the interior of the Kunstkammer is the wall mural of geometry with Leonhard Euler's lunar formulas.

Working from these foundations completed or on the way, the princess's program for the academy sought to increase the prestige and importance of its conference, decrease its bureaucracy so as to subordinate members directly to the director, and require academicians to report regularly to

[36] *The Memoirs of Princess Dashkova*, trans. and ed. by Kyril Fitzlyon, p. 206.
[37] By the time she left in 1794, she had earned a half million rubles. She left the academy with 100,000 rubles in the bank and the bookstore and library with 390,000 in the black. See Michael D. Gordin, "Arduous and Delicate Task," in Sue Ann Prince, ed., *The Princess & the Patriot*, p. 13.
[38] (1758 - 1825)

the conference on their current research and results. She began to educate and train more future scientists for the academy and Moscow University, recruiting fifty competent students for the academic gymnasium compared to the previous seventeen and funding them. Within three years she raised the number to eighty-nine. She also assigned more Russian students to attend European universities, particularly Leipzig, Heidelberg, Göttingen, and Edinburgh. Princess Dashkova insured that royal monopolies and privileges in crafts and manufacturing were strictly followed and that new results and discoveries in the sciences were kept secret until information on them was published in Russia. It was "to the academy's shame," she wrote, that observations and discoveries made "inside the country were communicated abroad before their publication in Russia and used by them [to their advantage] before they were here." [39] She removed the complete confusion in the management of the academy's editorial board and acquired fine type for the press. This managerial improvement and her redesign of book sales were but two of her reforms that responded to Euler's proposals. The printing of the two volumes of the quarto-sized transactions, *Nova acta*, annually had earlier been reduced to one volume and then for want of type suspended. The princess's administration and equipment acquisitions made it possible for the academy to resume regular publication of its *Nova acta*, mostly with articles left by Euler, and she had the first anthology of the writings of Michael Lomonosov published from 1784 to 1787. [40] These publications and the restructuring of the Russian book trade to make it more efficient increased academy revenues, which she funneled into improving laboratories and botanical gardens.

Princess Ekaterina added prizes in engineering and organized courses of popular lectures on geometry, mathematics, and natural history, which academicians had to deliver during the summer. These courses were given in Russian by native professors and free of charge, which made it possible for even the impoverished children of the Russian aristocracy to attend and benefit. Lecturers were paid two hundred rubles at the end of each course. Having the lectures in Russian was not only to allow all students, pupils, and lay people to learn from them but also to amplify scientific terminology in the Russian language and enter it into daily conversation. [41] Russian, the language of the common people, was slowly evolving beyond an elementary

[39] Michael D. Gordin, "Arduous and Delicate Task," in Sue Ann Prince, ed., *The Princess & the Patriot*, p. 16.
[40] Grand Chancellor Michael Vorontsov had been a friend and supporter of Lomonosov. Growing up with the grand chancellor's family, Princess Dashkova had known Lomonosov and his work since childhood.
[41] Alexander Woronzoff-Dashkoff, "Books Make the Woman," in Sue Ann Prince, ed., *The Princess & the Patriot*, p. 83.

discourse in the sciences. For example, the trilingual dictionary of Fedor Polikarpova, published in 1704, had lacked a definition for attraction from Newton, a term that became associated in Russia only with Enlightenment science.[42] In Catherinian Russia, Rumovskij's translation of Euler's three-volume *Letters to a German Princess* from French to Russian was a notable source of the reading public for the sciences. Its first edition appeared from 1768 to 1774; its second in 1785; and its third in 1790-91. In addition to encouraging the natural sciences, Princess Dashkova promoted research on the Russian language, dictionaries, and publications to enhance Russian secondary and higher education. In 1783 she commissioned Peter Pallas to design a uniform for all academy employees to wear. It was purple with light yellow piping and a light green necktie. The princess wanted to make academicians more a part of the state civil service though distinct from other groups and to suggest that science was developing as a profession.[43] Before her, Russia had not conferred the highest state honors on distinguished academicians in the sciences. From 1783 to 1794, Princess Dashkova recommended and had decorated eight academicians with the Order of St. Anna or that of St. Vladimir. Among the recipients was Johann Albrecht Euler, honored with the Order of St. Vladimir in 1787.

From the beginning of her directorship, a principal issue for Princess Dashkova was the recruitment and retention of talented mathematicians and physicists from western and central Europe. Apparently she discussed this matter and many other organizational problems during visits with Euler, who proposed candidates. After his death in 1783, his followers suggested members of the Euler circle dedicated to elaborating and refining the vast body of research that he left.

Notable among the scholars close to Euler and his circle was Nicholas Fuss.[44] In 1772 Euler had asked Daniel Bernoulli in Basel to choose a young mathematician to be his assistant. In July of that year Fuss arrived. He thereafter lived with Euler and served as his personal secretary, helping the blind genius by making computations for more than 160 memoirs. In 1783 he was elected an ordinary member or academician in mathematics. When he threatened to leave Russia, Princess Ekaterina doubled his salary to retain him. In the same way she kept the historian, ethnographer, and geographer Johann Gottlieb Georgi from departing.[45] In 1784 Fuss

[42] Valentin Boss, *Newton & Russia: The Early Influence, 1698 - 1796*, Cambridge, MA: Harvard University Press, 1972, p. 243.
[43] Michael D. Gordin, "Arduous and Delicate Task," in Sue Ann Prince, ed., *The Princess & the Patriot*, p. 12.
[44] (1755 - 1826)
[45] Johann Georgi (1729 - 1802) joined the academy in 1783.

married Albertina, the second daughter of Johann Albrecht Euler.[46] Fuss contributed to infinitary analysis, astronomy, geometry, the mechanics of elasticity, and the building of fortifications.[47]

Another prominent scholar recruited from Euler's circle abroad was the German mathematician and astronomer Friedrich Theodor von Schubert, whom the academy acquired in 1785. Schubert, having attended Greifswald and Göttingen universities, was well known in the 1780s for his research in cartography, celestial mechanics, spherical trigonometry and geometry. After his candidacy was considered in the early 1780s, Princess Dashkova in December 1784 invited him to St. Petersburg, and he arrived the next month as an adjunct in mathematics, director of the academic library, and chief of the "Minz-Kabinett" and the complete collection of astronomical and physical instruments and devices. He was asked to save and develop the maps created by Delisle and Euler. Schubert faced the most difficult task of putting the academic library in good order. Its problems stemmed largely from the great fire of December 1747, which had destroyed the observatory and severely damaged the Kunstkammer Museum containing the library. In January 1748 Euler wrote from Berlin to Delisle in Paris that the industrious efforts of cadets – and he could have added officers – had for the most part saved library books and journals, along with the holdings of the museum of Peter I.[48] Between 1754 and 1758 the Russian architect S. Chevakinsky restored the Kunstkammer except for the high tower, but the library remained disordered. Thirty-seven years after the fire, Schubert essentially finished the renovation of the library. Supported by relatively lavish funds from Princess Ekaterina, he catalogued the books there, searched for lost volumes, and ordered books and journals from western and central Europe.[49] In 1789 Schubert was elected an ordinary member of the academy and chairman of astronomy.

In the 1770s Daniel Bernoulli had informed Euler about the mathematical talent of his nephew Jacob II.[50] He was a student of Johann II and Daniel Bernoulli at the University of Basel. Possibly the idea of inviting Jacob II to Russia was discussed. After Daniel Bernoulli and Euler died

[46] (1766 - 1829). They had thirteen children. Their eldest son, Paul (1798 - 1855), succeeded his father as the permanent secretary of the academy from 1826 to 1855. He organized a partial first edition of an *Opera omnia* of Euler's works and wrote a biographical article on him.

[47] The permanent secretary of the academy conference from 1769 to 1800 was Johann Albrecht Euler, whom Nicolaus Fuss succeeded from 1800 to 1826.

[48] Leonhard Euler, *Opera omnia*, IVA.1, p. 104.

[49] Schubert was the great grandfather of Sofia Kovalevskaya (1850 - 1891) on her maternal side.

[50] (1759-1789)

Johann III Bernoulli, the director of Berlin Observatory from 1771 to 1786 and the eldest brother of Jacob II, wrote to Nicolaus Fuss to propose Jacob II for the Imperial Academy. Fuss enthusiastically recommended him to Princess Dashkova, who sent an offer to Jacob II in Venice. In 1786 Jacob II arrived in St. Petersburg as an adjunct in mathematics. The next year, the academicians elected him together with Schubert an ordinary member in mathematics and astronomy. While he had taken part in geographical and astronomical expeditions, his main scientific contributions were in various aspects of classical and applied mechanics: rotation motion of pendulums, elasticity theory, hydraulics, oscillation theory, and the hydrodynamics of fluid flow in tubes. His investigations of these problems expanded upon the brilliant results of Euler and Daniel Bernoulli. In 1787 Jacob II was also made a professor in mathematics and physics for the Military Cadet Corps in St. Petersburg. In 1789 he married Charlotta Euler,[51] Johann Albrecht Euler's fourth daughter, and so joined Fuss as a son-in-law of the younger Euler. But two months after his marriage he accidentally drowned in the Nevka-river, a branch of the Neva in St. Petersburg.[52] He was only thirty years old.

From 1783 to 1789, the Imperial Academy elevated its own status by electing famous foreign men of science and humanists as honorary academicians, members *honoris causa*. Princess Ekaterina and the Euler circle were the chief contributors of names.

In 1783 the princess proposed the appointment of the historian and rational cleric William Robertson, the principal of the University of Edinburgh and royal historiographer, whom she personally knew.[53] Together with his friend David Hume and Edward Gibbon, he formed the triumvirate of eminent British historians during the Enlightenment.[54] *A History of Scotland, 1542 - 1603*, published in 1759, was his most famous work. It underwent fourteen editions. It was followed by *A History of Charles V* in 1769 and *A History of the Discovery and Settlement of America* in 1777. Robertson's skillful search for original documents, his critical synthesis and cosmopoli-

[51] (1773-1831)

[52] His widow Charlotta married again in 1790 to John David Collins (1761-1833), a Scotsman and the pastor of the German Lutheran Church of St. Peter's in St. Petersburg. They had fourteen children. Their eldest son, Eduard Collins (1791-1840), a great-grandson of Leonhard Euler, was the adjunct in mathematics of the Imperial Academy from 1814 and ordinary academician in mathematics from 1820. Starting in 1824 he also taught mathematics in St. Peter's Lutheran School and from 1833 was its director.

[53] (1721 - 1793)

[54] See Stephen J. Brown, *William Robertson and the Expansion of Empire*, Cambridge: Cambridge University Press, 1997. These three and Voltaire were the foremost Enlightenment historians. No failings in Robertson's writing and research but the growth of knowledge led to the eventual neglect of his work.

tan settings, and especially his exploration of the general ideas of freedom, power, progress, and providence profoundly influenced the consciousness of Europe. The Imperial Academy had printed in the 1770s the Russian translation from the French of his two volumes on Charles V and would publish his *History of the Discovery and Settlement of America* in 1784.

The chemist Joseph Black, another Scot who had gained the attention of the princess during her travels, continued to correspond with her about diverse scientific phenomena. After the devastating Lisbon quake and tsunami in November 1755 in which as many as 100,000 people perished, earthquakes were a major topic throughout the scientific community. Debate flared over whether the Lisbon quake was a punishment for evil or there was a scientific explanation for it. As the academy conference secretary, Johann Albrecht Euler informed Black of his election to the academy. Crown Prince Paul, the future Paul I, later attended his lectures.

In 1786 two Germans were elected *honoris causa* by the academy. Both were familiar within the Euler circle. The mathematician, astronomer, and historian Abraham Gotthelf Kästner[55] of Göttingen University had corresponded with Euler and supervised Russian students at Göttingen. Johann Elert Bode,[56] named royal astronomer and director of the Berlin Observatory in 1786, belonged to Euler's astronomical school. He established the Titius-Bode Series showing nearly geometric distances between large planets, improved the accuracy of almanacs, and produced an atlas listing 17,000 stars. The Titius-Bode Series was to be important in astronomy through the nineteenth century. At the Berlin Observatory, Bode was a colleague and the successor of Johann III Bernoulli, who probably recommended him.

The German-born astronomer William Herschel, the discoverer in 1781 of Uranus and its two satellites, Oberon and Titania, was one of two eminent men of science elected members *honoris causa* in 1789. Euler had made some computations of the orbit of Uranus. Herschel, residing in England, provided a statistical foundation for stellar astronomy, observed star clusters and double nebulas in our galaxy, and from 1786 to 1789 constructed a reflector telescope twelve meters long. Later Empress Catherine II liked to observe the moon, stars, and planets with a Herschel telescope, a gift from George III of England. She had put the telescope at her country residence of Tsarskoe Selo and had Stepan Rumovskij, Euler's leading Russian student, as her personal astronomer.

[55] (1719 - 1800)
[56] (1747 - 1826)

On 2 November 1789, Benjamin Franklin[57] was voted *honoris causa*, the first American scientist so honored by the Imperial Academy.[58] Princess Ekaterina, having learned of him in Edinburgh and gained so favorable an impression of him during their encounter in Paris in 1781, nominated him. Russia's "armed neutrality" during the American War for Independence must have pleased Franklin. Catherine II, the initiator this policy, had refused the request of George III for troops to employ against the American colonists. After Princess Ekaterina returned to Russia, she and Franklin corresponded. In Paris Franklin met other Russians. In their number was Prince Dmitri A. Golitsin, the ambassador to Holland from 1769 and awarded membership *honoris causa* in 1778. His primary interest was in electricity. Several of his articles speak highly of Franklin's electrical unitary theory. At mid-century, Georg Richmann, Lomonosov, and Franz Aepinus of the Imperial Academy had extensively studied electricity. In a kite experiment in lightening conducted from his laboratory at the Imperial Academy following upon Franklin's work, Richmann, though aware of the dangers, was killed in July 1753. In December of that year Euler wrote a review praising the electrical research of Lomonosov, who organized a prize competition on electricity. Euler won first prize under the name of his son Johann Albrecht. He could not submit a thesis with his own name, since he was a member of the academy. He proposed an ether theory of electricity. Aepinus's *magnum opus*, *Tentamen theoriae electricitatis et magnetisimi* of 1759, bases electrical phenomena on Newton's action-at-a-distance rather than electrical atmospheres. Following Franklin's advice, Golitsin acquired for the academy electrical devices, probably including the electrophore, from the laboratory of Alessandro Volta in Pavia. In the 1780s these devices were crucial to the experiments of the Russian engineer and inventor Ivan Koulibin, like Wolfgang Krafft and Johann Albrecht Euler, an investigator of electricity.

In November 1789 Princess Ekaterina and the younger Euler sent letters informing Franklin of his selection *honoris causa* and sincerely apologizing for not having done this earlier. Russia knew him for his literary as well as his scientific works. In 1778 his writings had begun to be translated into Russian. *Poor Richard's Almanack*, the most popular, went through six editions. In 1791 the Russian poet, historian, and translator Nikolai Mikhailovich Karamzin published Franklin's *Autobiography*, which Franklin

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[57] (1706 - 1790)

[58] See E. Dvoichenko-Markoff, "Benjamin Franklin, the American Philosophical Society and the Russian Academy of Sciences," *Proc. American Phil. Soc.*, **91**, No. 3 (1947), pp. 250-257.

himself titled *Memoirs.*[59]

Early in 1789 the American Philosophical Society in Philadelphia, enjoying international prominence since 1769, unanimously elected Princess Dashkova a foreign member. Franklin, a leading founder of the society in 1743 and chosen its president, expressed in his nomination of the princess his friendship and esteem for her.[60] In April 1789 the society mailed Princess Dashkova a handsome diploma. She was the first woman and the second Russian chosen for the society,[61] the first being Prince Dmitri Golitsin. The society was to send her copies of all books that it published. But she did not immediately receive the packet from Franklin announcing the good news. From 1788 to 1790 Russia was at war with Sweden. In 1789 a Swedish ship intercepted the mails for Russia, including the Franklin packet. In turn, the Russian army captured the Swedish ship in the summer. Princess Dashkova explained to the empress the "nonsense" delaying the delivery of the packet. Her membership in APS added to her recognition outside Russia. Previously she had been elected a foreign member of science academies in Berlin, Dublin, Erlangen, and Stockholm. Even as her status of Enlightenment figure continued to rise, the fall of the Bastille in Paris in July 1789 and the ongoing French Revolution with its ideals of "liberté, egalité, and fraternité" was to lead in 1793 to a rupture in relations with the empress. While Princess Dashkova defended absolutism against the ideals of the French Revolution, she opposed Catherine II's harsh censorship and arrest of Russian authors from 1789 for writings deemed anti-monarchical or otherwise dangerous.[62]

From Domashnev's time, it was recognized that the old academy building, the former Princess Praskovea palace, no longer sufficed to house

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[59] (1766 - 1826). See A. G. Cross, *N. M. Karamzin: A Study of His Literary Career, 1783 - 1803*, Carbondale: Southern Illinois U. Press, 1971, and Samuel M. Lewis, *J. G. Herder and N. M. Karamzin*, Urbana-Champaign: U. Illinois Press, 1992.

[60] Karen Duval, "A Man Made to Measure: Benjamin Franklin: American *Philosophe*," in See Sue Ann Prince, ed., *The Princess & the Patriot*, pp. 64-65.

[61] See A. Woronzoff-Dashkoff, "Princess E. R. Dashkova: First Woman Member of the American Philosophical Society," *Proc. American Phil. Soc.*, **140**, No. 3 (1996), pp. 405-414.

[62] Among the authors arrested and exiled to Siberia were Alexander N. Radishev, author of *Voyage from Petersburg to Moscow*, published in 1790, and Nikolai Novikov. Radishev was a protégé of her favorite brother Aleksandr R. Vorontsov, who after the *Voyage* appeared was retired from Russian service. The time when Catherine's favorite fell into disfavor seems November 1793, when Princess Dashkova granted permission for the academic press to publish the playwright Jacob Kniazhin's critical drama *Vadim from Novgorod* in issue 39 of the journal *Russian Theater*. *Vadim*, which referred to the public assembly in medieval Novgorod, was sympathetic to republicanism. Aghast at the French Revolution with the execution of the king, Catherine opposed Kniazhin's views and decided to punish Princess Dashkova for not censoring them.

the academy's administrative, public, and research offices. A new main academy edifice was needed. But nothing was done, since the projected 100,000 rubles to finance its construction was not available. The reason that Princess Ekaterina gave Catherine in the fall of 1783 for beginning construction was that the academy required adequate space for the sale of books, for public lectures with paid admission, and for housing for servants of the academy. She would make the academy more than the traditional center for scientific research and show. Under her administration it was also to be a business enterprise earning the income to finance its principal architectural project. At the princess's insistence, the architect Giacomo Quarenghi began construction in 1783. Schubert, who was to stand nearest to Princess Ekaterina at the academy, supported its building.[63] Daily and sometimes twice daily she inquired about progress on the rising structure. Some of her messages were a bit imperious in their assumption that she knew better than anyone else what had to be done. The beautiful main academy building in neo-classical style on the Neva embankment near the Kunstkammer Museum was completed in 1789. Today it is the main office of the St. Petersburg branch of the Russian Academy of Sciences. Its presence is a memorial to the director who had done so much to take Russia into the future that the Enlightenment, in its struggle with the autocracy she served, was forging.

Acknowledgements: We are grateful to Ed Sandifer for providing photographs of the Imperial Academy's *Protokolii*, to John Glaus for his translations, and to Thomas R. West for his editorial comments.

[63] His son, General Theodor Friedrich von Schubert of the Russian army, remembered this in his memoirs, *Unter dem Doppeladler*, written in German from 1860 to 1865, and published in 1962.

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Leonhard Euler: Life, Work and Legacy
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Leonhard Euler and Philosophy

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In a treatise of 1783 Georg Christoph Lichtenberg makes a slightly premature inventory of the 18th century, listing all the historically significant events which he thought should be handed down to the next century. He wrote "I have seen Peter the Great, and Catherine, and Frederick, and Joseph, and Leibniz, and Newton, and Euler, and Winckelmann, and Mengs, and Harrison, and Cook, and Garrick."[1] It may be astonishing that the painter Mengs, the watch-maker Harrison, and also the actor Garrick were esteemed as such important persons in cultural history, but it may be more striking that philosophers so prominent as Locke, Hume, Wolff, and Kant are missing, even if the last had but modest influence at that time. The reason is that Lichtenberg's inventory did not concern philosophy, but mentioned Leibniz, Newton, and Euler only as scientists, though the 18th century was named the "philosophical" century.[2] Euler's outstanding position in Lichtenberg's list is not primarily due to his reputation as a philosopher. Only incidentally was he philosophically productive, and he did not pretend to be a philosopher, because he disdained the pretended philosophers. And his contemporaries scarcely esteemed or accepted him as a philosopher.[3] Maybe philosophy is in any case an answer and a reaction

[1] *Vermischte Gedanken über die aerostatischen Maschinen*, in: G. Chr. Lichtenberg, *Gesammelte Werke*, hrsg. v. W. Grenzmann, Bd. 2, Baden-Baden, s. a., p. 349 [transl. by W. Breidert].
[2] Andreas Speiser, *Leonhard Euler und die deutsche Philosophie*, Zürich 1934, p. 3.
[3] Otto Spiess, *Leonhard Euler*, Frauenfeld, Leipzig 1929, p. 120.

to earlier philosophy, but Euler's philosophising is in a very special way a dispute with other authors' philosophies. There is no peculiarly "Eulerian" philosophical question, no special problem worrying or torturing him, but, so to speak, an external motivation for his concern with philosophers and their false doctrines. This even applies to the "Euler-Kantian question," so named by Speiser:[4] "What can physics provoke in metaphysics?" Of course Euler is able to prove his originality by answering questions raised by other philosophers.

Euler's position in the history of philosophy is primarily characterized by three controversies:

(i) The dualism of body and soul developed by Descartes resulted in problems whose "solutions" provoked the monistic ontologies of mechanistic materialism on the one hand and of spiritualism (objective idealism) on the other hand.

(ii) In the theory of knowledge the rationalism of the Leibnizian-Wolffian school was irreconcilably opposed to the sensualism or empiricism propagated by the Lockians especially in Great Britain and the United States.

(iii) In natural philosophy the Cartesian concept of body, which defined extension to be the essential property of body, had to give way to the concept of body which was disseminated by Newton and Leibniz, including inertia or force in addition to extension. Nevertheless Newton's absolute space was opposed to the relativistic Leibnizian concept of space.

In the following I intend to describe Euler's point of view in these controversies.

In 1724 Euler wrote a paper, which is lost, dealing with the distinctions between the Cartesian and Newtonian philosophy.[5] Maybe therefore he was influenced by doctrines of these authors, e.g. Descartes' dualism and Newton's absolute space. As early as 1736 in a letter to C. L. G. Ehler and in 1738 in letters to G. B. Buelfinger (Bilfinger) he criticized Wolff's *Ontologia* and his *Cosmologia*, but at that time he did not publish that criticism. In 1741 Euler was compelled to write a letter to Wolff to dispel the rumour that he would like to belittle Wolff's prestige. But he expressed his deep respect to Wolff and simultaneously he expressed a critical attitude towards Wolffian monadism.[6]

[4] Speiser, l. c. (note 2), p. 9.

[5] Guenter Kroeber (ed.), *Leonhard Euler, Briefe an eine deutsche Prinzessin* (selection), Leipzig 1965, p. 13 ff.

[6] Leonhard Euler, *Opera omnia*, Series IV A, vol. I, Basel 1975, p. 466 (No. 2820).

In the German Enlightenment Wolff's rationalistic philosophy, with its goal of logical stringency, captured the attention of every philosophising reader. It is almost impossible to overestimate its appeal. Even Wolff's banishment from Halle during the reign of king Friedrich Wilhelm I signified nothing else than a change of venue for Wolff's activity.[7] When Friedrich II, who later appointed Euler to Berlin, called Wolff back to Halle, this come-back became a very triumphal procession, much to the dismay of the pietists.

In 1745 the *Berliner Akademie der Wissenschaften* chose its competition question asking either to expose or to refute or to demonstrate monadology; and in case of demonstration the physical laws of motion should be derived from that doctrine.[8] At that time Euler was the director of the mathematical class of the Academy, and so he was involved in judging the competition. Nevertheless he could not suppress his interest in it. Before the competition was finished he anonymously published his treatise on the problem of monads in a series of papers entitled *Gedancken von den Elementen der Cörper ...* [*Ideas on the elements of body ...*].[9] In this point Euler's behaviour was unfair. It became even more awkward when the Academy, owing to the numerous participants in the competition, extended the decision from just the philosophical class to a commission of all the classes, of which Euler was also a member. This jury could not supply an impartial award. And soon Euler was found to be the author of that anonymous treatise. His supporters, e.g. the prize winner Justi, defended Euler by the argument that it would have been necessary "to call the German scholars' attention to the competition in a peculiar way."[10] Even A. Speiser holds the opinion that Euler's treatise was nothing but the antithesis of monadology, "whose formulation he should not retain further after the prize-question was put."[11] Wolff disagreed with Euler's friends, and he tried to restrain Euler's influence with some letters to Maupertuis, who was the president of the Academy. At least some of the prize-essays, in which Leibniz was defended, were published together with Justi's prize-winning text. "So to speak the whole process was submitted to the philosophical

[7] Cf. J. Chr. Schwab, *Welches sind die wirklichen Fortschritte ..., Preisschriften ueber die Frage: Welche Fortschritte hat die Metaphysik ...*, Berlin, Reprint Darmstadt 1971, p. 19 f.
[8] E. Hoppe, K. Matter, J. J. Burckhardt, editors' introduction, in Leonhard Euler, *Opera omnia*, Series III, vol. 2,.p. XI.
[9] Leonhard Euler, *Opera omnia*, Series III, vol.2.
[10] Johann Heinrich Gottlob Justi, *Nichtigkeit aller Einwürfe und unhöflichen Anfälle ...*, Frankfurt, Leipzig 1748, p. 5.
[11] Loc. cit. [note 8].

public for revision."[12] Wolff reproached Euler for extending his recognition, which was well-deserved in mathematics, to other subjects with which he was not sufficiently acquainted.

Euler's motivation for his harsh attitude towards the contemporary philosophers became perhaps most evident in the title of his short theological pamphlet: *Rettung der Göttlichen Offenbarung gegen die Einwürfe der Freygeister* [*Rescue of the Divine Revelation from the objections of the Free-thinkers*] (1747).[13] In this treatise, among other things, the credibility of the Holy Scriptures is defended by a comparison with the credibility of science. It holds that even in science contradictions seem to exist, but no reasonable person would on this account doubt the sciences, even if it is impossible to resolve all difficulties.[14] In philosophy Euler takes up the pen only in cases where he is convinced that he has to protect the Holy Bible or the sciences against philosophers' attacks or against their false doctrines. "Going to the front against the monads, he defended his Christianity."[15] Similarly Ernst Mach spoke generally about the 18th century: "Theological questions were stimulated by everything, and they influenced everything."[16] Because Mach endeavoured to eliminate theological remarks from scientific writings, he was forced to ferret them out. In his view Euler belongs among those authors who mixed matters of "internal private life" with the subjects of science.

In ontology and theory of knowledge Euler attacks primarily three philosophical views: the Wolffian philosophy (monadists), mechanistic materialism, and idealism (spiritualism, solipsism). In his philosophical writings he often mentions Leibniz, the Wolffians, Newton and the Cartesians, whereas the names of particular idealists or materialists are not given. Though Euler's writings may give the impression that he explicitly refers to Berkeley,[17] it is open to question whether he had any acquaintance with Berkeley's philosophy. Indeed, as early as in the mid-18th century some of the compilations of philosophical positions include Berkeley as the (only) representantative of idealism. Sometimes Malebranche is named as

[12] Schwab, loc. cit. [note 7], p. 26.
[13] Leonhard Euler, *Opera omnia*, Series III, vol. 12, p. 267 ff.
[14] Ibidem, §§ 40-42.
[15] O. Spiess, loc. cit. [note 3].
[16] Ernst Mach, *Die Mechanik*, 9th ed., Leipzig 1933, Reprint Darmstadt 1963, p. 433.
[17] A. Speiser in: Leonhard Euler, Opera omnia, Series III, 11, p. XXIV; A. P. Juschkewitsch, *Euler und Lagrange über die Grundlagen der Analysis*, in: Kurt Schroeder (ed.), Sammelband der zu Ehren des 250. Geburtstages Leonhard Eulers der Deutschen Akademie der Wissenschaften zu Berlin vorgelegten Abhandlungen. Berlin 1959, p. 228; E. A. Fellmann, *Leonhard Euler*, in: Kindlers Enzyklopaedie *Die Grossen der Weltgeschichte*, Zürich 1975, p. 518.

a representative of solipsism,[18] but such remarks do not indicate a general acquaintance with Berkeley, because even Kant does not mention Berkeley in the first edition of the *Critique of Pure Reason* (1781), and later the "Refutation of Idealism" in the second edition of the "Critique of Pure Reason" does not prove any close acquaintance with Berkeley, who was best known through descriptions in the works of his opponents, namely Hume, Hamann and Beattie. Where Euler is dealing with the "English philosophers,"[19] he is not thinking of any sensualistic philosophers, but of the Newtonians. In Euler's time, the term "materialist" meant Epicurus, Hobbes, Coward, Spinoza, and Toland. Knutzen refers also to Dikaiarchos, mentioned by Cicero.

Idealism always had to struggle against the fact that at the first glance it seems to contradict the worldview of common sense. And when arguments are missing, ridicule props up quickly. In a similar way as Diogenes intended to "refute" the Eleatic philosophers, who denied the reality of motion, by walking up and down, Samuel Johnson responded to Berkeley's objective idealism by kicking a stone. Not far from such informal "arguments" we find another one maintaining that idealism cannot be reconciled with science. However Ernst Mach, a phenomenalist, was quite effective as a scientist. But there were still more arguments against idealism. Some very interesting reasoning stems from the opposite side, i.e. from rationalistic metaphysics, and it is based on Leibnizian optimism. It is to be found in Baumgarten's "Metaphysik" (§438): A world without any matter, as it was imagined by the idealists and solipsists, is no world of maximal content, therefore it is not one of the most perfect worlds, whereas God had to create the best of all possible worlds. Consequently God cannot create any world as it was imagined by the idealists. In the history of philosophy the point of that argument is the strange fact that Berkeley, according to his own account, introduced idealism in favour of theology. However, that situation shows that it was possible to attack idealism also for religious reasons. But it may be supposed that Euler was not acquainted with that argument, for nowhere did he refer to the consequence mentioned above and deduced from optimism, whereas he, too, holds our world to be the best of all possible worlds.

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[18] Martin Knutzen, *Systema causarum efficientium*, Editio altera, Lipsiae 1745, p. 72: "... cum Berckeleio ceteraque Idealistarum, Egoistarum et Pluralistarum cohorte ...". Jean Deschamps, Cours abrégé de la philosophie Wolffienne, tome II, Amsterdam, Leipzig 1747, p. 22: "... le célèbre George Berkeley ...".

[19] e.g. in: *Briefe an eine deutsche Prinzessin* (letter No. 68), in: Leonhard Euler, *Opera omnia*, Series III, 11, p. 147.

In the same way as the other authors[20] who reject idealism Euler had to concede that he was not equipped with "sufficient weapons" to beat that philosophy, though he liked to defeat it and never would accept that doctrine.[21]

Euler arranges Wolff's doctrines, listing eight points in his 76th letter to a German princess:

(i) Experience shows us all bodies perpetually changing their state;
(ii) Whatever is capable of changing the state of bodies is called force;
(iii) All bodies, therefore, are endowed with a force capable of changing their state;
(iv) Every body, therefore, is making a continual effort to change;
(v) Now, this force belongs to body, only so far as it contains matter;
(vi) It is therefore a property of matter to be continually changing its own state;
(vii) Matter is a compound of a multitude of parts, denominated the elements of matter; therefore,
(viii) As the compound can have nothing but what is founded in the nature of its elements, every elementary part must be endowed with the power of changing its own state.

These elements are simple entities, i.e. monads. Euler accepts the first two of these propositions, whereas in his view the third proposition contains some obscurity which leads to the other errors, for the force which changes the state of a body is always due to *another* body.

Euler uses this compressed exposition to emphasize the weakness of his opponents. The rationalistic metaphysicians were of the opinion that total division of a thing – Euler considered it to be impossible – would necessarily produce indivisible monads, which would even be unextended. That opinion was not considered to be an empirical result, rather it seemed to be urged by pure reason. It was held that a monad had no extension, but that its intension was more than nothing, and that it owns some states (actions, perceptions). In contrast Euler denies that there could exist anything without extension "in the world," and moreover that an extended thing (body) could thereby be constituted. In a letter to Goldbach (23 June/4 July 1747) and in the *Letters to a German Princess* (No. 123 ff.) Euler remarks that he considers infinite divisibility to be inconsistent with the idea of total division. Since he refuses to accept the distinction between substances and appearances based on them, which was an important point in monadology, he is dogmatically at cross-purposes with the dogmatists by his taking the concept of divisibility to be entailed in the concept of extension and

[20] e.g. Diderot, Beattie, Lenin.
[21] Letter [note 19] No. 97.

by his declaring the unextended to be nothing. Consequently Euler thinks that the spontaneous change of state inherent in the monads contradicts the law of inertia, for he cannot recognize that the law of inertia is valid only with regard to bodies, i.e. to *complexes* of monads, but never for a single monad. As a mathematician he should have been aware of the fact that qualities of a complex of monads may differ from the qualities of its elements, and he did know that the Wolffians never doubted the law of inertia, but on the contrary they considered it to be the most important law of nature.[22]

Euler feels provoked to poke fun at the idea of pre-established harmony: If the harmony between his body and his soul ended, he could accept any other body, even one of, say, a rhinoceros.[23] He contends that such a result would not be allowed by Leibnizian optimism. Once again, though a vagueness of the Leibnizian-Wolffian philosophy is revealed, for it is shown that, according to monadology, the body is superfluous. Of course, Euler is not very interested in this consequence, because he adheres to the reality of the bodies and to the mutual susceptibility of body and mind. One of his reasons for this conviction is made obvious in the letters No. 92 and No. 93: the nobility of mind. By the same passages it is plain to what extent he considers the nobility of mind, which consists primarily in freedom, to be threatened by the Wolffians. In his view it seemed to be unbelievable that his soul was nothing but an entity similar to the last particles of a body. This criticism is remarkable, because it is founded on interpreting his opponents in a materialistic way, whereas in the forgoing letter No. 76 they had been reproached with spiritualizing the body.

Leibniz and the Wolffians taught that the monads "have no windows," and that by the creation of monads all ideas within them were determined. The concept of pre-established harmony was invented to guarantee the concordance between these processes, but it is not needed, if the idea of missing windows is not consistently sustained. To rationalists it seemed unacceptable to make the soul dependent on the body by submitting the mind to a natural influence (*influxus*) of the body,[24] whereas they taught strict determinism with regard to the mental aspect of our actions. This determinism seemed to corrupt responsibility and morality. It is reported that the Prussian king's concern about the demoralizing effect of determinism was one of the reasons for banishing Wolff.[25] Euler seems to have been

[22] Cf. e.g. Deschamps, loc. cit. [note 18], tome I, 1743, pp. 274, 315-317.
[23] Letter [note 19] No. 83.
[24] Christian Wolff, *Vernuenfftige Gedancken von Gott, der Welt und der Seele des Menschen*, Preface to 2nd ed., quoted from the 8th ed., Halle 1741.
[25] Letter [note 19] No. 84.

moved by the same concern. Probably it was one of the reasons for him to refuse mechanistic materialism, which on the one hand seemed to threaten the existence of God, who was thought to be nothing but mind, and which on the other hand could corrupt the freedom of an acting human being. The postulates of practical reason make materialism unacceptable to Euler. Afterwards Kant, in the dialectic of *Critique of Pure Reason*, endeavoured to prove that it is possible to accept the totally determined causality of nature together with freedom (in the realms of *Noumena*).[26] Not only in the 18th century was Euler's dictum valid:[27] "The chapter on freedom is a stumbling-block in philosophy." He struggled for his dualism mainly against mixing the areas of self-determination (freedom of thinking) and external determination (necessitation by external forces). "I am the ruler of my thoughts."

Defending psychoanalysis Sigmund Freud declared that the resistance to new doctrines originates from general human narcissism, for an effect of psychoanalysis is that man is no longer "the ruler in the house" (of his soul).

Euler expresses this narcissism with very similar words. Postulating freedom being an essential quality of mind – just as postulating extension, inertia, and impenetrability being essential qualities of body – he was convinced of escaping from the problem of theodicy, for he thought that God himself had withdrawn the souls from his omnipotence, which is proved by the fact that they are able to sin. Obviously Euler does not recognize that the problem of theodicy consists in the incompatibility of omnipotence and sin. He considers the freedom of mind and consequently its responsibility with respect to sin to be indispensable, but he restricts the omnipotence of God.

Against the deterministic objections from theology and philosophy Euler defends the freedom of man by referring to the human ability of immediate sensation. He writes about a hypothetical journey to Magdeburg: "I feel it well enough that I am not forced to take it, and it is always under my control to take it or to stay at Berlin. But a pushed body adheres to a certain force necessarily, and you cannot say that it neither does nor that it does not enjoin that obedience."[28] "The mind is the ruler (of action)." Euler maintains that a person may err with respect to the question of whether another person is free or not free, but it would be impossible to

[26] Immanuel Kant, *Critique of Pure Reason*, A 558/B 586.
[27] Letter [note 19] No. 84.
[28] Letter [note 19] No. 85: "Ich fühle aber sehr wohl, dass ich nicht dazu gezwungen bin, und ich beherrsche es immer, diese Reise zu unternehmen oder in Berlin zu bleiben. Ein gestossener Körper folgt einer gewissen Kraft aber mit Notwendigkeit, und man kann nicht sagen, dass er diesen Gehorsam gebietet oder auch nicht."

have any doubt about one's own freedom. "He who feels free is free, indeed." An engine which considered itself being free would have some feeling and therefore it would have a soul, necessarily entailing freedom. God's foresight would not be contradictory to freedom, for, it is said, an action is not caused by foresight, but, on the contrary, the action is foreseen because it happens. The human mind is able to begin a series of events, but the series itself is an effect of God's order of creation. Thus God remains, strictly speaking, the ruler of all events in our world. Obviously Euler did not realize that he did not resolve but only presented the problem. He joins in the chorus adoring the creator's infinite perfection, whose work infinitely exceeds our understanding. Concerning the question about the origin of evil in the world Euler argues rationally in some cases, but in the end he resorts to the idea of an inconceivable mystery which goes beyond our intelligence but makes it possible for God's grace and creating power to be compatible with evil and sin in the world.[29] Indeed, freedom is restricted with regard to the realisation of action, but Euler holds that freedom of will is an essential quality of mind, which cannot be removed even by God: "Man remains at all time the ruler of volition."[30] Euler claims to be a defender of God's omnipotence and of human freedom. The mind is able to have an effect on a body, otherwise even God could not do that, and such a situation would encourage atheism.

Euler turns away from the monistic philosophy in his time and from monadology, and he returns to the Cartesian dualism of body and soul, but he does not accept the Cartesian concept of body. He refuses also the opinion that animals are mere automatic machines, and he reproaches the Wolffians for considering human beings in the same way. In letter No. 81 Euler explains "There is a special place in the brain where all the nerves come to an end, and just there the soul is seated or there it feels all its impressions which are effected on it by the senses." In letter No. 92 he says that an hour is not bound to a place. Then he continues: "Similarly I may say that my soul is neither in my head nor outside my head nor elsewhere, ... Therefore my soul does not exist at any place, but it operates at a certain place ..." In letter No. 83 he tries to ridicule pre-established harmony by the fiction of a connection between his soul and the body of a rhinoceros in Africa, but in letter No. 93 he argues for a stationary soul, and he thinks it is possible that immediately after death God could connect his soul with a body on the moon.

Since Euler returns to dualism, to him those problems that provoked the monistic philosophy return, along with the system of pre-established

[29] Letter [note 19] No. 89.
[30] Letter [note 19] No. 91.

harmony for either resolving them or avoiding them. He is well acquainted with these difficulties, and he refers to them in the letters again and again: If body and mind are based on two completely distinct substances, how could it be possible that the perceiving soul is able to "assume" something of the material world? Indeed, sensible perception needs the fulfilment of some conditions of the body, but nevertheless the picture on the retina is not yet the object of the seeing soul. The problem of the provenance of sensation, which is a hard problem in every theory of knowledge, especially in the Kantian theory, is treated in letter No. 82. Euler compares the soul with a man sitting in a dark room and seeing the things outside the room via a *camera obscura*. Similarly the soul is considering the ends of the nerves and receiving the impressions of the sense organs. "Though it is absolutely unknown to us what the similarity is between the impressions on the ends of nerves and the objects causing those impressions, these impressions are appropriate to deliver to us a very adequate idea of the objects." Descartes had at least endeavoured to argue for the adequacy of that idea, but in Euler's writings it is merely claimed. Obviously he does not advance to the Kantian question of whether there could be *any* similarity at all. For Euler it is certain in virtue of God's omnipotence that a connection between body and mind can exist and, indeed, does exist. The second principal question in the theory of knowledge is "*How* are they connected?" This question was treated by the atomists of antiquity and it is still discussed by modern brain scientists. Euler's answer is succinct: It is a great mystery (*grand mystère*)![31] Euler is famous for his clearness and distinctness in the foundation of mathematics and sciences, about whose fundamental explanations Schopenhauer says "... if you hear or read them, it is as if you exchange a bad telescope for a good one."[32] Yet here he, of all scholars, relies on "mystery" concerning the central point of the theory of knowledge!

At least Lichtenberg was not convinced by Euler's words, for he wrote with a critical view to Euler's *Letters*: "It seems to me that the concept of 'being' is something taken on credit of our thinking and that there will be nothing left any more if there are no thinking creatures. Though it sounds simple and though I would be laughed at for saying it publicly, nevertheless I think the ability to suppose such an idea is one of the greatest advantages of the human mind and properly speaking one of its strangest dispositions."[33]

[31] *Letters* [note 19] No. 80 and No. 97.
[32] *Die Welt als Wille und Vorstellung*, B. II, chap. 15.
[33] Loc. cit. [note 1], vol. I, p. 433 f.

In all honesty Euler does not endeavour to suppress the gap between body and mind, on the contrary he deliberately displays it, whereof Kant speaks highly.[34] By separating the areas some of the scholastic questions disappear, e.g. the question about the place and the hour of the Last Judgement, and whether an angel could occupy different places at the same time, or when God created the world.[35] Questions of such kind are based on an improper boundary crossing, and therefore they are absurd.

In addition to the reasons which Euler took from the theory of knowledge and from ethics for his aversion to the Wolffians he took some others from the philosophy of science. In the *Reflexions sur l'espace et le tems*[36] he starts by saying that the principles of mechanics were established so firmly that any philosophy of nature must be founded on them. Thus the relationship of service between physics and philosophy is reversed. Furthermore Euler recognizes the special position which is occupied by space and time in the system of the principal concepts within the theory of knowledge: We get the idea of space neither by sensation nor by abstraction, since a place in space is preserved even if we remove the body completely. We get the ideas of space and place only by virtue of (transcendental) reflexion.[37] Thus Euler reached one point of Kant's transcendental aesthetics, i.e. the empirical reality of space as one of the conditions of possible experience, but he did not arrive at the other point, i.e. transcendental ideality.

In the *Letters to a German Princess* Euler endeavoured to reduce the law of inertia to the theorem of sufficient reason, supposing the hypothesis that only one body exists and that this body is at rest. For that body there would be no reason to move in one direction rather than in another one,

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[34] De mundi sensibilis atque intelligibilis forma et principiis (1770), §§27 and 30, note.- Edmund Hoppe (*Die Philosophie Leonhard Eulers*, Gotha 1904, p. 166) wrote that Euler's name was not mentioned in Kant's works, but nowadays we know (primarily from H. E. Timerding, *Kant und Euler*, Kant-Studien 23, 1919, p. 18-64) more about the relationship between Kant and Euler, especially we know Kant's explicit and implicit references to Euler, which all of them are assenting. Meanwhile the covering letter from Kant to Euler attached to the book entitled *Die wahre Schaetzung der lebendigen Kraefte* is available (in Kroeber's edition of *Briefe an eine deutsche Prinzessin* [cf. note 5]). An old but very detailed description of Euler's philosophy is given in: Ernst Cassirer, *Das Erkenntnisproblem*, 3rd edition 1922, Reprint Darmstadt 1971, vol. II, p. 472-485 and 501-505.

[35] Indeed such questions were discussed, e. g. Albertus Magnus, *Ausgewählte Texte*, ed. A. Fries, Darmstadt 1981, p. 36.

[36] Written 1748, published 1750, in: Leonhard Euler, *Opera omnia*, Series III, vol. 2.

[37] Euler holds that a person denying the fact that we are able to get ideas only in virtue of "reflexion" would deceive (*tromperoit*) oneself. So Kant says the Leibnizian concept of space comes from a "deception of transcentental reflexion" (Critique of Pure Reason, A 275 / B 331; cf. also A 26 / B 42).

therefore it will be at rest until other bodies affect it. [38] This argument is not only founded on the theorem of sufficient reason but also, implicitly, on the reality of absolute space, since only on this supposition it is possible to consider one separated body as a resting body. [39] It seems Euler does not recognize that it might be possible to prove in the same way that all things exist everywhere, or nothing exists anywhere, because there is no reason for its existing (or non-existing) here rather than there.

In any case a validation or a judgement depends on various criteria. Therefore we will get different images if we consider Euler's work as a national achievement [40] or as a preparation for Kant, [41] whether we understand him as an important stimulator and innovator [42] or as a conservative thinker adhering to old things, [43] whether we consider him as someone provoking philosophical discussions [44] or as an apologist for Christianity. [45] Certainly it would be a mistake to try to characterize him by any single one of these aspects, but we should avoid a distorted picture, e. g. depict Euler as being a 'materialist'. In spite of Euler's realism, nowhere is there recognizable "a really materialistic answer, given by Euler, to the question about the relationship between body and mind, between matter and consciousness" [46] if one reads in his letter No. 80 "that the minds are the primary part of the world, and that the bodies are introduced to it merely to serve the minds". Euler not only had a mind, but he was also one of the most important spirits in the Eighteenth Century.

[38] *Letters* [note 19] No. 71 and No. 72.

[39] That thought is similar to another one uttered by George Berkeley in *De Motu* (§58) to prove the relativity of motion. In this case, it is true, he supposes the existence of a single body, but he does not suppose at the same time that it is at rest.

[40] Spiess, loc. cit. [note 3], p. 7 f.

[41] Timerding, loc. cit. [note 34], p. 18 f.

[42] Fellmann, loc. cit. [note 17], p. 519.

[43] A. Schopenhauer, *Die Welt als Wille und Vorstellung*, vol. I, B. 2, §25.

[44] Fellmann, loc. cit. [note 17], p. 519.

[45] Spiess, loc. cit. [note 3], p. 120.

[46] Kroeber, loc. cit. [note 5].

Leonhard Euler: Life, Work and Legacy
Robert E. Bradley and C. Edward Sandifer (Editors)

Images of Euler

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1. Introduction

Portraits have hundreds of uses and many facets. Portraits provide information on both the subject of the portrait and the artist. The portraits of Leonhard Euler (1707-1783) described herein are portraits made during his lifetime that are known to exist today. Only two of these original works were made while Euler sat for the artist. The first of these two was made in 1753 by an artist in the early years of his prolific career in portrait painting. This portrait has two copies made by the artist in 1756. In the second of the originals, made in 1778, another artist used a technique that could insure a true recording of Euler's face, and thus, we have the benefit today of knowing in as much precision as was available at the time, the likeness of Euler. In addition, this second portrait was subsequently copied by two copper engravers during Euler's lifetime (one of these copies of this portrait is included here in figure 7). The reproductive prints of the original portraits could be used then and now in printed material and, in particular, were used as covers of journals. The other portrait of Euler included herein, made in 1737, is an example of a work for which Euler did not sit for the artist.

A portrait of a famous relative of Euler's is also included to inform the reader about Euler's family. His father-in-law was a portrait painter who would certainly have made portraits of his own family, but these are not known now. The portrait of his wife's grandmother shows the style the artist might have used to portray Euler. Two other images of Euler, perhaps made in his lifetime, are not included. One is a 1768 (?) engraving of the 1756 portrait and one of a medal made in 1781 (?).

The author's choice of including only portraits of Euler made during his lifetime is to assure that the likenesses are not only an 18th century way of looking at people and objects, but also that they help in clarifying the social attitudes in that century. The artists who created works after 1783 would probably have derived their concepts of Euler from the images made during his lifetime as well as from information gained from those who knew Euler, but they would have added their own time periods' attitude toward a person of great fame. Many of these later works were commissions intended to venerate Euler as well as remember him. Posthumous works include oil paintings, drawings, medals, plaques, busts, and engravings. The engravings were collected by David Eugene Smith and a set can be seen in the Rare Book Room at Columbia University.

2. Maria Sibylla Merian

Euler wrote that his mother-in-law, Dorothea Maria (Graff) Merian Hendricks Gsell (1678-1743/5), painted butterflies and flowers. The significance of this sentence can be understood by noting that Dorothea Maria was the first woman commissioned by the Academy of Sciences in Petersburg founded by Peter the Great in 1724. Having learned to paint while traveling in the Dutch colony of Surinam with her mother from 1699 to1701, Dorothea's role in the czar's cabinet of curiosities was to give talks and arrange exhibits as well as paint butterflies, flowers and birds. She came to this position following the specific invitation of Peter that she and her husband, Georg Gsell (1673-1740), move to Petersburg the autumn after her mother died in 1717. Peter had been in Amsterdam at that time to buy art and other objects for his collection and in so doing utilized the skills of the Swiss painter and art dealer, Gsell. In the absence of a picture of Euler's mother-in-law, an engraving of Dorothea Maria's mother, Maria Sibylla Merian, executed no later than 1717 by Jacob Houbraken [1] , is shown here. This reproductive print is based on a drawing by Gsell, sometime after 1704, when he moved to Amsterdam.

Maria Sibylla Merian, (1647-1717), the grandmother of Euler's second wife, was an artist/scientist/business woman/publisher of considerable fame from the 17th century until today. She lived in Amsterdam after spend-

[1] From the collection of the Kunstmuseum in Basel.

Fig. 1. Maria Sibylla Merian, (1647-1717), Euler's wife's grandmother

ing two years studying and recording wildlife in Surinam. Georg and his two daughters from an earlier marriage boarded in the Merian household. He became the (second) husband of Merian's younger daughter Dorothea Maria (around 1716). After the move to Petersburg, Georg was appointed court painter and first curator of the Imperial Art Gallery founded in 1720. As a painter of the Academy of Science, he made drawings for the scientists there. Both he and Dorothea Maria gave art lessons. Portraits he executed now hang in the Hermitage and the Peter and Paul Cathedral in Petersburg.

Georg's portrait of his son-in-law, Isaäc le Long (1683-1762), was also engraved by Houbraken, so it can be easily concluded that he made drawings and perhaps paintings of his wife and his daughter, Catherine Gsell Euler (?1707-1773), whom Euler married in 1733, as well as of Euler himself. However none of these is known at this time.

3. Sokolov's mezzotint of Euler

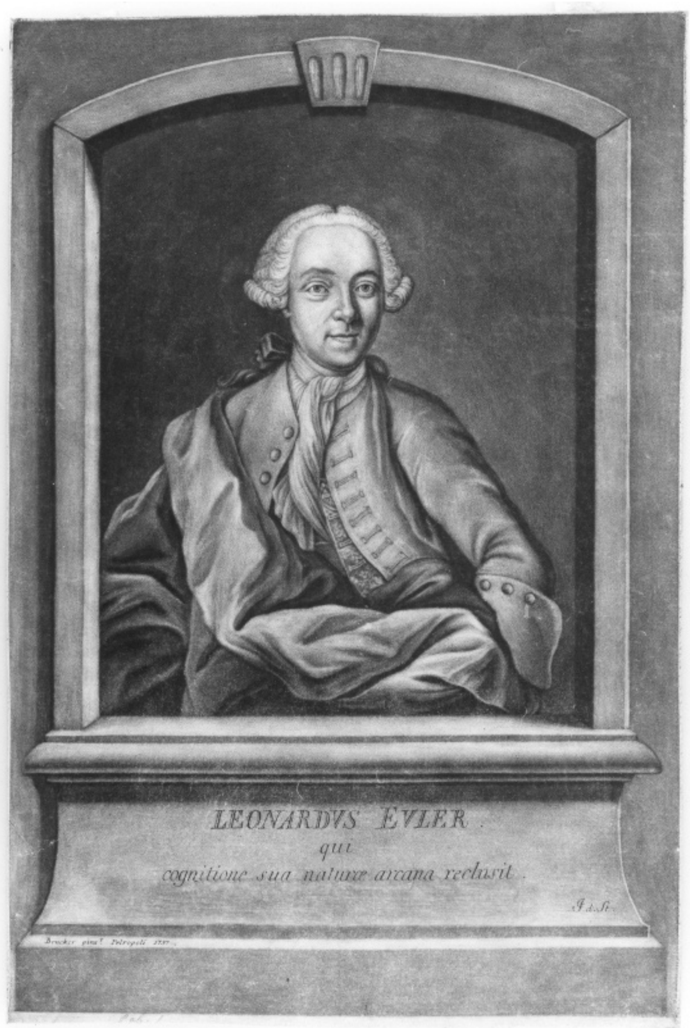

Fig. 2. Sokolov's 1737 mezzotint

Euler, dressed warmly in a bulky wrap over a jacket and an embroidered vest, is presented full-face in this mezzotint made by Vasilij Sokolov in

1737.[2] It is unclear exactly how old Euler was at the time he sat for the portrait in Petersburg as the original 1737 painting by Johann Georg Brucker is apparently lost. Brucker lived in Petersburg between 1733 and 1737. However, the mezzotint method of reproduction developed in the late 17th century faithfully reproduces both the tone and texture in a picture, even though in black and white, assuring that the print is very similar to the original oil, unlike other methods such as engraving.

Euler looks out from the picture apparently seeing with both eyes. The blindness that removed the sight in his right eye is reported to have taken place over time but the reason for it is not entirely clear. In his eulogy, Nicolas Fuss, Euler's assistant, reported that in 1735 Euler had a high fever and was near death. He states that the fever caused an abscess and Euler lost the sight of his right eye. Since this mezzotint is two years after the fever the viewer is being presented with an image that is incorrect since both eyes actually look the same. The artist apparently copied the left eye to draw the right eye, but did not copy the area around the eye.

Sometimes in portraits there is sufficient distinctive information to suggest a cause and an educated guess can be made. By examining the picture and noticing the heavy dark circle under the right eye and the heavy fold above the eye where the membrane keeps the fat back a trained ophthalmologist can offer a 21st century diagnosis even without knowing Euler's life history. The abscess, perhaps from a stye, would lead to orbital cellulitis, secondary to the infection. This would lead to pan-uveitis with retinal detachment and glaucoma, hence to phthisis and enophthalmus. In layman's words, the eye would get smaller and smaller and shrink like a raisin. As shown in later portraits this shrinkage did occur. However, the fact that Euler lived through this episode meant he was very lucky.

4. Handmann's Pastel Painting of 1753

This famous portrait of Euler by the Swiss painter Jakob Emanuel Handmann[3] (1718-1781) looks today as it did the day it was created in 1753[4]. The brightness of the color will remain throughout the life of the painting because the medium the artist used, pastel, does not fade or soil. Pastel

[2] This reproduction is courtesy of Sergey Androsov, Senior Curator, Department of Western European Art, The State Hermitage Museum. St. Petersburg, Russia.

[3] Information on the relation between Euler and Handmann and Handmann's paintings is from Thomas Freivogel, *Thomas Handmann, 1718-1781: ein Basler Porträtist im Bern den ausgehenden Rokoko*, Murten: Licorne-Verlag, 2002.

[4] Kunstmuseum, Basel, Switzerland.

Fig. 3. Handmann's Pastel Painting of 1753

is pure pigment, without liquid binder, so it does not crack or blister over time. However it is fragile because it is chalk and must be covered with glass so it can not be brushed off. Pastel painting began to be practiced around 1720. The technique is to stroke hard sticks of dry pigment across an abrasive paper, embedding the color in the surface. Here, Handmann handles the sticks delicately in the style of the studios where he had studied earlier in Paris.

Euler posed with his chest toward the front but with his head turned over his right shoulder so that his blind right eye is toward the wall in the picture. Euler's right eye has shrunken, and in later portraits the shrinkage is more obvious. He wears an ultramarine blue silk robe with black stripes and buttons, and wound around his head a loose blue and white silk cloth (as was the custom since he is not wearing a wig and his head is shaved).

Handmann, an accomplished painter of landscapes and mythological figures in the rococo style, became a prolific portrait painter of his country's citizens with more than 500 works completed in 36 years. These careful and realistic portraits of so many prominent citizens have aided scientists in determining the diseases that were common in Switzerland in the 18th century. After settling in his native land following study in France and Italy, Handmann made few foreign journeys. It was on his one journey to Berlin that this picture of Euler was made, one of eleven portraits he painted that year.

5. Handmann's oil painting of 1756

Fig. 4. Handmann's oil painting of 1756

In this picture Euler is portrayed wearing a bag wig tied with a black ribbon. This style was customary for fashionable men from the 1660s to the later 18th century for those who could afford this expensive item of personal grooming. Wigs required considerable upkeep as well: a barber to shave the head and to powder the wig, which was replaced or restyled every year.

Euler sits comfortably in a chair with his left forearm leaning on a desk holding a book. He wears a housecoat with a flaring collar over a crisp white jabot. His outstretched middle finger and index finger of his right hand point to mathematical computations in the open book. His folded little finger rests on the opposite page where there are mathematical diagrams.

In 1756, using the pastel painting as a model, Handmann made the oil painting shown[5] in figure 4. In 1756 (?), he painted another larger three-quarter portrait (figure 5), measuring 142 by 108 cm. Euler did not see Handmann after the artist's trip to Berlin in 1753, for Handmann did not leave Switzerland again and Euler did not return to his native land.

Euler's head in all three portraits is the same, showing a three-quarter view of the face with the quarter side turned to the right and the chest toward the surface of the picture. His eye that had become blind earlier is on the side toward the wall. The two oil portraits show neither any changes in Euler's countenance over the three or four years since the pastel was made nor any further shrinkage of the right eye that would have naturally occurred. The portraits are different, with a desk and books added to the oils as well as the inclusion of the sitter's hands. Thomas Freivogel, Hand-

[5] Museum an der Augustinergasse, Basel.

mann's biographer, has concluded that Handmann charged double price for painting hands in a portrait.

6. Handmann's large oil painting of 1756 (?)

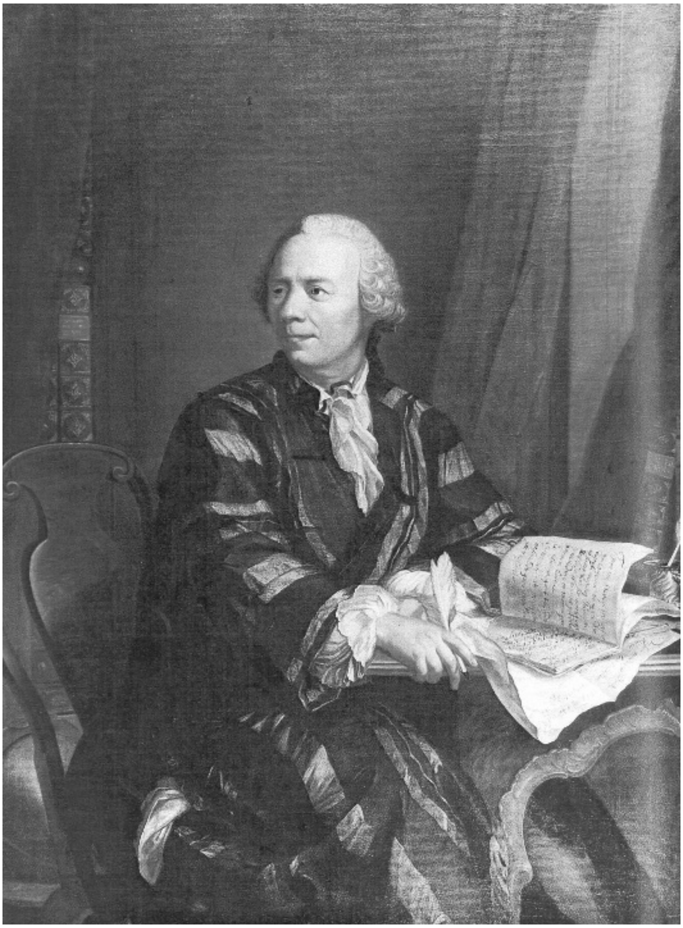

Fig. 5. Handmann's large oil painting of 1756 (?)

In this large three-quarter portrait, Euler sits erect on the front of a chair dressed in a long silk robe with wide blue-black stripes over white culottes and a white lacey jabot in which the collar is connected with a black tie.[6] His head is in a familiar pose with right eye to the wall, similar to that in Emmanuel Handmann's other 1756 portrait because the artist used his earlier pastel portrait of Euler to make this painting.

In the background on the left is an open bookshelf; there are olive-green draperies to the right and left. A globe is on the floor on the left behind the right side of the chair. Euler's right hand folds loosely over the front edge of a carved gilded writing table, holding a feather quill pen between his middle and index finger and apparently keeping a large paper spread under an open book from falling off the surface. The mathematical writing in the notebook is sufficiently large to see that it is in Latin. Euler's left forearm leans on the surface of the table which holds a silver ink well with another pen, two standing books, and a golden compass.

In this same year Handmann painted a half portrait of Euler's eldest son, Johann Albrecht Euler, age 22.[7] This was also copied from a pastel of 1753, but that picture is not known. Johann is dressed in a similar silk robe as Euler wears, but with the wide and narrow stripes being of green-blue, light brown and reddish. He wears a matching house cap, not a wig. In the background are a bookcase and the same draperies as in the portrait of Euler. Johann's left forearm leans on the desk next to an open snuff box. He holds a book open leafing with his right hand. At the top of the page the viewer can read LIII of DESSERTA[TIONES] DE NUTRITIONE [S.] 283. This book is by Johann Bernoulli, Euler's teacher.

7. Darbes' Painting of 1778

This often copied oil portrait of Euler with his left shoulder toward the viewer was painted in 1778 by Joseph Friedrich August Darbes (or d'Arbes) (1747-1810), a so-called Danish artist (born in Germany), who traveled widely painting portraits in oil, then pastel, and finally silver-point.[8] An oval frame around a bust without hands is the common format in his finished work. Here, Euler, pictured at age 71, no longer wears the formal clothes shown in the earlier portraits. His large hat and the fur collar on the coat were intended to keep the sitter warm.

The portrait records Euler's face and general appearance accurately. The usual portraiture technique of this artist was one of frankness. Wisps of gray

[6] Deutsches Museum, Munich.
[7] Private Collection.
[8] There are two portraits by Darbes, one at the Tretyakov Gallery in Moscow and the other at the Museum of Art and History in Geneva.

Fig. 6. Darbes' Painting of 1778

hair straggle down the side of Euler's neck. His forehead is creased and his cheeks and jowls are pulled down by gravity. There is deep thoughtfulness in his countenance. In the familiar pose with his blind right eye to the wall, Euler's left eye is shown but it is not a seeing eye. That eye was operated on for a cataract in 1772 but the follow-up treatment was not successful in keeping his vision. The viewer can be quite certain this is what Euler looked like in 1778.

Darbes studied in Copenhagen where he lived as a young boy and then

Fig. 7. Kütner's 1780 print, based on Darbes' 1778 painting, which Kütner himself commissioned.

moved to St. Petersburg in 1768, studying with Vigilius Erichser whose sharp focused style strongly influenced him. Before returning to St. Petersburg in 1773, he traveled in Germany, France and Poland. His earliest known work is dated 1774, a three-quarter portrait of Katherine II (Schloss Fredensborg). He continued to paint many portraits of royalty until his style went out of fashion in about 1790.

It was reported by J. G. Schadow in 1849 that Darbes, known for his friendly and jovial disposition, used a glass pane when painting in order to achieve the great likeness of his models. If he used this technique, he would be following the perspective method of Leonardo da Vinci where the artist's head does not move as he paints on a pane between his eye and the sitter.

8. Further Study

Since the attribution by G. B. Andreeva that the Moscow portrait was of Euler and was signed by Darbes was only made in 1984, those who are interested in how Euler actually looked can expect more portraits to appear in the future. Although numerous portraits were reported by Gustav Eneström in 1906 in *Bibliotheca Mathematica*, the majority of those listed were not made in Euler's lifetime. The changing styles of the various times of those made posthumously give a distinctively different impression of his profile and full face. A different model may have been used as the reproductive prints do not appear to have been widespread. These too can be studied to determine the changing perspective on a famous mathematician of both the artists and those who commissioned the artists. However, the reproductive prints by the excellent copper engravers Samuel Gottleib Kütner[9] (1747-1828) in 1780 and C. Darchow (active from 1782 to 1796) in 1782 of the Darbes portrait can be considered timely likenesses of Euler.

[9] The reproduction of Kütner's print is courtesy of Jennifer Leer, Rare Book Librarian, Columbia University.

Euler and Applications of Analytical Mathematics to Astronomy

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Introduction

As is well-known, the treatment of astronomy in Newton's *Principia* is, for the most part, geometrical in character. On occasion Newton has recourse to results derived not geometrically but by what we would call 'the calculus' - without, however, explaining his procedures. Cases of such results imported without explication occur, for instance, in his treatment of the 'Variation,'[1] and in the integration in Corollary I of Proposition 40 of Book I. Newton no doubt believed that the geometrical presentation was appropriate for making his main ideas accessible to likely readers among his contemporaries. But this preference for geometrical formulations did not arise solely from a concern for his readers.

In the 1660s Newton had engaged passionately in algebraic explorations. By 1675 this early interest had cooled; in John Collins' phrase, "he and Dr. Barrow &c beginning to thinke math$^{\text{all}}$ Speculations at least [last?] nice and dry, if not somewhat barren ..."[2] Later in the decade, attacking Descartes' treatment of solid loci, Newton wrote: "their [the Ancients']

[1] See my "Newton on the Moon's Variation and Apsidal Motion ...," in *Isaac Newton's Natural Philosophy*, eds. Jed Z. Buchwald and I. Bernard Cohen. Cambridge, MA: The MIT Press, 2001, p.139ff.
[2] *Correspondence of Isaac Newton*, 1:356.

method is more elegant by far than the Cartesian one. For he [Descartes] achieved the result by an algebraic calculus which, when transposed into words (following the practice of the Ancients in their writings) would prove to be so tedious and entangled as to provoke nausea, nor might it be understood."[3] In his *Geometria Curvilinea* (written ca.1680), Newton allows that analysis may be appropriate to some problems, synthesis or geometry to others, but his strictures against excessive algebraic complication remain strong: "Men of recent times, eager to add to the discoveries of the Ancients, have united the arithmetic of variables with geometry. Benefiting from that, progress has been broad and far-reaching if your eye is on the profuseness of output, but the advance is less of a blessing if you look at the complexity of the conclusions. For these computations, progressing by means of arithmetical operations alone, very often express in an intolerably roundabout way quantities which in geometry were designated by the drawing of a single line."[4] Not always or automatically, Newton is saying here, does a symbolic analysis provide the most appropriate means of gaining or conveying insight.

Amongst mathematicians on the Continent, in contrast, the idea of deducing the consequences of Newton's inverse-square law by means of the Leibnizian calculus would become an intensely pursued project. It was Leibniz's claim to have constituted out of the new infinitesimal calculus "an algorithm, whereby the imagination would be freed from the perpetual attention to figures."[5] Johann Bernoulli and his pupil Leonhard Euler would be foremost among those extending and clarifying the potentialities of the Leibnizian calculus. But already during the first decade of the 18$^{\text{th}}$ century the mathematician Pierre Varignon (1654-1722) had derived a number of theorems about central forces, some of them recognizably identical with Newton's results. For instance, assuming a constant force f acting in the direction of motion, he found that $\int f dx = \int\limits_0 v dv = \frac{1}{2}v^2$, where dx is the element of distance and v the final velocity attained when the initial velocity is zero; this is a case of Newton's proposition I.39. But when Leibniz asked Varignon to articulate the three-body problem (which Newton treats in proposition I.66) in Leibnizian style, Varignon succeeded only for the case in which the perturbing body is fixed in position. What he lacked – probably without recognizing the fact – was a developed calculus of the trigonometric functions, considered as functions of angles which could in turn be functions of time. The rules of this calculus – for instance

[3] *Mathematical Papers of Isaac Newton* 4:277.
[4] *Mathematical Papers of Isaac Newton*, 4:421.
[5] G.W. Leibniz, *Mathematische Schrifte*, ed. C. I. Gerhardt (Hildesheim: Olms, 1971), 5:393.

$$\frac{d}{dx}\sin x = \cos x, \qquad \int \cos x\, dx = \sin x,$$

– had been known to Newton. But nowhere did Newton set them out explicitly. Roger Cotes expressed some of them in a work that was published posthumously (*Harmonia mensurarum*, 1722), but his book was not widely known.

As Victor J. Katz has shown,[6] it was Euler who, among the mathematicians on the Continent, first became fluent in the calculus of trigonometric functions; and first laid out the rules of this calculus systematically. He did this in papers appearing from 1739 onwards. In 1736 he had completed his *Mechanica*, a work devoted to the algebraic formulation and solution of problems in Newtonian mechanics, but in that work the problem of planetary motion and the three-body problem are not broached. His appreciation of the importance of trigonometric functions came a few years later, when he undertook to solve certain differential equations, in particular linear differential equations with constant coefficients. The fact, obvious in hindsight, that the calculus of trigonometric functions is a key to the understanding of periodic phenomena, including the motions of planets and satellites, seems not to have been obvious to Continental mathematicians before Euler made it so. Euler was the first to embark on the formulation and solution of the perturbational problem - the key problem that had to be formulated and solved if Newton's inverse-square law was to be securely established as the basis for planetary and lunar theory.

1. Euler's first lunar tables, 1746

With the calculus of trigonometric functions in hand, Euler set about deriving the inequalities of the Moon; and from the inequalities so derived he constructed a set of lunar tables. These were published in his *Opuscula varii argumenti* in 1746. In the preface to the *Opuscula* he speaks of these tables as follows:

> I come now to my lunar tables. Their nature and basis of construction would require too much space to explain. I therefore only point out that they are derived from the theory of attraction which Newton with such happy success introduced into astronomy. Although it is claimed of several lunar tables that they are based on this theory, I dare to assert that the calculations to which this theory leads are so intricate, that such tables must be considered to differ greatly from the theory. Nor do I claim

[6] Victor J. Katz, "The Calculus of the Trigonometric Functions," *Historia Mathematica*, **14** (1987), 311-324.

> that I have included in these tables all the inequalities of motion which the theory implies. But I give all those equations which are detectable in observations and are above 1/2 arc-minute. For I was able to carry the calculation to the point of identifying the arguments of the individual inequalities, and I determined the true quantity of many of the equations by the theory alone, but some I was forced to determine by observations.

We will return later to the correction of theoretically derived terms by comparison with observation – a practice which stems from Euler, and became a standard procedure in the lunar theory. The trust in Newton's inverse-square law that Euler expresses here – and also in a number of other memoirs dating from before 1747 – was to crumble in the latter year.

2. Mutual perturbations of Jupiter and Saturn, 1748

Euler's first attempt to cope with *planetary* perturbations occurred in response to the Paris Academy's prize contest for 1748. The prize was offered for "a theory of Jupiter and Saturn explicating the inequalities that these planets appear to cause in each other's motions, especially about the time of their conjunction." Newton in his *Principia* had written of "a perturbation of the orbit of Saturn in every conjunction of this planet so sensible that astronomers have been at a loss concerning it." [7]

This formulation can mislead: the astronomers were not finding Saturn to be deviating from Keplerian theory more markedly at the time of conjunction than at other times. To be sure, in the time just before, during, and after conjunction, Saturn and Jupiter exchange much more energy than at other times. Prior to conjunction, Jupiter's gravitational attraction decelerates Saturn, subtracting energy from that planet's forward motion, causing it to fall into a lower orbit with – perhaps surprisingly to the uninitiated – a more rapid mean motion. After conjunction, the effect is just the opposite. If the effect before conjunction were just equal to the effect after conjunction, the net effect of the interactions before and after conjunction would be nil. But it is not quite so, because the two orbits are not concentric. If the conjunction occurs in a part of the zodiac where the two orbits are coming closer together, the effect after conjunction (because the two planets are then closer together) is greater than the effect before conjunction, hence the net effect is that Saturn rises into a higher orbit with a slower mean motion. If the conjunction occurs in a part of the zodiac where the two orbits are diverging, the net effect is that Saturn falls into a lower orbit with a more

[7] Newton, *Principia*, Book III, Proposition 13.

rapid mean motion. But these effects – a net slight slowing or slight speeding up – are not immediately discernible; they make themselves known only over time. Jupiter is affected in the way opposite to Saturn, so that when Saturn is speeding up, Jupiter is slowing down, and vice versa. Kepler in his *Rudolphine Tables* (1627) had determined Jupiter's and Saturn's mean motions by a comparison of observations reported by Ptolemy ca. 150 C.E. and by Tycho Brahe ca. 1600 C.E. During the 17$^{\text{th}}$ and 18$^{\text{th}}$ centuries astronomers found Jupiter moving on average more rapidly, and Saturn more slowly, than the Rudolphine numbers predicted. Laplace's analysis in 1785 at length showed that Saturn falls nearly 49 arc-minutes behind its long-term mean rate of motion, and then gets just as much ahead, the complete cycle from mean rate back to mean rate taking about 900 years. This is the largest perturbation in the solar system. During the same period Jupiter gets ahead of its overall mean rate of motion by about 20 arc-minutes, and then falls behind by the same amount before returning to its mean rate again. On this main oscillation, smaller oscillations are superimposed.[8] Astrologers were familiar with a 900 year periodicity as the time required for the conjunction to return to the same sign in the zodiac, but Laplace had to rediscover the period of 'the great inequality' from the algebra of the problem, which he managed to do only in 1785. For Euler, who died in 1783, the problem of Jupiter and Saturn would remain an unsolved enigma, disturbing his confidence in the inverse-square law.[9]

In response to the Paris Academy's announcement of its prize contest for 1748, Euler wrote two memoirs, both completed in mid-1747. In the first, which Euler presented to the Berlin Academy, he derived the differential equations for the problem of perturbation.[10] The second, a derivation of the perturbations of Saturn due to Jupiter, was submitted in the contest and awarded the prize, despite Euler's failure to account for the apparent slowing down of Saturn or speeding up of Jupiter.[11] The excellence of

[8] For a more complete explanation, see my article, "The Great Inequality of Jupiter and Saturn: from Kepler to Laplace," *Archive for History of Exact Sciences*, Vol.33 (1985), pp.24-36.

[9] Euler would have been baffled by our invocation, in this paragraph, of the word "energy" in its 19$^{\text{th}}$-century meaning. However, he could have understood the conclusions we have drawn on the basis of Kepler's third law. According to this law, the periods of the planets are as the 3/2 power of their mean solar distances. If a single planet is changing its mean solar distance, its period must also change in accordance with Kepler's law. The planet's period is inversely as its mean motion.

[10] [E112] "Recherches sur le movement des corps céléstes en générale," *Mémoires de l'Académie des Sciences de Berlin* **3** (1747), 93-143; Leonhard Euler, *Opera omnia*, ser.2, **25**, 1-44.

[11] [E120] "Recherches sur la question des inégalités du mouvement de Saturne et de Jupiter, sujet propose pour le prix de l'année 1748, par l'Académie Royale des Sciences

Euler's prize essay lay in the innovative methods he introduced for coping with planetary perturbations.

In the introductory sections of both papers, Euler reported that he had come to doubt the strict accuracy of Newton's inverse-square law, finding departures from it not only in the motions of Jupiter and Saturn, but also in the Moon's motion. It would be eventually necessary, he believed, to imagine new hypotheses and deduce their consequences. But in his prize paper he assumed the strict accuracy of the inverse-square law, and limited his aim to the deduction of the perturbations of Saturn due to Jupiter following from this assumption.

Euler formulated the algebraic problem with care. Only one of the assumptions he made would he later repudiate: taking the orbital plane of Jupiter as the reference plane; he imagined Jupiter's orbit as remaining unperturbed; but as he later realized (in a second prize paper of 1752 on the same subject[12]), whenever Jupiter perturbs Saturn, Saturn perturbs Jupiter, and these perturbations are best addressed as simultaneous and reciprocal. The tables of Jacques Cassini (Paris, 1740) put the inclination of the orbit of Saturn to Jupiter's orbital plane at about $1°15'$, an angle Euler initially thought might almost be ignored, but he took account of it nonetheless, and so made an important discovery, as we shall see.

In imitation of the astronomers, Euler used polar coordinates. He did not have the helpful device, first introduced by Lagrange, of indicial notation, but gave one letter to the radius vector of Jupiter, and an unrelated letter to the radius vector of Saturn, and similarly for other corresponding variables and constants. For our own and the reader's convenience, we shall use indicial notation here, representing Jupiter's radius vector and heliocentric longitude by the variables r, φ, and Saturn's polar coordinates by r',φ', z'. (The variable r', we note, is the *curtate* radius vector – the component of Saturn's radius vector lying in the plane of reference r, φ.) If we designate the components of force affecting Saturn's acceleration in the three coordinate directions as P, Q, R – namely P along r', Q at right angles to r' in the plane of reference, and R at right angles to the plane of reference – Euler's three equations of motion for Saturn take the form

de Paris," *Pièce qui a remporté le prix de l'Académie Royale des Sciences en 1748 sur les inégalités du mouvement de Saturne et de Jupiter* (Paris, 1749); Leonhard Euler, *Opera omnia*, ser.2, **25**, 45-157.

[12] [E384] "Recherches sur les irregularités du movement de Jupiter et de Saturne. Pièce qui a remporté le Prix proposé par l'Académie des Sciences, pour l'année 1752," in *Recueil des pieces qui ont remporté les prix de l'Académie des Sciences*, t.VII (1751-1661), Paris, 1769. Leonhard Euler, *Opera omnia*, ser.2, **26**.

$$d^2r' - r'd\varphi'^2 = -\frac{1}{2}Pdt^2 \tag{1}$$

$$2dr'd\varphi' + r'd^2\varphi' = -\frac{1}{2}Qdt^2 \tag{2}$$

$$d^2z' = -\frac{1}{2}Rdt^2 \tag{3}$$

The factor ½ on the right derives from Euler's writing the law of free fall as $\nu^2 = h$ rather than $\nu^2 = 2gh$, and taking the acceleration of free fall on the Earth's surface as the unit for measuring accelerative forces. However, he proceeded at once to substitute for $\frac{1}{2}dt^2$ the expression $a^3 (dM)^2 /S$, obtained from an equation like (1) applied to an imaginary Jupiter moving in a circle with radius equal to Jupiter's mean solar distance a, and with uniform angular motion M equal to Jupiter's mean longitudinal rate of motion, S being the mass of the Sun.

Euler included in P, Q, and R, besides the forces actually acting on Saturn, the components in the three coordinate directions of forces on the Sun from Jupiter and Saturn, understood as transferred to Saturn with directions reversed, in order that Saturn's motions might be referred to the Sun considered as at rest. (A geometer as acute as Daniel Bernoulli wanted to assume the Sun at rest, but did not see how to make it so in his theory.[13]) If we take p, p' as the masses of Jupiter and Saturn, ψ' as the latitude of Saturn above Jupiter's orbital plane, $\omega = \varphi - \varphi'$ as the difference in heliocentric longitude between Jupiter and Saturn, and v as the linear distance between Jupiter and Saturn, Euler's expressions for P, Q, and R will be:

$$P = \frac{(S+p')\cos^3\psi'}{r'^2} + \frac{pr'}{v^3} + \frac{p\cos\omega}{r^2} - \frac{pr\cos\omega}{v^3},$$

$$Q = \frac{p\sin\omega}{r^2} - \frac{pr\sin\omega}{v^3},$$

$$R = \frac{(S+p')\cos^2\psi'\sin\psi'}{r'^2} + \frac{pr'\tan\psi'}{v^3}.$$

Finally, dividing through by S, and giving special symbols to p/S and p'/S (we here use μ, μ',) Euler obtained for his equations (1) - (3) expressions of the form

$$d^2r' - r'(d\varphi')^2 = -a^3 (dM)^2 \left\{ \begin{array}{l} \frac{(1+\mu')\cos^3\psi'}{r'^2} + \frac{\mu r'}{v^3} \\ \frac{\mu\cos\omega}{r^2} - \frac{\mu r\cos\omega}{v^3} \end{array} \right\} \tag{4}$$

[13] See Fuss, *Correspondance Mathématique et Physique de quelques célèbres Géometres du XVIIIième Siècle*, T.II (St. Petersburg, 1843, Lettre XLVIII, pp.622-625.

$$2dr'd\varphi' + r'd^2\varphi' = -\mu a^3 (dM)^2 \sin\omega \left\{ \frac{1}{r^2} - \frac{r}{v^3} \right\} \tag{5}$$

$$d^2z' = -a^3 (dM)^2 \left\{ \frac{(1+\mu')\cos^2\psi' \sin\psi'}{r'^2} + \frac{\mu r' \tan\psi'}{v^3} \right\}. \tag{6}$$

Equation (6), however, Euler chose to transform. Astronomers were in the habit of using two parameters to specify the position of the orbital plane of a planet: the longitude of the ascending node of the orbit, where it rises to the north of the Ecliptic; and the inclination of the orbital plane to the Ecliptic. Since the orbital plane also passes through the center of the Sun, these parameters fix the orientation of the plane unambiguously. But in Euler's essay the plane of reference was the $r - \varphi$ plane. Given a force R on Saturn at right angles to the $r - \varphi$ plane, the resulting variations in latitude are reflected in variations in the node of Saturn on the $r - \varphi$ plane, and in the inclination of Saturn's orbital plane to the $r - \varphi$ plane; we label these two parameters π' and ρ'. At any moment,

$$z' = r' \sin(\varphi' - \pi') \tan\rho'$$

Since the orbital plane of Saturn is *almost* immobile, Euler wrote

$$dz' = dr' \sin(\varphi' - \pi') \tan\rho' + r'd\varphi' \cos(\varphi' - \pi') \tan\rho',$$

which assumes that π' and ρ' are invariable. But because of perturbation, the parameters π' and ρ' are in fact variable, and he therefore also wrote

$$dz' = dr' \sin(\varphi' - \pi') \tan\rho' + r'(d\varphi' - d\pi') \cos(\varphi' - \pi') \tan\rho' + \frac{r'd\rho' \sin(\varphi' - \pi')}{\cos^2\rho'}.$$

Defying logic, Euler took these two incompatible equations for dz' as simultaneously true; the first reflecting the near-invariability of π' and ρ' during the time dt in which the planet moves through the small distance $\left[(dr')^2 + (r'd\varphi')^2\right]^{1/2}$; the second reflecting the slow changes that π' and ρ' undergo because of perturbation. By eliminating dz' between the two equations, he obtained an expression for $d\rho'$:

$$d\rho' = \frac{d\pi' \sin\rho' \cos\rho'}{\tan(\varphi' - \pi')}.$$

Substituting in (3) his expression for R and an expression for d^2z' taking account of the variations $d\pi'$ and $d\rho'$ as well as dr' and $d\varphi'$, and making all possible reductions, he was able to extract two first-order equations for $d\pi'$ and $d\rho'$:

$$d\pi' = \frac{\mu a^3 \left(dM\right)^2 \sin\left(\varphi' - \pi'\right) \sin\left(\omega - \pi'\right)}{r' d\varphi'} \left\{ \frac{1}{r^2} - \frac{r}{v^3} \right\} \tag{7}$$

$$d\left(\ln\,\tan\rho'\right) = \frac{\mu a^3 \left(dM\right)^2 \cos\left(\varphi' - \pi'\right) \sin\left(\omega - \pi'\right)}{r' d\varphi'} \left\{ \frac{1}{r^2} - \frac{r}{v^3} \right\}. \tag{8}$$

Equations (7) and (8) represent the first beginnings of the analytical method known as "the variation of orbital elements." Euler himself undertook to develop it further, extending it to the changes in other orbital elements;[14] but it would receive its classical formulation from Lagrange, who, starting from the position of Euler's prize memoir of 1748, showed how the variations of the eccentricity and aphelion could be determined in a way exactly parallel to Euler's determination of the variations of the node and tangent of the inclination, then went on to apply the idea to the remaining two orbital elements, the mean motion and epoch.

The solution of equations (4), (5), (7), and (8) would be straightforward were it not for the quantity v^{-3}. The distance v between Jupiter and Saturn changes by a factor which can be as large as 3.418 as Jupiter moves from conjunction with Saturn to opposition; hence v^{-3} can change by a factor as large as $(3.418)^3 = 39.932$. How to express this variable factor in such a way that the terms involving it can be integrated?

Every term in which v^{-3} occurs is multiplied by the small factor μ, which Newton put at 1/1067. Euler therefore believed it sufficient to express v^{-3} in terms of the mean distances of Jupiter and Saturn, namely a and a':

$$v^{-3} = \left[a^2 + a'^2 - 2aa' \cos\omega\right]^{-3/2}.$$

By substituting $\alpha = a/a'$ and $g = 2\alpha/(1+\alpha^2)$, Euler obtained

$$v^{-3} = a'^{-3} \left(1+\alpha^2\right)^{-3/2} \left(1 - g\cos\omega\right)^{-3/2}.$$

His concern thus became that of finding a rational approximation to $(1 - g\cos\omega)^{-3/2}$ or more generally, $(1 - g\cos\omega)^{-s}$. An ordinary Taylor expan-

[14] Euler's contributions to the method appeared in his [E384] *Recherches sur les irregularities du movement de Jupiter et de Saturne. Pièce qui a remporté le Prix propose par l'Académie des Sciences.* T.VII (1751-1761), Paris, 1769; in the appendix to his [E187] *Theoria motus Lunae exhibens omnes eius inaequalitates, etc.* Berlin: Impensis Academiae Scientiarum Petropolitanae, 1753; in his [E232] "De motu corporum coelestium a viribus quibuscunque perturbato," in *Novi commentarii Academiae Scientiarum Imperialis Petropolitanae*, t.IV (1752-1753), pp.161-196, Leonhard Euler *Opera omnia* ser. II, **23**; and in his [E414] "Investigatio perturbationum quibus planetarum motus ob actionem eorum mutuam afficiuntur," which won the Paris Academy prize for 1756; see *Recueil des pièces qui ont remporté les prix de l'Académie des Sciences*, t.VIII, p. 138.

sion is unsatisfactory because, for Jupiter and Saturn, $g = 0.8404$, and the resulting Taylor series converges too slowly for practical use; moreover, the powers of cos ω in a Taylor series have to be transformed before terms in the series can be integrated. Euler therefore sought to obtain a series of the easily integrable form

$$q = A + B\cos\omega + C\cos 2\omega + D\cos 3\omega + \dots \tag{9}$$

He had introduced such a trigonometric series a few years earlier in a different context; in astronomy it will henceforth play a crucial role.

To obtain the coefficients of (9), Euler carried out a logarithmic differentiation:

$$\frac{dq}{d\omega}(1 - g\cos\omega) + sgq\sin\omega = 0.$$

Into this equation he substituted the values of q and $dq/d\omega$, then used trigonometric identities to reduce the products $\sin n\omega \times \cos\omega$ and $\cos n\omega \times \sin\omega$ to sums and differences of sines and cosines. He was thus able to show that the coefficients after the first two are given in terms of the two preceding coefficients, for instance,

$$C = \frac{2B - 2sgA}{(2-s)\,g}.$$

To obtain A and B he proceeded by approximation, using among other devices the method later known as "the method of special values," also as "harmonic analysis;" it is essentially a way of computing Fourier coefficients by numerical integration. Various methods of computing these coefficients were developed later by d'Alembert and Lagrange.

Euler solved equations (4) and (5) under several different simplifying assumptions: (A) that Jupiter's orbit is circular while Saturn's is pristinely (before the Jovian perturbations are introduced) circular; (B) that Jupiter's orbit is elliptical while Saturn's is pristinely circular; (C) that Jupiter's orbit is circular while Saturn's orbit is pristinely elliptical. Some of the terms resulting from these several initiatives were duplications. He thus obtained a good many terms in the longitude of Saturn that are indeed correct, but he did not discover the terms that cause Saturn's long-term inequalities, because he did not suspect the involvement of terms proportional to the third dimension in the products and powers of the eccentricities and inclinations of the two planets, that is, terms proportional to e^3, e'^3, e^2e', ee'^2, $e'\tan^2\rho'$, $e'\tan^2\rho'$. These products and powers are small and Euler supposed the terms in which they occur to be negligible. But the double integration that the differential equations require leads to their having the small divisor $(2M - 5M')^2 \approx 0°.00084224$/day, and calculation shows that they become observationally sizable.

In integrating (7) and (8), Euler assumed the variability of π' to be so slight that it could be treated as a constant. He obtained as the integral of (7)

$$\pi' = C' - \frac{\mu B\varphi}{4m^2h} + \text{oscillatory terms.} \tag{10}$$

Here C' is a constant of integration, and μ, B, m, and h are other constants. The import of the second term on the right of (10) is that the node of Saturn retrogresses on the orbital plane of Jupiter.

Euler realized that the same thing would happen to any planet perturbed by another planet: its node on the orbital plane of the perturbing planet would retrogress. Thus the node of the Earth's orbit will retrogress on the orbital planes of each of the other planets, and hence the Ecliptic, which is the orbital plane of the Earth, is in motion. Using Newton's values for the masses of those planets that have satellites, and hypothesizing the densities of Mercury, Venus, and Mars to be inversely as the square roots of their periods (an inexact hypothesis at best), Euler arrived at a figure for the diminution of the obliquity of the Ecliptic per century, namely −47".5, which is close to the present-day value, −46".8.[15] The goodness of the result is due to compensating errors. Nevertheless, Euler was the first to establish theoretically that the Ecliptic is in motion. A consequence of this discovery that he did not articulate is that, besides precession of the Equinox caused by the Moon's and Sun's attractions for the Earth's equatorial bulge (to be dealt with below), some precession of the Equinox is caused by motion of the Ecliptic due to planetary perturbation.[16]

In his memoir of 1748, Euler sought to correct his theoretically derived terms by a comparison with observations of heliocentric longitudes of Saturn, 95 of which, spanning the years from 1582 to 1745, had been listed by Jacques Cassini in his *Élémens d'astronomie.*[17] In deriving the theoretical terms, Euler had presupposed the orbital elements for Saturn given by Cassini in his *Tables astronomiques*;[18] but these elements are founded on the rules of Kepler, and therefore, Euler observed, "have need... of some correction, since the inequalities caused by Jupiter have been there enveloped in the eccentricity and position of the orbit of Saturn."[19] To the

[15] Euler, [E223] "De la variation de la latitude des étoiles fixes et de l'obliquité de l'écliptique," *Mémoires de l'Académie de Berlin* **10** (1754), 1756, 296-336.
[16] I do not know when clarity on this point emerged – perhaps not till after Laplace's discovery of the invariable plane.
[17] Paris, 1740; p.355 et sqq.
[18] Paris, 1740.
[19] Leonhard Euler, [E120] "Recherches sur la question des inégalités du mouvement de Saturne et de Jupiter ..." *Pièce qui a remporté le prix* ... 1749, *Opera Omnia*, **II, 25**, pp. 45-157. This passage is on p. 119.

longitude of Saturn in its Keplerian ellipse as derived from Cassini's tables Euler gave the expression

$$M' - 23552'' \sin E' + 168'' \sin 2E',$$

where, in accordance with Keplerian theory, M' is Saturn's mean longitude at the chosen epoch, E' is Saturn's eccentric anomaly as derived from M', $23552''$ is twice the eccentricity turned into arcseconds, and $168''$ is 5/4 the square of the eccentricity turned into arcseconds. Euler replaces M' by $M'+m+Nn$, where m is a constant correcting the mean longitude at epoch, N is the number of years since the epoch, and n is the correction of the mean movement per Julian year. The correction of E' he symbolized by dE', and replaced $\sin E'$ by $\sin E'+dE' \cos E'$, and $\sin 2E'$ by $\sin 2E'+2dE' \cos(dE')$. Also, if the eccentricity required correction, a quantity $\pm x$ had to be added to the coefficient $23552''$. The earlier Keplerian-style formula thus became

$$M' + m + Nn - (23525'' + x) \sin E' - 0.11405\,(dE') \cos E'$$
$$+168'' \sin 2E' + \frac{dE'}{600} \cos 2E'.$$

Since dE is expressed in arcseconds, the numerical coefficients containing dE (already converted to arcseconds) have here been multiplied by $2\pi/1296000''$ to change them back into parts of the radius.

In his observational comparisons, Euler aimed not only to correct orbital elements, but also to determine certain terms due to perturbation for which his theoretical calculation had given doubtful or impossible values. Thus for the term proportional to $\cos(\omega - E)$, Euler first obtained a coefficient with zero denominator (on account of his failure to allow for the motion of Jupiter's aphelion, caused by perturbation from Saturn); from the observations he hoped for a less perplexing result.

In all, Euler attempted to determine eight unknowns from the observations; eight simultaneous equations would therefore seem to suffice. "But since small errors committed both in the observations and in the calculation can produce large errors in the values of these letters, we must in this investigation choose with care the observations which will be the most proper for this purpose, in order that from the inevitable errors in the observations and in the calculation, there should result the least possible error in the values of the eight letters sought." [20]

Euler's actual procedure in combining observations involved arbitrary choices that could lead to different outcomes. That it led to only modest improvement in the agreement of his theory with observations was mainly due to inadequacies in his theory: it lacked large terms crucial to a correct

[20] *Ibid.*, p.122.

theory. But Euler's initiative here stimulated others to attempt similar undertakings, and with happier success. First of all, Tobias Mayer, in Chapter 13 of his "Abhandlung über die Umwälzung des Monds um seine Axe...", published in 1750,[21] applied Euler's idea in determining the inclination of the Moon's equator to the plane of the Ecliptic, and the longitude of the node in which these planes intersect. He followed Euler in combining equations so as to increase the importance of one or two terms and decrease that of all the rest, but far more systematically than Euler had done. Lagrange in his "Théorie de la libration de la lune..." remarked that Mayer had calculated the observations "with all the precision and elegance that one can desire..."[22] Laplace had in hand a copy of Lagrange's memoir by February 10, 1783, when he wrote Lagrange to thank him for the gift; and he undoubtedly studied Mayer's procedure with care, for he used a very similar procedure with eminent success in his "Théorie de Jupiter et de Saturne" of 1785. Delambre, who carried out a revision of the constants in the latter work by means of an extended comparison with observations, used the same method, which he always attributed to Mayer.

Meanwhile Mayer, relying on certain procedures in Euler's prize memoir of 1748 and making modifications in others, had developed a lunar theory which he proceeded to refine by comparison with observations, in particular with precise observations of the Moon's occultations of the star Aldebaran, carried out with the aid of a micrometer of his own design. The result, achieved by 1753, was a lunar theory accurate to about one arc-minute; it became the basis of the British Admiralty's *Nautical Almanac* in 1767. The *Almanac*'s lunar tables were corrected successively by Charles Mason in 1778 and 1780, by J.T. Bürg in 1806, and by K.B. Burckhardt in 1812 – in all cases partly by inclusion of new terms from the theory, but mainly by means of a comparison with a large number of the Greenwich observations. The combinations of observations involved were still not free from arbitrariness, and would not be so before the method of least squares was known and came to be applied *de rigueur*. Burckhardt's tables would remain the basis of the *Nautical Almanac* until 1862.

[21] *Kosmographische Nachrichten* (Nuremberg).

[22] *Nouveaux Mémoires de l'Académie royale des Sciences et Belles-Lettres de Berlin,* 1780/1782; *Oeuvres de Lagrange*, V. 6.

3. The precession of the Equinoxes and the mechanics of rigid bodies, 1751-1765

In early 1748 James Bradley announced his discovery of the nutation of the Earth's axis, a small wobble superimposed on the motion of that axis producing the precession of the Equinoxes. News of the discovery reached the Académie des Sciences in Paris during the summer of 1748. D'Alembert, who had been at work on his lunar theory, immediately put the theory aside, and set about investigating the following question: Are the precession and nutation derivable from Newton's inverse-square law? (The strict accuracy of Newton's law was just then in doubt, because Clairaut, d'Alembert, and Euler had all discovered that they could only derive from it about half the observed motion of the lunar apse.) Reviewing Newton's attempt to derive the precession, d'Alembert concluded that it was deeply flawed. He worked in haste, fearing (groundlessly as it turned out) to be forestalled by the English. His *Recherches sur la Précession des Equinoxes et sur la Nutation de l'Axe de la Terre dans le Systême Newtonian* was completed in May, 1749, and published in July. On July 20 d'Alembert sent off a copy to Euler.

Not until January 3, 1750, did Euler respond. He had, he said, long applied himself to this subject, but being unable to vanquish all the obstacles he met with, had at length abandoned the effort. Nor had he been able to follow the argument of d'Alembert's treatise, but seeing in general how d'Alembert surmounted the obstacles, he had recommenced the investigation in his own way, and has since carried it to a happy conclusion.[23]

D'Alembert's book is indeed difficult to read – "disorderly, full of typographical errors, and [containing] totally unintelligible diagrams," as Gabriel Cramer wrote the author. The crux of its argument lay in "d'Alembert's Principle," but d'Alembert's account of that principle was obscure and in fact flawed.[24] That d'Alembert's obtained the right result was due to compensating errors of sign.

Euler came to the reading of d'Alembert's treatise with a special perspective. He had long been attempting to formulate a theory of the mechanics of rotating bodies analogous to the mechanics of linear motion. The latter was founded on the Newtonian law of force, which Euler was the first to write in the form $F = ma$. The parallel law for rotation was $\tau = I\alpha$, where τ is the torque or moment of a force, I the sum $\sum mr^2$ or integral $\int r^2 dm$ which Euler named "moment of inertia," and α the angular ac-

[23] See my "D'Alembert *versus* Euler on the Precession of the Equinoxes and the Mechanics of Rigid Bodies," *Archive for History of Exact Sciences*, **37**, 233-273 for references and further details.

[24] See the analysis in *ibid.*, pp.244-245.

celeration. Euler had applied this formula to the rolling and pitching of a ship in his *Scientia navalis*, completed in 1740 but not published till 1749. This application was mediated by the hypothesis that, given three mutually perpendicular axes of symmetry in the ship (fore-and-aft, abeam, and vertically through the center of mass), the rotational motion about any one of these axes is undisturbed by the rotational motions about the other two. But the problem that had long baffled Euler was the following: Given an arbitrarily-shaped body turning about any axis, and acted upon by an oblique force, to find the change caused both in the axis of rotation and in the motion.

The problem can be resolved with the aid of D'Alembert's principle – if the latter is correctly understood. Here is a correct formulation.[25] Let $\boldsymbol{a}$ be the acceleration of a mass-element of a rigid body. Then $\boldsymbol{a} = \boldsymbol{a}_f + \boldsymbol{a}_c$, where $\boldsymbol{a}_f$ is the acceleration of this mass-element that would result from the externally applied forces, and $\boldsymbol{a}_c$ is the acceleration of this same element resulting from actions and constraints amongst the mass-elements. D'Alembert's principle means that the forces corresponding to the accelerations $\boldsymbol{a}_c$ form a system in static equilibrium, so that $\sum M\boldsymbol{a}_c = 0$, $\sum \boldsymbol{r} \times M\boldsymbol{a}_c = 0$, whence $\sum M(\boldsymbol{a} - \boldsymbol{a}_f) = 0$, $\sum \boldsymbol{r} \times M(\boldsymbol{a} - \boldsymbol{a}_f) = 0$ or in integral form, $\int dM(\boldsymbol{a} - \boldsymbol{a}_f) = 0$, $\int dM.\boldsymbol{r} \times (\boldsymbol{a} - \boldsymbol{a}_f) = 0$. It was because the mathematical development in d'Alembert's *Recherches* embodied – by grace of compensating errors of sign – the correct conditions for rotational equilibrium expressed in the last equation, that d'Alembert managed to arrive at differential equations from which the precession and nutation could be derived. D'Alembert's use of integrals to express such conditions was quite new; and it was undoubtedly from his reading of d'Alembert's treatise that Euler derived his own similar formulation.

In the essay on the precession that Euler presented to the Berlin Academy in March, 1750,[26] he computed the Earth's moment of inertia on the assumption that the Earth is a homogeneous sphere and also on the assumption that it is a sphere with denser core. In neither case did he take account of the flattening of the Earth, remarking that a small departure from sphericity changes the moment of inertia but little. But his neglect of the flattening meant that he could not determine separately the motion of the Earth's axis of figure and that of its instantaneous axis of rotation, as d'Alembert had done.

In computing the torques acting on the Earth due to the attractions of the Sun and the Moon, Euler used equatorial coordinates and assumed the

[25] Here we follow Truesdell, *The Rational Mechanics of Flexible or Elastic Bodies, 1638-1788*, in *Leonhard Euler, Opera omnia*, **II, 11** (2), pp.186-187.
[26] Leonhard Euler, *Opera omnia* **II, 29**: 92-123.

Earth to be spheroidal. His derivation is far more direct and perspicuous than d'Alembert's. To transform the resulting formulas from equatorial to ecliptic coordinates, he used the algorithms of spherical trigonometry and thus avoided much of the complicated geometry that d'Alembert had employed.[27]

The memoir is logically incomplete, in relying on a principle "explained elsewhere." Also, it assumes without proof that the Earth if set rotating about an axis slightly different from its axis of figure would continue to do so without variation unless acted upon by external forces. The "elsewhere" in which Euler explains the missing principle is his essay entitled "Discovery of a New Principle of Mechanics," presented to the Berlin Academy later in 1750.[28] Here Euler considered a rotating body whose center of mass is at rest, and which is subject to the action of external torques. He used a coordinate system at rest in absolute space, with whose z-axis the axis of rotation was initially coincident. By setting the total moments calculated from the kinematics of rotation equal to the components of any externally applied torque about the three coordinate axes, he obtained equations permitting the determination of the infinitely small changes in the components of the angular velocity during the time dt. The problem of the rotation of a rigid body about a free axis was thus in principle solved.[29]

The application, however, remained difficult, because the single coordinate system had to be shifted after each interval dt. To eliminate this difficulty, Euler in a new memoir presented in October 1751 introduced a second coordinate system fixed in the rotating body; to express the relation between the two systems, he introduced the now famous 'Eulerian angles.'[30] With this innovation, integrations became possible; the long-term development of the motion could be followed. Euler also showed from his equations that in any body there exists at least one axis about which the body can rotate without wobbling. In 1755, J.A. Segner went on to demonstrate that in any body there exist three mutually perpendicular axes about which it can rotate without wobbling. These axes Euler called "the principal axes."

A final discovery emerged in Euler's "Du mouvement de rotation des corps solids autour d'un axe variable," presented to the Berlin Academy

[27] For a more detailed account of Euler's derivation, see my article cited in n.23.

[28] Leonhard Euler, [E177] "Découvert d'un nouveau principe de Mécanique" *Mémoires de l'académie des sciences de Berlin* **6** (1750) 1752, p. 185-217, *Opera Omnia*,**II, 5**:81-108.

[29] A more complete explanation is provided in the article cited in n.23.

[30] Leonhard Euler, [E336] "Du mouvement d'un corps solide quelconque lorsqu'il tourne autour d'un axe mobile" *Mémoires de l'académie des sciences de Berlin* **16** (1760) 1767, p. 261-284, *Opera Omnia* **II, 8**: 313-356.

in November, 1758.[31] By choosing the principal axes as the coordinate axes fixed in the body, Euler found ("with surprise," he says) that he could obtain integral solutions of problems he had previously supposed to surpass the powers of the calculus. Thus, given a body on which a rotational motion had been imposed about a non-principal axis, he could now determine the continuation of its motion. Spurred on by this success, he now undertook to write his systematic treatise on the rotation of rigid bodies, *Theoria motus corporum solidorum seu rigidorum*, which was published in 1765 and established the standard terminology and techniques for dealing with the rotation of extended bodies.

4. The inverse-square law and the motion of the lunar apse

As indicated earlier, by the autumn of 1747, the three foremost mathematicians in Europe – Clairaut, Euler, and d'Alembert – had independently concluded that the inverse-square law yielded only about half the observed motion of the lunar apse. In a meeting of the Académie des Sciences in November Clairaut proposed amending the inverse-square law by adding a second term which would express a variation of gravitational force inversely as the fourth power of the distance. The proposal aroused a storm of protest, particularly from Georges-Louis LeClerc, Comte de Buffon, who insisted that a two-term law was 'metaphysically repugnant.'

From the beginning of his work on the lunar theory, Clairaut had expected to carry out a second-order calculation in order to improve the accuracy of the coefficients of the various inequalities. Such a second-order calculation was suggested by his mode of calculation. After deriving the equations of motion, a clever integration had given him

$$\frac{f^2}{Mr} = 1 - g\sin\varphi - c\cos\varphi + \sin\varphi\int\Omega\cos\varphi\cdot d\varphi - \cos\varphi\int\Omega\sin\varphi\cdot d\varphi. \quad (11)$$

Here f, M, g, c, are constants and Ω is a complicated expression involving the solar perturbing forces acting on the Moon in the radial and transverse directions. Into (11) he had then substituted the following expression for $\boldsymbol{r}$, the radius vector:

$$\boldsymbol{r} = \frac{k}{1 - e\cos m\varphi}, \quad (12)$$

[31] Leonhard Euler, [E292] "Du mouvement de rotation des corps solids autour d'un axe variable," *Mémoires de l'académie des sciences de Berlin* **14** (1758), 1765, p. 154-193, *Opera Omnia* **II, 8**: 200-235.

where k, e, and m are constants to be determined, and $m\varphi$ is the real anomaly, reaching 360° when the Moon returns to its apogee. This is the expression for an ellipse that rotates forward if $m > 1$. Its substitution would be justified, Clairaut said, if, k being identified with f^2M, the larger terms in the right member of (11) could be identified with the expression $(1 - e\cos m\varphi)$. The result of the first-order calculation appeared to fulfill this hope, for it gave him

$$\frac{k}{r} = 1 - e\cos m\varphi + \beta\cos\frac{2\varphi}{n} + \gamma\cos\left(\frac{2}{n} - m\right)\varphi + \delta\cos\left(\frac{2}{n} + m\right)\varphi \quad (13)$$

where n is the Moon's mean motion divided by the difference between the Moon's and Sun's mean motions, and β, γ, δ [b, g, d] evaluated in terms of other constants in the theory were 0.007090988, −0.00949705, and 0.00018361, respectively, hence small relative to e (known empirically to be about 0.05).

The second-order computation that suggested itself consisted in substituting (13), with e, m, β, γ, δ left as symbols, back into (11), and reevaluating the constants. Surprisingly, this led to new and impressive contributions to m, particularly from the term $\gamma\cos\left(\frac{2}{n} - m\right)\varphi$, which arose from the transverse component of the solar perturbing force. The constant m proved to be 0.99164, implying a monthly apsidal motion of $+3°2'6"$, just $2'5"$ shy of the empirical value Clairaut believed correct. He made a public announcement of this discovery in May of 1749.

On receiving Clairaut's news, Euler was amazed and troubled. He carefully reviewed his own derivation of the apsidal motion of the Moon, but could find no error in it. How had Clairaut achieved this result? Would Clairaut's derivation, if subjected to a close scrutiny, in fact prove valid?

At just this time the Petersburg Academy was planning its first prize contest. Euler, previously a resident member of that academy, and still an associate after leaving St. Petersburg to join the Berlin Academy in 1741, offered a list of suggestions for the topic of the contest, of which the first concerned the adequacy of the inverse-square law to account for the inequalities of the Moon. It was this topic that was chosen by an appointed committee of the Petersburg Academy; the question was framed as follows:

> "To demonstrate whether all the inequalities observed in lunar motion are in accordance with Newtonian theory – and if they are not, to demonstrate the true theory behind all these inequalities, such that the exact position of the Moon at any time can be computed by means of it." [32]

[32] Y.K. Kopelevich, "The Petersburg Astronomy Contest in 1751," *Soviet Astronomy - AJ*, Vol.9, January-February, 1966, p.653.

Euler also offered to be a member of the committee reviewing the essays submitted in the contest, and his offer was accepted.

By March 26, 1751, Euler had in hand four of the memoirs submitted in the contest, including one he recognized as Clairaut's, and he wrote to Clairaut to say

> "... it is with infinite satisfaction that I have read your piece, which I have waited for with such impatience. It is a magnificent piece of legerdemain, by which you have reduced all the angles entering the calculation to multiples of your angle ν [our angle φ], which renders all the terms at once integrable.... I must confess that in this respect your method is far preferable to the one I have used. However I see clearly that your method cannot give a different result for the movement of the apogee than mine; in which I have recently made some change, for having previously reduced all angles to the eccentric anomaly of the Moon, I have now found a way to introduce the true anomaly in its place. Thus while your final equation has as its two principal variables the distance of the Moon from the Earth and the true longitude, I have directed my analysis to the derivation of an equation between the longitude of the Moon and its true anomaly, which seems to me more suitable for the usage of astronomy." [33]

Euler's concern here to avoid variables like the radius vector $\boldsymbol{r}$, the observational determination of which was much less precise than that of the angular variable φ, is characteristic; we will encounter this phenomenological tendency again in his third lunar theory.

A second letter to Clairaut, dated April 10, 1751, is marked by a new note of elation:

> "I have the satisfaction of writing you that I am now altogether clear concerning the motion of the lunar apogee, and that I find it, as you do, entirely in agreement with Newton's theory. This investigation has drawn me into terrible calculations, and I have finally discovered the source of the insufficiency of the methods I had followed previously: it consisted in the incomplete determination of a constant of integration – an inconvenience to which your method is not subject. But since now two completely different methods lead to the same conclusion, no one will refuse to recognize the correctness of your research. For myself, knowing whereof I write, I felicitate you on this happy discovery, and I even dare to say that I regard this discovery as the most important and the most profound that has ever been made in mathematics. I ask your pardon a thousand times for having doubted the rightness of your retraction,

[33] G. Bigourdan, "Lettres inédites d'Euler à Clairaut," *Comptes rendus du Congrès des sociétés savants de Paris et des Départments tenu à Lille en 1928...*, pp.34-35.

but I believe that my stubbornness will render your victory all the more brilliant, and will protect it from all the attacks to which it might yet be subject. For it is very certain that there are very few persons who are capable of recognizing the correctness of your analysis, and I am obliged to confess that I would be yet in the same case, if I had not found a completely different method which led me to the same result. For this is how I am led there: instead of supposing the force of the Earth on the Moon to be m/x^2 for the distance x, I have expressed it by $m/x^2 - \mu$, with the aim of so determining μ that I should obtain the true motion of the apogee as given by observation. And I have at length found, contrary to my expectation, that this term must be supposed so small, that one can regard it without error as nothing; while according to my earlier opinion it should have turned out to be rather considerable." [34]

Clairaut's "Théorie de la lune déduite du seul principe de l'attraction reciproquement proportionelle aux quarrés des distances," which was awarded the prize of the St. Petersburg Academy in the contest of 1751, was published in St. Petersburg in 1752. [35] Euler's second lunar theory, the *Theoria motus lunae*, which as he finally revised it was intended primarily as a test, by an independent route, of Clairaut's claim that the observed motion of the lunar apse was in accord with Newton's inverse-square law, was published by the St. Petersburg Academy in 1753. [36]

5. Euler's later thoughts on celestial mechanics; his Third Lunar Theory

In his later years, Euler became increasingly critical of the methods in celestial mechanics he had done so much to establish earlier. In July 1762 he read to the Berlin Academy his "Nouvelle méthode de determiner les dérangemens dans le mouvement des corps célestes, causés par leur action mutuelle." [37] Here Euler advised that the attempt to achieve integrations satisfactory for all time be relinquished, and like Clairaut and his colleagues in determining the return of Halley's Comet for 1759, that the calculation be made directly from the differential equations. The position and velocity of a body being known for some moment of time, the increments to the position coordinates and velocity components were to be calculated from

[34] *Ibid.*, pp.36-37.
[35] St. Petersburg: Imprimerie de l'Académie Imperiale des Sciences, 1752.
[36] It is also found in Leonhard Euler, [E187] *Opera Omnia*, **II, 23** pp. 64-336.
[37] [E398] *Mémoires de l'Académie des Sciences de Berlin*,**19** (1763), 1770, 141-179. *Opera Omnia* ser. II vol. 26.

the differentials during successive small intervals of time. The result would be an ephemeris giving positions at successive equal intervals of time. Since the error could be expected to increase as the ephemeris was continued, at some point it would become necessary to have recourse to observations and begin over again. In a memoir of 1763, Euler showed how very exact positions and velocities could be obtained from a series of observations of the Moon on consecutive days by an application of the calculus of finite differences.[38] Laplace would later utilize this method in his work on the determination of cometary orbits.

In the early 1770s Euler added a further wrinkle: the use of a coordinate system rotating with the mean speed of the planet or Moon. He applied this proposal in the theory of the Moon that he submitted to the Paris Academy in its prize contest of 1772;[39] and also in a study of the perturbations in the motion of the Earth due to the action of Venus.[40] In both cases, his blindness having become total, Euler relied on other members of the Petersburg Academy for assistance with the numerical calculations. In the case of the second memoir, a special desideratum was to avoid use of trigonometric series to express the Venus-Earth distance. "In no way," Euler asserted, "can the irrational formula $v = \sqrt{a^2 - 2ab\cos\varphi + b^2}$ be resolved into a convergent series, as required if the integration is to be carried out in the usual way; for which reason we are compelled to determine our integrals mechanically."[41] For the Venus-Earth interaction, the parameter $g = 2ab/(a^2 + b^2) = 0.9497$; it was thus closer to 1 than in the case of any other two planets. Hence the series $A + B\cos\varphi + C\cos 2\varphi + \ldots$ for $\nu^{-3/2}$ would have had to be extended much farther than in the case of other pairs of planets to obtain a pre-specified degree of precision. Euler proposed instead to plot the functions to be integrated for 72 values of the angle φ between the *radii vectores* of Venus and the Earth, increasing by 5° increments; the points thus determined were to be connected by straight lines, and the areas of the resulting trapezia added up to obtain an approximation to the integral sought. These calculations were carried out

[38] [E401] "Nouvelle manière de comparer les observations de la Lune avec la théorie," *Mémoires de l'Académie des Sciences de Berlin*, **19** (1763), 1770, 221-234, *Opera Omnia* ser. II vol. 24, pp. 90-100.

[39] *Theoria motuum Lunae, nova methodo pertractata, una cum tabulis astronomicis, unde ad quodvis tempus loca Lunae expedite computari possunt, incredibili studio atque indefesso labore trium academicorum: Johannis Alberti Euler, Wolffgangi Ludovici Krafft, Johannis Andreae Lexell, opus dirigente Leonhardo Eulero*, St. Petersburg, 1772; also *Opera Omnia* ser. II, vol. 22, pp. 64-336.

[40] [E425] "De perturbatione motus Terrae ab actione Veneris oriunda," *Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae* **16** (1771), 1772 426-467. *Opera Omnia* ser. II vol. 26.

[41] *Ibid.*, 448.

by Johannis Andreas Lexell. The resulting table of perturbations differed shockingly from the widely accepted tables of Lacaille, which were based on Clairaut's calculation, using the ordinary trigonometric series, of the Venusian perturbations of the Earth. In a memoir published in 1778, Euler announced that, in relying on Lacaille's tables, astronomers would often be in error by $20''$ or $30''$ – an error leading to a disastrously wrong value of the Moon's longitude and hence to error in the navigator's determination of longitude at sea.[42]

Laplace, at the urging of Lalande, wrote to ask the opinion of Euler and Lexell as to the cause of the discrepancy. It was Lexell who responded: "The principal reason of this discrepancy is to be ascribed to an error I committed in the calculation. I judged it to be my part to correct what was erroneous, and at the same time to treat all this material with the greatest exactitude possible, so that no doubt will remain in the minds of astronomers that these diverse methods lead to completely concordant conclusions."[43] Lexell re-did the entire calculation, using steps of $1°$ rather than $5°$. Revising Lacaille's values for solar parallax and the mass of the Earth (he supposed the mass of Venus to be equal to the Earth's mass), he found that his values for the several major terms were proportional to Lacaille's, but smaller by the factor 0.702. The method of trigonometric series was thus vindicated.

One of the ideas instantiated in Euler's lunar theory of 1772 had occurred to him already in the mid-1760s, if we may suppose that the "Réflexions sur la variation de la lune" presented on April 17, 1766 to the Berlin Academy by J.A. Euler reflects the father's thinking.[44] young Euler began as follows:

> "Although there is reason to be content with the new tables of the Moon published by the late M. Mayer and the late M. Clairaut, since by means of these tables we can determine the place of the Moon almost as exactly as that of the Sun, the theory from which these tables were drawn is still far from the perfection that could be desired; very little progress, one must admit, has been made up to the present. The large number of equations that it is necessary to employ to determine the Moon's place to an arc-minute furnishes an evident proof of this, since it follows evidently that, to achieve a yet higher precision, the number of equations would

[42] [E511] "Réflexions sur les inégalités dans le movement de la terre, causes par l'action de Venus," *Acta Academiae Scientiarum Imperialis Petropolitanae*, pars prior pro anno 1778, 297-307. *Opera Omnia* ser. II, vol. 27.

[43] A. I.Lexell, "De perturbatione in motu telluris ab actione Veneris oriunda," *Acta Academiae Scientiarum Imperialis Petropolitanae*, pars posterior pro anno 1779, 359-360. The memoir as a whole occupies pp.359-392.

[44] *Histoire de l'Académie Royale des Sciences et Belles-Lettres*, Vol.22, pp.334-353.

have to be increased so greatly that employing them would no longer be practicable."

No general solution of the three-body problem having been achieved, Young Euler advised that the surest means of perfecting the lunar theory lies in simplifying the question as much as possible, and in making abstraction of several circumstances that concur in augmenting the number of inequalities. Thus some earlier theorists had made abstraction of the inclination of the lunar orbit to the Ecliptic and of the eccentricity of the solar orbit. They were thus limiting themselves to determining the inequalities dependent on **m**, the ratio of the Sun's mean motion to the synodic motion of the Moon, and on **e**, the eccentricity of the lunar orbit. (Young Euler fails to mention the inequalities dependent on the ratio of the Moon's to the Sun's parallax.) The general assumption here is that these several kinds of inequality are sufficiently small that they can be calculated independently of one another. Young Euler proposed carrying the simplification one step farther, and limiting his investigation to the inequalities dependent on **m**, that is, to those that may be included under the rubric of "the Variation." He asserted that, if a perfect solution of the problem of the Variation were obtained, hardly any further difficulty would remain in determining the true motion of the Moon.

In the case of the Variation, he obtained by calculations from the differential equations a formula accurate to 14 arc-seconds (it would be easy, he added, to increase the precision). It gives the Variation φ in the Moon's longitude as

$$\varphi = 0.010191 \sin 2\eta - 0.00007 \sin 4\eta ...,$$

where η is the difference between the mean longitudes of the Moon and the Sun.

Euler's *Theoria Motuum Lunae Nova Methoda Pertractata* [E418] of 1772 proposes an elaboration of these same ideas, with the addition of a rectangular coordinate system, of which the x- and y-axes rotate with the mean speed of the Moon so that the x-axis passes through the mean longitude of the Moon. The several inequalities are partitioned into orders and mixed orders, to be calculated separately. They are to be taken up in the following sequence:

(i) The Variation, i.e. the inequalities dependent solely on the mean elongation p of the Moon from the Sun.

(ii) The inequalities dependent on the eccentricity $k = 0.05450$ of the lunar orbit, and on k^2 and k^3.

(iii) The inequalities dependent on the eccentricity $\kappa = 0.01678$ of the solar orbit.

(iv) The parallactic inequalities, depending on a/a, the ratio of the distance of the Moon from the Earth to the distance of the Sun from the Earth, about 1/391.

(v) The reduction to the Ecliptic, dependent on the inclination $i = \sin 5°8'.5$.

There follow then the mixed orders, in which terms proportional, for instance, to κk, κk^2, ik, etc. are to be calculated. It is a *program* of calculation that Euler announces in this treatise; he suggests that the carrying out of it might take a year. The determination of such empirical constants as the eccentricity of the lunar orbit, he points out, is an especially difficult problem.

The question the Paris Academy had posed for its contest of 1772 was the cause of the Moon's secular acceleration, which had been discovered by Halley. Reviewing the several inequalities that the Sun causes in the Moon's motion, Euler found none that could account for a secular acceleration. On the other hand, an aether, such as Euler considered necessary for the propagation of light, would necessarily, Euler believed, be a source of friction in the motion of the celestial bodies; and long before he had demonstrated that planets and satellites subject to such friction would gradually fall into lower orbits with more rapid mean motions.[45]

Euler's program for developing the lunar theory was neglected for a century, but at last embarked upon by George William Hill, an assistant in the U.S. Nautical Almanac Office, in the late 1870s. Hill published a completely numerical solution for the "Variation curve" in 1878, and his calculation of the motion of the lunar apse insofar as it depends on the variation curve in 1877. Like Euler, he used rotating rectangular coordinates, but set the rate of rotation equal to the Sun's mean motion, thus obtaining a re-entrant, periodic orbit. The remaining calculations were carried out by Ernest W. Brown between 1892 and 1907, with the aid of a single (human) computer. Brown used Hill's numerical solution of the Variation curve as an intermediary orbit, but his development of the remainder of the theory was literal, using letters for the constants **m**, **e**, a/a, i. The fitting of the theory to observations was carried out at Yale University between 1908 and 1919; and the Hill-Brown lunar theory became the basis of the lunar ephemerides in the national almanacs beginning in 1923. Both Hill and Brown employed integrals of the equations of motion unknown to Euler. Their lunar theory also differed from the one projected by Euler because of their adoption of the exponential expression for sines and cosines of angles (an Eulerian innovation whose use in astronomy had been promoted by A.L.Cauchy).

[45] [E89] "De relaxatione motus planetarum," *Opuscula varii argumenti*, I, 1746, pp. 245-276, *Opera Omnia* Ser. II, vol. 31, pp. 195-220.

Whereas Euler had proposed seeking a theory accurate to 1 arc-minute, Brown set the goal of a precision of 0.01 arc-second.

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Leonhard Euler: Life, Work and Legacy
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Euler and Indian Astronomy

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1. Introduction: Indian astronomy in the Enlightenment

Even within the immense diversity of technical subjects covered by Leonhard Euler's collected works—ranging from musical dissonances to the construction of microscopes to orbital perturbations—the topic of traditional Sanskrit calendrics stands out as unusual. This was the subject of Euler's 1738 essay "De Indorum anno solari astronomico (On the solar astronomical year of the Indians)", published as an appendix to a treatise by the historian and philologist T. S. Bayer. The present paper discusses the source of this opusculum, and its relation to the Indian astronomical tradition and to Euler's other work.

Indian astronomy began to attract attention in European scholarly and scientific circles in the late seventeenth century, when French envoys to the Kingdom of Siam published descriptions of astronomical methods and data encountered there, which were ultimately derived from Sanskrit sources.[1] "Native" astronomical tables and texts also attracted the notice of researchers and missionaries in India itself, and subsequently of their cor-

[1] Some of these accounts were published in [13] and [9]. See also the references in the "Einleitung" by the editors of [6] in [7], Series II, vol. 30, pp. IX-X.

respondents in Europe. Historians tried to figure out the relationship between these Indian methods and the sciences of classical antiquity, while astronomers and mathematicians analyzed their technical features.

Questions of calendrics and chronology generally inspired the most interest among historians, since they had to be resolved before artefacts and texts could be satisfactorily dated. Astronomers, on the other hand, often hoped for records of ancient observations that could be used to test modern theories of celestial mechanics. Cultural prejudices of various kinds frequently affected interpretations: proponents of "Oriental wisdom" clashed with "classical chauvinists" who maintained the superiority of Greek and Roman civilization, as well as with defenders of biblical chronology. It took well over a century before these disputes were more or less resolved into a commonly accepted view of Indian mathematical astronomy as largely a creation of late antiquity, combining indigenous traditions with some models and units borrowed from Hellenistic Greek astronomy. (In particular, the zodiac and the week, along with the names of their component signs and weekdays, were adopted from Greek sources.)

In the 1730's, however, such controversies were still in their early stages, as historians strove to piece together the elements of the history and languages of the ancient Orient. The story of Euler's encounter with the Indian solar year illustrates several characteristic trends in this struggle for understanding. It also showcases the flexibility and soundness of Euler's mathematical common sense, while simultaneously revealing the limitations of common sense in analyzing texts from an unfamiliar tradition.

2. T. S. Bayer and his work

Theophilus (or Gottlieb) Siegfried Bayer (1694–1738) was one of the most active and ambitious scholars in the field of early Oriental studies, although ultimately not one of the most successful. A German Protestant from Königsberg, Bayer was originally interested in the history of the Eastern churches, a subject that led him to the study of Asia and, by the time he was twenty, to a passion for Sinology in particular. In the quarter-century or so of his short working life, in addition to his attempts at a complete linguistic analysis of Chinese characters, he wrote about South and West Asian languages, Central Asian history, numismatics, calendrical and mathematical traditions in Asia, classical and Church history, and a variety of related subjects.

In 1726 Bayer's fellow Königsberger Christian Goldbach obtained for him the chair of Greek and Roman Antiquities in the new Imperial Academy of Sciences at St. Petersburg; Bayer exchanged this post in 1735 for the professorship of Oriental Antiquities. [2] Up to his death in 1738, and for several years posthumously, his contributions almost singlehandedly represented the humanities or "historical class" in the Academy's journal, the *Commentarii academiae scientiarum imperialis Petropolitanae.* [3] In his researches Bayer, like many other Enlightenment Orientalists, was attempting to trace the contours of a vast historical landscape stretching from Rome to China, with a strong bias in favor of classical civilization as its most important feature. Thus his historical and philological studies combined bits and pieces of research on various cultures, often rather speculatively linked by his interpretations of the resemblances or relationships he perceived among them.

It was therefore typical that when Bayer wrote a monograph on the Hellenistic kingdom of Bactria in Central Asia (*Historia regni Graecorum Bactriani in qua simul Graecarum in India coloniarum vetus memoria explicatur* or "History of the Bactrian kingdom of the Greeks, in which also the ancient history of the Greek colonies in India is explained" [3], completed shortly before his death), he concluded it with a discussion of the relations between ancient Greek and Indian number systems, music, and calendrics. His descriptions of these topics in his table of contents indicate his conclusions about them:

> Whether the names of the Indian numbers [come] from the Romans, or the Persians, [or] indeed, both the Persian and Indian [names] from the Greeks; or rather, from a common stock of languages?... Wisdom of the Indians too much praised; the Indians received the names of numbers, together with arithmetic, from the Greeks; Greek arithmetic compared with Indian... The music of the Indians also seems to be received from the Greeks; the Metonic doctrine of time among the Bactrians, and other peoples both Indian and Chinese; the days of the week of the Indians from the Greeks... [4]

Moreover, Bayer added to the *Historia* two appendices explaining the traditional Indian calendar and timekeeping system. The explanations were derived from the interpretations of certain members of the Protestant mission at the Danish colony of Tranquebar, south of Madras, with whom Bayer regularly corresponded on Indological subjects. The two appendices

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[2] See [10], pp. 17, 20, and [1], pp. 27–28.

[3] The statement to this effect in [1], p. 52, is readily confirmed by the facsimile Tables of Contents for the *Commentarii* reproduced in [5], "Publication". Of the thirty "historical class" articles in ten volumes published between 1729 and 1750, only two were not written by Bayer.

[4] In this and all the subsequent translations from [3], my editorial additions and clarifications are in square brackets.

were written respectively by one of these correspondents, C. T. Walther[5] ("The Indian Doctrine of Time," [3], pp. 145–190, including some notes by Bayer), and by Bayer himself based on his correspondence with Walther and other Tranquebar missionaries ("Supplement (Paralipomena) to the Indian Doctrine of Time," [3], pp. 191–200). Bayer in his preface to the book stresses the importance of calendrical questions, extolling the sixteenth-century antiquarian J. J. Scaliger as the founder of the field of comparative chronology:

> There is no need for me to recommend this [the Supplement] to the intelligent. Joseph Scaliger [was] the first to realize how much this science, abstruse to the people, merited explanation. As this incomparable elder has not withheld even the smallest fragments, [shall] we not make the most of the deeper and fuller [knowledge] that is given into our hands? ([3], f. 3)

His interest in calendrics and astronomy had doubtless strengthened the bond between Bayer and some of his Academy colleagues in the sciences class. He kept up his acquaintance with Goldbach and developed a warm friendship with the French astronomer Joseph-Nicholas Delisle, probably sparked by the Bayers' sharing the Delisles' house for a while on their arrival in St. Petersburg. Although Bayer spoke no French and Delisle no German, the two scholars could communicate in Latin ([10], pp. 19, 21 n. 16).

3. Euler and Bayer

It is not known exactly how or when Bayer became acquainted with the brilliant young mathematician Leonhard Euler, who arrived in St. Petersburg in May 1727, shortly after his twentieth birthday. Probably Goldbach or Delisle, with both of whom Euler worked closely, made them known to each other. Little information survives concerning their interactions at the Academy, but it seems that Bayer occasionally consulted Euler about mathematical problems that cropped up in his historical researches.

A letter[6] from Bayer to Euler in January 1736 (apparently the only surviving correspondence between them) is said to contain a request for Euler's views on "two books, in which reference is made to the computation of the Malabar [Indians], and on the resemblance of the method for

[5] Christoph Theodosius Walther (1699–1741) served at the Tranquebar mission for about twenty years, and composed among other works a grammar of Tamil ([1], pp. 31–32 n. 2).

[6] See [7], Series IV A, vol. 1, p. 15.

calculating with fractions described in these books to that of the ancient Greeks and Romans". This doubtless relates to the comparison of Greek and Indian arithmetic that Bayer later discussed in the 1738 *Historia*.

Euler's own work in his early St. Petersburg years dealt mostly with problems in pure and applied mathematics. The "De Indorum anno", which appeared after Bayer's "Supplement" in the *Historia* ([3], pp. 201–213), was his first published foray into astronomy. [7] It appears to have been inspired by another request from Bayer, who clearly (as will be discussed below) did not understand some of the technical details in the computations he sought to explain. Bayer himself in his preface says merely that "Leonhard Euler, the noted mathematician and most closely linked to me by ties of collegiality, has now shown to some extent how welcome these [discoveries] must be to all" ([3], f. 3).

4. Indian calendrical methods and their representation in the appendices to Bayer's *Historia*

The traditional Indian calendar is luni-solar: that is, it keeps track both of the synodic or "moon-phase" months, like the Muslim calendar, and the solar year, like the Gregorian calendar. Since there are between twelve and thirteen full cycles of moon phases in one solar year, the synodic months over time slip more and more out of phase with the seasons. This is remedied, as in the traditional Jewish calendar, by periodic intercalation, or inserting of a "leap month" into certain years. However, the year of the Indian calendar is sidereal rather than tropical, meaning that it measures the time between successive conjunctions of the sun with a particular star, rather than the time between successive solstices or equinoxes. This particular position with respect to the fixed stars is defined as the zero-point of the sidereal zodiac. (Note that because Indian astronomy is geocentric, the year represents—not just notionally but physically—a complete revolution of the sun in a circular orbit about the stationary earth.)

The first lunar or synodic month of the year is considered to begin at the first new moon before the start of the solar year (although some variants of the calendar start the synodic month at full moon instead). The start of the solar year is also the start of the first solar month, a period measured from the sun's entry into one of the twelve 30° signs of the sidereal zodiac.

[7] I am indebted to Ed Sandifer for pointing this out, and also for drawing my attention to Euler's example of a year-length calculation in the *Introduction in analysin infinitorum*, discussed below.

The Indian calendar tracks not only the sidereal years and synodic and solar months, but also five different sequences of smaller time-units (hence the Sanskrit name for the calendar, "pañcāṅga" or "fivefold"). These are the civil weekday, starting usually at sunrise; the *tithi*, or one-thirtieth of a synodic month; the *karaṇa*, or half-*tithi*;[8] the *nakṣatra*, or "constellation", namely 1/27 of a sidereal month, the time it takes the moon to pass through the circle of 27 designated lunar constellations; and the *yoga*, the time during which the sum of the solar and lunar longitudes increases by the extent of a *nakṣatra* constellation, or $360°/27 = 13°20'$. Most of these time-units have ritual and/or astrological significance. The most important for common practical purposes is the day, which is sexagesimally divided into various smaller time-units.

The methods traditionally employed in Sanskrit astronomical texts for computing the start of a year rely ultimately on period relations involving very long cycles or eras. The mathematical principle is that of simple proportion. Namely, if the era contains an integer number R of solar revolutions or years, as well as an integer number D of civil days, then some given integer number r of years will elapse in the time of d civil days, where

$$\frac{d}{D} = \frac{r}{R}.$$

The "lord of the year", or weekday in which the start of the $(r+1)$th year falls, is then found from $d \mod 7$ plus the appropriate offset corresponding to the weekday of the start of the era, or "epoch" date. The first day of the week is Sunday.

Naturally, the value of the ratio D/R implies a particular length of the sidereal year in days. The size of this year-length, along with the choice of epoch/era, varies from one school of Indian astronomy to another. In the school of Āryabhaṭa, which was especially widespread in southern India, the preferred era was the "Mahāyuga" or "great era", sometimes denoted simply "yuga", containing $4,320,000$ years and $1,577,917,500$ civil days, implying a year-length of 365 days plus a sexagesimal fraction of approximately $0;15,31,15$ day. However, the Āryabhaṭa school generally reckons dates not from the start of the Mahāyuga but rather from the beginning of one of its subdivisions, the "Kaliyuga" or Fourth Age. The initial year of the Kaliyuga is taken to begin on a Friday falling in the year we know as

[8] Although there are consequently sixty *karaṇas* in a synodic month, there are only eleven distinct names for them, seven of which recur eight times each in every synodic month, while the remaining four appear only once. It is understandable that Bayer and his Tranquebar correspondents completely failed to grasp the rather recondite system of the *karaṇas*.

3102 BCE, a date traditionally identified with the legendary battle of the clans in the Sanskrit epic *Mahābhārata.*[9]

Another widely-used epoch in Indian astronomy, and more common in popular date-reckoning than the Kaliyuga epoch, is the start of the so-called "Śaka" era associated with the Śakas or Indo-Scythians, falling in 78 CE. (This era was doubtless especially interesting to Bayer due to its evident connection with the history of ancient Central Asia). Successive years are also frequently identified (particularly in southern India) by their position in what is known as the "Jupiter cycle", a recurring succession of sixty individually-named years. The first Jupiter cycle is held to have commenced after 3588 years of the Kaliyuga.

Walther's and Bayer's remarks explain the fundamentals of Indian time measurement and a basic procedure for finding the initial moment of a given year, insofar as they themselves understood these concepts. Their task was complicated by a number of factors. First, they were attempting to explain features from a variety of related but not identical calendrical systems. Second, they were comparing technical terms from different languages, including Tamil, Sanskrit or Hybrid Sanskrit ("Graenda", from Sanskrit "grantha", "verse" or "book"), and Deccani Urdu. Moreover, they included frequent attempts at comparisons with non-Indian traditions, appealing to sources in Greek, Persian, Hebrew, Chaldaean or Babylonian, and Chinese. It cannot have helped that most of the foreign words were freely and sometimes inconsistently transliterated (although Greek was represented in Greek characters, and Urdu, Persian, and Chaldaean in Arabic/Persian). And finally, they were apparently working from "cookbook" computational manuals and calendars with no access to the underlying models of Āryabhaṭa-school astronomical theory, except what they gleaned from the incomplete and/or imperfectly understood descriptions of the local practitioners with whom Walther and his colleagues studied in Tranquebar.

The following brief overviews describe the main subjects of these appendices and the points where Walther and/or Bayer came to grief in understanding their contents. Most of the comparisons with other chronological traditions are here ignored.

4.1. *Walther's "Indian Doctrine of Time"*

This work is divided into eleven sections, as follows:

I *On minutes.* The second sexagesimal subdivision of the day, equal to 24 seconds in the modern hour-minute-second system.

[9] For more detailed discussions of time measurement in the Āryabhaṭa school of Indian astronomy, see [12] and [4].

II *On hours.* The first sexagesimal subdivision of the day, equal to 24 of our minutes.

III *On days.* Civil days.

IV *On the week.* The seven-day week with weekday names derived from the planets, corresponding to the Hellenistic/modern system.

V *On months.* The solar months and names of zodiacal signs, corresponding to the Hellenistic/modern zodiac. The names of the twelve seasonal (lunar synodic) months, translated as the European month names from Aprilis to Martius. The names of fortnights or half-months.

VI *On the year.* The solar year, described as "common" (with 365 days) or "bissextile" (with 366 days). In fact, no Indian astronomical system, as far as I know, actually intercalates leap **days** instead of the traditional intercalary **months** mentioned above. It seems probable that Walther drew this conclusion from examining tables showing the initial weekdays of successive solar years, and noticing that the consecutive year-beginnings were either 365 or 366 days apart, without understanding the details of how they were calculated. This is supported by Walther's note on the topic:

> The method of computing of the Indians depends on tables devised by their predecessors, not on astronomical observation of that point [in time] at which the sun enters Aries. The beginning of the year occurs as the sun appears in the sign of Aries... The year of the Indians is Julian, into which the Egyptians under Roman domination changed their own Nabonassarean [365-day] year... ([3], p. 167)

Walther lists the initial dates and times of a few recent years in the Indian calendar:

42 .	Kîlaga	1728		1.15′	Tuesday
43 .	Saumja	1729	1 April, midday	16.46′	Saturday
44 .	Sadârana	1730	31 March, night	32.17′	Sunday
45 .	Wirôdigrudu	1731	31 March, night	47.48′	Monday
46 .	Paridâbi	1732	1 April, morning	3.20′	Wednesday

([3], p. 168)

VII *On the sixty-year cycle.* The names of the sixty years in the Jupiter cycle.

VIII *On the great cycle.* The Kaliyuga of 432000 years and the three other parts that make up the Mahāyuga. Bayer includes here a long note on the names and supposed nature of the "constellations" (*nakṣatras*), *karaṇas*, and *yogas*, since he believes them to have something to do with multi-year periods:

> After the 27 constellations the [Telugu Indians] set in their calendar the same number of Yoga and 11 Karana... But it is suffi-

> ciently clear from this calendar that Yoga and Karana are cycles of constellations... Yoga are assigned in their order to a particular day, together with the Indian hour and minute that bound it. Thus it happens that sometimes two Yoga come together in one day. But Karana are set in no such order; I have found no cause for this, as the nature of Karana is not yet investigated...
>
> These Yoga seem to me to indicate Indian periods. For 16000 years or $266\frac{2}{3}$ sexagesimal cycles multiplied by 27 make 432000 years or $7366\frac{2}{3}$ sexagesimal cycles, which is called the Kaliyuga period ([3], pp. 176–178).

IX *On the greatest cycle.* The Mahāyuga.

X *On other cycles and periods.* Indian names of large decimal powers.

XI *On the Indian calendar.* Its name *pañcāṅga* and its five components:

> The delimiters of the festivals are: 1. Tithi, fifteen parts of a half-month. 2. Day of the week. 3. Nakshatra, twenty-seven constellations. 4. Twenty-seven Yoga. 5. Eleven Karana, in astrological notions, deal with the determination of fortunate and unlucky days... ([3], p. 184)

Some translated excerpts and dates from an Indian calendar for 1730 are appended. Walther comments distrustfully on traditional Indian cosmological notions, such as the placement of the moon above the sun, the revolution of celestial bodies about Mount Meru, and the support of the earth and causation of eclipses by a great serpent:

> That Indian tables are not constructed, but received from others, is proved from [their] total ignorance of astronomy... How can those who are so ignorant of the causes of eclipses predict them in calendars, except from tables derived by the work of others? ([3], pp. 189–190)

4.2. *Bayer's "Supplement to the Indian Doctrine of Time"*

These few pages contain excerpts and summaries of letters from Walther and his colleagues, as follows:

– *Letter from Walther, 30 January 1732.* The calculation of the Indian year: an algorithm for finding the day and fraction of a day on which a given Indian solar year begins. The algorithm is not well understood, especially the subtraction of a mysterious constant 1237 (see the discussion in the following section). Walther's translation of a transliterated Sanskrit version of this procedure is as follows:

> 60 is multiplied by 20; they are added to the past year. When 9 plus 400 is added, you have the time of the Śaka kings. When 3179 is added to (this) year, you get the Kaliyuga [year]. The Kaliyuga [year] is multiplied by 365. Add a fourth of the (same) year. Again (this) year is multiplied by 5. 1237 is subtracted. You subtract 576 (per) day multiplied [the Sanskrit is more nearly "divided by 576 and added to the days"]. The sixth weekday [in Sanskrit, "Venus-day"] (is) the beginning of the days... ([3], p. 193)

– *Letter from Walther, 10 January 1735.* Comments about the "new year" algorithm and additional examples of recent year-beginnings from the calendar. Walther remarks:

> Our Indians do not require bissextile years or intercalary months, but fix the hour and minute of the first day in which the beginning of the year occurs. [They] do not readily teach anyone the mystery of their calculations; however, it is certain that they often transfer the start of the year to the following weekday. At any rate their year is greater than the true tropical solar year, whence the start of the Indian year occurs ten days later than ours. But I doubt whether this measure of time was received by the people before about the 600's. Because in the Roman Empire the Julian calendar was received by a wide territory, principally by the Egyptians, from whom I suppose our [Indians] also received it...
>
> [Bayer here remarks in a footnote: "I mistrusted this magnitude of the Indian year. But now see the whole thing fully explained below [by] Leonhard Euler..."]
>
> About the number 1237 I cannot indicate [anything], only it is certain that it is fixed... For [its] general form is employed in the calculation of all years.
>
> [Another footnote from Bayer: "At first I doubted whether this number was fixed, but now see below."] ([3], pp. 196–197)

– *Letter from Nicolaus Dal, Martin Bosse, Christian Friedrich Pressier and Walther, 30 December 1735.* Additional examples of year-beginnings from the calendar.

– *Letters from the same and from Johann Anton Sartorius, 11 January 1735.* Discussion of the lunar months accompanied by a list of months for 1734 containing 29 to 32 days.

4.3. *The calculation for the start of the year*

The "new year" algorithm that puzzled Walther and Bayer is actually fairly simple, but its steps are not entirely obvious. The technique is more or less as follows:

- Convert the desired year in the 60-year Jupiter cycle into the number of elapsed years of the Kaliyuga era.
- Multiply that year-number by the year-length of $365 + \frac{1}{4} + \frac{5}{576}$ days. That is the number of days elapsed between the start of the Kaliyuga and the start of the desired year. [10]
- Subtract $\frac{1237}{576}$ or $2\,\frac{85}{576}$ from the product. This step is less easy to interpret, but the integer part of the constant is probably a zero-point correction. Since the Kaliyuga began on a Friday, subtracting two days from the elapsed total effectively moves the zero-point to the following Sunday, the first day of the week. The source of the accompanying $\frac{85}{576}$ or $0;8,51,15$ of a day is not clear, but it may have something to do with the traditional terrestrial longitude correction. That is, the epoch moment is considered to have occurred at sunrise on the Indian prime meridian passing through Ujjain, which will correspond to a somewhat later time at a location east of the prime meridian, such as Tranquebar. However, the fraction seems too large to be completely accounted for in this manner.
- Divide this modified day-total by 7, and subtract (modulo 7) 1 from the remainder. The reason for subtracting 1 is not evident. Strictly speaking, if we wish to convert a result modulo 7 into one of a sequence of weekday-numbers beginning with 1 (Sunday) rather than zero, we should rather **add** 1 to the result. In effect, subtracting 1 from the result is equivalent to subtracting 2 from the weekday-number. This suggests the abovementioned Friday/Sunday correction for the start of the Kaliyuga—which, however, was already taken care of in the previous step. If this is not in fact an inadvertent repetition of the same epoch-weekday correction (which however is supported by the reference to "the sixth weekday" in Walther's translation of the rule), perhaps it represents an *ad hoc* correction to a weekday discrepancy in this particular calendar.
- A year-beginning falling in the second half of a day is pushed forward to the following weekday.

[10] These two steps, plus the reduction of the result modulo 7, constitute the complete "lord of the year" computation found in Āryabhaṭa-school astronomical works such as [4], verse 1.27.

5. Euler's interpretations in the "De Indorum anno"

Euler's essay [11] is not a comprehensive treatment of all the questions raised in the "Doctrine of Time", but rather intended primarily to clear up certain points, probably those which Bayer specifically asked him about. In 21 numbered paragraphs, Euler deals with the year-length and the difference between the Indian and Gregorian year, the technique for computing the moment of the start of a given year, the "transferring" of that initial moment to the subsequent weekday, and a few other points of interest. He begins with a straightforward analysis of the characteristics of the year, based on the data reported by Walther:

> 1. The Indians do not locate the start of any year, as is the custom with us, at the beginning of some day, but at that moment of time in which they consider the sun to arrive at a certain fixed point. But whether this point is the beginning of the sign Aries or instead the beginning of the constellation Aries is not sufficiently clear from the description, in which they say the year begins at that moment in which the sun enters Aries. But this endpoint may be defined not only from the amount of the year, but also from the beginning of some assigned year.
>
> 2. As for the length of this astronomical year of the Indians, it can be deduced from the beginnings of the years 1728, '29, '30, '31, '32 included on page 168. Certainly, by the calculation accepted among the Indians, in which they divide the day into 60 hours, the hour into 60 minutes, the minute into 60 seconds, it is clear enough that their year contains 365 days, 15 hours, 31 minutes and 15 seconds; which quantity, according to our manner of dividing time, produces 365 days, 6 hours, 12 minutes and 30 seconds.
>
> 3. If this quantity is reduced to days and parts of a day, the year of the Indians will be found to contain $365 + 1/4 + 5/576$ days, which quantity is thoroughly understood to have been accepted in their astronomical tables, from the method described on page 194 for computing the beginning of any year. The stated divisions by the number 576, which are often repeated, clearly proclaim it. This will appear more clearly hereafter, when I have derived from this length of the year the same rule of stated calculations that the Indians make use of.
>
> 4. Therefore the year of the Indians exceeds our tropical year comprising 365 days, 5 hours, $48'$, $57''$, and the excess is $23'$, $33''$...
>
> 6. The year of the Indians agrees accurately enough with our sidereal year, in which the sun returns to the same point in the heavens

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[11] See [6]. The translation there cited is the source of all the quotations from the "De Indorum anno" in the present paper.

with respect to the fixed stars, the length of which year is put by the astronomers at 365 days, 6 hours, 10 minutes, so that the year of the Indians differs from this year only by 2 minutes. This small error, owing to the inadequacy of observations which were retained [though] very ancient, is easy to forgive...

8. Since now the length of the year which the Indians accepted is known, it will be easy to determine from the given beginning of some year the beginning of the following one, by adding to the beginning of the elapsed year one weekday, 15 [sexagesimal] hours, 31′, 15″. So if the beginning of the year 1731 by the defined calculation is weekday 2, hour 47, 48′, 45″, the beginning of the following year must fall on weekday 4, hour 3, 20′, 0″.

9. Therefore by this rule those same beginnings of all years are easily found, which would be extracted by laborious calculation done in the prescribed manner...

10. The first day of the year according to the Indians is always the first day of the month April, but this [does] not always coincide with the weekday in which the beginning of the year occurs; but if the beginning of the year is celebrated in the daytime, they consider that same day (but if the beginning of the year happens [to fall] in the night, then the following day) as the first day of April...

Euler lists sample year-beginnings computed in this trivially simple way, and notes that they agree very well with the examples quoted from actual Indian calendars (see above). He then demonstrates the agreement of this method with the "mysterious" year-beginning algorithm quoted in Bayer's "Supplement". (He has evidently deduced the correct year-length partly from the constants in this algorithm as well as from Walther's calendar examples, since the latter show only two sexagesimal places.) Euler's explanation of the zero-point correction(s) is not entirely clear, but he obviously grasps the basic fact that they must relate to a time offset at the start of the Kaliyuga. After working an example, he notes some of the ways in which the algorithm could be simplified.

13. Although the beginning of any year can be found easily by the method used above, yet the rule described on p. 194 not only is not to be despised, but is very useful for [finding] the given beginning of any elapsed year. Also, just as a calculation of this sort must be undertaken with a specified year whose beginning is known, so in that rule the first year of the Kaliyuga is used, whose beginning should have occurred in the third weekday, hour 51, minute 8, second 45 by the Indian measure. Whence originates the number 1237, which is employed in the calculation.

14. Therefore for any proposed year, of which the beginning is to be

determined, first of all the time elapsed from the beginning of the first year of the Kaliyuga up to the proposed year is to be investigated, which is made by adding to the past year of the sexagesimal era the number of years elapsed from [the beginning of] the Śaka era up to the beginning of the sexagesimal era in which is the proposed year; if 3179 years are added to which, [there] results the interval of elapsed time from the first year of the Kaliyuga to the proposed year.

15. If the number of years so determined is multiplied by the amount of the year in days, namely $365 + 1/4 + 5/576$, there comes out the number of days elapsed from the first year of the Kaliyuga, from which number—because the beginning of the first year of the Kaliyuga occurs not in the beginning of the first day, but in hour 51, $8'$, $45''$ of it, and because the required weekday is sought not from the first, but from the sixth—the number $2\frac{85}{576}$ or $\frac{1237}{576}$ should be subtracted. Therefore if this sum is divided by 7 (or into sevenths), [and] however many can be made are subtracted, the integer part of the remainder will give the weekday in which the beginning of the proposed year occurs, counted from the sixth weekday. But the fractional part, converted to sexagesimal parts, will give first the hour, then the minute (first as well as second [part]) in which the beginning of the year occurs. So this calculation in accordance with the arithmetic rules most accurately agrees with the method of the Indians [previously] described, and hence the reasoning of the whole stated procedure is understood. But since from the rule as it is described it may hardly be clear in what way the fractions should be handled, we will demonstrate the matter by an example.

16. Therefore, let it be proposed to investigate the beginning of the present year 1736 after the manner of the Indians. Therefore the [position in the] cycle of the past year will be 49. And the calculation will be as follows:

60	
20	
1200	
49	
1249	past year of the Cycle
409	
1658	Śaka era
3179	

	4837	Kaliyuga by $365 + \frac{1}{4} + \frac{5}{576}$
	365	
	1765505	
	$1209\ \frac{1}{4}$	$\frac{1}{4}$ Kaliyuga
	$1766714\ \frac{1}{4}$	
	4837	
	5	
	24185	
	1237	by the rule
	22948	
576 }	$39\ \frac{484}{576}$	

Therefore $1766754\frac{52}{576}$ is the number of elapsed days; [when it is] divided by 7, the remainder will be $3\frac{52}{576} = 3\frac{13}{144}$ weekdays. So the beginning of the year occurs in weekday 3 counted from the sixth, that is, in weekday 2. And the fraction gives 5 hours and 25 minutes of that weekday, just as we stated above.

17. And this calculation can be rendered more easy and more brief in several ways; namely, where it should be multiplied by 365, multiplication by unity can be substituted in place of it, since 364 can be divided by 7. Then if the numbers to be divided by 576 have a common divisor with 576, the calculation can also be made easier by reducing fractions. But these are of no great moment. Moreover, if 1813 is subtracted in place of 1237, then the weekday in which the beginning of the year occurs will be immediately obtained.

Euler then undertakes to explain some other points that had apparently confused Bayer, namely the unequal length of calendar months and the nature of the Yoga time-unit. Again, Euler's confidence in the rationality of the underlying mathematics somewhat outruns his specialized knowledge: he interprets "months" as solar months *tout court*, and deduces the *yoga* to be 1/27 of a sidereal month (which is actually, as noted above, the definition of a *nakṣatra*). Finally, he illustrates his previously proposed simplification of the "new year" algorithm.

18. As to [what] pertains to the months of this Indian solar year, the number of days which is allotted to individual months does not seem to

me at all to be assigned at whim. For the Indians have as a month the space of time in which the sun traverses a twelfth part of the ecliptic, so that the length of a month depends on the speed of the sun. Therefore, since the sun progresses more slowly in the summer than in the winter, it is no wonder that the Indians make their summer months longer than the winter [ones]...

20. Therefore, since from the inequality of the months it is established [that] the inequality of the motion of the sun is not unknown to the Indians, it would be worthwhile to know what sort of table of solar equation they use, which, however it may be, will not be much different from our tables.

21. What the Yoga or 27 constellations of the zodiac are to the Indians also does not seem obscure to me. For from these constellations they form a month of the fourth type: this is clearly enough to be understood [as] periodic lunar months, which are completed in about 27 days. Wherefore, since the moon takes 27 days to go around the zodiac, one Yoga is seen to be a twenty-seventh part of the zodiac and an assemblage of stars existing in a space of this sort is without doubt one such constellation of which 27 are numbered in the zodiac. Consequently, when they allot these Yogas in the calendars, without doubt they wish to indicate by them the part of the zodiac where the moon is on some day; and since the moon sometimes in one day can enter into two constellations of this sort, it is no wonder if sometimes two such constellations are found written in one day.

Rule for computing the beginning of any year

First, the number of the sixty[-year cycle] in which the given year occurs, is multiplied by $5\frac{25}{48}$ and to the product is added $3\frac{71}{96}$. Then the cycle of the sought year is multiplied by $1 + 1/4 + 5/576$ and the product is added to the former; when this is done, the sum is divided by 7, and the remainder will indicate the weekday together with the hours and minutes in which time the beginning of the year occurs...

Euler's proposed "rule" is a pleasing specimen of arithmetic ingenuity, which can be explained as follows. If the number of elapsed sixty-year Jupiter cycles is c, and the current year is the yth year of the current cycle, then the original "new year" algorithm instructs us to compute the weekday w of the start of the current year as follows:

$$w = \left[(60c + 3588 + (y-1))\left(365 + \frac{1}{4} + \frac{5}{576}\right) - \frac{1237}{576} - 1\right] \mod 7.$$

These terms are just rearranged and then simplified by eliminating common factors and multiples of 7, as Euler suggested earlier, to make the calculation "more easy and more brief":

$$\begin{aligned} w = \Bigg[& 60c\left(365+\frac{149}{576}\right)+y\left(365+\frac{149}{576}\right) \\ & +(3588-1)\left(365+\frac{149}{576}\right)-\frac{1813}{576}\Bigg] \quad \text{mod } 7 \\ = \Bigg[& c\left(5+\frac{25}{48}\right)+y\left(1+\frac{149}{576}\right)+\left(3+\frac{71}{96}\right)\Bigg] \quad \text{mod } 7. \end{aligned}$$

To sum up, although Euler cannot have found these questions about the Indian calendar very interesting technically, his keen "mathematical nose" served him well in detecting simple, logical rationales for rather cryptically expressed concepts and rules. Occasionally his reliance on simplicity and logic led him slightly astray from the true explanations, which however he could not be expected to deduce from the limited information available to him concerning Sanskrit astronomy.

6. The impact of Euler's work

The "De Indorum anno", like the other Indological parts of Bayer's *Historia*, does not seem to have had any significant influence on the later historiography of Indian astronomy. Nor did it inspire Euler himself to explore the subject further, despite his passing expression of interest in Indian parameters such as the solar equation. Probably if Bayer had lived, he would have persuaded Euler to explain more such questions to him, although it is doubtful if the results would have been very useful without better comprehension of the sources on the Indological side.

In Euler's 1748 *Introductio in analysin infinitorum* (Introduction to the analysis of the infinite), the relation between the calendar and the year-length resurfaces in an example on the use of continued fractions:

> To express the ratio of the day to the mean solar year in the smallest numbers as closely as possible. Since this year is 365d, 5h, 48′, 55″, one year contains in fraction 365 $\frac{20935}{86400}$ days. Therefore the task is to expand this fraction, which will give the following [continued-fraction] quotients: 4, 7, 1, 6, 1, 2, 2, 4, from which these fractions are derived:

$\frac{0}{1}, \frac{1}{4}, \frac{7}{29}, \frac{8}{33}, \frac{55}{227}, \frac{63}{260}, \frac{181}{747}$, etc. Therefore the hours with [their] first and second parts, that exceed 365 days, in four years make about one day, whence the Julian calendar has its origin. But more exactly, 8 days are completed in 33 years, or in 747 years 181 days; whence [it] follows [that there are] 97 excess days in four hundred years. Therefore, since the Julian calendar adds 100 days [during] this interval, the Gregorian [calendar] converts in four centuries three bissextile years into common [ones].[12]

This example appears to be the only direct legacy in Euler's later work of his investigations into Indian astronomy. However, his collaboration in Bayer's chronological researches may have made him more susceptible to the "ancient data" fever so prevalent among eighteenth-century astronomers. More than ten years after Bayer's death, in a couple of letters to Kaspar Wettstein, Euler expressed his hopes and doubts about early astronomical observations—but this time from the Arabic tradition, not the Indian:

> [28 June 1749:] ...Monsieur le Monnier writes to me, that there is, at Leyden, an Arabic manuscript of Ibn-Jounis [ibn Yūnis]... which contains a History of Astronomical Observations... I am very impatient to see such a work which contains observations, that are not so old as those recorded by Ptolemy. For having carefully examined the modern observations of the sun with those of some centuries past, although I have not gone farther back than the fifteenth century, in which I have found Walther's observations made at Nuremberg: yet I have observed that the motion of the sun (or of the earth) is sensibly accelerated since that time; so that the years are shorter at present than formerly: the reason of which is very natural... [T]he effect of this resistance will gradually bring the planets nearer and nearer the sun; and as their orbits thereby become less, their periodical times will also be diminished... This then is a proof, purely physical, that the world, in its present state, must have had a beginning, and must have an end...
>
> [27 September 1749:] ...But I will be quite annoyed if M. Lemonnier is wrong about the contents of the Arabic manuscript that he told me about: because I have been counting on the observations that I would find there. I see that nowadays there is a great deal of work done on publishing the works of the ancient Arabs, but I would infinitely prefer works where they have given a detailed description of their observations to ones that contain only their conclusions. Of this latter type are their catalogues of fixed stars which are almost entirely useless to us, since the

[12] [5], document E101, part 1, ch. 18 ("On continued fractions"), Example 2, p. 320.

very observations from which they determined the places of the stars are unknown to us...[13]

Like many of his contemporaries, Euler recognized the great potential of early observational records (if sufficiently precise) for testing the accuracy of physical theories, and their cosmological implications, over long periods of time. And like them, he ultimately gave up on the prospect of data recovery from ancient texts, finding more scope among the moderns in the great eighteenth-century explosion of theory and experiment in applied mathematics.

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[13] [8], vol. 3, pp. 282–283; the two letters are respectively 2763 and 2764 in the index of Euler's correspondence, [7], Series 4 A, vol. 1, p. 458. The excerpt from the first letter is taken from E183, an English translation of a larger excerpt, available on-line at [5]. The excerpt from the second letter is my translation of the original French.

pour calculer les mouvements du Soleil et de la Lune. Traduit du siamois et depuis examinées et expliquées par M. Cassini [On the kingdom of Siam, vol. II. Names of days, months and years of the Siamese. Rules of Siamese astronomy for calculating the movements of the sun and the moon. Translated from the Siamese and then examined and explained by M. Cassini], Paris, 1691.

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Leonhard Euler: Life, Work and Legacy
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Euler and Kinematics

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1. Introduction

1.1. *Modern kinematics*

Ampère[1] introduced the name "kinematics" ("cinématique" in French) in 1834 in an essay in which he proposed a new classification of the sciences[2]. He defined kinematics as the sub-discipline of mechanics that deals with the properties of motion without taking the causes of the motion into consideration, in other words without considering masses and forces. In the course of the 19th century, kinematics became a coherent research area, encompassing the geometrical properties of motion, but also properties concerning velocities and accelerations. Kinematics was born in an area where mechanics, geometry and mechanical engineering meet. Nowadays, in particular in the science of machines and mechanisms, including the mechanical engineering part of robotics, the subject is still of considerable interest. Euler's contributions to kinematics belong to the pre-history of the subject. In order to put these contributions in perpective, I will first introduce some terminology and at the same time mention some fundamental results.

[1] A.-M. Ampère, *Essai sur la philosophie des sciences*, Paris, 1834.
[2] A classic modern text on kinematics is: O. Bottema, B. Roth, *Theoretical Kinematics*, North Holland Publishers, Amsterdam, 1979. For applications to design, see for example J. Michael McCarthy, *Geometric Design of Linkages*, Springer, 2000.

In *planar Euclidean kinematics* one studies in general coinciding Euclidean planes moving with respect to each other. In *spherical kinematics* one considers the motion of coinciding Euclidean spaces with a common fixed point. Often one restricts oneself in this case to the motion of coinciding spherical surfaces with respect to each other. Finally in *spatial Euclidean kinematics* one studies coinciding Euclidean spaces moving with respect to each other. Some important results are the following. In planar Euclidean kinematics, if we consider one Euclidean plane moving with respect to another, at each particular moment the motion is either an instantaneous translation or an instantaneous rotation.[3] If it is a rotation, its center is called the *instantaneous center of rotation* or *pole*. The analogous result in spherical kinematics says that at each instant the motion is an instantaneous rotation about an *axis of rotation, instantaneous.*[4] And in spatial kinematics in general at each instant there is an *instantaneous screw axis*[5] : the motion of two spaces with respect to each other is a combination of a translation along an axis and a rotation about this same axis.

If we exclude instantaneous translation, during a planar motion of a plane, considered as moving, with respect to another plane, considered as fixed, the instantaneous center of rotation describes curves in both planes. The locus of positions of the pole in the fixed plane is called the *fixed polhode*, while the locus of positions in the moving plane is called the *moving polhode*. During the motion, the moving polhode rolls without slipping on the fixed polhode. This means that in general planar motion can be defined by means of two curves rolling without slipping on each other. In spherical and spatial kinematics we have analogous results. In spherical kinematics motion in general can be defined by means of two (in general, non-circular) cones, called the *fixed and moving polhodes*, rolling without slipping on each other. In space, in general a motion can be defined by means of two ruled surfaces, the *fixed and moving axodes*, moving with respect to each other in a combination of rolling and slipping.[6]

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[3] Discovered by Johann Bernoulli. Cf. Johann Bernoulli, De centro spontaneo rotationis, In J. Bernoulli, *Opera omnia*, Tom. IV, Lausanne und Genf, 1742 (Nachdruck Hildesheim, 1968).

[4] Discovered by D'Alembert and Euler. See below.

[5] Discovered by Giulio Mozzi. Cf. Giulio Mozzi, *Discorso matematica sopra il rotamento momentaneo dei corpi*, Stamperia di Donato Campo, Napoli, 1763.

[6] Presumably the first to study this aspect of motion in a plane and in space (with or without a fixed point) was A.-L. Cauchy, who published in 1827: Sur les mouvements que peut prendre un système invariable, libre, ou assujetti à certaines conditions, Reprinted in *Oeuvres* IIe Série, Tome VII, pp. 94-120, Paris, 1899. The theorems are part of a complex of results that we owe mainly to French mathematicians like M. Chasles, L. Poinsot and O. Rodrigues.

Instantaneous kinematics deals with the properties of motion at a particular instant. Zero order properties concern positions, first order properties concern tangents and velocities and second order properties concern accelerations and radii of curvature of trajectories in the position under consideration. *Continuous kinematics* deals with complete motions. There is also a branch of kinematics called *discrete kinematics* in which one considers 2, 3, 4 or more discrete positions of a moving plane, moving sphere or moving space. A fundamental result in discrete spherical kinematics is that given two positions, one can always move from one of the two positions to the other one by means of a rotation about a unique axis of rotation that is determined by the two positions. Excluding translations, in the plane and in space two positions define analogously an axis of rotation and a screw axis, respectively. Discrete kinematics is useful in the design of machinery that should move a machine element through a number of predetermined positions.

1.2. *Euler and kinematics*

Before the middle of the 18^{th} century rigid body dynamics did not exist. There were no good methods to study the motion of a ship or the motion of the earth about their centers of gravity. Euler was a major contributor in this respect and his contributions to spherical and spatial kinematics are intimately connected to this work on dynamics. As for his contributions to spherical and spatial kinematics, he was the first to use perpendicular Cartesian coordinate systems to describe the motion of a rigid body in space. He introduced the so-called Euler angles and he showed the existence of an instantaneous axis of rotation in spherical kinematics. He was also the first to prove the existence of the axis of rotation in discrete two position spherical kinematics.

The background of Euler's contributions to kinematics is twofold. First of all Euler wrote two papers on the ideal shape of gear teeth. Secondly Euler contributed to spherical and spatial kinematics.

The papers on gear wheels are part of a development that started essentially with the investigation of the ordinary *cycloid*: the curve described by a point on the circumference of a circle when this circle rolls without slipping on a straight line. The curve offered challenging problems to mathematicians,[7] on the one hand, and, on the other hand, in the course of time the curve turned out to possess physical significance. This is im-

[7] Well known is Pascal's 1658 challenge to the mathematicians of his time. Using the pseudonym Amos Dettonville and offering 600 francs he proposed six problems on the cycloid that he had recently solved himself.

mediately clear if one considers some aspects of Huygens' work. In 1673 Christiaan Huygens (1629-1695) published his *Horologium Oscillatorium.* In the third part of the book Huygens introduced his theory of evolutes by means of which he proved that if the bob of a pendulum moves between two cycloidal-shaped plates, the bob is forced to move along an inverted cycloid and ideally will keep time uniformly, no matter how wide it swings. Huygens is the father of the theory of curvature of planar curves. The theory was born out of the concept of two curves related by "unrolling". It led to the notions of involute and evolute. The easiest way to define the relationship between evolute and involute is mechanical and visible in his isochronous pendulum clock: Let the evolute be given. Fit a thread to the shape of the given evolute and then unwind the thread from one end keeping the thread always pulled taut. The end of the thread then describes the involute. Huygens also considered the inverse relationship. He derives the evolute from a given involute as follows. If P and Q are two infinitesimally close points on a given involute, the lines perpendicular to the tangents in P and Q intersect in a point on the evolute. From a modern point of view the evolute is the set of all centers of curvature of the involute. The first major contribution to the geometry of gear wheels came from France. Charles E. L. Camus (1699-1768) published "Sur la figure des dents des roues et des ailes des pignons, pour render les horloges plus parfaits", *Histoire de l'Académie royale des sciences*, Paris, 1733, in which he showed that in order to get as output a uniform angular velocity from a uniform input angular velocity it is necessary that the shapes of the two teeth are such that they can be generated like epicycloids by rolling one and the same curve on two different circles. Euler's papers on the ideal shape of gear teeth are part of this development. Euler discovered an expression for the relationship that is nowadays called the *Euler-Savary formula*, a result concerning radii of curvature in instantaneous planar kinematics. It is remarkable that although Euler was merely studying a very specific subject, gear wheels, the Euler-Savary formula belongs from a modern point of view to planar theoretical kinematics and has general validity. Euler in this context also discovered so-called involute gearing, nowadays the most popular form of gearing.

2. Euler and instantaneous planar kinematics

2.1. *The Euler-Savary-formula*

If we have a planar motion at a particular instant, a modern kinematician thinks of a particular position of a moving polhode with respect to a fixed polhode. The point where the two curves touch each other is the instantaneous center of rotation or pole. Clearly an arbitrary point P of the moving plane describes a curve in the fixed plane. At the particular moment under consideration P coincides with a particular point of the curve that it describes. The tangent to the curve in this particular point can be constructed easily by means of the pole. How about the center of curvature in this particular point? Nineteenth century kinematicians have extensively studied the relation between the points of the moving plane and the corresponding centers of curvature of their trajectories in the fixed plane. This particular relation has many properties.

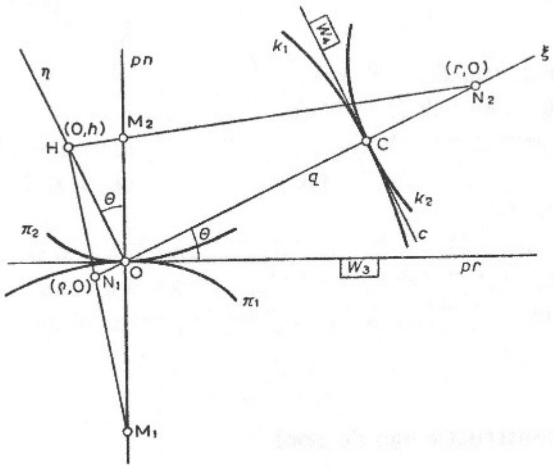

Fig. 1.

Consider Figure 1, which represents the situation at a particular moment. The fixed polhode is π_1. The moving polhode is π_2. The point O at which the two polhodes touch each other is the instantaneous center of rotation or pole at the moment that we are considering. k_2 is a curve in the moving plane. k_1 is the envelope in the fixed plane of the set of positions in the fixed plane of k_2. In the position that we are considering k_1 and k_2 touch in the point C. The points N_1, N_2, M_1 and M_2 are, respectively the centers of curvature of k_1, k_2, π_1 and π_2 corresponding to the points C and O. Let θ be the angle between the common tangent to the polhodes and the common perpendicular in C to k_1 and k_2. Then we have, in general, the following lovely relation:

$$\left(\frac{1}{ON_1} - \frac{1}{ON_2}\right) \cdot \sin\theta = \frac{1}{OM_1} - \frac{1}{OM_2} \tag{1}$$

This is the *Euler-Savary formula* or theorem. The variables ON_1, ON_2, OM_1 and OM_2 correspond to directed line segments; they have a sign. The pole, O, is the origin of a Cartesian coordinate system with pr as positive x-axis and pn as positive y-axis. Similarly O is also the origin of a Cartesian coordinate system $O\xi\eta$ with directed line segment OC defining the positive direction of the ξ-axis. The two systems have the same orientation. As for the signs of the variables in the Euler-Savary formula, ON_i is positive if moving from O to N_i is a move in the direction of the ξ-axis. OM_i is positive if moving from O to M_i is a move in the direction of the x-axis.

A modern proof of the Euler-Savary formula was given in 1970 by G. R. Veldkamp.[8]

2.2. *Euler and the Euler-Savary formula*

In Euler's first paper on gears, De aptissima figura rotarum dentibus tribuenda (E249 in *Opera omnia* II, 17, pp. 119-135), written in the first half of the 1750s, he phrased the condition that if the input rotation is uniform, the output rotation should be uniform as well. He proved that this condition implies that friction is inevitable. As for the shape of the teeth, Euler in this paper did not succeed in going beyond what Camus had already done: for example, if the teeth of one wheel are straight lines through the center of the wheel, the teeth of the other wheel should consist of an arc of an epicycloid. However, Euler's paper Supplementum de figura dentium rotarum (E330, *Opera omnia*, 17, pp. 196-219), written presumably ten years later is very original. In this paper Euler gave a formula that is equivalent with (1). Because Blanc and Haller in their preface to *Opera omnia* II, 17 quite extensively describe the contents of this particular paper, I will restrict myself to some supplementary remarks.

Euler did not study general planar motion at a particular instant; he studied the form of the teeth of gear wheels. The general validity of the formula that he discovered is an accidental spin-off of his research. This arises because, in general, just as for for first and second order properties,

[8] G. R. Veldkamp. *Kinematica*, Scheltema & Holkema, Amsterdam, 1970, pp. 70-72.

a planar motion at a particular instant can be represented by a circle rolling without slipping on another circle. This is exactly what we are dealing with when we have planar circular gear wheels satisfying Euler's condition of a constant velocity ratio.

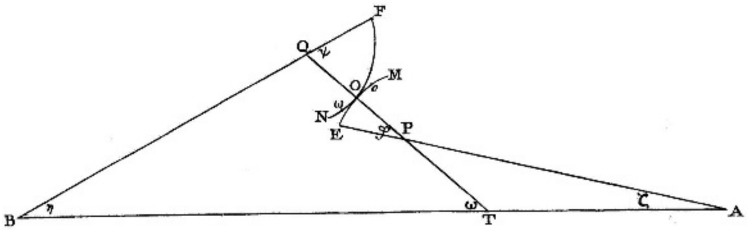

Fig. 2. $BT = a$, $AT = b$, $a + b = c$, $AP = p$, $PO = r$.

Euler started with Figure 2. The points A and B are the centers of the two wheels. EOM and FON are the two profiles of the teeth of the wheels. O is the point where the two profiles touch and the line perpendicular to the tangent in O cuts AB in the point T. When the gear wheels are functioning, a moment M_A about A yields a moment M_B about B. It is easy to see that at the instant under consideration the ratio of these two moments equals BT/AT. Euler argues that the condition of a constant velocity ratio implies that the ratio of these two moments must be constant, which leads to a kinematical result: the common normal in the point where the profiles touch each other intersects AB in a fixed point T. From a modern point of view T is the pole of the motion of the two gear wheels with respect to each other. The two polhodes of the motion of the two gear wheels with respect to each other are two circles, one with center A and one with center B. The two circles touch in T. Clearly Euler proved a kinematical result by means of a dynamical argument. This happened more often in the 18^{th} century. For example, Johann Bernoulli's discovery of the instantaneous center of rotation for planar motion occurred as follows. He discovered that after a rigid body in the plane is hit by an impulsive force of which the line of action does not pass through the center of gravity, the velocity distribution corresponds to a rotation about, as he said, "the spontaneous center of rotation." [9] So here as well a kinematical result was obtained by means of a dynamical argument.

After having established that the point T is fixed, Euler determined several relations between the parameters depicted in Figure 2 and differentiated. He basically considered a slight change in the position of the two

[9] De centro spontaneo rotationis, In J. Bernoulli, *Opera omnia*, Tom. IV, Lausanne, 1742.

profiles with respect to each other, using the fact that the common normal intersects AB always in the fixed point T. After some calculations this yields that $\frac{d\eta}{d\xi}$, the ratio of the angular velocities, is equal to $\frac{TA}{TB}$. Euler then derives the following expression that enabled him, in principle, to calculate in an arbitrary position the radius of curvature ρ' of profile NOM out of the parameters of profile EOM.

$$\rho' = c \cdot \cos\omega - r - p \cdot \cos\varphi - \frac{b^2 \cos\omega \cdot d(p \sin\varphi)}{c \cdot d(p \cdot \sin\phi) - a^2 d\varphi \cos\omega} \qquad (2)$$

N. B. As for the parameters, see Figure 2. In a particular position we can assume, without loss of generality, that profile EOM is a circle and that the center of curvature of profile NOM coincides with Q. Then $dp = 0$ and $\rho' = OQ$. If we, moreover, introduce the footpoints R and S of the perpendiculars from, repectively A and B, on the line PQ (See Figure 3), it is not very difficult to show that (2) implies

$$RT \cdot SQ \cdot TP + ST \cdot RP \cdot TQ = 0 \text{ or } \frac{RT \cdot TP}{RP} + \frac{ST \cdot TQ}{SQ} = 0 \qquad (3)$$

These are Euler's versions of the Euler-Savary formula, accompanied by a construction that enables the graphical determination of the center of curvature Q of NOM if the center of curvature p of EOM is given. It is not very difficult to show that (3) is equivalent to (1).[10]

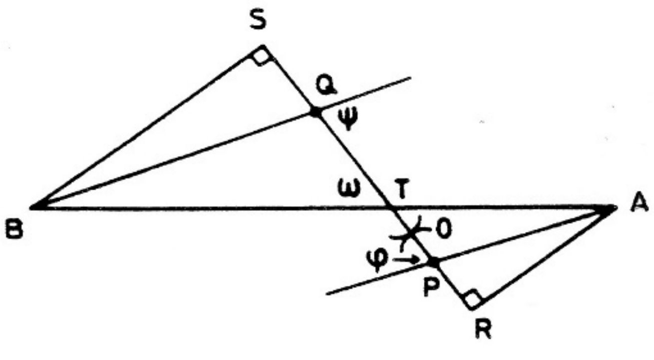

Fig. 3.

[10] Félix Savary (1779-1841) was the first to derive the Euler-Savary formula in its modern form. Savary's proof can be found in *Leçons et cours autographiés, Notes sur les machines*, par le professeur F. Savary, Ecole Polytechnique 1835-36 (unpublished lecture notes; available in the Bibliothèque Nationale in Paris).

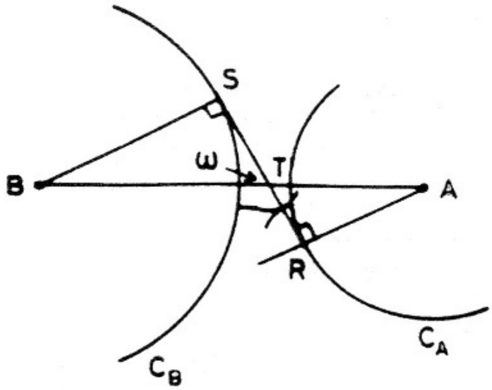

Fig. 4.

The formula has an amazing interpretation. It turns out that when p coincides with R, then Q coincides with S. And naturally Euler considered the possibility that this is the case during the entire motion. The profiles then are involutes of the circles C_B and C_A. See Figure 4. At this moment Euler discovered involute gearing.

3. Euler, acceleration, spherical and spatial kinematics

3.1. *Introduction*

Rigid body dynamics in space is essentially more difficult than the dynamics of planar systems. In the middle of the 18$^{\text{th}}$ century Euler was struggling to develop a satisfactory theory. And while he did so, he initiated the study of the different ways in which one can represent the position of a rigid body in space. Because the problems are difficult, the choice of the most suitable representation is crucial. One of Euler's first contributions, in E112, from 1749, was the idea to describe the moving body with respect to a rectangular Cartesian coordinate system. This was not a trivial move; earlier Euler had used intrinsic coordinates to deal with the trajectory of a moving point mass.[11] The idea to introduce another Cartesian coordinate

[11] Much later in the introduction to his book on the motion of rigid bodies, published in 1765, Euler wrote about the use of Cartesian coordinates: "These three velocities, that we attribute in our mind to the moving point, will make the whole calculation much easier, and because I have not used this means in my previous books on mechanics, I ran into very complicated calculations there" "Hae enim ternae celeritates, mente saltem

system, which moves with the body, came somewhat later. First, in his treatment of the motion of a body, Euler tried to exploit the kinematical discovery that a body moving about a fixed point possesses an instantaneous axis of rotation. However, after the introduction of two Cartesian frames of reference, coordinate transformations drew his attention. At this point his research in dynamics possesses overlap with algebraic problems. Faced by very complicated equations and calculations Euler first seems to have believed that the problem was in finding the best geometrical representation of the motion. In other words: doing the kinematics in the right way can reduce the complications. Yet, the discovery of the principal axes of inertia was in this respect just as important. Below I will discuss Euler's contributions to kinematics related to his work on rigid body dynamics in the following way. After a section on the notion of acceleration, I discuss the first occurrence, completely independent of mechanics, of the so-called Euler-angles. Euler used them in an investigation of the way in which the equation of a surface depends on the choice of the coordinate system. Then I discuss the separation of the motion of the center of gravity of a body and the motion about the center of gravity. After this I discuss Euler's work on the instantaneous axis of rotation and his different attempts to give the equations of motion of a rigid body a handy form.

3.2. *Acceleration: Euler and Newton's second law*

The Greeks did not possess the notion of non-uniform instantaneous velocity. In a development starting with the Merton College group in the 14^{th} century the notion of instantaneous velocity was formed. Casali and Oresme represented instantaneous velocity geometrically by means of line segments and Galilei applied the geometrical representation to his famous analysis of the uniformly accelerated movement of a falling object under the influence of gravity. In this analysis the velocity of the falling object in a vacuum is a linear function of time, which immediately leads to the parabolic trajectory of a bullet. In the 17^{th} century several other authors studied other examples of retarded or accelerated motion. Torricelli, for example, studied the trajectory of an object that possesses a constant horizontal velocity and vertically a velocity that is a quadratic function of time. And although Huygens, Newton and others studied difficult problems concerning the motion of points in a resisting medium, it took quite

puncto mobili tributae, totum negotium expedient; quo subsidio cum non sim usus in superioribus de Mecanica libris, in nimis intricatos calculos sum delapsus." p. 23 in E289, *Theoria motus corporum solidorum seu rigidorum* (Theory of the motion of solid or rigid bodies), first published in 1765 (*Opera omnia* II, 3).

some time before acceleration acquired a status comparable to the status of velocity. For example, in 1700, Varignon, in a paper [12] in which he applied the differential calculus to linear motion distinguished force, velocity, distance and time, denoted by y,v, x and t, respectively. Acceleration is not mentioned. Why? Varignon gives two general rules for linear motion.

Rule 1: $v = \frac{dx}{dt}$ and Rule 2: $y = \frac{dv}{dt}$

Because Varignon equates what we nowadays call acceleration with force he does not need a separate notion of acceleration. [13] The same happens in D'Alembert's *Traité de dynamique* of 1743. The "accelerating force" ϕ is defined as follows: $\phi dt = du$, where u and t are velocity and time respectively. And also when Euler in 1736 published his two volume *Mechanica sive motus scientia analytice exposita* (Mechanics or the science of motion exposed in an analytical way) on the motion of a moving mass point, he expressed a similar view: a force can be characterized by the change that it brings about in the motion of a point. [14]

If the motion of only one mass point is considered, we do not need next to the notion of force a separate notion of mass. We can apply Varignon's formula. However, as soon as we consider several masses simultaneously, we must be able to distingush them and we can no longer "hide" them in a notion of force. So instantaneous accceleration only became a notion that was clearly separated from the notion of force when rigid body dynamics was being developed. Euler played a crucial role in this development.

The early forms of Newton's law can only be understood if one realizes how the early physicists were measuring the different quantities that occur in their equations. Time was measured by pedulums, instantaneous velocity was measured by a length of fall. Forces, masses and weights were all

[12] M. Varignon, Manière générale de determiner les Forces, les Vtesses, les Espaces, & les Tems, une seule de ces quatre choses étant données [...], *Mémoires de l'Académie Royale*, 1700, pp. 22-27.

[13] In 1707 Varignon presented a second text to the Academy: "Des mouvemens, variés à volonté, comparés entre'eux & avec les uniformes", *Mémoires de L'Académie Royale* 1707. The main theorem is the following: the distance covered by a moving point with velocity v equals $\int vdt$. The theorem is illustrated by means of many examples in which the velocity v depends in different ways on the time t. Varignon distinguishes accelerated and retarded motions but the notion of acceleration does not in itself play a role.

[14] Euler and D'Alembert did not agree about the interpretation of the equation. For D'Alembert the notion of force was a derived notion, while it was for Euler a primary notion and consequently $\phi dt = du$ was for Euler a law and not merely definition of the notion of force. Cf. Véronique Le Ru, La force accélératrice: un exemple de définition contextuelle dans le *Traité de Dynamique* de d'Alembert, *Revue d'histoire des sciences* 47, 1994, pp. 475-494. The lemma on "accélération" in the *ENCYCLOPÉDIE*, which was written by d'Alembert, only confirms that acceleration was not seen as being on the same level as velocity.

measured by weights. Accelerations could not be measured directly and, consequently, "an examination of the literature shows a marked reluctance to speak of accelerations more than necessary." [15]

We will briefly consider two versions of Newton's second law in Euler's papers. Euler knew very well that the precise form of Newton's second law depends on the units of measurement. On pp. 478 - 479 of E842, Anleitung zur Naturlehre (Originally published in *Opera Postuma* 2, 1862, pp. 449-560; also in *Opera omnia* III, 1, pp. 16 - 180) Euler expressed Newton's second law is as follows:

$$dv = n \cdot \frac{p \cdot dt}{M} \tag{4}$$

dv is the increment of velocity, p is the force, t is time and M is the mass of the object. Euler pointed out that we are free in the way we measure v, p, t and M. However, the way we measure these quantities determines the value of n. Once we made up our mind how we want to measure v, p, t and M, we must determine n in a specific case. E112, Recherches sur le mouvement des corps célestes en général (Studies on the movement of celestial bodies in general) [16] is the first paper in which Euler uses rectangular Cartesian coordinates and decomposes Newton's second law with respect to the three axes. He considers the motion of a mass M. The "instantaneous change" of the motion of the body is then expressed with respect to each of the coordinates by means of the equation:

$$\frac{2dxx}{dt^2} = \frac{X}{M} \tag{5}$$

X is the "absolute" or "moving" force, and $\frac{X}{M}$ is the "accelerating force. Euler points out that the square of the velocity $\left(\frac{dx}{dt}\right)^2$ expresses the height corresponding to this velocity and that is why, he writes, the factor 2 occurs in the formula. It was quite common to measure instantaneous velocity by means of a length of fall. [17] This version of Newton's second law occurs frequently in Euler's work.

[15] J. Ravetz, The Representation of Physical Quanties in Eighteenth Century Physics, *Isis* 52, 1961, 7-20.

[16] Originally published in *Mémoires de l'académie des sciences de Berlin* 3, 1749, pp. 93-143 (*Opera omnia* II, Vol. 25, pp. 1 - 44).

[17] In the case of an object falling along the x-axis $X = M$, so Euler's formula (5) yields $2ddx = dt^2$ and this yields $dx/dt = t/2$ and $x = t^2/4$. Clearly the height is the square of the velocity at the end of the fall.

In E292, Du mouvement de rotation des corps solides autour d'un axe variable (On the rotation of rigid bodies about a variable axis) [18] Newton's law with respect to each of the three coordinates has the form

$$dv = \frac{2gpdt}{m} \tag{6}$$

Force p and mass m are both measured by means of corresponding weights on earth. Time t is measured in seconds. The velocity v, however, is measured by the distance covered in a second. This leads to the introduction of the factor g. One can easily check that g is the distance covered in one second by an object falling from a state of rest.

Although, for example, Lagrange on p. 232 of his *Mécanique analytique* of 1788 applied Newton's second law in the form $F = m \cdot \frac{dxx}{dt^2}$, that is without factors 2 or $2g$ in the accelerating force, only in the first half of the 19^{th} century, after Ampère in the 1830s introduced the word "kinematics" acceleration, defined in a purely kinematic way, became a separate object of investigation.

3.3. *Classification of surfaces*

It is remarkable that in chapter IV of the appendix to the second volume of his *Introductio in analysin infinitorum*, published in 1748, Euler, while he was investigating surfaces in space, came up with what were later called the *Euler angles*. Euler was investigating the way in which the equation of a surface depends on the choice of the coordinate system. There are subtle differences between Euler's and our views of a coordinate system. For example, the three perpendicular coordinates of a point do not primarily correspond to the three projections on the three coordinate planes. The expression "point M has the three perpendicular coordinates x, y, z" meant to Euler: Move from the origin A to the point P in the direction of the x-axis, such that $AP = x$, then move perpendicular to AP from P to Q in the direction of the y-axis such that $PQ = y$ and finally move perpendicular to the plane of APQ from Q in the direction of the z-axis over a distance z. This is how one reaches the point M. Euler's pictures do not contain the three coordinate axes either. They show the route $APQM$ from the origin to the point M consisting of the segments AP, PQ, QM. Such a figure defined the coordinate system. On the other hand everything that Euler did makes perfect sense to us. He knew perfectly well what a perpendicular coordinate system is. It is even so that without actually changing

[18] Presented in 1758 and originally published in *Mémoires de l'académie des sciences de Berlin* 14, 1765, pp. 154-193 (*Opera omnia* II. Vol. 8, 200-235).

his calculations much of what he does can be summarized more efficiently by means of vector and matrix calculus. This is very much so in his work on rigid body dynamics.

In order to find the representation of a surface with respect to another system, Euler changed a given (x_1,y_1,z_1) reference frame in three steps.

Step 1: A rotation about the z_1-axis plus a translation in the xy-plane. The result is the (x_2,y_2,z_2) reference frame. Transformation:

$$\begin{aligned} x_1 &= x_2 cos\zeta + y_2 sin\zeta - a \\ y_1 &= -x_2 sin\zeta + y_2 cos\zeta - b \\ z_1 &= z_2. \end{aligned}$$

Step 2: A rotation about the x_2-axis in the xy-plane. The result is the (x_3,y_3,z_3) reference frame. Transformation:

$$\begin{aligned} x_2 &= x_3 \\ y_2 &= y_3 cos\eta - z_3 sin\eta, \\ z_2 &= y_3 sin\eta + z_3 cos\eta \end{aligned} \tag{7}$$

Step 3: A rotation about the z_3-axis plus a translation in the xy-plane. Result: (x_4,y_4,z_4) reference frame. Transformation:

$$\begin{aligned} x_3 &= x_4 cos\theta + y_4 sin\theta - c \\ y_3 &= -x_4 sin\theta + y_4 cos\theta - d \\ z_3 &= z_4 \end{aligned} \tag{8}$$

Combination of these three steps give us the familiar equations:

$$\begin{aligned} x_1 &= x_4 A + y_4 B + z_2 C + f \\ y_1 &= x_4 D + y_4 E + z_2 F + g \\ z_1 &= x_4 G + y_4 H + z_2 I + h \end{aligned} \tag{9}$$

with

$$\begin{array}{lll} A = cos\zeta cos\theta - sin\zeta cos\eta sin\theta & B = cos\zeta sin\theta + sin\zeta cos\eta cos\theta & C = -sin\zeta sin\eta \\ D = -sin\zeta cos\theta - cos\zeta cos\eta sin\theta & E = -sin\zeta sin\theta + cos\zeta cos\eta cos\theta & F = -cos\zeta sin\eta \\ G = -sin\eta . sin\theta & H = sin\eta cos\theta & I = cos\eta \end{array}$$

Euler without further ado wrote that by means of these formula we can get the most general equation for a given surface. And indeed. Steps 1 and 2 enable us to give the original z-axis any direction. Moreover, step 3 enables us to give the other axes the required direction as well. Euler in fact used the insight that it is always possible to reach any position of a rigid body with a fixed point from any other position by means of three rotations. In his book Euler made limited use of the formulae. In Chapter V he used them to show that one can always eliminate the mixed and linear terms from the equation of a quadratic surface.

Although Euler's application was original, the idea was not. "Gimbals": concentric rings of the type used nowadays to keep a ship's compass in a horizontal position, were known already in Antiquity, both in China and in the Roman Empire. Often Cardano is mentioned as the inventor, but also Da Vinci was familiar with the idea. [19] The invention of the so-called "universal joint," before Euler, by Robert Hooke (1635-1703) is based on the same idea.

3.4. *Separating the progressive motion of the center of gravity from the rotatory motion.*

Euler's interest in rigid body dynamics was not only purely academic. A better understanding of the motion of a ship was potentially most useful. In the *Scientia navalis*, written between 1737 and 1740 and published in 1749, the first step in Euler's analysis of the motion of the ship is the separation of the motion of the center of gravity of the ship from the motion of the ship about its center of gravity. Euler would apply this separation consistently in his work in rigid body dynamics.

In the first volume of the *Scientia navalis* [20] Euler proves the validity of this separation as follows. He imagines that we apply to the center of gravity of the body an extra force equal and opposite to the one resulting from the composition of all forces as if they acted on the center of gravity. The center of gravity will then be at rest and the separation of the progressive motion of the center of gravity and the rotatory motion is justified because this extra force does not influence the motion of the body about the center of gravity. Later Euler gave a kinematical version of the argument for the separation of the progressive and rotatory motions. For example, in E177, Découverte d'un nouveau principe de mécanique (Dis-

[19] For more details see: Rudolf Franke, Zur Geschichte des Cardan-Gelenks, *Technikgeschichte* 46, 1979, pp. 3-19.
[20] Vol 1, Chapter 2, Section 128.

covery of a new principle of mechanics)[21] Euler first considered the motion of a point mass equal to the total mass of the body under the influence of the forces working on the body. Then he imagined that the space in which the body moves is subjected to a motion which is equal and opposite to the just determined motion of the center of gravity. The result is, according to Euler, the motion of the body independent of its progressive motion. Although Euler's arguments may not be completely convincing - the motion of a space in which a body moves can induce centrifugal and Coriolis forces - his intuition was, as usual, correct.

As for the motion about the center of gravity, in his treatment of the motion of a ship Euler first assumes a fixed axis of rotation. In this case the problem can be reduced to a planar problem: Euler in fact derives the formula $(\int r^2 dm) d\omega = M dt$. M is the total moment of the forces bringing about the rotation. Euler called $\int r^2 dm$ the "moment of intertia." However, if there is no fixed axis of rotation the situation gets more complicated. Euler assumed that in ships there are three perpendicular axes passing through the center of gravity such that the motion of the body about the center can be split in three rotational motions about these three axes. However, he did not get any further. After finishing the *Scientia navalis* it took Euler ten years to make a next move.

3.5. *The instantaneous axis of rotation.*

The problem to understand the motion of a ship undoubtedly stimulated the development of the dynamics of a rigid body in space. However, the problem to more precisely understand the rotation of the earth about its axis was also important: the precession of the equinoxes had to be explained. Newton gave the first explanation of this phenomenon[22] and D'Alembert in his *Recherches sur la Précession des Équinoxes et sur la Nutation de l'Axe de la Terre* of 1749 attempted to improve Newton's explanation. Euler wrote on the motion of the earth as well,[23] but his focus was a general theory.

Three papers are important in the development of kinematics and dynamics of a rigid body after the appearance of his *Introductio in analysin infinitorum.* They are E177, presented in 1750, E336 from 1751 and E292 from 1758. The papers have one thing in common. Although Euler intended

[21] Presented in 1750 and originally published in *Mémoires de l'académie des sciences de Berlin* 6, 1752, pp. 185-217 (O.O. II. Vol. 5, 81-108).
[22] *Principia*, Book 3, Prop. 39.
[23] Cf. Curtis Wilson, D'Alembert versus Euler on the Precession of the Equinoxes and the Mechanics of Rigid Bodies. *Archive for History of Exact Sciences* 37, 1987, pp 233-273.

to deduce the changes in the position and the velocity distribution from the given forces acting on the body, he starts with the inverse problem. The components of the accelerations of the points of the rigid body are determined with respect to a perpendicular Cartesian frame of reference. After multiplication with the corresponding mass element, the moments about three perpendicular axes are determined, followed by an integration over the body. Then the resulting moments are equated to the moments of the forces working on the body, which leads to versions of the equations of motion that carry Euler's name. In other respects the papers are different.

In E177 Euler considered the motion of a rigid body about a fixed point with respect to a perpendicular Cartesian frame of reference in order to be able to apply Newton's second law separately with respect to each of the coordinates. The next breakthrough was brought about by a kinematical result: the instantaneous axis of rotation. In order to study the velocity distribution Euler introduced a fixed Cartesian coordinate system in absolute space and assumed that a point Z of the body with coordinates x, y, z has the velocities P, Q, R in the direction of the axes. The components of the velocity P, Q and R, are functions of x, y and z. Euler intended to determine these functions.

Euler considered next to Z a point z with coordinates $x + dx$, $y + dy$, $z + dz$ and velocity components $P + dP$, $Q + dQ$ and $R + dR$. After time dt the position of these two points, Z and z, has changed but their distance has not. The coordinates of their positions have become, respectively, $x + Pdt$, $y + Qdt$, $z + Rdt$ and $x + dx + (P + dP)dt$, $y + dy + (Q + dQ)dt$, $z + dz + (R + dR)dt$.

Equating the distance of z and Z after time dt and before, while neglecting higher order terms, we get

$$dPdx + dQdy + dRdz = 0. \tag{10}$$

Euler assumes $dx = dy = 0$ and concludes that $dR = 0$, which means that R does not depend on z. Analogously p and Q do not depend on x and y, respectively. This enables Euler to write

$$\begin{aligned} dP &= Ady + Bdz, \\ dQ &= Cdz + Ddx, \\ dR &= Edx + Fdy. \end{aligned} \tag{11}$$

Substitution in (10) yields $D = -A$, $E = -B$ and $F = -C$. This implies

$$dP = Ady + Bdz$$
$$dQ = Cdz - Adx \qquad (12)$$
$$dR = -Bdx - Cdy$$

A and B cannot contain x, because p does not. C and A cannot contain y, because Q does not. So A only depends on z. Similarly B only depends on y and C only on x. Because $Ady + Bdz$ is an integrable differential, we have

$$\frac{dA}{dz} = \frac{dB}{dy} \qquad (13)$$

and, analogously

$$\frac{dC}{dx} = -\frac{dA}{dz}$$

and

$$\frac{dB}{dy} = \frac{dC}{dx}.$$

A, B and C are clearly three constants, that determine the velocities of all points at the moment t. Because the center of gravity is at rest and coincides with the origin, which implies that P, Q and R vanish in the origin, integration of (13) yields

$$P = Ay + Bz, Q = Cz - Ax, R = -Bx - Cy. \qquad (14)$$

Which points have velocity 0? The points for which

$$Ay + Bz = Cz - Ax = -Bx - Cy = 0. \qquad (15)$$

These points have coordinates Cu, $-Bu$, Au and they are on a straight line through the origin. All other points rotate about this axis with an angular velocity equal to

$$\sqrt{A^2 + B^2 + C^2}. \qquad (16)$$

3.6. *The rotation axis in discrete spherical kinematics*

Did Euler discover the instantaneous axis of rotation? The answer is that he shares the honor with D'Alembert who in his *Recherches sur la Précession* of 1749, on pp. 82-83, showed that at each instant the locus of points of the earth that are at rest with respect to the center of gravity of the earth is an axis of rotation. Euler's treatment concerns all rigid bodies and, moreover, Euler added the following nice geometrical proof, which gave a more general result: Given two different positions of a body with a fixed point, it is always possible to move the body from one of the two positions to the other one by means of a rotation about an axis through the fixed point. Euler considered in the moving rigid body a spherical surface of which the center coincides with the center of gravity. On this surface he considers an arc AB of a great circle that moves in a time dt to a position ab on another great circle. Clearly $AB = ab$. Now prolong BA and ba until they meet in the point C.

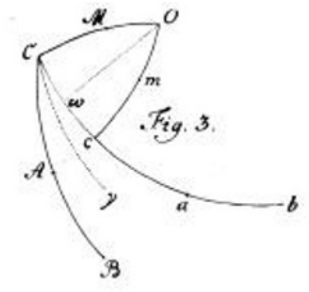

Fig. 5.

C moves in time dt to position c on the prolongation of ba. Now imagine a point M outside the great circle ABC. In time dt point M moves to a point m. CM and cm will intersect in a point O. If triangle cCO is isosceles, point O will not move in time dt. Then point O is on the instantaneous axis of rotation. However, this is a situation that we can bring about by choosing M in the right way. Spherical triangle cCO is isosceles if $\angle cCO = \angle CcO$.

Because we have $\angle cCO = \angle CcO = 180° - \angle acO = 180° - \angle ACO = 180° - \angle cCO - \angle ACc$, we can draw the conclusion that $\angle cCO = 90° - {}^1\!/_2\angle ACc$. M must be chosen such that CM is perpendicular to the angular bisector of the angle Acc. Then O is on the instantaneous axis of rotation.

Although Euler only discussed two positions of the spherical surface that are infinitesimally close, the argument also holds for two arbitrary positions. Euler discovered here in fact the rotation axis in discrete spherical kinematics, although he may not have been aware of it (see Section 3.10).

3.7. *Euler's first attempt to derive the equations of motion for a rigid body in space.*

After having proved in E177 the existence of an instantaneous axis of rotation, Euler turned to the accelerations. After having introduced the axis of rotation, it is only natural that he tried to exploit the existence of this axis. He expressed the ddx, ddy and ddz in terms of the position and motion of this axis. His starting point was the fixed perpendicular Cartesian reference frame with axes OA, OB, OC in absolute space and a given position of the axis of rotation. Euler defined the position of the axis by the coordinates: $x = \nu u$, $y = \mu u$, $z = \lambda u$. The angular velocity is $\sqrt{\nu^2 + \mu^2 + \lambda^2}$. Euler now considered a point in the body with coordinates x, y, z and he determined the components of the velocity of this point in the direction of the axes. He gets from (14)

$$dx = (\lambda y - \mu z)dt; dy = (\nu z - \lambda x)dt; dz = (\mu x - \nu y)dt, \tag{17}$$

which gives us after differentiation:[24]

$$\begin{aligned} ddx &= (yd\lambda - zd\mu)dt + (\lambda\nu z + \mu\nu y - (\lambda\lambda + \mu\mu)x)dt^2 \\ ddy &= (zd\nu - xd\lambda)dt + (\mu\nu x + \lambda\mu z - (\nu\nu + \lambda\lambda)y)dt^2 \\ ddz &= (xd\mu - yd\nu)dt + (\lambda\mu y + \lambda\nu x - (\mu\mu + \nu\nu)z)dt^2. \end{aligned} \tag{18}$$

Now Euler could apply Newton's second law in order to determine the components of the force needed to bring about such accelerations in a mass element dM:

$$\frac{2dM\,ddx}{dt^2}, \frac{2dM\,ddy}{dt^2}, \frac{2dM\,ddz}{dt^2}. \tag{19}$$

Euler integrated and after having introduced the moments of inertia about the axes, he got a first version of the three Euler equations of motion. In the end he gave three differential equations that describe the changes in the position of the axis of rotation through the center of gravity and the changes in the angular velocity about this axis. However, the solution of the equations leads to complicated calculations. So Euler assumes that the X-axis coincides with the axis of rotation. That makes life somewhat easier but for every other instant the coordinate axes must be changed. Although Euler had solved the problem in principle, in practice he had not yet reached a satisfactory solution.

3.8. *Euler gives the problem another try.*

Euler was not satisfied with the equations of motion that he had derived in E177. Soon he gave the problem another try. The result is in E336.[25] The

[24] Clearly we would nowadays write this, for example, in the form of the following vector equation: $\frac{dd\bar{x}}{dt^2} = \frac{d\bar{\omega}}{dt} \times \bar{x} + \bar{\omega} \times (\bar{\omega} \times \bar{x})$.

[25] E336 is in the 1760 volume of the memoirs of the Berlin Academy. That volume was printed in 1767. Charles Blanc assumed in the preface to Euler's *Opera omnia* II, 9.

first 21 pages are devoted to kinematics and one can feel Euler searching for the best way to represent the position, the velocities and the accelerations of the moving body. The moving earth as a major example is clearly very much on his mind. Euler first points out that in every rigid body we can fix an axis MK through the center of gravity O. Moreover we can fix in the body a meridian plane MLK. If the body is the earth, MK would be the line connecting the poles and the meridian plane MLK would be, for example, the meridian plane of Greenwich. Euler then chooses a perpendicular Cartesian coordinate system (x, y, z) in absolute space. The z-axis points at the zenith and the xy-plane is the plane of the ecliptic. The center of gravity O of the body coincides with the origin O of the (x, y, z) system. Then Euler introduces as follows three parameters p, q and r to fix a position of the body with respect to the coordinate system. Parameter q is the angle between the z-axis and the axis MK of the body. In Euler's interpretation this is the angle between the axis of the earth and the perpendicular to the ecliptic. In the case of the earth this parameter q is not a constant, but the nutation is extremely small. Parameter p corresponds to the rotation of the slanted earth axis about the perpendicular to the ecliptic. This parameter diminishes with about 50" every year, which causes the precession of the equinoxes. Finally parameter p measures, in the case of the earth, the rotation of the earth about the axis of the earth. So p grows by 360° every day.[26] Although Euler repeatedly refers to motion of the earth, his goal is a general theory.

Euler determines the position of a point Z inside the body by its distance s to the center of gravity and two angles u and v and afterwards he is in a position to determine the coordinates of point Z with respect to the (x, y, z) system. This gives him three expressions containing p, q, and r, and s, u and v. Euler subsequently first determines the instantaneous axis of rotation and the instantaneous angular velocity. After having done so, by way of illustration, he applies the formulae to the motion of the earth in order to determine angular velocity and the instantaneous axis of rotation that is the result of the daily rotation and the precession of the equinoxes.

Next he proceeds to determine with respect to the (x, y, z) coordinate system in absolute space the components $\frac{2ddx}{dt^2}$, $\frac{2ddy}{dt^2}$, $\frac{2ddz}{dt^2}$ of the accelerating force that works on a point mass Z. They are rather complicated expressions in, obviously, p, q, r, s, u and v. At this point Euler introduces

on p. xxiii that E336 was written after E292, that is many years after E177. However, it turns out that Euler must have written E336 soon after he wrote E177 and several years before he wrote E292. Cf. Curtis Wilson, D'Alembert versus Euler etc. *Archive for History of Exact Sciences* 37, 1987, pp 233-273, footnote 67.

[26] The three parameters p, q, and r can be interpreted as Euler's angles.

a perpendicular Cartesian coordinate system (x, y, z) fixed to the rigid body and for the first time he explicitly uses formulae that are equivalent to the ones that he had given in his *Introductio in analysin infinitorum* containing the "Euler-angles" (See Section 3.3) to find the components of the accelerating force with respect to this new Cartesian system. Euler determines the moments of the components. He integrates over the body and gets very complicated expressions. However, by replacing p, q and r by what are in fact the components P, Q and R of the direction of the instantaneous axis of rotation (and $\sqrt{P^2 + Q^2 + R^2}$ equal to the angular velocity) Euler succeeds in giving the equations of motion a rather regular, but still not simple form. Euler succeeds in deriving several nice results: he shows for example that whatever shape the body has, it is always possible find an axis about which the body can freely and uniformly. Euler derives a third degree equation and points out that it has at least one real root. Had he seen that the equation has always three real roots, he would have discovered the principal axes of inertia.

3.9. *E291 and E292: Euler's derivation of the equations of motion for a rigid body in space*

Several years later, Euler described the discovery of the three perpendicular axes of intertia of a body in E291.[27] They are three perpendicular lines going through the center of inertia that are such that the body can rotate freely about them in the sense that the centrifugal forces neutralize each other (*Opera omnia* II, 8, pp. 192). Soon afterwards, Euler gave the dynamics of a rigid body renewed attention in E292, Du movement de rotation des corps solides autour d'un axe variable (written in or before 1758 and published in 1765; *Opera omnia* II, 8, pp. 200-235). The lovely idea in E292 is simple: let the axes of the reference system in absolute space - at the instant under consideration - coincide with the principle axes of inertia in the body. The application of the idea is successful. Euler could derive a simple version of the equations that carry his name and he succeeded in solving several problems that he could not solve before.

Euler's approach in E292 as for the kinematics is not different from his approach in E177. The main difference with E177 is that Euler introduced the angular velocity explicitly in the equations. In E177 Euler had defined the position of the axis by the coordinates: $x = \nu u$, $y = \mu u$, $z = \lambda u$, such

[27] The discovery was first published by Euler's friend Andreas Segner in his *Specimen theoriae turbinum*, Halle 1755. See: p. 266 of Curtis Wilson, D'Alembert versus Euler on the Precession of the Equinoxes and the Mechanics of Rigid Bodies. *Archive for History of Exact Sciences* 37, 1987, pp 233-273.

that the angular velocity is $\sqrt{\nu^2+\mu^2+\lambda^2}$. In E292 Euler chose perpendicular fixed axes in absolute space: IA, IB, IC and described the position of the axis of rotation IO by the angles $AIO=\alpha$, $BIO=\beta$, $CIO=\gamma$.[28] The angular velocity is ω. Then he derived equations similar to (18) for the acceleration distribution in the body. Newton's law is then applied in the form (6) to find the components of the forces needed to bring about these accelerations. After having determined the moments about the axis of the components of these forces, Euler can integrate these moments over the whole body. The assumption that the axes of inertia coincide with the axes in absolute space pays off immediately.

Euler subsequently considered a point Z in the body with coordinates x, y, z, with respect to the axes IA, IB, IC respectively. Through Z Euler took three perpendicular axes Za, Zb, Zc, parallel to IA, IB and IC. Then he determined the components of the velocity of Z in the direction of the axes. He gets the nowadays very well-known equations of motion

$$\begin{aligned} P &= Maa\cdot\frac{d.\omega\cos\alpha}{2gdt}+M\,(cc-bb)\cdot\frac{\omega\omega\cos\beta\cos\gamma}{2g},\\ Q &= Mbb\cdot\frac{d.\omega\cos\beta}{2gdt}+M(aa-cc)\cdot\frac{\omega\omega\cos\gamma\cos\alpha}{2g},\\ R &= Mcc\cdot\frac{d.\omega\cos\gamma}{2gdt}+M(bb-aa)\cdot\frac{\omega\omega\cos\alpha\cos\beta}{2g}. \end{aligned} \tag{20}$$

The constants aa, bb and cc denote the moments of inertia about the principal axes.

These equations meant for Euler a considerable step forward. It immediately enables him, for example, to determine the instantaneous axis of rotation, immediately after a body at rest is subjected to arbitrary forces. Because at the instant under consideration $\omega=0$, the equations can be solved easily.

3.10. *Other formulae for the arbitrary change of position of a rigid body.*

In 1775 Euler published E478,in which he reconsidered the problem of the most general way of describing a change of position of a rigid body.[29] He introduced rectangular Cartesian coordinates (x, y, z) in absolute space and asked himself what happens to the coordinates (p, q, r) of an arbitrary

[28] In E177 he had already briefly experimented with this characterization of the axis of rotation.

[29] E478, Formulae generales pro translatione quacunque corporum rigidorum, *Novi commentarii academiae scientiarum Petropolitanae* 20 (1775), 1776, pp. 189-207. Also in *Opera omnia* II, 9, pp. 84-98.

point of the rigid body, when the body changes its position. Suppose the new position of the origin is $(f,\ g,\ h)$. By considering points with coordinates $(p,0,0)$, $(0,q,0)$, and $(0,0,r)$ and pointing out that a straight line remains a straight line, Euler draws the conclusion that the formulae for the transformation necessarily have the following form.

$$\begin{aligned} x &= f + Fp + F'q + F''r, \\ y &= g + Gp + G'q + G''r, \\ z &= h + Hp + H''q + H''r. \end{aligned} \tag{21}$$

The constants f, g, h, F, F', F'', G, G', G'', H, H', H'' depend on the change of position. Moreover, the distance of the points $(p,0,0)$, $(0,q,0)$, and $(0,0,r)$ to the point that initially coincides with the origin must remain the same. This implies

$$F^2 + G^2 + H^2 = 1, F'^2 + G'^2 + H'^2 = 1 \ \text{ and } \ \mathrm{F}''^2 + \mathrm{G}''^2 + \mathrm{H}''^2 = 1. \tag{22}$$

Putting

$$\begin{array}{lll} F = sin\zeta & F' = sin\zeta' & F'' = sin\zeta'' \\ G = cos\zeta sin\eta & G' = cos\zeta' sin\eta' & G = cos\zeta'' sin\eta'' \\ H = cos\zeta cos\eta & H = cos\zeta' cos\eta' & H = cos\zeta'' cos\eta'' \end{array} \tag{23}$$

Euler expressed the change of position in terms of the new coordinates of the point coinciding initially with the origin and six angles. He also showed that the values of the three angles η , η', η'' suffice to determine the others. Euler was aware of what we would nowadays call the orthonormality conditions for the columns of the transformation matrix envolved in (21). However, here, he did not give the similar conditions for the rows of the matrix. He would do this later in E407 presented in 1770, that we will discuss in the next section. For the first time Euler asks here the question whether for an arbitrary change of position such that the point coinciding with the origin remains fixed, it is possible to bring about the same change of position by means of a rotation about a line. Starting from (21) with $f = g = h = 0$, he attacks the problem analytically, but he does not succeed. However, Euler gives a geometrical proof, similar to the one we discussed in section 3.6. It is possible that he did not realize that his proof in E177 for the instantaneous case is also valid in the discrete case.

3.11. *An algebraic problem that is notable for some quite extraordinary relations*

In E407, Problema Algebraicum ob affectiones prorsus singulares memorabile, presented to the Academy in Petersburg in 1770,[30] Euler considers the following algebraic problem. Determine 9 numbers

$$\begin{matrix} A & B & C \\ D & E & F \\ G & H & I \end{matrix} \tag{24}$$

such that the following 12 conditions are satisfied:

$$\begin{array}{llll} 1. & AA+BB+CC = 1; & 4. & AB+DE+GH = 0; \\ 2. & BB+EE+HH = 1; & 5. & AC+DF+GI = 0; \\ 3. & CC+FF+II = 1; & 6. & BC+EF+HI = 0; \\ \\ 7. & AA+DD+GG = 1; & 10. & AD+BE+CF = 0; \\ 8. & DD+EE+FF = 1; & 11. & AG+BH+CI = 0; \\ 9. & GG+HH+II = 1; & 12. & DG+EH+FI = 0. \end{array} \tag{25}$$

Euler compared squares like (24) satisfying conditions 1 through 12 to magic squares. Completely analytically Euler derived expressions for the 9 unknowns in terms of the three Euler-angles. Actually the expressions are exactly the ones of (9). It is interesting that Euler then turned to the 4- and 5-dimensional cases, which he also explicitly solved. In fact he showed that in the n-dimensional case we need $n(n-1)/2$ parameters. Euler's method, also in the 4- and 5-dimensional cases is the method he applied in his *Introductio in analysin infinitorum* (see section 3.3 of this paper): He generated the solutions by means of series of successive transformations that each only affect two of the variables. Euler wrote this paper as a contribution to algebra and number theory. The last sections are devoted to rational solutions of the equations. In Section 33 he pointed out that if p, q, r, and s are four arbitrary numbers, and $p^2+q^2+r^2+s^2=u$, the following formulae represent a solution to the problem.

[30] *Novi Commentarii academiae scientiarum Petropolitanea* 15, 1771, pp. 75-106 and *Opera omnia* I, 6, pp. 287-315.

$$A = \frac{p.p + qq - rr - ss}{u} \quad B = \frac{2qr - 2ps}{u} \quad C = \frac{2qs - 2pr}{u}$$

$$D = \frac{2qr - 2ps}{u} \quad E = \frac{pp - qq + rr - ss}{u} \quad F = \frac{2pq + 2rs}{u}$$

$$G = \frac{2qs + 2pr}{u} \quad H = \frac{2rs - 2pq}{u} \quad I = \frac{pp - qq - rr + ss}{u} \tag{26}$$

Seventy years later, in 1840, Olinde Rodrigues showed that an arbitrary change of position of a body with a fixed point can be represented in this way.[31] That is why the parameters in (26), with $u = 1$, are nowadays called the Euler-Rodrigues parameters for the representation of motion.[32]

4. Final remarks

Euler's work belongs to the pre-history of kinematics. Kinematics was not yet an independent research area and as we have seen Euler's contributions were all made in a non-kinematical context: a mechanical engineering problem, a geometrical problem, a dynamical problem or an algebraic or number theoretical problem. Although it is not complete,[33] the more or less coherent survey of kinematics in Euler's work that we have given illustrates this.

Euler's focus was not on kinematics, which is part of the explanation why Euler missed some results that we find rather obvious. For example, the separation of the progressive motion of the center of gravity and the rotatory motion about this center was very functional in Euler's work. Yet it prevented Euler from realizing that in general at each instant the velocity distribution of a body moving in space is an instantaneous screw motion.

[31] Olinde Rodrigues, Des lois géometriques qui régissent les déplacements d'un système solide dans l'espace, et de la variation des coordonnées provenant de ces déplacemenst considerés indépendemmant des causes qui peuvent les produire, *Journal de mathématiques pures et appliques* 5, 1840, pp. 380-440.
[32] Cf. Malcolm D. Shuster, A Survey of Attitude Representations, *The Journal of the Astronautical Sciences* 41, 1993, pp. 439-517.
[33] For example, I have not discussed E825, De motu corporum circa punctum fixum mobilium (*Opera Postuma* 2, 1862, pp. 43-62 and in *Opera omnia: Series* 2, Volume 9, pp. 413 - 441).

This result was first published in 1763 by the Italian Giulio Mozzi.[34] Actually it is easy to see that the composition of a translation T and a rotation R about an axis yields a screw motion: Choose a plane p perpendicular to the axis of R. Decompose the translation in a component T_1 in plane p and a component T_2 perpendicular to P. Composition of R and T_1 yields a rotation R' about an axis parallel to T_2. R' and T_2 together represent a screw motion. There is a similarity with Newton who in 1666 studied the construction of tangents to kinematically defined curves. In this context Newton distinguished three kinds of instantaneous motion: translation, rotation and the composition of a rotation and a translation.[35] It is extremely obvious that in a plane a combination of a rotation and a translation is a rotation: draw from the center of the rotation a line perpendicular to the direction of the translation. On this line there is a point which has a velocity opposite and equal to the translatory velocity. This is the instantaneous center of rotation. Newton did not see it. His focus was elsewhere.

At the occasion of Euler‘s 250$^{\text{th}}$ birthday, in 1959, the German mathematician Wilhelm Blaschke (1885-1962), well-known for important work in differential geometry, wrote a paper called “Euler und die Kinematik” (Euler and Kinematics)[36] in which he used quaternions to derive some basic results in instantaneous spherical and spatial Euclidean kinematics. The paper is not about history with the exception of the first sentence in which Blaschke says: “It is maybe little known that the quaternions were first identified by L. Euler in a letter to Goldbach written on May 4, 1748.” The only link between this letter and quaternions lies in the fact that in the letter he mentioned what is nowadays called Euler’s four squares theorem:

If $m = a^2 + b^2 + c^2 + d^2$ and $n = p^2 + q^2 + r^2 + s^2$ then $mn = A^2 + B^2 + C^2 + D^2$, if $A = ap+bq+cr+ds$; $B = aq-bp-cs+dr$; $C = ar+bs-cp-dq$ and $D = as - br + cq - dp$.

For a reader who is familiar with quaternions, the theorem in this form expresses the fact that the absolute value of the product of the quaternions $a - bi - cj - dk$ and $p + qi + rj + sk$ is equal to the product of their absolute values. However, it makes no sense whatsoever to say that Euler

[34] Marco Ceccarelli, Screw axis defined by Giulio Mozzi in 1763 and early studies on helicoidal motion, *Mechanism and Machine Theory* 35, 2000, pp. 761-770. Cf. footnote 5 of this paper.

[35] I. Newton, *The mathematical papers of I. Newton* (Edited by D. T. Whiteside), Vol. I, Cambridge, 1967, pp. 390-391.

[36] Wilhelm Blaschke, Euler und die Kinematik, in Kurt Schrder, *Sammelband der zu Ehren des 250. Geburtstages LEONHARD EULERs der Deutschen Akademie der Wissenschaften zu Berlin vorgelegten Abhandlungen,* Akademie-Verlag. Berlin, 1959, pp. 35-41.

"identified" the quaternions in this letter. Blaschke was an outstanding geometer, but this claim is absurd.

Yet there are other and even more striking links between Euler's work and the theory of quaternions. For example, if we use the unit quaternion $p + qi + rj + sk$ to describe the motion of a body about a fixed point, the matrix (26) in Section 4 is exactly the transposed of the rotation matrix for the same motion.

Leonhard Euler: Life, Work and Legacy
Robert E. Bradley and C. Edward Sandifer (Editors)

Euler on Rigid Bodies

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1. Introduction

In his paper "Découverte d'un nouveau principe de mécanique" [E177], published in 1752, Euler derives the equations of motion for a rigid body. According to Clifford Truesdell, "Few indeed are works contributing so much to mechanics as this one paper." Euler argues that the principles of Mechanics which were commonly accepted at that time were insufficient to solve this problem in full generality. Consequently, he proposes a new principle, which he takes as the fundamental axiom on which all of Mechanics is to be founded: it is the application of "Newton's Second Law" $\mathbf{F} = M\mathbf{a}$ (written in Cartesian coordinates) to each infinitesimal element of the body. Not until several years later does Euler recognize that this one principle is itself insufficient; one needs, in addition, the principle that the applied torque is equal to the rate of change of rotational momentum. In the first part of this paper, I will describe how Euler applied his new principle to solve the problem of rigid bodies, and how he finessed his lack of the principle of rotational momentum.

Having obtained the general equations for rigid-body motion, Euler, like a good teacher, shows how to apply them to a concrete example, namely, a homogeneous ball rotating about a fixed axis, which is subject to an external torque. But after having worked out the resulting motion on the basis of his equations, Euler then wants to give an alternative, more elementary, demonstration, starting from first principles. Here, however, he gets into trouble. His alternative method is not correct, and he arrives finally at the correct result only by making another, compensating, error. It's a little hard, in fact, to see just where he went wrong, since his reasoning seems rather loose, involving manipulations with infinitesimals. In the second part, I will try to clarify the situation, and explain just what Euler's error was.

2. Cancellation of forces

A recent ponderous volume, the *Reader's Guide to the History of Science*, [H] has articles on many scientists and some non-scientists, including even Ernst Mach and John Stuart Mill —but no article on Leonhard Euler. In fact, the index to this work has only two entries for Euler. One concerns an alleged "statistical failure" on Euler's part in connection with the inequalities of the motion of Jupiter and Saturn; the other, in the article on Maupertuis (!), mentions Euler's support for the principle of least action. (There are also, however, several references to a mysterious "Hans Euler". On inspection, these prove to be references to Leonhard. [1])

Clearly, Euler's achievements in physical science deserve to be better-known. In this paper, I will discuss Euler's derivation of the laws of motion for a rigid body. In particular, I will consider the two papers: "Découverte d'un nouveau principe de mécanique", [E177], published in 1752 and "Nova methodus motum corporum rigidorum determinandi", [E479], of 1776.

Since some readers may be unfamiliar with the equations for rigid-body motion, let's see briefly what they say. We first observe, with Euler [E177, §5], that we can separate the motion of the *center of mass* of the body from its *rotational* motion about that center of mass. The former can be determined from "Newton's Second Law," $\mathbf{F} = M\mathbf{a}$, and need not be dealt with here.

Let us denote by $\mathbf{H}$ the rotational momentum of the body, and by $\mathbf{L}$ the applied torque acting on the body. (We will assume here that $\mathbf{H}$ and $\mathbf{L}$ are computed with respect to the center of mass of the body.) A basic principle of Mechanics (stated clearly by Euler in [E479, §§28-29]; see below) is the principle of *Balance of Rotational Momentum*: $\mathbf{L} = \dot{\mathbf{H}}$; the applied torque equals the rate of change of rotational momentum. (Here I use a dot to indicate the derivative with respect to time. Note that, by the same token,

[1] There actually was a Hans Euler, rather Hans von Euler-Chelpin, who won the Nobel Prize for Chemistry in 1929. Leonhard's great-great-grandfather was named Hans, according to Juškevič in the *Dictionary of Scientific Biography.*

we can write the relation $\mathbf{F} = M\mathbf{a}$ in the form $\mathbf{F} = \dot{\mathbf{p}}$, where $\mathbf{p} = M\mathbf{v}$ is the linear momentum, and refer to it as the principle of *Balance of Linear Momentum*.)

For a rigid body, its rotational momentum depends linearly on its *angular velocity vector* $\boldsymbol{\omega}$. In other words, there is a linear transformation $\mathbf{I}$ (in effect, a three-by-three matrix), such that $\mathbf{H} = \mathbf{I}\,\boldsymbol{\omega}$. Here the letter '$\mathbf{I}$' does *not* denote the identity transformation; $\mathbf{I}$ is called the *inertia tensor*. It can be shown that $\mathbf{I}$ is *symmetric* and *positive definite*.

Thus, to apply the principle of Balance of Rotational Momentum, we must compute the derivative $(\mathbf{I}\,\boldsymbol{\omega})^{\cdot}$. Using a little calculus and linear algebra, it can be shown that $(\mathbf{I}\,\boldsymbol{\omega})^{\cdot} = \mathbf{I}\,\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}\,\boldsymbol{\omega}$. Therefore, substituting into the equation $\mathbf{L} = \dot{\mathbf{H}}$, we get $\mathbf{L} = \mathbf{I}\,\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}\,\boldsymbol{\omega}$. This is *Euler's equation* for the motion of a rigid body.

Since the tensor $\mathbf{I}$ is symmetric, it can be diagonalized. Its eigenvalues are called the *principal moments of inertia* of the rigid body; the corresponding eigenspaces are the *principal axes*. Expressing Euler's equations with respect to the principal axes, we get

$$\begin{aligned}
L_1 &= I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 \\
L_2 &= I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 \\
L_3 &= I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2.
\end{aligned}$$

Here the I_i are the principal moments of inertia, the ω_i are the components of the angular velocity vector, and the L_i are the components of the applied torque.

Now, in 1750 (when E177 was presented to the Berlin Academy), Euler didn't know how to diagonalize the inertia tensor.[2] Consequently, when he wrote down these equations, he had to include the terms coming from the off-diagonal elements of $\mathbf{I}$. Thus, in E177, §55, he writes the equations in the form

[2] Segner showed in 1755 (*Specimen theoriae turbinum*, Halle) that every rigid body has three principal axes.

$$\text{I. } \frac{Pa}{2M} = \frac{ff\,d\lambda}{dt} - \frac{nn\,d\mu}{dt} - \frac{mm\,d\nu}{dt} + \lambda\nu nn - \lambda\nu mm - (\mu\mu - \nu\nu)ll + \mu\nu(hh - gg),$$

$$\text{II. } \frac{Qa}{2M} = \frac{gg\,d\mu}{dt} - \frac{ll\,d\nu}{dt} - \frac{nn\,d\lambda}{dt} + \lambda\mu ll - \mu\nu nn - (\nu\nu - \lambda\lambda)mm + \lambda\nu(ff - hh),$$

$$\text{III. } \frac{Ra}{2M} = \frac{hh\,d\nu}{dt} - \frac{mm\,d\lambda}{dt} - \frac{ll\,d\mu}{dt} + \mu\nu mm - \lambda\nu ll - (\lambda\lambda - \mu\mu)nn + \lambda\mu(gg - ff).$$

Here M is the mass of the body, ν, μ, λ are the components of the angular velocity vector, Pa, Qa, Ra are the components of the applied torque, Mff, Mgg, Mhh are the diagonal terms of the inertia tensor, and $-Mll$, $-Mmm$, $-Mnn$ are the off-diagonal terms. [3]

Euler's equations form a system of nonlinear ordinary differential equations, which are hard to solve; and Euler does not know how to solve them, in general. In some particular cases, they can be integrated in terms of elliptic functions; see, for example, Whittaker's *Analytical Dynamics*, Chapter VI.

One particular case that Euler considers is that of *free motion*; that is, the applied torque is zero. Even then, the motion can be quite complicated, so Euler specializes further to the case in which the instantaneous axis of rotation (the line spanned by $\boldsymbol{\omega}$) is *steady* —that is, not varying with time. In that case, the derivative $\dot{\boldsymbol{\omega}}$ must be parallel to $\boldsymbol{\omega}$, and hence $\mathbf{I}\,\dot{\boldsymbol{\omega}}$ must be parallel to $\mathbf{I}\,\boldsymbol{\omega}$. On the other hand, $\boldsymbol{\omega} \times \mathbf{I}\,\boldsymbol{\omega}$ is clearly *perpendicular* to $\mathbf{I}\,\boldsymbol{\omega}$, so that in the equation $0 = \mathbf{I}\,\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}\,\boldsymbol{\omega}$ the two vectors on the right-hand

[3] The factors of 2 in the denominators on the left-hand side come from the particular units Euler uses. He explains these in §21. First, M is the *weight* of the body near the surface of the Earth, so that, in our terms, $M = mg$, where m is what we call the mass and g is the acceleration of gravity. Further, Euler chooses units of time t and distance x so that if dx/dt is the speed attained by a body falling through a height v, then $v = (dx/dt)^2$. It follows easily that, in Euler's units, the value of g is $g = \frac{1}{2}$. (Clearly, this normalization of g involves units of *both* time and distance — or rather, as Euler remarks, the relation between them. See [T1, p. XLIII]; though I do not understand what Truesdell means there by "the ratio of the units of length and time".) I imagine that Euler might have found the constant 2 in his equation as follows. Write "Newton's Second Law" in the form $F = kM(d^2x/dt^2)$, where F is the force and k is a constant of proportionality. Taking F to be the force of gravity, we have $F = M$, and the distance v fallen in time t will be $t^2/(2k)$. The speed attained will be t/k. Setting $t^2/(2k) = (t/k)^2$, we find that $k = 2$.
Later, in [E479, §20], Euler takes the units of time to be seconds.

side are perpendicular, and hence *each* must be 0. From $\boldsymbol{\omega} \times \mathbf{I}\,\boldsymbol{\omega} = 0$, we see that $\mathbf{I}\,\boldsymbol{\omega}$ must be parallel to $\boldsymbol{\omega}$, whence $\boldsymbol{\omega}$ must be an *eigenvector* of the inertia tensor. In other words, a steady axis of free rotation must be a principal axis. This, in fact, is the first result Euler derives in [E177]; see §15. He works it out from scratch, before he even has the general equations of motion.

We also see that $\mathbf{I}\,\dot{\boldsymbol{\omega}} = 0$. Since $\mathbf{I}$ is *nonsingular*, this implies that $\dot{\boldsymbol{\omega}} = 0$; in other words, not only the *direction* of the angular velocity vector, but also the *angular speed* must be constant. In his 1776 paper [E479, §47], Euler conjectures that this must be so (in the special case in which all the principal moments of inertia are equal), but is unable to prove it.

It is known today that, if the principal moments of inertia are different, then the free rotations about the axes corresponding to the largest and smallest moments are stable, whereas the rotation about the other axis is *unstable*. Since Euler does not mention this fact, I presume that he did not know it. I have been able to trace this idea back to sometime in the 19^{th} century, but I do not know its history.

As Euler points out in E177, the case in which the instantaneous axis of rotation is *not* steady had not been treated adequately before.[4] Euler's first problem is to give a mathematical *description* of such a motion.

Let's first work this out in modern terms. A rotation of the body about its center of mass is given by an orthogonal transformation $\mathbf{Q}$. Differentiating the relation $\mathbf{Q}\mathbf{Q}^T = \mathbf{1}$ (where $\mathbf{1}$ is the identity transformation), we get $\dot{\mathbf{Q}}\mathbf{Q}^T + \mathbf{Q}\dot{\mathbf{Q}}^T = 0$, whence $\dot{\mathbf{Q}}\mathbf{Q}^T = -\mathbf{Q}\dot{\mathbf{Q}}^T = -(\dot{\mathbf{Q}}\mathbf{Q}^T)^T$. Letting $\mathbf{A} = \dot{\mathbf{Q}}\mathbf{Q}^T$, we see that $\mathbf{A} = -\mathbf{A}^T$; in other words, the transformation $\mathbf{A}$ is *skew-symmetric*. Clifford Truesdell calls the transformation $\mathbf{A}$ the *spin* of the motion [T4, p. 48].

Now, in three-dimensional space, there is a representation theorem for skew-symmetric transformations: there exists a unique vector $\boldsymbol{\omega}$ such that $\mathbf{A}\mathbf{v} = \boldsymbol{\omega} \times \mathbf{v}$ for every vector $\mathbf{v}$. Of course, $\boldsymbol{\omega}$ is the *angular velocity vector*. If we write the matrix of $\mathbf{A}$ as

$$\mathbf{A} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix},$$

[4] Of course, the problem of the precession of the equinoxes, the subject of a 1749 memoir of d'Alembert (*Recherches sur la Précession des Équinoxes et sur la Nutation de l'Axe de la Terre dans le Systême Newtonian*, Paris), falls under this heading. See Curtis Wilson, "D'Alembert versus Euler on the Precession of the Equinoxes and the Mechanics of Rigid Bodies", *Archive for History of Exact Sciences*, **37**, 1987, pp. 223–273. I thank Ryoichi Nakata for this reference.

then

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \nu \\ \mu \\ \lambda \end{bmatrix},$$

the latter expression being Euler's. Euler constructs the angular velocity vector in [E177, §§26–35]. Of course, Euler does not have the abstract concept of a *vector*, in our sense, but he knows the *direction* of $\boldsymbol{\omega}$ — he describes it as the direction of the line $x = \nu u, y = \mu u, z = \lambda u$ — and he also knows that the angular speed is $\sqrt{\nu^2 + \mu^2 + \lambda^2}$, so that he has both the magnitude and direction of $\boldsymbol{\omega}$.

By 1776, Euler has gone even further. In [E479, §22], he writes

$$\begin{aligned} x &= f + FX + F'Y + F''Z \\ y &= g + GX + G'Y + G''Z \\ z &= h + HX + H'Y + H''Z, \end{aligned}$$

where X, Y, Z are the initial coordinates of some point of the body, and x, y, z are its coordinates at time t. The coefficients F, G, H, F', …, H'' are the matrix elements of our transformation **Q**! Furthermore, Euler knows that this matrix is an orthogonal matrix: in §23 he writes the relations

$$\begin{array}{llllllll} FF & +GG & +HH & = 1, & FF' & +GG' & +HH' & = 0, \\ F'F' & +G'G' & +H'H' & = 1, & F'F'' & +G'G'' & +H'H'' & = 0, \\ F''F'' & +G''G'' & +H''H'' & = 1, & FF'' & +GG'' & +HH'' & = 0, \end{array}$$

which are just the conditions of orthogonality.

Since 1 is an eigenvalue of **Q**, it has an eigenspace consisting of fixed points —the (cumulative) axis of rotation. The transformation **Q** can be described geometrically by giving the direction cosines α, β, γ of this axis, and the angle φ of rotation about the axis. In §22, Euler writes down the equations

$$F = \cos^2\alpha + \sin^2\alpha\cos\varphi$$
$$G = \cos\alpha\cos\beta(1-\cos\varphi) + \cos\gamma\sin\varphi$$
$$H = \cos\alpha\cos\gamma(1-\cos\varphi) - \cos\beta\sin\varphi$$

$$F' = \cos\alpha\cos\beta(1-\cos\varphi) - \cos\gamma\sin\varphi$$
$$G' = \cos^2\beta + \sin^2\beta\cos\varphi$$
$$H' = \cos\beta\cos\gamma(1-\cos\varphi) + \cos\alpha\sin\varphi$$

$$F'' = \cos\alpha\cos\gamma(1-\cos\varphi) + \cos\beta\sin\varphi$$
$$G'' = \cos\beta\cos\gamma(1-\cos\varphi) - \cos\alpha\sin\varphi$$
$$H'' = \cos^2\gamma + \sin^2\gamma\cos\varphi,$$

which express the components of $\mathbf{Q}$ explicitly in terms of the geometric quantities $\alpha, \beta, \gamma, \varphi$. He derives these equations using spherical trigonometry.

Once Euler is able to describe the motion, he needs to apply some principle of dynamics to relate the motion to the applied force. According to him, however, the previously known principles of mechanics were insufficient to solve this problem. Hence he must search out and apply a new principle. In [E177, §22], he proposes the "general and fundamental principle of all of Mechanics":

$$\text{I. } 2M\,ddx = P\,dt^2, \qquad \text{II. } 2M\,ddy = Q\,dt^2, \qquad \text{III. } 2M\,ddz = R\,dt^2.$$

Of course, this is just "Newton's Second Law", $\mathbf{F} = M\mathbf{a}$! How can Euler suggest that this is a new principle?

This point has been dealt with extensively by Clifford Truesdell in his *Essays in the History of Mechanics*,[5] so I will be brief. Newton in his 1687 *Principia* stated as his "Second Law" a principle equivalent to $\mathbf{F} = M\mathbf{a}$.[6] He applied it, for example, to determine the orbit of a planet under an inverse-square law of gravity. However, when he came to consider problems about fluid motion in Book II of the *Principia*, he did not solve them

[5] [T2]; Lectures II: "A Program toward Rediscovering the Rational Mechanics of the Age of Reason" and III: "Reactions of Late Baroque Mechanics to Success, Conjecture, Error, and Failure in Newton's Principia".

[6] "Mutationem motus proportionalem esse vi motrici impressae, & fieri secundum lineam rectam qua vis illa imprimitur"; see [KC, vol. 1, p. 54]. In my opinion, this means $\int_{t_0}^{t_1} \mathbf{F}(t)\,dt = M\mathbf{v}(t_1) - M\mathbf{v}(t_0)$. In other words, Newton is here using the word 'vis' to mean the time-integral of force, what today is called "impulse".

by applying the "Second Law". Indeed, the different particles of a fluid typically have *different* accelerations, so how would $\mathbf{F} = M\mathbf{a}$ even apply?

In the time between Newton and Euler, many questions in mechanics were studied by Leibniz, the Bernoullis, Clairaut and d'Alembert. They applied various principles to solve these problems: using reversed accelerations to reduce to a problem of equilibrium; the law of the lever; the principle of virtual work; conservation of energy. Ryoichi Nakata, in his talk at the Euler Conference 2002, described how the problem of a particle in a rotating tube was solved by Clairaut, d'Alembert, and Daniel Bernoulli.[7] They did not begin by simply writing down $\mathbf{F} = M\mathbf{a}$; apparently, they didn't even think of doing so! It is obvious today that $\mathbf{F} = M\mathbf{a}$ is a fundamental principle of Mechanics, because Euler has taught us how to apply it. It evidently was far from obvious in 1750.

In [E177, §20], Euler writes, "Consider an infinitely small body, or one whose mass is contained in a single point, that mass being $= M$." Although the language here suggests that, by an "infinitely small body", Euler means what today we call a "point-mass" (what physicists call a "particle"), I think that Euler has in mind also the case in which the body has an *infinitesimal* mass —in other words, that it is what today we would call an "element of integration." In fact, elsewhere in E177, Euler denotes its mass by 'dM', and finds the total mass by integration.

Thus, Euler has now seen how to apply $\mathbf{F} = M\mathbf{a}$ to a fluid — or, as in the present paper E177, to a rigid body, not all of whose points have the same acceleration. Euler's innovation is to see that $\mathbf{F} = M\mathbf{a}$ applies to each infinitesimal part of the body. We might write this principle in the form $d\mathbf{F} = \ddot{\mathbf{r}}\, dM$, where $\mathbf{r}$ is the position vector of a point of the body.

We can now derive the principle of Balance of Rotational Momentum as follows. The rotational momentum $\mathbf{H}$ is defined as $\mathbf{H} = \int_{\mathcal{B}} \mathbf{r} \times \dot{\mathbf{r}}\, dM$, where the integral is over the body $\mathcal{B}$. From $d\mathbf{F} = \ddot{\mathbf{r}}\, dM$, we get $\mathbf{r} \times d\mathbf{F} = \mathbf{r} \times \ddot{\mathbf{r}}\, dM$, hence $\mathbf{L} = \int_{\mathcal{B}} \mathbf{r} \times d\mathbf{F} = \int_{\mathcal{B}} \mathbf{r} \times \ddot{\mathbf{r}}\, dM = \dot{\mathbf{H}}$. In fact, Euler carries out essentially this same calculation (without using vector notation, of course) in [E177, §§43–49].

But there is an error here! If we apply $d\mathbf{F} = \ddot{\mathbf{r}}\, dM$ to each infinitesimal part of the body, the force $d\mathbf{F}$ on that part must include *all* the forces acting on it. As Euler saw, these include not only the external forces acting on the body, but also the *internal* forces which are necessary in order to maintain rigidity. However, in doing our calculation, we don't want to include the *internal* forces —because we don't know what they are!

[7] See his paper, "Analysis of motion of a rotating tube including a material point by Johann Bernoulli, Daniel Bernoulli, Clairaut, d'Alembert and Euler", presented at the Euler Conference 2002. See also [N].

How does Euler deal with this difficulty? In §42, he says, "Now it is to be remarked that the internal forces mutually cancel one another, so that the continuation of the motion requires only the external forces, to the extent that these forces do not mutually cancel."

But how does he *know* that the internal forces must cancel out? Euler doesn't say.

In 1983, van der Waerden published a paper titled "Eulers Herleitung des Drehimpulssatzes" ("Euler's derivation of the theorem on rotational momentum") [V]. Van der Waerden, responding to earlier discussions by Clifford Truesdell,[8] suggested that Euler had in mind an argument which we first find in print in a textbook by Poisson, published in 1833.[9] Poisson had considered the case of a finite collection of point-masses. If one assumes "Newton's Third Law", so that the mutual forces between each pair of point-masses are equal in magnitude but opposite in direction, and if, in addition, one assumes that these forces are *central*, in other words, directed along the line joining the points, then it is a matter of simple algebra, as Poisson showed, to deduce that the mutual forces make no net contribution to the applied torque on the whole collection of point-masses —in other words, the internal forces "cancel out," just as Euler claimed.

Is this really what Euler had in mind? Did he really, first of all, consider a rigid body to be a finite collection of point masses? Actually, we have seen that he denoted the mass of an infinitesimal part of the body by 'dM', and obtained the total mass by integration. Van der Waerden acknowledges that Euler appears to treat the rigid body as a continuum, but he argues that Euler knew (or believed) that the body was "really" just a finite collection of corpuscles, and that he used the continuum model as "just an approximation", in the manner of modern physicists.[10]

I think that this is a very anachronistic interpretation. Although 18^{th}-century scientists sometimes speculated that matter was formed of small corpuscles —Euler himself made an attempt to develop a kinetic theory of gases— the corpuscular nature of matter certainly never attained the dogmatic status it enjoyed in the 20^{th} century. Furthermore, Euler never hints that the mutual forces between the parts of the body must be *central*, as is required for Poisson's proof. Finally, as Curtis Wilson points out [W2, p. 400], Euler consistently rejected the idea of action at a distance.

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[8] [T3], Lectures III: "Reactions of Late Baroque Mechanics" and V: "Whence the Law of Moment of Momentum?".

[9] *Traité de Mécanique*, Paris, 1833, §§552–554.

[10] "Er weiss aber sehr wohl, dass das *nur eine Näherung* ist" (van der Waerden's italics); [V, p. 280].

Truesdell, on the other hand, interpreted Euler's meaning as follows: "Since a body does not spontaneously assume any motion in virtue of whatever internal forces there may be within it, these do not contribute to any of its motions as a whole."[11] Though this may indeed have been Euler's thought, he never, as far as I know, states it explicitly, so that this analysis, like any other, must remain conjectural.[12]

By 1775, Euler has of course learned how to diagonalize the inertia tensor. As the rigid body rotates, however, its principal axes move with it. Consequently, if we choose our coordinate axes so that they lie along the principal axes at some particular instant, they will, in general, no longer do so during the succeeding instants. In his 1752 paper, Euler had tried to simplify his equations by choosing one of the coordinate axes to be the axis of rotation. But "it is necessary for each instant to change the position of the three axes OA, OB, OC in order that OA always coincide with the axis of rotation, and then we will be obliged to calculate anew for each instant the values ll, mm, nn, ff, gg, hh, because they will vary continually as a result of the change in the position of the body with respect to the three axes" [E177, §57].

In E479, Euler gets around this difficulty by describing the motion of the body in terms of the *initial* position of each of its body-points, using (in effect) the mathematics of orthogonal transformations, as we have seen above.[13] Then it becomes possible to choose the coordinate axes for this *initial* position to be principal axes.

But what about the cancellation of the internal forces? As Truesdell has emphasized, Euler in E479 states the principle of Balance of Linear Momentum ($\mathbf{F} = M\mathbf{a}$) and the principle of Balance of Rotational Momentum as parallel principles of mechanics (§29):

$$\begin{array}{llll}
\text{I.} & \displaystyle\int dM\left(\frac{ddx}{dt^2}\right) = iP & \text{IV.} & \displaystyle\int x\,dM\left(\frac{ddy}{dt^2}\right) - \int y\,dM\left(\frac{ddz}{dt^2}\right) = iS \\
\text{II.} & \displaystyle\int dM\left(\frac{ddy}{dt^2}\right) = iQ & \text{V.} & \displaystyle\int x\,dM\left(\frac{ddz}{dt^2}\right) - \int y\,dM\left(\frac{ddx}{dt^2}\right) = iT \\
\text{III.} & \displaystyle\int dM\left(\frac{ddz}{dt^2}\right) = iR & \text{VI.} & \displaystyle\int x\,dM\left(\frac{ddx}{dt^2}\right) - \int y\,dM\left(\frac{ddy}{dt^2}\right) = iU.
\end{array}$$

[11] "Reactions of Late Baroque Mechanics", [T3, p. 171].

[12] An assumption similar to 's was made a few years earlier by Daniel Bernoulli, ("Nouveau probleme de mécanique résolu par Mr. Daniel Bernoulli", Berlin, 1745; reprinted in *Die Werke von Daniel Bernoulli*, vol. 3, ed. D. Speiser et al., Birkhäuser, 1987, pp. 179–196) in connection with the motion of a particle sliding in a rigid tube. See *Essays in the History of Mechanics*, p. 254; see also [N].

[13] In Hydrodynamics, this description is called "Lagrangean"; see Lamb, *Hydrodynamics*, 6^{th} edition, p. 2.

(The two sets of equations are printed side-by-side like this in the original publication in the proceedings of the St. Petersburg Academy; see *Essays in the History of Mechanics*, p. 261, where this page is reproduced.[14] In the reprint in the *Opera omnia*, the six equations are printed one above the other. The constant i is a scale factor corresponding to a different choice of units from those Euler had used in E177.[15])

How does Euler derive these equations in E479? His justification is (§27): "it follows from the principles of motion" *("per principia motus necesse est")*! According to Truesdell, Euler has recognized that the two principles, Balance of Linear Momentum and Balance of Rotational Momentum, are two independent axioms of mechanics, and hence no "derivation" of the latter from the former is possible. Nor is it necessary to postulate anything about the internal forces.

It is odd, however, that, having insisted so strongly in E177 that the principle of Linear Momentum was the *only* principle for all of mechanics,[16] Euler does not now go back to that declaration and say explicitly that he had been mistaken, and that in fact *two* principles are required.

It appears to me that another *possible* interpretation of Euler's procedure in E479 is that he considered that he had derived the principle of Balance of Rotational Momentum long ago, and that it was not necessary to revisit that derivation.

But here, too, all we can do is speculate, since Euler (uncharacteristically) is very laconic. I think, however, that Truesdell is quite correct to say that Euler here recogizes these two principles —at least in practice— as the two fundamental principles of mechanics.

Of course, the question is really indeterminate. The axioms for mechanics, like those of any other branch of mathematics, can be set up in more than one way. In Truesdell's own book on Rational Mechanics, he derives *both* Balance of Linear Momentum and Balance of Rotational Momentum from an axiom asserting the "frame-indifference" of *work* [T4, vol. I, p. 62, 2$^{\text{d}}$ ed.].

Neither the principle of Balance of Linear Momentum nor the principle of Balance of Rotational Momentum was completely new with Euler. As

[14] The entire article is now available online at the Euler Archive, `www.eulerarchive.org`.

[15] See §27. There, Euler defines $i = 2g$, where "g denotes the height from which a heavy object falls in one second". In other words, Euler's i is the same as our customary "acceleration of gravity". Multiplying $\mathbf{F} = m\mathbf{a}$ (where m is the *mass*) by i, we get $i\mathbf{F} = M\mathbf{a}$, where M is the *weight* of the body.

[16] *E.g.*, §19: "We find ordinarily several such principles... but I remark that all these principles reduce to a single one, which can be regarded as the unique foundation of all of Mechanics.... And it is on that one principle that all the other principles must be established...."

we have seen, the first had been stated in a somewhat different form by Newton, who, for his part, asserted that it had been known to Galileo, Wren, Wallis, and Huygens. The second comes ultimately from the "Law of the Lever", which goes back at least to Archimedes. But it is Euler's merit to have seen the importance and the central position of these principles in Mechanics as a whole. Thus, it seems to me that Truesdell's proposal that they be called *Euler's Two Laws of Mechanics* is just.

And a man who singled out, formulated, and set up the basic principles of the science of Mechanics —not to mention that he showed how to apply these principles to solve a variety of mechanical problems— such a man deserves to be mentioned in the History of Science.

3. An Error of Euler on Rigid Bodies

indignor quandoque bonus dormitat Homerus

—Horace, *Ars Poeticae*, 359

Having, in E177, derived, for the first time, the equations of motion for a rigid body, Euler, like a good teacher, shows how to apply them to a particular example, namely, a homogeneous ball rotating about a fixed axis, which is subject to an external torque. But after having worked out the resulting motion on the basis of his equations, Euler then wants to give an alternative, more elementary, demonstration, starting from first principles. Here, however, he gets into trouble. His alternative method is not correct, and he arrives finally at the correct result only by making another, compensating, error. It's a little hard, in fact, to see just where he went wrong, since his reasoning seems rather loose, involving manipulations with infinitesimals. In this part, I will try to clarify the situation, and explain just what Euler's error was.

Recall that Euler's equations, in modern notation, are

$$\begin{aligned} L_1 &= I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 \\ L_2 &= I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 \\ L_3 &= I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2, \end{aligned}$$

where the I_i are the *principal moments of inertia* of the body, the ω_i are the components of the angular velocity vector of the body, and the L_i are the components of the applied torque. The simple form of these equations, in comparison with Euler's original version in [E177, §55], is the result of a more felicitous choice of coordinate system.

Starting in §59, Euler applies his equations to a particular example: a homogeneous ball, rotating, initially, about a fixed axis, but subject to an external torque.[17]

Euler's own treatment of the case of the rotating ball is somewhat obscure because of an awkward choice of coordinates. Using modern notation, we can easily derive Euler's result as follows. For a homogeneous ball, all the principal moments of inertia are equal because of symmetry: $I_1 = I_2 = I_3 = I$, say. (It is an elementary exercise in calculus to show that $I = \frac{2}{5}MR^2$, where M is the mass of the ball and R is its radius.)

In Euler's equations, then, the nonlinear terms drop out and the equations themselves decouple. In vector notation, we can write

$$\mathbf{L} = I\dot{\boldsymbol{\omega}},$$

where $\mathbf{L}$ is the torque vector and $\boldsymbol{\omega}$ is the angular velocity vector. This equation is easy to solve: taking the torque $\mathbf{L}$ to be *constant*, we get

$$\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \frac{\mathbf{L}t}{I},$$

where $\boldsymbol{\omega}_0$ is the initial angular velocity vector. (In fact, Euler solves the equation only for an *infinitesimal* time t, so he does not have to make any assumption about whether $\mathbf{L}$ is constant or not.)

We see that $\boldsymbol{\omega}$ begins to turn in the direction of the *torque* (not in the direction of the applied *force*). If, as Euler assumes, the applied force $\mathbf{F}$ is applied at the surface of the ball, tangent to the pole of the initial rotation, then $\mathbf{L} = \mathbf{R} \times \mathbf{F}$, where $\mathbf{R}$ is the radius vector, so that the torque $\mathbf{L}$ is perpendicular to $\mathbf{F}$. Euler is interested in the angle φ that the axis of instantaneous rotation at time t (which is the direction of the angular velocity vector $\boldsymbol{\omega}$) makes with the initial axis of rotation. For small t (again, Euler assumes that t is *infinitesimal*), the angle φ will be small, and hence

$$\varphi \approx \tan\varphi = \frac{Lt}{\omega_0 I},$$

(where L is the magnitude of $\mathbf{L}$ and ω_0 is the magnitude of $\boldsymbol{\omega}_0$).

At the end of E177 (§§62–63), Euler wants to give a more direct, elementary, derivation of this result.[18] To do this, he considers two separate

[17] As we noted above, Euler must have had in mind d'Alembert's recent work on the precession of the equinoxes.

[18] Ryoichi Nakata pointed out to me that the *Opera omnia* reprint of this paper contains several errors in these sections, in comparison with the original Berlin publication, which is available online at the Euler Archive, and also at

`www.bbaw.de/bibliothek/digital/index.html`

The Berlin version makes more sense, but still contains the basic error discussed below.

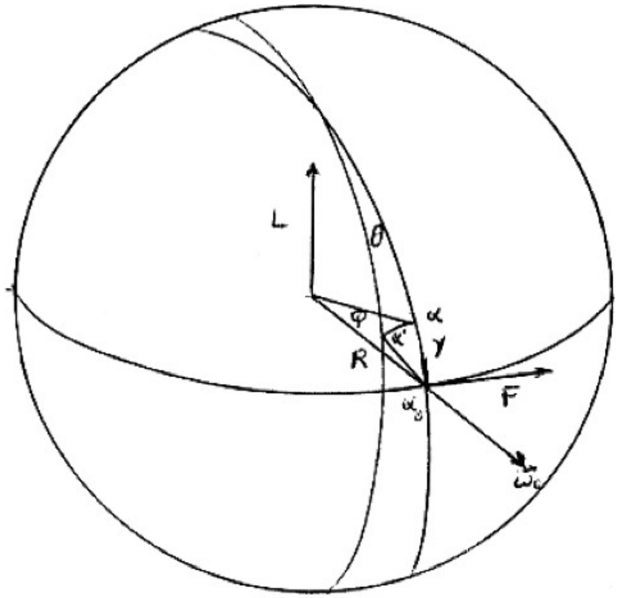

Fig. 1.

motions of the ball. The first is the initial rotation with angular velocity $\boldsymbol{\omega}_0$. The second is the motion that would result from the applied torque **L** under the assumption that the ball was initially at rest. For infinitesimal time t, Euler assumes that these two motions can be *added.* (From our modern point of view, the instantaneous angular velocities can be added because they are vectors.) Let α_0 be the pole of the initial instantaneous rotation and α be the pole of the instantaneous rotation at time t (see the figure). Let γ stand for the great-circle distance from α_0 to α. By virtue of the initial rotation, α will be rotated to α' in the small time t. To first order, the arc $\widehat{\alpha\alpha'}$ will be given by

$$\widehat{\alpha\alpha'} = \gamma\omega_0 t.$$

Now Euler claims that the *second* motion —the one generated by the torque **L** in a ball initially at rest— must rotate the point α' precisely back to α, since α is supposed to be pole of the instantaneous rotation at time t. But, starting from rest, the torque **L** will turn the ball through an angle

$$\theta = \frac{1}{2}\frac{L}{I}t^2$$

in time t. So since, to first order, the points α and α' are at a distance R from the axis of this rotation, we must have

$$\widehat{\alpha\alpha'} = \frac{1}{2}\frac{L}{I}t^2 R.$$

Comparing these two expressions for $\widehat{\alpha\alpha'}$, we can solve for γ.

But here something strange happens. Euler *omits* the factor $\frac{1}{2}$ in the second equation for $\widehat{\alpha\alpha'}$. Thus, the equation he gets is

$$\gamma\omega_0 t = \frac{L}{I}t^2 R,$$

so that

$$\gamma = \frac{Lt}{\omega_0 I} \cdot R.$$

Since it is clear that $\gamma = \varphi R$, this agrees with our previous result.

But something is wrong here! How can Euler get the correct result if he has mistakenly dropped a factor of $\frac{1}{2}$? The answer is that Euler's method is wrong. It is not correct to determine α by the property that it must remain invariant under the superposition of the two motions. Rather, α, the pole of the instantaneous rotation at time t, is the point on the surface of the ball which is *instantaneously at rest.* This is not the same property.

To see this more clearly, consider the following example. Suppose that we put the points of the real line in motion, in such a way that the position x' after time t of the point which was initially at x will be $x' = x + xt - \frac{1}{2}t^2$. This motion is a superposition of two motions. The first, represented by the term xt, is a motion in which each point x of the line moves away from the origin at a constant velocity x. Since points which were initially farther from the origin have greater velocity, in this motion the line is expanding. The second motion, represented by the term $-\frac{1}{2}t^2$, is a *rigid* motion, to the *left*, which has uniform acceleration.

Now we can ask, first, which point is instantaneously at rest at time t? The answer, clearly, is $x = t$, the point which makes $dx'/dt = 0$. On the other hand, we can ask which point, at time t, occupies the same position that it had initially? This requires that $xt - \frac{1}{2}t^2 = 0$. Of course, at time $t = 0$, *all* the points are in their initial position. But if $t > 0$ we can divide by t, to get $x = \frac{1}{2}t$. The two answers are different (and hence so are the questions). But if we had dropped the factor $\frac{1}{2}$ in the equation $x' = x + xt - \frac{1}{2}t^2$, then when we solved the second problem, we would have obtained the correct answer to the first. This is essentially what Euler has done.

How should Euler have solved the problem correctly? Instead of balancing *distances*, he should have balanced *velocities.* Thus, the point α has a horizontal velocity $\gamma\omega_0$ to the left coming from the ball's initial rotation. On the other hand, the torque **L** generates, after time t, a velocity $(L/I)tR$ in the other direction. Since the point α must be instantaneously at rest, these velocities must balance, and we have

$$\gamma\omega_0 = \frac{L}{I}tR,$$

whence

$$\gamma = \frac{Lt}{\omega_0 I}R,$$

and we are done.

According to Clifford Truesdell:

> "In true science, there are mistakes. In mathematics, a mistake may be found by anyone — by a freshman, by an arrogant colleague from another department, by a failure in his profession who up to that time had never done anything correct in his life. A mathematician does not like to be wrong, but he may *be* wrong, and when he is, there is no doubt and no excuse. The truth is the truth, demonstrable, and independent of persons. In other human endeavors, truth is contingent. The politician, the lawyer, the physician, the general, the university official are all modest men, more modest than most mathematicians; they are the first to admit, in theory, that they are fallible, capable of error, perhaps even that in the (carefully unspecified) past they have not always been right; but today, here and now, they are *never* wrong, and nothing can make them wrong. The patient can die in agony, the army can be killed off to a man, the nation can be annexed and enslaved, the university can be overrun by a horde of students, but the error, if any there were, lies elsewhere than with those in command." [T2, pp. 102–103]

Euler was a mathematician.

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Leonhard Euler: Life, Work and Legacy
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Euler's Analysis Textbooks

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The most important precalculus and calculus textbooks of the eighteenth century were the series written in Latin by Leonhard Euler: the two volumes of the *Introductio in Analysin Infinitorum* (*Introduction to Analysis of the Infinite*, 1748) [E101,E102], the *Institutiones Calculi Differentialis* (*Basic Principles of the Differential Calculus*, 1755) [E212], and the three volumes of the *Institutiones Calculi Integralis* (*Basic Principles of the Integral Calculus*, 1768–1770) [E342,E366,E385]. These are the first "modern" texts, in the sense that today's mathematicians can read them relatively easily without having to translate older notions such as that of fluxions into modern terms. In particular, they use what looks like modern notation; of course, the reason for this is that Euler introduced much of our current notation for the calculus. Nevertheless, when one considers these texts in detail, one sees that they are not so "modern." There is much that is missing that would be included in a current text covering the same basic material and there is much that Euler includes that is missing from today's texts. And some of Euler's methods are quite different from modern ones. Thus, in our survey of these basic texts, we will consider both the similarities and the differences between Euler's work and what one finds in texts written some two hundred fifty years later.

In what follows, quotations are taken from the Blanton translations of the *Introductio* and the first part of the *Differential Calculus*.

1. Introduction to Analysis of the Infinite

Book I of the *Introductio*, Euler's "precalculus" text, was an attempt to develop those topics "which are absolutely required for analysis" so that the reader "almost imperceptibly becomes acquainted with the idea of the infinite." [p. v] The "idea of the infinite" is certainly a critical idea in the study of the calculus itself. But, since for us and for Euler, analysis is concerned with functions, Euler began his work with that topic. In fact, making functions the central topic of the book represented a change in viewpoint in the history of analysis. Newton and Leibniz, in their development of calculus, dealt with "curves." And the title of the first calculus book, by the Marquis de l'Hospital, was, after all, *Analysis of infinitely small quantities for the understanding of curves.* Euler, in his studies of differential equations in the 1730s, had gradually come to the conclusion that the notion of "function" should be the basis of analysis. And so Chapter 1 of the *Introductio* opens with a definition of the term: "A **function** of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities."[p. 3] To understand this definition, we need to be clear on the meaning of the various terms in it. First of all, a "variable quantity" is "one which can take on any value," that is, any numerical value. (Later, we will see that Euler usually included complex numbers as well as real numbers under the term "any value.") Second, "constant quantities" are those which "always keep the same value." Finally, since "analytic expression," seems to mean "formula" the statement as to how these formulas are to be formed, "in any way whatsoever," can only be understood by considering his further discussion.

Euler explicitly classified functions into two types, algebraic and transcendental. The former are formed from the variables and constants by addition, subtraction, multiplication, division, raising to a power, extraction of roots, and the solution of an equation. The latter are those defined by exponentials, logarithms, trigonometry, and, more generally, by integrals. Because integrals could not be discussed in a precalculus work, the transcendental functions discussed in the *Introductio* were limited to the special cases of trigonometric, exponential and logarithmic functions. Furthermore, as the remainder of the *Introductio* showed, "in any way whatsoever" includes the notion of infinite series, infinite products, and infinite continued fractions. Thus, as will become apparent, an important tool in Euler's discussion of functions is that of a power series.

Euler also made a distinction between "single-valued" and "multiple-valued" functions. Today, it is only the former that is called a "function."

By the latter, Euler meant an "analytic expression" that "for some value substituted for the variable z, the function determines several values."[p. 7] For example, for Euler the expression $\sqrt{z}$ represented a two-valued function and the expression $\arcsin z$ represented an infinite-valued function. He noted that, in general, Z is an n-valued function of z if Z satisfies a polynomial equation of degree n, where the coefficients are all single-valued functions of z. On the other hand, the function $z^{1/3}$ - although strictly speaking a three-valued function - can be thought of as single-valued since there is only a single real value corresponding to each real number z. Despite his discussion of multiple-valued functions, Euler concentrated, in the remainder of the book, on single-valued functions, what we call "functions."

In the next several chapters, Euler discussed some basic results about algebraic functions. For example, he noted that if f is a root of the polynomial $p(z)$, then $z - f$ is a factor of $p(z)$. Therefore, if the roots of $Z = Az^n + Bz^{n-1} + Cz^{n-2} + \cdots + K$ are $f, g, h, \ldots$, then Z can be factored as $Z = A(z-f)(z-g)(z-h)\cdots$. Similarly, if $Z = A + Bz + Cz^2 + \cdots + Kz^n$ has roots $f, g, h, \ldots$, then Z can be factored as

$$Z = A\left(1 - \frac{z}{f}\right)\left(1 - \frac{z}{g}\right)\left(1 - \frac{z}{h}\right)\cdots$$

(Note that Euler did not use subscripts, and we will try to stick as closely to Euler's notation as possible. Nevertheless, we realize that in results such as these, subscripts would make the meaning much clearer for us.)

Euler then proceeded to deal with various kinds of factoring of real polynomials. It appears that he is working towards at least a statement of the fundamental theorem of algebra, the theorem that in the eighteenth century was given as: Any polynomial with real coefficients can be factored into the product of real linear and/or real quadratic factors. But, although he wrote a paper in 1746 [E170] where he gave what he felt was a complete proof of this result, he did not do so in the *Introductio*. What he did do is show, for example, that complex linear factors of a real polynomial always occur in pairs whose product is real, that if a polynomial is the product of four complex linear factors, then it can also be represented as the product of two real quadratic factors, and, by use of the Intermediate Value Theorem, that any polynomial of odd degree has at least one real linear factor. Euler's chief goal in dealing with factoring, however, was to demonstrate the partial fraction decomposition of rational functions, a task necessary for later use in integral calculus.

After next considering how to simplify functions by appropriate substitutions, he then explained how to represent rational functions as power series. This again is preliminary to later work in developing power series for transcendental functions. As Euler writes, "Since the nature of polynomial functions is very well understood, if other functions can be expressed by different powers of z in such a way that they are put in the form $A+Bz+Cz^2+Dz^3+\cdots$, then they seem to be in the best form for the mind to grasp their nature, even though the number of terms is infinite." [p. 50]

In fact, Euler claimed that any function can be expressed as an infinite series – and much of the text is devoted to giving examples to convince the reader of the truth of that statement. Among the first of these examples is the representation of $(1+Z)^m$ as a power series, where m can be any rational number and Z is any polynomial function of z:

$$(1+Z)^m = 1 + \frac{m}{1}Z + \frac{m(m-1)}{1\cdot 2}Z^2 + \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}Z^3 + \cdots$$

Although he made frequent use of the binomial theorem later on, he did not discuss the convergence of this series, nor, in general, the convergence of any power series.

The central chapters of Book I of the *Introductio*, and those which were to prove most influential, are the chapters dealing with the exponential, logarithmic, and trigonometric functions, for it is there that Euler introduced the notations and concepts which were to make obsolete all the discussions of such functions in earlier texts. All modern treatments of these functions are in some sense derived from those of Euler. Thus Euler defined exponential functions as powers in which exponents are variable and then — and this is a first — defined logarithms in terms of these. Namely, if $a^z = y$, Euler defined z to be the logarithm of y with base a. The basic properties of the logarithm function are then derived from those of the exponential.

Using these properties, Euler showed how one can calculate a logarithm. As an example, he calculated $\log_{10} 5$. (In the rest of this calculation, we will simply write log for $\log_{10}$.) Beginning with $A=1$ and $B=10$, he first calculated $C=\sqrt{AB}=\sqrt{10}=3.16227$. By the properties of the logarithm, $\log C = \frac{1}{2}(\log A + \log B) = \frac{1}{2}\cdot 1 = 0.5$. Similarly, he calculated $D=\sqrt{BC}=\sqrt{31.6227}=5.623413$ and then noted that $\log D = \frac{1}{2}(\log B + \log C) = 0.75$. He continued in this manner for 26 values, eventually finding $X = 4.999997$, $\log X = 0.6989697$, $Y = 5.000003$, $\log Y = 0.6989702$, and then $Z=\sqrt{XY}$, with $Z = 5.000000$ and $\log Z = 0.6989700$. Euler noted that this is essentially the original method of Briggs and Vlacq, but that, as he would show shortly, there are much more efficient ways of performing this calcululation.

Before getting to those methods of calculation, Euler demonstrated the standard method of converting from logarithms in one base to logarithms in a second and then pointed out that logarithm tables are "of great use in carrying out numerical computations."[p. 84] To demonstrate this latter point, he presented a few problems. For example, "Since after the flood all men descended from a population of six, if we suppose that the population after two hundred years was 1,000,000, we would like to find the annual rate of growth."[p. 86] To solve this, he assumed that the yearly increase was $1/x$, so after two hundred years, the population can be expressed as

$$\left(\frac{1+x}{x}\right)^{200} 6 = 1,000,000.$$

It follows that

$$\log\left(\frac{1+x}{x}\right) = \frac{1}{200}\log\left(\frac{1,000,000}{6}\right) = \frac{1}{200}(5.2218487) = 0.0261092.$$

He then found the "antilog" of 0.0261092 to be 1.061963, so he could solve for x, which equals approximately 16. In other words, the yearly population incrased by 1/16. However, he notes, if this rate were to continue for 400 years, the population would be 166,666,666,666, and "the whole earth would never be able to sustain that population." (Note that this problem was written well before Malthus's famous essay.) In another problem he demonstrated that an annual rate of growth of only $1/144$ would be sufficient to double the human population in a century, thus concluding that "it is quite ridiculous for the incredulous to object that in such a short space of time [i.e. from Biblical creation to the present] the whole earth could not be populated beginning with a single man."[p. 87]

As one final example, we look at a problem in which Euler used logarithms to find the value of an unknown exponent: "A certain man borrowed 400,000 florins at the usurious rate of five percent annual interest. Suppose that each year he repays 25,000 florins. The question is how long will it be before the debt is repain completely."[p. 88] Setting $a = 400,000$, $b = 25,000$, and $n = 1.05$, Euler showed by use of the sum of a geometric series that after x years, the man owes $n^x a - (n^x b - b)/(n-1)$ florins. When the debt is paid, this value should be zero, so the resulting equation becomes:

$$n^x = \frac{b}{b-(n-1)a}.$$

By taking logarithms, we get

$$x = \frac{\log b - \log(b-(n-1)a)}{\log n} \approx 33 \text{ years.}$$

After dealing with basic functional aspects of the logarithm and exponential, Euler next developed their power series for an arbitrary base a by use of the binomial theorem. His technique made important use of both "infinitely small" and "infinitely large" numbers. These concepts have diappeared from modern mathematics, partly because their use led to numerous inconsistencies. However, Euler rarely erred when he used them. For example, he noted that since $a^0 = 1$, it follows that $a^\omega = 1+\psi$ where both ω and ψ are infinitely small. Therefore, ψ must be some multiple of ω, depending on a, and

$$a^\omega = 1 + k\omega \qquad \text{or} \qquad \omega = \log_a(1 + k\omega).$$

Euler noted next that for any j, $a^{j\omega} = (1 + k\omega)^j$, and, expanding the right side by the binomial theorem, that

$$a^{j\omega} = 1 + \frac{j}{1}k\omega + \frac{j(j-1)}{1\cdot 2}k^2\omega^2 + \frac{j(j-1)(j-2)}{1\cdot 2\cdot 3}k^3\omega^3 + \cdots$$

If j is taken to equal z/ω, where z is finite, then j is infinitely large and $\omega = z/j$. The series now becomes

$$a^z = 1 + \frac{1}{1}kz + \frac{1(j-1)}{1\cdot 2j}k^2z^2 + \frac{1(j-1)(j-2)}{1\cdot 2j\cdot 3j}k^3z^3 + \cdots$$

Because j is infinitely large, $(j - n)/j = 1$ for any positive integer n. The expansion then reduces to the series

$$a^z = 1 + \frac{kz}{1} + \frac{k^2z^2}{1\cdot 2} + \frac{k^3z^3}{1\cdot 2\cdot 3} + \cdots$$

where k depends on the base a. Euler also noted that the equation $\omega = \log_a(1+kw)$ implies that if $(1+kw)^j = 1+x$, then $\log_a(1+x) = j\omega$. Since then $k\omega = (1 + x)^{1/j} - 1$, it follows that

$$\log_a(1 + x) = \frac{j}{k}(1 + x)^{\frac{1}{j}} - \frac{j}{k}.$$

Another clever use of the binomial theorem finally allowed him to derive the series

$$\log_a(1 + x) = \frac{1}{k}\left(\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right).$$

The choice of $k = 1$, or equivalently, $a = e$, gave the standard power series for e^z and $\ln z$. The latter series, and one that is easily derived from it, namely

$$\ln\left(\frac{1+x}{1-x}\right) = \frac{2x}{1} + \frac{2x^3}{3} + \frac{2x^5}{5} + \cdots$$

can then be used, as promised earlier, to efficiently calculate logarithms to the base e – and Euler proudly displayed values for the logarithms of the first 10 positive integers to 25 decimal places.

Euler's treatment of "transcendental quantities which arise from the circle"[p. 101] is the first textbook discussion of the trigonometric functions which deals with these quantities as functions having numerical values, rather than as lines in a circle of a certain radius. Euler did not, in fact, give any new definition of the sine and cosine. He merely noted that he would always consider the sine and cosine of an arc z to be defined in terms of a circle of radius 1. All basic properties of the sine and cosine, including the addition and periodicity properties, are assumed known, although Euler did derive some relatively complicated identities that do not usually appear in today's texts. For example, starting with the "sum-to-product" rules, he derived

$$\left(\tan\left(\frac{a+b}{2}\right)\right)^2 = \frac{(\sin a + \sin b)}{(\sin a - \sin b)}\frac{(\cos b - \cos a)}{(\cos a + \cos b)}.$$

More importantly, he derived the power series for the sine and cosine through use of the binomial theorem and complex numbers.

Euler began by deriving the cases $n = 2$ and $n = 3$ of DeMoivre's formula and then quoted the general result: $(\cos z \pm i \sin z)^n = \cos nz \pm i \sin nz$. He then concluded that

$$\cos nz = \frac{(\cos z + i \sin z)^n + (\cos z - i \sin z)^n}{2}$$

and, by expanding the right side, that

$$\cos nz = (\cos z)^n - \frac{n(n-1)}{1 \cdot 2}(\cos z)^{n-2}(\sin z)^2 + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4}(\cos z)^{n-4}(\sin z)^4 + \cdots$$

Again letting z be infinitely small, n infinitely large, and $nz = v$ finite, it follows from $\sin z = z$ and $\cos z = 1$ that

$$\cos v = 1 - \frac{v^2}{1 \cdot 2} + \frac{v^4}{1 \cdot 2 \cdot 3 \cdot 4} - \cdots$$

Similarly, Euler derived the power series for the sine. Having given the value of π to 127 decimal places, Euler then used these power series to show how to calculate the value of the sine and cosine for any fractional multiple m/n of $\pi/2$.

Virtually as an aside, Euler derived the formulas relating complex exponentials to sines and cosines: $e^{\pm iv} = \cos v \pm i \sin v$, then used these to develop the classic power series for the arctangent:

$$\arctan t = \frac{t}{1} - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \cdots.$$

Noting that this series implies that $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \cdots$ but that this series "hardly converges," he then manipulated with the tangent function to give a much more rapidly converging series for $\pi/4$, namely, $\pi/4 = \arctan(1/2) + \arctan(1/3)$.

Furthermore, in his chapter on trinomial factors, Euler essentially found the expressions for the complex nth roots of unity. He began by noting that the real trinomial $p^2 - mz + q^2z^2$ is irreducible over the real numbers and therefore has complex linear factors whenever $m^2 < 4p^2q^2$, or, $\frac{m}{2pq} < 1$. In this case, $\frac{m}{2pq}$ is the cosine of some angle ϕ, so $m = 2pq\cos\phi$, and we can write the original polynomial in the form $p^2 - 2pqz\cos\phi + q^2z^2$, whose linear factors are $qz - p(\cos\phi \pm i\sin\phi)$. It follows that the two zeros of the polynomial are $z = (p/q)(\cos\phi \pm i\sin\phi)$. In the special case of irreducible quadratic factors of $a^n - z^n = 0$, Euler noted that if $p^2 - 2pqz\cos\phi + q^2z^2$ is such a factor, and if $r = p/q$, then $r^n\cos n\phi = a^n$ and $r^n\sin n\phi = 0$. It follows that $n\phi$ is an even multiple of π, so $\cos n\phi = 1$ and $r = a$. The trinomial factor then becomes $a^2 - 2az\cos\frac{2k\pi}{n} + z^2$, with roots

$$z = a\left(\cos\frac{2k\pi}{n} \pm i\sin\frac{2k\pi}{n}\right).$$

Although Euler did not look at the special case $a = 1$, he did consider explicitly the cases where $n = 1, 2, 3, 4, 5, 6$, factoring each of the polynomials $a^n - z^n$ for those values of n into their irreducible factors and thus determining the nth roots of a^n for those values of n.

The remainder of volume one of the *Introductio* includes much else about infinite processes, including infinite products as well as infinite series. For example, Euler factored the hyperbolic sine and cosine functions (although he did not name these) as

$$\frac{e^x - e^{-x}}{2} = x\left(1 + \frac{x^2}{\pi^2}\right)\left(1 + \frac{x^2}{4\pi^2}\right)\left(1 + \frac{x^2}{9\pi^2}\right)\cdots$$

and

$$\frac{e^x + e^{-x}}{2} = \left(\frac{1 + 4x^2}{\pi^2}\right)\left(\frac{1 + 4x^2}{9\pi^2}\right)\left(\frac{1 + 4x^2}{25\pi^2}\right)\cdots.$$

Because he could also write

$$\frac{e^x - e^{-x}}{2} = x\left(1 + \frac{x^2}{1\cdot 2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{x^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} + \cdots\right)$$

and

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} + \frac{x^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \cdots,$$

he could use the relationship between roots and coefficients of a polynomial equation (extended to power series) to calculate the infinite sums

$$\sum_{n=1}^{\infty}\frac{1}{n^{2k}}$$

and

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{2k}}.$$

The simplest of these formulas gave a solution to the question of Johann Bernoulli as to the sum of the reciprocal squares of the integers, that is,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

But he could also show, for example, that the sum of the reciprocal fourth powers equals $\frac{\pi^4}{90}$ and that the sum of the reciprocal squares of the odd integers is $\frac{\pi^2}{8}$. Similarly, Euler derived Wallis's infinite product formula

$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}$$

by using infinite product representations of the sine and the cosine functions. Euler also considered the equality

$$\prod_{p} \frac{1}{1 - \frac{1}{p^n}} = \sum_{m} \frac{1}{m^n},$$

where the product is taken over all primes and the sum over all positive integers. This product and sum, both generalized to the case where n is any complex number s, are today called the Riemann zeta function of the variable s, the study of which has led to much new mathematics. Finally, Euler devoted the whole of chapter 16 to an introductory study of the theory of partitions, the number of ways one can write a given number as the sum of positive numbers. It turns out that the answer depends on the coefficients of a power series determined by an infinite product of reciprocals of polynomials of the form $1 - x^n$.

With volume one being completely devoted to "pure analysis," Euler decided that he also needed a volume devoted to what we now call analytic geometry, since "analysis is ordinarily developed in such a way that its application to geometry is shown."[p. vi] Thus, he began this second volume by dealing with curves given by functions. As was customary at the time, Euler used only a single axis, not our standard two. The "variable quantity" x (or, the abscissa) is laid out along a horizontal straight line, while the dependent quantity y is simply determined at each point along that horizontal line by erecting a perpendicular (the ordinate) of the appropriate length, above the line if y is positive and below if y is negative. Euler noted that it is also possible to have the ordinate oblique to the axis of abscissas. The curve that represents the function is then constructed by connecting the tips of the perpendicular straight lines y. As Euler wrote, "any function of x is translated into geometry and determines a line, either straight or curved, whose nature is dependent on the nature of the function."[p. 5]

Recall, however, that Euler distinguished between multiple-valued and single-valued functions. Thus, in his introductory discussion of curves in general, he showed how this distinction is reflected in the geometry. For a single-valued function, "to each abscissa there corresponds a unique ordinate," and therefore the curve extends indefinitely with the axis. But for multiple-valued functions, each abscissa x corresponds to two or more ordinates y. Euler then gave several examples, with figures, explaining the various kinds of situations which could occur, including situations where for some values of x there are no real values of y at all. So in some cases, the curve can be a closed curve; the curve could be simple or it could cross itself; it may even have several parts, some of which are closed while others go off toward infinity.

After his initial general discussions, Euler considered separately curves of first order (i.e., straight lines), curves of second order (i.e., conic sections), curves of third order, and curves of fourth order. Euler gave the general equation of a straight line in the form $\alpha + \beta x + \gamma y = 0$, noting also that the line is actually determined by the two ratios $\beta : \alpha$ and $\gamma : \alpha$. Thus, two points suffice to determine exactly one straight line. Interestingly, Euler gave no geometric interpretations of the coefficients in the equation of a straight line; there is nothing about slope or intercepts. However, he did note that to find where the line intersects the axis, one simply sets $y = 0$ and solves.

A curve of second order is given by the equation $\alpha + \beta x + \gamma y + \delta x^2 + \epsilon xy + \zeta y^2 = 0$ and, for the same reason as before, the curve is really determined by five ratios or, to put it another way, five points completely determine such a curve. With that in mind, Euler discussed various properties of second order curves in general, including such concepts as conjugate diameters, foci, parameters, vertices, and a method of constructing a tangent. He noted, in fact, that Newton used many of these properties of the conic sections to solve problems in the *Principia.* After the generalities, Euler showed how to recognize the three types of conic sections, the ellipse, the parabola, and the hyperbola, noting that the essential difference "lies in the number of branches which go to infinity."[p. 83] The ellipse has no part going to infinity; the parabola has two branches going to infinity; while the hyperbola has four. He then derived the basic properties of these three types, using their equations rather than the sectioning of a cone. Later in volume two, Euler classified and discussed both third order and fourth order curves, and gave a brief discussion of the curves defined by the transcendental functions discussed in volume one, the exponential (or logarithmic) curves and the trigonometric curves.

Euler concluded the *Introductio* with a systematic treatment of the study of quadric surfaces in three-dimensional space. Euler used a single coordinate plane, with only one axis defined on it, and represented the third coordinate by the perpendicular distance from a point to that plane. But he did remark that it was possible to use three coordinate planes and often described a surface by means of its trace in various such planes. He gave the equation for a plane in three space as $\alpha x + \beta y + \gamma z = a$ but described the meaning of the coefficients only in terms of the cosine of the angle θ between that plane and the xy-plane: $\cos\theta = \gamma/\sqrt{\alpha^2 + \beta^2 + \gamma^2}$. In his discussion of the quadric surfaces themselves, Euler began by noting that the general second degree equation in three variables can be reduced by a change of coordinates to one of the forms $Ax^2 + By^2 + Cz^2 = a^2$, $Ax^2 + By^2 = Cz$, or $Ax^2 = By$. The relationships among the coefficients then determined the type of surface: ellipsoid, elliptic or hyperbolic paraboloid, elliptic or hyperbolic hyperboloid (now called they hyperboloids of one and two sheets, respectively), cone, and parabolic cylinder.

2. Basic Principles of the Differential Calculus

Although much of volume one of the *Introductio* was largely concerned with series, Euler considered this material as the algebra necessary for the calculus. He discussed the calculus itself in his *Institutiones Calculi Differentialis* of 1755. That work began with his definition of the differential calculus: "[It] is a method for determining the ratio of the vanishing increments that any functions take on when the variable, of which they are functions, is given a vanishing increment."[p. vii] Euler had already given a definition of "function" in the *Introductio*, but here he generalized it somewhat: "Those quantities that depend on others in this way, namely, those that undergo a change when others change, are called functions of these quantities. This definition applies rather widely and includes all ways in which one quantity can be determined by others."[p. vi] Thus Euler no longer required a function to be an "analytic expression." The reason for this change is perhaps connected to the controversy among Euler, D'Alembert and Daniel Bernoulli over the vibrating string problem and the nature of possible solutions to partial differential equations coming from physical situations. Euler was, naturally, well aware of the many applications of the differential calculus to geometry. He wrote, however, that in this regard "I have nothing new to offer, and this is all the less to be required, since in other works I have treated this subject so fully."[p. xi] Thus, he decided to keep the *Differential Calculus* as a work of pure analysis, so that there was no need for any diagrams. Similarly, Euler did not deal with the relationship to the subject to physics.

Because calculus has to do with ratios of "vanishing increments," Euler began with a discussion of increments in general, that is, with finite differences. Given a sequence of values of the variable, say x, $x+\omega$, $x+2\omega$, ... and the corresponding values of the function y, y', y'', ..., Euler considers various sequences of finite differences. The first differences are $\Delta y = y'-y, \Delta y' = y''-y'$, $\Delta y'' = y'''-y''$, ...; the second differences are $\Delta\Delta y = \Delta y' - \Delta y$, $\Delta\Delta y' = \Delta y'' - \Delta y'$, ...; third and higher differences are defined analogously. For example, if $y = x^2$, then $y' = (x+\omega)^2$ and $\Delta y = 2\omega x + \omega^2$, $\Delta\Delta y = 2\omega^2$, while the third and higher differences are all 0. Using various techniques, including expansion in series, Euler calculated the differences for all of the standard elementary functions. Furthermore, using the sum Σ to denote the inverse of the Δ operation, he derived various formulas for that operation as well. Thus, because $\Delta x = \omega$, it followed that $\Sigma\omega = x$ and that $\Sigma 1 = x/\omega$. Similarly, because $\Delta x^2 = 2\omega x + \omega^2$, it followed that $\Sigma(2\omega x + \omega^2) = x^2$ and that

$$\Sigma x = \frac{x^2}{2\omega} - \Sigma\frac{\omega}{2} = \frac{x^2}{2\omega} - \frac{x}{2}.$$

Euler then easily developed rules for Σ from the corresponding rules for Δ. Rather than discuss the rules for finite differences, however, it will be more useful to discuss Euler's rules for differentials.

"The analysis of the infinite ... is nothing but a special case of the method of differences ... wherein the differences are infinitely small, while previously the differences were assumed to be finite."[p. 64] Euler's rules for calculating with these infinitely small quantities, the differentials, produce the standard formulas of the differential calculus. For example, if $y = x^n$, then $y' = (x+dx)^n = x^n + nx^{n-1}dx + \frac{n(n-1)}{1\cdot 2}x^{n-2}dx^2 + \cdots$. Thus $dy = y' - y = nx^{n-1}dx + \frac{n(n-1)}{1\cdot 2}n^{n-2}dx^2 + \cdots$. "In this expression the second term and all succeeding terms vanish in the presence of the first term."[p. 77] Thus $d(x^n) = nx^{n-1}dx$. It should be noted here that Euler intended his argument to apply not just to positive integral powers of x, but to arbitrary powers. The binomial theorem, after all, applies to all powers. Thus the expansion of $(x+dx)^n$ does not necessarily represent a finite sum; it may well represent an infinite series. Euler therefore concluded immediately that $d(\frac{1}{x^m}) = -\frac{m\,dx}{x^{m+1}}$ and, more generally, that $d(x^{\mu/\nu}) = (\mu/\nu)x^{(\mu-\nu)/\nu}dx$.

Euler did not give an explicit statement of the modern chain rule, but did deal with special cases as the need arose. Thus if p is a function of x whose differential is dp, then $d(p^n) = np^{n-1}dp$. Euler's derivation of the product rule was the same as the original derivation of Leibniz, but his derivation of the quotient rule was more original. He expanded $1/(q+dq)$ into the power series

$$\frac{1}{q+dq} = \frac{1}{q}\left(1 - \frac{dq}{q} + \frac{dq^2}{q^2} - \cdots\right),$$

neglected the higher order terms, and then wrote

$$\frac{p+dp}{q+dq} = (p+dp)\left(\frac{1}{q} - \frac{dq}{q^2}\right) = \frac{p}{q} - \frac{p\,dq}{q^2} + \frac{dp}{q} - \frac{dp\,dq}{q^2}.$$

It follows, since the second order differential $dp\,dq$ vanishes with respect to the first order ones, that

$$d\left(\frac{p}{q}\right) = \frac{p+dp}{q+dq} - \frac{p}{q} = \frac{dp}{q} - \frac{p\,dq}{q^2} = \frac{q\,dp - p\,dq}{q^2}.$$

The differential of the logarithm requires the power series derived in the *Introductio*. If $y = \ln x$, then

$$dy = \ln(x+dx) - \ln(x) = \ln\left(1 + \frac{dx}{x}\right) = \frac{dx}{x} - \frac{dx^2}{2x^2} + \frac{dx^3}{3x^3} - \cdots$$

Dispensing with the higher order differentials immediately gave Euler the formula $d(\ln x) = \frac{dx}{x}$. With the basic rule determined, Euler discussed numerous examples which make use of the properties of logarithms. For example, he showed that if

$$y = \frac{1}{2}\ln\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\right),$$

then, since $y = \frac{1}{2}\ln(\sqrt{1+x^2}+x) - \frac{1}{2}\ln(\sqrt{1+x^2}-x)$, we have

$$dy = \frac{\frac{1}{2}\,dx}{\sqrt{1+x^2}} + \frac{\frac{1}{2}\,dx}{\sqrt{1+x^2}} = \frac{dx}{\sqrt{1+x^2}}.$$

Euler calculated the differential of the exponential function two ways, first using the logarithm and second from basic principles. Thus, if $y = a^x$, one can take the logarithm of both sides to get $\ln y = x \ln a$. Taking differentials then gives $dy/y = dx \ln a$, so that $dy = y\,dx \ln a = a^x\,dx \ln a$. On the other hand, to calculate the differential of $y = a^x$ directly, Euler noted that $dy = a^{x+dx} - a^x = a^x(a^{dx} - 1)$ and then expanded the expression inside the parentheses by using power series:

$$a^{dx} = 1 + dx \ln a + \frac{dx^2(\ln a)^2}{2} + \cdots$$

Neglecting higher order differentials gives $a^{dx} - 1 = dx \ln a$, so $dy = a^x\,dx \ln a$, as before.

Curiously, Euler next proceeded to calculate the differential of the arcsine rather than the sine, with his initial approach being through complex numbers. Substituting $y = \arcsin x$ into the formula $e^{iy} = \cos y + i \sin y$ gives

$e^{iy} = \sqrt{1-x^2} + i\,x$. It follows that $y = \frac{1}{i}\ln(\sqrt{1-x^2} + i\,x)$ and therefore that

$$dy = d(\arcsin x) = \frac{1}{i}\frac{1}{\sqrt{1-x^2}+i\,x}\left(\frac{-x}{\sqrt{1-x^2}}+i\right)dx = \frac{dx}{\sqrt{1-x^2}}.$$

On the other hand, he noted that if $y = \arcsin x$, then $x = \sin y$. Therefore $x + dx = \sin(y + dy) = \sin y \cos\,dy + \cos y \sin\,dy$. But as dy vanishes, the arc becomes equal to its sine and its cosine becomes equal to 1. Therefore $x + dx = \sin(y + dy) = \sin y + dy \cos y$. Since $\cos y = \sqrt{1-x^2}$, it follows that $dx = dy\sqrt{1-x^2}$ or, again,

$$dy = \frac{dx}{\sqrt{1-x^2}}.$$

But rather than note at this point that $d(\sin x) = \cos x\,dx$, Euler, several pages later, did a new computation: $d(\sin x) = \sin(x + dx) - \sin x = \sin x \cos dx + \cos x \sin dx - \sin x$. He then recalled his series expansions of the sine and cosine and, again rejecting higher order terms, noted that $\cos dx = 1$ and $\sin dx = dx$. It follows that $d(\sin x) = \cos x\,dx$ as desired. (To be fair to Euler, at this point he also noted that this result could be easily derived – without power series – from the previous calculation of the differential of the arcsine. As a consummate textbook writer, Euler delighted in being able to derive results in several different ways.)

The central concepts of Euler's chapter on functions of two or more variables, as in the case of functions of one variable, are that of the differential and the differential coefficient. Euler showed, chiefly through the use of examples, that if V is a function of the two variables x and y, then dV, the change in V resulting from the changes x to $x + dx$ and y to $y + dy$, was given by $dV = p\,dx + q\,dy$ where p, q are the differential coefficients resulting from leaving y and x constant, respectively. There is naturally no difficulty in calculating p or q, since, in modern terms, $p = \partial V/\partial x$ and $q = \partial V/\partial y$. One merely applies the rules already derived, treating one or the other variable as a constant. Euler showed further, by an algebraic argument involving differentials, that the "mixed partial derivatives" are equal. It follows, then, that if $dV = p\,dx + q\,dy$, then $\partial p/\partial y = \partial q/\partial x$. But Euler also claimed, with a nod to the upcoming volume on integral calculus, that if this equality is true for the differential $p\,dx + q\,dy$, then "from the principles of integration," the differential comes from differentiating some function V.

Euler's text has many other features, including an introduction to differential equations, in which he shows how to generate these from a given equation in two variables, a discussion of the Taylor series, a chapter on various methods of converting functions to power series, an extensive discussion on finding the sums of various series, including those for the sums of the various powers of the integers, and a variety of ways of finding the roots of equations numerically. The remainder of the discussion here, however, will center on Euler's two chapters on finding maxima and minima. Recall that there are no diagrams in the text and therefore no pictures of curves possessing maxima or minima. Everything is done analytically. But Euler began the discussion by distinguishing between an absolute maximum, a value greater than any other of the function, and a local maximum, a value of y taken at $x = f$ which is greater than any other value of y for x "near" f on either side.

Euler derived the basic criteria for a function to have a maximum or minimum value at $x = \alpha$, in terms of the first and second derivatives, by the use of the Maclaurin series. But Euler bolstered his methodology with numerous examples and often sought to generalize. Thus, after considering maxima and minima for several specific polynomials, he discussed in some detail the case of an arbitrary polynomial $y = x^n + Ax^{n-1} + Bx^{n-2} + \cdots + D$. After dealing with several cases of rational functions, he considered the more general rational function

$$\frac{(\alpha + \beta x)^m}{(\gamma + \delta x)^n}.$$

After discussion of the lack of a power series for $x^{2/3}$ around 0, and therefore, the necessity of formulating some different criteria for a maximum or minimum, he dealt with the more general case $x^{2pz/(2q-1)}$. Most of Euler's examples are of algebraic functions, but he concluded with a few examples using transcendental functions, including the functions $x^{1/x}$ and $x \sin x$, both of which required detailed numerical work to arrive at an exact solution for an extreme value.

For extrema of functions V of two variables, Euler began by considering the special case of functions of the form $X + Y$ where X is a function solely of x and Y of y. In that case, a pair of values (x_0, y_0) such that x_0 is a maximum for X and y_0 a maximum for Y clearly gives a maximum for $X + Y$. For the more general case, Euler realized, by holding each variable constant in turn, that an extreme value of V can only occur when the differential $dV = P\,dx + Q\,dy = 0$, therefore only when both $P = \partial V/\partial x = 0$ and $Q = \partial V/\partial y = 0$. The question of determining whether a point (x_0, y_0) where both first partial derivatives vanish produces a maximum, a minimum, or neither is more difficult and, in fact, Euler failed to give complete results. He claimed, in fact, that if $\frac{\partial^2 V}{\partial x^2}$ and $\frac{\partial^2 V}{\partial y^2}$ are both positive at (x_0, y_0), then the function V has a minimum there, and if they are both negative, there is a maximum. Euler gave several examples illustrating the method, including $V = x^3 + ay^2 - bxy + cx$. He noted that an extreme value would occur when

$$x = \frac{b^2 \pm \sqrt{b^4 - 48a^2c}}{12a},$$

as long as $b^2 - 48a^2c > 0$. Furthermore, since $\frac{\partial^2 V}{\partial x^2} = 6x$ and $\frac{\partial^2 V}{\partial y^2} = 2a$, he claimed that when $a > 0$ and both possible values of x are positive, then the two extreme values are both minima. In particular, in Euler's special case where $a = 1$, $b = 3$, and $c = 3/2$, his criteria imply that $V = x^3 + y^2 - 3xy + (3/2)x$ has a minimum both when $x = 1$, $y = 3/2$ and when $x = 1/2$, $y = 3/4$. Unfortunately, Euler was wrong; the latter point is not a minimum, but a saddle point.

3. Basic Principles of the Integral Calculus

Euler began the final part of his trilogy in analysis, the *Institutiones Calculi Integralis*, with a definition of integral calculus. It is the method of finding, from a given relation of differentials of certain quantities, the quantities themselves. Namely, for Euler as it was for Johann Bernoulli, integration is the inverse of differentiation rather than the determination of an area. Thus the first part of the work dealt with techniques for integrating (finding antiderivatives of) functions of various types while the remainder of the text dealt with the solutions of differential equations. Although Euler began his section on techniques with such standard results as

$$\int ax^n\,dx = \frac{a}{n+1}x^{n+1} + C$$

for $n \neq -1$ and

$$\int \frac{a\,dx}{x} = a \ln x + C = \ln cx^a,$$

he quickly moved on to many types of integrals, some being familiar while others of types not usually covered in today's texts. Thus, he notes that to integrate any rational function, it suffices to integrate functions of the form

$$\frac{A}{(a+bx)^n} \qquad \text{and} \qquad \frac{A+Bx}{(a^2 - 2abx\cos\zeta + b^2x^2)^n},$$

using the same trigonometric form of an irreducible quadratic that he had discussed in the *Introductio*. The first type of integral is straightforward. For the second, he began with the special case $n = 1$:

$$\int \frac{(A+Bx)\,dx}{a^2 - 2abx\cos\zeta + b^2x^2} = \frac{B}{ab^2}\ln(a^2 - 2abx\cos\zeta + b^2x^2) + \frac{Ab + Ba\cos\zeta}{ab^2\sin\zeta}\arctan\frac{bx - a\cos\zeta}{a\sin\zeta}.$$

In this example, as in the others discussed below, Euler considered various special cases before generalizing. And then, once he had his general integral results, he often specialized again, frequently calculating the same integral in more than one way.

To integrate functions involving square roots, Euler used substitution, although not our modern trigonometric substitutions. For example, to integrate

$$\frac{dx}{\sqrt{\alpha + \beta x + \gamma x^2}},$$

he considered two cases depending on whether the quadratic polynomial factored into two real factors or not. In the first case, he assumed the factorization was $(a + bx)(f + gx)$. Then, "to remove the irrationality," he set $(a + bx)(f + gx) = (a + bx)^2 z^2$, or $(f + gx) = (a + bx)z^2$. Solving for x gives $x = (az^2 - f)/(g - bz^2)$, and therefore

$$dx = \frac{2(ag - bf)z\,dz}{(g - bz^2)^2} \qquad \text{and} \qquad \frac{dx}{\sqrt{(a + bx)(f + gx)}} = \frac{2\,dz}{g - bz^2}.$$

Assuming $g > 0$, we then have that if $b > 0$, the integral is

$$\frac{1}{\sqrt{bg}} \ln\left(\frac{\sqrt{g} + z\sqrt{b}}{\sqrt{g} - z\sqrt{b}}\right),$$

while if $b < 0$, the integral is

$$\frac{2}{\sqrt{bg}} \arctan\left(\frac{z\sqrt{b}}{\sqrt{g}}\right).$$

An analogous substitution works in the case where the original quadratic polynomial is irreducible over the real numbers.

Euler next considered integration by the use of infinite series, Newton's favorite technique. To integrate functions involving logarithms, he invoked the technique of what we call integration by parts. As he described this technique, if the function V can be factored as $V = PQ$, and if the integral $\int P\,dx = S$ is known, then from $P\,dx = dS$, we get $V\,dx = PQ\,dx = Q\,dS$. Thus, since $d(QS) = Q\,dS + S\,dQ$, we have $\int V\,dx = QS - \int S\,dQ$. He immediately applied this rule to integrating functions of the form $x^n \ln x$:

$$\begin{aligned}\int x^n \ln x\,dx &= \frac{1}{n+1}x^{n+1}\ln x - \int \frac{1}{n+1}x^{n+1}\,d(\ln x)\\ &= \frac{1}{n+1}x^{n+1}\left(\ln x - \frac{1}{n+1}\right).\end{aligned}$$

In another chapter, Euler dealt with numerous procedures for integration of powers of trigonometric functions. For example, he used the substitution

$\cos\phi = \frac{1-x^2}{1+x^2}$, $\sin\phi = \frac{2x}{1+x^2}$ to convert rational functions involving sines and cosines to ordinary rational functions. Thus, he showed that, in the case where $a > b$,

$$\begin{aligned}\int \frac{d\phi}{a+b\cos\phi} &= \int \frac{2\,dx}{a+b+(a-b)x^2}\\ &= \frac{2}{\sqrt{a^2-b^2}}\arctan\frac{(a-b)x}{\sqrt{a^2-b^2}}\\ &= \frac{1}{\sqrt{a^2-b^2}}\arctan\frac{\sin\phi\sqrt{a^2-b^2}}{a\cos\phi+b}.\end{aligned}$$

Although normally Euler just calculated antiderivatives, occasionally he calculated what we would call a "definite integral." Thus, he first demonstrated the reduction formula

$$\int \frac{x^{m+1}\,dx}{\sqrt{1-x^2}} = \frac{m}{m+1}\int \frac{x^{m-1}\,dx}{\sqrt{1-x^2}} - \frac{1}{m+1}x^m\sqrt{1-x^2}.$$

Then, noting that the second term on the right vanished at both $x = 0$ and $x = 1$ and that

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2} \qquad \text{and} \qquad \int_0^1 \frac{x\,dx}{\sqrt{1-x^2}} = 1,$$

he concluded that

$$\int_0^1 \frac{x^{2n}\,dx}{\sqrt{1-x^2}} = \frac{1\cdot3\cdot5\cdots(2n-1)\cdot\pi}{2\cdot4\cdot6\cdots2n\cdot2}$$

and

$$\int_0^1 \frac{x^{2n+1}\,dx}{\sqrt{1-x^2}} = \frac{2\cdot4\cdot6\cdots2n}{3\cdot5\cdot7\cdots(2n+1)}.$$

(Note that Euler himself did not write limits of integration; they are put there for clarity.)

After considering these various techniques of integration, Euler moved on to deal with methods of solving differential equations. Euler solved the general first order linear equation $dy + Py\,dx = Q\,dx$ (or, in modern terms, $y' + Py = Q$) by separation of variables to get

$$y = e^{-\int P\,dx}\int e^{\int P\,dx}Q\,dx.$$

As examples of this, he solved $dy + y\,dx = ax^n\,dx$ for various values of n. For $n = 3$, he found that $y = Ce^{-x} + x^3 - 3x^2 + 6x - 6$. As promised earlier, he showed how to integrate $P\,dx + Q\,dy$ in the "exact" case where $\partial P/\partial y = \partial Q/\partial x$, again following the general discussion with numerous examples. He demonstrated how to find integrating factors in the case where

$P\,dx + Q\,dy$ is not exact, again detailing the method through various examples. He considered many cases of second and higher order differential equations, including the linear case with constant coefficients, which required the solving of a polynomial equation. Finally, Euler concluded the book with a long discussion of methods of solving partial differential equations.

4. Conclusions

The *Integral Calculus*, the *Differential Calculus* and the first part of the *Introductio* are texts in pure analysis, so much so that, as mentioned earlier, Euler does not even deal with applications to geometry, let alone physics. This is, perhaps, especially surprising in the *Integral Calculus* since the original motivation for the solution of differential equations came from physical questions, questions that in fact led Euler to some of these methods of solution in the 1730s and 1740s. So the modern reader may well be surprised that in the *Differential Calculus* there are no tangent lines or normal lines, no tangent planes, no study of curvature — all topics with which Euler was fully conversant in 1740 but which only appear in some of his geometrical works. And in the *Integral Calculus* there is no mention of the vibrating string problem or various other vibration problems that had led Euler to "invent" the trigonometric functions in the 1730s, nor is there any calculation of areas nor any material on lengths of curves, or volumes, or surface areas of solids. And then, although Euler presented an extraordinary number of methods to find antiderivatives, the central technique of modern texts for determining areas, the fundamental theorem of calculus, did not appear. That is not to say that Euler did not know how to calculate areas using antiderivatives. Euler in fact did so in various papers. But since geometrical ideas are not present in the calculus texts, there is no definition of the area under a curve as a function, and therefore no call for the derivative of such a function. And until an independent definition of area could be provided, as Cauchy did in the 1820s, this fundamental relationship between derivatives and integrals, discovered by Newton and Leibniz, could not be "fundamental."

It appears that, with the exception of the second part of the *Introductio*, which was filled with graphs, Euler had an abiding belief that "pure mathematics" had no need of diagrams. One could understand everything that was needed by pure manipulation of symbols according to the rules that he and others developed. (Of course, the modern theoretical underpinnings of the calculus based on an understanding of the real number system do not appear here. What proofs there are are based on the use of infinitely large and infinitely small quantities.) Euler seemed further to believe, like Euclid two thousand years earlier, that it was unnecessary to help students learn analysis by showing them the motivations for the various techniques – a far cry from the standard modern opinions on the teaching of the subject. That is not to say that Euler was not a good pedagogue. He was very patient with his readers, frequently explaining every step in an argument while also demonstrating the same result in several ways. And he certainly used motivations from the sciences in many of his other papers.

It is common to consider Euler's analysis texts as the most influential texts of the eighteenth century. But it is difficult to quantify this influence. They are surely among the works to which Laplace referred when he wrote, "Read Euler; read Euler. He is the master of us all." But to figure out who actually did read them is difficult. Certainly, the *Introductio* was read frequently. For not only were Euler's notation and methods taken up in numerous analysis texts that followed, but also the book itself saw several reprints even during the eighteenth century as well as translations toward the end of that century into both French (twice) and German. On the other hand, the *Differential Calculus* only has a single German translation – in 1790 – while there are no eighteenth century translations of the *Integral Calculus*. The first translation, into German, appeared in 1828-30.

Certainly the techniques that Euler developed to determine derivatives and integrals continued to appear in other texts, but his use of infinitesimals as a basis for the calculus was gradually replaced by the idea of a limit, beginning with the ideas of Jean d'Alembert as expressed in his articles in the French *Encyclopeédie*. But it was the French Revolution and the influence of Napoleon which really changed everything. Suddenly, with the aristocracy removed in France, and greatly weakened elsewhere, there was a great need for educating a new class of students who were entering the sciences. And it was this need that inspired the writing of many new texts in the vernacular, texts which replaced those of Euler and were the direct ancestors of the texts of today.

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Leonhard Euler: Life, Work and Legacy
Robert E. Bradley and C. Edward Sandifer (Editors)

Euler and the Calculus of Variations

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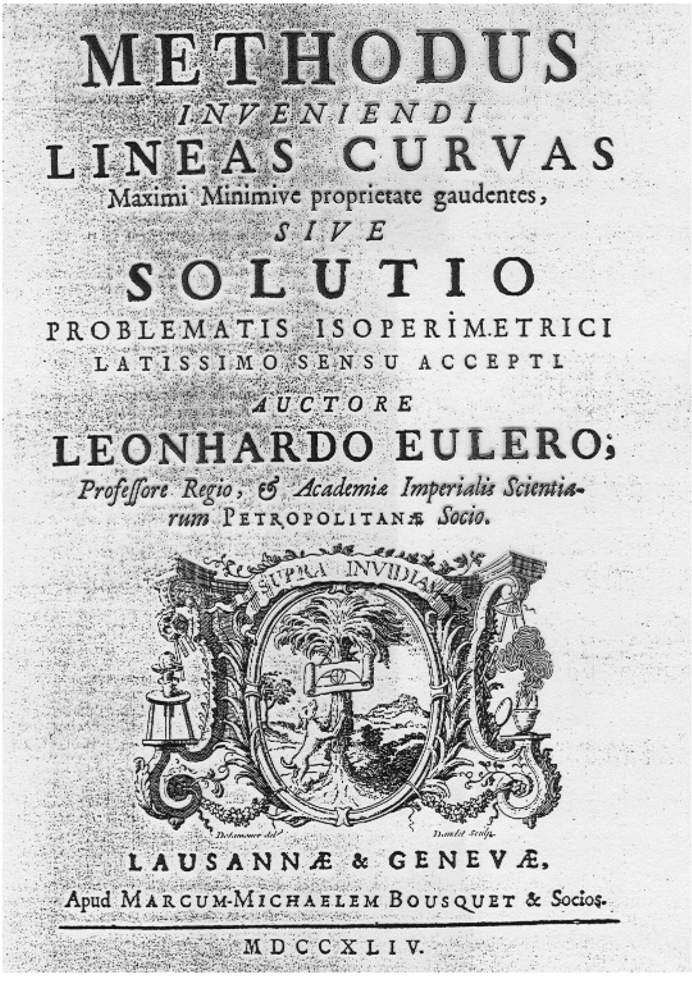
Dedicated to the memory of Michael Raith, Basel (1944-2005): a fine historian and good colleague who died too young at the age of 61.

1. Expository remarks

Extremal thinking is old. With a twinkle in his eye, the contemporary Polish mathematician Krysztof Maurin (born 1923) speaks of a mythical origin, because soon after the Fall, mankind began to minimize, maximize, and criticize [M]. A more realistic beginning could be Dido's remarkable Isoperimetric Problem in antiquity (about 900 BC), by which Dido simultaneously founded Carthage and invented the Calculus of Variations. However, we will begin our profane history with the Bernoulli brothers in 1696. For the moment, by the Calculus of Variations we mean the treatment, by any method, of problems like those posed by Jacob and Johann Bernoulli (1654-1704 and 1667-1748 respectively). Leonhard Euler (1707-1783) collected and extended these problems "in one of the most wonderful books that ever has been written about a mathematical subject." [1] . This

[1] "[die] Methodus, eines der schönsten mathematischen Werke, die je geschrieben worden sind" [Ca, p. ix]

was how Constantin Carathéodory (1873-1950) described Euler's *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes sive Solutio Problematis Isoperimetrici latissimo sensu accepti* (The art of finding curved lines which enjoy some property of maximum or minimum or the Solution of the isoperimetric problem taken in its widest sense) [E65]. The *Methodus inveniendi* does not present the Calculus of Variations in the form we are familiar with. The reader will also notice that Euler did not speak of the "Calculus of Variations" but of "isoperimetric problems," but this will be part of the story that will be told in this article.


METHODUS
INVENIENDI
LINEAS CURVAS
Maximi Minimive proprietate gaudentes,
SIVE
SOLUTIO
PROBLEMATIS ISOPERIMETRICI
LATISSIMO SENSU ACCEPTI.
AUCTORE
LEONHARDO EULERO;
Professore Regio, & Academiæ Imperialis Scientiarum PETROPOLITANÆ *Socio.*

LAUSANNÆ & GENEVÆ,
Apud MARCUM-MICHAELEM BOUSQUET & Socios.
MDCCXLIV.

Fig. 1. The title page of the *Methodus inveniendi*, Lausanne, 1744.

Before starting the historical discussion, I would like to make some remarks from a modern viewpoint to give the reader a basic knowledge of the problems arising in this paper. In the first place I will ask: *What is the Calculus of Variations?* In his title, *Methodus inveniendi*, Euler already gave an excellent answer, albeit in a baroque style. This was watered down by later authors, but was resurrected by David Hilbert (1862-1943) who gave the following definition:

> Given a set of mathematical objects a, b, c, ... and a relation so that each element a is associated with a real number N_a, then look for the element or elements which have a minimal or maximal associated number ... if there is any. [2]

In the classical Calculus of Variations, the elements are curves or functions. In general, the relation is expressed by a definite integral like the following, for which we seek a function $y = y(x)$ of one variable x:

$$J(y) = \int_a^b Z(x, y, y')\, dx.$$

Hilbert's abstract definition, which ultimately repeats Euler's detailed explanation, leaves us no way of avoiding these questions:

i) Is there any solution (any curve or function) at all?

ii) If so, how may one find such a solution? In the first place the question is above all to find necessary conditions for a solution.

iii) If we have found a curve (function) to consider as a solution, how may one prove that the curve in question is, in whole or at least in part, an actual solution?

Of course, in modern mathematics we would thoroughly discuss the analytical assumptions of the problem. In Euler's time, such questions were not considered.

Existence of a solution

Hilbert's explanation makes it abundantly clear that a given problem does not necessarily have a solution. Take for N_a the sequence of the reciprocals of the natural numbers - then the minimum 0 is not an admissible solution and is to be excluded. A more interesting problem is a question

[2] "Gegeben sind irgendwelche mathem. Dinge. Jedem ist in bestimmter geg. Weise eine reelle Zahl zugeordent. Man soll das Ding oder solche Dinge heraussuchen, denen die kleinste oder größte Zahl zugeordnet ist" ... "falls eine solche existiert." *Vorlesung Variationsrechnung* (WS 1904/05), Mathematisches Institut of the University of Göttingen, p. 4f. See also the chapter "Calcul des variations" by M. Lecat in the French edition of the German *Encyklopädie der mathematischen Wissenschaften* (ed. J. Molk), ser. II/6.31, Paris 1913, p. 1.

proposed by Soichi Kakeya (1886-1947) in 1917: Find a planar figure of least area in which a needle can be completely turned around by continuous movement to assume the opposite direction. Common sense unconditionally suggests the existence of such a figure, but in 1927 Abram S. Besicovitch (1891-1970) found a surprising result: there are admissible figures a of arbitrary small area N_a and hence one cannot find a figure of minimal area. [HT, p. 91]

How may one find a solution?

In the classical Calculus of Variations, extremal thinking is supported by infinitesimal reasoning. Because it is most natural to view variational integrals as functions of functions, the differential calculus is a cornerstone of the Calculus of Variations, an opinion frequently expressed by Hilbert in his lectures: "die Variationsrechnung, die so der Differentialrechnung als Fortführung und Verallgemeinerung an die Seite tritt (the Calculus of Variations accompanies the differential calculus as an extension and generalization)";[3] and by Jacques Hadamard (1865-1963) "Le calcul des variations est, pour les opérations fonctionnelles, ce que le calcul différentielle est pour les fonctions (The Calculus of Variations is for functionals what the differential calculus is for functions)." [H, 4:2260] So we can expect similarities between the differential d and the variation δ, but we will also have differences.

In general, the classical methods – mainly the vanishing of the first variation, $\delta J(y^0) = 0$ – presuppose the existence of a solution and therefore all resulting consequences will only provide *necessary* conditions for a solution. Such a curve or function is a candidate to be an actual solution. However, if we are convinced that the problem actually has a solution, as Euler was, then the necessary condition is also a sufficient one. In Adolf Kneser's terminology a curve C_0 represented by an equation $y = y^0(x)$ or a function $y = y^0(x)$ satisfying the necessary condition $\delta J = 0$ is called an extremal (*Extremale* [K1, p. 24]).

[3] Mechanik, winter term 1905/06, lecture notes prepared by E. Hellinger, Mathematisches Institut of the Universität Göttingen, p. 122; similar in Flächentheorie, summer term 1900, Hilbert's own notes, Niedersächsische Staats- und Universitätsbibliothek Göttingen, Handschriftenabteilung, Cod. Ms. D. Hilbert 557; "We call the Calculus of Variations the differential calculus of functions (... wenn wir die Variationsrechnung die Differentialrechnung der Funktionen nennen)", Gewöhnliche Differentialgleichungen, summer term 1912, Mathematisches Institut of the Universität Göttingen, p. 138.

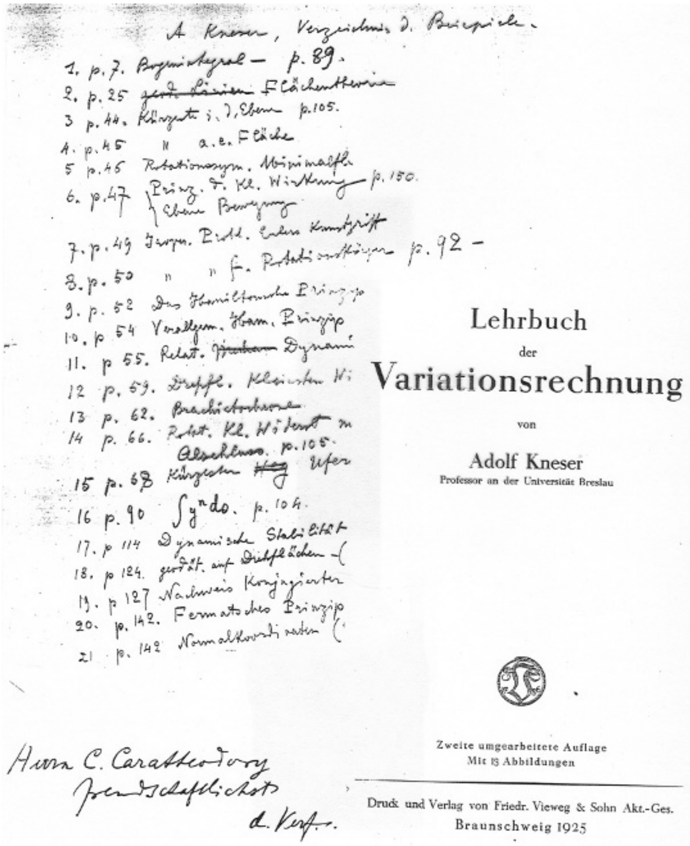


Fig. 2. Title page of Carathéodory's copy of the second edition of [K1], autographed by Kneser. Listed in Carathéodory's hand are the the first 21 of Kneser's 34 examples.

Sufficiency proofs

To produce evidence that an extremal is an actual solution (which gives a minimizer or a maximizer), one has to take into account whether the extremal is a solution only in part or as a whole. Mathematicians of the 18th century often regarded such questions naively, i.e. accepted the existence *a priori*; and Euler did too. An exception, not repeated until the late 19th century, was Johann Bernoulli in his proof that the Brachistochrone is indeed the curve of shortest descent [B1]. Guillaume de Saint-Jacques de Silvabella (1722-1801) in his paper "On the body of least resistance"[S] published in 1760 noticed that a curve which causes the first variation to vanish will not, in general will be an extremum. Then in 1786 Adrien-

Marie Legendre (1752-1833) had some doubts as to whether the second variation is sufficient to give an extremum (as the second derivative does in differential calculus) [Leg], a problem which was finally solved by Karl Weierstrass (1815-1897) in 1879 [We].

There were further differences. On a sphere the great circles are extremals (here called geodesic curves), however an extremal containing both poles is a shortest line only on arcs between the poles. Compared with the tasks of differential calculus, we have a completely new phenomenon here, due to the extension.

Moreover, in a variational problem we have to consider what would constitute admissible functions (curves). This question is an important part of formulating the problem. Unfortunately the "natural" classes of admissible functions of a problem will not automatically be identical with those classes we presuppose when using differential techniques. Two commonly used classes and the underlying function spaces in geometry and mechanics are C^2 and C^1 respectively,[4] but they are not at all the natural setting for tackling variational problems (think of optical problems with refraction also.) Euler will notice a paradox of the Calculus of Variations in about 1770, which we will deal with in more detail in section 6 of this paper, with its roots in the *a priori* use of certain admissible curves (functions) falsely regarded as the natural ones [E735]. As mentioned above, since Weierstrass we have known some sufficient conditions for both these function spaces and the related extrema, for which Adolf Kneser (1862-1930) coined the terms strong and weak extrema in 1900 (*starkes* and *schwaches Extremum* [K1, p. 24]).

2. Prehistory

The most important papers before Euler's research started are:

1686 Newton, "Motion in a resisting medium" (unpublished);

1696 Jacob and Johann Bernoulli, the Brachistochrone Problem and the related isoperimetric problems, classical method of variation [B2];

1701 Jacob Bernoulli, *Analysis magni problematis* [B2, pp. 485-505];

1715 Taylor, *Methodus incrementorum directa et inversa* (London);

1716 Hermann, *Phoronomia* (Amsterdam, actually printed in 1715);

1718 Johann Bernoulli, Remarques [B2, pp. 527-568].

[4] Function spaces the elements of which are twice and once continuously differentiable.

3. General remarks on Euler's work

Euler wrote some of his best papers about the Calculus of Variations. There is no consensus as to how many he wrote on the subject. Gustav Eneström (1852-1923) in his *Verzeichnis* [En] including the so called Eneström Index published in three parts between 1910 and 1913, gives 10 titles. Three years later Maurice Lecat (1884-1951) in his *Bibliographie* [Lec] of 1916 catalogued 33 titles, admittedly including the *Lettres une Princess d'Allemagne (Letters to a German Princess)* [E343,E344,E417] and letters by Euler to Daniel Bernoulli (1700-1782). The two volumes of the *Euleri Opera omnia* devoted to the Calculus of Variations (ser. I, vols. 24-25) contain the *Methodus inveniendi* [E65] and 19 papers. If we add the 10 titles of ser. II, vol. 5, which deal with the Principle of Least Action, we have in total 30 pieces. Furthermore, we should also take into account Euler's first paper "*Constructio linearum isochronarum in medio quocunque resistente* (Construction of an isochrone line in a resistant medium)" [E1], the "*Anleitung zur Naturlehre* (Instruction in natural sciences)" [E842] and some others, so in the widest sense about 35 papers might be an acceptable number. Incidentally, almost all of these papers were published in St. Petersburg, with the exception of those papers on the Principle of Least Action published in Berlin because the corresponding interactions were focused on the Berlin Academy and Euler lived in Berlin at that time.

We can divide Euler's contributions to the Calculus of Variations into three periods:

i) *First Period*: from Euler's first publication on the Brachistochrone Problem in a resistant medium [E1], dealing with a variational theme, which came out in 1726, and about 1732, when he began to develop his own approach to the Calculus of Variations,

ii) *Second Period*: from about 1740, when his theory began to take shape until it culminated in his masterpiece *Methodus inveniendi*,

iii) *Third Period*: from 1755 when, thanks to Joseph Louis de Lagrange (1736-1813), Euler had the initial idea for the analytic treatment and its foundation.

There is also a fourth period until 1818 when Euler's Calculus of Variations continued to appear posthumously, but we will not detail that here.

On the other hand, besides this chronological ordering of the developmental stages, we may divide Euler's papers by considering important ideas serving as a guiding principle. Concerning this evolution, we have three (or four) fundamental themes:

i) the early Petersburg papers as the roots of later developments,

ii) variational methods,

iii) Langrange's calculus, and applications of the Calculus of Variations, especially in

iv) the principle of least action.

Finally, we should mention the extraordinary number of examples Euler gave and for which he actually computed solutions. Carathéodory was fond of problems, which he regarded as the true substance of mathematics. In total he listed 66 kinds of problems in Euler's work and his appreciation of examples is shared by Adolf Kneser, who remarked that there is always merit in adding a further example to Euler's list. [5]

We now begin our systematic treatment of Euler's work.

4. First period

In the early days of the Calculus of Variations we see Johann Bernoulli's challenge problem about the curve of quickest descent, without resistance or friction. In this spirit Euler published his first publication of three pages written at the age of 18. In a resistant medium he posed the problem of the line of quickest descent: "*Constructio linearum isochronarum in medio quocunque resistente*" [E1]. One year later in the *Acta* Jacob Hermann (1678-1733) gave a solution "*Theoria generalis motuum* (General theory of motion)" (1727, but published in 1729), which was incorrect. It was not until Hermann had left St. Petersburg in 1731 that Euler informed his colleague of this error. However, Hermann had no opportunity to correct the error since he died in 1733. It was up to Euler to attempt to improve on Hermann's paper. However, Daniel Bernoulli soon informed his friend in a letter of September 12, 1736 that Euler's improved solution still was not correct.

The first paper on shortest lines on a general surface $F(x, y, z) = 0$ (not in parametric form) was published by Euler under the title "*De linea brevissima in superficie quacunque duo quaelibet puncta jungente* (On the shortest line on an arbitrary surface connecting any two points)" [E9] appeared in the St. Petersburg *Commentarii* volume for 1728, but was not published until 1732. In 1697, when Johann Bernoulli posed new provocations for his brother, there was a remarkable discussion of what kind of (convex) surface could be under consideration at all, i.e. which kind of

[5] "Aber auch in der Variationsrechnung gilt das Wort Jacobis: es ist immer ein Fortschritt, wenn man den Beispielen Eulers ein neues hinzuzufügen weiß." [K2, p. 29] Kneser refers to a well-known quotation of Jacobi: "that any progress [*Fortschritt*] in the theory of partial differential equations must lead to progress in mechanics too" (lectures on dynamics). Compare also Carathéodory in [Ca, p. x].

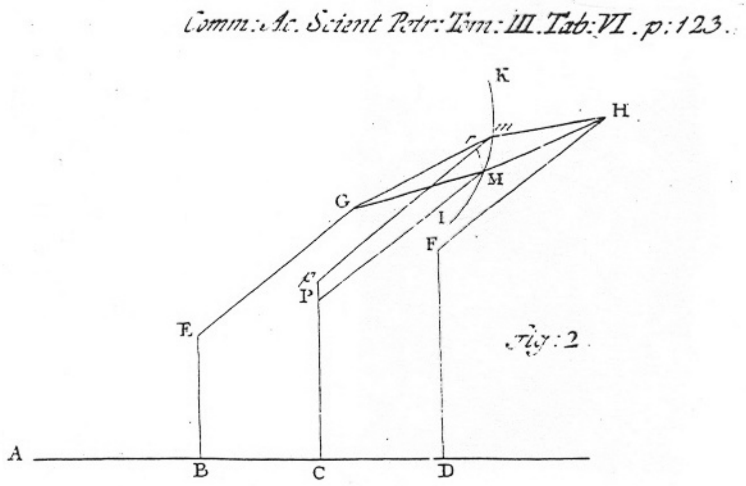


Fig. 3. Figure 2 from *De lina brevissima ...* [E9].

surface of revolution [T, pp. 117-131]. Notice at this time there was no concept of analytic function, and therefore no concept of analytic surface either. For a convex surface Euler gave a simple mechanical solution: fix a string at one point and pull it taut in the direction of the other. Obviously, this method fails in the case of non-convex surfaces and that is why Euler developed an infinitesimal method for general surfaces, whereby in the tangent plane the line GMH composed of two straight lines GM and MH is to be minimized; see figure 3.

This method is equivalent to a geometrical theorem on an osculating plane of a geodesic (i.e. extremal space curve on a surface) at a point P, developed but not published by Johann Bernoulli in 1698. This theorem states that the osculating plane intersects the tangent plane to the surface at P at a right angle. Shortest or geodesic lines can be characterized by this property of osculation. Obviously, Bernoulli did not teach this theorem to his disciple, and, more generally, we may infer that in Basel Euler was not yet involved in variational problems. Incidentally, it was a letter of Johann Bernoulli to his son Daniel in 1727 that drew Euler's attention to this subject.

In 1753, Euler used the theory of shortest lines for the foundation of spherical trigonometry, as well as for the extension of such a trigonometry from the spherical surface to general surfaces [E214,E215]. Thanks to a theorem of Pierre Ossian Bonnet (1819-1892), geodesics on concave surfaces are always shortest lines. This is not the case in general and required further investigation (Jacobi theory). Moreover, in 1736 Euler published his *Mechanica* [E15,E16], in which he developed an analytic geometry of space and in which geodesics were characterized by the osculating plane. Moreover, Euler stated that inertial motion follows either straight lines in a plane or, more generally, geodesics on surfaces.

In 1738 we have an important milestone on Euler's way to the *Methodus*, the paper "*Problematis isoperimetrici* (Isoperimetric problems taken in the widest sense)" [E27], already written in 1732. Euler's variational method has its roots in Jacob Bernoulli's variational process, developed in 1697 in the study of isoperimetric problems. However, where the Bernoulli brothers solved specific problems, the disciple Euler mastered and extended the method and ultimately he began to look for a general theory. We see this intention in the methodical procedure he pursued in his investigations, for example he divided the problems into groups based on the side condition and laid down different kinds of variation for each group. Of course, because side conditions depend on the choice of the coordinate system, such a classification is only relative.

Quite naturally, having mastered the method of Jacob Bernoulli, Euler generalized the isoperimetric problems Jacob Bernoulli had dealt with, especially those in which arc length also appeared among the independent variables. In Euler's general investigation of isoperimetric problems, all of the variables x, y, s ($s =$ arc length) enjoyed equal rights, which should not be the case because of the side condition $ds^2 = dx^2 + dy^2$ and, moreover, because of the fixed length of admissible curves (the isoperimetric condition). In the next paper "*Curvarum maximi minimive proprietate gaudientium inventio nova et facilis* (New and simple invention of curved lines which enjoy some property of maximum or minimum)" [E56], we will see, Euler partially corrected the mistake.

We also find remarks on the independent integral, the integrand of which is a total differential and therefore depends not on the path of integration but only on the endpoints. Hilbert later used such integrals to give a very elegant two- or three-line sufficiency proof for the case of strong extrema - a royal road. [6] Here Euler correctly pointed out that no variational problems emerge from such integrands. Rather interestingly, he maintained that, in the case of a plane, for any differential form $\Omega = A(x,y)\,dx + B(x,y)\,dy$

[6] *Mathematische Probleme*, Lecture at the Paris Meeting in 1900, problem 13. In: *Nachrichten der Akademie der Wissenschaften in Göttingen*, 1900, pp. 253-297. English translation by M. Winson in: *Bull. AMS*, **8** (1901/02), pp. 437-479, reprinted in *Bull. AMS (New Series)*, **37** (2000), pp. 407-436.

there exists an integrating factor F, an Euler multiplier, which generates a total differential $F\Omega = \Pi$ with $d\Pi = 0$.

In conclusion we remark that there is a similar paper by Alexis Claude Clairaut (1713-1765) "*Sur quelque questions de maximis et minimis* (On several question of maximum and minimum)" [Cl], independently written in 1733 in which the side conditions are delivered by a force field, related to his investigations on the shape of earth. [7]

Euler made progress in the process of generalization and in 1736 he wrote the paper "*Curvarum maximi minimive proprietate gautentium inventio nova et facilis* (New and easy invention to find curves having a maximum or minimum property)" [E56], published in 1741, in which he tried to unify old results. In dealing with 40 problems he sought one general result including his 24 specific cases. As previously noted, he became aware that for some variational problems with constraints he was wrong, but he only noticed this fact after he had already written 33 paragraphs. So he briefly mentioned the fact in the introductory sentences and gave the corrections in detail, but only in the last four paragraphs. This was typical behavior for the busy Euler – the manuscript had probably already been sent to the printer and he only partially remembered the manuscript. Moreover, he had aready begun or was just about to begin the proof-reading of his textbook *Methodus* [E65]. Among the 24 expressions for the first variations there are only 9 (nos. I-VI, XIII-XIV) which are correct.

Between the two papers E27 and E56 (i.e. between 1732 and 1736) Euler was engaged with the Brachistochrone Problem in a resistant medium. He wrote "*De linea celerrimi descensus in medio quocunque resistente* (On the curve of fastest descent in whatever resistant medium)" [E42], which inspired him to allow even differential equations as constraints. Ultimately, in "*Curvarum maximi minimive proprietate gautentium inventio* (The finding of curves enjoying properties of maximum or minimum)," [E56] he gave corrected results. The Brachistochrone paper "*De linea celerrimi descensus in medio quocunque resistente*" was published in volume 7 of the Petersburg *Commentarii*, incidentally the same volume in which Euler used the notation $f(x)$ for a function f of x for the first time [E44]. In the last pages of E56 Euler made yet another very important remark. There had previously been no doubt concerning a principle used by Jacob Bernoulli in 1697 and later on by others to derive differential equations for solutions of variational problems: If any curve possesses a maximum or minimum property then each part of the curve (especially any infinitesimal part) enjoys

[7] See also Daniel Bernoulli's report to Euler on such knowledge in Paris, which he wrote there on his journey back to Basel on September 23, 1733; "dass dergleichen problemata den hiesigen Mathematicis nicht schwer fallen".

this property too. However, Euler recognized that the principle is not, in general, true for variational problems with constraints.

5. Second period

In 1744, at the age of 37, Euler published the *Methodus inveniendi* [E65], a landmark in the history of mathematics, with which he created the new branch of mathematics we now call the Calculus of Variations, although the name came later. Euler changed the subject from a discussion of special cases to that of very general classes of problems. Above all, in this textbook he set up a general analytic apparatus for writing down the so-called Euler or Euler-Lagrange differential equations, thus extending the methods of the Bernoulli brothers to a general theory of the first variation.

The book consists of six chapters and two very important addenda. Before sketching the contents of this book, I will mention the *Scientia navalis* [E110,E111] published in 1749 but already written in 1738. Some of the important results can already be found there, but this has not yet been investigated thoroughly.

In the *Methodus,* Euler considered a general variational problem for one function y of one variable x, in which the integrand Z was allowed to involve derivatives of y of arbitrary order,

$$J(y) = \int Z(x, y, y', y'', \ldots)\, dx.$$

Euler regarded the variational integral J as an infinite sum so the variables and their variations could be inserted into the sum. He gave all changes under consideration in tables. Euler expressed the infinitesimal changes both of the functions (curves) under consideration and of their derivatives (i.e. extremal and admissible functions/curves) and then calculated the infinitesimal change of the variational integral, the *valor differentialis formulae.*

It is noteworthy that Euler did not make a finite approximation and then carry out limiting processes. Rather, he operated completely in the spirit of the 18th century and its use of infinitesimals. A corresponding approximation that substituting finite quantities for infinitesimals is easily done and corresponds to our understanding. In 1907 Adolf Kneser showed that this procedure, which is preferable by modern standards, is indeed correct [K2].

Let us look at this in more detail. Supposing the existence of a curve C_0 that is an actual minimizer, Euler looked at the difference in the variational integral for any admissible curve C and the minimizer C_0: $\Delta J = J(C) -$

$J(C_0)$. Because of Euler's technique of variation, the difference between the extremal and the admissible curve appears only in a few points. If there are no constraints, Euler varies only a single point. In the case of one constraint, Euler varies in two successive points, etc. In other words, in the expression of ΔJ only a few terms were affected and so the question reverts to an ordinary extremal problem of the differential calculus that is already well-known: minimize the *valor differentialis formulae*. Ultimately, for each point of the extremal, Euler arrived at an equation for the differentials, the well-known Euler differential equation, as a necessary condition that a solution must satisfy.

Euler set up a single algorithm that worked for both minima and maxima. To decide which kind of extremum for a given curve C_0 was, he used a practical test and considered the sign of $\Delta J = J(C^*) - J(C_0)$ for some other admissible curve C^*. It was that simple! Today, with a better understanding of the existence of solutions, in particular of the question whether there is a solution at all, we do not need to discuss why such a test will often fail. Also Euler did not completely understand the significance of the endpoints of the curves. In modern terms, he did not understand the corresponding boundary value problem of the Euler equation. Therefore he also did not look for the so-called transversality conditions, which play an important role in problems with free boundaries and, more generally, in the field theory which arose principally from Fermat's principle in optics. In chapter 3, Euler took up isoperimetric problems involving curve length. Carathéodory remarked that, despite some incomplete results here, we have first-class results of such a kind even Euler did not get too often. [8] In the next chapter Euler showed that his necessary condition, the Euler equation remains invariant under transformations of the coordinates. This means that despite his geometric reasoning, which rests on special coordinates, the results are general. Thus the underlying figures are only convenient geometrical visualizations. This observation was also made later by Joseph Louis de Lagrange in his *Méchanique analitique* of 1788 [La2].

Initially, Euler based his research on a geometrical foundation, but later on, as "Analysis Incarnate," he considered the variables as abstract quantities. In a period of transition from geometry to analysis Euler himself remarked: "It is thus possible to reduce problems of the theory of curves to problems belonging to pure analysis. And conversely, every problem of this kind proposed in pure analysis can be considered and solved as a problem of the theory of curves." [9]

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[8] " . . . stellen die Resultate des Kapitels III seines Buches eine Spitzenleistung dar, wie sie auch einem Euler nicht allzuoft geglückt ist." [Ca, p. XXII].

[9] "Corollarium 8: Hoc ergo pacto quaestiones ad doctrinam linearum curvarum perti-

The Bernoulli brothers used diagrams in their papers, and in his early essays (including the *Methodus inveniendi*) their successor Euler did the same. However he did it more and more for the purpose of illustration and in the end Euler presented the Calculus of Variations without any diagrams as he generally did in all his analysis books throughout his career. Of course, the second volume of the *Introductio* [E102], dealing with analytic geometry, is an exception. Ultimately, this transition has its logical roots in an analytic function concept, which was first introduced by Johann Bernoulli in 1697 in the course of his quarrels with his brother Jacob. As Johann Bernoulli's student, Euler started from the very beginning in 1727 with an analytic function expression representing the function concept which he then extended step by step. To quote Craig Fraser (born 1951): "Although the theme of analysis was well established at that time [about 1730] there was in [Euler's] work something new, the beginning of an explicit awareness of the distinction between analytical and geometrical methods and an emphasis on the desirability of the former in proving theorems of the calculus." [F1, p. 63]

In the first appendix to the *Methodus inveniendi* [E65], "On elastic curves," Euler dealt with problems Jacob Bernoulli had already considered, but Daniel Bernoulli was most influential on Euler in this matter and encouraged Euler to deal with elastic lines by means of a variational problem. Euler was successful and found nine types of elastic lines and the buckling theorem too. About a century ago Max Born (1882-1970) was called upon to lecture in Felix Klein's seminar more or less by chance. He got into trouble with "Divine Felix" (1849-1925) and in the end wrote a remarkable Ph.D. thesis with Hilbert on elastic lines, using only Hilbert's lecture on the calculus of variations. [10]

In the second appendix "On the motion of bodies in a non-resisting medium, determined by the method of maxima and minima" we find the first publication of the Principle of Least Action, usually attributed to Pierre Louis Moreau de Maupertuis (1698-1759). The problem is famous, even notorious. it is the starting point for Joseph Lagrange, William Rowan Hamilton (1805-1865), Carl Gustav Jacob Jacobi (1804-1851) and others, in their foundation of mechanics and further branches of physics.

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nentes ad Analysin puram revocari possunt. Atque vicissim, si huius generis quaesti in Analysi pura sit proposita, ea ad doctrinam de lineis curvis poterit referiri ac resolvi." [E65, I, §32]

[10] *Untersuchungen über die Stabilität der elastischen Linie in Ebene und Raum.* Dissertation Göttingen 1906. Also in: *Ausgewählte Abhandlungen*, Bd. 1. Göttingen: Vandenhoeck & Ruprecht 1963, pp. 1-22.

6. Third period

In August 1755 Euler received a letter from the 19-year-old Lagrange, in which Lagrange announced the presentation of the Calculus of Variations in a purely analytic form, using his δ-algorithm. Indeed, Lagrange saw how to reduce the entire process he had learnt from Euler's papers to a purely analytic apparatus, which functioned almost automatically. Lagrange referred to Euler's remark in the *Methodus*, which encouraged him to develop the new technique: "A method is therefore desired, free of geometric ... solutions." [11] Euler immediately adopted the new method and in a lecture to the Berlin Academy in 1756, "M. Euler a lû *Elementa calculi variationum* (Mr. Euler read 'Elements of the calculus of variations')". [Wi, p. 226] He confessed that he "had meditated a long time" on this subject but "the glory of the first discovery was reserved to the very penetrating geometer of Turin, Lagrange, who having used analysis alone, has clearly attained the very same solution which the author [Euler] had deduced by geometrical considerations." [12]

Euler waited until Lagrange had published on the subject in 1762 in the "*Essai d'une nouvelle méthode* (Essay on a new method)," [La1] before he committed his lecture "Elementa calculi variationum" [E296] to print, so as not to rob Lagrange of his glory. Indeed, it was only Lagrange's method that Euler called Calculus of Variations. Among mathematicians this custom lasted until the 1850s.

Lagrange relied on algorithmic and algebraic properties and the success of his δ-formalism was immediately regarded as justification of his approach. Moreover, in their early investigations, both Lagrange and Euler viewed the variational operator as an additional and new operation in higher analysis. In this spirit, in "De calculo variationum," a detailed appendix to his Integral Calculus text, the *Institutiones Calculi integralis* [E385], Euler summarized the results of the Calculus of Variations in terms of the variational operator. Here we find such results as the variation of the dependent as well as of the independent variable and even the variation of double integrals.

However in about 1771 Euler had a remarkable insight into how the formal Lagrange calculus could be reduced to well-known methods of the differential calculus. In his "Methodus nova et facilis calculum variationum tractandi" (New and simple method to deal with the calculus of variations) [E420] he described a kind of trick which has been used ever since. Euler

[11] *Desideratur itaque Methodus a resolutione geometric ... libera.* [E65, II, §39]

[12] Euler expressed his esteem of Lagrange in the paper "Elementa calculi variationum" [E296], related to the lecture delivered in 1756 but published a decade later.

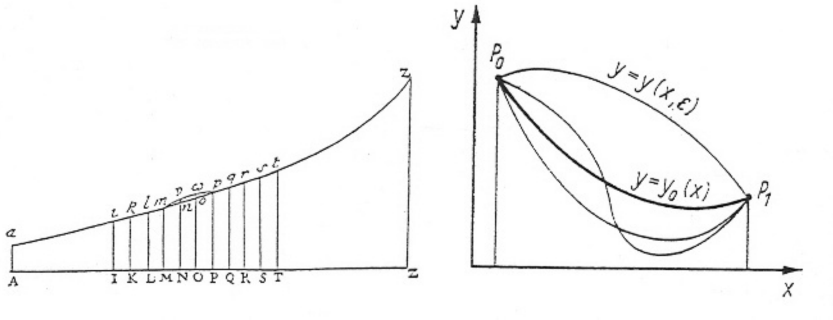

Fig. 4. Left: the pointiwse variation, from *Methodus inveniendi* [E65]. Right: the author's illustration of the embedding trick used in "Methodus novae . . . " [E420].

embedded the extremal $y = y^0(x)$ under consideration in a family of admissible curves $y = y(x, \varepsilon)$ with $y = y(x, 0) = y^0(x)$ (see figure 4). The variation of y is given by the partial derivative with respect to ε for $\varepsilon = 0$: $\delta y = \partial y(x, 0)/\partial\varepsilon$. Now the curve (function) is varied as a whole and not in some points only. The modern proof of the Euler equations uses the so-called Fundamental lemma which for a long time was regarded as obvious. The first proof was given by Paul Du Bois-Reymond (1831-1889) in 1879 [D]. More recent developments lead to the concept of weak solutions of differential equations (Sergeij L. Sobolev (1908-1989), Charles B. Morrey (1907-1984), *et al*) and to the theory of distributions (Laurent Schwartz, (1915-2006)).

Using partial integration in the Lagrange calculus, the integrand of the variational integral is partly shifted onto the boundary of the domain under consideration, thereby appearing in the form of boundary conditions. We have seen that Euler did not completely understand boundary conditions. In his time, these difficulties were also justified by the lack of integral theorems in the two and three dimensions (later developed by George Green (1793-1841), George Stokes (1819-1903), Carl Friedrich Gauss (1777-1855), Michail V. Ostrogradski (1801-1862)).

In the 1779 paper "De insigni paradoxo (On an outstanding paradox)" [E735], Euler made a surprising discovery, which he spoke of as a paradox in the Calculus of Variations. In modern terms, Euler considered a variational problem with a real analytic integrand and fixed boundaries. Surprisingly, the minimizer is not a function of the class C^2 or even of the class C^1, but a Lipschitz function which is not a solution of the necessary Euler equations. In the example, the solution of the Euler-Lagrange equation was a relative minimizer only, whereas Euler got the absolute minimum using a combination of line segments. Euler naturally considered functions of the class C^2, but his example shows that the spaces C^1 and C^2 are by no means the natural classes in which every variational problem should be tackled. In 1831, in a winning prize paper "Determinatio superficiei minimae rotatione curvae data duo puncta jungentis circa datum axem ortae (Determination of the minimal surface of rotation generated by a curve which links two given points)," [13] Benjamin Goldschmidt (1807-1851), encouraged by C. F. Gauss, investigated the phenomena for the catenoid, the surface of revolution of the catenary. In some cases one does not get a surface, but two discs joined by a straight line. Half a century later, in lectures on the calculus of variations, Hermann Amandus Schwarz (1843-1921) discussed this matter in great detail (as was his habit). His lecture courses of 1896 and 1898 were recorded in notes taken by John Charles Fields (1863-1932) and preserved in the Archives of the University of Toronto. Harris Hancock (1867-1943) has documented Schwarz's remarks in three articles and a book. [14]

We conclude with a question of Adolf Kneser posed at the Euler Conference in 1907: "Why do we rummage in rubble for some antiques?" [15] He and I give the same answer: "To enrich the *ars inveniendi*, to explain the methods by excellent examples, and last but not least to appreciate the intellectual company." [16] I hope you have gotten a glimpse of all these.

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[16] "Es sei nicht nur wichtig, daß die Geschichte jedem sein Recht gebe ..., sondern ihr Zweck sei, daß die ars inveniendi gemehrt und die Methode durch hervorragende Beispiele erläutert werde. ... Am geistigen Verkehr mit großen Männern wollen wir uns erheben." [K2, p. 34]

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Euler, D'Alembert and the Logarithm Function

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1. Introduction

Leonhard Euler and Jean d'Alembert were never able to agree on the proper way to extend the domain of definition of the logarithm function to negative numbers. Beginning late in 1746, they debated the matter vigorously in their private correspondence, but their dispute did not become public until 1761. At that time, there was a difference of opinion in the mathematical community as to whose position was correct, but by 1800 even French mathematicians had abandoned d'Alembert's point of view [E 1980, p. 19]. Euler's description of the logarithm as an infinitely multivalued function of the complex numbers, which he worked out sometime between 1743 and 1746, is precisely the function as we define it today. In modern notation, it is

$$z \longmapsto \ln|z| + i(\arg(z) + 2\pi n),$$

where n is an arbitrary integer and ln is the ordinary logarithm function on positive real numbers.

The aim of this paper is to consider both sides of this debate on the logarithm function, placed in the proper historical context, and to examine the papers that each of the disputants subsequently published on the subject.

Cajori wrote a comprehensive multi-part history of exponential and logarithmic functions early in the twentieth century [Cj], which included a discussion of Euler's letters to d'Alembert concerning logarithms. However, Cajori only had access to a much smaller portion of the Euler-d'Alembert correspondence than what is now available in the *Opera omnia* [E 1980]. In particular, Cajori had none of d'Alembert's letters, so he could not do justice to d'Alembert's side of the discussion. Nor did he know of the letter by Euler in which the topic first came up. At this time, we can be confident that all but one of the letters concerning logarithms have been discovered.

2. Euler and D'Alembert

D'Alembert is generally not as well known to mathematicians as is Euler. However, those in the humanities are much more likely to be familiar with him than with Euler, since he was an editor of the *Encyclopédie* and was a man of letters, rising to the rank of *secrétaire perpétuel* of the French Academy. However, in his early years, he was a successful and highly original mathematician, and his scientific credentials lent a certain legitimacy to the encyclopedic venture and to the status of the circle of *philosophes* of which he was a member. Those familiar with his mathematical career may have some inkling of his difficult relationship with Euler, since their quarrels are part of the folklore of mathematics. In the nineteenth century, for example, Libri wrote in the *Journal des Savants* [L] that they "frequently engaged in battle" (*"Ces deux grands géomètres eurent souvent à lutter ensemble ..."*). Riemann discussed their conflict over the wave equation (*"der Streit zwischen Euler und d'Alembert"*] in the historical survey of trigonometric series in his *Habilitationschrift* [R].

Euler was genial and unpretentious, and never boasted about his discoveries; this was noted by Condorcet in his *Eloge* [Co, p. 56-8], and the observations have often been repeated often by later biographers. Moreover, Euler was apparently less concerned about matters of priority than many of his contemporaries. There were incidents in which he ceded priority for discoveries that were made independently by others, even when the priority rightly belonged to him. For example, in a letter to Stirling of 27 July 1738 [E 1975, p. 433-4] discussing what we now call the Euler-Maclaurin formula, he wrote "I have very little desire for anything to be detracted from the fame of the celebrated Mr. Maclaurin since he probably came upon the theorem for summing series before me" [Tw, p. 146]. There was a similar case involving d'Alembert. However, we will also encounter a case in which Euler ought to have acknowledged d'Alembert's priority, but failed to do so.

By contrast, d'Alembert was vain and combative. By the late 1750s, he "had embroiled himself with nearly all other geometers at home and abroad" [Tr, p. 274]. Therefore, it should not be surprising that he quarreled with even such a genial person as Euler. However, d'Alembert had a particular axe to grind with Euler as a result of the great disappointment he suffered in the Berlin Prize competition for 1750.

The jury for the 1750 Berlin contest, chaired by Euler, determined that none of the entries submitted for that year's prize competition were worthy, and so the prize was remanded to 1752 [Wn, p. 150]. D'Alembert declined to enter the 1752 competition and instead published his entry himself [A3]. Then in November of the same year, the young astronomer Augustin Nathanael Grischow was summarily dismissed from the Berlin Academy "for furtively contracting an engagement with the Russian Academy" [Wn, p. 157]. Grischow (not to be confused with his father Augustin (1683-1749), also an astronomer with the Berlin Academy) had been one of the three judges of the 1750 competition [*ibid*, p. 146]. He was also an acquaintance of d'Alembert. No doubt humiliated by the Academy's actions, he made trouble for his former colleagues by revealing to d'Alembert and to others in Parisian society his version of the events that had led to the rejection of all the entries in that competition [E 1980, p. 313]. Whatever may actually have happened behind closed doors, d'Alembert came away with the firm belief that Euler had recognized his entry and convinced Grischow and the other judge (Kies) that the paper, which they considered to be the front-runner, had not sufficiently answered the question set for the competition.

Like other prize competitions of this time, entries in the Berlin competition were anonymous, identified only by a motto or *dévise*. Nevertheless, Euler could easily have identified d'Alembert's distinctive mathematical style, so there is at least some credibility to the story. In any case, d'Alembert believed that he had been treated unfairly and broke off his correspondence with Euler, which had been proceeding amiably since August 1746. For most of the rest of his career, d'Alembert treated Euler as an adversary.

This might have marked the end of their relationship, at least until the mid-1750s when the controversy over the wave equation erupted. However, in 1751, when the Berlin Academy's journal for the year 1749 appeared, d'Alembert was greatly disturbed to find within its pages four memoirs by Euler concerning subjects they had discussed in their correspondence. "D'Alembert quickly took alarm. All of his work was being stolen! Even . . . his book on the equinoxes, had not received a single mention from Euler" [H, p. 50].

D'Alembert wrote to the Berlin Academy in June of 1752 with priority claims for three of these articles [E 1980, pp. 337-350]. Euler immediately ceded priority for the problem of the precession of the equinoxes in a brief notice that appeared in the next volume of Berlin journal [E180]. In this note he explained that his intention was only to give an alternate proof of the results in [A2], and he apologized for not acknowledging d'Alembert's priority in his article. Wilson argues that Euler's proof is novel and more powerful, and that it subsequently led him to the discovery of very general principles governing the motion of rigid bodies [Wl]. In the same notice, Euler also ceded priority to d'Alembert for the discovery of an algebraic equation whose graph exhibits a cuspidal point of the second kind, even though the priority rightly belonged to him; for more on this controversy, see [Br]. Euler judged d'Alembert's third priority claim, related to the Fundamental Theorem of Algebra and the representation of complex numbers, to be without merit and never acknowledged it.

D'Alembert took exception to one more of Euler's articles in the 1749 volume of the Berlin *Mémoires*, but priority was not the issue here. Instead, he objected to Euler's paper on logarithms of negative and complex numbers on on the grounds of mathematical correctness . As we will see, these objections fell on deaf ears, and d'Alembert ultimately published his opinions on the subject himself.

Over the course of the next 11 years, Euler and d'Alembert had no direct contact with one another, although disagreement over the vibrating string problem provoked a number of polemical articles in which they, along with Lagrange and Daniel Bernoulli, engaged in a lively debate; see [Tr, pp. 254-281]. d'Alembert also demanded satisfaction on the third of his priority claims, but to no avail. Then in 1763, d'Alembert visited Berlin and Potsdam as the guest of Frederick the Great, who was courting him for the vacant presidency of the Berlin Academy. He and Euler met face to face for the first and only time in July of that year and at the same time resumed a polite correspondence. Although d'Alembert never accepted the King's offer to head up the academy, the unspoken subtext of this final portion of their correspondence was the very real prospect that Euler might end up as a subordinate to his former antagonist.

When Euler left Berlin for St. Petersburg in 1766, his correspondence with d'Alembert ceased. Whereas the debate over the logarithm function had occupied much of their early letters, the matter was never mentioned during the 1763-66 portion of their correspondence.

3. A History of the Logarithm Function

The logarithm function can be defined in at least two ways: it is the inverse of the exponential function and the antiderivative of the hyperbola. This dichotomy, which is important enough to be the basis for separate editions of some modern calculus texts, was a complicating factor in the debate between Euler and d'Alembert. During the course of their correspondence, d'Alembert would shift from one definition to the other as he raised a seemingly endless stream of arguments in defense of his contention that $\log(-x) = \log x$ for positive real numbers x.

There is, in fact, a third way to define the logarithm function, which was also mentioned by d'Alembert in the correspondence. This is the definition that d'Alembert actually adopted in his only published article on logarithms [A4]. It is the original conception of the logarithm, as given in 1614 by John Napier, based on the relationship between arithmetic and geometric series. Napier imagined two points flowing along parallel lines, the first of whose velocity diminished geometrically while the second point moved at a constant speed, representing an arithmetic sequence. Napier's own description can be found in translation in [Ca1, p. 282-289].

The logarithm as area under the hyperbola has its origin in the work of Gregory of St. Vincent and his *Opus Geometricum*, published in 1647. He studied the area under the hyperbola using his techniques of infinitesimal analysis. Building on St. Vincent's results, two years later Alfonso Antonio de Sarasa recognized the logarithmic property of hyperbolic areas [Ca2, p. 258-259]. Thus, the signed area under the hyperbola between $x = 1$ and $x = a > 0$ is the natural or 'hyperbolic' logarithm of a. This would give rise to the differential equation

$$d(\log x) = \frac{dx}{x}$$

mentioned by Euler and d'Alembert in their correspondence.

The logarithm as the inverse of the exponential function originated with John Wallis' *Algebra* of 1685 [Cj, p. 37]. However, this point of view was not popularized until the publication of Euler's *Introductio in analysin infinitorum* in 1748. In any case, the real logarithm function was well understood by the end of the seventeenth century. The extension of the domain of definition to negative and complex numbers was a task for the eighteenth century. Like so many mathematical advances in that century, it was a problem that was ultimately solved by Leonhard Euler.

Euler had already come across the assertion that $\log(-x) = \log x$ long before he began corresponding with d'Alembert. It was a position that Johann Bernoulli had expressed in his correspondence with Leibniz in 1712-

13 and had repeated to Euler in 1727-29. In the first case, Leibniz countered that the logarithm of a negative number is "impossible" [Cj, part 2], but in the Bernoulli-Euler correspondence, Euler was able to make some real progress on the problem.

Euler had been mentored by Bernoulli during his student days at the University of Basel. After Euler moved to St. Petersburg in 1727, the two corresponded regularly for nearly two decades, until shortly before Bernoulli's death. This correspondence is collected in [E 1998].

In November 1727 Euler wrote:

> By chance, I find myself struggling with the equation $y = (-1)^x$. It is extremely difficult to determine what figure it describes. It is sometimes positive, sometimes negative and sometimes imaginary, so it seems to me that it does not make a steady line, but rather has infinitely many points equally spaced at 1 unit on either side of the axis. [E 1998, p. 78]

Bernoulli responded to Euler's musings in January 1728:

> You ask what $(-1)^x$ signifies. I judge it thusly: if $y = (-n)^x$ then $\log y = x \log(-n)$, and so
> $$\frac{dy}{y} = dx \log(-n).$$
> However, $\log(-n) = \log(+n)$, for it is generally true that
> $$d \log(-z) = \frac{-dz}{-z} = \frac{dz}{z} = d \log z.$$
> From this it follows that $\log(-z) = \log(z)$, so
> $$\frac{dy}{y} = dx \log(+n).$$
> Integrating, one has $\log y = x \log n$, and from this it follows that $y = n^x = 1^x$, and so $y = 1$. [E 1998, p. 83]

[We are using the modern notation $\log y$, but Bernoulli and Euler always wrote ly. Parentheses were also added in some places in this passage for clarity.]

Euler must have been dissatisfied with this answer, which flies in the face of common sense in the case of odd integers. In his next letter, he admitted that there are arguments in favor of $\log(-x) = \log x$: "If $\log xx = z$, then $\frac{1}{2}z = \log \sqrt{xx}$, but $\sqrt{xx}$ is as much $-x$ as x, so $\frac{1}{2}z$ is $\log x$ and $\log -x$." [E 1998, p. 88] Euler observes that one may counter that xx has two logarithms, "but whoever claims two, ought to claim an infinite number." [*ibid*] Although Euler doesn't follow up on this at the time, it is interesting to see him already considering the bold claim that the logarithm is infinitely multi-valued.

Still, Euler was not ready to accept $\log x = \log(-x)$. He pointed out that this does not follow from the equality of the differentials $d\log x = d\log(-x)$. On the other hand, both the calculus and the laws of logarithms are consistent with $\log(-x) = \log x + \log(-1)$, "hence one cannot conclude the equality of $\log(-x)$ and $\log x$ without first showing that $\log(-1)$ is 0." [E 1998, p. 88]

To investigate the possibility that $\log(-1) = 0$, Euler referred Bernoulli to his own result on the quadrature of the circle from 1702 [Be]: given a sector of a circle of radius a having sine y and cosine x, Bernoulli had shown the area of the sector to be

$$\frac{aa}{4\sqrt{-1}} \log \frac{x + y\sqrt{-1}}{x - y\sqrt{-1}}. \tag{1}$$

So Euler considered the first quadrant, for which $x = 0$ and the area is

$$\frac{aa}{4\sqrt{-1}} \log(-1).$$

Euler reasoned that since the area is a finite quantity, if it were true that $\log(-1) = 0$, it would necessarily follow that $\sqrt{-1} = 0$, whence $1 = 0$.

Bernoulli did not give a satisfactory response to this objection and Euler soon let the matter drop. Unlike d'Alembert, who would debate the same position with Euler two decades later, Euler was too diplomatic to press his case when his correspondent clearly wished to move on.

Returning to Bernoulli's formula (1) with $x = 0$, we note that since the area of one quadrant is $\pi a^2/4$, what actually follows is that

$$\log(-1) = \pi\sqrt{-1}.$$

Euler always gave credit for this formula to Bernoulli, even though he himself was the first to write it explicitly [E 1980, p. 16]. The identity $\mathrm{e}^{i\pi} + 1 = 0$, which we now call Euler's Identity [1], is an immediate consequence as soon as one accepts the logarithm as the inverse of the exponential function.

4. The Introductio

Euler's two-volume *Introductio in analysin infinitorum* [E101-2] was the first of his three great analysis textbooks. Had he actually written it in 1748, he could have included a complete and correct description of the complex logarithm function. However, the manuscript was completed some time in 1743 or 1744, and it was during the long delay at the printer that

[1] Curiously, it seems that Euler never wrote down this relation, although the more general $\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta$ was given in the *Introductio*.

Euler cracked the logarithm problem. Cajori [1913, part 3] gives 1745 as the date of the completion of the *Introductio*, but more recent scholarship places the date no later than 1744, as indicated by letters between Euler and Cramer [E 1975, p. 92].

In Book I, chapter 6 of the *Introductio*, Euler described the properties of the exponential function $y = a^z$ for a real variable z. He then defined the logarithm of y to base a is to be z, and derived the laws of logarithms from the properties of the exponential function. In the next chapter, he found the usual series representations for a^x and $\log_a(1 + x)$ without the use of differential calculus. This is achieved by considering a^ω for ω infinitely small. Since $a^0 = 1$, a^ω must differ from 1 by an infinitely small quantity, say ψ. If we let $\psi = k\omega$, where k depends on the choice of a, then

$$a^\omega = 1 + k\omega \quad \text{and} \quad \omega = \log_a(1 + k\omega).$$

The series formulations arise from the application of the binomial theorem and the clever manipulation of both infinitely large and infinitely small quantities. Finally, the base e is the value of a which gives $k = 1$.

Euler clearly stated that the logarithm function is only defined for positive real numbers in Book I, chapter 6. However, in paragraph 103 in the same chapter, he mentioned that the logarithm of a negative quantity is imaginary (which should not be interpreted as purely imaginary in the modern sense, but simply as a complex number).

Although Euler eventually extended the definition of the exponential, sine, and cosine functions to the set of all complex numbers, he did not similarly extend the domain of the logarithm function. In Book II, chapter 21, he did discuss some of the "paradoxes" that make such an extension difficult. For example, since

$$1 = (-1)^2 = \left(\frac{-1 \pm \sqrt{-3}}{2}\right)^3 = (\pm\sqrt{-1})^4 = \cdots$$

it follows that

$$2\log(-1),\ 3\log\frac{-1+\sqrt{-3}}{2},\ 4\log\sqrt{-1}, \cdots$$

must all be equal to $\log 1$. This might be used to suggest that $\log x = 0$ for all complex numbers of unit modulus. However, Euler knew Bernoulli's formula (1), although he did not mention it in the *Introductio*. Nevertheless, he rejected the possibility that all of these logarithms are 0. As a consequence, he could only conclude that $\log 1$ had infinitely many values. Euler made this plausible by considering that $x = \log a$ is a root of the "infinite degree" equation

$$a = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots,$$

thus, an infinite number of solutions is to be expected.

In summary, Euler gave an elegant presentation of the real logarithm function as the inverse of the exponential in Book I of the *Introductio.* His incomplete description of the complex logarithm function in Book II represented the state of his research in 1743 or 1744. By the time he began corresponding with d'Alembert in 1746, his solution was complete.

5. The Debate between Euler and d'Alembert

The Euler-d'Alembert correspondence began on August 3, 1746, with a brief letter written by d'Alembert to Euler. The numbering of their letters used here was assigned by Juškevič and Taton [E 1980]. d'Alembert did not mention logarithms in letter 1. Euler's response and d'Alembert's follow-up are both lost, but Euler's letter is nevertheless catalogued as number 2, since there are records of its sale by a Paris antiquarian. The next letter initiated their discussion of logarithms.

5.1. *Letter 3 – December 29, 1746, from Euler*

This letter was unknown to Cajori. Euler opened it with some remarks on fluid dynamics. He then praised d'Alembert for an article on the integral calculus that he had submitted for publication [A1]. It contained d'Alembert's attempt to prove the Fundamental Theorem of Algebra and Euler would include it in the *Mémoires* of the Berlin Academy for 1746. After positive words about much of the paper, Euler said

> However, you must permit me to be in disagreement with your feelings on the subject of $\log(-x)$, which you believe not to be an imaginary number. The argument you advance is drawn from the logarithmic differential equation $dy = dx/x$, with which you wish to prove that the curve has two equal branches emanating from the asymptote, due to the fact that the equation remains the same whether one takes x to be positive or negative. [E 1980, p. 252]

That is, d'Alembert had claimed that $\log(-x) = \log x$ for positive numbers x. In response, Euler sketched a description of his complex logarithm function, although he did not provide d'Alembert with any proofs. He first noted that from the equality of the differentials of $\log(-x)$ and $\log x$ one may only conclude that $\log(-x)$ and $\log x$ differ by a constant "which is effectively $= \log(-1)$," and that the equality of differentials says nothing about whether or not that quantity is real. About that quantity, he said,

> ... I believe that I have proved that it is imaginary and that it is $= \pi(1 \pm 2n)\sqrt{-1}$, where π indicates the circumference of a circle of diameter $= 1$, and n any whole number whatsoever.
>
> For I have shown that, just as every sine responds to an infinity of arcs of the circle, so the logarithm of every number has an infinite number of different values, amongst which there is only one that is real when the number is positive, but when the number is negative all the values are imaginary. Therefore $\log 1 = \pi(0 \pm 2n)\sqrt{-1}$, n denoting any whole number whatsoever, and letting $n = 0$ we will obtain the ordinary logarithm $\log 1 = 0$. In the same way we have $\log a = \log a + \pi(0 \pm 2n)\sqrt{-1}$, where $\log a$ in the latter part indicates the ordinary logarithm of a. Now $\log(-a) = \log a + \pi(1 \pm 2n)\sqrt{-1}$, all of whose values are imaginary numbers. [E 1980, p. 252-253]

5.2. *Letter 5 – January 29, 1747, from d'Alembert*

Letter 4, written by d'Alembert on 6 January 1747, crossed in the mail with Euler's letter 3 and made no mention of logarithms. Early in letter 5, d'Alembert thanked Euler for his kind words concerning [A1] and continued:

> ... With regard to $\log(-x)$, everything you tell me on this matter disturbs me a great deal. I have not given the matter as much thought as you have, and as I wish, if possible, not to propose anything unless it is quite certain, I would appreciate it if you would cross out of my treatise the portion where it is discussed, if it has not already gone to the printer. [E 1980, p. 257]

Although d'Alembert was too shrewd to let the claim stand in his paper if there were a reasonable chance that he was in error, he was still keen to debate the matter privately. He never fully accepted Euler's account and maintained that $\log(-x) = \log x$ until the end of his days. He wrote "even though your reasoning is very keen and wise, yet I admit, Sir, that I am not entirely convinced," [E 1980, p. 257-258] so he began a tenacious debate that would occupy the balance of this letter and most of each of the six letters that followed. Consequently, the debate over the nature of the logarithm function is the largest single topic of discussion in the Euler-d'Alembert correspondence.

In this letter, d'Alembert raised three numbered objections. The most interesting of these is an argument involving the area under the hyperbola. It is most easily described using modern notation and definite integrals, but d'Alembert actually sketched the graph of the hyperbola in the body of the letter and used geometric language for his argument. He simply asserted that if $a > 0$, then

$$\log(-a) = \int_1^{-a} \frac{dx}{x}. \tag{2}$$

He then reasoned that

$$\begin{aligned}\log(-a) &= \int_{-1}^{-a} \frac{dx}{x} + \int_0^{-1} \frac{dx}{x} + \int_1^0 \frac{dx}{x} \\ &= \int_{-1}^{-a} \frac{dx}{x} \\ &= \int_1^a \frac{dx}{x} \\ &= \log a\end{aligned}$$

Of course, the assertion 2, when combined with the cancellation of infinite areas in the second step, is tantamount to *defining* $\log(-a)$ to be $\log a$. Although eighteenth century analysts were not particularly sensitive to issues of convergence, d'Alembert often was, as was Euler. In an attempt to justify the cancellation, d'Alembert wrote:

> That which confirms the real value of the ordinate while taking x negative, is that if one had been given the curve whose equation is $dy = dx/x^3$, i.e. whose ordinates were equal to the areas $\int dx/x^3$ and whose ordinate y was $= 0$ given $x = 1$, one could make same case for this curve as I have for the logarithm and, following the same line of reasoning, determine that a negative x would correspond to a real y, as it does in fact. Because the integration gives $y = -\frac{1}{2x^2} + \frac{1}{2}$. [E 1980, p. 258]

D'Alembert's other points do not merit much consideration, but it is interesting to note that one of them had the flavor of Napier's construction, involving "... two geometric progressions, ... the terms of one of which are positive and the other negative, and to suppose that to each of the two of these progressions, there corresponds the same arithmetic progression." [E 1980, p. 258-9]

5.3. *Letter 6 – March 24, 1747, from d'Alembert*

Euler clearly responded to letter 5, but there is no record of it, so d'Alembert's reply is the next letter in the sequence. In his missing letter from February or March of 1747, Euler appears to have addressed the following point, raised obliquely by d'Alembert in his second objection above. The antiderivative of $y = x^{-n}$ has two branches and is symmetric about the y-axis for odd natural numbers $n = 3, 5, 7, \ldots$, so why should this not also be so for $n = 1$? In the missing letter, Euler evidently proposed the example

$$y = \sqrt{x} + \sqrt{x\sqrt{x+a}},$$

as a curve whose behavior is suddenly different at one particular value of the parameter. In this example, when $a > 0$ the curve has two branches in the first quadrant, meeting at the origin, and is symmetric about the x-axis. However, when $a = 0$ there are no branches in the fourth quadrant and the curve exhibits a cusp at the origin, known as a cuspical point of the second kind or the second species; see [Br] for more about the importance of this curve in the professional relationship between Euler and d'Alembert.

Most of this letter concerns the logarithm debate, and d'Alembert has six numbered points. In both of the first two points, d'Alembert fails to appreciate the role of Euler's example in probing his reasoning process. Instead, he dwells on the irrelevant fact that neither this particular example, nor anything similar to it, would refute his claim that $\log(-x) = \log x$. In another one of his points, d'Alembert considers the logarithm to be the inverse of the exponential for the first time in the correspondence. Yet another point indicates nothing less than a failure of imagination on d'Alembert's part:

> You agree that the equation $dx = dy/y$ proves that the logarithm of $-y$ and that of y differ by only a constant, but you claim that this constant is imaginary. Now this seems to me quite difficult to conceive of, for if you imagine any function of y you wish, which is the logarithm of y, and making y negative in this function, it becomes the logarithm of $-y$. That is to say, according to you, we need to find a function of y which, in making y negative, does not change its value, except that it gives birth, all of a sudden, to an imaginary constant. Well, I declare that I am unable to conceive such a function. [E 1980, p. 261]

Without a doubt, his strongest argument in this letter is his last one:

> All difficulties reduce, it seems to me, to knowing the value of $\log(-1)$. Now why may we not prove this to be 0 by the following reasoning? $-1 = 1/-1$, so $\log(-1) = \log 1 - \log(-1)$. Thus $2\log(-1) = \log 1 = 0$. Thus $\log(-1) = 0$. [E 1980, p. 261]

5.4. *Letter 7 – April 15, 1747, from Euler*

Euler's response to letter 6 is an important document. It reads very much like a draft version of major portions of his paper [E807], which we will consider in detail in the next section. In [Cj, part 3], this letter is the first letter considered, whereas Euler and d'Alembert had in fact already exchanged four letters on logarithms and raised numerous arguments.

In this letter, Euler addressed most of d'Alembert's objections from letter 6, including the argument that $\log(-1) = 0$. In this passage, he assumed that d'Alembert was familiar with Bernoulli's formula (1):

> With the same reasoning that you use to prove $\log(-1) = 0$, you could equally well prove that $\log\sqrt{-1} = 0$ for since $\sqrt{-1}\cdot\sqrt{-1} = -1$, you would have $\log\sqrt{-1}+\log\sqrt{-1} = \log(-1)$, i.e. $2\log\sqrt{-1} = \log(-1) = \frac{1}{2}\log(+1)$ and thus $\log\sqrt{-1} = \frac{1}{4}\log 1 = 0$. If you do not approve of this reasoning, you will agree that the first is no more convincing. Now, you must at least agree that logarithms of imaginary numbers are not real, for otherwise the expression $\log\sqrt{-1}/\sqrt{-1}$ could not express the quadrature of the circle. [E 1980, p. 264]

Having dealt with d'Alembert's objections, Euler now shares more of his insight into the matter. We observe that towards the end of this passage, Euler uses the symbol π, whose meaning has not yet become standardized, to represent the quantity we denote by 2π, and so the imaginary components of the complex numbers appear to us to be too small by half.

> To explain this better, let 0, α, β, γ, δ, ϵ, ζ, η, θ, ι, κ etc. be the logarithms of unity, and I say that values of $\log(-1)$ will be
>
> $$\frac{\alpha}{2}, \frac{\gamma}{2}, \frac{\epsilon}{2}, \frac{\eta}{2} \quad \text{etc.}$$
>
> all imaginary, such that the double of each appears among the logarithms of $+1$. However, it does not follow that the half of each of the values of $\log(+1)$ is found among the $\log(-1)$, since -1 is but one value of $\sqrt{+1}$, the other being $+1$, whose logarithms are
>
> $$\frac{0}{2}; \frac{\beta}{2}; \frac{\delta}{2}; \frac{\zeta}{2} \quad \text{etc.}$$
>
> which are precisely the same as 0, α, β, γ, δ, ϵ, ζ, etc. For
>
> $$\frac{\beta}{2} = \alpha, \quad \frac{\delta}{2} = \beta, \quad \frac{\zeta}{2} = \gamma, \quad \frac{\theta}{2} = \delta, \quad \text{etc.}$$
>
> Similarly, as the three cube roots of 1 are
>
> $$1, \quad \frac{-1+\sqrt{-3}}{2} \quad \text{and} \quad \frac{-1-\sqrt{-3}}{2},$$
>
> the logarithms of these three roots will be
>
> $$\log 1 = \frac{0}{3}, \quad \frac{\gamma}{3}, \quad \frac{\zeta}{3}, \quad \frac{\iota}{3}, \quad \frac{\mu}{3} \quad \text{etc.},$$

> the same as 0, α, β, γ, δ, ϵ, etc.,
>
> $$\log\left(\frac{-1+\sqrt{-3}}{2}\right) = \frac{\alpha}{3},\quad \frac{\delta}{3},\quad \frac{\eta}{3},\quad \frac{\kappa}{3},\quad \frac{\nu}{3}\quad \text{etc.,}$$
>
> $$\log\left(\frac{-1-\sqrt{-3}}{2}\right) = \frac{\beta}{3},\quad \frac{\epsilon}{3},\quad \frac{\theta}{3},\quad \frac{\kappa}{3},\quad \frac{\nu}{3}\quad \text{etc.,}$$
>
> and these letters α, β, γ, δ, ϵ, etc. are not based on pure conjecture; I have had the honor giving you their actual values. For if π is the circumference of the circle whose radius is $= 1$ [sic], then the values of $\log(+1)$ are 0; $\pm\pi\sqrt{-1}$; $\pm 2\pi\sqrt{-1}$; $\pm 3\pi\sqrt{-1}$; $\pm 4\pi\sqrt{-1}$; $\pm 5\pi\sqrt{-1}$; etc. of $\log(-1)$ are $\pm\frac{\pi}{2}\sqrt{-1}$; $\pm\frac{3\pi}{2}\sqrt{-1}$; $\pm\frac{5\pi}{2}\sqrt{-1}$; etc. [E 1980, p. 265-6]

5.5. *Letter 8 – April 26, 1747, from d'Alembert*

At this stage in the correspondence, most of the important points have already been made. One of d'Alembert objections indicates that he is not willing to follow Euler in adopting the power series definition of the exponential function, for "I had objected that if $x = 1/2$, then e^x has two values, one positive and the other negative." [E 1980, p. 267] He goes on to say "the reduction of quantities to series often expresses their values incorrectly..." It is curious that d'Alembert expresses concerns about convergence with respect to the exponential series, and yet he is happy to cancel improper integrals in an earlier letter. Another notable passage from this letter is the following, where he first raises a point that he will come back to in subsequent letters.

> It is true that the formula involving sines gives those values for $\log(-1)$, but is it quite clear that this formula gives all of the values of $\log(-1)$? It is this that I do not yet see ... [E 1980, p. 268]

5.6. *Letter 9 – August 19, 1747, from Euler*

Among the interesting points in this letter is Euler's confirmation that he was able to remove the paragraph concerning $\log(-x)$ from d'Alembert's paper [A1] and his announcement that he has forwarded his memoir [E807] to the Academy, "... where I believe I have put this matter to rest; at least for my part, I have not the least difficulty with it, whereas I had previously been extremely perplexed [E 1980, p. 271]."

5.7. *The Debate Winds Down*

In letter 10, d'Alembert makes the curious comment that "from my point of view, log 1 is completely indeterminate." [E 1980, p. 274] Euler opens his brief Letter 11, of December 30, 1747, with the following passage:

> I have learned from Mr. de Maupertuis that you wish to suspend your work in mathematics for a little while, in order to reestablish your health, which has been considerably weakened by your great efforts. I approve so heartily of this resolution, for which I wish you all attendant success, that I do not wish to trouble you with a discussion of imaginary logarithms, although I can't think of much else that I could add to this matter that I have not already mentioned, and I really doubt whether my paper on this subject will be able to allay the doubts which you have taken the trouble to describe to me. [E 1980, p. 273]

Although Euler's intention to drop the subject is clear, d'Alembert brings up logarithms early in 1748 in letters 12 and 14. During the summer of the same year, the *Introductio* was finally published, and Euler sent d'Alembert a copy. In letter 16 of September 7, d'Alembert reacted to Euler's brief mention of the problems with logarithms of negative numbers in Book II. However, Euler had no appetite for continuing the discussion of logarithms, and in letter 17, of September 28, he said "the matter of imaginary logarithms is no longer so familiar to me that I may rigorously respond to your latest remarks ..." [E 1980, p. 294] In letter 18, of 27 October 1748, d'Alembert got in a brief last word, and the debate came to an end.

6. Euler's First Memoir

On September 7, 1747, in the midst of his debate with d'Alembert, Euler read his paper *Sur les logarithmes des nombres négatifs et imaginaires* (On the logarithms of Negative and Imaginary Numbers) [E807] to the Berlin Academy. The article was not published during Euler's lifetime, but only in his *Opera postuma* in 1862.

The paper consists of 34 paragraphs. The first seven are devoted to a discussion of the debate between Leibniz and Johann Bernoulli, available since 1745 in their published correspondence. The next nine contain a general discussion of the problem, including the nature of curves with multiple branches and, in §13, the definition of the logarithm as the inverse of the exponential function. The series for e^x is given in §16, and Euler stresses that this will be the definitive arbiter of the value of e^x, even in the case of square roots.

> This series being regarded in analysis as completely equivalent to the expression e^x, there can be no doubt that its value is determined as soon as x is assigned a given value, for the series is always convergent, no matter how large the number we substitute for x. And for this reason, we are right to hold that insofar as the expression e^x represents the number whose logarithm is $= x$, there can never be any ambiguity in it, and its value is always unique and positive, whatever fraction we take for x, so that even if x is a fraction such as $1/2$, the expression for e^x will have but a single positive value. [E807, p. 274]

During the course of the next seven paragraphs, Euler discusses the paradoxes of logarithms. He includes and elaborates upon the observations he made in Book II of the *Introductio*. He describes the various hypotheses and pieces of evidence that led him to conclude the the number 1 must have infinitely many logarithms, using the same notation he used in letter 7 to d'Alembert.

In §24, he begins his proof that every number has an infinitude of logarithms. He notes the important correspondence between logarithms and arcs of a circle, and says "we are more familiar with the circle than with the logarithmic curve and, for this reason, the consideration of the circle will bring us to a more perfect understanding of logarithms than even the logarithmic curve will."

Euler begins §25 as follows: "Consider an arbitrary arc φ of a circle of radius $= 1$. Let x be the sine of this arc and y the cosine ...." In the discussion that follows, we will interchange the roles of x and y so that $x = \cos\varphi$ and $y = \sin\varphi$, as a modern reader would expect, and we will also use i and log in place of $\sqrt{-1}$ and l. In all other details, the argument is as given by Euler.

Consider an arbitrary arc φ of the unit circle, with $x = \cos\varphi$ and $y = \sin\varphi$, so that $x = \sqrt{1-y^2}$. Clearly, any arc of the form $\pm 2n\pi + \varphi$ has the same sine and cosine, where n is an arbitrary natural number. Now

$$d\varphi = \frac{dy}{x} = \frac{dy}{\sqrt{1-y^2}}.$$

This observation apparently needed no further justification to readers familiar with Leibniz' calculus of differentials. For a demonstration, consider the diagram in figure 1: since the tangent $d\varphi$ is perpendicular to the radius by elementary geometry, it follows that the differential triangle is similar to the right triangle involving x and y. Thus, $d\varphi : 1 :: dy : x :: dx : y$.

Now let $y = iz$, and we will have

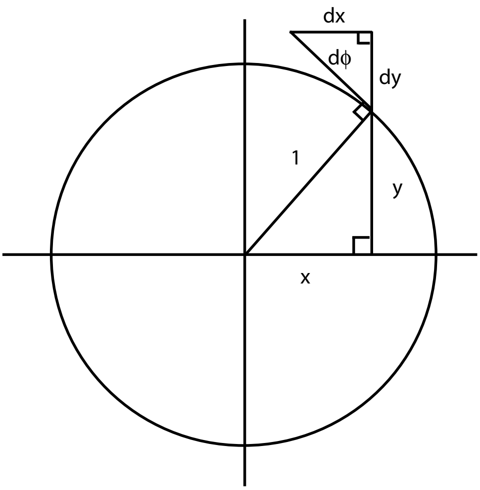

Fig. 1. The unit circle and its differential triangle

$$d\varphi = \frac{i\,dz}{\sqrt{1+z^2}}.$$

Euler assumed familiarity with the indefinite integral

$$\int \frac{dz}{\sqrt{1+z^2}} = \log(\sqrt{1+z^2}+z) + C,$$

so that

$$\varphi = \int \frac{i\,dz}{\sqrt{1+z^2}} = i\log(\sqrt{1-y^2}+\frac{y}{i}) + C.$$

By considering $y = 0$, it is clear that $C = 0$. So

$$\begin{aligned}\varphi &= i\log(x-iy)\\ &= \frac{1}{i}\log[(x-iy)^{-1}]\\ &= \frac{1}{i}\log(x+iy).\end{aligned}$$

In §26, Euler notes that the formula holds for any other arc φ with the same sine and cosine, so for any integer n, we have

$$\varphi + 2n\pi = \frac{1}{i}\log(x+iy),$$

and hence

$$\log(x+iy) = i(\varphi + 2n\pi),$$

"from which it is clear that to any number $x + iy$ there corresponds an infinity of logarithms." Substituting back for x and y, we have the general formula for the logarithm of any complex number of unit modulus:

$$\log(\cos\varphi + i\sin\varphi) = i(\varphi + 2n\pi).$$

7. Euler's Second Memoir

When it came time to publish, Euler replaced his first memoir on logarithms with a different memoir, *"De la controverse entre Mrs. Leibnitz et Bernoulli sur les logarithmes des nombres négatifs et imaginaires"* (On the Controversy between Messers. Leibniz and Bernoulli concerning the Logarithms of Negative and Imaginary Numbers) [E168], which appeared in the journal of the Berlin Academy for 1749. It was 60% longer than the original and had somewhat more detail concerning the Leibniz-Bernoulli controversy. However, there is significant difference in the mathematical exposition of the two papers, and this accounts for most of the difference in length.

At two points in their correspondence (letters 5 and 10), d'Alembert declared himself unconvinced by Euler's arguments involving "formulas for the arcs of circles" [E 1980, p. 259]. Perhaps this is why Euler took an entirely different approach in his second article. Alternately, it may have been the use of the integral calculus in [E807] that felt unsatisfactory to Euler, since the logarithm ought to be an elementary topic; this explanation was suggested in [Cj, part 3]. It is likely that Euler also wanted his exposition to be similar in style and notation to his *Introductio*, where it almost certainly would have been included had he discovered his results in time. Cajori's theory is certainly compatible with this.

After preliminary discussion, Euler proves the theorem that any given quantity has infinitely many logarithms. Following his exposition in the *Introductio*, he observes that $\log(1+\omega) = \omega$ for any infinitely small quantity ω, so that

$$\log(1 + \omega)^n = n\omega.$$

If n is finite, then $(1+\omega)^n$ differs from 1 only by an infinitely small quantity. So let n be infinitely large and

$$x = (1 + \omega)^n. \tag{3}$$

Let $y = \log x$, so that we we have $y = n\omega$. To find y, we first solve equation (3) to get

$$\omega = x^{1/n} - 1.$$

From this it follows that

$$\log x = y = nx^{1/n} - n.$$

The equivalent modern statement is formulated in terms of limits. Euler anticipated this as follows:

> From this it is clear that the value of the formula $nx^{\frac{1}{n}} - n$ will approach the logarithm of x all the more, the larger the number n is taken, and if we let n be an infinite number, this formula will give the true value of the logarithm of x. [E168, pp. 156-157]

To complete the proof, Euler observes that

> ... as it is certain that $x^{\frac{1}{2}}$ has two values, $x^{\frac{1}{3}}$ three, $x^{\frac{1}{4}}$ four, and so on, it is equally certain that $x^{\frac{1}{n}}$ must have an infinity of different values. As a consequence, this infinity of distinct values of $x^{\frac{1}{n}}$ will produce an infinity of distinct values of $\log x$, so that the number x must have an infinity of logarithms. QED [E168, p. 157]

Having proved the theorem, Euler now solves the problems of finding all the logarithms of a given positive number, of a given negative number and finally of an arbitrary 'imaginary number,' i.e. a complex number. Of course, the results are the same as in [E807]. The arguments are longer and feel more complicated. They certainly seem more contrived to the modern reader, since they involve subtle properties of infinitely large and infinitely small quantities. Unlike the first memoir, which is entirely accessible to an undergraduate mathematics major, the second memoir is incompatible with modern notions of analysis.

8. Conclusion: D'Alembert's Memoir

Because of the publication lag in the *Mémoires* of the Berlin Academy, d'Alembert did not see Euler's second memoir on logarithms until after he had stopped corresponding with him in 1751. Shortly afterwards, when d'Alembert disputed Euler's priority on three other articles in this volume, he also expressed his disagreement with the substance of Euler's logarithms paper, which is distinguished by being the only one of the four disputed articles to have been discussed at such significant length in their correspondence.

If one considers d'Alembert's penchant for self-promotion and demonstrated appetite for controversy, it seems entirely plausible to that he was disappointed not to be mentioned by name in Euler's logarithms article, for both the publicity that it would have afforded him, and for the presumed opportunity to publish a rebuttal. At the time Euler composed the article, however, he and d'Alembert were still corresponding amiably, so I believe that Euler, convinced that d'Alembert's position was demonstrably wrong, suppressed any mention of his name only because he wished to spare him embarrassment. The publication of Bernoulli's correspondence in 1745 afforded him a literary device for discussing d'Alembert's position while attributing it to Bernoulli.

Whatever the case, d'Alembert did submit a response to Euler's article. The manuscript of *"Sur les logarithmes des quantités négatives,"* was dated 16 June 1752. There is no mention of this manuscript in the *Registres* of the Academy [Wn] and we may infer that Euler decided that it was not fit for publication on the grounds of both its mathematical content and its polemical tone. Instead, d'Alembert eventually published a slightly edited version of this essay in the first volume of his *Opuscules mathématiques*, described in [Tr, p. 274] as "collections of papers having little or no solid content, not of a quality or style fit for a learned journal, but nevertheless sufficient, with the renown of d'Alembert's name among the 'semi-learned,' to be sold successfully by a commercial publisher."

d'Alembert began his article with the following definition, in the spirit of Napier: "We call logarithms an arbitrary series of numbers in arithmetic progression, corresponding to an arbitrary series of numbers in geometric progression." [A4, p. 181-2] However, this is immediately followed by a discussion of the exponential curve, and later in the paper d'Alenbert freely uses the logarithmic differential equation when it suits his needs. Almost all of the arguments in this paper had already been put forward in his correspondence with Euler.

In his introductory remarks, d'Alembert described the structure of his essay in the following terms.

> I will first set forth the purely metaphysical reasons which permit one to consider logarithms of negative numbers as being real. To these reasons, I will add others that are purely geometrical, which seem to me to constitute a proof, and finally, I will respond to objections. [A4, p. 183]

D'Alembert's 'metaphysical' arguments are actually mathematical in nature; he seems to use the designation for arguments that are meant to persuade, but fall short of being a proof. For example, he argues by analogy with algebraic curves that a graph should only pass from real into complex values by virtue of having 'radical pairs.' Thus, it is difficult to concieve of x "passing abruptly from $-\infty$ to the imaginary" [p. 184] as y becomes negative in the exponential curve, since there is but a single value of x corresponding to each positive y, all the more so since x does not take a finite value when $y = 0$.

It is interesting that these metaphysical arguments come ahead of the mathematical ones, almost as though rhetorical and legalistic arguments should be as important to mathematical discourse as proofs and counterexamples. There is indeed a sort of scholastic tone to the whole piece: it is almost as though his real goal here, as in his letters, was to propose objections that would stump Euler.

However, it would be a mistake to dismiss d'Alembert as a mathematical crank. His record of other achievements in mathematics stands on its own merits. Also, d'Alembert probably deserves some credit for a rather modern point of view that a function may be defined arbitrarily, whereas Euler proceeds as though the logarithm function were somehow given *a priori*. Of course, Euler's patiently argued position on logarithms is more persuasive to the modern reader, but the whole notion of an infinitely multi-valued function must have seemed very odd to any mathematician in the 1740s. We know that Euler himself struggled mightily with the difficulties presented by the paradoxes of logarithms of negative numbers, before concluding that every number has infinitely many logarithms.

Acknowledgement: Translations from the correspondence were jointly done with John S. D. Glaus. Other translations are by the author.

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Leonhard Euler: Life, Work and Legacy
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Some Facets of Euler's Work on Series

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1. Introduction

Much of Euler's most important work involves series. When the Editors of the *Opera omnia* prepared the Series I volumes 14, 15, 16 and 16*, they used a definition that includes infinite products and continued fractions as well as the more traditional topics in progressions and infinite sums. If we use their definition, then Euler published 81 papers on the subject. Only number theory, with 96 papers, yields a larger count. However, this analysis understates the importance of series in Euler's work. They form much of the basis of Euler's analytic worldview. Euler used series as the foundation of his differential calculus, and as a basic tool in many other topics. Sometimes, it is only because of a decision by the Editors of the *Opera omnia* that we might think an article is mostly about series rather than something else, like differential equations or geometry or elliptic integrals. There are a number of landmarks in Euler's contributions involving series. We list a few of them:

(i) Evaluation of $\zeta(2)$ in 1735, thus solving the so-called Basel problem;
(ii) What Euler called the "interpolation of the hypergeometric series," what we now call the factorial numbers in 1729, thus discovering what

we now call the Gamma function;

(iii) The gradual discovery of the Euler-Maclaurin summation formula, beginning about 1732;

(iv) The application of series to number theory, leading, among other things, to generating functions and the 1741 solution to Philip Naudé's problem on the partition of integers;

(v) The product-sum formula for the Zeta function, discovered in 1737;

(vi) The development of rapidly-converging series to enable the practical preparation of tables of logarithms, trigonometric functions and the logarithms of trigonometric functions, starting about 1738 and continuing throughout his life;

(vii) The identification of certain constants and their usefulness, including γ (the so-called Euler-Mascheroni constant), e, π, and sequences of coefficients called the Bernoulli numbers and the Euler numbers, starting about 1736;

(viii) All of the fundamental properties of continued fractions in 1737, with extensions and clarifications later on.

Many readers will want to add to this list.

There are at least two excellent and comprehensive accounts of Euler's work in series. The first is Georg Faber's 112-page "Übersicht über die Bände 14, 15, 16, 16* der ersten Serie," [F] that serves as the Editor's preface to the three volumes (in four parts) on series in the *Opera omnia.* It was written in 1935.

Faber divides his summary into several parts.

(i) 14 articles on summation formulas, especially Bernoulli numbers,

(ii) 10 articles on the Gamma function,

(iii) 13 articles on trigonometric series and series involving trigonometric functions,

(iv) 9 articles on binomial series and binomial coefficients,

(v) 17 articles on other functions and series,

(vi) 11 articles on continued fractions and infinite products,

(vii) 7 articles involving the number π.

Like most of the Editors' Introductions in the *Opera omnia*, this is a masterpiece of 20^{th} century scholarship. It is thoroughly and meticulously researched and contains information almost impossible to find anywhere else. However, it was written in 1935, and based on the vision for the *Opera omnia* that was formulated about 1905. That vision reflects an 19^{th} century intellectual theory that relies heavily on taxonomy and classification. In this era of relational databases, connectivity and network analysis, there are too many questions that a taxonomical historiography cannot answer. It may be too much to hope that the Editors of the *Opera omnia* might be able to add new introductions to these key parts of the work in future editions.

A somewhat more modern account of Euler's early work in series is given by Josef Ehrenfried Hofmann [H] in "Um Eulers erste Reihenstudien," written in 1957. This is a copiously footnoted 80-page article in a *Sammelband* in which the other twenty-five articles average only 10 pages. Hofmann traces the roots and motivation of Euler's work through the greatest mathematicians of earlier times, like Fermat, Leibniz and several Bernoullis, as well as many lesser luminaries like Goldbach, Riccati, Bilfinger, Stirling, Maclaurin and Hermann. He uses Euler's correspondence, as well as the work of his contemporaries, to trace the birth and development of much of Euler's work on series through the First St. Petersburg period.

Both of these are excellent, though neither is recent, and they reflect the historiography and sentiments of their times. Both are also considerably longer than this piece can be, so this chapter must have limited but focused ambitions.

The first part of this essay will treat Euler's recognition of the ubiquity and utility of the Bernoulli numbers. It is a demonstration of how the act of giving a name to an object, the sequence of Bernoulli numbers, made that object a powerful tool that connected previously unrelated mathematical results.

The second part will complement the first, describing the many circumlocutions of notation Euler used to do what modern mathematicians would do with indicial notations like subscripts and superscripts. This will demonstrate how the lack of a convenient and powerful notation distorted certain kinds of mathematical developments of the era, and it will not lead to a resolution and epiphany the way the first part does.

The two parts are related in other ways as well. Both will rely almost exclusively on the work of Euler himself, with little reference to the contributions of his contemporaries. The first part will follow Faber's classification very closely; it will use almost exclusively the articles Faber identifies as relating to Bernoulli numbers. The second part, in contrast, will cut across the taxonomy of the *Opera omnia* and use sources from a variety of volumes.

2. Euler and the Bernoulli numbers

2.1. *Prehistory*

Bernoulli numbers are a sequence of rational numbers that arise in a dazzling variety of applications in analysis, numerical analysis and number theory. When Charles Babbage designed the Analytical Engine in the

19^{th} century, one of the most important tasks he hoped the Engine would perform was the calculation of Bernoulli numbers.


JACOBI BERNOULLI,
Profeſſ. Baſil. & utriuſque Societ. Reg. Scientiar. Gall. & Pruſſ. Sodal.
MATHEMATICI CELEBERRIMI,
ARS CONJECTANDI,
OPUS POSTHUMUM.
Accedit
TRACTATUS
DE SERIEBUS INFINITIS,
Et EPISTOLA Gallicé ſcripta
DE LUDO PILÆ RETICULARIS.

BASILEÆ,
Impenſis THURNISIORUM, Fratrum.
cIↃ IↃcc xiii.

Fig. 1. Title page from Bernoulli's *Ars Conjectandi*

The first few Bernoulli numbers are $B_0 = 1$, $B_1 = \frac{-1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = \frac{-1}{30}$, $B_5 = 0$, $B_6 = \frac{1}{42}$.

After B_1 all Bernoulli numbers with odd index are zero, and the non-zero ones alternate in sign. They first appeared in 1713 in Jakob Bernoulli's pioneering work on probability, *Ars Conjectandi.* Jakob Bernoulli (1654-1705) was the older brother of Johann Bernoulli (1667-1748), who was, in turn, Euler's teacher and mentor at the University of Basel.

Bernoulli was studying sums of powers of consecutive integers, like sums of squares,

$$1 + 4 + 9 + 16 + 25 = 55$$

or sums of cubes

$$1+8+27+64+125+216+343=784.$$

In modern notation (Bernoulli did not use subscripts, nor did he use Σ for summations or ! for factorials) Bernoulli found that

$$\sum_{k=1}^{n-1} k^p = \sum_{k=0}^{p} \frac{B_k}{k!} \frac{p!}{(p+1-k)!} n^{p+1-k}$$

If n is large and p is small, then the left hand side is a sum of a relative large number of relatively small powers. If we know the necessary Bernoulli numbers then the sum on the right is simpler to evaluate than the sum on the left. Jakob Bernoulli himself is said [G+S] to have used this formula to find the sum of the tenth powers of numbers 1 to 1000 in "less than half of a quarter of an hour." The answer is a 32-digit number.

Just two years after *Ars conjectandi*, in 1715, Brook Taylor (1685-1731) published his book *Methodus incrementorum directa et inversa.* It contains what is today called Taylor's Theorem. Though clearly others of the era had similar or equivalent results, Euler himself called such series "Mr. Taylor's series." Soon, great varieties of functions were expanded into power series, but at the time, apparently nobody noticed that the coefficients in the expansion $\frac{x}{e^x-1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}$ involved the very same sequence that Bernoulli had found. Bernoulli numbers are also involved in the expansions of several other functions, including $\tan x$, $\frac{x}{\sin x}$, $\log\left(\frac{\sin x}{x}\right)$, among others.

2.2. *1729-1736: The Euler-Maclaurin summation formula*

More than 25 years pass between the time Euler has his first ideas related to Bernoulli numbers and the time Euler sees that relationship. In his very first letter to Goldbach [J+W], dated 13 October 1729, Euler mentions "interpolating the sums" of the harmonic series. By this, Euler means he gives meaning to expressions like $\sum_{k=1}^{n} \frac{1}{k}$ when n is not an integer. The next year he published his results in *De summatione innumerabilium progressionum* [E20]. There, Euler observes that, for integer values of n,

$$\frac{1-x^n}{1-x} = 1+x+x^2+\cdots+x^{n-1}.$$

Integrating from 0 to 1 gives

$$\int_0^1 \frac{1-x^n}{1-x}dx = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}.$$

Since the expression on the left is well defined even if n is not an integer, Euler declares that it interpolates the harmonic series. In particular, if

$n = \frac{1}{2}$, we get the value $2 - 2\ln 2$. This is the value that Euler announced, without proof, in his 1729 letter to Goldbach.

This is significant to the present discussion because it inspired Euler to compare integrals with series. In the same article, Euler was able to represent $\sum_{k=1}^{n} \frac{1}{k^2}$ as a double integral, which, though he could not evaluate it exactly, he was able to approximate as n became infinite, and thus get a six-decimal place approximation to the solution to the Basel problem.

Two years later, in the next issue of the *Commentarii* of the St. Petersburg Academy, Euler extended his results from E20 in *Methodus generalis summandi progressiones* [E25]. There he gives his first version of what we now call the Euler-Maclaurin summation formula as follows:

"If the general term of index n of a progression is given by t, and if s is the sum of all the terms up to t, then

$$t = \frac{ds}{1dn} - \frac{dds}{1\cdot 2dn^2} + \frac{d^3s}{1\cdot 2\cdot 3dn^3} - \frac{d^4s}{1\cdot 2\cdot 3\cdot 4dn^4} + \text{etc.}$$

in which equation dn is constant. This equation can be transmuted into the equation

$$s = \int tdn + \alpha t + \frac{\beta dt}{dn} + \frac{\gamma d^2t}{dn^2} + \frac{\delta d^3t}{dn^3} + \text{etc.},$$

in which the coefficients α, β, γ, etc. have the values

$$\begin{aligned}
\alpha &= \frac{1}{2}\\
\beta &= \frac{\alpha}{2} - \frac{1}{6}\\
\gamma &= \frac{\beta}{2} - \frac{\alpha}{6} + \frac{1}{24}\\
\delta &= \frac{\gamma}{2} - \frac{\beta}{6} + \frac{\alpha}{24} - \frac{1}{120}\\
\varepsilon &= \frac{\delta}{2} - \frac{\gamma}{6} + \frac{\beta}{24} - \frac{\alpha}{120} + \frac{1}{720}
\end{aligned}$$

etc."

This is all the explanation of the formula that Euler gives us. There is no proof other than claiming that the "equation can be transmuted." Modern proofs of the Euler-Maclaurin formula are usually based on an analysis of the error term for the trapezoid rule, and on properties of Bernoulli polynomials. Such a proof would have been impossible for Euler in the 1730s.

In E25, Euler applied his new results only to summing certain kinds of sequences, especially those for which the numerators formed a geometric sequence and the denominators an arithmetic or polynomial sequence. Then between 1734 and 1736, Euler wrote four more papers, E43, E46, E47 and E55 that extended and clarified these results. We will look more closely at E46 and E47 in the second part of this essay.

In 1742, Colin Maclaurin independently discovered the same formula during his studies of quadrature. The approaches of Maclaurin and Euler combine to show how integration can be used to approximate series and, conversely, series can approximate integrals.

2.3. *1735: The Basel problem*

At the end of E20, Euler used his new methods to estimate the value of the sum of the squares of the integers, the solution of the so-called Basel problem, as 1.644934. To get such an accurate answer by summing the series itself would require over 10,000 terms. He later extended his approximation to 19 decimal places. Euler, as the best calculator of his era, eventually recognized the value as $\frac{\pi^2}{6}$, and, with this key clue, he solved the Basel problem in 1735 in *De summis serierum reciprocarum* [E41]. Dunham [D] gives a clear description of Euler's wonderful solution.

It is a mark of a good problem and of a good solution that the solution gives more than was asked in the problem. This was certainly the case in Euler's solution to the Basel problem, as Euler gives not only the sum of the reciprocals of the squares, but also the sum of the reciprocals of any even power of the integers. He shows that

$$[\zeta(2n)] = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = A_{2n}\pi^{2n},$$

and he gives exact values for A_{2n} through $n = 6$, and a recursive relation for generating more coefficients. Later, in the *Introductio in analysin infinitorum* [E101] he gives coefficients through $n = 13$. At the time, Euler did not recognize that these coefficients were related to Bernoulli's numbers and to the coefficients in the Euler-Maclaurin summation formula by

$$\sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{2^{2n}|B_{2n}|\pi^{2n}}{2\,(2n)!}$$

2.4. *1734: Gamma: The Euler-Mascheroni constant*

The number we now call the Euler-Mascheroni constant and denote by gamma (γ) made its first appearance in *De progressionibus harmonicis observations* [E43]. Here, Euler uses the Euler-Maclaurin formula to sum series in which the numerators form geometric series and the denominators form arithmetic series. He does not yet realize the full power of the Euler-Maclaurin formula, and seems to think that the most interesting application is accelerating the convergence of series for logarithms. In the course of his calculations, he writes

$$1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{i}=\ln(i+1)+0.577218$$

where i is an infinite number.

When Euler re-visits gamma in *Inventio summae cuiusque seriei ex dato termino generali* [E47], Euler calculates the sum of the first million terms of the harmonic series to 16 decimal places. Ignoring issues of convergence, he further describes the sum of the entire harmonic series as

$$= \ln\infty + 0.5772156649015329.$$

At this stage, nobody recognized the special significance of this constant. Euler treated things as though any series had an associated constant that describes the difference between its sum and its associated integral. Euler works an example involving the odd harmonic series. To find the associated constant, he writes the series $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\text{etc.}$ and assumes that it is equal to "Const. $+\frac{1}{2}\ln\infty$." He tells us to double this series, then subtract the harmonic series, and get $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\text{etc.}$ He recognizes this as $\ln 2$, and substitutes back to get

$$\ln 2 = 2\,\text{const.} + \ln\infty - \ln\infty - 0.577215\,\text{etc.}$$

From this, he finds that the constant associated with the odd harmonic series is 0.6351814227307392. Euler examines the constants associated with several other series as well, but he seems to take no notice of the other relations between their constants and the constant for the harmonic series.

Except for one episode in 1755, which we will describe later, gamma research was mostly idle until 1790, when Mascheroni published his *Adnotationes ad Calculum integralem Euleri* [M]. Mascheroni gives the value of gamma as a diverging series, $\gamma = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k}$, and shows how its value arises in series expansions of integrals like $\int \frac{e^x\,dx}{x}$, $\int \frac{dx}{\ln x}$ and $\int \frac{dx}{\ln(\ln x)}$. It is because he recognized these additional properties of the constant that gamma is known as the Euler-Mascheroni constant.

2.5. *1755: Making the trees into a forest*

Let us turn now to 1755, when Euler published his *Institutiones calculi differentialis* [E212]. At that time, most of the results above were known, but their links to Bernoulli numbers were not yet recognized. To summarize the known results, we list:

(i) Bernoulli's own results on summing powers of integers. Bernoulli showed how this involved Bernoulli numbers, hence the name,
(ii) The Euler-Maclaurin summation formula,
(iii) Taylor series for various functions,
(iv) Euler's evaluation of $\zeta(2n)$,
(v) The value of the Euler-Mascheroni constant and its relation to the harmonic series.

Then, through all the trees, Euler sees the forest. It must have been a wonderful feeling to see how so many different aspects of mathematics are linked through these mysterious Bernoulli numbers.

Euler devotes almost all of chapters 5 and 6 of Part 2 of his *Calculus differentialis* to results related to Bernoulli numbers, and on page 420 (page 321 of the *Opera omnia* edition) he attributes them to Jakob Bernoulli and calls them *Bernoulli numbers.* Though de Moivre was the first one to call this sequence the Bernoulli numbers, when Euler joined him in adopting the terminology, it certified their importance.

Unfortunately, only Part 1 of the *Calculus differentialis* has been translated into English, so readers who want to enjoy it in Euler's words must either brave the Latin or find a copy of the rare 1790 German translation.

Euler begins his chapter 5, "Investigation of the sums of series from their general term" with a quick treatment of Bernoulli's results on summing sequences of powers. Then he repeats his own results from the 1730s [E25] on the Euler-Maclaurin formula and gives the recursive relation on the coefficients in that formula. Euler does not mention Maclaurin, so he is probably unaware of his work on the subject.

Then he shows how those coefficients arise from the Taylor series expansions of $\frac{x}{1-e^{-x}}$ and $\frac{1}{2}\cot\left(\frac{1}{2}x\right)$.

Eventually, after quite a bit of work, he lists the Bernoulli numbers, naming them after Bernoulli in the process, and shows how they are related to the coefficients in the Euler-Maclaurin formula.

This done, he extends occurrence of Bernoulli numbers in the expansion of $\frac{1}{2}\cot\left(\frac{1}{2}x\right)$ to the more general form $\frac{\pi}{n}\cot\left(\frac{m\pi}{n}\right)$ and uses that to relate Bernoulli numbers to the values of $\zeta(2n)$. To end the theoretical parts of his exposition, he gives some of the properties of the Bernoulli polynomials and notes that Bernoulli numbers grow faster than any geometric series.

Euler spends the rest of these two chapters doing applications of Bernoulli numbers, including calculating the Euler-Mascheroni constant, γ, to 15 decimal places.

All this is rather unexpected in a textbook on differential calculus.

2.6. *On names*

The *Book of Genesis* recounts the story that God commanded Adam to give names to all the beasts, and in doing so gave people dominion over the animals. This is often interpreted metaphorically as describing the power of giving and knowing names. The story of naming the Bernoulli numbers is consistent with that metaphor, for once the Bernoulli numbers had a name, their diverse occurrences could be recognized, organized, manipulated and understood. Having a name, they made sense.

Simon Singh [S] quotes Andrew Wiles as describing the process of mathematical discovery with the colorful words "You enter the first room of the mansion and it's completely dark. You stumble around bumping into the furniture but gradually you learn where each piece of furniture is. Finally, after six months or so, you find the light switch, you turn it on, and suddenly it's all illuminated." It must have been something like this for Euler, when he saw how the "furniture" was arranged around the Bernoulli numbers.

3. Euler and the lack of subscripts

Modern definitions of sequences and series are almost always given in terms of a list of numbers denoted with subscripts, like x_n where n is an integer, usually starting with $n = 0$ or $n = 1$. For example, above we spoke of the Bernoulli numbers

$$B_0 = 1,\ B_1 = \frac{-1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0,\ B_4 = \frac{-1}{30},\ B_5 = 0,\ B_6 = \frac{1}{42}.$$

It is by use of such subscripts that we easily show how the numbers that occur in the Riemann zeta function

$$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{2^{2n}|B_{2n}|\pi^{2n}}{2\,(2n)!}$$

are related to the numbers that occur in Bernoulli's problem of summing powers,

$$\sum_{k=1}^{n-1} k^p = \sum_{k=0}^{p} \frac{B_k}{k!} \frac{p!}{(p+1-k)!} n^{p+1-k}$$

or in the coefficients in the Euler-Maclaurin summation formula, sometimes written as

$$\sum_{k=1}^{n-1} f_k = \int_0^n f(k)dk - \frac{1}{2}\left[f(0) + f(n)\right] + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(n) - f^{(2k-1)}(0)\right]$$

Without subscripts, or some tools that could perform the same functions, Euler was not able to express such relations so compactly and elegantly. We plan to spend the rest of this essay surveying Euler's notation for series, paying particular attention to the difficulties he had and the *ad hoc* notations he developed. We will also notice how often he came very close to developing a general indicial notation, and we will see that these notations appear in a broad variety of topics.

This part of the paper should complement the first part in that it will demonstrate that a powerful notation can be as illuminating as a unifying terminology. It should contrast the first part in that the naming of the Bernoulli numbers was success, but that Euler failed to discover a powerful indicial notation. Moreover, it is different from the first part, which focused on a relatively few works of Euler, all classified in the *Opera omnia* as papers on Series. This second part will rely on a larger number of works on Series. We emphasize that in addition to Euler's works on Series, we could have made most of the same points by drawing on papers on Number Theory, Combinations and Probability and Calculus of Variations, Elliptic Integrals, Geometry and Mechanics.

3.1. *1730: The Gamma function*

Euler's first paper on series was *De progressionibus transcendentibus seu quarum termini generales algebraice dari nequeunt*, On transcendental progressions or those whose general term cannot be given by algebraic expressions [E19], written in 1730 and published in 1738. It was based on material that Euler had included in his very first letter to Goldbach, written in 1729, and in which he showed how to "interpolate the hypergeometric series $1+1\cdot 2+1\cdot 2\cdot 3+1\cdot 2\cdot 3\cdot 4+$etc." We recognize these as the factorial numbers. Euler seems to have a bit of trouble thinking of a list of numbers, and is more comfortable connecting them as a sum, even if the resulting series diverges.

Euler tells us that the general term of this progression can be given by the expression

$$\frac{1\cdot 2^n}{1+n}\cdot\frac{2^{1-n}\cdot 3^n}{2+n}\cdot\frac{3^{1-n}\cdot 4^n}{3+n}\cdot\frac{4^{1-n}\cdot 5^n}{4+n}\cdot \text{etc.}$$

It is a remarkable expression, and worthy of more attention, but here we are interested in how he distinguishes particular terms. He speaks of the "term with exponent n." He takes $n = 2$ and gets the "termino secondo 2," and $n = 3$ to get "termino tertio 6." He also speaks in the title of the article of the *termini generales*, general terms.

Even at this early date, we see that he has the vocabulary to distinguish particular terms and to speak of general terms.

3.2. *1732: Euler sums series of polynomials*

As we noted above, Euler first used the series that he later called the Bernoulli numbers in 1732 in *Methodus generalis summandi progressions* [E25]. There he denotes the Bernoulli numbers with lower case Greek letters, α, β, γ, δ, ϵ, etc.

In other parts of E25, Euler seems fairly comfortable using the indices that exponents provide, but he does not attach them to arbitrary coefficients, only coefficients that can be given explicitly in terms of the index. Thus we see expressions like

$$\alpha x^a + \beta x^{a+b} + \gamma x^{a+2b} + \delta x^{a+3b} + \text{etc.}$$

and

$$2x^{a+b} + 3x^{a+2b} + ... + nx^{a+(n-1)b}.$$

3.3. *1734: The Basel problem*

Two years later, in 1734, Euler solved the Basel problem [E41]. Here for the first time he discusses several different sequences of coefficients simultaneously. The first one is an arbitrary sequence of roots of an equation. He denotes the roots with capital Roman letters, A, B, C, D, etc. and then makes the brilliant step of writing the equation two ways, first as a Taylor series, and then as an infinite product,

$$\left(1 - \frac{s}{A}\right)\left(1 - \frac{s}{B}\right)\left(1 - \frac{s}{C}\right)\left(1 - \frac{s}{D}\right) \text{etc.}$$

Then he also works with an arbitrary series

$$a + b + c + d + e + f + \text{etc.}$$

the sum of which he denotes with a Greek lower case α. The sum of products of distinct terms taken two at a time he denotes β, taken three at a time by γ, etc.

He introduces a fourth sequence, P, Q, R, S, as the sum of first powers, second powers, third powers, etc. of the terms a, b, c, etc., and he gives relations among α, β, γ, δ, etc., and P, Q, R, S, etc.

Finally, he denotes the values $\zeta(2)$, $\zeta(4)$, $\zeta(6)$, etc. by P', Q', R', etc.

This practice, different alphabets - or different parts of the alphabet - for different sequences, becomes Euler's standard practice for the next several decades. Sometimes he even resorts to the old German Fraktur alphabet, but readers of the *Opera omnia* ought to be careful. Sometimes, like in the *Introductio*, Euler denotes two different sequences with the symbols A, B, C, etc., and the Editors of the *Opera omnia* have chosen to render one of these sequences in Fraktur.

3.4. *1736: Integrals and sums*

Two years later, in a short article [E46], Euler tries to give a geometrically based proof of the Euler-Maclaurin summation formula. He is summing a series $a + b + c + d + e + f +$ etc., but this time he uses x to denote "the term of index n." He takes a sequence of abscissas A, B, C, etc., really meaning them to be 1, 2, 3, etc. To approximate the sum of the series by an integral, he finds a function y that "naturally expresses" the series at the points A, B, C, etc. Finally, he completes the rectangle aAB with the point β, rectangle bBC with γ, etc. This is illustrated in Euler's Fig. 1, our Figure 2.

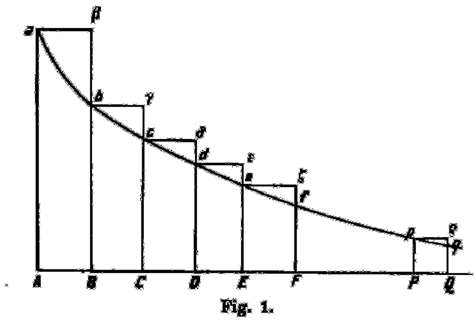

Fig. 2. Euler's Figure 1, approximating a sum with a larger integral

Here, the modern eye sees a summation that approximates an integral. Euler means it to be an integral that can approximate the sum of a series.

Similarly, he finds a function x that naturally expresses the values β, γ, δ, etc. This gives a curve that fits outside the rectangles, as shown in

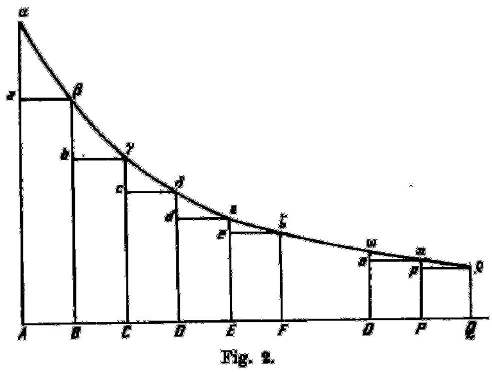

Fig. 3. Euler's Figure 2, approximating a sum with a smaller integral

Euler's Fig. 2, our Figure 3. Euler concludes that, with appropriate bounds on the integrals

$$\int ydn < a + b + c + ... + x < \int xdn$$

The innovation here is that x denotes the general term of index n. Note how Euler over-burdens the symbol x. This is common.

To the modern reader, Euler's figures look like Riemann's upper sums and lower sums, though Bernhard Riemann won't be born until 1826. Euler, though, was using integrals to approximate series, where Riemann used series to approximate integrals.

3.5. *1736: Better general terms*

Later that same year [E47], Euler is speaking of a general term X of index x given by a formula involving x. He considers forms $X = x^n$, as well as the harmonic series, $X = \frac{1}{x}$ and sums of reciprocals of squares and cubes, $X = \frac{1}{x^2}$ and $X = \frac{1}{x^3}$.

This is getting closer to the idea of a sequence of numbers X with an index x. He also uses a notation $\int X$ to denote what we would write $\sum_{i=1}^{x} X(i)$, so, for example, $\int x^1 = \frac{x^2}{2} + \frac{x}{2}$.

3.6. *1736: Euler-Maclaurin formula works for all functions*

Also in 1736, Euler realizes that the Euler-Maclaurin formula works for arbitrary functions, not just the powers of x that he had considered to this point. He starts with the finite sum of a progression for which the indices are given by an arithmetic progression instead of just the sequence of natural numbers. He represents this in a new way as

$$\overset{a}{A} + \overset{a+b}{B} + \overset{a+2b}{C} + \ldots + \overset{x}{X} = S.$$

Here, upper case letters represent the values to be summed, and the lower case letters above represent the indices of each term. He is thinking of the capital letters as the values we get when we substitute the lower case letters into some function.

Euler continues to use this notation in a paper he wrote in 1739 [E122] where he elaborates on his ideas on the Gamma function [E19] and the interpolation of sequences of products. Here he writes

$$(f \overset{1}{+} g) + (f+g)\overset{2}{(}f+2g) + (f+g)\,(f \overset{3}{+} 2g)\,(f+3g) + \text{etc.}$$

He speaks of the term with index n, and, in particular, is interested in the case $n = \frac{1}{2}$. He still has the vocabulary to speak of a particular term, but his notation is not very efficient.

3.7. *1748: Introductio in analysin infinitorum*

The *Introductio* [E101] is often called Euler's greatest work. Though it was published in 1748, and it was being revised as late as 1747, it was mostly written several years earlier and probably essentially complete in 1742. The first book of the *Introductio* presents the material about series that a student should understand before learning calculus. The second book is about curves.

In the *Introductio*, series are almost always written as $A + Bz + Cz^2 + Dz^3 + \text{etc.}$ If there is a recursive relation among the coefficients, as in the series expansion of $\frac{1}{1-\alpha z-\beta z^2-\gamma z^3}$, then he does not use indices, but instead writes

$$\begin{aligned} D &= \alpha C + \beta B + \gamma A \\ E &= \alpha D + \beta C + \gamma B \\ F &= \alpha E + \beta D + \gamma C \\ &\text{etc.} \end{aligned}$$

3.8. *1750: Matching the terms with the indices*

In 1750 or 1751, Euler seemed to understand that he was having trouble with the relations between the terms and the indices of a series. The paper he wrote on this issue [E189] is interesting for other reasons, since Euler does an analysis that closely resembles Fourier series. He is also interested in the interpolation of series, as in E19 and E122. Here, though, we will look at his treatment of indices. After some introductory remarks, he describes the series 1 + 2 + 3 + etc. as having the value x for its term of index x. He seeks to clarify this with some examples given in a tabular form. He tells us that the sequence of logarithms can be described as

$$\text{indices } 1\ 10^1\ 10^2\ 10^3\ 10^4\ 10^5 \text{ etc.}$$

$$\text{terms } 0\quad 1\quad 2\quad 3\quad 4\quad 5 \text{ etc.}$$

Later when y is the value of a term of index x, he uses y' to denote the term of index $(x + 1)$. He will enhance this notation in 1755 in the *Calculus differentialis* [E212] where he will further write y'' for the term of index $(x + 2)$, y''' for the term after that, etc.

3.9. *1761: Continued fractions and the tangent function*

Early in the 1760s, Euler uses continued fractions to evaluate certain tangents. [E280] In addition to the usual series of Roman and Greek letters, Euler also uses Fractur. He brings back his supra-script notation from E122 when he gives the convergents of a continued fraction as

$$\overset{\alpha}{\frac{1}{0}},\ \overset{\beta}{\frac{(a)}{1}},\ \overset{\gamma}{\frac{(a,b)}{(b)}},\ \overset{\delta}{\frac{(a,b,c)}{(b,c)}},\ \overset{\varepsilon}{\frac{(a,b,c,d)}{(b,c,d)}},\ \overset{\zeta}{\frac{(a,b,c,d,e)}{(b,c,d,e)}},\ \text{etc.}$$

The Greek letters denote the convergents. The parenthesis notation representing the numerators and denominators is described by a recursive definition:

$$\begin{aligned}
(a) &= a \\
(a,b) &= a\,(b) + 1 = b\,(a) + 1 \\
(a,b,c) &= a\,(b,c) + (c) = c\,(a,b) + (a) \\
(a,b,c,d) &= a\,(b,c,d) + (c,d) = d\,(a,b,c) + (a,b) \\
(a,b,c,d,e) &= a\,(b,c,d,e) + (c,d,e) = e\,(a,b,c,d) + (a,b,c) \\
&\text{etc.}
\end{aligned}$$

Though this might not be any clearer if written in a notation involving subscripts and indices, it certainly would be more concise.

3.10. *1769: Operating on the terms of a series*

By the end of the 1760s, Euler was ready to return to questions involving the Bernoulli numbers. He gives the Bernoulli numbers as $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{42}$, $\frac{1}{30}$, $\frac{5}{66}$, etc., what we would call the absolute values of B_2, B_4, B_6, B_8, etc. He wants to study the recursive relations on the coefficients $\zeta(2n)$ that he found in E41 in his solution to the Basel problem. Recall that, in modern notation,

$$[\zeta(2n)] = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = A_{2n}\pi^{2n}$$

where

$$A_{2n} = \frac{2^{2n}|B_{2n}|}{2(2n)!}$$

He finds that rather than consider the coefficients, B_{2n}, he would rather consider $\frac{B_{2n}}{2n+2}$, so he asks us to construct a new sequence that he gets by dividing the Bernoulli numbers by the sequence 6, 10, 14, 18, 22, etc. He names the new series using the Fraktur alphabet, and shows how that series is related to the series A, B, C, etc. that he uses to denote the coefficients on $\zeta(2n)$.

This seems to be the first time he performs such an operation on a sequence.

3.11. *1772: An ad hoc function notation for sequences*

Just three years later, in 1772, Euler marshals all his powers to try to solve some difficult outstanding problems [E432], like the sum of the reciprocals of the odd powers of integers. For the most part, he is not successful, but not for his lack of ideas or effort. He brings back the tabular index and term notation we saw in E189.

What is more exciting, though, is that he uses two series expressed and manipulated by their indices. He writes one sequence, $S(1)$, $S(2)$, $S(3)$, etc., and another one $\Sigma(1)$, $\Sigma(2)$, $\Sigma(3)$, etc. He gives explicit values for $S(n)$ for $n = 1$, 3, 5 and 7, but does not take note that in general for n odd, he has $S(n) = \frac{1}{n(n+1)2^n}$. Some of his use of this sequence is illustrated in the image below, extracted from the original article.

14. Hinc deducimur ad iſtam ſeriem infinitam generalem, quae illas ſeries numericas omnes in ſe complectitur:

$$\frac{1}{n\cdot 2^{2n}} + \frac{n}{(n+1)\,2^{2n+2}} + \frac{(n+1)(n+2)}{2(n+2)\,2^{2n+4}} + \frac{(n+2)(n+3)(n+4)}{2\cdot 3\cdot (n+3)\,2^{2n+6}} + \frac{(n+3)(n+4)(n+5)(n+6)}{2\cdot 3\cdot 4\,(n+4)\,2^{2n+8}} + \text{etc.}$$

cuius igitur ſummam inueſtigari oportet. Quodſi enim huius ſeriei ſummam in genere hoc ſigno $S(n)$ indicemus, habebimus

$$Z = -\tfrac{1}{8} + S(1) + 2\alpha\pi^2\left(\tfrac{1}{4\cdot 2^4} - S(3)\right) + 2\beta\pi^4\left(\tfrac{1}{6\cdot 2^6} - S(5)\right) + 2\gamma\pi^6\left(\tfrac{1}{8\cdot 2^8} - S(7)\right) + 2\delta\pi^8\left(\tfrac{1}{10\cdot 2^{10}} - S(9)\right) + \text{etc.}$$

Fig. 4. A passage from E432 showing Euler's use of a function-like notation to denote series

He goes on to recycle the notation $S(n)$ to denote a sequence of polynomials in x and he takes $\Sigma(n)$ to be the derivative of $S(n)$.

This is the closest Euler has come to a functioning indicial notation for sequences, but he does not seem to recognize its utility. He seems now to be able to denote a sequence by $S(n)$, but only if its formula is sufficiently complicated to make the simplification worthwhile, but then he does not consider expressions like $S(n+1)$ or $S(n+2)$.

3.12. *1773: Recursively defined sequences*

The next year, in 1773, Euler studied sequences defined by second order recursive relations, and wrote a paper with the intimidating title *Insignes proprietates serierum sub hoc termino generali contentarum* $x = \frac{1}{2}\left(a + \frac{b}{\sqrt{k}}\right)\left(p + 1\sqrt{k}\right)^n + \frac{1}{2}\left(a - \frac{b}{\sqrt{k}}\right)\left(p - 1\sqrt{k}\right)^n$, "Special properties of series with general term given by $\cdots$" [E453]. This title is not as horrible as it seems, since such formulas describe the nth term of a sequence defined by a second order linear recurrence relation. The Fibonacci sequence is one such sequence.

Euler makes some substitutions so that he is considering the sequence given by $f \cdot v^n + g \cdot u^n$. He denotes this sequence by [0], [1], [2], [3], etc. and the nth term by $[n]$. He considers several terms simultaneously as he finds a relation among $[n]$, $[n+1]$ and $[n+2]$. Later, he performs even more complicated operations on his indices and considers problems involving $[n]$, $[n+\nu]$ and $[n+2\nu]$ and involving $[n]$ and $[2n]$

Still later he considers a different sequence given by $['n] = f \cdot v^n - g \cdot u^n$.

This is a better use of indices than we saw in E432, but without assigning a name, like S, to the sequence, as he had in E432, denoting it only by a symbol, he does not get full use of his innovation.

3.13. *1776: Properties of binomial coefficients*

As we move past Eneström number 550, we get to the papers published after Euler's death in 1783. Late in his life, Euler's productivity, measured in number of papers written, increased dramatically, as did the time between the writing of a paper and its eventual publication. Also, the order in which the papers that interest us here were published is somewhat different from the order in which they were written. From here on, we will describe them in the order in which they were written.

Moreover, for the last 15 years of his life, Euler was almost completely blind. He wrote papers by dictating them to his assistants, who included Lexell, Georgi, Fuss, Krafft, J. A. Euler, Golovin [P], who also presented his papers to the Academy. Their help made 1776 Euler's most prolific year. That year he wrote 56 articles, more than one a week. Given these circumstances, it is possible that Euler's late innovations in notation were in fact as much the ideas of one or more of his assistants as they were Euler's. At present, we have no way of telling for sure.

In 1776, Euler was studying properties of binomial coefficients [E584], but the paper did not get published until 1785, two years after his death. A few years earlier, Euler had introduced the notation $\left[\frac{n}{p}\right]$ to denote the binomial coefficient we now call "n choose p."

In E584, Euler writes one formula as $\int \left[\frac{m}{x}\right]\left[\frac{n}{p+x}\right] = \left[\frac{m+n}{n-p}\right]$ and $\int \left[\frac{n}{x}\right]^2 = \left[\frac{2n}{n}\right]$ Euler uses an integral sign where we use a sigma to denote summation, and he does not have a notation to indicate which of the symbols denotes the index, x in this formula, nor to describe the values x is meant to take. He has to give this information in text rather than in his notation The modern way to write the first formula is $\sum_x \binom{m}{x}\binom{n}{p+x} = \binom{m+n}{n-p}$, where x ranges either from 0 to m or from 0 to $n-p$, whichever range is smaller. This is Euler's best use of an index yet.

3.14. *1776: Symbols for equations and factorials*

In another paper written in 1776 [E652], but this one not published until 1793, Euler uses $\Delta : n$ to denote $n!$, and then, as he had done in E19, extends it to fractional values of n.

He also assigns names to equations, like:

$$I. \qquad \Delta : n = \frac{1}{n+1}(1+\alpha)^n$$

and a similar equation II. Then uses equation labels in formulas to write

$$\frac{\text{II}}{\text{I}} = \frac{2}{n+2} \cdot \frac{(2+\alpha)^n}{(1+\alpha)^n}$$

3.15. *1777: Formulas involving specific terms*

In 1777, the year Gauss was born, Euler's productivity fell to only 43 papers. In one of them [E703], which does not appear until 1798, he examines properties of the cosine expansion of $\frac{b}{1+e\cos\varphi}$ into a series that Euler denotes with Γ, defining the symbol by

$$\Gamma : \varphi = A + B\cos\varphi + C\cos 2\varphi + \text{etc.}$$

Euler has a specific application in mind. Here, e represents the eccentricity of an orbiting body. This particular series plays a key role in his work in orbital mechanics, as described in the article by Curtis Wilson, elsewhere in this volume. Its relation to Fourier series is clear and interesting, and is worth a closer look. Here, though, we are concerned with notations.

Euler introduces an indicial notation for the coefficients of Γ, writing $A = (0)$, $B = (1)$, $C = (2)$, etc. With this notation, he can describe the values of Γ at particular points, like

$$\begin{aligned}\frac{1}{8}\Gamma : 0 + \frac{1}{8}\Gamma : \pi &= A + I + R + \text{etc.} \\ &= (0) + (8) + (16) + (24) + (32) + (40) + \text{etc.}\end{aligned}$$

He also writes

$$\Gamma : \varphi = (0) + (1)\cos\varphi + (2)\cos 2\varphi + (3)\cos 3\varphi + (4)\cos 4\varphi + \text{etc.}$$

This leads to some difficulties, though, when he writes

$$S = (1+n)(0) + (n+1)(2n) + (n+1)(4n) + \text{etc.}$$

where the factors $(n+1)$ are taken to be numbers, and the factors (0), $(2n)$, $(4n)$, etc. are taken to be the coefficients in the series Γ.

This is part of a series of articles including Eneström numbers 246, 686, 704, 747 and 810, among others dealing with the same series, but Euler only uses such a well-developed indicial notation in the sequel to this paper, E704.

In two related papers, E709 and E710, Euler becomes bolder with his notation, using Δ as the name of a function, and various zodiacal signs as the values of series and integrals.

3.16. *1780:* Σ *and* Π *for sums and products*

In one of 37 papers Euler wrote in 1780 [E613], Euler denotes the terms of a series by (1), (2), (3), (4), etc., and the general term by (x). He denotes the sum of the first x terms of the series by $\sum : x = (1) + (2) + (3) + ... + (x)$. He further writes the first differences as $\Delta 1$, $\Delta 2$, $\Delta 3$, etc. and second differences as $\Delta^2 1$, $\Delta^2 2$, $\Delta^2 3$, etc., and higher differences similarly. With this notation, he can write things like $\sum : (x+1) = \sum x + (x+1)$. He uses similar notation for products, but here he takes his factors to be A, B, C, etc. Then $\Pi : 1 = A$, $\Pi : 2 = AB$, $\Pi : 3 = ABC$, etc. This paper was published in 1787.

3.17. *The rest of the story*

By the time he died in 1783, Euler had developed at various times several different *ad hoc* indicial notations, but these notations were not generally adopted by the mathematical community. Though it is not really the point of this essay, one may ask when the subscript notation arose. Curtis Wilson writes in a forthcoming article that the use of indices arises in Lagrange before 1800, but in 1805, Legendre was still using lower case letters to describe series [C]. Gauss used an *ad hoc* notation in 1810 to describe his pivot method for systems of linear equations, but in 1822, F. D. Budan uses a fully-developed doubly-indexed system of subscripts and superscripts in his paper on the solution of numerical equations. (See [C], p. 233-235.) It seems that, after a long childhood, indicial notation matured and spread very rapidly in the second decade of the 19^{th} century.

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Leonhard Euler: Life, Work and Legacy
Robert E. Bradley and C. Edward Sandifer (Editors)

The Geometry of Leonhard Euler

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1. Introduction

Lagrange is supposed to have said: "If one wishes to study geometry, then one must read Euler" (see [10]). Although Euler was not primarily a geometer—those of his works classified as geometrical span a mere 1600 or so pages, volumes I.26-29 of the *Opera Omnia*—nevertheless there is ample justification for Lagrange's statement.

Among Euler's most well-remembered contributions to geometry are the following: he was the first to recognize clearly the importance of geodesic lines; he developed the theory of surfaces, and a general theory of space curves. In fact he is recognized, along with Monge and Gauss, as one of the three founders of differential geometry. His comprehensive studies of spherical trigonometry (E.524, 698) established the notation and approach still in use today. He classified second-degree plane curves, and did early work on the classification of curves of the third degree. Euler excelled in the development of coordinate systems for specific purposes; he applied the techniques of calculus to analyze geometrical problems posed in traditional language, proved the relationship between vertices, edges and faces for convex polyhedra (E.230, 231), and laid the foundations for graph theory with his solution of the Königsberg Bridge problem (E.53). In an era when classical Euclidean geometry was of minor importance for many leading mathematicians, Euler made notable contributions to the subject: among many other things, he proved that the three classic triangle centers of Greek geometry – the circumcenter, orthocenter and centroid – are collinear (E.325), thus sparking a renaissance in classical geometrical studies that steadily gained momentum throughout the nineteenth century.

Previous general discussions of Euler's geometry have tended to focus on well-known high points. Fellmann gives brief mention of the foregoing contributions in [9], while Dunham [5] covers fewer high points, but offers detailed summaries of several of Euler's demonstrations, including the Euler Line proof and the solution to the Königsberg Bridge Problem.

On the one hand, it is not my intent in this article to give extended blow-by-blow accounts of a few of Euler's most memorable arguments; nor, on the other hand, will I duplicate Fellmann's comprehensive summary of Euler's important geometrical contributions. Instead I aim to complement these earlier accounts by following something of a mathematical stream of consciousness through the Eulerian geometrical corpus, gently steering clear, for the most part, of its most celebrated results. I want to convey a sense of the everyday sorts of geometrical problems that attracted Euler's attention, and to shed light on the methods Euler considered, in his era of Analysis, to be appropriate for geometers. Occasionally I will linger over a few demonstrations or constructions whose elegance justifies their rescue from obscurity.

2. The Tour

2.1. *Reciprocal Trajectories*

We begin with a brief notice of Euler's earliest geometrical work. The problem at hand is of the type traditional in geometry up to Euler's time and beyond: namely, to find geometric objects possessing certain stated properties. Specifically, one wants to find a curve and an axis so that, when the curve is reflected across the axis and translated parallel to that axis by an arbitrary amount, the image intersects the original curve at a constant given angle, usually specified to be a right angle. Such curves were known as *reciprocal trajectories.* The problem was apparently first posed by the English mathematician Henry Pemberton (1694-1771), and was assigned to Euler by his mentor Johann Bernoulli, with the probable intention of sharpening Euler's calculus skills.

Euler's first article on the subject is E.3, "A method for finding algebraic reciprocal trajectories." As the title suggests, the aim is not merely to find curves, but to find curves that may be described by some algebraic formula. (The distinction between algebraic and non-algebraic curves is one that Euler keeps up through all of his geometrical work.) Euler offers two methods for constructing reciprocal trajectories, the first due to Johann Bernoulli and a second, quite similar, one of his own devising. Both constructions are typical of solutions to curve-finding problems throughout Euler's career, in that the desired curve is a locus derived from the motion of a point along a suitably chosen initial curve, that might be called a 'seed curve.' We describe the Bernoulli construction here. The seed curve, which Euler discovered, is defined implicitly by the equation

$$y^2 + \frac{2}{3}a^2 = a\sqrt[3]{ax^2},$$

for any $a > 0$. This curve has vertex at $B = (a\sqrt{\frac{8}{27}}, 0)$ and the x-axis for an axis of symmetry. Letting M be any point on this curve, one draws the horizontal segment MN headed toward the y-axis, so that MN is congruent to the arc length along the seed curve from B to M. The locus of the endpoint N, under the motion of M is said to be a reciprocal trajectory (with the axis of reflection being the vertical line through B). Of course, in order to obtain an algebraic reciprocal trajectory, one desires the seed curve to be rectifiable, in the sense that the arc length BM can be expressed algebraically. Euler shows that the arc length is, in fact,

$$\frac{y^3}{a^2} + y,$$

where y is the y-coordinate of M. From this it easily follows that the reciprocal trajectory has equation

$$x = \frac{\left(y^2 + \frac{2}{3}a^2\right)^{\frac{3}{2}}}{a^2} - y - \frac{y^3}{a^2}.$$

Euler then shows that this trajectory is the graph of an implicit equation of degree four, and he goes on to investigate the problem of finding rectifiable reciprocal trajectories of arbitrarily high degree. Euler returned to reciprocal trajectories several times during his career, providing more construction methods and identifying ever more general classes of solutions. He once said (in E.85) that from the reciprocal trajectory problem "Analysis trulyappears to have received augmentation that is not to be despised."

2.2. *Reciprocal Problems, Catoptrical Curves and Curves of Constant Width*

Among the problems which require one to find curves endowed with a given property, Euler considered several that he eventually termed 'reciprocal' problems, in a different sense than the problem of reciprocal trajectories.[1] The idea is that when a particular type of curve satisfies some given property, the reciprocal problem for this property is to find other curves satisfying that property.[2] For example, an ellipse has the property that, when one draws segments from its foci to any point M on the ellipse, the two segments make the same angle to the tangent line to the ellipse at M. The corresponding reciprocal problem, appearing as Problem 1 in E.771, is stated as follows: "Given two points A and B, to find a curve so situated that, when line segments MA and MB are drawn from any point M on the curve, both segments are inclined equally to the curve." Of course, ellipses having foci A and B are one obvious set of solutions. When one considers the reflection property of the hyperbola, one can also recognize that hyperbolae with A and B as foci are also solutions. Euler is also able to show, using calculus, that ellipses and hyperbolae are the only solutions.

Another example: Problem 2 in E.771 is the reciprocal problem corresponding to a property obtained by tweaking the defining property of an ellipse: given a line l and a fixed point A on l, to find a curve so that, for any point M on the curve, when you start from A along segment AM and reflect off the curve, meeting l again at some point O, the sum $AM + MO$ is some constant a. Euler is able to show that parabolas, ellipses and hyperbolae provide the only solutions. He offers three demonstrations, the last of which, said to be "without calculus", is actually a non-rigorous argument making use of infinitesimals that is worth a look, as it may shed some light on Euler's discovery methods.

In figure 1, m is a point on the curve (not shown) that is supposed to be "very close" (*proximum)* to M. Am reflects to o on l, and p and q are the feet of perpendiculars dropped from M and m respectively. Since M and m are close, the angle of incidence at M is nearly equal to the the angle of incidence at m, which in turn is nearly equal to angle Mmp. Hence the angle of reflection at M is nearly equal to the angle of incidence at m. The former angle is nearly equal to mMq, and the latter angle is nearly Mmp, hence angle Mmp is nearly equal to angle mMq. This is enough to force

[1] This terminology first appears rather late, in E.771: "Solution of three rather difficult problems pertaining to the inverse method of tangents." (The inverse method of tangents is simply the application of integral calculus to determine the equation of a curve from given properties of tangents to it. It is the "inverse" of differentiation.)

[2] Actually, the most important reciprocal problem that Euler considered involves surfaces rather than curves. In E419 "On solids whose surfaces may be unfolded onto a plane," Euler notes that cylinders and cones may be flattened out – "developed" – onto a plane, and asks what other kinds of surfaces have this property. E419 laid important foundations for differential geometry; see the article by Karin Reich elsewhere in this volume.

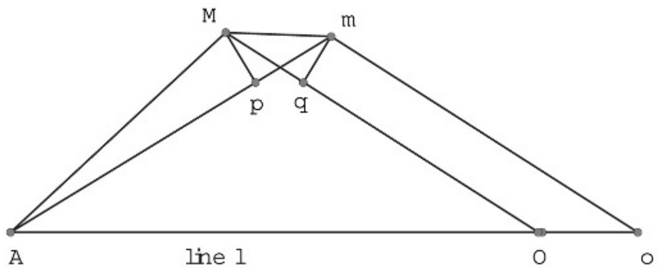

Fig. 1. E771, Problem 2

the near congruence of right triangles Mmp and mMq, which share the hypotenuse Mm. It follows that $Mq \approx mp$.

Also, since

$$AM + MO = a = Am + mo,$$

we have

$$Am - AM = MO - mo.$$

But $mp \approx Am - AM$, and $Mq = MO - Oq$. Since $Mq \approx mp$, we deduce that $MO - mo \approx MO - Oq$, and so

$$Oq \approx mo.$$

These quantities don't look nearly equal, at least not in the diagram shown. Euler concludes that the near equality can occur in one of two ways: either all reflections, including MO and mo, are perpendicular to the line l, in which case the curve is a parabola with focus A and axis of symmetry perpendicular to l, or else o always coincides with O, in which case we have either an ellipse or a hyperbola, with A and O as foci.

Euler appears to have been especially drawn to reciprocal problems relating to conics. The first published paper examining such problems in depth is E.83, "On certain properties of conic sections that are common to infinitely many other curves", which appeared in 1745. Euler notes near the beginning: "It is clear that those properties by which the conic sections are defined, are so proper to them that they cannot be held in common with any other curve; but one does encounter, besides these, other properties, for which it is difficult to decide if they are proper to conic sections, or not." One such problem, which Euler says was inspired by a correspondence with Clairaut[3], is as follows. Say that you have a curve with perpendicular diameters AB and aC. (See figure 2.) Suppose further that

[3] Clairaut to Euler, 11 October 1741 and 4 January 1742.

when you pick any point M on the curve, and construct the segment Cm to a point m on the curve so that Cm is parallel to the tangent line MT at M, then the triangle MCm has a constant area, equal to the triangle ACa.

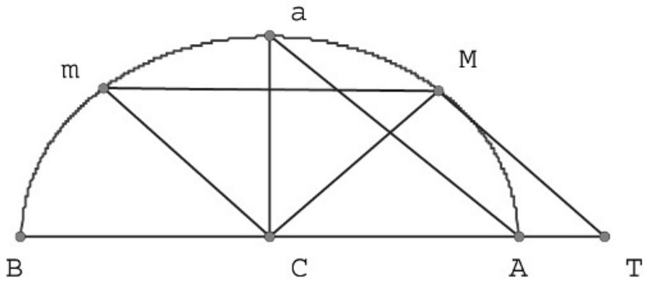

Fig. 2. E83, Problem 2

The problem is to find all such curves. Ellipses are only one class of solutions: Euler finds many others that are not conic sections at all. In fact the solutions are given in terms of a parameter z as follows

$$x^2 = \frac{c^2\,(1+z)}{(1-z^2)\,\frac{dZ}{dz} + Zz},$$

$$y^2 = \frac{c^2\,(1+z)\left((1+z)\,\frac{dZ}{dz} - Z\right)^2}{(1-z^2)\,\frac{dZ}{dz} + Zz},$$

where Z is any odd function of z, and c^2 is the area of triangle ACa. Euler points out that in the simplest case, where $Z = \alpha z$ for some constant α, we have ellipses.

Perhaps the most important conic reciprocal problem that Euler considered is that of the "catoptrix." Posed anonymously by Euler in 1745 in the Nova Acta Eruditorum, the problem runs as follows: "Placing any given point F that you please, to find all curves of the following nature: that any ray proceeding from F returns to that same point F after a twofold reflection in points M and N [of the curve]." See the figure 3.

A curve satisfying this property is known as a catoptrix. In view of their reflection property, ellipses obviously are catoptrices, but Euler wants to see "all" others. Euler did not give his colleagues much time to work: the following year he presented, in the same journal, his "Solution of the catoptrix problem" (E.85). In this short paper, Euler first gives differential equations for the solution curves, and then a locus method, in the same

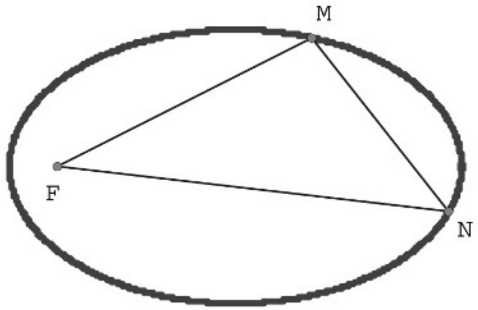

Fig. 3. The Catoptrix

spirit as the earlier construction of reciprocal trajetories in E.3, in which a catoptrix is traced out as the locus determined by the motion of a point along any "seed curve" that has 180 degree rotational symmetry. In the special case that the seed curve is a straight line, the catoptrix will be an ellipse.

In E.85, Euler does not solve the differential equations, nor does he demonstrate the correctness of his rather complex locus construction: "I will announce these solutions with their analysis hidden, lest I deny to others the opportunity of exercising their own powers on this most elegant problem – whose solution promises to be most useful for Analysis – and of contributing thereby their own 'mosaic stone' to the growth of science." [4] Detailed solutions followed two years later in E.106.

Nobody seems to remember the catoptrix these days, but nearly every mathematician has heard, if only in passing, of Reuleaux curves. A Reuleaux curve, also known as a curve of *constant width*, is a convex curve such that when one encloses it between any two parallel tangent lines, the distance between the tangents is constant. Such curves can be rolled along a flat surface without changing their height; also, the cover of a manhole constructed as a Reuleaux curve will not fall into the hole, no matter how it is rotated. Although Reuleaux curves appear quite early – one can find Reuleaux-shaped windows (not the obvious circular ones) in the Notre Dame Cathedral – Leonhard Euler appears to have made the first rigorous study of them in E.513, "On triangular curves", published in 1778.

[4] In E.85 Euler also remarks, somewhat mysteriously, that the catoptrix problem is related to the problem of reciprocal trajectories, in that both problems require that "from a given relation, which agrees [*competat*] equally in two points of a curve, continuous curved lines, endowed with this relation, are to be discovered."

A triangular curve is a plane figure having three curved sides, such that the tangent lines at the point of intersection of any two of the sides coincide with one another. Thus the sum of the angles of a triangular curve is zero degrees. After introducing triangular curves, Euler's first move is to define another class of curves which arise "by evolution" from them. Here is how evolution works. Take a line segment sufficiently long to complete the following process: placing this segment tangent to the curve at one of its vertices, so that one of its endpoints X lies on or outside of the curve, roll the rigid segment along one of the available sides until it is tangent to the next vertex. Continue rolling in this way, in the same direction, until you have gone around the curve twice. Euler proves easily that X will have returned to its original location, and that the locus of X under this evolution is a curve of constant width. (Euler called Reuleaux curves *Orbiformes*, because with respect to their width they resemble circles.)

But what is Euler's primary interest in orbiformes? It is just that they are seed curves in an easy construction of catoptrices!

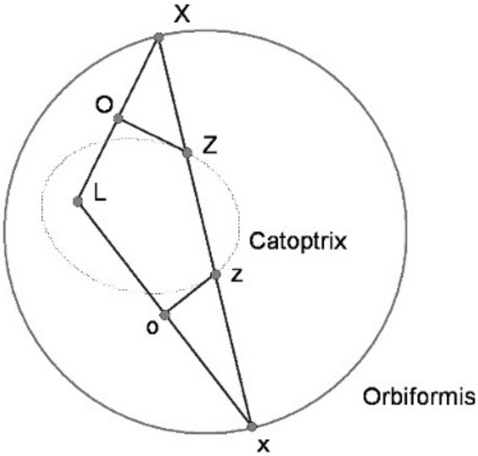

Fig. 4. The Orbiformis

Taking X to be any point on the orbiformis in figure 4, form the normal segment Xx. This segment will also be normal at point x, and will have constant length. Let O and o be midpoints of segments LX and Lx respectively, and erect perpendiculars OZ and oz from LX and Lx, meeting Xx at Z and z respectively. The locus of Z under the motion of X will form a catoptrix, where segment LZ gets reflected to Zz, which in turn reflects back to L. (In the special case where the seed orbiformis is a circle, the catoptrix is an ellipse with foci at L and the center of the circle.) Euler leaves out the demonstration, "lest this treatment be drawn out excessively," and devotes the rest of the paper to finding triangular curves.

2.3. *Trigonoscopy*

Curves of constant width are a well-remembered means to a now-obscure end – a property that they share with the Euler line. In E. 325, "An easy solution of certain rather difficult geometrical problems", published in 1765, Euler shows, using a sensible combination of analytic and synthetic methods, that the orthocenter, the centroid, and circumcenter of any triangle are collinear and that the centroid is half as far from the circumcenter as it is from the orthocenter. [5] What is now forgotten is that Euler did this in order to facilitate the solution of a construction problem he had set for himself: given the distances between the orthocenter E, the centroid F, the circumcenter G and incenter H of a triangle, to determine the triangle itself (up to congruency). Euler does not specify what it means to "determine" the triangle: an algebraic solution for the sides in terms of the distances appears to satisfy him. Although simplified considerably by knowledge of distance relationships provided by the Euler line, the algebra involved is still rather back-breaking, and Euler must introduce several *ad hoc*, non-obvious substitutions of the sort for which he was notorious in his day. [6]

We will outline the procedure by which Euler recovers the side lengths a, b, and c of the original triangle. Because of the relationship of the points on the Euler line, we need only to be given the distances

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[5] The circumcenter is the intersection of the perpendicular bisectors of the sides of the triangle; the orthocenter is the intersection of the altitudes, and the incenter is the intersection of the the angle bisectors. The circumcenter is the center of the circumsribing circle; the incenter is the center of the inscribed circle. The centroid is the intersection of the medians of the triangle. Though Euler does not mention it, from some of his calculations in this article one can easily deduce what is commonly known as *Euler's Triangle Theorem:* If d is the distance between the circumcenter and the incenter of a triangle, and if R is its circumradius and r its inradius, then $d^2 = R(R-2r)$.

[6] However, these computations and the construction problem itself were significant for the later study, by Jacobi and others, of "invariant quantities" associated with a triangle. See the articles on elementary geometry and triangle geometry in [6], and look for a forthcoming paper "The Porisms of Poncelet" by John McCleary, whose private communication is the source of this note.

$$f = GH$$
$$g = FH$$
$$h = FG$$

Euler sets

$$p = a + b + c$$
$$q = ab + ac + bc$$
$$r = abc$$

Note that a, b, c are the roots of the cubic equation

$$z^3 - pz^2 + qz - r = 0,$$

so we are reduced to finding p, q, r. Even so, this is a mighty task, requiring further substitutions. One solution, with those substitutions removed, is given below as an an example:

$$p = \sqrt{\frac{27f^4}{3g^2 + 6h^2 - 2f^2} - 12f^2 - 15g^2 + 6h}.$$

Solutions for q and r are of a similar nature, in that they may be constructed from f, g, h with straightedge and compass. (Euler does not point this out explicitly, though he seems to have it in mind.)

We are now back to the cubic equation. In certain cases it will factor easily. In one case, when H also lies on the Euler line, Euler finds that the triangle is isosceles with

$$a = b = \frac{\sqrt{3}fh\,(4f - 3h)}{f - 3h},$$
$$c = \frac{\sqrt{3}h\,(2f - 3h)\,(4f - 3h)}{f - 3h}.$$

In the case where $EG = EH$, Euler also discovers that the cubic will factor, and he again finds the triangle side lengths.

The general case, of course, is not so simple, but Euler works out a numerical example, attacking the cubic by the method of François Viète, in which solutions are found by trisecting a certain angle. "And thus the problem is solved easily enough through the trisection of an angle", he concludes. Indeed, it can be shown, although Euler does not mention it, that his original construction problem is equivalent to the classical trisection problem, in the sense that, when one is armed with straightedge, a compass and a device to solve Euler's problem, one can trisect any angle, and vice versa.

Euler's interest in triangle construction problems may have stemmed from his acquaintance with Phillipe Naudé (1684-1745), the president of

Berlin Academy of Sciences prior to Euler. Naudé was very much taken up with what he called *trigonoscopy,* the science of reconstructing triangles and quadrilaterals from given bits of information about them. At any rate, E.325 is a trigonoscopic paper, and the later article E. 749 "Certain Geometrical and Spherical Matters" addresses another problem in trigonoscopy. Here Euler stipulates that three cevians [7] Aa, Bb, and Cc of a triangle ABC are concurrent at a point O, as shown in figure 5. If one is given only the lengths AO, Oa, BO, Ob, CO, Oc, can one reconstruct the triangle up to congruence?

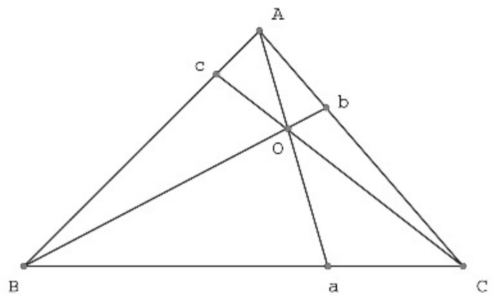

Fig. 5. A Problem from E749

One may think of this as a generalization of another problem, familiar to Naudé and other mathematicians of the time: namely, to reconstruct a triangle from its medians – it being known that the medians cut one another in 2:1 ratios. As an auxiliary to his solution, Euler establishes the following interesting relation:

$$\frac{AO}{Oa}\cdot\frac{BO}{Ob}\cdot\frac{CO}{Oc}=\frac{AO}{Oa}+\frac{BO}{Ob}+\frac{CO}{Oc}+2.$$

For this he gives a fairly long trigonometric proof, followed by a sequence of successively simpler proofs involving only synthetic geometry. [8] There are two reconstructions of the triangle: a fairly involved algebraic determination of the angles formed by the cevian segments about O, and a purely geometric construction of the triangle itself.

[7] A *cevian* is a line containing a vertex of a triangle.

[8] For the sake of completeness, he also proves the relation for spherical triangles! The relation itself, which desrves to be better known, has interesting mathematical extensions that are studied in [1] and [11].

2.4. *A Digression on Method*

At this juncture we pause to offer some observations on Euler's methods of demonstration. Certainly most of his geometrical articles, accounting for nearly three of the four volumes in the *Opera Omnia*, involve quite extensive use of calculus: indeed, some of the curve-finding problems that Euler tackled were interesting for him primarily because of the scope they afforded to the study of differential equations. Even in Volume 26, which is restricted to matters treated without calculus, Euler makes extensive use of algebra and analytic geometry. In common with other leading mathematicians of his time, he appears to have no interest in restricting himself to time-honored construction methods such as compass and straightedge, nor does he exhibit any special preference for synthetic methods of demonstration.

One exception to this pattern is E. 135, appearing in 1750 and entitled "Various Geometrical Demonstrations." Euler begins by considering the following problem posed by Fermat:

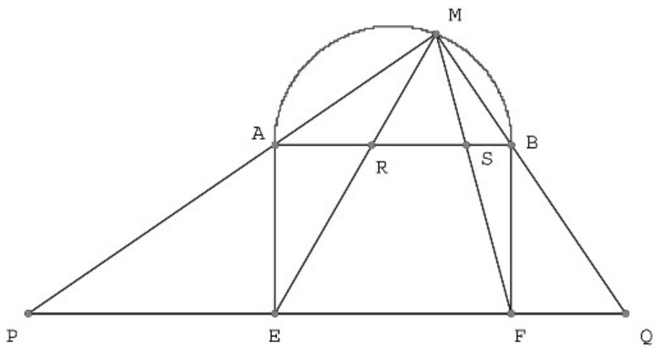

Fig. 6. A Problem posed by Fermat in E135

Consider a semicircle with a rectangle $ABFE$ erected on its diameter AB as shown in figure 6, where AE is $\frac{AB}{\sqrt{2}}$ in length. From any point M on the semicircle, form segments AE and AF that intersect AB in points R and S respectively. Then

$$AS^2 + BR^2 = AB^2.$$

Fermat had demanded a purely geometric solution to his problem. Euler remarks (Section 1): "By means of Analysis it is not difficult to ascertain the truth of this [theorem], and to extract from that a [geometric] demonstration would not be at all difficult, but most of the demonstrations of this sort stink so much of Analysis that they can scarcely be understood – even by those who are skilled in this art! We therefore require, of this proposition brought forth by Fermat, a geometric demonstration, composed in the manner of the ancient geometers, so that it may be understood even by those who are not acquainted with Analysis." And so Euler proceeds. He starts with the following Lemma (recognized later as an ingredient in the concept of *separation* in inversive geometry [9]).

Lemma 1. (Linear Separation) For any collinear points X, Y, Z, W given in that order along their line,

$$XW \cdot YZ + XY \cdot WZ = XZ \cdot WY.$$

One proof is given by algebra, and another by dissection of an appropriately chosen rectangle.

Now to the solution of Fermat's challenge: referring to the diagram, one first demonstrates the similarity of triangles PEA and BFQ, whence

$$PE \cdot QF = AE \cdot BF = AE^2.$$

Since

$$2 \cdot AE^2 = AB^2 = EF^2,$$

we get

$$2 \cdot PE \cdot QF = EF^2.$$

Again by similar triangles, we get the same relation "lifted" to the base of the semi-circle:

$$2 \cdot AR \cdot BS = RS^2.$$

Combining this with the relation

$$AB \cdot RS + AR \cdot BS = AS \cdot BR$$

provided by the lemma, a little algebra results in $AS^2 + BR^2 = AB^2$, as desired.

---

[9] Readers will note that the resemblance to *Ptolemy's Theorem*: when the four points X, Y, Z, W are arranged in this order around a circle, the same relation holds. (Later, in E.601, Euler offered new proofs of Ptolemy's Theorem, and even generalized it.)
In inversive geometry, two pairs of points XZ and YW are said to *separate* one another if every circle through X and Z intersects or coincides with every circle through Y and W. When pairs of points sepate one another, the four points are either collinear or cocyclic. It is known that for any set of four points X, Y, Z, W, the inequality

$$XW \cdot YZ + XY \cdot WZ \geq XZ \cdot WY$$

holds, with equality iff XZ and YW separate one another. See [4].

The remainder of the paper, and by far the larger part of it, is an appendix of sorts. Therein we find various other geometrical theorems, beginning with Heron's area formula for tiriangles and culminating in the following new result on quadrilaterals: in figure 7, where P and Q are midpoints of the diagonals, we have

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4PQ^2.$$

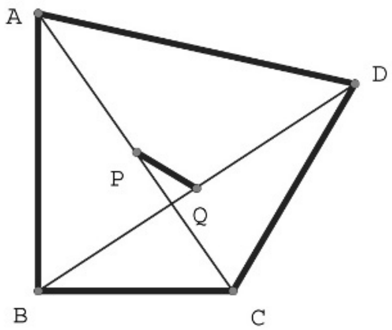

Fig. 7.

All of the foregoing are proved using classical synthetic methods, in which, says Euler, "no trace of Analysis may be discerned." [10] It is significant that Euler situates this explicitly synthetic work within the context of a challenge, from another mathematician, to limit himself to synthetic techniques. [11]

For the most part, Euler was not averse to using any method at hand, so long as it could shed light on a geometrical problem. For example, in E.73, "Solution of a geometrical problem concerning lunes formed by circles", Euler considers a problem that he attributes to Christian Goldbach. A *lune* is the area within a given circle that remains after its intersection with another circle is taken away. If two circles C_1 and C_2 intersect one another, two lunes are formed. The problem is to find a pair of congruent line segments, one in each lune, with each line segment having one of its endpoints on C_1 and the other endpoint on C_2, so that the each line segment cuts the same area away from the lune in which it lies. Of course there are literally countless solutions. Euler says that Daniel Bernoulli had already

[10] Probably the most noteworthy proof is of Heron's Theorem; it is much simpler than any previous argument. See [5] for a summary.

[11] An English translation of "Variae Demonstrationes Geometriae" by Adam Glover is now available in the online Euler Archives under E135.

given an easy geometrical solution [12], but that it was only a particular one. Euler proposes to tackle the problem with "analysis" (which in this case means trigonometry and analytic geometry rather than calculus). He notes that analysis is often viewed by mathematicians as an inappropriate way to solve a geometrical problem, but he counters (see Section 1): "That which is inconvenient about Analysis, although [such inconvenience] is customarily alleged in the case of many geometrical problems, is to be imputed not so much to Analysis itself as to the Analysts!" Perhaps Euler had seen too many needlessly cluttered, obscure analytic demonstrations. [13] For the problem at hand, Euler says that Analysis "is greatly to be preferred to the geometrical method, since in working through [the Analysis] I will produce a fully general solution, which with a geometrical method would scarcely be capable of being set forth."

2.5. *Looking Ahead*

The Linear Separation Lemma of E.325 was not the only occasion in which Leonhard Euler anticipated nineteenth-century developments in classical plane geometry. In the paper E.693 "On the Center of Similitude", presented to the St. Petersburg Academy of Sciences in 1777, Euler takes up a matter that his successors recognized as fundamental to the geometry of transformations.

Euler begins by considering two segments in the plane: a longer segment AB and a shorter one ab, and he seeks a point G, which he calls the "center of similitude" of the two segments, for which triangle GAB is similar to triangle Gab.

Euler's construction of G boils down to the following, illustrated in figure 8. Extend segments AB and ab until they meet at some point O. (If they don't, the problem becomes very simple: the desired point G is the intersection of lines Aa and Bb.) Now, as in the figure above, make a circle containing OA, and a, and another circle containing O, B, and b. These cir-

[12] Euler says that it may be found in the *Exercitationes quaedam Mathematicae* of Daniel Bernoulli, published in in Venice (in 1724, according to the *Opera Omnia* editor Andreas Speiser.)

[13] Some of Euler's contemporaries noticed the same thing. Condorcet writes in his *Eulogy on Euler* [2]: "... on examining the works of the great geometricians of the last age, even of those to whom algebra is indebted for the most important discoveries, we shall see how little they were accusomed to handle this very weapon, which has been brought to such a state of perfection; and it is impossible to refuse to Euler the praise of having effected a revolution which renders algebraic analysis a mode of calculation luminous, universal, of general application, and of easy acquisition." (From the partial English translation in [8].)

cle meet at O and at another point G, the desired center of similitude. We leave the demonstration, which involves a little work with angles inscribed in circles, to the reader.

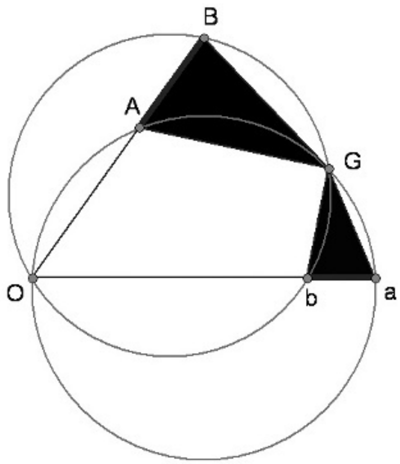

Fig. 8. The Center of Similitude

The significance of G is clear when one considers it from the point of view of geometric transformations. Define a *similitude* as a mapping $X \to X'$ of the plane to itself, so that every segment AB is mapped to a segment $A'B'$ in such a way that $p = \frac{AB}{A'B'}$ is constant. A similitude is so called because it maps every triangle ABC onto a similar triangle $A'B'C'$. The similitude is said to be *direct* if the orientation of the image triangle $A'B'C'$ is the same as that of ABC; otherwise it is said to be *indirect*. If $p < 1$, then a similitude is *contractive*. It is well known that every contractive similitude has a unique invariant point. However, there are always two contractive similitudes mapping AB onto ab: one direct and one indirect. Euler's construction locates the invariant point of the direct similitude.

Euler attempts to extend his work to three dimensions. Unfortunately he equivocates on what he means when he says that two solid figures are "related similarly" to a point, and this equivocation dooms his efforts. The unfortunate result constitute the only serious non-computational error that I have found to date in all of Euler's geometrical works. [14]

[14] A more detailed treatment of Euler's equivocation and its consequences may be found in my article "Leonhard Euler on the Center of Similitude", posted in the online Euler Archives under E693.

Of course Euler, in company with other eighteenth century mathematicans, was aware of geometrical transformations, but generally in the context of problems in which transformation is an explicit element. The clear example is cartography. Euler actually made maps of Russia, and discussed them in E. 492; in E.390 he uses complex functions to describe conformal transformations from one plane to another; in E.490 and E.491 he shows that there is no congruent mapping of a sphere into a plane.

Let us close with another look at conics. Given any triangle ABC, there exists an ellipse that passes through each of its vertices, and has as its center the centroid of ABC. Also, among all ellipses that pass through A, B, and C, it has the smallest area. It is called the *Steiner ellipse* for ABC and its discovery is commonly attributed to Steiner himself. Although the name should probably stand – after all, the Steiner ellipse also contains the Steiner point of ABC – Euler admirers will be pleased to know that Euler himself came across this ellipse first, in E.692, "The solution of a most curious problem, in which one seeks, among all ellipses that can be circumscribed about a given triangle, the one whose area is the smallest of them all", presented to the St. Petersburg Academy of Sciences in 1777 but not published until 1795. Euler's elegant solution permits him to observe that the area of this ellipse is

$$\frac{2\pi ac \sin\omega}{3\sqrt{3}},$$

where a and c are two sides of the triangle and ω is the included angle; also, that the tangent to the ellipse at each vertex of the triangle is parallel to the side opposite that vertex.

A similar problem considered by Euler (in E.691) is to find the ellipse of minimal area circumscribing a given parallelogram. Having previously solved the problem in the case of a rectangle,[15] Euler attacks the general case of any quadrilateral, employing an oblique-angled coordinate system based on three points of the quadrilateral. He makes enough headway to reach a complete solution for the case of a parallelogram, finding that for a parallelogram $ACBD$ with diagonals AB and CD, the minimal ellipse has its center at the intersection of the diagonals, and the tangents to the endpoints of one diagonal, say AB, are parallel to the other diagonal CD. Once again, Euler is skirting around later geometrical developments: modern-day geometers would solve the problem more efficiently, using projectivities to reduce the problem about parallelograms to one about squares.

Nevertheless, how many geometers of today would have the stomach to tackle the central problem of E.563, namely: to find the ellipse of minimal

[15] See E.563, "On the minimal ellipse to be circumscribed about a given parallelogram."

circumference circumscribing a given rectangle? Of course, in order to begin work on this problem, Euler needs an expression for the circumference of an ellipse. With a bit of infinitesimal analysis, Euler arrives at the following: for the ellipse with standard equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

if we set $a^2 + b^2 = c^2$ and $\frac{a^2-b^2}{a^2+b^2} = n$, then the perimeter is

$$\sqrt{2}\pi c \left(1 - \frac{1 \cdot 1}{4 \cdot 4} n^2 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{4 \cdot 4 \cdot 8 \cdot 8} n^4 - \frac{1 \cdot 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{4 \cdot 4 \cdot 8 \cdot 8 \cdot 12 \cdot 12} n^6 - \ldots \right).$$ [16]

Further progress ensues from relating this expression, as a function of n, to the solution of a certain Riccati equation. Ultimately, as might be expected, Euler is unable to provide a closed-form equation for the minimal ellipse, but he is more than happy to work out a few numerical examples.

3. Conclusion

I hope that the foregoing journey through Euler's work has provided some evidence that his reputation as a geometer should not rest primarily on a short list of celebrated results. The free combination of methods, from classical synthetic to algebraic to analytic – as illustrated in the minimal perimeter problem and many others – constitute, on the whole, Euler's most notable contribution to geometry. Condorcet, in his *Eulogy on Euler* (see [8]), has rightly praised this achievement:

> "Thus, at certain epochs, when, after strenuous exertions, the mathematical sciences seemed to have exhausted all the resources of genius, and to have reached the *ne plus ultra* of their career; all at once a new method of calculation is introduced, and the face of the science is wholly changed. We find it immediately, and with inconceivable rapidity, enriching the sphere of knowledge, by a solution of an incredible number of important problems, which geometers had not dared to attempt, intimidated by the difficulty, if not the physical impossibility, of pursuing calculation to

[16] Euler remarks in passing that, in the special case where $b = 0$, we get

$$\frac{2\sqrt{2}}{\pi} = 1 - \frac{1 \cdot 1}{4 \cdot 4} - \ldots$$

which transforms to

$$\frac{\pi}{2\sqrt{2}} = \frac{4 \cdot 4}{3 \cdot 5} \cdot \frac{8 \cdot 8}{7 \cdot 9} \cdot \frac{12 \cdot 12}{11 \cdot 13} \cdot \ldots.$$

real issue. Justice would, perhaps, demand, in favor of the man who invented and introduced these methods, and who first taught their use and application, a share in the glory of all those who have practiced them with success; he has, at least, claims upon their gratitude, which cannot be contested without crime."

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Robert E. Bradley and C. Edward Sandifer (Editors)

Cyclotomy: From Euler through Vandermonde to Gauss

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The word "cyclotomy" is of Greek origin and means "division of the circle." As a mathematical term it denotes the subdivision of a full circle line into a given number of equal parts. Consider the unit circle $x^2 + y^2 = 1$ in the Euclidean plane with Cartesian coordinates (x, y). If this circle is divided into n equal parts beginning with the point $(1, 0)$ then the other division points will have coordinates $(\cos \frac{2\pi \cdot k}{n}, \sin \frac{2\pi \cdot k}{n})$ where k runs from 1 to $(n-1)$. All those points form the edges of a regular n-sided polygon. It is well-known that by means of the imaginary quantity $i := \sqrt{-1}$ one can prove the formula

$$(\cos \alpha + i \cdot \sin \alpha)^n = \cos n\alpha + i \cdot \sin n\alpha \tag{1}$$

which is usually called de Moivre's formula. But in the form (1) it is due to Leonhard Euler (1707-1783), see [Euler 1748], cap. VIII. In particular, the n arguments $\alpha = \frac{2\pi \cdot k}{n}$ with $0 \le k \le n-1$ provide us with the n powers $1, \zeta_n, \zeta_n^2, \ldots, \zeta_n^{n-1}$ of the complex number $\zeta_n := \cos \frac{2\pi}{n} + i \cdot \sin \frac{2\pi}{n}$:

$$\zeta_n^k = \cos \frac{2\pi \cdot k}{n} + i \cdot \sin \frac{2\pi \cdot k}{n} \qquad (0 \le k \le n-1) \tag{2}$$

satisfying the equation

$$x^n - 1 = 0. \tag{3}$$

This means that Eqn. (3) has exactly n roots which are given in the transcendental form (2) and which are the powers of one of them, namely ζ_n. For these powers we shall adopt the name *nth roots of unity* common today among mathematicians. If the exponent i is prime to n then ζ_n^i is called a *primitive* nth root of unity since its powers $1, \zeta_n^i, \zeta_n^{2i}, \zeta_n^{3i}, \ldots$ run through all nth roots of unity. [1]

This way, the geometric problem of cyclotomy was entirely reduced to the algebraic problem of solving the equations $x^n - 1 = 0$ in complex numbers. On the other hand, in the 18th century the dominating problem in the theory of equations was still the problem of solving equations "algebraically" or "by radicals" which meant by the extraction of roots. Euler also devoted some papers to this topic and he was very well aware that roots like $\sqrt[n]{A}$ are only determined up to factors which are nth roots of unity. These ambiguities led him to investigate the "binomial" equation $x^n - 1 = 0$, [Euler 1751], § § 38-48. It was Euler who succeeded in solving the equations $x^n - 1 = 0$ for $n \leq 10$ in terms of radicals with indices $< n$. In his treatise [Vandermonde 1774] Alexandre-Théophile Vandermonde (1735-1796) overcame the difficulties occuring in the case $n = 11$ in a truly pioneering way. Eventually, Carl Friedrich Gauss (1777-1855) was the brilliant architect of a fully-fledged cyclotomy theory which among other topics solved the equations (3) by radicals at least for prime number exponents n, [Gauss 1801], Sectio septima.

In the present paper we intend to display the evolution of ideas from Euler to Gauss. Our first section is devoted entirely to Euler, in particular to his solution of $x^7 - 1 = 0$ by square and cubic roots. The second section elucidates Vandermonde'sVandermonde innovations in the theory of equations which parallel the work of Joseph-Louis Lagrange (1736-1813) in many aspects. Vandermonde however far excelled Euler as well as Lagrange by his solution of $x^{11} - 1 = 0$. We shall expose a version of "pre-Gaussian" cyclotomy theory based on the ideas and tools of Euler, Vandermonde and Lagrange. [2] Our third section mainly discusses Gauss's relationship to Vandermonde. On the example of $x^{17} - 1 = 0$ we are going to make visible the difference between the cyclotomy theories of these two mathematicians. It is very remarkable that Vandermonde's ideas also allow us to construct

[1] The primitive nth roots of unity should not be confused with the *primitive roots* $a \not\equiv 0 \pmod{n}$ for prime numbers n which bear their name since the powers $1, a, a^2, a^3, \ldots$ run through all residue classes $\not\equiv 0 \pmod{n}$, in other words, a mod n should be a generator of the prime residue class group mod n.

[2] In the conceptual framework of our recent set-theoretic mathematics this version amounts to studying the maximal totally real subfields $\mathbb{Q}(\zeta_n + \zeta_n^{-1})$ of $\mathbb{Q}(\zeta_n)$ first and after that considering the fields $\mathbb{Q}(\zeta_n)$ as quadratic extensions of $\mathbb{Q}(\zeta_n + \zeta_n^{-1})$.

the regular 17-gon by ruler and compass.[3] Our considerations will be self-contained and entirely based on Vandermonde's theory of quartic equations and straightforward calculations. This possibility seems to have gone unnoticed until now. This very much begs the question whether Gauss became acquainted with Vandermonde'sVandermonde *Mémoire* before or after publishing his *Disquisitiones Arithmeticae* and whether Vandermonde could have exerted some effect on Gauss. Our main thesis is that Gauss was *not* esssentially influenced by Vandermonde's algebraic work in contrast to conjectures formulated by Henri Lebesgue (1875-1941) and adopted or reproduced by other authors without further examination, see [Lebesgue 1940], [Jones 1991], [Waerden], p. 79.

In the end some concluding remarks should show the reader how to link our considerations to Gauss's theory of cyclotomy.

Comments on mathematical facts in modern terms and notations will, as a rule, not be given in the text but in footnotes.

1. Euler

Here we are going to explain in detail how Euler solved the equations $x^n - 1 = 0$ by radicals for $n \leq 10$. We skip the rather easy cases $n \leq 4$ and pass at once to the equation $x^5 - 1 = 0$. Division by $x - 1$ gives the new equation $x^4 + x^3 + x^2 + x + 1 = 0$ which is mirror-symmetric with regard to the middle term x^2.

1.1. *Reciprocal Equations*

Equations of this type were already considered by Euler in his paper [Euler 1738], § 10 seq. There he is dealing with mirror-symmetric equations of the general form

$$y^{2n} + a \cdot y^{2n-1} + b \cdot y^{2n-2} + \ldots + p \cdot y^n + \ldots + b \cdot y^2 + a \cdot y + 1 = 0, \quad (4)$$

especially for $n = 2, 3, 4, 5$, and is calling them *reciprocal equations* since they do not change their form when y is replaced by $\frac{1}{y}$.[4] Now Euler observes that the left-hand side of Eqn. (4) is a product of n quadratic factors

$$y^2 + \alpha \cdot y + 1, y^2 + \beta \cdot y + 1, y^2 + \gamma \cdot y + 1, y^2 + \delta \cdot y + 1, \text{etc.}$$

---

[3] It should be emphasized that our construction does *not* explicitly use a primitive root mod17 as a generator of all residue classes $\not\equiv 0 \pmod{17}$, contrary to Gauss.

[4] *Aequationes huiusmodi, quae posito* $\frac{1}{y}$ *loco y formam non mutant, voco reciprocas.* [Euler 1738], § 11.

where the coefficients α, β, γ, δ, etc. satisfy an equation of degree n the coefficients of which are nothing but linear combinations of the coefficients of Eqn. (4) with some explicitly known rational integers. The idea behind the factorization of Eqn. (4) can be exposed as follows. Divide Eqn. (4) by y^n and write the resulting equation in the form

$$(y^n + y^{-n}) + a \cdot (y^{n-1} + y^{-n+1}) + b \cdot (y^{n-2} + y^{-n+2}) + \ldots + p = 0.$$

The kth power sum $y^k + y^{-k}$ $(0 \leq k \leq n)$ is symmetric in y and y^{-1}, hence a polynomial in the elementary symmetric polynomials $z := y + y^{-1}$ and $y \cdot y^{-1} = 1$ according to the recursive relation

$$y^{k+1} + \frac{1}{y^{k+1}} = \left(y^k + \frac{1}{y^k}\right) \cdot z - \left(y^{k-1} + \frac{1}{y^{k-1}}\right)$$

(or by the so-called Girard-Newton formulas for power sums). Hence the auxiliary quantity $z := y + y^{-1}$ satisfies an equation

$$z^n + a' \cdot z^{n-1} + b' \cdot z^{n-2} + \ldots + p' = 0 \tag{5}$$

where the coefficients $a', b', \ldots, p'$ are linear combinations of $a, b, c, \ldots, p$ with rational integer coefficients. Denote by $-\alpha$, $-\beta$, $-\gamma$, $-\delta$, etc. the roots of Eqn. (5), in other words

$$z^n + a' \cdot z^{n-1} + b' \cdot z^{n-2} + \ldots + p' = (z + \alpha) \cdot (z + \beta) \cdot (z + \gamma) \cdot (z + \delta) \cdots .$$

In view of $z = y + y^{-1}$ it is obvious that every root of Eqn. (4) satisfies one of the quadratic equations

$$y^2 + \alpha \cdot y + 1 = 0,\ y^2 + \beta \cdot y + 1 = 0,$$
$$y^2 + \gamma \cdot y + 1 = 0,\ y^2 + \delta \cdot y + 1 = 0,\ \text{etc.}$$

whence one gets the desired factorization of Eqn. (4). Below we shall encounter the reciprocal equations once more in Vandermonde's work.

1.2. *Roots of Unity*

After these preparations it is rather easy to solve the reciprocal equation $x^4 + x^3 + x^2 + x + 1 = 0$ by radicals. We put $u := -(x + x^{-1})$ and obtain $x^2 + x^{-2} = u^2 - 2$. In summary $u^2 - u - 1 = 0$ with the roots $p := \frac{1+\sqrt{5}}{2}$ and $q := \frac{1-\sqrt{5}}{2}$. Then one has to solve the two quadratic equations $x^2 + p \cdot x + 1 = 0$ and $x^2 + q \cdot x + 1 = 0$ but that does not cause any problems, [Euler 1751], § 40. It turns out that all 5th roots of unity are rational functions (with rational coefficients) of $\sqrt{5}$ and the square root $\sqrt{-10 + 2\sqrt{5}}$:

$$x = \frac{1}{4}\left[-1 - \delta_1\sqrt{5} + \delta_2\sqrt{-10 - \delta_1 2\sqrt{5}}\right] \tag{6}$$

with $\delta_1, \delta_2 = \pm 1$, [Euler 1751], § 40. [5] Notice that

$$\sqrt{-10+2\sqrt{5}} \cdot \sqrt{-10-2\sqrt{5}} = 4\sqrt{5}.$$

The case $n = 6$ is settled by the factorization

$$x^6 - 1 = (x^2 - 1) \cdot (x^2 + x + 1) \cdot (x^2 - x + 1),$$

[Euler 1751], § 41.

The case $n = 8$ can be dealt with by the remark that $x^8 - 1 = (x^2)^4 - 1 = 0$, in other words, we have to extract square roots from the solutions of $x^4 - 1 = 0$. Euler's formulas show that all 8th roots of unity are rational functions of $\sqrt{-1}$ and $\sqrt{2}$ (with rational coefficients). In a similar way, Euler solves the equation $x^9 - 1 = 0$ by extracting cubic roots from the third roots of unity. Last but not least the equation $x^{10} - 1 = 0$ is solved via the factorization $x^{10} - 1 = (x^5 - 1) \cdot (x^5 + 1)$, [Euler 1751], § 48.

The remaining case $n = 7$ requires a great amount of calculations. First Euler follows the general approach to reciprocal equations and then simplifies the resulting expressions of the 7th roots of unity carrying out some further subtle calculations. He factorizes the equation

$$\frac{x^7 - 1}{x - 1} = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$$

into three quadratic equations

$$x^2 + p \cdot x + 1 = 0, \quad x^2 + q \cdot x + 1 = 0, \quad x^2 + r \cdot x + 1 = 0 \tag{7}$$

where p, q, r are the roots of the cubic equation

$$F(u) := u^3 - u^2 - 2u + 1 = 0. \tag{8}$$

Now he applies Cardano's formula to this equation and obtains the explicit expressions (in our abbreviations)

$$\begin{aligned} p &= \frac{1}{3}\left[1 + \sqrt[3]{A} + \sqrt[3]{A'}\right], \quad q = \frac{1}{3}\left[1 + \rho \cdot \sqrt[3]{A} + \rho^2 \cdot \sqrt[3]{A'}\right], \\ r &= \frac{1}{3}\left[1 + \rho^2 \cdot \sqrt[3]{A} + \rho \cdot \sqrt[3]{A'}\right] \end{aligned} \tag{9}$$

with

[5] In today's terms we have $\mathbb{Q}(\zeta_5) = \mathbb{Q}(\sqrt{5})(\sqrt{-10+2\sqrt{5}})$ according to Euler. The cyclotomic extension $\mathbb{Q}(\zeta_5)/\mathbb{Q}$ is cyclic of degree 4 which can be proved by means of Gauss's theory.

$$\rho = \frac{-1+\sqrt{-3}}{2}, \qquad A = 7\cdot\frac{-1+3\cdot\sqrt{-3}}{2},$$
$$A' = 7\cdot\frac{-1-3\cdot\sqrt{-3}}{2}, \ \sqrt[3]{A}\cdot\sqrt[3]{A'} = 7.$$

After that Euler goes on to solve Eqns. (7). For instance, the first equation in (7) has the roots

$$x = \frac{-p\pm\sqrt{p^2-4}}{2}. \tag{10}$$

Substituting q, r instead of p we obtain the roots of the other two equations. At this point there arises the problem of how to extract square roots from expressions which in their turn are sums of certain cubic radicals. Euler tackles the explicit calculation of the square roots $\sqrt{p^2-4}$, $\sqrt{q^2-4}$, $\sqrt{r^2-4}$ putting $v := \sqrt{u^2-4}$ and indicating the cubic equation for v^2, i. e. the equation of degree 6 for v:

$$G(v) := v^6 + 7v^4 + 14v^2 + 7 = 0 \tag{11}$$

with the six roots $\pm\sqrt{p^2-4}$, $\pm\sqrt{q^2-4}$, $\pm\sqrt{r^2-4}$. This equation splits up into two cubic equations according to

$$(v-\sqrt{p^2-4})(v-\sqrt{q^2-4})(v-\sqrt{r^2-4}) =: v^3 + p'v^2 + q'v + r' = 0$$

and

$$(v+\sqrt{p^2-4})(v+\sqrt{q^2-4})(v+\sqrt{r^2-4}) =: v^3 - p'v^2 + q'v - r' = 0.$$

But Euler's choice of a suitable triple $(\sqrt{p^2-4}, \pm\sqrt{q^2-4}, \pm\sqrt{r^2-4})$ out of 4 possibles ones looks quite arbitrary and calls for a systematic procedure. The crucial point is that this choice can be made such that the coefficients p', q', r' take the values

$$p' = \sqrt{-7}, \qquad q' = 0, \qquad r' = \sqrt{-7}$$

which results from the comparison of coefficients in the product of the last two equations and in (11). Eventually these coefficients require a new quadratic irrationality only. [6] To sum up we have two cubic equations

$$v^3 \pm \sqrt{-7}\cdot v^2 \pm \sqrt{-7} = 0$$

[6] The right factorization of $G(v)$ is closely tied up to the quadratic residues mod 7 and will not be discussed here in detail. It seems that Euler was not aware of this fact. Anyway we have $-r'^2 = 7$, $\sqrt{p^2-4}\cdot\sqrt{q^2-4}\cdot\sqrt{r^2-4} = -\sqrt{-7}$. With regard to Eqns. (10) one concludes from there that $\sqrt{-7}$ is a linear combination of 7th roots of unity with rational integer coefficients. This remarkable fact was generalized by Gauss to all prime exponents n instead of 7, [Gauss 1801], art. 356. In recent terms it means the embedding of all quadratic number fields in cyclotomic fields. See also our subsection 2.5 after Eqn. (58).

and Cardano's formula gives us six values of v. These values turn out to be rational functions (with rational coefficients) of $\sqrt{-3}$, $\sqrt{-7}$ and the cubic radicals $\sqrt[3]{B}$, $\sqrt[3]{B'}$ with

$$B = \alpha^2\sqrt{-7}, \quad B' = \alpha'^2\sqrt{-7}, \quad \sqrt[3]{B} \cdot \sqrt[3]{B'} = -7$$

and

$$\alpha = \frac{-1+3\sqrt{-3}}{2}, \quad \alpha' = \frac{-1-3\sqrt{-3}}{2}, \quad \alpha \cdot \alpha' = 7 = -(\sqrt{-7})^2.$$

Furthermore it is obvious how to remove the factor $\sqrt{-7}$ from B and B':

$$\sqrt[3]{B} = \frac{\alpha}{\sqrt{-7}} \cdot \sqrt[3]{7\alpha'}, \quad \sqrt[3]{B'} = \frac{\alpha'}{\sqrt{-7}} \cdot \sqrt[3]{7\alpha}$$

with the relation

$$\sqrt[3]{7\alpha'} \cdot \sqrt[3]{7\alpha} = 7.$$

Euler's definite formulas show that the 7th roots of unity are rational functions (with rational coefficients) of the two quadratic irrationalities $\sqrt{-3}$, $\sqrt{-7}$ and a *single* cubic radical $\sqrt[3]{7\alpha}$ only. [7]

What can be objected to in Euler's exposition? As to the 7th roots of unity his formulas are correct but incomplete at one point. He does not mention the relations

$$\sqrt[3]{A} \cdot \sqrt[3]{A'} = 7, \quad \sqrt[3]{B} \cdot \sqrt[3]{B'} = -7, \quad \sqrt[3]{7\alpha'} \cdot \sqrt[3]{7\alpha} = 7$$

which are indispensable in order to obtain the actual roots of the cubic equations in question. However, we don't hesitate to suppose that Euler had these relations in mind as well, cf. [Euler 1738], § § 3-4, [Euler 1770], part II, section 1, § 12.

More generally, in his 1738 and 1751 papers beyond the cubic and quartic equations Euler does *not* address the important question in full generality of how to restrict the ambiguities in expressions like $\sqrt[n]{A} + \sqrt[n]{B} + \sqrt[n]{C} + \sqrt[n]{D} + \ldots$ for the roots of an equation of degree n. This question arises in a natural way since every nth root $\sqrt[n]{A}$, etc. takes n values. Only later on in his 1764 paper Euler discusses equations with solutions of the special form $\omega + \mathcal{A}\sqrt[n]{v} + \mathcal{B}\,(\sqrt[n]{v})^2 + \mathcal{C}\,(\sqrt[n]{v})^3 + \ldots + \mathcal{O}\,(\sqrt[n]{v})^{n-1}$ and obtains all

[7] In today's terminology this means that the cyclotomic number field $\mathbb{Q}(\zeta_7, \sqrt{-3}) = \mathbb{Q}(\zeta_7, \zeta_6) = \mathbb{Q}(\zeta_{42})$ is identical with the field $\mathbb{Q}(\sqrt{-7}, \sqrt{-3}, \sqrt[3]{7\alpha})$. The Galois extension $\mathbb{Q}(\zeta_7)/\mathbb{Q}$ is cyclic of degree 6 which can be deduced from Gauss's theory. Hence this extension is the composite of $\mathbb{Q}(\sqrt{-7})/\mathbb{Q}$ and a uniquely determined cyclic extension $L/\mathbb{Q}$ of degree 3. L is nothing but the splitting field of $u^3 - u^2 - 2u - 1$, and the "cycle" of explicit relations $q = p^2 - 2, r = q^2 - 2, p = r^2 - 2$ shows that $L = \mathbb{Q}(p, q, r) = \mathbb{Q}(p) = \mathbb{Q}(q) = \mathbb{Q}(r)$. We have $L(\sqrt{-3}) = \mathbb{Q}(\sqrt{-3}, \sqrt[3]{7\alpha})$ which is a cyclic extension of $\mathbb{Q}$ of degree 6. By the way, the equality $7 = \alpha \cdot \alpha'$ yields the unique prime factor decomposition of 7 in the euclidean domain $\mathbb{Z}[(-1+\sqrt{-3})/2]$.

solutions by multiplying $(\sqrt[n]{v})^i$ by the ith power of a nth root of unity $(1 \leq i \leq n-1)$, [Euler 1764], esp. § 13, [Breuer 1921], [Euler 1928], pp. 65-94 (annotations by the editor), [Maǐstrova 1985].

How could one improve on Euler's exposition using Eulerian tools only? In hindsight it is tempting to describe all occuring quantities in terms of roots of unity. Let ζ be a primitive 7th root of unity. Thus we have

$$p = -\zeta - \zeta^6,\ q = -\zeta^2 - \zeta^5, r = -\zeta^4 - \zeta^3 \tag{12}$$

and

$$p^2 - 4 = (\zeta - \zeta^6)^2,\ q^2 - 4 = (\zeta^2 - \zeta^5)^2,\ r^2 - 4 = (\zeta^4 - \zeta^3)^2. \tag{13}$$

Now it is rather easy to write down two suitable factors of $G(v)$:

$$(v - (\zeta - \zeta^6))(v - (\zeta^2 - \zeta^5))(v - (\zeta^4 - \zeta^3)) = v^3 + \sqrt{-7}v^2 + \sqrt{-7} \tag{14}$$

and

$$(v + (\zeta - \zeta^6))(v + (\zeta^2 - \zeta^5))(v + (\zeta^4 - \zeta^3)) = v^3 - \sqrt{-7}v^2 - \sqrt{-7}. \tag{15}$$

The equality

$$(\zeta - \zeta^6)(\zeta^2 - \zeta^5)(\zeta^4 - \zeta^3) = -\sqrt{-7} \tag{16}$$

allows us to solve Eqn. (14) *without* a repeated application of Cardano's formula contrary to Euler. Indeed each of the three products

$$(\zeta - \zeta^6)(\zeta^2 - \zeta^5),\ (\zeta^2 - \zeta^5)(\zeta^4 - \zeta^3),\ (\zeta^4 - \zeta^3)(\zeta - \zeta^6) \tag{17}$$

is symmetric in ζ, ζ^{-1}, hence a polynomial in p or q or r as we want. For instance, we have

$$(\zeta - \zeta^6)(\zeta^2 - \zeta^5) = (\zeta - \zeta^6)^2(\zeta + \zeta^6) = -(p^2 - 4)p. \tag{18}$$

This implies $\zeta^4 - \zeta^3 = \frac{\sqrt{-7}}{(p^2-4)p}$ immediately and it only remains for us to calculate the reciprocal values of p and $p^2 - 4$ which is an easy standard exercise. [8] We will further comment on those equations from a more general point of view in our next section on Vandermonde.

After his splendid results published in 1751 Euler was not able to settle the case of the 11th roots of unity. He said that

> "indeed the eleven roots of the equation $x^{11} - 1 = 0$ cannot be calculated with the help of the accompanying equation of degree 5; since its solution is hidden hitherto we should stop here." [9]

For us now it will be the right moment to pass to Vandermonde's work and his solution of $x^{11} - 1 = 0$.

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[8] We know the equations (8) and (11) satisfied by p and $\sqrt{p^2 - 4}$, resp., and can write $(p^2 - p - 2)p + 1 = 0$ and $((p^2 - 4)^2 + 7(p^2 - 4) + 14)(p^2 - 4) + 7 = 0$.

[9] *At vero radices undecim aequationis $x^{11} - 1 = 0$ exhiberi non possunt ope aequationis quinque dimensionum; cuius resolutio cum adhuc lateat, hic subsistere debemus.* [Euler 1751], § 48.

2. Vandermonde

Without any doubt Vandermonde was an outstanding mathematician of his time, which is confirmed by Lebesgue's deserving and authoritative biography, [Lebesgue 1940], [Jones 1991]. He published only four mathematical papers among which there were two significant ones. The geometrico-topological paper *Remarques sur des problèmes de situation* (1771) aroused Gauss's interest and let him speak of "the geometer Vandermonde held in high esteem by me", [Olbers 1900], p. 103 [10]. As far as we know this is the first documented mentioning of Vandermonde by Gauss.

Here we will mainly be concerned with Vandermonde's extensive *Mémoire sur la résolution des équations* which was read before the Paris Academy in November 1770 but was not published until 1774. Apparently, the British mathematician Edward Waring (1734-1798) was the first who appreciated Vandermonde's contributions to the theory of equations and praised his acumen, [Waring 1782], Praefatio, pp. XXIV-XXV. More than two decades later Lagrange commented on Vandermonde's theory of equations rather extensively, [Lagrange 1808], notes XIII, XIV. He said:

> "Thus one may say that Vandermonde is the first who had crossed the limits within which the solution of equations of 2 terms was constricted", [11]

Augustin-Louis Cauchy (1789-1857) in two of his papers on rational functions and permutations referred to Vandermonde as well, [Cauchy 1815a,b]. No less a mathematician than Leopold Kronecker (1823-1891) praised the memoir in the words:

> "With Vandermonde's memoir on the resolution of equations, presented in 1770 to the Parisian Academy, began a new blossoming of algebra; the profundity of the view which is expressed in such clear words as in this work, arouses nothing less than our astonishment." [12]

Meanwhile, several detailed overviews of Vandermonde's algebraic work were published, e. g., [Loewy 1918], [Lebesgue 1940], [Wussing 1969], 2. Kap., [Nový 1973], pp. 36-41, [Edwards 1984], § § 15-16, § § 22-23, [Waer-

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[10] *Dieser bisher fast ganz brach liegende Gegenstand, über den wir nur einige Fragmente von Euler und einem von mir hochgeschätzten Geometer Vandermonde haben (...)* Letter to Olbers, October 12, 1802. Apparently, Lebesgue was ignorant of this letter.

[11] *On peut donc dire que Vandermonde est le premier qui ait franchi les limites dans lesquelles la résolution des équations à deux termes se trouvait resserrée.* Loc. cit. Note XIV, § 33. Lebesgue apparently did not know this assessment of Vandermonde's work.

[12] *Mit Vandermonde's im Jahre 1770 der Pariser Akademie vorgelegten Abhandlung über die Auflösung der Gleichungen beginnt der neue Aufschwung der Algebra; die Tiefe der Auffassung, welche sich in dieser Arbeit in so klaren Worten ausspricht, erregt geradezu unser Erstaunen.* (Preface to [Itzigsohn 1888]. Cf. [Neumann 2006], § 2.)

den 1985], pp. 77-79, [Tignol 1988], chap. 11. Here we shall concentrate our attention on his theory of equations and its application to cyclotomic equations.

2.1. *Resolvents*

In the introductory sentences of his treatise Vandermonde mentioned the papers [Euler 1764] and [Bézout 1764] as the most significant ones of the recent past. His own most notable innovation can be described as follows:

> "Starting with the well known solution of quadratic and cubic equations, Vandermonde develops general principles upon which the solution of equations may be based. (...) Vandermonde now asks whether the general equation of degree n can be solved by a similar expression
>
> $$\frac{1}{n}[x_1 + \cdots + x_n + \sqrt[n]{(r_1 x_1 + \cdots + r_n x_n)^n} + \cdots \cdots + \sqrt[n]{(r_1^{n-1} x_1 + \cdots + r_n^{n-1} x_n)^n}\ ]$$
>
> in which $r_1, \ldots, r_n$ are the nth roots of unity." [Waerden 1985], p. 77, cf. [Vandermonde 1774], § VI.

Of course, $x_1, \ldots, x_n$ denote the roots of the given equation. Vandermonde and Lagrange introduced (or should one say: invented?) expressions like

$$\Delta^{(i)} := r_1^i x_1 + \ldots + r_n^i x_n, \qquad (1 \le i \le n), \tag{19}$$

independently of each other and almost at the same time, [Lagrange 1770-1771], § 69. Nowadays these expressions usually are called "Lagrange resolvents". The Eqns. (19) can be regarded as a system of linear equations for $x_1, \ldots, x_n$ which has the solution

$$x_k = \frac{1}{n} \cdot \sum_{i=1}^{n} r_k^{-i} \cdot \Delta^{(i)} = \frac{1}{n} \cdot \left[(x_1 + \ldots + x_n) + \sum_{i=1}^{n-1} r_k^{-i} \cdot \Delta^{(i)} \right], \tag{20}$$

for $1 \le k \le n$. For $n \le 7$ the reader will also find these explicit solutions in Vandermonde in §§ VII-X.

Vandermonde and Lagrange alike observed that in (19) the whole sum is multiplied by a nth root of unity if on the $x_1, \ldots, x_n$ a suitable cyclic permutation, (say) σ, or one of its powers is performed (this permutation depends on the sequence $(r_1, \ldots, r_n)$).[13] Therefore the nth power of each

[13] The simplest way to see that is to number the roots $x_1, \ldots, x_n$ such that $r_k = \rho^{k-1}$ where ρ denotes a primitive nth root of unity ($1 \le k \le n$) as Lagrange did. Under this assumption σ can be taken as $x_1 \mapsto x_2, \ldots, x_n \mapsto x_1$. For odd prime numbers $n = 2m+1$ Vandermonde prefers to choose $r_1 = 1, r_2 = \rho, r_3 = \rho^{-1}, r_4 = \rho^2, r_5 = \rho^{-2}, \ldots, r_{2m} = \rho^m, r_{2m+1} = \rho^{-m}$, [Vandermonde 1774], § XI.

of the expressions (19) remains unchanged under σ. We put

$$V^{(i)} := (\Delta^{(i)})^n = (r_1^i x_1 + \ldots + r_n^i x_n)^n \qquad (1 \leq i \leq n). \tag{21}$$

In many examples in § XXXVI Vandermonde is "expanding" the $V^{(i)}$'s in sums of simple σ-invariant components which he calls "partial types" (*types partiels*) and each of which can be calculated separately. Lagrange's exposition is much more extensive than Vandermonde's and explains the important step from (19) to (21) as follows. For any i, consider the values of $\Delta^{(i)}$ under all $n!$ permutations of $x_1, \ldots, x_n$. These $n!$ values can be arranged in groups of n values each consisting of $\Delta, \rho \cdot \Delta, \rho^2 \cdot \Delta, \ldots, \rho^{n-1} \cdot \Delta$ where ρ denotes a primitive nth root of unity. Let Δ' run through all $n!$ values of $\Delta^{(i)}$. Then the polynomial $\prod (x - \Delta')$ splits up into factors $\prod_{i=0}^{n-1} (x - \rho^i \Delta) = x^n - \Delta^n$. In other words $\prod (x - \Delta')$ is actually a polynomial in x^n with the roots $(\Delta')^n$. Therefore, the quantity $V^{(i)}$ satisfies an equation of degree $(n-1)!$ the coefficients of which are rational functions of the coefficients of the given equation and the nth roots of unity.

Combining Eqns. (20) and (21) we obtain Vandermonde's and Lagrange's approach

$$x = \frac{1}{n} \cdot \left[(x_1 + \ldots + x_n) + \sqrt[n]{V^{(1)}} + \ldots + \sqrt[n]{V^{(n-1)}} \right] \tag{22}$$

where the nth roots are to be suitably chosen. Vandermonde calls the right-hand side of (22) a "function which one could say equals *any* root depending on the meaning attributed to that function." [Vandermonde 1774, § IV.] Though for $n > 4$ neither Vandermonde nor Lagrange addresses the question of how to choose the n suitable $(n-1)$-tuples $(\sqrt[n]{V^{(1)}}, \ldots, \sqrt[n]{V^{(n-1)}})$ out of the n^{n-1} possible ones. But they are well aware of the following partial answer to this question: if, in (22), $(\sqrt[n]{V^{(1)}}, \ldots, \sqrt[n]{V^{(n-1)}})$ gives us a solution then $(\rho \cdot \sqrt[n]{V^{(1)}}, \rho^2 \cdot \sqrt[n]{V^{(2)}}, \ldots, \rho^{n-1} \cdot \sqrt[n]{V^{(n-1)}})$ will do as well where ρ denotes any nth root of unity, see Eqn. (20) and [Vandermonde 1774, § § VII-X.] It was only Gauss in case of cyclotomy and Niels Henrik Abel (1802-1829) under more general assumptions who settled that question based on the remark that the products $\Delta^{(i)} \cdot (\Delta^{(1)})^{n-i}, (2 \leq i \leq n-1)$ as well as the $V^{(i)}$'s are invariant under the same cyclic permutations of $x_1, \ldots, x_n$, [Gauss 1801], art. 360.III, [Gauss 1863], [Abel 1829], formulas (38)-(42), cf. [Neumann 2006], § § 2, 4. In our subsection 2.3 we shall use this remark in order to complement Vandermonde's most spectacular achievement.

For the actual calculation of the $V^{(i)}$'s Vandermonde proves the main theorem on symmetric polynomials and then is able to obtain the known solutions of the cubic and quartic equations anew, [Vandermonde 1774], §§

V, VII, XII-XIII, XIX-XXII. As to the general equations of degree $n > 4$ he does not arrive at any substantial steps towards an "algebraic" solution.

For the composite degrees $n = 4, 6, 8, 9$ Vandermonde indicates modified "explicit" solutions in §§ XIII-XVII. In these cases there are indices i having a common divisor $d > 1$ with n, say $n = n'd, i = i'd$. Then formally we can write $\sqrt[n]{(\Delta^{(i)})^n} = \sqrt[n']{(\Delta^{(i)})^{n'}}$. Now the expression $\Delta^{(i)}$ looks simpler than, e. g., $\Delta^{(1)}$ insofar as it contains the dth powers of the nth roots of unity only, in other words the n'th roots of unity only. The n roots $x_1, \ldots, x_n$ are arranged in n' groups of d summands each. These groups of d summands have the form

$$x_k + x_{k+n'} + x_{k+2n'} + \cdots + x_{k+(d-1)n'} \qquad (1 \leq k \leq n'). \tag{23}$$

Starting from $\Delta^{(2)}$ or $\Delta^{(3)}$ or $\Delta^{(4)}$, resp., Vandermonde succeeds to build up explicit solutions after permuting the $x_1, \ldots, x_n$ suitably in $\Delta^{(2)}$ or $\Delta^{(3)}$ or $\Delta^{(4)}$, resp. Certain sums of the special kind (23) occur three decades later in Gauss's cyclotomy theory again and are baptized by Gauss as *periods*, [Gauss 1801], art. 343.[14] This fact had led Lebesgue to the conjecture that for Gauss the concept of period could have been suggested by his early reading of Vandermonde, [Lebesgue 1940], pp. 33-34, 38. From a purely mathematical point of view this hypothesis looks at least well admissible but we reject it for various reasons which will be discussed in our next section on Gauss.

2.2. *Quartic Equations and 5th Roots of Unity*

As announced above for the quartic equation

$$(x-a)(x-b)(x-c)(x-d) = x^4 + Nx^3 + Px^2 + Qx + R = 0$$

with the roots a, b, c, d Vandermonde indicates the solutions in the modified and elegant form

$$\begin{aligned} x = \frac{1}{4}\Big[&\epsilon_1\sqrt{(a+b-c-d)^2} + \epsilon_2\sqrt{(a+c-b-d)^2} \\ &+ \epsilon_3\sqrt{(a+d-b-c)^2}\Big] + \frac{1}{4}(-N) \end{aligned} \tag{24}$$

with $\epsilon_1, \epsilon_2, \epsilon_3 = \pm 1$ and $\epsilon_1\epsilon_2\epsilon_3 = +1$, [Vandermonde 1774], § XIII. The same formulas can be found with Lagrange, [Lagrange 1770-1771], § 32. Vandermonde leaves it at that although the conditions imposed on $\epsilon_1, \epsilon_2, \epsilon_3$

[14] Precisely speaking the Gaussian "periods" are always sums of roots of unity, and the number of terms is the number of these roots. In Gauss's language Vandermonde's cyclotomy theory is based on the "periods of 2 terms" $r^i + r^{-i}$.

are still insufficient to select the admissible triples of square roots as realized by Lagrange. But the latter author shows the way out when we take into account that the quantity

$$\Pi := (a+b-c-d)(a+c-b-d)(a+d-b-c)$$

is invariant under all permutations of a, b, c, d. Hence $\Pi = -N^3+4NP-8Q$ is known from the given equation.[15] Therefore we should stipulate that

$$\sqrt{(a+b-c-d)^2} \cdot \sqrt{(a+c-b-d)^2} \cdot \sqrt{(a+d-b-c)^2} = \Pi.$$

On the other hand, the squares $(a+b-c-d)^2, (a+c-b-d)^2, (a+d-b-c)^2$ are only permuted with each other when we permute a, b, c, d in all possible ways. Therefore those squares satisfy an equation of third degree the coefficients of which can be calculated by means of the given equation, [Vandermonde 1774], § XVI. We shall use this fact in our last section to construct the regular 17-gon by ruler and compass circumventing the primitive roots (mod 17) and closely following Vandermonde's ideas.

Without loss of generality in the given equation we can assume $N = 0$. Then according to Eqn. (24) the solutions are sums of three square roots. This fact was as early as 1738 published by Euler who wrote

$$x = \sqrt{A} + \sqrt{B} + \sqrt{C}$$

and deduced a cubic equation for A, B, C, [Euler 1738], § 5. Moreover, he derived the equality $\sqrt{A} \cdot \sqrt{B} \cdot \sqrt{C} = -Q/8$ which coincides with Lagrange's condition (notice the factor 1/4 in Eqn. (24)). As in Eqn. (24) Euler indicated the three remaining roots in the form

$$\sqrt{A} - \sqrt{B} - \sqrt{C}, \quad \sqrt{B} - \sqrt{A} - \sqrt{C}, \quad \sqrt{C} - \sqrt{A} - \sqrt{B}$$

which indeed means nothing but Eqn. (24).

Vandermonde uses his approach to quartic equations just described above in order to solve the 5th cyclotomic equation

$$\frac{x^5-1}{x-1} = x^4 + x^3 + x^2 + x + 1 = 0.$$

In this case he obtains a reducible auxiliary equation of degree 3 which eventually gives

$$x = \frac{1}{4}\left[-1 + \epsilon_1\sqrt{5} + \epsilon_2\sqrt{-5+2\sqrt{5}} + \epsilon_3\sqrt{-5-2\sqrt{5}}\right] \qquad (25)$$

[15] Lagrange himself erroneously wrote $+N^3-4NP+8Q$ which was realized by the editor J.-A. Serret.

with $\epsilon_1, \epsilon_2, \epsilon_3 = \pm 1$ and $\epsilon_1\epsilon_2\epsilon_3 = +1$, [Vandermonde 1774], § XXIII. Here the square roots should be chosen such that

$$\sqrt{5} \cdot \sqrt{-5+2\sqrt{5}} \cdot \sqrt{-5-2\sqrt{5}} = -5.$$

Of course, this result should coincide with (6). This follows indeed from

$$\left[\sqrt{-5+2\sqrt{5}} \pm \sqrt{-5-2\sqrt{5}}\right]^2 = -10 \pm 2\sqrt{5}.$$

2.3. *11th Roots of Unity*

The unquestionable apex of Vandermonde's theory is the representation of the 11th roots of unity by means of radicals which has no counterpart in Lagrange's work at that time, see [Vandermonde 1774], § XXXV. Vandermonde's own exposition was very sketchy whereas some three decades later Lagrange at length commented on Vandermonde's solution of $x^{11} - 1 = 0$ and $x^5 - 1 = 0$, [Lagrange 1808], Note XIV. First of all, for the reciprocal equation

$$\frac{x^{11}-1}{x-1} = x^{10} + x^9 + \ldots + x + 1 = 0$$

with the roots $r, r^2, \ldots, r^{10}$ Vandermonde calculates the auxiliary equation

$$x^5 - x^4 - 4x^3 + 3x^2 + 3x - 1 = 0$$

with the five roots

$$a := -r - r^{10},\ b := -r^2 - r^9,\ c := -r^3 - r^8,\ d := -r^4 - r^7,\ e := -r^5 - r^6.$$

Then he goes on to write down a series of intrinsic relations among those roots:

$$a^2 = -b+2,\ b^2 = -d+2,\ c^2 = -e+2,\ d^2 = -c+2,\ e^2 = -a+2, \quad (26)$$

$$ab = -a-c, \qquad bc = -a-e, \qquad cd = -a-d, \qquad de = -a-b, \quad (27)$$

$$ac = -b-d, \qquad bd = -b-e, \qquad ce = -b-c, \quad (28)$$

$$ad = -c-e, \qquad be = -c-d, \quad (29)$$

$$ae = -d-e, \quad (30)$$

$$a+b+c+d+e-1 = 0. \quad (31)$$

[16] Vandermonde's further constructions are based on his fundamental observation that *the relations (26)-(31) are only permuted with each other*

[16] In recent terminology these relations entail that the subdomain $\mathbb{Z}[a,b,c,d,e] = \mathbb{Z}[a] = \mathbb{Z}[b] = \mathbb{Z}[c] = \mathbb{Z}[d] = \mathbb{Z}[e]$ of the number field $\mathbb{Q}(a,b,c,d,e) = \mathbb{Q}(a) = \mathbb{Q}(b) = \mathbb{Q}(c) = \mathbb{Q}(d) = \mathbb{Q}(e)$ is the (torsion-free, hence free) abelian group $\mathbb{Z}\cdot 1 + \mathbb{Z}\cdot a + \mathbb{Z}\cdot b + \mathbb{Z}\cdot c + \mathbb{Z}\cdot d + \mathbb{Z}\cdot e$ with the relation (31). Its rank is 5 which follows from the irreducibility of $x^{10} + x^9 + \ldots + x + 1$ according to [Gauss 1801], art. 341.

when one applies the cyclic permutation $(abdce) = \begin{pmatrix} a & b & c & d & e \\ b & d & e & c & a \end{pmatrix}$ *and its powers to* a, b, c, d, e. To use his resolvents (19) he has to take into consideration the 5th roots of unity $1, \rho, \rho^2, \rho^3, \rho^4$ where ρ denotes a primitive 5th root of unity. That is why he introduces the expressions

$$\begin{aligned}\Theta^{(i)} &:= a + \rho^i \cdot b + \rho^{-i} \cdot d + \rho^{2i} \cdot c + \rho^{-2i} \cdot e \\ &= a + \rho^i \cdot b + \rho^{4i} \cdot d + \rho^{2i} \cdot c + \rho^{3i} \cdot e \end{aligned} \tag{32}$$

$(1 \le i \le 4)$ which are multiplied by ρ^{-i} when one performs the cyclic permutation $(abdce)$. Vandermonde's own notations are

$$\Delta^{(1)} := \Theta^{(1)}, \quad \Delta^{(2)} := \Theta^{(4)}, \quad \Delta^{(3)} := \Theta^{(2)}, \quad \Delta^{(4)} := \Theta^{(3)}. \tag{33}$$

Hence the quantities

$$V^{(i)} := (\Delta^{(i)})^5 \tag{34}$$

are invariant under $(a\,b\,d\,c\,e)$ and Vandermonde concludes from his explicit calculations that all $V^{(i)}$'s are linear combinations of $1, \rho, \rho^2, \rho^3, \rho^4$ with rational integer coefficients. Then the Eqns. (22) and (34) enable him to display a, b, c, d, e by means of iterated square roots and 5th roots.

Here we want to emphasize how important the relations (26) are and first we are going to prove the following assertion.

Lemma.

Let

$$f(x) = (x - x_1)(x - x_2) \cdots (x - x_n)$$

be a polynomial such that there is a rational function $\vartheta(X)$ with the property

$$x_2 = \vartheta(x_1),\ \ldots,\ x_{i+1} = \vartheta(x_i),\ \ldots,\ x_1 = \vartheta(x_n).$$

Let $W(x_1, \ldots, x_n)$ be a rational function the coefficients of which are considered to be invariant under the cyclic permutation $(x_1 \ldots x_n)$. Suppose that, moreover, W is invariant under the cyclic permutation $(x_1 \ldots x_n)$. [17] Then W depends only on the elementary symmetric polynomials of $x_1, \ldots, x_n$, i. e. on the coefficients of $f(x)$.

[17] The reader of today should notice that a permutation of $x_1, \ldots, x_n$ need not be an automorphism of the splitting field of $f(x)$. For instance, with the function $\vartheta(X) = 1/(1 - X)$ and the assumptions x_1 be an arbitrary element $\neq 0, 1$ of the base field, $x_2 = \vartheta(x_1) = 1/(1 - x_1), x_3 = \vartheta(x_2) = (x_1 - 1)/x_1$ the polynomial $f(x)$ splits into linear factors. Our remark shows that the Lemma is of purely combinatorial nature and does not pertain to Galois theory properly.

Proof. Put $W'(X) := W(X, \vartheta(X), \vartheta^2(X), \vartheta^3(X), \dots, \vartheta^{n-1}(X))$. Then we have $W'(x_1) = W(x_1, \dots, x_n)$. The invariance of W gives us $W'(x_1) = W'(x_2) = \cdots = W'(x_n)$ and further

$$W = \frac{1}{n} \cdot (W'(x_1) + \cdots + W'(x_n)).$$

The right-hand side is symmetric in $x_1, \dots, x_n$. This means that W depends only on the elementary symmetric polynomials of $x_1, \dots, x_n$ which was to be proved. □

By the way, the proof is modeled on the arguments in Abel's paper [Abel 1829]. In Vandermonde's case the premises of the Lemma are satisfied by $\vartheta(X) = -X^2 + 2$ in view of (26). Now we can reproduce his results on the $V^{(i)}$'s and are additionally able to handle the relations between the radicals $\sqrt[5]{V^{(i)}}$.

Relations (26)-(31) allow us to write any rational function $W(a, b, c, d, e)$ in the form

$$W = A \cdot a + B \cdot b + C \cdot c + D \cdot d + E \cdot e + F. \tag{35}$$

For our numerical calculations the following corollary will be of some use.

Corollary.

If $W = A \cdot a + B \cdot b + C \cdot c + D \cdot d + E \cdot e + F$ is invariant under the cyclic permutation $(abdce)$ then

$$W = \frac{1}{5} \cdot (A + B + C + D + E) + F. \tag{36}$$

(See [Lebesgue 1940], pp. 35-36.)

Proof. Applying the permutation $(abdce)$ and its powers to W we obtain the additional equalities

$$W = A \cdot b + B \cdot d + C \cdot e + D \cdot c + E \cdot a + F \tag{37}$$

$$W = A \cdot d + B \cdot c + C \cdot a + D \cdot e + E \cdot b + F \tag{38}$$

$$W = A \cdot c + B \cdot e + C \cdot b + D \cdot a + E \cdot d + F \tag{39}$$

$$W = A \cdot e + B \cdot a + C \cdot d + D \cdot b + E \cdot c + F. \tag{40}$$

Taking the average of the right-hand sides of (35), (37)-(40) we get

$$W = \frac{1}{5} \cdot (A + B + C + D + E)(a + b + c + d + e) + F$$

and with regard to (31) we have the formula (36).□

Of course, the explicit determination of the $V^{(i)}$'s requires a great amount of calculations. Vandermonde relies on his expansions of the $V^{(i)}$'s in sums of $(abdce)$-invariant terms exposed in § XXVIII of his treatise. Here we indicate the final results only:

$$V^{(1)} = 196 + 130\rho - 90\rho^4 - 255\rho^2 + 20\rho^3 \tag{41}$$

$$V^{(2)} = 196 + 130\rho^4 - 90\rho - 255\rho^3 + 20\rho^2 \tag{42}$$
$$V^{(3)} = 196 + 130\rho^2 - 90\rho^3 - 255\rho^4 + 20\rho \tag{43}$$
$$V^{(4)} = 196 + 130\rho^3 - 90\rho^2 - 255\rho + 20\rho^4. \tag{44}$$

The quantities x_1, x_2, x_3, x_4, x_5 are now expressions of the form

$$x = \frac{1}{5}\left[1 + \sqrt[5]{V^{(1)}} + \sqrt[5]{V^{(2)}} + \sqrt[5]{V^{(3)}} + \sqrt[5]{V^{(4)}}\right]. \tag{45}$$

After inserting the explicit values of the 5th roots of unity Vandermonde obtains

$$\sqrt[5]{V^{(1)}} = \sqrt[5]{\frac{11}{4}\left(89 + 25\sqrt{5} - 5\sqrt{-5+2\sqrt{5}} + 45\sqrt{-5-2\sqrt{5}}\right)} \tag{46}$$

$$\sqrt[5]{V^{(2)}} = \sqrt[5]{\frac{11}{4}\left(89 + 25\sqrt{5} + 5\sqrt{-5+2\sqrt{5}} - 45\sqrt{-5-2\sqrt{5}}\right)} \tag{47}$$

$$\sqrt[5]{V^{(3)}} = \sqrt[5]{\frac{11}{4}\left(89 - 25\sqrt{5} - 5\sqrt{-5+2\sqrt{5}} - 45\sqrt{-5-2\sqrt{5}}\right)} \tag{48}$$

$$\sqrt[5]{V^{(4)}} = \sqrt[5]{\frac{11}{4}\left(89 - 25\sqrt{5} + 5\sqrt{-5+2\sqrt{5}} + 45\sqrt{-5-2\sqrt{5}}\right)}. \tag{49}$$

He leaves it at that and does not take care of the ambiguities of the radicals. It is not difficult to calculate

$$\Delta^{(1)} \cdot \Delta^{(2)} = \Delta^{(3)} \cdot \Delta^{(4)} = 11.$$

This means that the four quantities $\Delta^{(1)}, \Delta^{(2)}, \Delta^{(3)}, \Delta^{(4)}$ are complex numbers of modulus $\sqrt{11}$ since a, b, c, d, e are real numbers and either of the two couples $(\Delta^{(1)}, \Delta^{(2)})$, $(\Delta^{(3)}, \Delta^{(4)})$ consists of numbers which are complex conjugate to each other. For the radicals we have the relations

$$\sqrt[5]{V^{(1)}} \cdot \sqrt[5]{V^{(2)}} = \sqrt[5]{V^{(3)}} \cdot \sqrt[5]{V^{(4)}} = 11. \tag{50}$$

Moreover, we have further relations at our disposal since the products $(\Delta^{(1)})^2 \cdot \Delta^{(4)}$ and $(\Delta^{(1)})^3 \cdot \Delta^{(3)}$ turn out to be $(abdce)$-invariant as well. For our purposes it will be sufficient to calculate $(\Delta^{(1)})^2 \cdot \Delta^{(4)}$ since we know already $(\Delta^{(1)})^5$. It is not difficult to expand our product of three factors in a sum of 7 $(abdce)$-invariant terms following an idea of Vandermonde at that. We skip the details of the calculations and indicate the final result:

$$\begin{aligned}(\Delta^{(1)})^2 \cdot \Delta^{(4)} &= S(a^3) + (\rho^2 + 2\rho^4)S(ab^2) + (\rho^4 + 2\rho^3)S(ad^2) + \\ &\quad + (\rho + 2\rho^2)S(ac^2) + (\rho^3 + 2\rho)S(ae^2) \\ &\quad + (2 + 2\rho^2 + 2\rho^3)S(abd) + (2 + 2\rho + 2\rho^4)S(abc) \\ &= 11(-2\rho + 2\rho^2 + \rho^3).\end{aligned}$$

In these equations the symbols $S(...)$ denote the sums taken over all expressions which result from the argument after applying the cyclic permutation $(abdce)$ and its powers to that argument.[18] For the radicals in Eqn. (45) we obtain the relation

$$\left(\sqrt[5]{V^{(1)}}\right)^2 \cdot \sqrt[5]{V^{(4)}} = 11(-2\rho + 2\rho^2 + \rho^3). \tag{51}$$

The equations (50) and (51) taken together show us that all radicals $\sqrt[5]{V^{(i)}}$ are uniquely determined by the value of $\sqrt[5]{V^{(1)}}$. Hence the formula (45) yields five values of x only as it should be expected.

2.4. *"Vandermonde's Condition"*

In § § VI, XXXVI Vandermonde affirms to the reader that for primes $n = 2m+1$ the auxiliary equation of degree m associated with $x^n - 1 = 0$ "can always be solved easily". It seems he formed this opinion on the examples for $n \leq 11$. Was Vandermonde really right ? How could he himself have generalized his method from $n = 11$ to other primes as well ? We are going to expose his basic ideas in as much generality as is possible without abandoning the framework of his treatise.

Let $n = 2m+1$ be an odd prime and r be a primitive nth root of unity. We define m quantities in an *upper numbering* as follows:

$$x^{(k)} := -r^k - r^{-k} \quad (1 \leq k \leq m)$$

and some of them in a *lower numbering*:

$$x_i := -r^{(2^i)} - r^{-(2^i)} \quad (0 \leq i).$$

Now it is evident that Vandermonde's method hinges on the relations (26). In our setting we have $x_i^2 = -x_{i+1} + 2$. From there we conclude that Vandermonde's method will work *if and only if the x_i will run through the whole set $\{x^{(1)}, \dots, x^{(m)}\}$*. This can happen *if and only if there are precisely m unordered pairs* $\{2^i \bmod n, -2^i \bmod n\}$. The latter condition is in turn equivalent to the condition that m be the least positive exponent k with $2^k \equiv \pm 1 \pmod n$ or, in other words, $2^m \equiv \pm 1 \pmod n$ and $2^k \not\equiv \pm 1$

[18] The equality $(-2\rho + 2\rho^2 + \rho^3)(-2\rho^4 + 2\rho^3 + \rho^2) = 11$ shows us that the complex number $(\Delta^{(1)})^2 \cdot \Delta^{(4)}$ has the correct modulus $\left(\sqrt{11}\right)^3$. The factorization of 11 can be refined such that one obtains the unique prime decomposition of 11 in the euclidean domain $\mathbb{Z}[\rho]$:

$$11 = (\rho^2 + \rho^3 - \rho^4)(1 + \rho^2 - \rho^3)(\rho^3 + \rho^2 - \rho)(1 + \rho^3 - \rho^2).$$

(mod n) for $0 < k < m$. For sake of brevity we shall call this *"Vandermonde's condition"*.

The special prime numbers $n = 2m + 1$ such that m is also a prime number satisfy this condition. Indeed for the prime number 5 we have $2^2 \equiv -1 \pmod 5$. Further for primes $n = 2m + 1 > 5$ the primes m are odd. In this case we can prove even more. For *every* g, $2 \le g \le m$, either $g \bmod n$ or $(-g) \bmod n$ is a primitive root mod n. Indeed, $g^2 \equiv 1 \pmod n$ is excluded, thus $g \bmod n$ has order m or $2m$. If $g^m \equiv 1 \pmod n$ then $(-g)^m = -g^m \equiv -1 \pmod n$ and $(-g) \bmod n$ is a primitive root mod n. [19] In particular, if 2 is a primitive root mod n then we have $2^m \equiv -1 \pmod n$, and a congruence $2^k \equiv \pm 1 \pmod n$ with $0 < k < m$ is excluded. On the other hand, if (-2) is a primitive root mod n then we have $(-2)^m \equiv -1 \pmod n$, hence $2^m \equiv 1 \pmod n$, and a congruence $2^k \equiv \pm 1 \pmod n$ with $0 < k < m$ is impossible since m is odd.

Thus the class of primes just discussed contains 25 primes < 1000, namely

$$5, 7, 11, 23, 47, 59, 83, 107, 167, 179, 227, 263, 347,$$
$$359, 383, 467, 479, 503, 563, 587, 719, 839, 863, 887, 983.$$

Besides we have 75 further primes < 1000

$$3, 13, 19, 29, 37, 53, 61, 67, 71, 79, 101, 103, 131, 139, 149,$$
$$163, 173, 181, 191, 197, 199, 211, 239, 269, 271, 293, 311,$$
$$317, 349, 367, 373, 379, 389, 419, 421, 443, 461, 463, 487,$$
$$491, 509, 523, 541, 547, 557, 599, 607, 613, 619, 647, 653,$$
$$659, 661, 677, 701, 709, 743, 751, 757, 773, 787, 797, 821,$$
$$823, 827, 829, 853, 859, 877, 883, 907, 941, 947, 967, 991$$

for which Vandermonde's condition can be checked immediately. [20] Among the primes < 100 only the numbers 17, 31, 41, 43, 73, 89, 97 do not satisfy Vandermonde's condition. In particular, 17 is missing there. Nevertheless one can construct the regular 17-gon by ruler and compass following Vandermonde's ideas in a slightly *modified* way as we will show in our last section. A further remark refers to the so-called Fermat primes of the form $F_k = 2^{(2^k)} + 1$ like 3, 5, 17, 257 and 65 537 (further instances are not

[19] In his paper [Loewy 1918], p. 192, Loewy overlooked this fact and referred unnecessarily to [Abel 1829], § 3, theorem, in order to underpin his arguments.

[20] This sequence of primes was very quickly computed with the help of the tables [Jacobi 1956]. For every prime $n < 1000$ these tables contain the least positive primitive root mod n g, the map $i \mapsto N \equiv g^i \pmod n$ (table of "numeri"), the inverse map $N \mapsto ind(N)$ (table of "indices") and two further tables, $ind(x) \mapsto ind(x+1)$ and $ind(x) \mapsto ind(x-1)$.

known until now). For $k > 1$ we have $2^{(2^k)} = (-2)^{(2^k)} \equiv -1 \pmod{F_k}$ where the exponent 2^k is less than $1/2 \cdot (F_k - 1) = 2^{(2^k - 1)}$. In other words, for a Fermat prime with $k \geq 2$ Vandermonde's method would never work immediately.

2.5. *"Pre-Gaussian" Cyclotomy Theory*

Despite of all limitations in Vandermonde's work we feel entitled to agree with Alfred Loewy (1873-1935) who said that

> "Vandermonde was the first who saw and carried out the right method to solve the equation $x^{2m+1} = 1$ for prime numbers $2m + 1$", [Loewy 1918], p. 194. [21]

Let $n = 2m + 1$ be a prime *without further restrictions for the time being* and r a primitive nth root of unity. Here we are going to develop a fragment of cyclotomy theory entirely based on the ideas and tools of Euler, Vandermonde and Lagrange. In short we will expose a kind of "pre-Gaussian" cyclotomy theory. Throughout we will abstain from the conscious use of primitive roots mod n for arbitrary primes n, in other words, from the use of a powerful tool due only to Gauss.

With the notations just introduced above the m quantities $x^{(1)} = -r - r^{-1}, \dots, x^{(m)} = -r^m - r^{-m}$ in the upper numbering satisfy an equation

$$\begin{aligned} F(x) = & x^m - x^{m-1} - (m-1)x^{m-2} + \\ & + (m-2)x^{m-3} + \frac{(m-2)(m-3)}{1 \cdot 2} x^{m-4} - \cdots \\ = & 0, \end{aligned} \tag{52}$$

[Vandermonde 1774], § VI. All $x^{(i)}$'s are power sums of r, r^{-1}, therefore polynomials of each other with rational integer coefficients since all nth roots of unity $r, r^2, \dots, r^{2m} \neq 1$ are primitive and can replace each other. The nth roots of unity $r, r^{-1}, r^2, r^{-2}, \dots, r^m, r^{-m}$ in this order just are the roots of m quadratic equations

$$y^2 + x^{(i)} \cdot y + 1 = 0 \qquad (1 \leq i \leq m), \tag{53}$$

this means

$$(x^{(i)})^2 - 4 = (r^i - r^{-i})^2, \; y = \frac{-x^{(i)} \pm \sqrt{(x^{(i)})^2 - 4}}{2} = r^i \text{ or } r^{-i}. \tag{54}$$

[21] *Vandermonde hat demnach, wie man wohl sagen kann, als erster die richtige Methode zur Auflösung der Gleichung $x^{2m+1} = 1$ für primzahliges $2m + 1$ erkannt und durchgeführt.*

Conversely, assume for the moment that only the equations (52) and (53) would be given. Denote for any i the solutions of (53) by r, r^{-1}. Then $x^{(i)} = -r - r^{-1}$. Inserting this equality in Eqn. (52) we will get

$$\frac{r^n - 1}{r - 1} = r^{n-1} + r^{n-2} + \ldots + r + 1 = 0.$$

This means that r, r^{-1} really are nth roots of unity and the m disjoint pairs $\{r^k, r^{-k}\}$ exhaust all nth roots of unity $\neq 1$.

Moreover we observe that for any pair (i, j) of indices the quotient

$$\frac{r^i - r^{-i}}{r^j - r^{-j}} = \frac{(r^i - r^{-i})(r^j - r^{-j})}{(r^j - r^{-j})^2} = \frac{(r^i - r^{-i})(r^j - r^{-j})}{(x^{(j)})^2 - 4}$$

is *symmetric* in r, r^{-1}, hence a rational function of $r + r^{-1} = -x^{(1)}$ with rational coefficients. This means that any two square roots $\sqrt{(x^{(i)})^2 - 4}$, $\sqrt{(x^{(j)})^2 - 4}$ differ from each other only by a rational function of $x^{(1)}$ with rational coefficients. In particular, every nth root of unity is a rational function (with rational coefficients) of $x^{(1)}$ and a *single* quadratic radical $\sqrt{(x^{(i)})^2 - 4}$ where i can be chosen arbitrarily. Anticipating some further considerations, we inform the reader that below, for odd m, this result will be reinforced considerably; insofar as in this case the radicals $\sqrt{(x^{(i)})^2 - 4}$ can be replaced by $\sqrt{-n}$. This explains Euler's successful treatment of the 7th roots of unity.

Euler's equation (11) can also be generalized to the equation satisfied by the $2m$ quantities $\pm\sqrt{(x^{(i)})^2 - 4}$. The polynomial

$$\begin{aligned} G(x) &:= \prod_{i=1}^{m} \left(x - \sqrt{(x^{(i)})^2 - 4}\right)\left(x + \sqrt{(x^{(i)})^2 - 4}\right) \\ &= \prod_{i=1}^{m} (x^2 - (x^{(i)})^2 + 4) \end{aligned} \tag{55}$$

has coefficients which are symmetric in $x^{(1)}, \ldots, x^{(m)}$ and, therefore, rational integers. Thus $G(x)$ shows the desired properties.

Furthermore we have

$$(r^i - r^{-i})^2 = -(1 - r^{2i})(1 - r^{-2i}) \tag{56}$$

for all exponents i. Multiplying all these m equations we obtain

$$\begin{aligned} \left(\prod_{i=1}^{m}(r^i - r^{-i})\right)^2 &= (-1)^m \cdot \prod_{i=1}^{m}(1 - r^{2i})(1 - r^{-2i}) \\ &= (-1)^m \cdot (x^{2m} + x^{2m-1} + \cdots + x + 1)|_{x\mapsto 1} \\ &= (-1)^m \cdot n \end{aligned} \tag{57}$$

since $r^2, r^{-2}, r^4, r^{-4}, \dots, r^{2m}, r^{-2m}$ are all nth roots of unity $\neq 1$. Taking the square root and using Eqns. (54) we see that

$$\prod_{i=1}^{m} \sqrt{(x^{(i)})^2 - 4} = \pm \prod_{i=1}^{m} (r^i - r^{-i}) = \pm\sqrt{(-1)^m \cdot n}\,. \tag{58}$$

The last equality shows us that *the square root* $\sqrt{(-1)^m \cdot n}$ *is a linear combination of the nth roots of unity with rational integer coefficients.* This remarkable (and momentous) fact could very well have been proved by Euler, Lagrange or Vandermonde but it wasn't. The case $n = 3$ is obvious whereas the cases $n = 5, 7$ are settled implicitly in the calculations of the nth roots of unity with Euler, Lagrange and Vandermonde. The proof in the general case is due to Gauss (whose proof is different from the one given here), [Gauss 1801], art. 356.

A closer inspection exhibits that the product $\prod_{i=1}^{m}(r^i - r^{-i})$ has the form of a polynomial $H(r, r^{-1})$ in r, r^{-1} with rational integer coefficients and the property $H(r^{-1}, r) = (-1)^m \cdot H(r, r^{-1})$. From there it follows that for *even* m $H(r, r^{-1})$ is symmetric in r, r^{-1}, hence a polynomial in $r + r^{-1} = -x^{(1)}$ and $r \cdot r^{-1} = 1$ with rational integer coefficients. In other words, *for even m the quadratic radical* $\sqrt{(-1)^m \cdot n} = \sqrt{n}$ *is already a polynomial in* $x^{(1)}$ *or in any other* $x^{(i)}$. For *odd m* we have $H(r^{-1}, r) = -H(r, r^{-1})$ and $H(r, r^{-1})$ takes the special form

$$\begin{aligned}\pm\sqrt{-n} = H(r, r^{-1}) &= (r - r^{-1}) \cdot S(r, r^{-1}) = \\ &= (r - r^{-1}) \cdot P(x^{(1)}) = \pm\sqrt{(x^{(1)})^2 - 4} \cdot P(x^{(1)})\end{aligned}$$

with a symmetric polynomial S and some polynomial P. From there we deduce that the two square roots $\sqrt{-n}$ and $\sqrt{(x^{(1)})^2 - 4}$ differ from each other only by a rational function of $x^{(1)}$. In summary, *for odd m the nth roots of unity are rational functions of* $x^{(1)}$ *(or of any other* $x^{(i)}$*) and* $\sqrt{-n}$ *with rational coefficients.* In a more explicit manner we can write

$$r^i - r^{-i} = \pm \frac{\sqrt{-n}}{\prod_{j \neq i}(r^j - r^{-j})}.$$

The numerator of this quotient is symmetric in r, r^{-1}, therefore a polynomial in $x^{(1)}$. Together with the definition $x^{(i)} := -r^i - r^{-i}$ this gives us the desired result since $x^{(i)}$ is a polynomial in $x^{(1)}$.

2.6. *"Vandermonde's Condition" Again*

Now in order to treat $x^n - 1 = 0$ or, more precisely, the equation (52) using Vandermonde's ideas one has to impose on n the restriction that

$2^m \equiv \pm 1 \pmod{n}$ and $2^k \not\equiv \pm 1 \pmod{n}$ for $0 < k < m$. The series of the primes in question contains 100 primes < 1000 and begins as follows:

$$3, 5, 7, 11, 13, 19, 23, 29, 37, 47, 53, 59, 61, 67, 71, 79, 83, \ldots$$

Our aim is to solve Eqn. (52) by mth roots of unity and radicals of index m. Under Vandermonde's condition we can switch to the lower numbering and on the analogy of (26)-(31) we have the "cyclic" relations

$$x_i^2 = -x_{i+1} + 2 \qquad (1 \leq i \leq m-1), \qquad x_m^2 = -x_1 + 2$$

as well as $m(m-1)/2$ formulas for the products $x_i \cdot x_j \; (i < j)$. All these relations are only permuted with each other when one performs the cyclic permutation $(x_1 \ldots x_m)$. Let ρ be a primitive mth root of unity. Then one has to form the expressions

$$\Delta^{(i)} = x_1 + \rho^i \cdot x_2 + \cdots + \rho^{(m-1)i} \cdot x_m \qquad (1 \leq i \leq m-1)$$

and their mth powers $V^{(i)}$ which turn out to be invariant under the permutation $(x_1 \ldots x_m)$. Now we can apply the Lemma and see the $V^{(i)}$'s to be linear combinations of the mth roots of unity with rational integer coefficients. Unfortunately, Vandermonde himself proves this fact only for $n = 5, 11$ in § § XXIII, XXXV. That is why here we have interpolated the Lemma in our comment on Vandermonde's text.

All products $\Delta^{(i)} \cdot \Delta^{(m-i)}$ are also invariant under the permutation $(x_1 \ldots x_m)$, and it is not difficult to expand them in sums of $(x_1 \ldots x_m)$-invariant terms which can be calculated separately.

$$\begin{aligned}\Delta^{(i)} \cdot \Delta^{(m-i)} &= (x_1^2 + \cdots + x_m^2) \\ &\quad + \sum_{j=1}^{[\frac{m-1}{2}]} (\rho^{ji} + \rho^{-ji}) \cdot (x_1 x_{j+1} + x_2 x_{j+2} + \cdots + x_m x_j) \\ &\quad + \frac{1}{2}(1 + (-1)^m)(-1)^i (x_1 x_{[\frac{m}{2}]} + \cdots) \\ &= (-1 + 2m) \; + \; (-1)(-2) = n.\end{aligned}$$

These equalities show us that all resolvents $\Delta^{(i)}$ are $\neq 0$. More precisely, $\Delta^{(m-i)}$ is the complex conjugate of $\Delta^{(i)}$ since $x_1, \ldots, x_m$ are real numbers, hence $\Delta^{(i)}$ is a complex number of modulus $\sqrt{n}$.

On the analogy of (45) we obtain "explicit" solutions of (26). First of all, with regard to $x_{i+1} = -x_i^2 + 2$ we see that

$$x_{i+1}^2 - 4 = x_i^2 \cdot (x_i^2 - 4), \; \left(\frac{-x_i \pm \sqrt{x_i^2 - 4}}{2} \right)^2 = \frac{-x_{i+1} \mp \sqrt{x_{i+1}^2 - 4}}{2}.$$

Therefore, beginning with the solutions r, r^{-1} of $y^2 + x_1 \cdot y + 1 = 0$ we can arrange all solutions of the quadratic equations in the sequence $r, r^{-1}, r^2, r^{-2}, r^4, r^{-4}, \ldots$.

The polynomial (55) can be specified very easily using Eqn. (52). With respect to the quantities $\pm\sqrt{x_i^2 - 4} = \pm\sqrt{-x_{i+1} - 2}$ we put $x^2 := -x_{i+1} - 2$ and form the polynomial

$$(x^2 + x_1 + 2)(x^2 + x_2 + 2) \cdots (x^2 + x_m + 2) =: G(x)$$

which can be derived immediately from (52):

$$G(x) = (-1)^m \cdot F(-x^2 - 2) = (-1)^m \cdot F(-x^2 - 2). \qquad (59)$$

Here $F(x)$ denotes the polynomial in (52) with the zeroes $x_1, \ldots, x_m$.

2.7. *7th Roots of Unity*

To round off this section on Vandermonde we would like to treat the equation $x^n - 1 = 0$ for $n = 7$ following Vandermonde's ideas whereas the cases $n = 3, 5$ are left to the reader. With a primitive 7th root of unity r and a primitive 3rd root of unity $\rho = \frac{-1+\sqrt{-3}}{2}$ we define

$$x_1 = -r - r^6,\ x_2 = -r^2 - r^5,\ x_3 = -r^4 - r^3,$$
$$\Delta^{(1)} = x_1 + \rho \cdot x_2 + \rho^2 \cdot x_3, \qquad \Delta^{(2)} = x_1 + \rho^2 \cdot x_2 + \rho \cdot x_3.$$

In our case Eqn. (52) takes the form

$$F(x) := x^3 - x^2 - 2x + 1 = 0$$

which of course is nothing but Euler's equation (8). The quantity $\Delta^{(1)} \cdot \Delta^{(2)}$ is a symmetric polynomial in x_1, x_2, x_3:

$$\Delta^{(1)} \cdot \Delta^{(2)} = x_1^2 + x_2^2 + x_3^2 - x_1 \cdot x_2 - x_2 \cdot x_3 - x_3 \cdot x_1 = 7.$$

In § III Vandermonde indicates the expansion of $(\Delta^{(1)})^3$ in $(x_1 x_2 x_3)$-invariant components which reads as follows:

$$(\Delta^{(1)})^3 = (x_1^3 + x_2^3 + x_3^3) \quad + \quad 3\rho(x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_1) + \\ + \quad 3\rho^2(x_1 x_2^2 + x_2 x_3^2 + x_3 x_1^2) \quad + \quad 6 x_1 x_2 x_3.$$

Now by means of the specific relations

$$x_1 x_2 = -x_1 - x_3, \qquad x_2 x_3 = -x_2 - x_1, \qquad x_3 x_1 = -x_3 - x_2 \qquad (60)$$

one calculates each component of $(\Delta^{(1)})^3$ and $(\Delta^{(2)})^3$, resp., rather easily and obtains

$$(\Delta^{(1)})^3 = (-7) \cdot \frac{1 + 3\sqrt{-3}}{2}, \qquad (\Delta^{(2)})^3 = (-7) \cdot \frac{1 - 3\sqrt{-3}}{2}.$$

This way Euler's solutions of (8) occur here again. In the next step Euler's equation (11) satisfied by $\pm\sqrt{-x_1-2}, \pm\sqrt{-x_2-2}, \pm\sqrt{-x_3-2}$ shows up when we use Eqn. (59):

$$0 = G(x) = -F(-x^2-2) = x^6 + 7x^4 + 14x^2 + 7.$$

Furthermore, our previous general considerations show that the 7th roots of unity are rational functions of $\sqrt{-7}, \sqrt{-3}$ (via the 3rd roots of unity) and a single cubic radical. Thus the results of Euler's calculations are confirmed. We see that the approach to $x^7 - 1 = 0$ à la Vandermonde is at least conceptually superior to Euler's due to the use of the relations (60).

It is to be regretted that Vandermonde did not consider the case $n = 13$ in detail. In this case he could have observed the phenomenon of "periods of 4 or 6 terms" according to Eqn. (19) not yet occuring in the former cases $n = 7, 11$. Let r be a primitive 13th root of unity and $\rho = \frac{1+\sqrt{-3}}{2}$ be a primitive 6th root of unity. In our previous notations we obtain six resolvents $\Delta^{(1)}, \ldots, \Delta^{(6)}$ among which there are the quantities

$$\Delta^{(2)} = (x_1 + x_4) + \rho^2(x_2 + x_5) + \rho^4(x_3 + x_6),$$
$$\Delta^{(4)} = (x_1 + x_4) + \rho^4(x_2 + x_5) + \rho^2(x_3 + x_6)$$

and

$$\Delta^{(3)} = (x_1 + x_3 + x_5) + \rho^3(x_2 + x_4 + x_6).$$

In these expressions the terms are summed according to the powers of ρ, thus new combinations of the 13th roots of unity show up like the "periods of 4 terms"

$$x_1 + x_4 = r^{(2^0)} + r^{-(2^0)} + r^{(2^3)} + r^{-(2^3)},$$
$$x_2 + x_5 = r^{(2^1)} + r^{-(2^1)} + r^{(2^4)} + r^{-(2^4)},$$
$$x_3 + x_6 = r^{(2^2)} + r^{-(2^2)} + r^{(2^5)} + r^{-(2^5)},$$

and the "periods of 6 terms"

$$x_1 + x_3 + x_5 = r^{(2^0)} + r^{-(2^0)} + r^{(2^2)} + r^{-(2^2)} + r^{(2^4)} + r^{-(2^4)},$$
$$x_2 + x_4 + x_6 = r^{(2^1)} + r^{-(2^1)} + r^{(2^3)} + r^{-(2^3)} + r^{(2^5)} + r^{-(2^5)}.$$

The reader should notice that here in each sum the total number of terms is counted as the number of roots of unity involved in the sum but not with respect to the pairs (r^i, r^{-i}).

3. Gauss

How did Gauss come to study the cyclotomy? Could he have been influenced by Vandermonde? Before discussing these questions we are going to look at his biography and the mathematical writings to which he had access in his early years. Gauss spent his childhood and early youth in Braunschweig (Brunswick) where in 1792 at age of 15 he entered the Collegium Carolinum which was a semi-academic science-oriented institution preparing young men for a career as well-qualified loyal bureaucrats and military personnel. The library of the Carolinum was unusually good and gave Gauss access to many of the best and most advanced textbooks in mathematics and the sciences and to classics like Isaac Newton's (1643-1727) writings and John Wallis's (1616-1703) "A Treatise of Algebra". But it should be stressed that rather recent investigations into the extant catalogues of the Carolinum library have shown this library did *not* have writings of Pierre de Fermat (1601/1607? - 1665), Euler, Waring, Vandermonde, Lagrange and Adrien-Marie Legendre (1752-1833), let alone the publications of the learned academies in Berlin, Paris, St. Petersburg and London, [Küssner 1979], pp. 32-40. In October 1795 Gauss moved to Göttingen and enrolled as a student of classical philology and mathematics at the university. Gauss was a zealous user of the excellent Göttingen library which was one of the best all over Europe in its time. At last he could study the masters like Euler and Lagrange and their treatises on mathematics and the sciences in the publications of the European academies. G. Waldo Dunnington (1906-1974) compiled an impressive record of the books which Gauss had borrowed from the library, though in this record the summer semester 1796 is missing, [Dunnington 2004], pp. 398-404. Little is known about Gauss's first semester but beginning with March 30, 1796, we have his invaluable mathematical diary (*Notizenjournal*) where a great many of entries refer to number theory and algebra, [Gauss 1796-1814].

3.1. *The 17-Gon*

It is well-known and often quoted that Gauss's first entry in his diary reads as follows:

> "The principles upon which the division of the circle depends, and geometrical divisibility of the same into seventeen parts, etc.", [Gauss 1796-1814].[22]

After almost three weeks, on April 18, 1796, he had written a short announcement of his discoveries in cyclotomy whereas this communication was published under the date of June 1, 1796, [Gauss 1863-1933], vol. I, p. 3. Gauss emphasized the then completely unexpected constructibility of

[22] English translation quoted after [Dunnington 2004], p. 469. Original in Latin: *Principia quibus innititur sectio circuli, ac divisibilitas eiusdem geometrica in septemdecim partes etc.*.

the regular 17-gon and some other polygons by ruler and compass. Moreover, he wrote that these results are only “a corollary of a theory which is not yet complete”. The “complete theory” was eventually exposed in his first major opus *Disquisitiones arithmeticae* (1801), more precisely, in the seventh section of this work, [Gauss 1801]. Unfortunately, little is known about when, how and why Gauss began to study the problems of cyclotomy. He was hardly led to his theory by geometrical problems, probably algebraic and arithmetical questions got the theory going, cf. [Bachmann 1911], pp. 32-40. This opinion is backed by Gauss's letter to the mathematician, physicist and astronomer Christian Ludwig Gerling (1788-1864) from January 6, 1819, where he said that as early as in his first semester in Göttingen he had obtained an important result in cyclotomy *before* he discovered the constructibility of the 17-gon, [Gauss 1863-1933], vol. X/1, p. 125. The result in question concerns the equation $x^n - 1 = 0$ for prime numbers $n = 2m + 1$. Gauss had seen that it would be appropriate to subdivide the nth roots of unity ($\neq 1$) $r, r^2, \ldots, r^{n-1}$ into *two groups* depending on whether the exponent with r be a quadratic residue mod n or a quadratic non-residue mod n. The sum over either group is nothing but a period of m terms in his later terminology. These sums

$$\tau_1 = \sum_{i=1}^{m} r^{i^2}, \qquad \tau_2 = \sum_{i=1}^{m} r^{h \cdot i^2} \tag{61}$$

where h denotes a quadratic non-residue mod n were thoroughly studied by Gauss, and they turned out to be quadratic irrationalities. Especially, he found the quadratic equation

$$x^2 + x - (-1)^m \frac{m}{2} = 0$$

satisfied by the two sums. Moreover, he could prove that

$$\frac{4(x^n - 1)}{x - 1} = Y(x)^2 - (-1)^m n Z(x)^2$$

for some polynomials Y, Z with rational integer coefficients. To these results he alluded in his letter to Gerling, and this means he had discovered no less than the close and extremely important connections between cyclotomy, quadratic irrationalities and quadratic residues! These facts fit very well with Gauss's studies on quadratic residues at that time and his efforts to prove the reciprocity law of quadratic residues. Eventually, his second entry in the diary on April 8, 1796, testified the first complete proof of that fundamental law. A systematic exposition of his first results in cyclotomy was given by Gauss in 1801, as he indicated to Gerling, [Gauss 1801], art. 124, 356, 357.

Now one could ask, how did Gauss then proceed to discover the constructibility of the 17-gon? In the case of $x^{17} - 1 = 0$ we have the two sums τ_1, τ_2 of 8 terms each according to Eqn. (61). We share Paul Bachmann's (1837-1920) opinion that most probably in a flash of genius Gauss had seen how further to subdivide the sums of 8 terms each into 2 suitable sums of 4 terms each, further into sums of 2 terms each and to end up with the 17th roots of unity, [Bachmann 1911], p. 40. The principle of the iterated subdivisions is of arithmetical nature, and that is what Gauss had called "the interconnection of all roots on arithmetical grounds" (*Zusammenhang aller Wurzeln unter einander nach arithmetischen Gründen*) in his letter to Gerling, see also [Reich 2003]. The basic fact which Gauss had used was the existence of primitive roots mod n for any prime number n, i. e. the existence of such residue classes g mod n that the powers $1, g, g^2, g^3, \dots, g^{n-2}$ run through all residue classes mod n. This fact was first formulated by Euler and Johann Heinrich Lambert (1728-1777) but its first complete proof is due to Gauss, [Gauss 1801], art. 55-56 (with comments on Euler and Lambert). Those primitive roots mod n allowed Gauss to order the nth roots of unity in the series $r, r^g, r^{(g^2)}, r^{(g^3)}, \dots, r^{(g^{n-2})}$ where each member just is the gth power of the preceding one. It is this series which gives us the "right" order of the roots of unity for *any* prime number! For $n = 17$ one can choose $g = 3$ mod 17 whereas Vandermonde's methods do not work because $2^4 \equiv -1 \pmod{17}$. In the general case let $n - 1 = e \cdot f$ be a factorization. A *period of f terms* is determined by a geometric sub-progression of 1, g, g^2, g^3, …, g^{n-2} mod n with quotient g^e mod n. This means a period of f terms is a sum

$$\eta_h = r^h + r^{hg^e} + r^{hg^{2e}} + \cdots + r^{hg^{(f-1)e}} \tag{62}$$

where h denotes an arbitrary exponent $\not\equiv 0 \pmod{n}$. There is every reason to believe that Gauss first developed the theory of periods and only after that the solution of $x^n - 1 = 0$ by means of Lagrange (-Vandermonde) resolvents. One should notice that only half of a year later, on September 17, 1796, Gauss made a note in his diary of the expressions coincident with the Lagrange (-Vandermonde) resolvents. Like Vandermonde he hoped then for "a new method by means of which it will be possible to investigate, and perhaps try to invent, the universal solution of equations."[23] The application to the cyclotomic equations is mentioned in January 1797 and July 1797.

[23] English translation quoted after [Dunnington 2004], p. 472. Complete original in Latin. *Nova methodus qua resolutionem aequationum universalem investigare forsitanque invenire licebit. Scil[icet] transm[utetur] aeq[uatio] in aliam, cuius radices* $\alpha\rho' + \beta\rho'' + \gamma\rho''' + \cdots$, *ubi* $\sqrt[n]{1} = \alpha, \beta, \gamma$ *etc. et n numerus aequationis gradum denotans.*

We have no evidence whatsoever that Vandermonde could have known or used the existence of primitive roots mod n in the general case. At least this marks the decisive difference between him and Gauss. Even in the simplest cases $n = 5, 11$ Vandermonde did not mention nor use that 2 is a primitive root mod n. Nevertheless Lebesgue made the assertion that Gauss followed Vandermonde "step by step" in his exposition of the cyclotomy theory but there "he perfected Vandermonde very much". For instance, as to the periods, Lebesgue says that "the method is that of Vandermonde, the results are those of Gauss", [Lebesgue 1940], p. 38. We maintain that this judgement goes too far since Vandermonde like Euler confined himself to the sums $r+r^{-1}$, i. e. to periods of 2 terms in Gauss's sense, the introduction of which is clearly suggested by the reciprocal equations without any further sophistication.

Let us return for a while to the regular 17-gon as promised above. Lebesgue claimed that Vandermonde "had not understood the full importance of his method", and he attempted to convince the reader that the constructibility of the 17-gon could have been derived rather transparently from Vandermonde's method [Lebesgue 1940], p. 42. For "that method would have given Vandermonde the roots of $x^{17} - 1 = 0$ by means of radicals of index 16, therefore, by means of superposition of square roots". Apparently, here Lebesgue alluded to Eqn. (22) applied to $\frac{x^{17}-1}{x-1} = x^{16} + x^{15} + \cdots + x + 1 = 0$, in other words to a representation of the 17th roots of unity by sums of radicals $\sqrt[16]{V^{(i)}}$. But the formation of the $V^{(i)}$'s is inseparably tied up to a suitable order of the 17th roots of unity $\neq 1$. Otherwise one could not prove that the $V^{(i)}$'s would actually be linear combinations of the 16th roots of unity with rational coefficients. Hence one is in urgent need of a primitive root mod 17 which cannot be found with Vandermonde.

This situation raises a question: are there ways different from Lebesgue's which do prove the constructibility of the regular 17-gon à la Vandermonde? Our answer will be *affirmative*, and we are going to pursue such a way. Of course Vandermonde's *Mémoire* suggests to us to start with Eqn. (52) of degree 8

$$x^8 - x^7 - 7x^6 + 6x^5 + 15x^4 - 10x^3 - 10x^2 + 4x + 1 = 0 \qquad (63)$$

which has the roots $-r^i - r^{-i}, (1 \leq i \leq 8)$, for a primitive 17th root of unity r. We shall use a suitable lower numbering of these roots which looks as follows:

$$\begin{aligned} x_1 &= -r - r^{-1}, \quad x_2 = -r^2 - r^{-2}, \\ x_3 &= -r^4 - r^{-4}, \ \ x_4 = -r^8 - r^{-8}, \end{aligned} \qquad (64)$$

$$x_5 = -r^6 - r^{-6},\ x_6 = -r^{12} - r^{-12},$$
$$x_7 = -r^7 - r^{-7},\ x_8 = -r^3 - r^{-3}. \tag{65}$$

For the first line we obtain

$$x_{i+1} = -x_i^2 + 2\ (1 \le i \le 3), \qquad x_1 = -x_4^2 + 2, \tag{66}$$

whereas the second line gives us

$$x_{i+1} = -x_i^2 + 2\ (5 \le i \le 7), \qquad x_5 = -x_8^2 + 2. \tag{67}$$

Moreover, we have

$$x_1 + x_2 + \cdots + x_7 + x_8 - 1 = 0. \tag{68}$$

Obviously these relations correspond to Eqns. (26) and (31). All of the 28 products $x_i x_j, (1 \le i < j \le 8)$, have the form $(-x_k - x_l)$ which follows immediately from the definitions. But these products mingle the two lines (64) and (65) with each other, for instance, one has $x_1 x_2 = -x_1 - x_8, x_5 x_6 = -x_5 - x_1$. Thus there is no complete analogue to (27)-(30). Moreover, we can jump from the first group (64) to the second one (65) according to

$$x_{i+4} = -r^{6i} - r^{-6i} = -x_i^6 + 6x_i^4 - 9x_i^2 - 6 \qquad (1 \le i \le 4) \tag{69}$$

and it is possible to jump back according to

$$x_{i-4} = -r^{3i} - r^{-3i} = x_i^3 - 3x_i \qquad (5 \le i \le 8). \tag{70}$$

Still the two quadruples (x_1, x_2, x_3, x_4) and (x_5, x_6, x_7, x_8) are each accessible to Vandermonde's basic ideas. One can factorize Eqn. (63) of degree 8 in two equations each of degree 4 with the roots x_1, x_2, x_3, x_4 and x_5, x_6, x_7, x_8, resp. Calculating the coefficients of those equations comes down to forming the products $x_i x_j$ which are known from the definitions. The resulting equations are

$$\begin{aligned}&(x - x_1)(x - x_2)(x - x_3)(x - x_4)\\ = \quad & x^4 + \tau_1 x^3 - (\tau_1 + 2)x^2 - (2\tau_1 + 3)x - 1 = 0\end{aligned} \tag{71}$$

and

$$\begin{aligned}&(x - x_5)(x - x_6)(x - x_7)(x - x_8)\\ = \quad & x^4 + \tau_2 x^3 - (\tau_2 + 2)x^2 - (2\tau_2 + 3)x - 1 = 0\end{aligned} \tag{72}$$

with the abbreviations

$$\begin{aligned}\tau_1 &= -x_1 - x_2 - x_3 - x_4\\ &= r + r^2 + r^4 + r^8 + r^9 + r^{13} + r^{15} + r^{16}\end{aligned} \tag{73}$$

and

$$\begin{aligned}\tau_2 &= -x_5 - x_6 - x_7 - x_8 \\ &= r^3 + r^6 + r^{12} + r^7 + r^{10} + r^5 + r^{11} + r^{14}.\end{aligned} \tag{74}$$

These quantities coincide with the ones in Eqn. (61) which is checked very easily. One calculates without difficulty

$$\tau_1 + \tau_2 = -1, \qquad \tau_1 \cdot \tau_2 = -4 \tag{75}$$

and this gives us immediately

$$\tau_1 = -1/2 + 1/2\sqrt{17}, \qquad \tau_2 = -1/2 - 1/2\sqrt{17}. \tag{76}$$

Thus the coefficients in the Eqns. (71) and (72) require the quadratic irrationality $\sqrt{17}$ only. Further each of those equations can be solved by Vandermonde's methods, and it remains for us to show that this could indeed be done by iterated square roots. Obviously it will suffice to solve one of the Eqns. (71), (72). To this end we fall back upon Vandermonde's solution (24) of quartic equations and obtain for Eqn. (71)

$$x = \frac{1}{4}\left[1 \quad + \quad \epsilon_1\sqrt{\alpha} \quad + \quad \epsilon_2\sqrt{\beta} \quad + \quad \epsilon_3\sqrt{\gamma}\right] \tag{77}$$

with $\epsilon_1, \epsilon_2, \epsilon_3 = \pm 1$, $\epsilon_1\epsilon_2\epsilon_3 = +1$, the three quantities

$$\begin{aligned}\alpha &= (x_1 + x_2 - x_3 - x_4)^2, \\ \beta &= (x_1 + x_3 - x_2 - x_4)^2, \\ \gamma &= (x_1 + x_4 - x_2 - x_3)^2\end{aligned}$$

and the condition

$$\sqrt{\alpha} \cdot \sqrt{\beta} \cdot \sqrt{\gamma} = 7\tau_1 + 12. \tag{78}$$

α, β, γ satisfy a cubic equation the coefficients of which are polynomials in τ_1 with rational integer coefficients with regard to (71). In our case the relations (61) allow us to apply the Lemma of our subsection 2.3 to the sequence (x_1, x_2, x_3, x_4). Of course the reference to the Lemma is unnecessary in principle but it tells us *what* we will have to calculate. In particular, we see that α, γ are permuted with each other under the cyclic permutation $(x_1x_2x_3x_4)$ whereas the quantities

$$\beta \quad \text{and} \quad \frac{\alpha - \gamma}{\sqrt{\beta}} = \frac{4(x_1 - x_3)(x_2 - x_4)}{x_1 + x_3 - x_2 - x_4}$$

are left fixed. Hence the latter ones are rational functions of τ_1 with rational coefficients. Straightforward calculations show us that

$$\alpha = A + B\sqrt{-\tau_1 + 8}, \quad \beta = -\tau_1 + 8, \quad \gamma = A - B\sqrt{-\tau_1 + 8}$$

with polynomials A, B in τ_1 with rational coefficients.

Thus the cubic equation with the roots α, β, γ turns out to be *reducible* if one admits coefficients of the form $s + t\sqrt{17}$ with rational numbers s, t. From Eqn. (78) it follows that $\sqrt{\alpha} \cdot \sqrt{\gamma}$ has the form $C + D\sqrt{-\tau_1 + 8}$ with polynomials C, D in τ_1 with rational coefficients. Hence $\sqrt{\gamma}$ can be written as $(E + F\sqrt{-\tau_1 + 8}) \cdot \sqrt{\alpha}$ where E, F denote some polynomials in τ_1 with rational coefficients.

In summary, in order to obtain x_1, x_2, x_3, x_4 we can make do with only the three square roots $\sqrt{17}$, $\sqrt{-\tau_1 + 8}$ and $\sqrt{\alpha} = \sqrt{A + B\sqrt{-\tau_1 + 8}}$. A 17th root of unity is obtained after solving a further quadratic equation the coefficients of which are polynomials in the square roots already constructed. This shows us that *all we need to produce the 17th roots of unity is to solve a chain of quadratic equations.* As a consequence it is possible to construct the regular 17-gon by ruler and compass.

Our last question concerning the 17-gon to be answered is: how to link our construction à la Vandermonde to Gauss's construction? [Gauss 1801, art. 365], [Reich 2003] The key to the answer is the union of the two sequences (64) and (65) in a *single* sequence induced by the iteration of the map $r \mapsto r^6$. This way we go forth and back between (64) and (65), and the new sequence of 8 terms will be:

$$x_1, \quad x_5, \quad x_2, \quad x_6, \quad x_3, \quad x_7, \quad x_4, \quad x_8. \tag{79}$$

We can do it this way since 6 mod 17 is a square root of 2 mod 17 and therefore turns out to be a primitive root mod 17, which can be verified easily. The exponents occuring in the first sequence (64) are just the quadratic residues mod 17 whereas the exponents occuring in the second sequence (65) are the quadratic non-residues mod 17. The x_i's are Gauss's periods of 2 terms × (-1). Moreover, the sequence (79) is a "cycle" insofar as the successor of every member x_i has the form $\vartheta(x_i) = -x_i^6 + 6x_i^4 - 9x_i^2 - 6$ with the polynomial $\vartheta(X) = -X^6 + 6X^4 - 9X^2 - 6$. The cycle is actually closed since $\vartheta(x_8) = x_1$. We remember that the sums $x_1 + x_2 + x_3 + x_4 = -\tau_1$ and $x_5 + x_6 + x_7 + x_8 = -\tau_2$ are the two Gaussian periods of 8 terms × (-1). The 4 sums $x_1 + x_3$, $x_2 + x_4$, $x_5 + x_7$, $x_6 + x_8$ do not occur explicitly in our considerations à la Vandermonde, but they coincide with the 4 Gaussian periods of 4 terms × (-1). Gauss's systematic use of these periods is a further advantage of his construction over our construction in addition to the conscious use of a primitive root mod 17. The successive subdivision of the periods into periods of fewer terms gives Gauss a clear guideline to obtain the quadratic equations to be solved. Indeed this allows him to avoid equations of degree 4 or 8 which occur in our approach to $x^{17} - 1 = 0$. Especially, the equalities

$$(x_1 + x_3)(x_2 + x_4) = (x_5 + x_7)(x_6 + x_8) = -1$$

give Gauss the equations

$$(x-(x_1+x_3))(x-(x_2+x_4)) = x^2+\tau_1 x - 1 = 0$$
$$(x-(x_5+x_7))(x-(x_6+x_8)) = x^2+\tau_2 x - 1 = 0.$$

As soon as the periods of 4 terms will be known one would be able to calculate the periods of 2 terms in view of the equtions

$$(x-x_1)(x-x_3) = x^2-(x_1+x_3)x-(x_6+x_8) = 0,$$
$$(x-x_2)(x-x_4) = x^2-(x_2+x_4)x-(x_5+x_7) = 0.$$

For further details of Gauss' cyclotomy theory we refer the reader, of course, to the *Disquisitiones Arithmeticae* and its important unfinished and posthumously published continuation *Disquisitionum circa aequationes puras ulterior evolutio*, [Gauss 1801], [Gauss 1863]. The structure of Gauss' theory is elucidated in Richard Dedekind's (1831-1916) excellent review of Bachmann's book [Bachmann 1872] where Dedekind, in particular, emphasized how important the concept of irreducibility is in this theory and the development of algebra following Gauss, [Dedekind 1873], see also [Neumann 2006].

3.2. *Gauss and Vandermonde*

Gauss happened to know at least Vandermonde's geometrico-topological paper *Remarques sur des problèmes de situation* (1771) and to think highly of it which is testified twice, namely by his letter to the physician and astronomer Wilhelm Olbers (1758-1840) on October 12, 1802, and a note dated from January 22, 1833, in his papers, see [Olbers 1900], p. 103, [Gauss 1863-1933], vol. V, p. 605, vol. X/2, Abhandl. 4, pp. 46-48, 58. As far as we know, nowhere else can we find any trace of Gauss's reading of Vandermonde. Neither Dunnington's list of books that Gauss borrowed from the Göttingen University Library during the years 1795-1798 nor Karin Reich's recent investigations into Gauss's relations with France give any further direct hints of his preoccupation with Vandermonde. [Dunnington 2004, pp. 398-404], [Reich 1996] Only Dunnington's list for the date of January 4, 1797, notices that Gauss borrowed Waring's *Meditationes Algebraicae* from the Göttingen University Library, [Dunnington 2004], p. 400. [24] Waring's book is not mentioned in Martha Küssner's monograph on Gauss's and his "world of books", [Küssner 1979]. Moreover, in the

[24] The Göttingen University Library has copies of the second (1770) and the third (1782) editions of Waring's book (third edition with the shelf mark 4 Math. II, 9069 < 3 >, HG-FB). As far as I could see Waring, unfortunately, did not indicate where Vandermonde's paper was published.

extant personal scientific library of Gauss (kept in the Gauss archives of Niedersächsische Staats- und Universitätsbibliothek Göttingen) there are no writings of Vandermonde.[25]

Though a special detail is of some interest to our considerations. Vandermonde's paper on "problems of situation" mentioned in the introductory sentence of this subsection was printed in one volume together with his algebraic *Mémoire*. Any user of this volume could hardly overlook Vandermonde's algebraic treatise, and the old Göttingen University Library does have a copy of that volume (with the shelf mark Phys. Math. III 2550). There is every reason to believe that this very copy was in Gauss's own hands. Thus one must assume that he became acquainted with Vandermonde's algebraic treatise not later than in 1802. Paul Stäckel (1862-1919), expert on all kinds of Gaussiana, discoverer of the *Notizenjournal* and co-editor of the *Werke*, even held the opinion that Gauss knew Vandermonde's *Mémoire* when he was writing his *Disquisitiones Arithmeticae*, see [Loewy], p. 195, [Gauss 1863-1933], vol. X/2, Abhandl. 4, p. 58.

Above all, we share Stäckel's view that Gauss in his theory of cyclotomy was then not influenced by Vandermonde. In favour of this view Stäckel refers the reader to Gauss's letter to Gerling in 1819, discussed in the preceding subsection. In our opinion the peculiar construction of the 17-gon based on Vandermonde's ideas could very well have been carried out by any mathematician who would have studied Vandermonde's *Mémoire* thoroughly and, first of all, who would have *asked* how to construct the 17-gon. But with Gauss the construction of the 17-gon appears to have been more a fortunate breakthrough on a broad background ("a corollary of an incomplete theory") than the final completion of a construction which Gauss would have striven for.

Already at several places in the present paper we discussed Lebesgue's attempts to make plausible direct links of Gauss to Vandermonde. In summary we could not find those attempts convincing since they do not fit the known sources.

A special comment should be given on the situation in 1808. In that year Lagrange published the new editon of his voluminous *Traité de la résolution des équations numériques de tous les degrés* where *inter alia* he reproduced at length Vandermonde's solution of $x^{11} - 1 = 0$ (though supplemented by the use of $\sqrt{-11}$), [Lagrange 1808], Note XIV, § § 28-36. On the other hand, in that same year Gauss was writing a manuscript on "pure equations" continuing the *Disquisitiones Arithmeticae* but left unfinished. [Gauss 1863] We know this detail from a letter of Gauss to Olbers from July 3, 1808, where Gauss acknowledged to have received a copy of Lagrange's treatise

[25] Communicated by Karin Reich to the author, e-mail from June 8, 2004.

and gave some critical comments on this work. At the latest at that time Gauss should have drawn his attention to Vandermonde's solution of $x^{11} - 1 = 0$. The more it is surprising that Gauss in his manuscript did *not* mention Vandermonde although there he also treated the equation $x^{11} - 1 = 0$ applying his own theory to it, [Gauss 1863], artt. 13, 17, see also [Bachmann 1872], pp. 96-98.

There remains the question: why did Gauss not quote Vandermonde's *Mémoire* in his published writings or in his papers? Our present knowledge of the diary, of his letters and other documents allows us to give one answer only: we will probably never know.

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Leonhard Euler: Life, Work and Legacy
Robert E. Bradley and C. Edward Sandifer (Editors)

Euler and Number Theory: A Study in Mathematical Invention

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1. Introduction

When discussing the history of mathematics, André Weil once said that "its first use for us is to put or to keep before our eyes 'illustrious examples' of first-rate mathematical work" [1, p. 204] to provide useful insights into the process of mathematical research. Here we present one such example: that of Euler, who turned number theory from an amateur's playground to a vital part of mathematics.

2. Fermat and Number Theory

Euler's work in number theory began with Fermat's conjectures. In Euler's time, these conjectures could be found in three main sources. The last to be written, but the first to be printed, were the letters that made up the *Commercium Epistolicum* (1658); next were Fermat's notes on Bachet's edition of Diophantus's *Arithmetic*, which were compiled by Fermat's son Samuel (1670); finally, Fermat's own treatises and some of his correspondence which appeared in *Varia Opera* (1679).

The *Commercium Epistolicum* came about because of a chance meeting between Fermat and Kenelm Digby. Digby was, among other things, adventurer, courtier, agent provocateur, alchemist, and brewer, but his importance to Fermat was his relationship to the scientists and scholars who would later form the Royal Society. During a diplomatic visit to France in 1656, Digby suffered an attack of kidney stones, and retired to southern France for his health. It was around this time that he met Fermat, and the two began to correspond. Digby played the role of an English Mersenne, passing letters between Fermat, Frenicle de Bessy, van Schooten, Wallis, and Brouncker. Although only a small fraction of the *Commercium* dealt with the theory of numbers, it would be the first widespread publication of Fermat's conjectures.

Fermat chose to model his version of number theory after Diophantus's *Arithmetic*, which posed problems and provided a specific solution, usually based on clever algebraic manipulations. For example, Book II, Problem 9 asked to express a number that was the sum of two squares as a sum of two other squares. Diophantus took the number $13 = 2^2 + 3^2$, and assumed (in modern notation) $13 = (x+2)^2 + (2x-3)^2$. This gave rise to a homogeneous quadratic equation with two rational solutions. Unfortunately problems like this gave Wallis and others the impression that number theory consisted only of clever calculations and algebraic reductions, and thus held little in the way of real mathematical interest.

Letter XXXVII (April 7, 1658) contains the first appearance of actual theorems (as opposed to problems):

Conjecture 1. There is no right triangle with rational side lengths whose area is the square of a rational number.

This statement is equivalent to "Fermat's Last Theorem" for $n = 4$. The conjecture immediately following is the "Last Theorem" for the $n = 3$ case:

Conjecture 2. There is no integral cube that is the sum of two rational cubes.

Fermat provided no proof of either proposition.

In his last letter (June 1658) Fermat presented propositions for which he claimed he had a "most sound proof." Among these propositions were:

Conjecture 3. Every number is a square or the sum of two, three, or four squares.

Conjecture 4. Every prime p of the form $p = 4n + 1$ is the sum of two squares.

Conjecture 5. Every prime p of the form $p = 3n + 1$ is the sum of a square and three times another square.

Conjecture 6. Every prime p of the form $p = 8n + 1$ or $p = 8n + 3$ is the sum of a square and twice another square.

Fermat declined to provide the proofs. He might have hoped other mathematicians would find their own proofs and in the process discover (as he had) the joys of number theory.

Fermat also posed some unsolved problems. To enlist Wallis's help, he turned to flattery:

> We know that Archimedes did not scorn the propositions of Conon, which were certainly true, but unproven, eventually setting forth true and most subtle demonstrations. Why therefore can we not hope for similar help, a French Conon from an English Archimedes? [4, Vol. I, p. 404]

The propositions which he claimed true "in the fashion of Conon" were the following:

Conjecture 7. Numbers of the form $2^{2^n} + 1$ are prime for all n.

Today numbers of this type are known as Fermat numbers, with $F_n = 2^{2^n} + 1$.

Conjecture 8. If p is a prime number of the form $p = 8n - 1$, then $2p$ can be written as the sum of three squares.

Conjecture 9. The product of two prime numbers ending in 3 or 7 and of the form $4n + 3$ is the sum of a square and five times another square.

Wallis declined to pursue number theory, and he speaks for all his contemporaries when he writes in Letter VII:

> ...I looked upon problems of this nature (of which it is easy to contrive a great many in a little time,) to have more in them of labour than either of Use or Difficulty.[8, Vol. II, p. 766]

Wallis later expressed a belief that nothing of significance rested on the truth or falsity of these conjectures [8, Vol. II, p. 782].

Fermat died in 1665. The notes he made in Bachet's edition of *Diophantus* were compiled by his son Samuel, and published in 1670. We will only repeat only one conjecture from Fermat's *Diophantus*; it is, of course:

Conjecture 10. The equation $x^n + y^n = z^n$ has no solution in integers for $n > 2$.

What is interesting is that this particular claim appears only in the Diophantus, and nowhere else in Fermat's writing. Indeed, except for Conjecture 2 in the *Commercium*, the only references to the Last Theorem are indirect, as in his challenge to other mathematicians to *find* a cube (or fourth power) that is the sum of two cubes (or two fourth powers). It is probable that Fermat had a proof (or had the outline of a proof) for the $n = 3$ and $n = 4$ cases; it seems unlikely that he had a general proof.

The last major source of Fermat's number theory appeared in 1679, when the *Varia Opera* appeared. This work includes some of Fermat's correspondence, as well as most of his mathematical treatises. In it, Fermat repeats Conjecture 7 in several letters, though it is only in the *Dissertation tripartie* that we see why Fermat considered the conjecture important. Fermat argued that the problem of inserting $p - 1$ mean proportionals between two given numbers would be maximally difficult if p was prime; hence he was interested in a formula that would generate arbitrarily large primes. Fermat's argument is spurious, but it is interesting to note that Fermat numbers do play a role in constructibility in a manner entirely different from that envisioned by Fermat.

If we judge the value of a mathematical discovery by how often it is used in subsequent investigations (the "citation" principle), then Fermat's most important discovery appeared in the *Varia Opera* as part of a letter dated October 18, 1640 to Frenicle de Bessy:

Conjecture 11. Every prime number divides a power, minus one, of any given number, and the aforementioned power is always a divisor of the prime minus one.

In modern terms if p is prime, then p divides $a^m - 1$ for some m, and in that case m divides $p - 1$. Fermat omitted an obvious requirement, that p and a have no common factors. This is a more general form of what is usually referred to as Fermat's Little (or Lesser) Theorem.

Fermat's attempt to interest his contemporaries in number theory failed, and while his failure cannot be attributed to any single cause, we will point out three key aspects of Fermat's presentation. First, he presented number theory as a set of problems to be solved, rather than generalizations to be made, so it appeared as a collection of unrelated results. Moreover, Fermat gave no reason why number theory might be important, so mathematicians were drawn to other fields, such as analysis, where the applications were much more apparent. Finally, where Fermat presented a proposition, he did so without proof, but at the same time claimed possession of a proof, so number theory appeared to be a guessing game between Fermat and other mathematicians. Thus for sixty-five years, number theory languished.

3. Goldbach and Euler

Christian Goldbach would stimulate Euler's interest in number theory, but only after attempting and failing to interest others in the subject. On December 18, 1723 Goldbach posed a Diophantine problem to Daniel Bernoulli: To find four numbers such that the pairwise product of any two, plus 1, was a square (this is similar to Diophantus's Book III, Problem 10: to find three numbers so that their pairwise products, added to a given number, was square). Then on February 2, 1724 Goldbach posed a variant: given one number, to find three more so the pairwise products, plus 1, were squares. Like Fermat, Goldbach posed problems rather than suggested general results, though in his letter of September 13, 1724 to Bernoulli, Goldbach mentioned that Jacques Ozanam (1640-1717) proved the difference of two fourth powers could not be a square (again, equivalent to Fermat's Last Theorem for $n = 4$).[1] Bernoulli had little interest in pursuing number theory, though on June 29, 1728 he wrote to Goldbach with one of his results:

> I will finish with a problem which appears to me very curious and which I have solved. Thus: to find two unequal numbers x and y so that $x^y = y^x$. There is one solution among the whole numbers, namely $x = 2$ and $y = 4$ (because $2^4 = 4^2$), but one can give an infinite number of broken [real] numbers which solve this problem. There are other [questions] of this type of which I will say nothing [5, vol. II, p. 262].

Thus like Fermat's correspondents, Bernoulli took only a passing interest in number theory.

Goldbach finally found a willing investigator in Euler, though it took some effort. As a parting note in a letter of December 1, 1729, Goldbach asked:

[1] Ozanam was the author of a number of excellent mathematics texts, including one that helped to found recreational mathematics.

> Do you know of Fermat's observation that all numbers of the form $2^{2^{x-1}}+1$, such as 3, 5, 17, etc., are prime, something that he himself was unable to show, and no one after him has shown [5, Vol. I, p. 10].

Euler, like so many of Fermat's correspondents, thought the result unimportant: "Probably nothing can be discovered from this observation of Fermat" [5, Vol. I, p. 18]. Indeed, since the result had been obtained empirically, Euler doubted its validity (or the validity of any result obtained solely on the basis of scientific induction).

Goldbach was persistent, however, and tried to encourage Euler to work on the problem by suggesting means of approach. Goldbach's suggestions were of varying quality. On May 22, 1730 he noted that the remainders, when squares of the terms in an arithmetic sequence were divided by a prime number, formed a periodic sequence, an observation that Euler would use later (though not in connection with the Fermat numbers). He offered the additional observation that if $p \neq 2^n$, then $2^p + 1$ had divisors; he gave $2^{84} + 1$, with divisor 17, as an example.

Meanwhile Euler began to study Fermat's work, and on June 4, 1730 Euler wrote to Goldbach expressing some enthusiasm for number theory. Euler's attention was caught by the "not inelegant theorem" that every number could be expressed as the sum of four squares. Euler mentions other Fermat conjectures on the resolution of numbers as the sum of polygonal numbers and cubes, "whose proofs would contribute greatly to analysis" [5, vol. I, p. 24]; hence number theory, while worthy of pursuit on its own merits, could also shed useful insight into other areas of mathematics.

Goldbach's next "contribution" to the investigation of the Fermat numbers was on June 26, 1730 (June 15 O.S.):

> It is likely that the least divisor (1 and the number itself not being considered as divisors for this purpose) of any number of the form $a^{2^x}+1$ is of the form $n^{2^x}+1$, but this has not yet been completely examined, except in a single case, namely $x = 1$, which is easy to demonstrate [5, vol. I, p. 26].

Hence, Goldbach notes, a proof of Fermat's conjecture would follow: if it is true that the least divisor $n^{2^x}+1$ is of the form $a^{2^x}+1$, then if $n = 2$, then a can only be 1 or 2, but if $a = 1$, then $1^{2^x}+1 = 2$, which does not divide $2^{2^x}+1$, and if $a = 2$, then the least divisor is the number itself, which is thus prime.

Euler pointed out almost immediately (June 25—presumably new style) that Goldbach's claim is untrue: if $a = 34$ (where $34^2 + 1 = 1157$ has least divisor 13), $a = 76$ (where $76^2 + 1 = 5777$ with least divisor 53), and numerous other cases.

Despite Goldbach's help, Euler made progress and on November 25, 1731 Euler announced a crucial discovery:

> Finally consider the formula $2^n - 1$, which cannot be prime unless n is prime, and consider the cases where $2^n - 1$ is not prime, although n is. These exceptions are $n = 11$, $n = 23$, $n = 83$, and all the remaining primes less than 100 make $2^n - 1$ prime.[2] Indeed, $2^{11} - 1$ can be divided by 23, $2^{23} - 1$ by 47, $2^{83} - 1$ by 167. Upon this is based the not inelegant theorem: $2^n - 1$ can always be divided by $n + 1$, whenever $n + 1$ is a prime number. Thus $2^{22} - 1$ can be divided by 23. Often as well $2^{n/2} - 1$, and indeed $2^{n/4} - 1$ etc., can be divided by $n + 1$, and from this the investigation of the case where $2^n - 1$ is prime is not difficult [5, vol. I, p. 59-60].

Euler is announcing a restricted form of Fermat's Little Theorem, namely that if p is prime, then $2^{p-1} - 1$ is divisible by p. As a more general form of this result appeared in Fermat's *Varia Opera*, it seems that Euler's reading of Fermat's works has to this point been only cursory. As we shall see, Fermat's Little Theorem is the easiest and most general path to finding factors of the Fermat numbers, and Euler would return to the Little Theorem many times during his number theoretic investigations.

In the meantime, Euler discovered that Fermat's conjecture was in fact false, and presented his results to the Academy on September 26, 1732. "Observationes de theoremate quodam Fermatiano aliisque ad numeros primos spectantibus" (E26) was the first of nearly 100 papers on number theory published by Euler; though mathematically insignificant, it hints at things to come.

E26 begins with a discussion of the possible factors of a^n+1. Euler begins by stating two propositions:

(i) If $n = 2m + 1$, then $a^n + 1$ has a factor of $a + 1$.
(ii) If $n = p(2m + 1)$, then $a^n + 1$ has a factor of $a^p + 1$.

Euler gave no proof, but these follow easily from straightforward factorization of $a^n + 1$.

Thus in order for $a^n + 1$ to be prime, n must be a power of 2 and, of course, a must be even. These conditions are necessary but not sufficient, and Euler gives several counterexamples:

(i) $a^2 + 1$ has a factor of 5 whenever $a = 5b \pm 3$.
(ii) $30^2 + 1$ has divisor 17 and $50^2 + 1$ has divisor 41.
(iii) $10^4 + 1$ has divisor 73.
(iv) $6^8 + 1$ has divisor 17.
(v) $6^{128} + 1$ has divisor 257.

What insight might we gain from this list of counter-examples and the disproof of Conjecture 7? From the first counter-example, the observant reader will note

[2] Euler omits $n = 37$, though in an earlier letter he noted 223 divides $2^{37} - 1$.

$$(5b \pm 3)^2 + 1 = 25b^2 \pm 30b + 3^2 + 1$$

Hence it seems that if one wishes $a^2 + 1$ to be divisible by some prime p, one need only let $a = (pb \pm c)$ where $c^2 + 1$ is divisible by p. For small primes p, trial and error would suffice to find c so $c^2 + 1$ is divisible by p. Thus since $4^2 + 1 = 17$, we have $a^2 + 1$ divisible by 17 whenever $a = 17k \pm 4$; since $9^2 + 1 = 82$ is divisible by 41, we have $a^2 + 1$ divisible by 41 whenever $a = 41k \pm 9$. Thus the first two counter-examples can be viewed as direct results of naive number theory—uninteresting results of the very type dismissed by Wallis and others.

But what of examples iii through v, and Euler's factorization of $F_5 = 2^{2^5} + 1$? It seems likely that Euler had already suspected the validity of Conjecture 11 and used it to find potential factors.

If a prime p divides $a^n + 1$, then p divides $(a^n + 1)(a^n - 1) = a^{2n} - 1$, and thus by Conjecture 11 $2n$ is a divisor of $p - 1$. Thus p is a prime of the form $2nk + 1$. Hence the possible factors of $10^4 + 1$ are primes of the form $8k + 1$: the first few primes of this form are 17, 41, and 73, and three trial divisions suffice to find a factor. For $6^8 + 1$ the possible factors are primes of the form $16k + 1$: 17 is the first prime of this form, and a single trial division suffices to find a factor. Finally $6^{128} + 1$ might have prime factors of the form $256k + 1$, and 257 is again the first of these.

For $2^{2^5} + 1 = 2^{32} + 1$, Euler only needed to examine primes of the form $64k + 1$. The first few primes of this form are 193, 257, 449, 577, and the actual factor 641. Thus five trial divisions would have sufficed to find the factor; "Hence [the Fermat numbers are] not a solution to the problem of finding a prime that exceeds any given number" [3, Series 1, Vol. II, p. 3]. It is interesting to note that Fermat could have found a factor of $2^{2^5} + 1$, and the computations were well within his capabilities (and if not, within those of Frenicle de Bessy, a more assiduous calculator). At least one of them should have been capable of disproving Conjecture 7. That they failed to do so is a minor mystery.

Did Euler in fact use Fermat's Little Theorem to find a factor of F_5? Euler's fame as a calculator makes it plausible that trial division was the method used to find the factor 641. However, there are two pieces of evidence that support Euler's use of Fermat's Little Theorem. The first is that Euler mentions that he found the factor through "a long method" that opened the way for similar problems to be resolved: this suggests a general method like Conjecture 11 rather than a method like trial division; the "length" in this case would encompass the discovery of Conjecture 11 as well as its application to finding potential factors. More definitively (though perhaps less reliably), we will see that fifteen years later Euler claims that Conjecture 11 was precisely how he found the factor 641.

At the end of E26, Euler notes that he believes (but has not yet proven) that if a, b are not divisible by a prime $n+1$, then $a^n - b^n$ is divisible by $n+1$. Consequently $2^n - 1$ is divisible by the prime $n+1$, which is a specific instance of Conjecture 11. Euler concludes E26 with six "theorems" (*theorema*) he believes valid, but had not yet obtained a proof. The first was:

Conjecture 12. If n is prime, then all powers with an exponent of $n-1$ will leave a remainder of 0 or 1 when divided by n.

This is what most books on elementary number theory call Fermat's Little Theorem: namely that if p is prime and a is relatively prime to p, then $a^{p-1} \equiv 1 \bmod p$. We shall refer to this particular conjecture as Euler's form of Fermat's Little Theorem.

In addition, Euler stated some generalizations of Conjecture 12:

Conjecture 13. If n is prime, then any number raised to the power $n^{m-1}(n-1)$, divided by n^m, will have remainder 0 or 1.

Conjecture 14. If m, n, p, q , ... are distinct primes not dividing a, and A is the least common multiple of $m-1$, $n-1$, $p-1$, $q-1$, ..., then a^A divided by $mnpq\cdots$ leaves a remainder of 0 or 1.

4. Fermat's Little Theorem: First Proof

If Fermat had a proof of Conjecture 11 (or its special case, Conjecture 12), he did not write it down. Leibniz proved the theorem some time before 1683, but the proof only appears in manuscript and was not brought to light until 1894 [2, Vol. I, p. 59]. Thus Euler was the first to publish a proof. He presented "Theorematum quorundam ad numeros primos spectantium demonstratio" (E54) to the St. Petersburg Academy on August 2, 1736.

By now Euler was firmly convinced that number theory was a mathematical discipline worth pursuing. He is rather less enamored of Fermat's methods, however, and criticizes Fermat's lack of proof and reliance on (scientific) induction: after all, Fermat's conjecture on the primality of numbers of the form $2^{2^n}+1$ seemed well-supported by observation, but it was nonetheless false. This casts doubt on the validity of *all* conjectures based on observation.

Euler proves his form of Fermat's Little Theorem by induction; this may be the first induction proof to appear in post-Newtonian mathematics (induction had already appeared in some of Pascal's work and in the work of Levi ben Gerson, as well as Leonardo of Pisa's *Liber Quadratorum*). Moreover, he not only gives a proof, but carefully elaborates upon the process by which the proof came about; thus E54 is a good example of mathematical epistemology.

First, Euler shows that $2^{p-1} - 1$ is divisible by any prime p; this follows from the binomial expansion

$$(1+1)^{p-1} = 1 + \frac{p-1}{1} + \frac{p-1}{1}\frac{p-2}{2} + \frac{p-1}{1}\frac{p-2}{2}\frac{p-3}{3} + \dots$$

Since there are p terms in the series, the number of terms is odd; subtracting 1 leaves an even number of terms, which Euler proceeds to group pairwise:

$$\frac{p-1}{1} + \frac{p-1}{1}\frac{p-2}{2} + \frac{p-1}{1}\frac{p-2}{2}\frac{p-3}{3} + \dots = \frac{p}{1}\frac{p-1}{2} + \frac{p}{1}\frac{p-1}{2}\frac{p-2}{3}\frac{p-3}{4} + \dots$$

Since p is an odd number, the last term is $\frac{p}{1}\frac{p-1}{2}\frac{p-2}{3}\frac{p-3}{4}\cdots\frac{2}{p-1} = p$, and thus every term in the series is divisible by p. Thus $2^{p-1} - 1$ is divisible by p.

Unfortunately the proof as given is not amenable to generalization, so Euler sought a different proof inspired by a corollary: If $2^{p-1} - 1$ is divisible by some prime p, so is $2^p - 2$; conversely, if p divides $2^p - 2$ and $p \neq 2$, then p must divide $2^{p-1} - 1$. This time the binomial expansion gives us:

$$(1+1)^p - 2 = 1 + \frac{p}{1} + \frac{p}{1}\frac{p-1}{2} + \frac{p}{1}\frac{p-1}{2}\frac{p-3}{3} + \dots + 1 - 2$$

where all the remaining terms obviously have a factor of p. If p is taken to be an odd prime, then since p divides $2^p - 2 = 2(2^{p-1} - 1)$ and p does not divide 2, then it must divide $2^{p-1} - 1$.

Note that this proof emerged from a corollary to the main result. Although listing corollaries is not new, Euler was more diligent than most in providing an exhaustive listing of the consequences of a theorem. As we shall see, some of the corollaries were trivial and we might classify them as examples, but others played important roles in the proofs of later theorems.

After proving this corollary Euler notes that if p divides $2^{p-1} - 1$, then p will also divide $2^{k(p-1)} - 1$, and thus p divides $4^{p-1} - 1$, $8^{p-1} - 1$, $16^{p-1} - 1$, and so on. It seems that Euler is considering the following path to the proof: we need only show that a prime p divides $a^{p-1} - 1$ when a is prime; consequently p will divide $a^{p-1} - 1$ when a is a power of a prime. If we can then show that p divides $a^{p-1} - 1$ when a is a product of primes or powers of primes, we are done.

The task seems daunting, but the very first step along this path to the proof will reveal a shortcut. We must first prove $3^{p-1} - 1$ is divisible by some prime p (not equal to 3). Fortunately Euler has given two proofs of the divisibility of $2^{p-1}-1$; we can use the second to show, first, if p divides 3^p-3 and $p \neq 3$, then p divides $3^{p-1}-1$. Again we use the binomial expansion:

$$(1+2)^p = 1 + \frac{p}{1}2 + \frac{p}{1}\frac{p-1}{2}4 + \frac{p}{1}\frac{p-1}{2}\frac{p-2}{3}8 + \ldots + 2^p$$

Since every term is divisible by p except the first and last, we have $3^p - 2^p - 1$ divisible by p. But $3^p - 2^p - 1 = 3^p - 3 - (2^p - 2)$, and p divides the last two terms, so p must also divide $3^p - 3$.

We might be tempted to prove the theorem for $a = 5$, but note instead that the proof for $3 = 2 + 1$ depended on the validity of the theorem for 2. This gives us an induction step: If $a^p - a$ can be divided by a prime p, then so can $(a+1)^p - (a+1)$. The proof of the induction step follows by binomial expansion, and the proof of Fermat's Little Theorem follows immediately. As a postscript to E54, we note that the induction proof of Fermat's Little Theorem is not well-known, so it is periodically rediscovered by mathematicians great (Laplace and Cauchy and insignificant ([7]).

5. Fermat's Little Theorem: Second Proof

In 1740, pro-Slavic elements gained control of the Russian government, and a purge of the pro-German elements which had dominated Russia for a generation was inevitable. For this and other reasons Euler accepted a position at the Berlin Academy of the Sciences, where he would spend the next twenty-six years. However, Euler maintained his membership in the St. Petersburg Academy, and continued to correspond with Goldbach; much of his work in number theory in this period would be communicated to Goldbach first and only later presented to the Berlin Academy.

Euler's second proof of Fermat's Little Theorem first appeared in a letter to Goldbach dated March 6, 1742. First Euler proved that any prime p divided $(a+b)^p - a^p - b^p$ using the binomial expansion of $(a+b)^p$. If $a = b = 1$, this implied that any prime p divided $2^p - 2$, and if $p \neq 2$, then p divided $2^{p-1} - 1$. If $a = 2$, $b = 1$, then p divides $3^p - 2^p - 1$, but since p divides $2^p - 2$, then p must also divide $3^p - 3$ and again, if $p \neq 3$, p divides $3^{p-1} - 1$.

Next, Euler shows that if p divides $a^p - a$, then p divides $(a+1)^p - a - 1$; this is the induction step from his first proof of Fermat's Little Theorem. Thus if p does not divide a, then p divides $a^{p-1} - 1$ and Fermat's Little Theorem, as stated by Euler, follows.

Euler then proves an important result, which will eventually lead to a proof of Conjecture 4: If p is a prime of the form $4n-1$, it *cannot* divide the sum of two squares a, b that are relatively prime to p. This follows because $p = 4n-1$ must divide $a^{4n-2} - b^{4n-2}$, and thus it cannot divide $a^{4n-2} + b^{4n-2}$ (since if it did, it could divide their sum and their differences, and thus p would divide both $2a^{4n-2}$ and $2b^{4n-2}$, which is impossible if p is assumed relatively prime to a, b). Since $4n-2 = 2(2n-1)$ (Euler calls this an "odd-even" number, a reference to Greek number theory), then $a^{4n-2} + b^{4n-2}$ has a factor of $a^2 + b^2$. Thus p cannot divide $a^2 + b^2$. Conversely, any prime divisor of the sum of two squares must be a prime of the form $4n + 1$.

These results and proofs, substantially unchanged, were presented to the Berlin Academy on March 23, 1747 as "Theoremata circa divisores numerorum" (E134). Euler opens E134 with a defense of number theory as a legitimate area for mathematical research. In support of this viewpoint, Euler points to the existence of seemingly true but as-yet-unproven propositions in number theory: this establishes the superiority of number theory over, say, geometry, since (by Euler's argument) the more abstruse truths are also those harder to prove. That these truths seem unimportant misses the point: not only is there value in knowing *any* truth, but the very act of proof may bring to light methods of proof that can be used in other problems, an idea he first stated in his June 4, 1730 letter to Goldbach.

The proof of Fermat's Little Theorem, and that the prime factors of $a^2 + b^2$ must be of the form $4n + 1$, are essentially the same as those in his letter to Goldbach. Euler continues E134 by classifying potential divisors of $a^4 + b^4$ (2 or primes of the form $8n + 1$) and $a^8 + b^8$ (2 or primes of the form $16n + 1$). With these two specific cases dealt with, the generalization to prime divisors of the form $a^{2^m} + b^{2^m}$ is transparent: the only possible divisors are 2 or primes of the form $p = 2^{m+1}n + 1$. Hence in the case of Fermat's claim that $2^{2^5} + 1$, one has only to examine primes of the form $64k + 1$; this, he claims, is how he found the factor 641.

E134 concludes with a number of results on power residues. It is the first extensive treatment of the subject, and Euler's Theorem 11 is the first to provide a general solution to the congruence $x^m \equiv 1 \bmod p$, namely: If $a = f^2 \pm (2m+1)\alpha$ where $p = 2m+1$ is prime and f, α are arbitrary, then p divides $a^m - 1$. Euler then gives six corollaries to this result, then three examples, which he solves (we will leave the solutions as an exercise for the reader):

(i) Find a so $a^2 \pm 1$ is divisible by 5.
(ii) Find a so $a^3 \pm 1$ is divisible by 7.
(iii) Find a so $a^5 \pm 1$ is divisible by 11.

As with his previous works, Euler begins the process of generalization by proving a specific instance: in this case (Theorem 12) if $a = f^3 \pm (3m+1)\alpha$, with $p = 3m + 1$ prime, then $a^m - 1$ is divisible by $3m + 1$. The proof of these two cases allows the generalization to be made (Theorem 13): If $a = f^n \pm (mn + 1)\alpha$, where $p = mn + 1$ is prime, then $a^m - 1$ is divisible by $mn + 1$.

6. The Sum of Four Squares

On June 17, 1751 Euler presented "Demonstratio theorematis Fermatiani omnem numerum sive integrum sive fractum esse summam quatuor pauciorumve quadratorum" (E242) to the Berlin Academy. In it Euler makes significant progress towards proving Fermat's Conjecture 3, though he in fact proves that all rational numbers can be written as the sum of four rational squares. Lagrange would provide the finishing touches in a 1770 paper, though his proof was subsequently improved by Euler a few years later.

Of greater importance is that E242 foreshadows the development of group theory, a theme that will carry Euler through the next phase of his number theoretic work. In particular, Euler proved a restricted form of Fermat's Little Theorem based not, as in Euler's first two proofs, on the binomial expansion, but on group theoretic properties.

Euler considers the remainders when the squares of the integers are divided by some prime p. If we consider the sequence of squares

$$1, 4, 9, \ldots, p^2, (p+1)^2, \ldots, 4p^2, (2p+1)^2, \ldots$$

it is clear (Theorem 2) that the remainders upon division by p form a sequence with period p. These remainders will necessarily omit some numbers less than p; in particular, if we consider the first $p - 1$ squares

$$1, 4, 9, 16, \ldots, (p-4)^2, (p-3)^2, (p-2)^2, (p-1)^2$$

it is clear that the terms equidistant from the ends have the same remainder on division by p. Thus we need only consider the remainders when the squares of the numbers 1, 2, 3, ... $\frac{p-1}{2}$ are divided by p. Euler claims that there are exactly $\frac{p-1}{2}$ remainders. Euler did not show this, but a proof is trivial: if two of the squares have the same remainder, then their difference is divisible by p, so either the sum or difference of their roots is divisible by p; however, this is impossible since the sum and difference are both less than p.

After a few more pages Euler proves that if r is any remainder, then all powers of r are also remainders (Theorem 5). Since there are only finitely many remainders, then (Corollary 3) an infinite number of powers of r must have equal remainders on division by p. Taking two of these powers with equal remainders, r^m and r^n, then p must divide their difference $r^m - r^n$, and consequently p must divide $r^n(r^{m-n} - 1)$; hence (since $r < p$), p must divide $r^\lambda - 1$ for some λ. This is a restricted form of Fermat's Little Theorem.

7. Fermat's Little Theorem: Third Proof

Euler converted the basic idea in this proof of a restricted form of Fermat's Little Theorem into a proof of the full theorem, presented on February 13, 1755 to the Berlin Academy. There are three noteworthy facts about "Theoremata circa residua ex divisione potestatum relicta" (E262). First, Euler identifies the theorem as one of Fermat's, which he had not done previously. In addition, Euler proves the theorem in the form it was originally stated by Fermat. Third and most important, the paper takes several crucial steps towards the development of group theory.

The format of E262 is similar to Euler's other papers on number theory: each theorem is followed by a number of corollaries, and the corollaries are usually the basis of the proof for a later theorem. Euler focuses on the remainders when a power of a is divided by a prime p. If p does not divide a, then p cannot divide any power of a (Theorem 1); thus, since there are only $p - 1$ possible remainders, then the terms of the infinite sequence $1, a, a^2, a^3, \ldots,$ must include some terms with the same remainder on division by p (Corollary 2). Several results on the behavior of the remainders follow; Euler then shows how these properties can be used to show that the remainder when 7^{160} is divided by 641 is equal to 640 "or -1." It is possible (though by no means certain) that this, rather than direct division, was how Euler identified 257 as a factor of $6^{128} + 1$.

With Theorem 3 Euler takes an important step: If a is relatively prime to p, a prime number, then there exists an infinite number of terms in the geometric sequence $1, a, a^2, a^3 \ldots$ which will have a remainder of 1 when divided by p, and the exponents of these terms will form an arithmetic sequence. The proof is straightforward: Since there must be at least two terms with the same remainder, say a^μ and a^ν (where we may assume $\mu > \nu$), then their difference $a^\mu - a^\nu = a^\mu(a^{\mu-\nu} - 1)$ is divisible by p; since no power of a is divisible by p, then $a^{\mu-\nu} - 1$ must be. Letting $\lambda = \mu - \nu$, then every term in the sequence

$$1, a^\lambda, a^{2\lambda}, a^{3\lambda}, a^{4\lambda}, a^{5\lambda}, a^{6\lambda} \ldots$$

must also leave a remainder of 1 upon division by p.

Most of the proofs that follow this point in E262 are based on closure properties and counting arguments, and place minimal reliance on symbolic manipulation. For example, to show (Theorem 7) that if λ is the least power for which $a^\lambda \equiv 1 \bmod p$, then the remainders when the terms of the sequence

$$a, a^2, a^3, \ldots, a^{\lambda-1}$$

must be all different, Euler assumes that two are the equal, a^μ and a^ν, where we may assume $\nu < \mu < \lambda$; hence $a^\mu - a^\nu = a^\nu(a^{\mu-\nu} - 1)$ is divisible by p, and so is $a^{\mu-\nu}$, which is impossible since $\lambda > \mu - \nu$ was assumed the least power to leave a remainder of 1 when divided by p. Moreover, since $a^{n\lambda} \equiv a^\lambda \bmod p$, then the sequence of remainders repeats itself with period λ (Theorem 8). Consequently if p is a prime number and all numbers less than p appear as remainders, then $\lambda = p - 1$ (Theorem 9).

The proof of Theorem 10 foreshadows another important idea in group theory: that of a coset of a subgroup. In this case, Euler proves that if the number of remainders λ is less than $p - 1$, then the number of non-remainders is at least as great as the number of remainders. This follows because if we consider the remainders when the terms of the sequence

$$1, a, a^2, a^3, \ldots, a^{\lambda-1}$$

are divided by p, the λ remainders are distinct. By assumption there exists at least one non-remainder k; then the terms of the sequence

$$k, ak, a^2k, a^3k, \ldots, a^{\lambda-1}k$$

are also distinct and non-remainders. Following this are a number of corollaries and theorems of the form: If $\lambda < \frac{p-1}{n}$, then $\lambda \leq \frac{p-1}{n+1}$. It follows that λ must be a divisor of $p-1$ (Theorem 13)—Fermat's Little Theorem in the form stated by Fermat. As a consequence we have Fermat's Little Theorem in the form stated by Euler (Theorem 14), that if p is prime and not a divisor of a, then $a^{p-1} \equiv 1 \bmod p$. Euler, incidentally, did not credit Fermat with having conjectured that this might be true for some divisor λ of $p-1$. As with E134, Euler proves Fermat's Little Theorem partway through the paper and devotes the rest of the paper to the theory of power residues.

8. Fermat's Little Theorem: Fourth Proof

On June 8, 1758 Euler presented "Theoremata arithmetic nova methodo demonstrata" (E271) which includes his fourth proof of Fermat's Little Theorem, though Euler actually proves a generalization now called the Euler-Fermat Theorem. Like E262, the method of proof involves little symbolic algebra and many group theoretic ideas. It is instructive to compare the proofs in E262 with those in E271.

E262 began by considering the remainders when a geometric progression 1, a, a^2, ... where a was divided by p. E271 begins by considering the remainders when an *arithmetic* progression a, $a+d$, $a+2d$, ... was divided by p.

Many of the results in E271 are analogous to those in E262: for example, there must be terms with the same remainder. However there is a crucial difference: if n is relatively prime to the difference d, then the n terms in the arithmetic sequence from a to $a+(n-1)d$ must, upon division by n, yield every number less than n as a remainder (Theorem 1); we have no such guarantee of a complete set of remainders with the geometric sequence.

Euler takes advantage of the one-to-one correspondence between the remainders 0, 1, 2, ..., $n-1$ and the terms of the arithmetic sequence a, $a+d$, $a+2d$, ..., $a+(n-1)d$ in Theorem 2. If some remainder r is relatively prime to n, then the corresponding term of the sequence $a+\nu d$ is also relatively prime to n; if r and n have a common factor, then so do $a+\nu d$ and n. Hence the number of terms in the sequence a, $a+d$, $a+2d$, ..., $a+(n-1)d$ relatively prime to n is equal to the number of numbers less than n that are relatively prime to n.

Since the number of numbers less than n that are relatively prime to n seems to be important, Euler spends the next few pages delving into the properties of what is now called the Euler ϕ-function (a notation first used by Gauss). We note in passing that, although arithmetic sequences led Euler to the ϕ-function, Euler then abandoned arithmetic sequences and returned to the geometric progressions of E262.

The proof of the Euler-Fermat theorem begins with Theorem 7, which is a repeat of a result from E262, namely that given any x relatively prime to N, there must be some least power ν where x^ν leaves a remainder of 1 when divided by N. Euler then shows (Theorem 8) that the remainders when the sequence 1, x, x^2, x^3, ... are divided by N are closed under multiplication and exponentiation; the proof is by straightforward algebraic manipulation. Thus (Theorem 9) the number of distinct remainders when the powers of x are divided by N is either equal to the number of numbers less than N that are relatively prime to it, or is a divisor of this number; this is a proof based on the coset idea. This implies (Theorem 10—the Euler-Fermat Theorem) that x^ν leaves a remainder of 1 when ν equals the number of numbers less than N that are relatively prime to N, or some divisor of this number. Since (in modern terms) if N is prime, then $\phi(N) = N-1$, this implies Fermat's Little Theorem in the form originally stated by Fermat; as in E262, Euler did not indicate that Fermat suggested this might be true for ν that divided $\phi(N)$.

9. Results on Quadratic Forms

Let us now turn to quadratic forms (see Fermat's Conjectures 4, 5, and 6). On September 9, 1741 Euler communicated further "curious properties" he had discovered about numbers of the form $a^2 \pm mb^2$; additional results followed on August 28, 1742, though he was as yet unable to prove any of them. Most of these conjectures would appear (still unproven) in "Theoremata circa divisores numerorum in hac forma $paa \pm qbb$ contentorum" (E164), presented in 1747 to the Berlin Academy. Euler lists some 59 theorems on quadratic forms but only a handful of proofs. Many of Euler's conjectures coincide with Fermat's: Euler's Theorem 2 is Fermat's Conjecture 4; Theorem 5 is Conjecture 6, and Theorem 8 is an alternate form of Conjecture 5. One is reminded uncomfortably of Wallis's complaint: the conjectures do not seem particularly profound, and little of consequence seems to hinge on their truth or falsity.

On May 6, 1747 Euler wrote to Goldbach and announced:

> I can now prove that, I. All prime numbers of the form $4n+1$ are the sum of two squares, and also II. All non-primes of the form $4n+1$, provided they have no divisors of the form $4n-1$, are also the sum of two squares [5, Vol. I, p. 415].

Euler's proof is as follows: first, he shows that the product of two numbers that are each the sum of two squares is likewise the sum of two squares; next, if a number that is the sum of two squares is divisible by another number that is the sum of two squares, then their quotient is likewise the sum of two squares. Both proofs are based on symbolic manipulation. Next Euler shows that if a number divides the sum of two squares relatively prime to one another, then the number itself is the sum of two squares. This is proven using Fermat's method of infinite descent, and is one of the few places where Euler used this particular method.

Finally, Euler is ready to prove the main result, that all primes p of the form $4n+1$ are the sum of two squares. By Fermat's Little Theorem, p must divide $a^{4n} - b^{4n}$, so p must divide exactly one of $a^{2n} + b^{2n}$ or $a^{2n} - b^{2n}$. Euler claims (but is as yet unable to prove) that there must be a pair of relatively prime numbers a, b where p cannot divide $a^{2n} - b^{2n}$; consequently there exists a sum of squares that p divides, and hence p itself must be the sum of two squares.

This proof, again largely unchanged, was presented to the Berlin Academy on March 20, 1749 as "De numeris, qui sunt aggregata duorum quadratorum" (E228). One especially interesting feature about E228 is that Euler begins by listing all numbers less than 200 that are the sum of two squares as well as the numbers less than 200 that are not the sum of two squares;

he then uses observations on this list to help establish some conjectures (which he then proves).

For example, it is trivial to show algebraically that if $N = a^2 + b^2$, then N must either be divisible by 4, or of the form $8n+1$ or $8n+2$. We know, of course, not to expect the converse to be true, but we also know that denying the automatic validity of the converse is a learned, not instinctive, response. Euler simply pointed to the list to provide counterexamples to the converse.

It took a little over a year for Euler to complete the proof; again, the first appearance of the final proof was in a letter to Goldbach (April 12, 1749). In order to prove that there must be some a, b for which p does not divide $a^{2n} - b^{2n}$, Euler considers the sequence

$$1, 2^{2n}, 3^{2n}, 4^{2n}, \ldots, (4n)^{2n}$$

Then the first differences

$$2^{2n} - 1, 3^{2n} - 2^{2n}, 4^{2n} - 3^{2n}, \ldots, (4n)^{2n} - (4n-1)^{2n}$$

cannot all be divisible by p, since if they are, then all the differences are so divisible, and in particular the $2n$th difference will be. But the $2n$th difference will equal $(2n)!$ which cannot be divisible by p (since by assumption $p = 4n+1$ is prime). Subsequently Euler presented "Demonstratio theorematis Fermatiani omnem numerum primum formae $4n+1$ esse summam duorum quadratorum" (E241) to the Academy (October 15, 1750)

Euler's next major results on quadratic forms make up one of his more interesting papers, "Specimen de usu observationum in mathesi pura" (E256), presented to the Berlin Academy on November 22, 1753. Here Euler makes his method of approaching a problem very clear; ironically, Euler is not able to get very far in his investigations!

Literally translated, E256 is "Examples of the use of observation in pure mathematics." Among other things, Euler is showing that observation (and conjecture) play a crucial role in the development of a mathematical idea. In E256 Euler lists all numbers less than 500 of the form $a^2 + 2b^2$. From this list he makes eight observations; Observation 7 is the same as Fermat's Conjecture 6. Euler then proves many of these observations, mainly through clever symbolic manipulation, though Euler's proof of Theorem 9 (no number of the form $2a^2 + b^2$ is divisible by a prime not of that form) uses Fermat's method of infinite descent. The stumbling block is showing that every prime p must divide some number of the form $2a^2 + b^2$.

At this point Euler notes that the proof applies to $ma^2 + b^2$ only as long as $\frac{m+1}{4}$ does not exceed 4; worse yet, since $3(1)^2 + 1^2 = 4$ has prime divisor $2 \neq 3a^2 + b^2$, the method only works for $m = 1$ (the sum of two squares) and $m = 2$ (the sum of a square and twice another square). Thus by the end, Euler notes he is unable to make significant headway on Fermat's Conjecture 6.

It took six more years before Euler made further progress on quadratic forms. "Supplementum quorundam theorematum arithmeticorum, quae in nonnullis demonstrationibus supponuntur" (E272) was presented to the St. Petersburg Academy on October 15, 1759. As in the earlier papers, Euler is able to show, through clever algebraic manipulation, that every prime divisor of a number of the form $a^2 + 3b^2$ is itself of the same form. Thus if p is a prime of the form $6n+1$, then p must divide $a^{6n} - b^{6n}$, and so p divides either $a^{2n} - b^{2n}$ or $a^{4n} + a^{2n}b^{2n} + b^{4n}$. If p divides the latter factor, we are done, since $f^2+fg+g^2 = \left(f + \frac{1}{2}g\right)^2 + 3\left(\frac{1}{2}g\right)^2$. Euler then considers the differences

$$2^{2n} - 1, 3^{2n} - 1, 4^{2n} - 1, \ldots, (6n)^{2n} - 1$$

which cannot all be divisible by p, since in that case the $2n$th differences would also be divisible by p, and the $2n$th differences are $(2n)!$.

E272 is near the beginning of significant slowdown in Euler's work on number theory. In 26 years Euler published 25 papers on number theory; during the 1750s alone Euler presented 13 papers on the subject. But during the 1760s, Euler would only present four papers on number theory (one of which had been originally presented in the 1750s). It was not until 1770 that Euler resumed his earlier pace in publishing papers on number theory. That year marked the publication of his *Algebra*, which contained a proof of the impossibility of integer solutions to $x^3 + y^3 = z^3$.

10. Fermat's Last Theorem

In his correspondence, Fermat claimed or implied the impossibility for $n = 3$ and $n = 4$, but only a sketch for the proof of the $n = 4$ case has been found. The only reference to the famous "last theorem" in its full generality occurs in the *Diophantus* of 1670, and it is not until February 25, 1748 that Euler mentions Fermat's Last Theorem:

> Fermat says in his *Observations on Diophantus* that the equation $x^n = y^n + z^n$ is impossible among the rationals, except for the cases $n = 1$ and $n = 2$; that is, a sum of two cubes cannot be a cube, nor can the sum of two biquadrates be a biquadrate, nor in general can the sum of two higher powers equal a like power [5, Vol. I, p. 446].

Euler had already proven the $n = 4$ case. On June 23 and August 16, 1738, he presented "Theorematum quorundam arithmeticorum demonstrationes" (E98). The theorem in question is that neither the sum nor difference of two biquadratics could be a square (i.e., $x^4 \pm y^4 = z^2$ has no integral solutions). This particular theorem has two corollaries of consequences: first, it implies the impossibility of integral solutions to $x^4 + y^4 = z^4$. Second, it proves Fermat's Conjecture 1. Euler noted that a proof of this last had already been given by Frenicle de Bessy, but it depended on properties of right triangles and was so obscure and convoluted that it took considerable effort to understand. Thus Euler sought and presented a clearer and more analytic proof. The proof is largely accomplished through symbolic algebra and a few parity arguments. There are very few new methods in this proof.

Euler claimed a proof of the $n = 3$ case as early as August 4, 1753, but this proof did not appear until his *Algebra* (1770), and the proof presented is incomplete. Portions of the proof in the *Algebra* are reminiscent of work done by Euler in E272; indeed, Euler noted a possible connection between his work in E272 and a proof of Fermat's Last Theorem. The connection for the $n = 3$ case is this: if $x^3 \pm y^3 = z^3$, then $z^3 = (x \pm y)(x^2 \pm xy + y^2)$. But this second factor is $x^2 \pm xy + y^2 = x^2 \pm xy + \frac{1}{4}y^2 + \frac{3}{4}y^2 = \left(x \pm \frac{y}{2}\right) + 3\left(\frac{1}{2}y\right)^2$. Hence the sum or difference of two cubes factors into a product consisting of the sum or difference of their roots, and a number expressible as $p^2 + 3q^2$. We may assume that x, y, and z have no common factors; hence one factor of z must be a number of the form $p^2 + 3q^2$. Euler assumed without proof that numbers of this form are products of primes of this form. In a like manner, other quadratic forms may play a role as factors of numbers of the form $x^n \pm y^n$. Unfortunately neither Euler nor anyone else would be able to convert this speculation into a viable proof.

For the $n = 3$ case Euler factored $p^2 + 3q^2$ as $(p + q\sqrt{-3})(p - q\sqrt{-3})$. Through the use of this variant of Gaussian integers, and an implicit assumption of unique factorization, Euler was able to construct a descent proof that showed the impossibility of the $n = 3$ case. As in his proof for the $n = 4$ case, Euler's proof rested primarily on symbolic manipulations and parity arguments, and introduced little in the way of new methods.

The *Algebra* marked Euler's return to number theory, and soon he would return to and even exceed the pace he set before his slow decade of the 1760s. But while the pre-1760 papers broke new ground, established powerful new tools, and reinvented the subject as one fit for serious mathematical investigation, the post-1770 papers were, by and large, of little significance. Fortunately Euler's work had inspired a new convert to number theory: his successor at Berlin, Joseph Louis Lagrange. Lagrange and later Legendre would carry the investigation of number theory forward until the age of Gauss.

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Euler and Lotteries

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1. Introduction

Generally there are two kinds of lotteries. The first, and probably the oldest, is to have m tickets sold to m players, where the tickets are uniquely numbered, say 1, 2, $\ldots, m$. A winning number is determined by drawing one of the m numbers at random. The second kind, typically called a lotto, is to have each of the m players choose r numbers from a set of n numbers. At the draw t numbers are chosen at random without replacement from the n available. A winning number is determined by a player matching the selected combination chosen. Euler analyzed both kinds of lotteries.

The first style of lottery dates from at least the fifteenth century or earlier [Ewen 1972]. Earliest evidence is from the Low Countries, particularly cities such as Ghent, Utrecht and Bruges. The first French lottery dates from the 1530s and the first English one from the 1560s. In this kind of lottery as it evolved in the sixteenth and seventeenth centuries, winning numbers were chosen using two lottery wheels, as illustrated in William Hogarth's famous engraving *The Lottery*[1] . One wheel contained the m tickets and the other wheel contained m slips, some with a prize to be awarded written on it and the remainder blank indicating no prize. A lottery with, for example, one

[1] The engraving may be viewed on various internet sites. See for example, http://www.library.northwestern.edu/spec/hogarth/Politics2.html.

large prize, ten secondary prizes and one hundred third prizes would have 111 non- blank slips, one for each of the 111 prizes to be awarded, and $m - 111$ blanks. Both wheels were rotated. To determine the prize allocation, a lottery ticket from one wheel was drawn without replacement and matched to a slip drawn from the other wheel, again without replacement. This continued until all the prizes were distributed. Drawings from this type of lottery, which for convenience will be called a blanks and prizes lottery, could take several days or weeks.

Modern lotteries of this type have tickets printed in multiples of 10^b for b typically of value 3, 4, 5, or 6. Each digit of the ticket takes one of the numbers 0 through 9. Prizes are awarded by matching the last two digits of the ticket, the last three and so on to matching all the digits for the largest prize. This simplifies the drawing of the winning numbers. Winning numbers can be drawn by choosing independently at random b integers from 0 through 9 using b machines with numbered balls in them.

The lotto was an Italian invention. In its original form five ($= t$) numbers were chosen at random and without replacement from one hundred ($= n$) probably by mixing one hundred numbered pieces of rolled parchment in a box and having an unbiased observer draw five. Players could bet on correctly guessing any one, two, three, four or all five of the five numbers drawn. These bets were called *extrait*, *ambe*, *terne*, *quarterne* and *quine* respectively. Tradition has it that the lotto grew out of betting on the outcomes of elections by lot in Genoa in the early seventeenth century. [Bellhouse 1991] has assessed the evidence and found it conflicting; seventeenth century sources on the Genoese election system show that there were only two people elected out of a pool of 100 and it was not by lot. To add to the mythology, the lotto's invention was attributed to a man named Benedetto Gentile [Ewen 1972]. Interestingly, *benedetto* is the Italian word for "blessed" and *gentile* is the word for "kind" so that the inventor's name was probably invented to reflect an avid player's perception of this game (blessed) and his appeal to the goddess of fortune (kind). Part of the mythology was in place by the 1660s. A broadside [Anon 1662] promoting a Genoese-style lottery in England claimed that the lottery was invented by the state of Genoa "for their pleasure and fortune in the choice of their senators."

A common modern lotto that is run in Canada and several European countries is the 6/49 in which six numbers are chosen from 49. One difference from the Genoese lotto is that the modern player chooses six numbers for a ticket and wins prizes depending upon how many of the player's numbers turn up in the draw rather than betting on one or two, or more, specific numbers to show. The other major difference is that in the modern lotto the prize distribution is based on a pari-mutuel system while in the

Genoese version there were fixed payoffs for each of the bets. The Genoese system could lead to bankruptcy of the lottery promoter if a large prize were won and only a few tickets were sold.

2. Euler's First Analysis

Whatever the true origins of the lotto were, it spread throughout Europe during the mid-eighteenth century. It first showed up in Vienna in 1752 [Palgrave 1912] under Empress Maria Teresa of Austria. France was next to take on this lottery; it was initially proposed by two Italians, writer and librettist Ranieri de' Calzabigi (also Calsabigi) and the adventurer Giacomo Casanova [Serwer 2005], [Stigler 2003]. Calzabigi was the first director of the French lotto, taking a cut of the proceeds, and Casanova received a pension and six sales offices as his reward. The lottery ran under the name *Loterie de l'École militaire*. Prussia, under Frederick II, followed in 1763 with the implementation of its lotto; the lottery ran out of Berlin. There was again a Calzabigi and Casanova connection. Ranieri's younger brother, Anton Maria, was the one who convinced Frederick to start the lottery [Serwer 2005]. Casanova arrived in Berlin two years after the beginning of the lottery. According to Casanova [1970], the Berlin lotto "was doing well, and fortunately had never suffered an unlucky drawing." By the time of Casanova's arrival, Frederick was trying to put the financial risk of the lottery operation onto Calzabigi and Casanova offered Calzabigi some help in trying to convince Frederick to continue to underwrite the risk. The last draw in 1765 lost Frederick twenty thousand crowns [Casanova 1970]. The next year Frederick had a new director of the Prussian lottery, the music critic, theorist and composer, Friedrich Wilhelm Marpurg [Brown 2005].

From Prussia the lotto spread through several other German states. The Austrian lottery is still running today under the name of Zahlenlotto; the French lottery ceased operation in 1836 and the Prussian lottery in 1810. By the eighteenth century, n was reduced from 100 to 90 and t remained at 5.

In the same year that Prussia ran its first lotto, Euler read a paper before the Berlin Academy giving a detailed and general analysis of it.[2] Bradley [Bradley 2004] has given a concise description of Euler's analysis of this lotto as well as Euler's later writings on this lotto. Euler's paper was published posthumously [E812]. One of the basic results that Euler

[2] Euler's papers and letters dealing with problems in probability, including lotteries, have been translated into English by Richard J. Pulskamp and can be found on the internet at http://cerebro.xu.edu/math/Sources/Euler/.

obtained was to find a formula for winning the bet of correctly guessing r of the t numbers chosen out of the total of n available at the lottery draw. He did this for the cases $r = 1 \ldots 5$ since $r = 1$ is the *extrait* bet, $r = 2$ is the *ambe* bet, and so on. Euler showed that the probability of correctly guessing r is given by

$$\frac{\prod_{i=1}^{r} (t - i + 1)}{\prod_{i=1}^{r} (n - i + 1)}.$$

This is the reduced form of the hypergeometric probability

$$\frac{\binom{r}{r}\binom{n-r}{t-r}}{\binom{n}{t}}$$

after expanding the combinatorial symbols and simplifying. Euler argued directly to the reduced form using products of conditional probabilities. Euler then went beyond the common simple bets and considered all possible betting outcomes that could be obtained from this game. Suppose at the draw t numbers are drawn from n and the player bets on r numbers to show. Euler found the probability that a player would be correct on $s = 0, 1, 2, \ldots, r$ of his numbers to show, again for the cases $r \leq 5$. Euler obtained, again in reduced form, the hypergeometric probabilities

$$\frac{\binom{r}{s}\binom{n-r}{t-s}}{\binom{n}{t}} \tag{1}$$

Using these probability calculations, Euler calculated three practical scenarios for payouts on all the bets and took into account the possibility of taking a profit for the lottery promoters. The payout on any bet was proportional to the inverse of the probability of winning that bet.

Euler's 1763 presentation of his analysis of the lotto before the Berlin Academy was probably motivated by the running of the first Prussian lotto that year. The results that Euler obtained all had a practical motivation in the context of a generalization of the 5/90 lottery bounded by the typical bets of the day on this lotto. Further, the origins of Euler's interest in the lotto probably stem from a practical question posed to him by his employer Frederick II in 1749. In a letter dated September 15 of that year [Euler 1986], Frederick asked Euler to examine a lottery scheme that had been proposed to him by another Italian, this one named Roccolini. Euler's response, which came two days later, contains all the elements and issues that he covered in 1763 but in a more simplified fashion.

There were previously published analyses of the Genoese lotto with at least two in the eighteenth century, [Bernoulli 1709] and [de Bessy 1729]. In Nicolaus Bernoulli's case the analysis formed part of his doctoral thesis on the use of probability in law. For Frenicle de Bessy, writing in the 1660s or 70s, it was an example he used to illustrate combinatorial calculations. Both obtained the result in (1) for $n = 100$, $t = 5$, $r = 5$ and $1 \leq s \leq 5$, i.e. only the cases of a player picking five numbers and correctly matching some or all of the five numbers chosen at the draw. Based on the payouts that were known in his time Bernoulli also computed the expected return in the lottery run in his day. Euler seems to have been unaware of these earlier analyses.

Why was the 1763 paper not published in Euler's lifetime? Was Euler informed of the earlier work and then chose not to publish? That seems unlikely since Euler's results were more general in scope than the earlier work. There are at least two other possible explanations. The first is that Euler may have presented his 1763 paper only to celebrate or advertise the first Prussian lotto and never intended to have his analysis printed. In a later comment on probability calculations regarding the prize structure in the lotto [E338], Euler said that probability calculations for the prizes were quite easy to obtain from basic probability arguments. Euler's initial analysis had been lying around since 1749 when he responded to his king's request; now that the lotto was actually being run it was time to make these basic and simple results public. Another explanation is through Euler's strained relationship with Frederick II, which is described in, for example, [Cajori 1927]. Euler handled most of the administrative work in the Berlin Academy after the death of the Academy's president Maupertuis in 1759. Despite Euler's work, Frederick allowed the presidency to remain open and then made himself president in 1763 after failing to convince D'Alembert to take the position. That same year Frederick wrote to Euler about another lottery proposition. In the letter [Euler 1986], Frederick asked Euler "not [to] make a scandal of it again" [3] with respect to the calculations he made on it. Perhaps Frederick was angered by Euler's 1763 presentation of his lotto analysis. It revealed what could or should be paid as prizes; and that

[3] From Pulskamp's translation of the letter.

almost certainly differed from what was actually paid, which probably was a much lesser amount. Consequently, the paper was not published.

3. Euler's Later Analyses

Euler's first analysis of the lotto was highly practical in the sense that it analyzed an actual lottery in play by giving rules for determining the payoffs in the prize structure. His next paper on the lotto was read before the Berlin Academy in 1765 [E338]. It bore no relation to any of the typical bets being made in the lotto and so was, perhaps, safe on political grounds. It was also an analysis of a challenging mathematical problem using the lotto as a model. The problem was to determine the probability of seeing sequences of numbers in the lotto draw [E338]. Again in the lotto, at any draw, t numbers are chosen at random without replacement from the n available; a lotto player can bet on seeing $r \leq t$ of the numbers from the set of n numbers.

A sequence of length l is any run of l consecutive numbers in the numbers drawn. If t numbers are drawn for the lotto there is only one possible sequence of length t, two possible sequences, one of length $t-1$ and another of length 1 and so on to t possible sequences each of length 1 (or equivalently no sequences of consecutive numbers). The enumeration of sequences is a problem in partitions. Euler refers to the different types of partitions as "species". For $t = 5$ and general n the possible types of partitions are in

Euler's 'Species' Number	Species Descriptions	Lengths l	Total Number of Sequences	Number of distinct Types of Sequences
I	$a, a+1, a+2, a+3, a+4$	5	1	1
II	$a, a+1, a+2, a+3, b$	$4, 1$	1	2
III	$a, a+1, a+2, b, b+1$	$3, 2$	1	2
IV	$a, a+1, a+2, b, c$	$3, 1, 1$	3	2
V	$a, a+1, b, b+1, c$	$2, 2, 1$	3	2
VI	$a, a+1, b, c, d$	$2, 1, 1, 1$	4	2
VII	a, b, c, d, e	$1, 1, 1, 1, 1$	5	1

Table 1
The enumeration of species for $t = 5$

table 1, where a, b, c, d and e are integers such that $1 \le a, b, c, d, e \le n$.

The probability of seeing a certain type of "species" or partition appear is the number of ways in which the partition can occur divided by the number of selections of the t numbers from n. Euler proceeds to find these probabilities. On a case-by-case basis Euler evaluated all the probabilities for each type of partition for general n and $t = 2, 3, 4, 5, 6$. Once he worked his way up to $t = 6$, Euler then gave a general formula for the probability. Suppose that the lotto draw, of t numbers from n, yields a partition in which there are k sequences with only p distinct types of them. Suppose further that in the partition there are α_i sequences each of length l_i , where $i = 1, \ldots, p$. Then $\sum_{i=1}^{p} \alpha_i l_i = t$ and $\sum_{i=1}^{p} \alpha_i = k$. Euler's formula for the probability that this particular partition will show is

$$\frac{\prod_{j=1}^{k} (n - t - j + 2) \Big/ \prod_{i=1}^{p} \alpha_i!}{\binom{n}{t}}. \tag{2}$$

For example, in the table above take Euler's "Species [partition] IV". In this case $p = 2$, $k = 3$ with $\alpha_1 = 1$, $l_1 = 3$, $\alpha_2 = 2$, and $l_2 = 1$. Then $1 \cdot 3 + 2 \cdot 1 = 5 = t$, $1 + 2 = 3 = k$ and the probability in (2) reduces to

$$\frac{3 \cdot 4 \cdot 5(n-5)(n-6)}{n(n-1)(n-2)(n-3)},$$

as obtained by Euler for this special case. As noted in [Bradley 2004] prior to this lottery paper Euler had already studied partitions extensively.

Euler's next research into lotteries was again motivated by a practical problem. In 1763 a proposal was made to Frederick II for a lottery to be run in Cleves, one of the territories under Prussian control. Frederick wrote to Euler about the proposal to ask his opinion of it. It was a blanks and prizes lottery with a twist to it. There were five classes, or sets of drawings, of the lottery over the course of a year. The lottery was to be run yearly for ten years. In each of the five draws there were 50,000 ticket numbers with 8,000 prizes. The twist was that at the end of the five draws if a ticket number had never been selected for a prize it was given a small consolation prize for up to 30,000 tickets. Euler noted the uncertainty involved for the lottery promoter since there could be as few as 10,000 consolation prizes if all the numbers were different over the five draws and as high as 42,000 if the same set of 8000 numbers were drawn five times in a row. Euler also noted that it was highly unlikely that the extremes would occur and thought that the number of consolation prizes was more likely to be around the middle value of 26,000.

Although Euler's analysis of this lottery was published in the proceedings of the Berlin Academy [E412], it was not submitted to the Academy until 1769, after Euler had returned to St. Petersburg. This may have reflected Frederick's possible unease with a public analysis of a lottery in operation. Euler's practical motivation for his analysis that he gave in his paper was a thinly disguised version of the 1763 proposal given to him by Frederick. Euler stated that the lottery proposal shown to him had five classes with 10,000 tickets and 1,000 prizes, again with a consolation prize if the ticket number had not been drawn in any of the five draws. He immediately generalized the problem to k classes with m tickets and p prizes. He then went on to calculate the complete probability distribution for the number of consolation prize winners at the end of the k draws. From a very complex distribution he showed that the expected payoff for a consolation prize valued at 1 per winner is very simply expressed as

$$(m-p)\left(\frac{m-p}{m}\right)^{k-1}. \tag{3}$$

For the lottery that Frederick initially asked Euler about, the expected number of winners, obtained on setting $m = 50,000$ and $p = 8,000$ in (3), is 20,991 rather than the middle value 26,000. Euler generalized the problem further by allowing the number of prizes to vary over the draws.

Euler continued to use the lotto as a model for problems in probability. In 1785 he considered the following problem [E600]. Suppose a lotto in which t numbers are chosen without replacement from n is run d times. What is the probability that at least $n-x$ of the numbers show in the d draws of the lottery? Again, this is not motivated by a practical gambling problem. Rather, it is a difficult mathematical problem given in a real context. Euler showed that the probability is given by

$$\frac{\binom{n}{t}^d + \sum_{i=1}^{n-x-t} (-1)^i \binom{x+i-1}{x}\binom{n}{x+i}\binom{n-x-i}{t}^d}{\binom{n}{t}^d}.$$

Euler's method of demonstration of the results in the problem differs from his earlier work on lotteries. In his first three papers, Euler started from the simpler cases, derived the results and moved to the general case. In this paper Euler starts with the general solution and then provides some examples.

The general problem tackled by Euler in this last paper originates in Problems 18 and 19 of De Moivre[De Moivre 1711]. There De Moivre used a single die as his model so that for his situation $t = 1$ and n is the number of faces on the die. Further, De Moivre considered the cases in which a specified number of the faces on the die show. [Todhunter 1865] (p. 160) notes that, besides Euler, a number of authors looked at the problem over the years 1772-1795. Euler wrote his paper after he had become blind so that he may not have been familiar with the papers on the same problem written in the 1770s.

Euler carried out some other work in probability, analysing the card games Treize and Pharaon, for example. He also investigated problems in the calculation of life annuities. Probability and its applications were not Euler's main mathematical interest, making up only a very small fraction of his total research output. What he did produce in the area of lotteries was of high quality, tackling both practical problems for actual lotteries that were run and difficult mathematical problems posed within a gambling framework.

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Euler's Science of Combinations

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In addition to all his other mathematical achievements, Euler discovered the first significant results in many fields of modern combinatorics. In this chapter, we survey this work spread over some fourteen publications.

In the first section, we consider Euler's work on partitions of integers, focusing on three articles and a book that span his career. The second section addresses various types of squares that Euler considered — magic, Graeco-Latin, and chessboards. The final section samples Euler's contributions to the study of binomial coefficients, the Catalan numbers, derangements, and the Josephus problem. We omit the bridges of Königsberg and the polyhedral formula as they are treated elsewhere in this volume.

1. Partitions

In 1699 Leibniz wrote to Johann Bernoulli asking about "divulsions of integers," now called partitions. A basic problem is determining the number $p(n)$ of ways that a positive integer n can be written as the sum of positive integers; for example, $p(4) = 5$, corresponding to the sums 4, $3 + 1$, $2 + 2$, $2+1+1$ and $1+1+1+1$. Variations of this basic problem ask for partitions of n into a given number of parts, or into distinct parts, odd parts, etc. For example, we can write the number 10 as the sum of exactly three positive numbers in eight ways,

$$\begin{array}{cccc} 8+1+1 & 7+2+1 & 6+3+1 & 6+2+2 \\ 5+4+1 & 5+3+2 & 4+4+2 & 4+3+3 \end{array}$$

Notice that four of these are partitions with distinct parts.

The first publication on partitions of integers came from a presentation that Euler made in 1741 to the St. Petersburg Academy [E158]. Euler answered two questions posed by Philip Naudé and stated what became known as the pentagonal number theorem. We present his arguments from a later publication, his celebrated *Introductio in Analysin Infinitorum* [E101], in which his main method of proof is generating functions; Euler often repeated his results on partitions, in some cases providing multiple proofs.

Naudé's Question 1: In how many ways can the number 50 be written as the sum of seven distinct positive integers? To answer this, Euler considered the following infinite product in x and z, organized in increasing powers of z.

$$\begin{aligned}(1+xz)\ &(1+x^2z)(1+x^3z)(1+x^4z)(1+x^5z)(1+x^6z)\cdots \\ = &\ 1+z(x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+\cdots) \\ &+z^2(x^3+x^4+2x^5+2x^6+3x^7+3x^8+4x^9+4x^{10}+\cdots) \\ &+z^3(x^6+x^7+2x^8+3x^9+4x^{10}+5x^{11}+7x^{12}+\cdots) \\ &+\cdots\end{aligned} \tag{1}$$

What does a term such as $4x^{10}z^3$ indicate? Each $x^{10}z^3$-term arises from one of the four products $x^7z\cdot x^2z\cdot xz$, $x^6z\cdot x^3z\cdot xz$, $x^5z\cdot x^4z\cdot xz$, and $x^5z\cdot x^3z\cdot x^2z$. These products correspond to the above four ways that we can write 10 as a partition of three distinct positive integers.

However, we do not want to have to compute the coefficient of $x^{50}z^7$ from the terms of this infinite product. Writing $m^{(\mu)i}$ for the number of ways of writing m as the sum of μ "inequal" integers, Euler established the following recurrence relation:

$$(m+\mu)^{(\mu)i} = m^{(\mu)i} + m^{(\mu-1)i}$$

With this, it is not hard to compute 522 as the answer to Naudé's first question.

Naudé's Question 2: In how many ways can the number 50 be written as the sum of seven positive integers, equal or unequal? Here Euler considered another infinite product in x and z, this time with factors in the denominator.

$$\begin{aligned}
&\frac{1}{(1-xz)\,(1-x^2z)\,(1-x^3z)\cdots} \\
&\quad = \left(\frac{1}{1-xz}\right)\left(\frac{1}{1-x^2z}\right)\left(\frac{1}{1-x^3z}\right)\cdots \\
&\quad = (1+xz+x^2z^2+x^3z^3+\cdots)(1+x^2z+x^4z^2+\cdots)\cdots \\
&\quad = 1+z(x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+\cdots) \\
&\qquad +z^2(x^2+x^3+2x^4+2x^5+3x^6+3x^7+4x^8+4x^9+\cdots) \\
&\qquad +z^3(x^3+x^4+2x^5+3x^6+4x^7+5x^8+7x^9+8x^{10}+\cdots) \\
&\qquad +\cdots
\end{aligned} \tag{2}$$

Here, a term such as $8x^{10}z^3$ indicates the eight ways (given above) that we can write 10 as the sum of three positive integers. Again, there is a recurrence relation. Writing $m^{(\mu)}$ when the parts need not be distinct, Euler established the following equation:

$$m^{(\mu)} = (m-\mu)^{(\mu)} + (m-1)^{(\mu-1)}$$

¿From this, we can determine that the answer to Naudé's second question is 8496. But Euler first established a connection between the two questions. He deduced the number of partitions with μ distinct parts from the formula

$$m^{(\mu)i} = \left(m - \frac{\mu(\mu-1)}{2}\right)^{(\mu)}$$

— in particular, the number of unrestricted seven-part partitions of 50 is equal to the number of distinct seven-part partitions of $50 + 21 = 71$. Euler also discussed the connection between the numbers of parts in a and the maximum number of parts; for example, 8496 is also the number of partitions of $50 - 7 = 43$ that use only the numbers $1, 2, \ldots, 7$.

This revolutionary paper ends with a celebrated formula that Euler had mentioned in his correspondence (see [B]), but had not yet proved. If we let $z = 1$ in equation (2) we can combine the expressions in x to give

$$\begin{aligned}
&\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots} \\
&\quad = 1+x+2x^2+3x^3+5x^4+7x^6+11x^7+15x^8+22x^9+\cdots
\end{aligned}$$

where the coefficient of x^k is the total number of unrestricted partitions of k. But now consider the reciprocal of this infinite product. From extensive computations Euler concluded that

$$\begin{aligned} P &= (1-x)(1-x^2)(1-x^3)\cdots \\ &= 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} \\ &\quad + x^{22} + x^{26} - x^{35} - x^{40} + x^{51} + \cdots \end{aligned} \tag{3}$$

where the exponents are the generalized pentagonal numbers, $(3k^2 \pm k)/2$. This result is now known as Euler's pentagonal number theorem.

Euler devoted a chapter of his 1748 *Introductio* [E101] to partitions, expanding on the results from the previous article. It includes one of the most striking and elegant applications of generating functions to partitions. Letting $z = 1$ in equation (1) and combining the expressions in x we obtain

$$\begin{aligned} Q &= (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6)\cdots \\ &= 1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + 6x^8 + 8x^9 + \cdots \end{aligned}$$

where the coefficient of x^k is the total number of partitions of k into distinct parts. Again, we consider the reciprocal of this infinite product. With the infinite product P as defined in equation (3), we note that the terms of

$$PQ = (1-x^2)(1-x^4)(1-x^6)\cdots$$

are factors of P, so that we can divide P by PQ. This leaves

$$\frac{P}{PQ} = \frac{(1-x)(1-x^2)(1-x^3)\cdots}{(1-x^2)(1-x^4)(1-x^6)\cdots} = (1-x)(1-x^3)(1-x^5)\cdots = \frac{1}{Q}$$

so that Q can now be written as

$$\begin{aligned} Q &= \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots} \\ &= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^3}\right)\left(\frac{1}{1-x^5}\right)\cdots \\ &= (1 + x + x^2 + x^3 + \cdots)(1 + x^3 + x^6 + x^9 + \cdots)\cdots \\ &= (1 + x + x^{1+1} + x^{1+1+1} + \cdots)(1 + x^3 + x^{3+3} + x^{3+3+3} + \cdots)\cdots \end{aligned}$$

in which the coefficient of x^k gives the number of partitions of k into odd parts, not necessarily distinct. This proves a surprising theorem:

> *The number of partitions of k into distinct parts equals the number of partitions of k into odd parts.*

As an example, note that there are six partitions of the number 8 into distinct parts (8, 7 + 1, 6 + 2, 5 + 3, 5 + 2 + 1, and 4 + 3 + 1) and six partitions of 8 into odd parts (7 + 1, 5 + 3, 5 + 1 + 1 + 1, 3 + 3 + 1 + 1, 3 + 1 + 1 + 1 + 1 + 1, and 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1).

The chapter concludes with generating-function proofs of the facts that each positive integer can be expressed uniquely as a sum of distinct powers of 2 and can also be uniquely expressed as a sum and difference of distinct powers of 3.

Euler continued his exploration of partitions with a paper presented in early 1750 [E191]. This is his longest article on partitions, filled not so much with new material, but rather with numerous examples and tables. Writing $n^{(\infty)}$ for what is now known as $p(n)$, he established that

$$n^{(\infty)} = (n-1)^{(1)} + (n-2)^{(2)} + (n-3)^{(3)} + \cdots + (n-n)^{(n)}$$

which he then used recursively to suggest a recurrence relation for $n^{(\infty)}$; this also follows from the still unproved pentagonal number theorem (3).

$$\begin{aligned} n^{(\infty)} = \ & (n-1)^{(\infty)} + (n-2)^{(\infty)} - (n-5)^{(\infty)} \\ & -(n-7)^{(\infty)} + (n-12)^{(\infty)} + \cdots \end{aligned}$$

Andrews [A] asserts that "No one has ever found a more efficient algorithm for computing $p(N)$. It computes a full table of values of $p(n)$ for $n \leq N$ in time $O(N^{3/2})$."

Later in 1750, in a letter to Christian Goldbach, Euler finally proved the pentagonal number theorem (3). He eventually published two proofs, and also considered properties of the pentagonal numbers themselves, such as that each pentagonal number is one-third of a triangular number; see Bell [B] for a detailed account of this pentagonal-number result throughout Euler's work. Interestingly, the function that sums the divisors of a number — for example, $\int 10 = 1 + 2 + 5 + 10 = 18$ in Euler's notation — shares almost the same recurrence relation; Euler also devoted several articles to this divisor function.

Euler returned to partitions once more in a presentation of 1768 [E394], in which he combined two previous restrictions on partitions – the number of parts and how large each part can be. The running example for much of the article used only $1, 2, \ldots, 6$ as parts, and Euler's computation of the coefficients in $(x + x^2 + \cdots + x^6)^n$ was simplified by the use of various recurrence relations; for example, Euler established that

$$n^{(6)} = \frac{(n-1)(n-1)^{(6)} - (48-n)(n-6)^{(6)} - (43-n)(n-7)^{(6)}}{n-6}$$

as an example of the formulas that can be derived from these methods.

The article also considers the problem of partitions with varying constraints. In particular, Euler considered three-part partitions where the first part is 6 or less, the second is 8 or less, and the third is 12 or less. The 576 resulting partitions of the numbers from 3 to 26 are specified by the coefficients of

$$(1 + x + \cdots + x^6)(1 + x + \cdots + x^8)(1 + x + \cdots + x^{12})$$

whose computations are simplified by generating-function techniques. The coefficients for x^k, $k = 3, 4, \ldots, 14$ are as follows:

exponent	3	4	5	6	7	8	9	10	11	12	13	14
coefficient	1	3	6	10	15	21	27	33	38	42	45	47

The coefficients of x^k for $k = 15, 16, \ldots, 26$ are the reverse of these, from 47 down to 1.

Although Euler was not the first mathematician to consider generating functions or partitions of integers — De Moivre [dM] had used generating functions in 1718 to analyze multiple-step recurrence relations — he was the first to treat them in a thorough and general way. A thorough early history of partitions, drawing on some of Euler's correspondence and including work of his contemporaries, is given in Dickson [D]. Generating functions have since become an essential tool in combinatorics and number theory, "a clothes line on which we hang up a sequence of numbers for display" (see Wilf [W]). Even though Hardy and Ramanujan obtained a stunning exact formula for the partition number $p(n)$, the theory of partitions has remained a very active area of research with many impressive results and many outstanding problems (see Andrews and Eriksson [AE]). Even so, we can still agree with Andrews [A] that "Almost every discovery in partitions owes something to Euler's beginnings."

2. Squares

In 1776, Euler delivered a short article *On magic squares* to the St. Petersburg Academy [E795]. Such an arrangement of integers, already long familiar, is an $n \times n$ square with the numbers $1, 2, \ldots, n^2$ arranged in such a way that the numbers in each row, each column, and each of the two diagonals have the same sum. After discussing what this sum must be, Euler introduced Latin and Greek letters to help him to analyze magic squares: each Latin letter stands for a multiple of n from 0 to $n(n-1)$ and each Greek letter has a value from 1 to n. With each individual cell assigned both a Latin and a Greek letter in such a way that no pair is repeated, he was able to determine values for the letters so that the sums give a magic square. An example of such a 3×3 square is given in Table 1. Note that the left-hand square has each letter occurring exactly once in each row and column, and is an example of what is now known as a Graeco-Latin square (because of Euler's notation).

A 4×4 non-Graeco-Latin square appears in Table 2, with the associated magic square obtained by using the specified values. Although the square is

$a\gamma$	$b\beta$	$c\alpha$
$b\alpha$	$c\gamma$	$a\beta$
$c\beta$	$a\alpha$	$b\gamma$

2	9	4
7	5	3
6	1	8

Table 1
Graeco-Latin square for $n = 3$ and associated magic square from $a = 0, b = 6, c = 3; \alpha = 1, \beta = 3, \gamma = 2$.

$a\alpha$	$a\delta$	$d\beta$	$d\gamma$
$d\alpha$	$d\delta$	$a\beta$	$a\gamma$
$b\delta$	$b\alpha$	$c\gamma$	$c\beta$
$c\delta$	$c\alpha$	$b\gamma$	$b\beta$

1	4	14	15
13	16	2	3
8	5	11	10
12	9	7	6

Table 2
Non-Graeco-Latin square for $n = 4$ and associated magic square from $a = 0, b = 4, c = 8, d = 12; \alpha = 1, \beta = 2, \gamma = 3, \delta = 4$.

not Graeco-Latin (notice that α appears twice in the first column, a twice in the first row), it still yields a magic square. Euler's article closes with descriptions (depending on whether n is even or odd) of how to construct magic squares of any size.

Euler remained interested in the problem of Graeco-Latin squares. Three years later, in 1779, he presented one of his longest published papers, *Investigations on a new type of magic square* [E530]. It begins with the celebrated "36-officers problem:"

> *Six regiments are each represented with six officers, one per rank — can they be placed in a 6 by 6 formation such that there is one officer of each regiment in each row and column, and one officer of each rank in each row and column?*

Euler claimed that the answer is no, and embarked on a thorough study of Graeco-Latin and Latin squares.

By the second page, Euler had replaced his Graeco-Latin notation with pairs of numbers, the second written in superscript; we give an example in Table 3. He gave several general methods for building Latin squares, of which the "double march" is illustrated in Table 3 on the right — notice how the square divides into four smaller Latin squares involving 1 and 2 or 3 and 4. There are also single, triple, and quadruple marches.

He also used these methods (and others) to build Graeco-Latin squares of odd order and orders that are multiples of 4; however, none of the methods produced a 6 × 6 Graeco-Latin square. Euler suspected that there are none, claiming that he would have come across one in his investigation if any existed, while recognizing that an exhaustive search would be very lengthy. There is no formal conjecture of the general case, but he stated

1^1	4^3	2^4	3^2
2^2	3^4	1^3	4^1
3^3	2^1	4^2	1^4
4^4	1^2	3^1	2^3

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

Table 3
Revised notation for Graeco-Latin squares, and example of a "double march."

that a Graeco-Latin square of order $4k + 2$ would have to be "completely irregular" and seemed to doubt that there are such. Euler's article also includes enumeration of Latin squares of small orders under certain conditions and a discussion of collections of Latin squares any two of which can be combined into a Graeco-Latin square. For example, notice in Table 3 that the Latin square of base numbers on the left can be combined with the Latin square on the right to make a Graeco-Latin square, and the same is true for the Latin square of superscript numbers.

An exhaustive search in the next century verified Euler's conjecture for 6×6 Graeco-Latin squares, showing that Euler was right about the 36 officers problem. However, the general $4k + 2$ conjecture was shown to be false in 1960 by Bose, Shrikhande and Parker (see Klyve and Stemkoski [KS] for details); this result was so unexpected that it was reported on the front page of the *New York Times*. A related research area is that of finding "mutually orthogonal Latin squares:" are there $n - 1$ Latin squares of size $n \times n$ with the property that any two of them constitute a Graeco-Latin square? (The three 4×4 Latin squares of Table 3 are an example.) This is an area of contemporary research (see Mullen [M]).

Recently, there have been uninformed claims in the media that Euler invented the popular number puzzle Sudoku in which 9×9 Latin squares satisfy the additional requirement that no number should be repeated in the principal 3×3 subsquares. While a completed Sudoku puzzle is a Latin square, none of Euler's 9×9 examples of Latin squares has the form of a Sudoku puzzle. The closest he came were the 4×4 examples of Table 3 (each set of numbers in the left-hand square), which coincidentally have the additional structure that each 2×2 corner contains 1, 2, 3, and 4.

However, Euler did write on a topic in recreational mathematics that relates to squares. His paper *Solution of a curious question that does not seem to have been subject to any analysis* is based on a 1759 presentation to the Berlin Academy about knight's tours on chess boards of various sizes [E309]. The question is how to have a knight make its L-shaped moves around the board and visit each square exactly once (now known as a Hamiltonian cycle!). Euler demonstrated many such tours on standard 8×8 and other size boards, often producing tours with high degrees of symmetry. A knight's tour is shown by labeling consecutive positions, as in Tables 4 and 5.

37	62	43	56	35	60	41	50
44	55	36	61	42	49	34	59
63	38	53	46	57	40	51	48
54	45	64	39	52	47	58	33
1	26	15	20	7	32	13	22
16	19	8	25	14	21	6	31
27	2	17	10	29	4	23	12
18	9	28	3	24	11	30	5

Table 4
Closed knight's tour of an 8×8 board with half-turn symmetry about the center of the board.

For an $n \times n$ board, the labels are $1, 2, \ldots, n^2$ — could a knight's tour give rise to a magic square? This is not a question Euler posed; the closest such path given in the article is the 5×5 example of Table 5: the diagonals, as well as rows and columns including the center, all sum to 65, but this seems coincidental. Computers have recently been used to conclude that there are no 8×8 "Euler knight tours," but if the requirement about diagonal sums is removed, then there are 140 such tours (see Jelliss [J]).

7	12	17	22	5
18	23	6	11	16
13	8	25	4	21
24	19	2	15	10
1	14	9	20	3

Table 5
Non-closed knight's tour of a 5×5 board.

3. Other Topics

Binomial coefficients

Many of Euler's articles incorporate binomial coefficients; here we highlight three articles that consider properties of these numbers with integer arguments.

In a 1776 presentation to the St. Petersburg Academy [E575], primarily about integrals, Euler collected several facts about binomial coefficients, using notations very similar to those used today. One primary result, in modern notation, is the following equation:

$$\binom{n}{0}\binom{p}{q}+\binom{n}{1}\binom{p}{q+1}+\cdots=\binom{p+n}{q+n}$$

Later in the same year, Euler presented an article generalizing binomial coefficients to higher-degree polynomials [E584]. He first reviewed the relationship between

$$\binom{n}{p}$$

and the coefficients of $(1+z)^n$, and properties such as the sum of squares (a special case of the preceding formula) and

$$\binom{n+1}{p+1}=\binom{n}{p}+\binom{n}{p+1}$$

Euler then moved on to trinomial, quadrinomial, and higher-order coefficients. In particular, the coefficients of $(1+z+zz+z^3)^n$ (to use his notation for squares) for small values of n are given in the following partial table, where the columns correspond to the degree of z.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	1													
1	1	1	1	1										
2	1	2	3	4	3	2	1							
3	1	3	6	10	12	12	10	6	3	1				
4	1	4	10	20	31	40	44	40	31	20	10	4	1	
5	1	5	15	35	65	101	135	155	155	135	101	65	35	15
6	1	6	21	56	120	216	336	456	546	580	546	*etc.*		

Table 6
Coefficient of z^k in $(1+z+z^2+z^3)^n$, the quadrinomial coefficients.

He showed that these coefficients, indexed here with 4, satisfy

$$\binom{n+1}{p+3}_4=\binom{n}{p+3}_4+\binom{n}{p+2}_4+\binom{n}{p+1}_4+\binom{n}{p}_4$$

and the general relationship

$$\binom{n}{0}_4\binom{m}{0}_4+\binom{n}{1}_4\binom{m}{1}_4+\cdots=\binom{n+m}{3n}_4.$$

In 1778, Euler returned to these coefficients in another presentation in the same venue [E709]. By writing

$$(1 + z + zz + z^3)^n = (1 + z(1 + z + zz))^n,$$

he related the quadrinomial coefficients to the binomial and trinomial ones. For example, again writing subscripts for the degree so that binomial coefficients are indexed by 2, we have

$$\begin{aligned}\binom{n}{4}_4 &= \binom{n}{4}_2\binom{4}{0}_3 + \binom{n}{3}_2\binom{3}{1}_3 + \binom{n}{2}_2\binom{2}{2}_3 + \binom{n}{1}_2\binom{1}{3}_3 \\ &= \binom{n}{4}_2 + 3\binom{n}{3}_2 + 3\binom{n}{2}_2\end{aligned}$$

Catalan numbers

In a letter of 4 September 1751 to Christian Goldbach [EG], Euler discussed the problem of finding the number of different ways that a polygon can be broken into triangles using diagonals. After considering several examples, he gave the formula

$$\frac{2 \cdot 6 \cdot 10 \cdot 14 \cdot 18 \cdot 22 \cdots (4n-10)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdots (n-1)}$$

which we now call the (n-2)nd Catalan number, usually written as

$$\frac{1}{n-1}\binom{2n-4}{n-2}.$$

Euler closed his letter with the generating function associated with this sequence, a topic that he and Goldbach discussed in subsequent correspondence:

$$1 + 2a + 5a^2 + 14a^3 + 42a^4 + 132a^5 + \cdots = \frac{1 - 2a - \sqrt{1-4a}}{2a^2}$$

Derangements

Many of Euler's articles discuss probability and games of chance, especially lotteries. One that is relevant here is his *Calculation of the probability in the game of coincidence* [E201], published in 1753. Two players have identical decks of cards, shuffled, which they turn over one at a time. If they turn over the same card at any turn, the first player wins and the game ends. The second player wins only if the cards are different at every turn. Euler explained that this is equivalent to numbering the cards and checking to see if the second player turns over card n on turn n, and showed that the probability of the second player winning is

$$\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \cdots = \frac{1}{e}$$

In 1779, Euler returned to this topic with *The solution of a curious question in the science of combinations*, presented to the St. Petersburg Academy [E738]. No longer motivated by the card game, he asked for the number of ways that the sequence $a, b, c, d, e, \ldots$, can be reordered such that no letter is in its original position. We will write $D(n)$ for this, suggesting the later name "derangement" for such an ordering. Euler derived the following two identities, and showed their equivalence:

$$D(n) = (n-1)(D(n-1) + D(n-2)),$$
$$D(n) = nD(n-1) + (-1)^n$$

The Josephus problem

We close with another topic in recreational mathematics, a staple of discrete mathematics textbooks. Suppose that n people stand in a circle. Moving clockwise, we remove every kth person. Which person is the last to be removed? This is known as the Josephus Flavius problem, named for the Jewish historian and general and an intricate suicide pact which left him the last man standing. In *Observations about a new and singular type of progression*, presented to the St. Petersburg Academy in 1771 [E476], Euler included several tables of data. For instance, with fifteen people, removing every fourth one gives the following order of removal:

$$4, 8, 12, 1, 6, 11, 2, 9, 15, 10, 5, 3, 7, 14, 13$$

He then analyzed the general problem to develop a recursive procedure for determining the number of the last person removed. In many cases the recursive step is just adding the number skipped to the previous answer. To demonstrate that the procedure is feasible, Euler gave the computations to show that if there are 5000 people and every ninth person is removed, then the last one standing is number 4897.

Note: Euler's publications are cited below by their Eneström number. All are reprinted in *Leonhard Euleri Opera omnia*, abbreviated *OO*. Most are available electronically at *The Euler Archive*, http://eulerarchive.org, which also links to some English translations.

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Leonhard Euler: Life, Work and Legacy
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The Truth about Königsberg

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Euler's 1736 paper on the bridges of Königsberg is widely regarded as the earliest contribution to graph theory–yet Euler's solution made no mention of graphs. In this paper [1] we place Euler's views on the Knigsberg problem in their historical context, present his method of solution, and trace the development of the present-day solution.

1. What Euler didn't do

A well-known recreational puzzle concerns the bridges of Königsberg. It is claimed that in the early eighteenth century the citizens of Königsberg used to spend their Sunday afternoons walking around their beautiful city. The city itself consisted of four land areas separated by branches of the river Pregel over which there were seven bridges, as illustrated in Figure 1.

---

[1] This article has previously appeared in *The College Mathematics Journal*, **35**(3), pp. 198–207. It won The Mathematical Association of America's George Pólya Award in 2005. It is included here with the kind permission of The Mathematical Association of America.

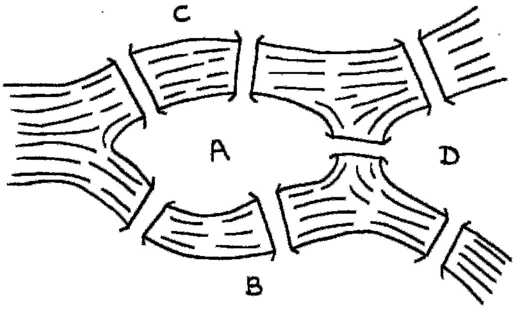

Fig. 1. Königsberg

The problem that the citizens set themselves was to walk around the city, crossing each of the seven bridges exactly once and, if possible, returning to their starting point.

If you look in some books on recreational mathematics, or listen to some graph-theorists who should know better, you will 'learn' that Leonhard Euler investigated the Königsberg bridges problem by drawing a graph of the city, as in Figure 2, with a vertex representing each of the four land areas and an edge representing each of the seven bridges. The problem is then to find a trail in this graph that passes along each edge just once.

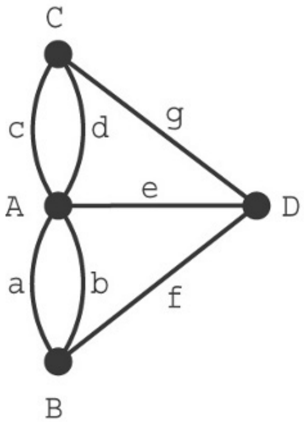

Fig. 2. The Königsberg graph

But Euler didn't draw the graph in Figure 2—graphs of this kind didn't make their first appearance until the second half of the nineteenth century. So what exactly did Euler do?

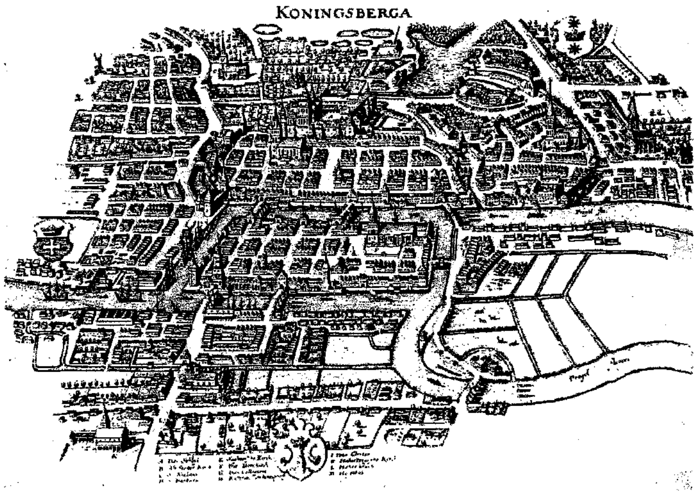

Fig. 3. Seventeenth-century Königsberg

2. The Königsberg bridges problem

In 1254 the Teutonic knights founded the Prussian city of Königsberg (literally, king's mountain). With its strategic position on the river Pregel, it became a trading center and an important medieval city. The river flowed around the island of Kneiphof (literally, pub yard) and divided the city into four regions connected by seven bridges: Blacksmith's bridge, Connecting bridge, High bridge, Green bridge, Honey bridge, Merchant's bridge, and Wooden bridge: Figure 3 shows a seventeenth-century map of the city. Königsberg later became the capital of East Prussia and more recently became the Russian city of Kaliningrad, while the river Pregel was renamed Pregolya.

In 1727 Leonhard Euler began working at the Academy of Sciences in St Petersburg. He presented a paper to his colleagues on 26 August 1735 on the solution of 'a problem relating to the geometry of position': this was the Königsberg bridges problem. He also addressed the generalized problem: given any division of a river into branches and any arrangement of bridges, is there a general method for determining whether such a route exists?

In 1736 Euler wrote up his solution in his celebrated paper in the *Commentarii Academiae Scientiarum Imperialis Petropolitanae* under the title 'Solutio problematis ad geometriam situs pertinentis' [2], numbered E53 in the Eneström index. Euler's diagram of the Königsberg bridges appears in Figure 4. Although dated 1736, Euler's paper was not actually published until 1741, and was later reprinted in the new edition of the *Commentarii* (*Novi Commentarii* ...) which appeared in 1752.

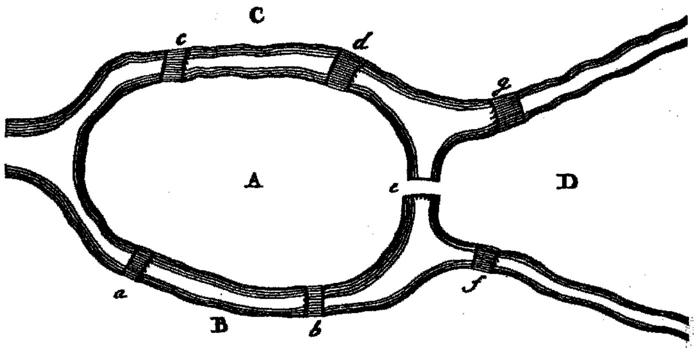

Fig. 4. Diagram from Euler's 1736 paper

A full English translation of this paper appears in several places—for example, in [1] and [6]. The paper begins:

> 1. In addition to that branch of geometry which is concerned with distances, and which has always received the greatest attention, there is another branch, hitherto almost unknown, which Leibniz first mentioned, calling it the geometry of position [*Geometriam situs*]. This branch is concerned only with the determination of position and its properties; it does not involve distances, nor calculations made with them. It has not yet been satisfactorily determined what kinds of problem are relevant to this geometry of position, or what methods should be used in solving them. Hence, when a problem was recently mentioned which seemed geometrical but was so constructed that it did not require the measurement of distances, nor did calculation help at all, I had no doubt that it was concerned with the geometry of position—especially as its solution involved only position, and no calculation was of any use. I have therefore decided to give here the method which I have found for solving this problem, as an example of the geometry of position.
>
> 2. The problem, which I am told is widely known, is as follows: in Königsberg ...

This reference to Leibniz and the geometry of position dates back to 8 September 1679, when the mathematician and philosopher Gottfried Wilhelm Leibniz wrote to Christiaan Huygens as follows [5]:

> I am not content with algebra, in that it yields neither the shortest proofs nor the most beautiful constructions of geometry. Consequently,

in view of this, I consider that we need yet another kind of analysis, geometric or linear, which deals directly with position, as algebra deals with magnitudes ...

Leibniz introduced the term *analysis situs* (or *geometria situs*), meaning the analysis of situation or position, to introduce this new area of study. Although it is sometimes claimed that Leibniz had vector analysis in mind when he coined this phrase (see, for example, [8] and [11]), it was widely interpreted by his eighteenth-century followers as referring to topics that we now consider 'topological'—that is, geometrical in nature, but with no reference to metrical ideas such as distance, length or angle.

3. Euler's Königsberg letters

It is not known how Euler became aware of the Königsberg bridges problem. However, as we shall see, three letters from the Archive Collection of the Academy of Sciences in St Petersburg [3] shed some light on his interest in the problem (see also [10]).

Carl Leonhard Gottlieb Ehler was the mayor of Danzig in Prussia (now Gdansk in Poland), some 80 miles west of Königsberg. He corresponded with Euler from 1735 to 1742, acting as intermediary for Heinrich Kühn, a local mathematics professor. Their initial communication has not been recovered, but a letter of 9 March 1736 indicates they had discussed the problem and its relation to the 'calculus of position':

> You would render to me and our friend Kühn a most valuable service, putting us greatly in your debt, most learned Sir, if you would send us the solution, which you know well, to the problem of the seven Königsberg bridges, together with a proof. It would prove to be an outstanding example of the calculus of position [*Calculi Situs*], worthy of your great genius. I have added a sketch of the said bridges ...

Euler replied to Ehler on 3 April 1736, outlining more clearly his own attitude to the problem and its solution:

> ... Thus you see, most noble Sir, how this type of solution bears little relationship to mathematics, and I do not understand why you expect a mathematician to produce it, rather than anyone else, for the solution is based on reason alone, and its discovery does not depend on any mathematical principle. Because of this, I do not know why even questions which bear so little relationship to mathematics are solved more quickly by mathematicians than by others. In the meantime, most noble Sir, you have assigned this question to the geometry of position, but I am igno-

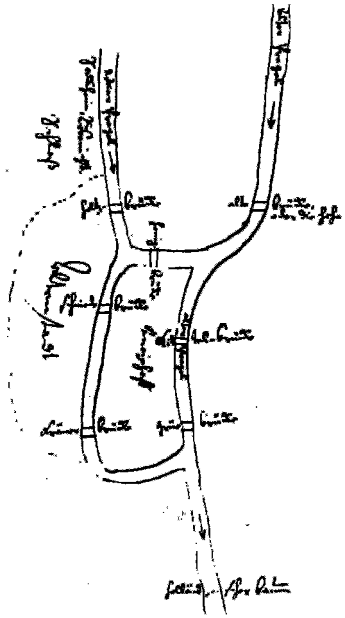

Fig. 5. Ehler's letter to Euler

> rant as to what this new discipline involves, and as to which types of problem Leibniz and Wolff expected to see expressed in this way ...

Around the same time, on 13 March 1736, Euler wrote to Giovanni Marinoni, an Italian mathematician and engineer who lived in Vienna and was Court Astronomer in the court of Kaiser Leopold I. He introduced the problem as follows (see Figure 6):

> A problem was posed to me about an island in the city of Königsberg, surrounded by a river spanned by seven bridges, and I was asked whether someone could traverse the separate bridges in a connected walk in such a way that each bridge is crossed only once. I was informed that hitherto no-one had demonstrated the possibility of doing this, or shown that it is impossible. This question is so banal, but seemed to me worthy of attention in that geometry, nor algebra, nor even the art of counting was sufficient to solve it. In view of this, it occurred to me to wonder whether it belonged to the geometry of position [*geometriam Situs*], which Leibniz had once so much longed for. And so, after some deliberation, I obtained a simple, yet completely established, rule with whose help one can immediately decide for all examples of this kind, with any number of bridges in any arrangement, whether such a round trip is possible, or not ...

4. Euler's 1736 paper

Euler's paper is divided into twenty-one numbered paragraphs, of which the first ascribes the problem to the geometry of position as we saw above, the next eight are devoted to the solution of the Königsberg bridges problem itself, and the remainder are concerned with the general problem. More specifically, paragraphs 2–21 deal with the following topics (see also [12]):

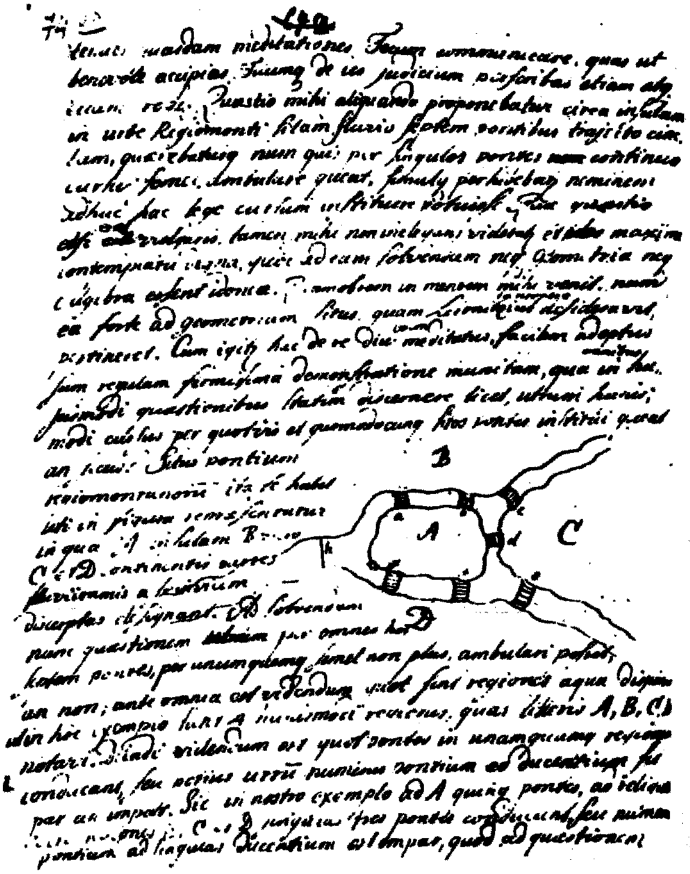

Fig. 6. Euler's letter to Marinoni

Paragraph 2. Euler described the problem of the Königsberg bridges and its generalization: 'whatever be the arrangement and division of the river into branches, and however many bridges there be, can one find out whether or not it is possible to cross each bridge exactly once?'

Paragraph 3. In principle, the original problem could be solved exhaustively by checking all possible paths, but Euler dismissed this as 'laborious' and impossible for configurations with more bridges.

Paragraphs 4–7. The first simplification is to record paths by the land regions rather than bridges. Using the notation in Figure 4, going south from Kneiphof would be notated AB whether one used the Green Bridge or the Blacksmith's Bridge. The final path notation will need to include an adjacent A and B twice; the particular assignment of bridges a and b is irrelevant. A path signified by n letters corresponds to crossing $n-1$ bridges, so a solution to the Königsberg problem requires an eight-letter path with two adjacent A/B pairs, two adjacent A/C pairs, one adjacent A/D pair, etc.

Paragraph 8. What is the relation between the number of bridges connecting a land mass and the number of times the corresponding letter occurs in the path? Euler developed the answer from a simpler example (see Figure 7). If there is an odd number k of bridges, then the letter must appear $(k+1)/2$ times.

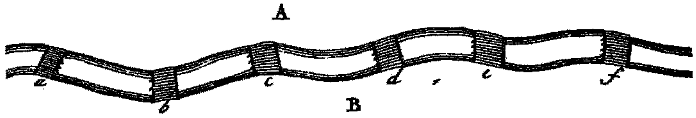

Fig. 7. A simple case

Paragraph 9. This is enough to establish the impossibility of the desired Königsberg tour. Since Kneiphof is connected by five bridges, the path must contain three As. Similarly, there must be two Bs, two Cs, and two Ds. In *Paragraph 14*, Euler records these data in a table.

region	A	B	C	D
bridges	5	3	3	3
frequency	3	2	2	2

Summing the final row gives nine required letters, but a path using each of the seven bridges exactly once can have only eight letters. Thus there can be no Königsberg tour.

Paragraphs 10–12. Euler continued his analysis from *Paragraph 8:* if there is an even number k of bridges connecting a land mass, then the corresponding letter appears $k/2+1$ times if the path begins in that region, and $k/2$ times otherwise.

Paragraphs 13–15. The general problem can now be addressed. To illustrate the method Euler constructed an example with two islands, four rivers, and fifteen bridges (see Figure 8).

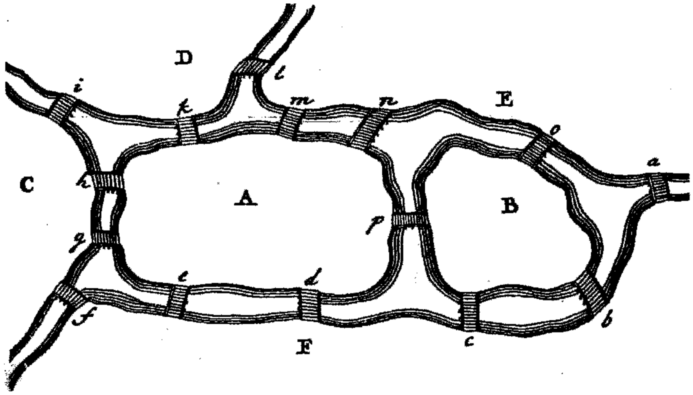

Fig. 8. A more complicated example

This system has the following table, where an asterisk indicates a region with an even number of bridges.

region	A^*	B^*	C^*	D	E	F^*
bridges	8	4	4	3	5	6
frequency	4	2	2	2	3	3

The frequencies of the letters in a successful path are determined by the rules for even and odd numbers of bridges, developed above. Since there can be only one initial region, he records $k/2$ for the asterisked regions. If the frequency sum is one less than the required number of letters, there is a path using each bridge exactly once that begins in an asterisked region. If the frequency sum equals the required number of letters, there is a path that begins in an unasterisked region. This latter possibility is the case here: the frequency sum is 16, exactly the number of letters required for a path using 15 bridges. Euler exhibited a particular path, including the bridges:

$$E\ a\ F\ b\ B\ c\ F\ d\ A\ e\ F\ f\ C\ g\ A\ h\ C\ i\ D\ k\ A\ m\ E\ n\ A\ p\ B\ o\ E\ l\ D.$$

Paragraph 16–19. Euler continued with a simpler technique, observing that:

> ... the number of bridges written next to the letters A, B, C, etc. together add up to twice the total number of bridges. The reason for this

> is that, in the calculation where every bridge leading to a given area is counted, each bridge is counted twice, once for each of the two areas which it joins.

This is the earliest version known of what is now called the *handshaking lemma*. It follows that in the bridge sum, there must be an even number of odd summands.

Paragraph 20. Euler stated his main conclusions:

> If there are more than two areas to which an odd number of bridges lead, then such a journey is impossible.
> If, however, the number of bridges is odd for exactly two areas, then the journey is possible if it starts in either of these two areas.
>
> If, finally, there are no areas to which an odd number of bridges lead, then the required journey can be accomplished starting from any area.

Paragraph 21. Euler concluded by saying:

> When it has been determined that such a journey can be made, one still has to find how it should be arranged. For this I use the following rule: let those pairs of bridges which lead from one area to another be mentally removed, thereby considerably reducing the number of bridges; it is then an easy task to construct the required route across the remaining bridges, and the bridges which have been removed will not significantly alter the route found, as will become clear after a little thought. I do not therefore think it worthwhile to give any further details concerning the finding of the routes.

Note that this final paragraph does not prove the existence of a journey when one is possible, apparently because Euler did not consider it necessary. So Euler provided a rigorous proof only for the first of the three conclusions. The first satisfactory proof of the other two results did not appear until 1871, in a posthumous paper by Carl Hierholzer (see [1] and [4]).

5. The modern solution

The approach mentioned in the first section developed through diagram-tracing puzzles discussed by Louis Poinsot [7] and others in the early-nineteenth century. The object is to determine whether a figure can be drawn with a single stroke of the pen in such a way that no edge is repeated. Considering the figure to be drawn as a graph, the general conditions in *Paragraph 20* take the following form:

If there are more than two vertices of odd degree, then such a drawing is impossible.

If, however, exactly two vertices have odd degree, then the drawing is possible if it starts with either of these two vertices.

If, finally, there are no vertices of odd degree, then the required drawing can be accomplished starting from any vertex.

So the 4-vertex graph shown in Figure 2, with one vertex of degree 5 and three vertices of degree 3, cannot be drawn with a single stroke of the pen so that no edge is repeated. In contemporary terminology, we say that this graph is not Eulerian. The arrangement of bridges in Figure 8 can be similarly represented by the graph in Figure 9, with six vertices and fifteen edges. Exactly two vertices (E and D) have odd degree, so there is a drawing that starts at E and ends at D, as we saw above. This is sometimes called an Eulerian trail.

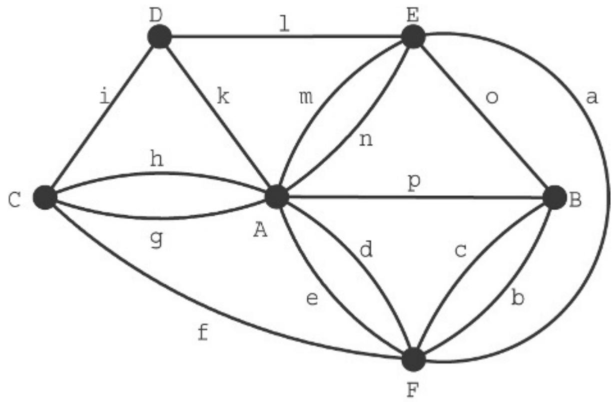

Fig. 9. The graph of the bridges in Figure 8

However, it was some time until the connection was made between Euler's work and diagram-tracing puzzles. The 'Königsberg graph' of Figure 2 made its first appearance in W. W. Rouse Ball's *Mathematical Recreations and Problems of Past and Present Times* [9] in 1892.

Background information, including English translations of the papers of Euler [2] and Hierholzer [4], can be found in [1]; an English translation of Euler's paper also appears in [6].

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Leonhard Euler: Life, Work and Legacy
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The Polyhedral Formula

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On November 14, 1750 Leonhard Euler sent a letter from Berlin to his friend Christian Goldbach in St. Petersburg announcing his discovery of a simple relationship between the features on a polyhedron [19, p. 332–3]. This observation, now known as Euler's polyhedral formula, is one of the most beloved theorems in mathematics. (A 1990 survey of mathematicians found the polyhedral formula to be the second most beautiful theorem in all of mathematics [40].) Euler stated the theorem as follows [11] [1].

THEOREM: *In every solid enclosed by plane faces, the number of faces along with the number of solid angles exceeds the number of edges by two.*

This relationship is typically expressed as $F - E + V = 2$ where F, E, and V denote the number of faces, edges, and vertices of a polyhedron.

Euler wrote two papers on the polyhedral formula, both published in 1758. The first paper, written in 1750, contains the statement of the theorem [10], and the second, written the following year, contains his proof [11] (henceforth we shall refer to them by their Eneström index numbers, E230 and E231, respectively). Euler wrote these two papers because he was interested in classifying all polyhedra. He wanted to develop the theory of *stereometry* (solid geometry) just as it had been developed for *planimetry* (planar geometry). He did not achieve his goal of classifying all polyhedra, and he never returned to this topic after publishing these two papers.

Euler's seemingly elementary observation proved to be an important theorem in mathematics that was generalized in many directions. The ideas

[1] A full English translation of [11] can be found at [12].

contained in Euler's formula were later extended to polyhedra with non-trivial topology, polyhedra in higher dimensions, planar graphs, topological of surfaces, other topological spaces, and abstract algebraic entities. From these generalizations countless applications were found.

In this paper we present Euler's proof of the polyhedral formula. We look closely at his hypotheses and his proof, discuss the flaw in his argument and show how it can be repaired. We also present the related work of other mathematicians prior to 1850. It is during this period that the theory for polyhedra develops, whereas after 1850 the focus becomes much more topological.

1. The polyhedral formula

In his letter to Goldbach, Euler wrote, "It astonishes me that these general properties of stereometry have not, as far as I know, been noticed by anyone else" [19]. In 1750 all of the accumulated knowledge about polyhedra was metric. There were many theorems and formulas about volume, surface area, angle measures, inscribability, etc. No one prior to Euler (except, as we will see, Descartes) looked at polyhedra with an eye toward their combinatorial properties.

It was not only the formula that went unnoticed prior to 1750. In this same letter Euler described "the junctures where two faces come together along their sides, which, for lack of an accepted term, I call *acies*" [19]. Until he gave them a name, no one had explicitly referred to the edges of a polyhedron. *Acies* is a Latin term which is comonly used for the sharp edge of a weapon, a beam of light, or an army lined up for battle. Giving a name to this feature may seem to be a trivial point, but it is not. It is a crucial observation that the edge of a polyhedron is an important feature to count. For the faces of a polyhedron Euler uses the well established Latin term *hedra*, which translates to face or base. He refers to the vertices of a polyhedron as *angulus solidus*, or solid angles.

It is clear that Euler understood the importance of these three features. In E230 he wrote [10]:

> Therefore three kinds of bounds are to be considered in any solid body; namely 1) points, 2) lines and 3) surfaces, or, with the names specially used for this purpose: 1) solid angles, 2) edges and 3) faces. These three kinds of bounds completely determine the solid.

Viewed in this light we see that Euler's formula is a way of relating objects of different dimensions – the zero-dimensional vertices, one-dimensional edges, and two-dimensional faces. This theorem and Euler's 1736 solution to the Bridges of Königsberg problem [9] were among the earliest contributions to the young field of *analysis situs*, or topology.

In E230 Euler begins his study of stereometry. In this paper he states the polyhedral formula, and he verifies that it holds for a variety of polyhedra, but he is unable to give a proof. In E231 he recalls:

> After the consideration of many types of solids I came to the point where I understood that the properties which I had perceived in them clearly extended to all solids, even if it was not possible for me to show this in a rigorous proof. Thus, I thought that those properties should be included in that class of truths which we can, at any rate, acknowledge, but which it is not possible to prove.

Then in E231 he gives a proof. The idea of the proof is to cut away vertices, one at a time, until four vertices remain. This excision is done in such a way that $F-E+V$ remains unchanged at each step. At the end the resulting polyhedron is a triangular pyramid which satisfies the polyhedral formula. Thus the original polyhedron does as well. We now give Euler's argument in more detail.

Begin with a polyhedron P having F faces, E edges, and V vertices. Choose any vertex O of P. We must remove O in such a way that that the resulting polyhedron P' has $V-1$ vertices. O can be any vertex of P, but if P is a pyramid we may wish to avoid choosing O as the apex, for in this case P' will collapse into a polygon (although Euler remarks that a polygon, thought of as a polyhedron with two faces, still satisfies the formula).

Remove O by cutting away triangular pyramids. For each excised pyramid one vertex is O and the other three are vertices adjacent to O in P. A simple case is shown in Figure 1.

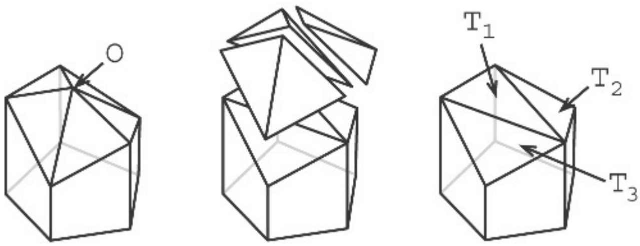

Fig. 1. Removing the vertex O by cutting away pyramids

More specifically, if the degree of O is n, then the n vertices adjacent to O form a (perhaps nonplanar) n-gon. By adding $n-3$ diagonals we triangulate this polygon into $n-2$ triangles, $T_1, \ldots, T_{n-2}$. The $n-2$ pyramids we cut away have the triangles T_i as bases and the vertex O as the apex. Notice that since there are several ways to triangulate a polygon, this decomposition is not unique.

We must now determine the number of faces and edges in P'. First we make two simplifying assumptions: that all of the faces of P meeting at O are triangular, and that no pair of neighboring triangles T_i and T_{i+1} are coplanar (the polyhedron in Figure 1 has both of these properties). In this case, we cut away the n faces meeting at O and added back the $n-2$ triangles, $T_1, ..., T_{n-2}$. Likewise we cut away the n edges meeting at O and added back the $n-3$ edges between the T_i. Thus, P' has $F-n+(n-2) = F-2$ faces and $E-n+(n-3) = E-3$ edges.

Now, consider the case that there are ν nontriangular faces meeting at O. When the triangular pyramids are removed they cut through these ν faces, and in each case leave behind part of a face and create a new edge (see Figure 2). So, we remove n faces and add back $n-2+\nu$, and we remove n edges and add back $n-3+\nu$. Thus P' has $F-2+\nu$ faces and $E-3+\nu$ edges.

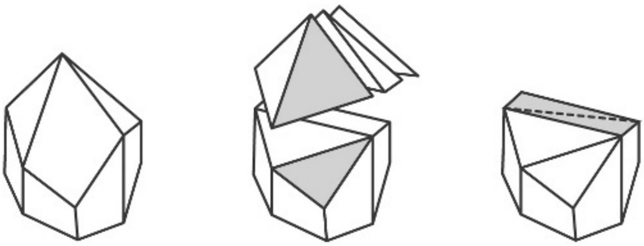

Fig. 2. When this vertex is removed we find $\nu = 1$ (middle) and $\mu = 1$ (right).

Suppose that among the triangles $T_1, \ldots, T_{n-2}$ on P' there are μ pairs of coplanar neighbors (see Figure 2). Each pair of such neighbors merge to form a single face, thus we lose one edge and one face. So, in total we add back μ fewer faces and μ fewer edges. Thus, P' has $F-2+\nu-\mu$ faces and $E-3+\nu-\mu$ edges.

Although the numbers of edges and faces may go up or down when a vertex is removed, the difference between the number of edges and the number of faces decreases by one,

$$(E-3+\nu-\mu)-(F-2+\nu-\mu) = E-F-1.$$

Continue cutting away vertices in this way, removing n in total, until only 4 remain. Thus we obtain a triangular pyramid (with 4 faces and 6 edges). The difference in the number of edges and faces is $E-F-n = 6-4 = 2$ and the number of vertices is $V-n = 4$. Solving for n and substituting we have $E-F-(V-4) = 2$, or $F-E+V = 2$.

Although we omit the proof here, Euler uses this same technique to prove a second theorem, that the sum of all the plane angles of a polyhedron is $2\pi(V-2)$ (a *plane angle* is an angle in the polygon forming a face of the polyhedron). In E230 Euler proved that this theorem is equivalent to the polyhedral formula. He was the first mathematician to publish the angle sum formula, but, as we will see, it was known to Descartes.

2. The flaw and the repair

In 1924 Henri Lebesgue pointed out that Euler was not sufficiently careful when he gave his proof of the polyhedral formula [22]. The first problem is that he never defines the objects he is studying. The second problem is that he is too casual when describing the decomposition process. As we will see, Euler's proof fails for both convex and for nonconvex polyhedra. However, in the case of convex polyhedra, Euler's proof can be salvaged.

Euler does not use the word polyhedron. Instead he refers to "solids enclosed by plane faces" (*solida hedris planis inclusa*). We could take this phrase to be synonymous with polyhedron, but in 1750 there was no explicitly-stated definition of polyhedron either. As Poincaré wrote, "the objects occupying mathematicians were long ill defined; we thought we knew them because we represented them with the senses or the imagination; but we had of them only a rough image and not a precise concept upon which reasoning could take hold" [30]. It is reasonable to believe that Euler, like the Greeks, made the unstated assumption that every polyhedron is convex. It was not until the nineteenth century that mathematicians attempted to formulate a precise definition. One should consult Lakatos' excellent book [21] for an extended discussion of the many attempts to define polyhedron.

Convexity is important for Euler's decomposition algorithm. It may be impossible to cut away a vertex when the polyhedron is not convex in its vicinity (such as the vertices around the waist of the hourglass in Figure 3). It may be impossible to remove a locally convex vertex when the polyhedron is nonconvex (such as the apex of the polyhedron in the center). As a worst-case scenario it may be impossible to remove any single vertex using Euler's method. In the third polyhedron in Figure 3 the vertices located in the indentations cannot be removed at all, and when the vertex located at the center of a star is removed the number of vertices decreases by six, not by one.

Problems may arise for convex polyhedra as well. Euler does not give instructions for how to decompose a polyhedron. Instead he presents a few

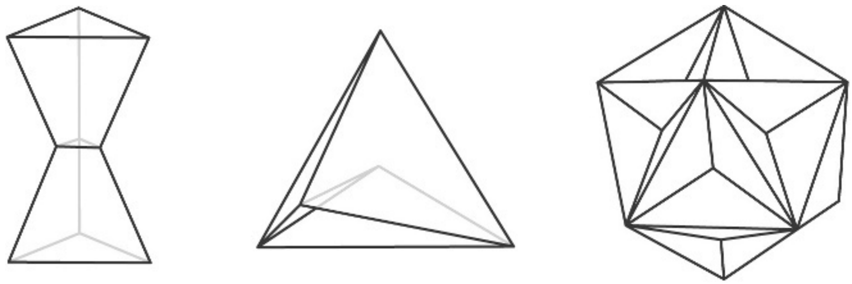

Fig. 3. Various counterexamples to Euler's method of proof.

examples and describes how to decompose these polyhedra. The following example shows that after a vertex is removed, a convex polyhedron may become nonconvex. Here, the vertex to be removed, O, has four adjacent vertices A, B, C, and D. He writes:

> This can be done in two ways (Fig. 3 [our Figure 4]): two pyramids will have to be cut away, either $OABC$ and $OACD$ or $OABD$ and $OBCD$. And if points A, B, C, D are not in the same plane the resulting solids will have a different shape accordingly.

Fig. 4. Removing O yields a convex (middle) and a nonconvex (right) polyhedron.

It is not difficult to see that if A, B, C, and D are not coplanar, then one of resulting solids will not be convex. He does not acknowledge that one decomposition is acceptable and the other is not. This example shows that Euler was not concerned, or not aware of issues of convexity when drafting his proof.

Worse still, Lebesgue showed that it is possible to apply Euler's algorithm to a convex polyhedron and obtain a degenerate polyhedron that fails to satisfy the polyhedral formula. In Figure 5 we see that one choice yields a polyhedron while the other choice yields two polyhedra joined along an edge. Similarly, we may obtain two polyhedra joined at a vertex or two disjoint polyhedra (see Figure 6). None of these polyhedra are topological balls. However, like the vertex O in Figure 5, the vertices labeled O in

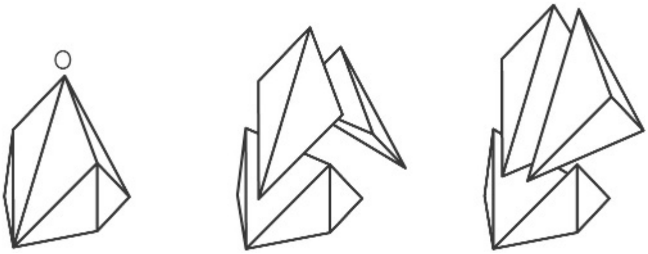

Fig. 5. Euler's technique, applied to the polyhedron on the left may (middle) or may not (right) produce a polyhedron.

Figure 6 can be removed in such a way that the resulting polyhedron is convex.

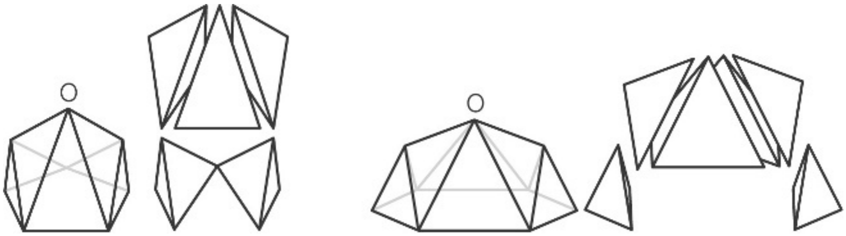

Fig. 6. More degenerate polyhedra

Indeed, as Samelson shows in [33], given any convex polyhedron we can decompose it in the way Euler intended. The only stipulation is that at each stage in the decomposition, the pyramids must be chosen strategically, not arbitrarily. Essentially, this amounts to finding a triangulation $T_1, ..., T_{n-2}$ that preserves the convexity of the polyhedron.

Recall that the *convex hull* of a set is the smallest convex set containing this set. It is easy to see that a polyhedron is convex if and only if it is the convex hull of its vertex set. Thus, if we take the vertex set for P, remove the vertex O, and then take the convex hull, we obtain a convex polyhedron P'. Doing so creates a convex cap in place of the removed vertex. This cap may produce the desired triangulation $T_1, ..., T_{n-2}$, but in some cases (corresponding to $\mu \neq 0$) some of the new faces may have more than three sides. These faces may be triangulated arbitrarily. Notice that it is only in this case that the choices can be made in the removal of O. Even in this case the resulting polyhedron P' is unique.

3. Legendre's proof

The first rigorous proof of the polyhedral formula was given by Adrien-Marie Legendre in 1794. The proof appeared in the first edition of his popular textbook *Élements de Géométrie* [23]. His elegant proof is not a reworking of Euler's proof, but presents a completely new and unexpected approach. The proof is not a combinatorial proof, but instead it uses metric properties of spheres.

The key ingredient in the proof is a theorem proved independently by Thomas Harriot in 1603 [28] (he did not publish the result) and Albert Girard in 1629 [16]. They showed that a geodesic triangle on a sphere of radius r with interior angles a, b, and c has area

$$A = r^2(a + b + c - \pi).$$

More generally, a geodesic polygon with interior angles $a_1, a_2, \ldots, a_n$ has area

$$A = r^2(a_1 + \ldots a_n - (n-2)\pi).$$

To prove Euler's formula, place the polyhedron inside a sphere (which we assume to be the unit sphere) and project the edges and vertices onto the sphere from the sphere's center. In this way the faces of the polyhedron project to geodesic polygons. The sphere has area 4π, but the area can also be computed by summing the areas of the F geodesic polygons. By the Harriot-Girard theorem, the area is

$$4\pi = \sum_{i=1}^{I} a_i + \sum_{j=1}^{F} (n_j - 2)\pi = \sum_{i=1}^{I} a_i + \pi \sum_{j=1}^{F} n_j - 2\pi F.$$

where the first sum is taken over all interior angles of all of the geodesic polygons. Since the sum of the interior angles that meet at a vertex is 2π we have $\Sigma a_i = 2\pi V$. Since each edge borders two faces $\pi \Sigma n_j = 2\pi E$. Thus we obtain

$$4\pi = 2\pi F - 2\pi E + 2\pi V.$$

Dividing by 2π we obtain Euler's formula.

Legendre, like Euler, assumed his polyhedron was convex. That way we can take any point inside the polyhedron to be the center of the sphere. However, in 1810 in the appendix to [31] Louis Poinsot remarked that Legendre's proof applies without alteration to any polyhedron that has such a central point from which the projection can be made (so-called star-convex polyhedra). Thus, Poinsot was the first person to explicitly show that some nonconvex polyhedra satisfy Euler's formula.

4. The exceptions of Lhuilier, Hessel, and Poinsot

At the beginning of the nineteenth century mathematicians were trying to come to grips with Euler's formula. They wanted to determine exactly which polyhedra satisfied Euler's formula, or using the terminology of Johann Friedrich Christian Hessel, which polyhedra were *Eulerian*.

Some stated the polyhedral formula only for convex polyhedra, not knowing or not caring that it held more generally. E. de Jonquières wrote that, "in invoking Legendre, and like high authorities, one only fosters a widely spread prejudice that has captured even some of the best intellects: that the domain of validity of the Euler theorem consists only of convex polyhedra" [7]. For, as D. M. Y. Sommerville writes, "convexity is to a certain extent accidental, and a convex polyhedron might be transformed, for example, by a dent or by pushing in one or more of the vertices, into a nonconvex polyhedron with the same configurational numbers" [36]. Others erred in the other extreme by stating that it applied to all polyhedra.

The first few decades of the nineteenth century saw several examples of non-Eulerian polyhedra. In 1811 the Swiss mathematician Simon-Antoine-Jean Lhuilier wrote a long paper on polyhedra [24] and submitted a memoir to Joseph Diaz Gergonne's journal *Annales de Mathématiques*, but it was too long to print. (It is amusing to note that *l'huilier* means "the oilcan" or "the one who oils," thus Lhuilier may be called "The Oiler.") In 1813 Gergonne published his own shortened account of Lhuilier's paper [25] and included in it ideas of his own.

In this memoir Lhuilier gives three classes of counterexamples to Euler's formula, or *exceptions* as he called them. An example of each type of exception is shown in Figure 7. The first polyhedron has a face that is not simply connected—it is topologically an annulus. Lhuilier remarked that every "inner polygon" within a face would increase by one the quantity $F - E + V$. The second polyhedron has the shape of a polyhedral torus. Lhuilier observed that each "tunnel" decreased the alternating sum by two. Finally, the third example is a cube with a cube-shaped cavity in the interior. This exception was inspired by a mineral in the collection of his friend Professor Pictet that had a colored crystal suspended inside a clear crystal (In 1832 Hessel also found such a crystal—in his case he identified it as a lead sulphide cube within a calcium chloride crystal [17].) Each cavity increases the alternating sum by two.

Thus Lhuilier proposed a modified version of the polyhedral formula. A polyhedron with T tunnels, C cavities, and P inner polygons satisfies

$$F - E + V = 2 - 2T + P + 2C.$$

(This is the earliest incarnation of the topological theorem that the Euler

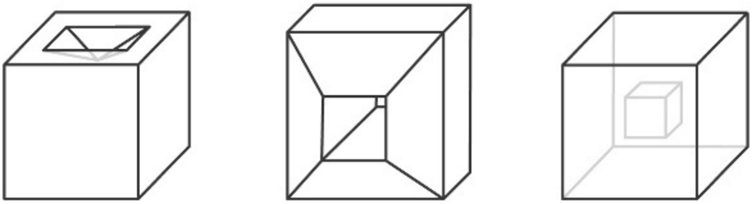

Fig. 7. Lhuilier's exceptions: annular faces, tunnels, and cavities

characteristic of a topological surface of genus g is $2 - 2g$)

In his account of Lhuilier's work Gergonne wrote, "one will easily be convinced that *Euler's Theorem* is true in general for all polyhedra, whether they are convex or not, except for those instances that will be specified" [25]. However, there are exceptions that do not fit comfortably into Lhuilier's three classes. In Figure 8 we see a polyhedron with a face possessing two inner polygons that share a common vertex; a polyhedron with a branched tunnel in it; a polyhedron with a torus-shaped cavity; and a polyhedral torus without an obvious tunnel.

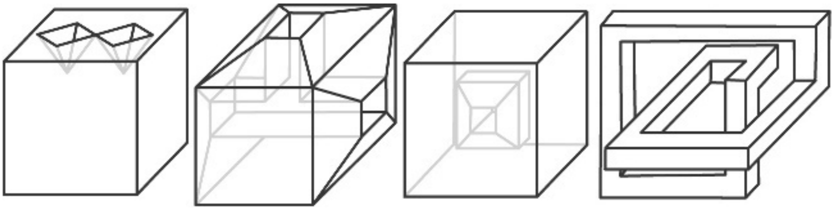

Fig. 8. Complicated polyhedra

In 1832 Hessel, a mineralogist who is most well-known for his mathematical investigation of symmetry classes of minerals [2], presented five exceptions to the polyhedra formula [17]. Shortly after submitting the paper he learned of Lhuilier's memoir from two decades earlier and discovered that three of his five exceptions coincided with Lhuilier's. Hessel believed that many people were unaware of these important exceptions, so he decided not to withdraw the publication [21]. His two new exceptions are shown in Figure 9. One is a polyhedron formed from two polyhedra joined along an edge and the other is a polyhedra formed from two polyhedra joined at a vertex. It is debatable whether these figures should be classified as polyhedra, but they certainly fail to satisfy the polyhedral formula.

In 1810 Poinsot wrote about the four star polyhedra shown in Figure 10 [31]. Unbeknownst to Poinsot, two of the four star polyhedra, the great and small stellated dodecahedra, can be found in Kepler's *Harmonice Mundi* from 1619 [20], and prior to that they appeared in artwork by Wentzel

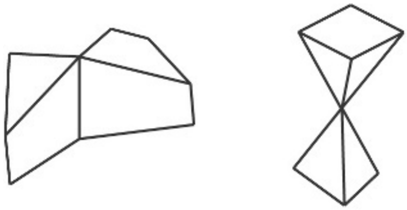

Fig. 9. Hessel's exceptions to the polyhedral formula

Jamnitzer and Paolo Uccello, respectively. Poinsot was the first to present the other two polyhedra, the great dodecahedron and the great icosahedron, in a mathematical context, although the former is also seen in the artwork of Jamnitzer. This collection of four polyhedra is now referred to as the *Kepler-Poinsot polyhedra*.

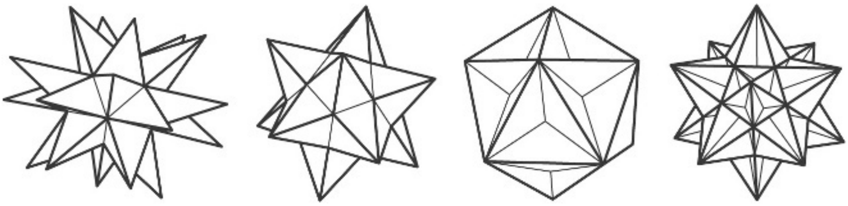

Fig. 10. The Kepler-Poinsot polyhedra: great and small stellated dodecahedra, the great dodecahedron, and the great icosahedron

The Kepler-Poinsot polyhedra are not exceptions to Euler's formula if they are viewed as nonconvex polyhedra formed from triangular faces. However, both Kepler and Poinsot imagined that these polyhedra had self-intersecting faces and were in fact regular. For instance, when we view the great dodecahedron as a polyhedron with 12 pentagonal faces it does not satisfy the polyhedral formula (it has 30 edges and 12 vertices, thus $12 - 30 + 12 = -6$). The other three polyhedra are also exceptions. We now know that the Kepler-Poinsot polyhedra do not obey the polyhedral formula because they are not topological spheres.

5. Cauchy's proof

The first two of Augustin Louis Cauchy's many mathematical papers concerned the theory of polyhedra. They were completed in 1811 and 1812 while he was a engineer working at the harbor of Cherbourg, before he began his mathematical career. He proved that the four Kepler-Poinsot polyhedra were unique [3]. He proved his famous rigidity theorem—a convex polyhedron is completely determined by its faces [4]. He also gave a new proof of Euler's polyhedral formula and extended it in several new, important directions [3]. Both papers appeared in 1813.

The first notable feature that distinguishes Cauchy's proof from Euler's and Legendre's is that it applies to polyhedra that are hollow, not solid. Despite what some historians contend, Cauchy still viewed polyhedra as solid, but his proof used the "convex surface of a polyhedron."

In Cauchy's proof we begin by choosing a face, and then "by transporting onto this face all the other vertices without changing their number, one will obtain a plane figure made up of several polygons contained in a given contour" [3] (Figure 11).

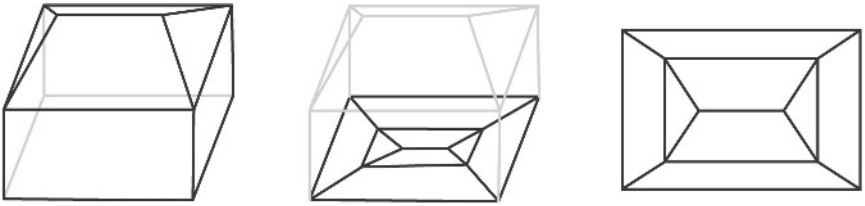

Fig. 11. Cauchy projected the polyhedron into the bottom face.

In 1813 Gergonne describes this process as follows: "Take a polyhedron, one of its faces being transparent; and imagine that the eye approaches this face from the outside so closely that it can perceive the inside of all the other faces; this is always possible when the polyhedron is convex. The things being so arranged, let us imagine that on the plane of the transparent face a perspective is made of the set of all the others" [25]. Lakatos puts a modern spin on Gergonne's idea by suggesting that a camera be placed near the removed face, then the network will appear on the photographic print [21].

Thus, Cauchy realized that it is sufficient to relate numbers of faces, edges, and vertices in this planar network of polygons, or what we would today call a planar graph or map. Cauchy proved that every such graph satisfies $F - E + V = 1$. Then it is easy to complete the proof of the polyhedral formula since the graph has the same number of edges and vertices as the polyhedron and it has one fewer face.

Cauchy begins his proof by triangulating the graph (see Figure 12). He argues that doing so not change the quantity $F-E+V$. Then, "we remove successively the various triangles, so that only one remains in the end, starting with those that border the external contour, and then removing only those which, by earlier reductions, have one or two sides belonging to that contour" [3]. In one case the triangle can be removed by taking away one edge and no vertices (such as the removal of triangle number 1), and in the other case the triangle can be eliminated by removing two edges and a vertex (such as the removal of triangle number 2). In either case, the quantity $F - E + V$ remains unchanged. Thus, since $F - E + V = 1$ for the final trangle, $F - E + V = 1$ for the original graph.

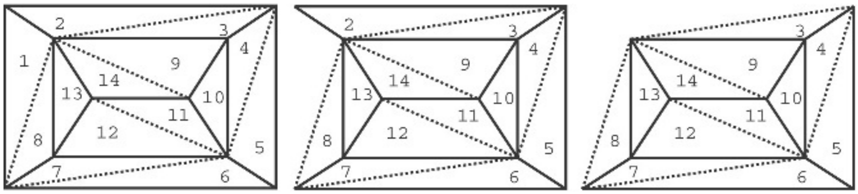

Fig. 12. The order of triangle removal from the triangulated graph

Cauchy's proof was later criticized. Just as Euler ran into trouble by failing to give explicit instructions on what order to remove the pyramids, Cauchy does not give instructions on how to cut away the triangles. If we are not careful it is possible to follow Cauchy's algorithm and obtain a disconnected graph, for which relation fails to hold (see Figure 13).

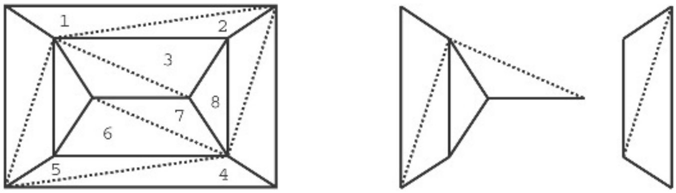

Fig. 13. Cauchy's method can yield a degenerate polygon

Freudenthal writes the following about : "In nearly all cases he left the final form of his discoveries to the next generation. In all that Cauchy achieved there is an unusual lack of profundity... He was the most superficial of the great mathematicians, the one who had a sure feeling for what was simple and fundamental without realizing it" [15]. Cauchy's proof of the polyhedral formula is an apt example of this. The proof applies to very broad class of polyhedra. Using the language of today, the polyhedron must be a topological sphere and have simply connected faces. These properties are guaranteed by convexity, but convexity is by no means necessary. In the statement of his theorem he omits the word convex, giving the impression that he realized the power of the proof. However, in the proof he explicitly states that he is considering convex polyhedra. He never addresses this inconsistency. Some historians, such as Steinitz [38] and Lakatos [21], claim that Cauchy knew his proof applied to some or perhaps all nonconvex polyhedra but this is not clear from what he wrote.

Regardless of whether he recognized that the result could be extended easily to some nonconvex polyhedra, it was quickly seen by others. In 1813, the same year that Cauchy's paper was published, Gergonne gave his own proof of Euler's formula. Afterward he wrote, "one might prefer still, with reason, the beautiful proof of Mr. Cauchy, who has the precious advantage of not assuming that the polyhedron is convex" [25].

Just like Cauchy did not recognize the full strength of his proof for polyhedra, he also did not see the full strength of his theorem for graphs. This theorem was generalized by Cayley in 1861 [5] who showed that it applies to graphs with curved edges (this fact was noticed independently by Listing in 1861 [27] and Jordan in 1866 [18]). Moreover, Cauchy proved the theorem for any collection of polygons contained in a polygonal outline. We now know that it applies to any connected planar graph.

In this same paper Cauchy gives a glimpse of the higher-dimensional generalization of Euler's formula. He proves that if faces, edges, and vertices are inserted into the interior of a convex polyhedron dividing it into P convex polyhedra and if the total number of faces, edges, and vertices (including those in the interior) is F, E, and V, then they satisfy $-P + F - E + V = 1$. This equality shows that the Euler characteristic of the 3-ball is 1. In 1855 Schläfli generalized Cauchy's result to polytopes (as they are now called) of all dimensions. [35].

6. Von Staudt's proof

The first half of the nineteenth century saw several exceptions to Euler's formula and many new proofs. We will not give an account of all of the proofs here (see e.g., [25,37]). All of the proofs that appeared before 1847 apply comfortably to convex polyhedra, and in some cases they can be extended to a broader class of polyhedra. However, no one had yet given a broad classification of Eulerian polyhedra. It was in this year that Georg Christian von Staudt, in his book *Geometrie der Lage* [39], finally gave a very general set of criteria that describe Eulerian polyhedra. Von Staudt's criteria for the polyhedra, which he assumed were hollow, are:

(i) It is possible to get from any vertex to any other vertex by a path of edges.

(ii) Every simple closed path of edges divides the polyhedron into two components.

He then gave a beautiful argument that proved that any polyhedron satisfying these hypotheses is Eulerian. We now give a brief sketch of von Staudt's proof (using modern terminology).

Create a spanning tree for the edges of the polyhedron. Such a tree is shown in second image in Figure 14 as a thick solid line. This tree has V vertices. By property (1) the tree is connected, thus it contains $V-1$ edges.

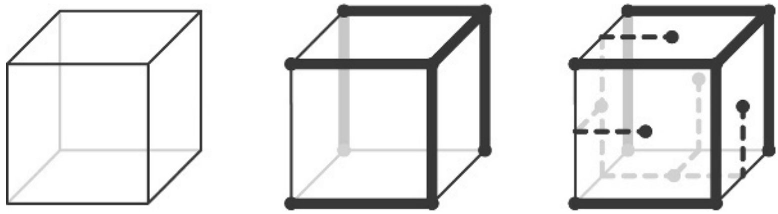

Fig. 14. Von Staudt's algorithm applied to a cube

Now, place a new vertex inside each face. Draw a dashed edge from one face to an adjacent face whenever the two faces are not separated by an edge of the first tree. Property (ii) implies that this graph is connected. Moreover, this graph is a tree, for if it contained a circuit, then by property (ii) the original path would not be connected. Since this tree has F vertices, it has $F-1$ edges. Every edge in the original polyhedron is either in the original spanning tree or is crossed by a dashed edge. Thus the number of edges of the polyhedron is:

$$E = (V-1) + (F-1).$$

Rearranging terms we obtain $F - E + V = 2$.

7. Prehistory of the polyhedral formula: Descartes' lost notes

By 1860 the polyhedral formula was well-known and it had Euler's name firmly attached to it. It was in this year that Foucher de Careil discovered a note hand-written by Gottfried Leibniz indicating that René Descartes knew the polyhedral formula in approximately 1630, 120 years before Euler's proof.

The story of the document now called *De Solidorum Elementis* is fascinating and unlikely. Descartes died in Sweden in 1650 while visiting Queen Christina. After his death his personal effects were shipped back to Paris, but they were nearly lost when the boat wrecked in the Seine. After his unpublished manuscripts were hung to dry, they were made available for public inspection. During a visit to Paris in 1675-6 Leibniz copied some of Descartes' notes, including *De Solidorum Elementis*. Descartes' document was never seen again and Leibniz's copy was lost until its discovery in a dusty cupboard of the Royal Library of Hanover in 1860. (For more details see [13].)

This document contains Descartes' observations on polyhedra. It has the angle sum formula that appeared in Euler's E230 and E231. It also has a relation between the numbers of plane angles, faces, and vertices (P, F, and V, respectively), $P = 2F + 2V - 4$. As Euler proved, the first formula is equivalent to the polyhedral formula, and second can be transformed into the polyhedral formula by substituting $P = 2E$.

Some historians contend that Descartes' knowledge of these relations entitles him to credit for discovering the polyhedral formula. As de Jonquières wrote, "It cannot be denied then that he knew it, since it is a deduction so direct and so simple, we say so intuitive, from the two theorems that he had just stated" [13]. Today we frequently encounter the polyhedral formula called the Descartes-Euler formula.

Other historians point to the importance of edges in the polyhedral formula. They argue that polyhedral formula is a theorem about dimension—that it must relate the numbers of cells of 0, 1, and 2 dimensions. There is no indication that Descartes viewed polyhedra in this way. Lebesgue, after carefully examining the manuscript, wrote, "Descartes did not enunciate the theorem; he did not see it" [22].

8. After 1850

As Pont wrote, "After one hundred years of history (1750–1850), the theorem of Euler traversed the various stages allocated to an honest theorem: empirical appearance, approximate statement, proof in a particular case, exact statement, generalization" [32]. However, during this time no one noticed the topological significance of Euler's formula. This observation came in 1861 in a long work by Johann Benedict Listing, a student of Gauss [27]. Listing is known as one of the early pioneers of topology. We can thank him for coining the term "topology" in the title of his 1847 *Vorstudien zur topologie* [26] and for co-discovering the Möbius strip (along with Möbius).

In the second half of the nineteenth century the mathematical subject called topology began to take shape with contributions from Jordan, Riemann, Möbius, Klein, Betti, Dyck, and others. In a series of papers starting in 1895 Poincaré unveiled the blueprint for the field of algebraic topology and gave the first truly modern interpretation of Euler's formula [29]. The alternating sum, now referred to as Euler characteristic (or the Euler-Poincaré characteristic), is one of the most fundamental topological invariants. The myriad applications of the Euler characteristic are far too numerous to list.

In order to appreciate the current state of algebraic topology it is important to recognize the important contributions of Euler and the other mathematicians who studied polyhedra from 1750 to 1850.

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Leonhard Euler: Life, Work and Legacy
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On the Recognition of Euler among the French, 1790-1830

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1. The rise of Paris to mathematical eminence

When Jean d'Alembert, Daniel Bernoulli and Leonhard Euler died within 18 months of each other in 1782 and 1783, a generation of researchers passed and the inheritance came to their principal successors. This change also established Paris as by far the main mathematical centre, a status that it was to enjoy until well into the 1820s. The initial figures, some already well recognized, were C. Bossut (1730-1814), G. Monge (1746-1818), P.S. Laplace (1749-1827) and A.M. Legendre (1752-1833); and their ranks were augmented considerably in 1787 when J.L. Lagrange (1736-1813) moved to the Paris Academy from his post at Berlin's equivalent.

The French maintained their dominating status in mathematics during the period treated here, not only by the happenstance of producing major figures but especially for the explicit encouragement given to science and engineering by the new regime that followed the revolution of 1789. The *Ecole Polytechnique*, founded in 1794, was an important focus, partly for its employment of all the figures named above as teachers or as graduation examiners. This next generation included J.B.J. Delambre (1749-1822), L. Carnot (1753-1823), G. Riche de Prony (1755-1839), J.B.J. Fourier (1768-1830), J.N.P. Hachette (1769-1834), L. Puissant (1769-1843) and A.M. Ampère (1775-1836); apart from Delambre, all were also involved with the school for some period and purpose.

An especially important reason for the success of the school was its policy of recruiting talented students. Several of them went on to pursue important careers in mathematics, engineering and science, some within the school itself, some at the various other schools and professional institutions (especially in civil and military engineering) available in the country, and some at both. These ranks include J.B. Biot (1774-1862), L. Malus (1775-1812), L. Poinsot (1777-1855), C.L.M.H. Navier, (1785-1836), S.D. Poisson (1781-1840), A.J. Fresnel (1788-1827), J.V. Poncelet (1788-1867), A.L. Cauchy (1789-1857) and G.G. Coriolis (1792-1843).

The task addressed here is to describe the place accorded to the contributions of Euler by this cohort, and also by many contemporaries who made up the community in total. The primary and historical literature on the achievements involved is far too vast to be treated in detail. The most substantial single source is [Grattan-Guinness 1990a], where many historical items are cited and recommended for further information. Historical articles on several of the major books are provided in [Grattan-Guinness 2005, chs. 12-14, 16-21, 24-28]. To my knowledge the question of Euler's influence on the French has been treated explicitly, though briefly, only in [Grattan-Guinness 1983], [Grattan-Guinness 1985a]; but the literature on Euler and/or the early 19th century contains many individual pieces of evidence on his place and influence.

2. Varieties in the calculus and mechanics

For all mathematicians of the mid and late 18th century, by far the main part of mathematics was the calculus and its use in mechanics. Three traditions of the calculus were in place: Newton's fluxions (though largely confined to British mathematicians); the differential and integral theory, set out by Leibniz and modified by Euler with his addition of the differential coefficient, the forerunner of our derivative; and Lagrange's version, reducing the calculus to a branch of algebra by assuming that a mathematical function $f(x + h)$ could always be expanded as a power series in h, with the 'derived functions' defined from the coefficients of h. In each tradition the integral was specified ('defined' is too strong a term) as some sort of inverse of the fluxion, differential or derivative. A fourth tradition of founding the calculus was to be created by Cauchy, as we see in section 9 below.

As well as the core topics of differentiation and integration, an impressive body of knowledge lay in the general theory of solutions of ordinary and partial differential equations. In addition, many particular methods were found to solve equations of various kinds. These achievements, especially the second, led to quite a wide range of special functions and infinite series; they also encouraged the theory of polynomial and other equations, in particular properties of their roots.

Mechanics lay alongside this empire, and indeed constituted an even more enormous subject in its own right. It too had three traditions in place by the late 18th century [Grattan-Guinness 1990b]: one based upon central forces and Newton's various laws; another relying upon 'live forces' and their relationship to work; and a third, often called 'analytical', where principles such as d'Alembert's, least action and virtual velocities held sway.

The range of phenomena and artefacts handled within mechanics can be fairly divided into five branches; the descriptive italicised adjectives that follow are mine. Proceeding from the rather large to the very small, we have *celestial* mechanics, where all heavenly bodies were treated as mass-points; *planetary*, where the major questions included the shapes of these bodies, now taken to be extended, and related topics such as Lunar theory, topography and the tides; *corporeal*, including the basic principles of the subject (as required by the three traditions just mentioned), and topics such as sound, elasticity theory and fluid mechanics; *engineering*, covering, for example, friction studies of various kinds, machines including water-wheels and turbines, and structures such as arches; and a little work on *molecular* structure.

3. Euler's place: preliminary remarks

The question addressed here concerns the ways in which Euler's contributions were to be adopted, adapted or maybe ignored by the cohort of French mathematicians from around 1790 up to 1830. Two parts of that output need to be noted: the very many papers and books that appeared by the time of his death; and the papers, around 100 in number, that the Saint Petersburg Academy published posthumously in their *Mémoires* until 1830. I shall not take account of his manuscripts or letters, as the former were not then available while the latter were known only to their correspondents and maybe a few others.

Firstly, let us note that all sorts of specific results due to Euler were known, stated in textbooks and monographs. In addition, some of his main results or methods had become part of the mathematical furniture; the calculus with the differential coefficient, for example, and the exposition of perturbation theory in celestial mechanics by expanding the principal variables in infinite trigonometric series. So the place of Euler was sure, although often not explicitly stated; as time went by, the newer authors may not have known that Euler was their original source for notions that they learnt from textbooks and other writings by intermediate authors. In addition, at that time references to others' works were not given systematically in science in general, and the tradition of ending a paper or book with a list of works did not commence until the late 19th century. Since lack of evidence is not evidence of lack, the place of Euler is doubtless underestimated in the account to follow.

4. Euler or Lagrange in the calculus and analysis?

These two mathematicians were arguably the two main sources and influences on others, at least into the early 19th century. In analysis and especially the calculus there were specific differences in their approaches. In all aspects of mathematics Lagrange was an algebraist; that is, not just did he use algebra, like everybody else, but he sought to *reduce* mathematical theories to algebraic principles. His reliance upon Taylor series mentioned above is typical (and important); the claim was that *only* the normal algebraic operations were needed to develop the calculus. (He allowed for exceptional cases when a function and/or its derivatives took infinite values.) An important theory adjoint to the differential and integral realms was the calculus of variations: Euler had made important innovations, and indeed the name is due to him, but its generality and algebraic formulation owed most to Lagrange. He publicised his approach in the books *Théorie des fonctions analytiques* (1797), and *Leçons sur le calcul des fonctions* (1804 and later editions), which were based upon his teaching at the *Ecole Polytechnique.*

Now the calculus was a much broader subject than (common) algebra, so that the algebraic brief had to be extended. Fulfilling this aim encouraged some followers (more than Lagrange himself) in the development of algebras that were new in the sense that their 'objects' were neither numbers nor geometrical magnitudes. One algebra was that of differential operators, based upon forming the operations of differentiation as $D := d/dx$ and integration $\int := 1/D$, where '1' is the identity operator. The other was functional equations, such as $f(x)f(y) = f(x+y)$ (to take a very simple example), where the unknown is the function f.

The measure of support for Lagrange's approach and these algebras was quite well supported by the French, and it can be seen as an eclipse of Euler.

But Euler's version of the differential and integral calculus retained its great popularity, especially in applications; it was used in some way in almost all the contexts reviewed below. However, the balance was different in the general theory of ordinary and partial differential equations. There Euler had made important contributions – for example, on singular solutions – but Lagrange had rather taken over with his theory of general solutions of various kinds, which had been further developed by Laplace, Monge and others.

So the inheritance of the calculus for the French was several-sided. Two substantial and contrasting monographs published in the later 1790s exemplify the differences.

The physicist J.A.J. Cousin wrote the first one in two volumes. In his second volume he covered quite a wide range of differential equations, and so gave Euler's contributions quite a reasonable coverage; but in the first volume he praised the use of limits and judged the differential method to be imperfect; he even interpreted 'dy/dx' as the limiting value of the difference quotient [Cousin 1796, vol. 1, esp. pp. 151-153].

A different balance comes from Lagrange's successor as professor at the *Ecole Polytechnique*, namely Lacroix. His mathematical mentor was neither Euler nor Lagrange but the Marquis de Condorcet (1743-1794), not a major mathematician but a significant representative of Enlightenment philosophy. Together they prepared an edition of the *Lettres* (1787-1789), a few years after Condorcet prepared the eloge of Euler for the old *Académie* [Condorcet 1786]. Later Lacroix wrote a very praising article on Euler for a general multi-volume biography project [Lacroix 1815].

One of the effects on Lacroix of this kind of philosophy was the desire to present all theories, not just one preferred approach. His major publication was a huge *Traité du calcul différentiel et du calcul integral*, which appeared in two three-volume editions [Lacroix 1797-1800], [Lacroix 1810-1819]. Interested in the history of mathematics, he endowed his book with a level of scholarship unique for that time: in the table of contents he listed the many original texts that he had consulted, and he finished the book with a combined name and subject index. The latter is particularly useful, as it shows that the entry for Euler is about 50 percent longer than that for Lagrange. The main difference lay in the mass of particular results about series and functions, from which Lacroix reported quite a large selection; otherwise he cited both men for their versions of the calculus and contributions to the general theory of ordinary and partial differential equations.

In the latter context Lacroix added a long and interesting footnote on the notations for the multi-variate calculus. He found unnecessary Euler's use of brackets to denote partial differential coefficients (such as '(dz/dx)'), but he was quite critical of the capacity of Lagrange's primes to distinguish apart partial derivatives [Lacroix 1797-1800, vol. 3, pp. 10-12]. In the second edition he elevated the passage to the main text of the first volume [Lacroix 1810-1819, vol. 1, pp. 242-246]. Elsewhere in his book he introduced the name 'differential coefficient' [Lacroix 1797-1800, vol. 1, p. 98]; Euler had not used any special name.

5. Euler or Lagrange in mechanics?

Lagrange's intent in mechanics was similar; again he wanted to algebraise it, as the title of his treatise, *Méchanique analitique* [Lagrange 1788], makes clear. The long-suffering word 'analytic' meant 'algebraic' here, as he stressed in his oft-quoted remark in the preface that 'one will not find Figures in this Work:' algebra was the only guarantee of the generality and rigour with clarity that major branches of mathematics should exhibit. Hence he drew upon principles such as d'Alembert's, least action and 'virtual velocities', which could be formulated algebraically, including using the calculus of variations; from them he obtained the basic equations for dynamics that are now named after him. Other principles, in particular, Newton's laws and the conservation of energy, came out as theorems. He also included in his book some short historical passages, which became more influential than they deserved.

Again the reaction was mixed. The theory was capable of extension, notably with the theory of 'Lagrange-Poisson' brackets to solve canonically the equations of motion, which the aged Laplace and his young disciple Poisson developed between 1808 and 1810. But in general the approach showed its strength best in the systematic exposition and assembly of results *already* found; it was not normally conducive to the creative side of mechanics. There Newton's and the energy/work traditions were kinder, especially for their ready appeal to geometry and spatial situations.

The position of Euler was rather peculiar. He made substantial use of Newtonian mechanics: indeed, some of its normal features are actually due to him, such as the notion of the mass-point, and taking the second law exclusively in the form $F = ma$ and applying it in *any* direction. (For some reason Lagrange attributed this last innovation to the Scot Colin MacLaurin [Lagrange 1788, pt. 2, sec. 1, art. 3]. Euler's handling of perturbation theory was mentioned in section 3, and he adapted Newton's laws in continuum mechanics, especially in fluid mechanics and elasticity theory. However, in the 1740s he had also been one of the main advocates of the (new) principle of least action, upon which he failed to draw in these and other later contributions!

In addition to the Newtonian and analytical traditions in mechanics, a third approach based upon energy and work was put forward as a general one by Carnot in the 1780s; in contrast to Lagrange's belief that dynamics was reducible to statics, he stressed dynamics and mechanical situations involving impact. The role of Euler here was modest, though he had used 'quantity of action' in connection with his advocacy of the principle of least action. Several engineer scientists connected with the *Ecole Polytechnique* furthered Carnot's tradition with enthusiasm: Hachette, Navier, Coriolis (who coined 'work') and Poncelet stand out.

These men will have gained some insights as students from the engineer de Prony, who was with Lagrange a founder professor of mathematics at the *Ecole Polytechnique*, and led the teaching of mechanics. Further, while Lagrange taught his calculus for only a few years, de Prony held his chair for 20 years, when he switched to graduation examiner.

De Prony published several volumes of his lecture courses at the school, of which the first, *Mécanique philosophique* [de Prony 1800], is the best known and most interesting, although incomplete. In this work he made explicit his attachment to Enlightenment philosophy, in particular classifying parts of mechanics in various ways: synoptic tables, and especially the division of almost all of the right hand pages into four columns listing the notations, definitions, theorems and problems. At first glance the book seems to be very Lagrangian: lots of algebra, and no diagrams. But the algebra is not variational but rather trigonometry, especially to express components of notions such as force and moment. In his preface he acknowledged his sources: major writings by Lagrange and Laplace, but 'The principal works that have furnished me with my material are those of Euler' [de Prony 1800, vii].

The same features apply also to de Prony's later textbooks, even the *Leçons de mécanique analytique* [de Prony 1810-1815] (complete, and admitting quite a few diagrams). However, in his final Part on machines he made little use of Euler's writings on science and technology, which for some reason were little used by anybody.

A broadly similar impression about Euler comes from the second edition of Bossut's treatise on hydrodynamics [Bossut 1796]. In his lengthy 'Preliminary discourse' he reviewed much of the literature of the 18th century, with a notable emphasis on the concerns of engineers; he even cited one of Euler's books on the navigation of vessels. As with de Prony, his discussion of the basic equations both of equilibrium and motion of fluids avoided variational techniques.

Other notable texts came from graduates of the school. The first was Poinsot's book *Elémens de statique* ([Poinsot 1803, and many later editions]. His opening chapter was an important account of the 'couple' (his word), a major feature of statics of which nobody had previously taken proper account. Then he considered various general conditions for equilibrium that drew upon virtual velocities, and also included a detailed survey of machines, which belonged most closely to the energy/work tradition; and throughout he made much use of diagrams.

In 1811 Poinsot's non-friend Poisson put out the first edition of his *Traité de mécanique* based upon his teaching at the *Ecole Polytechnique*. One might expect to read a version of Lagrange's treatise for learners, but this is not so: the calculus of variations was used sparely. Following a long practice among the French, his basic principle was d'Alembert's, which served not only as a fundament for analytic mechanics but also as the justification for Newton's laws; Poisson used especially the second law fairly often (but never mentioned Newton once). At least that theory was presented; Poinsot's recent book on mechanics was ignored completely.

6. On Laplace and his own place

The mixed picture is evident also with our third major figure: Laplace [Gillispie 1998]. His reputation, already high, rose still spectacularly when he began to publish his *Traité de mécanique céleste*, of which the first four volumes appeared as [Laplace 1799-1805]. This work was authoritative for all aspects of celestial and planetary mechanics, and also for many aspects of the calculus, and some series and solutions of various partial differential equations.

Laplace is credited with the remark: 'Read Euler, he is the master of us all'. [1] Both Euler and Lagrange featured strongly in his work (though he did not much fancy the new algebras). For example, he made much use of Euler's use of trigonometric series; but he also adapted Lagrange's marvellous attempt to prove mathematically that the planetary system was stable, a decisive rejection of Euler's (and also Newton's) view that God was responsible for stability. Among other sources, for Lunar theory he made most use of d'Alembert's formulation. Many of the analyses were as much his own as anybody else's. For example, for the theory of equipotential surfaces and the attraction of a heavenly body to an external point he solved the partial differential equation now named after him and solved it with the help of functions that also took his name in the 19th century but

[1] Unfortunately our closest source for Laplace's remark is [Libri 1846]. This article, part of a review of an edition of 18th-century mathematical correspondence, is cited in the biographical article on Euler [Anonymous 1857], which therefore may also be by Libri.

then became called 'Legendre functions'; the use of both names reflects a pretty competitive situation since the 1770s.

A striking feature of Laplace's opening sections was his use of one of Euler's posthumous papers: [Euler 1793], in which Euler had proved that torque obeyed the same kind of linearity that obtained with moments. Laplace quickly used it to define the invariable plane of a system of point masses (the planetary system being the case most in mind) in terms of maximal torque [Laplace 1799-1805, Book 1, art. 20]. This use of Euler is not only noteworthy in its own right; it is also a very rare case of anybody using a posthumous paper by Euler.

7. Laplace's programme of molecular physics, and the alternatives

In the last volume of his treatise Laplace began to make public his growing interest in physics, which did not have a high status in science at that time. He analysed the path of light through the atmosphere, and then added two lengthy supplements to the fourth volume on capillarity. Common to both analyses was a principle that "all" phenomena, mechanical or physical, were to be interpreted as actions between the elementary 'molecules' of which the pertaining bodies were presumed to be composed. He based upon it an ambitious programme for physics, for which he recruited several able younger colleagues, mostly graduates of the *Ecole Polytechnique*.

In particular, Laplace adopted a corpuscular theory of light, which was then the more popular type of theory among French physicists (see, for example [Haüy 1806, vol. 2, 134-401]; it was to be the most successful part of his programme. The most important theorist and experimentalist was Malus; among other achievements, he saw that the principle of least action could be used to explain double refraction (an insight that Laplace was to purloin and extend), and he coined the word 'polarisation' because he assumed that the moving particles of light oscillated about an internal axis, like the poles of a magnet.

This kind of theory was bad news for fans of Euler, who had adopted a wave theory of light. He had studied especially reflection, refraction and aberration, the latter leading to an interesting exchange about achromatic lenses with the Englishman John Dollond, who upheld Newton's corpuscular theory [Speiser 1962] – the only detail in which Euler was mentioned in Haüy's long account of optics just cited. But bad news turned to good, in that from the mid 1810s onwards Fresnel began to elaborate such a theory. He construed light to be the result of disturbance from equilibrium of the tiny particles in the assumedly punctiform aether. He appealed to analogies with mechanics whenever useful: principles such as the cosine law of decomposition, and 'energy' conservation for double refraction. However, in his papers and letters he referred only once to Euler's theory, and then in passing [Fresnel 1822, art. 1].

By the 1820s even Laplace was admitting the quality of Fresnel's theory; it was the main confrontation of his programme. However, it was not the first, which had occurred over heat diffusion. From the mid 1800s Fourier had much exercised himself over this topic, giving it the first extensive mathematical treatment. Philosophically he adhered to a kind of positivism (the word that Auguste Comte was to coin, with Fourier much in mind): heat was heat, to be exchanged with its opposite, cold. He had derived the diffusion equation by the normal (Eulerian) version of the differential and integral calculus, and took no interest in the molecularist re-derivation that Laplace offered in 1809: the loyal Poisson was to pursue the idea from the mid 1810s onwards, but nobody took much interest in it.

So Euler was present in Fourier's derivation of the equations, but not in the preferred solutions: trigonometric series (not to be confused with Euler's technique in celestial mechanics) for finite bodies, and integrals for infinite ones. Now Euler had found the Fourier series in [Euler 1798] as a mathematical exercise (as Lacroix was to point out to Fourier), but he did not exploit it physically; in particular, he never changed his stance of the 1740s when in the famous debate over the analysis of the vibrating string he had preferred the functional solution of the wave equation, a kind of solution that was normally favoured at that time.

The other main developments in the new mathematical physics lay in the study of 'electricity' (mostly electrostatics) and magnetism. Here the Laplacians enjoyed some success, thanks mainly to the efforts of Poisson (around 1812 and 1824 respectively). However, molecularism took a limited role; Poisson made much use of the electric and magnetic fluids that were supposed to exist. Euler had said little technical about either subject, though several of the *Lettres* treated magnetism. Further, naturally he did not anticipate electromagnetism, which was to obsess Ampère from 1820 to around 1827 – and to attract little interest among the Laplacians. However, once again his form of the calculus was preferred there, and indeed was extended importantly into line and surface integrals; Poisson had already made some use of the latter in his contributions to magnetism.

8. Continuum mechanics, molecular and otherwise

Contemporary with these innovations in mathematical physics, mechanics continued to be studied in all its branches. The most important parts not yet treated were fluid mechanics and elasticity theory, where the Parisian talent for rivalry was well to the fore, especially between Poisson, Cauchy and Navier, with Fourier sniping on occasion.

On fluid mechanics, in the 1750s Euler had applied Newton's second law to a differential parallelepiped and using also his own notion of pressure; Lagrange had later substituted the method that came to be known as 'the history of the particle', which made use of the calculus of variations. The results of both men were restricted to shallow fluid bodies. Shortly after the death of Lagrange in 1813, the mathematical and physical class of the *Institut* posed a prize problem for 1814 on the propagation of waves in a deep fluid body. Cauchy won it; Poisson, already a member of the class, contributed two papers at the same time. He based his treatment upon Euler's method, while Cauchy drew upon Lagrange's; one might have expected the preferences to be the other way round. As usual, Cauchy produced the more profound results (in particular, he found Fourier's integral theorem, in apparent independence of Fourier), but not especially because of his use of Lagrange's method.

In elasticity theory, the class had already run in 1811 a problem on the motion of an elastic lamina. This problem, partly inspired by the sand experiments of the Austrian acoustician Ernst Chladni, seems to have been tailored for Poisson, who was then not yet a member of the class, to produce another Laplacian molecular exercise. He did produce one eventually, but the prize was won by Sophie Germain, after three versions and important help from Lagrange and Legendre. The basic ideas, however, were hers, and drew upon Euler's work.

More significant developments began in the late 1810s, with a string of papers from Navier, Cauchy and Poisson. Navier worked his way from elastic rods and planes to solids and also viscous fluids. Some of Euler's assumptions were used in the formation of the equations, but for solution he appealed to Fourier's new methods. Poisson predictably was very molecularist. Cauchy as usual eclipsed everybody, with a long string of analyses in terms of stress and strain (to use the names which William Rankine was to introduce). Some of the models were molecular while others not, and it is not easy to tell why each type was chosen. He then adapted his method to study dispersion within the framework of Fresnel's optics. As usual he was spare in references, and he may not have drawn much upon either Euler or Lagrange.

9. A new tradition for the calculus: the impact of Cauchy

Student at the *Ecole Polytechnique* in the mid 1800s, a decade later Cauchy was appointed professor of analysis and mechanics there in the changes that accompanied the restoration of the Catholic monarchy, to which he was fanatically attached. His teaching was disliked by students and staff for its inappropriate content for an engineering school, and also for his failure to coordinate with other courses; but mathematically it was of immense importance.

Cauchy formulated a fourth version of the calculus. It was grounded upon a proper *theory* of limits that itself was based upon the careful studies of infinite sequences of values and not just the modestly developed *notions* that his predecessors had achieved. In its terms he defined the derivative as the limit of the difference quotient and the integral as the limit of a sequence of partition sums, and he allowed in both cases for the possibility that the limit did not in fact exist. As one offshoot, by means of counter-examples he refuted in 1822 Lagrange's belief in the universality of the Taylor expansion. His new version (which was not motivated by these counter-examples) was bad news for all predecessors, Euler included; but it gradually became adopted worldwide, especially among those mathematicians who stressed rigour. However, Euler's version long continued to retain its high status among those figures concerned with applications, who included Cauchy's colleagues at the school.

As part of his reliance upon a theory of limits, Cauchy also revised the theory of functions and of infinite series, defining continuity of the former and convergence of the latter in terms of the proven existence of the limiting value. All previous criteria were substantially revised; in particular, much of Euler's production of sums of series was rejected as illegitimate, and only from the end of the 19th century was it rehabilitated within the theory of summability and formal power series.

10. Three smaller topics

10.1. *Geometry*

Euler's *Introductio in analysin infinitorum* (1748) was divided into two distinct volumes. The first one covered many aspects of (real- and some complex-variable) analysis and the theory of functions, and became a standard reference for these topics. The second one helped substantially to launch analytic and coordinate planar and solid geometry [Boyer 1956, chs.

7-8]. Among French mathematicians Monge and Puissant were active, and [Lacroix 1799b] and [Biot 1802] produced textbooks that appeared in many later editions. The subject also featured in the treatises on the calculus by Cousin and Lacroix mentioned in section 4. Once again Lagrange tried to algebraise the theory, but this time with limited success.

In addition, the *Introductio* itself was translated into French and published as [Euler 1796-1797]. The task was fulfilled by the school-teacher J.B. Labey, who later also published a translation of the *Lettres*.

10.2. *Number theory*

Number theory was a very recondite subject, with few practitioners; however, three of them were Euler, then Lagrange, then Legendre. In his *Essai sur la théorie des nombres* [Legendre 1798, and later editions] Legendre treated the algebraic side of the subject. The topics covered included reduction of quadratic forms, sums of squares, cyclotomy, reciprocity properties, certain equations and their roots, and Fermat's 'last' and other theorems. In his preface he duly praised Euler, and acknowledged Lagrange (and also C.F. Gauss in the later editions). This was valuable tribute from the community of French mathematicians; but it was a small one, since Legendre was its only regular practitioner to the subject in the period treated here.

10.3. *Probability and mathematical statistics*

Some of Euler's contributions to analysis bore upon these topics: in particular, the beta and gamma functions and the hypergeometric series. In addition, he wrote on the errors of observation, games, tontines and lotteries, and mortality and annuity tables [Sheynin 1972]. Some of this work lay in interactions with contemporaries, especially Bernoulli and Lagrange. However, few French worked in these fields; mainly Laplace and some Poisson, with a short burst from Fourier and a textbook by Lacroix. They do not appear to have made much use of Euler's offerings, which seem *never* to have been much used.

11. Three general surveys

I complete this appraisal with three French sources of the 1800s that give us further insight about Euler's status. The first is the massive *Histoire des mathématiques* of E. Montucla. He died in 1799, just as he was writing and proof-reading a Part of the third volume. The project was taken over by Montucla's friend the astronomer J.J. Lalande, who edited the surviving manuscripts and wrote the rest himself, drawing on colleagues such as Lacroix for certain sections. The third and a fourth volume appeared, as [Montucla 1802]. The books contain a huge amount of information, although often surprisingly spare of symbols.

A noteworthy feature of the volumes is the entry for Euler in their index: 'Euler, the greatest geometer of the eighteenth century' [Montucla 1802, vol. 4, 678]; no other figure was characterised in such a way, though the names of persons in the index are poorly furnished. Further, while Euler was mentioned a lot, some details are missing. Take, for example, the Part in progress when Montucla died, a long and rather untidy account of the calculus and analysis during the 18th century; he noted Lagrange's approach to the calculus [Montucla 1802, vol. 3, 260-270], but he did not describe Euler's introduction of the differential coefficient. Again, in the next Part, on optics, his waval theory was mentioned less than one might have expected.

In the last two Parts, on mechanics and machines, the text adopted Lagrange's *Méchanique analitique* as the main guide, including its little historical essays. So, while Euler duly appeared in the discussions of some of the basic principles, his later contributions were rather summarily treated. Further, as usual he did not feature in (Lalande's) review of machines and technology, even though the bibliographical information there was quite extensive. Carnot was also omitted; de Prony's engineering treatise *Nòuvelle architecture hydraulique* [de Prony 1790-1796] was a leading source.

Also published in 1802 was our second source, the second volume of a much shorter (but also prosodic) history of mathematics written by Bossut. Most of the book treated the history of the calculus from its creation by Newton and Leibniz, and much of the text was taken up with applications [Bossut 1802]. Euler featured a fair amount in the parts of the book covering his career period, though perhaps less than one might have expected. But he was better served than was Lagrange, since Bossut adopted the peculiar policy of omitting all figures then still living. Perhaps in response to the criticisms, a few years later he issued an expanded version of his book, coming right up to date. Euler featured rather more than before, especially in applications (including optics and some technology); he had more page entries than anybody else in the index, and was praised on several occasions [Bossut 1810, esp. pp. 148-150]. But he still left out the differential coefficient.

Our last source also comes from that time. As a permanent secretary of the scientific class of the *Institut de France*, in 1809 Delambre had to present to Emperor Napoléon a book-length survey of progress in the 'mathematical sciences' (pure and applied) during the past glorious 20 years. As for Lalande, Lacroix helped him with the purer mathematical sections [Delambre 1810]. Lagrange and Laplace were naturally the leading authors, but Euler was next, with more entries than even for Legendre; he was mentioned over a score of times, of which several were more than passing references. However, the balance again was rather askew: reasonable for the calculus and mechanics, but nothing on technology, or on cartography.

12. Concluding remark

As one might expect, Euler was a major background figure for the French in the period treated here, and for several topics he was a good deal more prominent. The pure mathematics seems to have been the most visible part of his achievement, and several parts of his work in celestial, planetary and continuum mechanics; the technology survived much less well.

The main figure 'in between' Euler and his French successors is Lagrange, their senior member from 1787 until his death in 1813. More importantly, he differed from Euler substantially on the adopted principles of both the calculus and mechanics. The 'competition' between them is hard to evaluate. Lagrange had put forward impressive general theories, but their utility was limited, especially in applications or the creative sides of theories. But the use of Newton's laws in mechanics can only be seen as a partial affirmation of Euler's position, where nominally the principle of least action should have been as prominent as it was with Lagrange. Finally, while Lagrange's published references to Euler were rather slender, on his deathbed he praised Euler to the skies: 'read Euler, because in his writings all is clear, well calculated, because they teem with beautiful examples, and because one must always study the sources' [Grattan-Guinness 1985b, art. 4].

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Leonhard Euler: Life, Work and Legacy
Robert E. Bradley and C. Edward Sandifer (Editors)

Euler's Influence on the Birth of Vector Mechanics

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1. Introduction

In the course of his examination of Euler's paper "A more accurate development of the formulae found for the equilibrium and motion of flexible threads" [E608], C. Truesdell wrote:

> "The reader will have remarked Euler's mastery of the methods of vectorial algebra; the formulae we have presented are shortened by the use of vector symbols, but the operations indicated are those used by Euler." [Truesdell 1960, p. 383]

There is, of course, some exaggeration in this statement. If we agree that vector calculus is a theory of the composition of directed segments expressed by means of an algebraic symbolism, it is a matter of fact that there is no trace of such ideas in the writings of Euler. On the contrary, as we will see, there are many reasons to believe that Euler did not fully understand the vectorial character of the entities and the operations that occur in his purely algebraic calculations. [1]

[1] In this article I use the term "vector" quite freely. It would have been more appropriate to employ everywhere the locution "directed segment," for this is what these early authors had in their minds, but its usage would have led to a cumbersome mode of expression. The difference is significant: a vector is, strictly speaking, an element of a vector space.

Truesdell had previously given a different and more just estimation of the role of Euler in the development of vector calculus:

> "The expression of the laws of motion in *rectangular Cartesian co-ordinates* is also of the greatest importance. Today this possibility is so obvious that many scientists seem to believe that Newton himself used Cartesian co-ordinates, but of course this is not so. [...] The importance of the use of Cartesian co-ordinates lies deeper than in mere simplicity; in these co-ordinates the addition of vectors located at different points is so natural as to become customary at once, and the possibility of performing this addition lies at the heart of the classical conception of space-time." [Truesdell 1960, p. 252]

In fact, Euler has the merit of having constantly referred all quantities to rectangular axes fixed in space in his works from about 1750 onward. It is clear that from the Cartesian representation of physical quantities their vectorial character can easily be judged; let us recall Heaviside's observation:

> "I ought to also add that the invention of quaternions must be regarded as a most remarkable feat of human ingenuity. Vector analysis, without quaternions, could have been found by any mathematician by carefully examining the mechanics of the Cartesian mathematics, but to find out quaternions required a genius." [Heaviside 1892, vol. 2, p. 557]

However, the task of extracting the concept of vector from analytic geometry and mechanics turned out to be more difficult than Heaviside had imagined.

2. On Euler's conception of vectors

What then was Euler's conception of the geometrical representation of vectors? Perhaps, a clearer account can be found at the beginning of his "Attempt at a metaphysical demonstration of the general principle of equilibrium" [E200], where he defines the concept of force:

> "One calls force everything that can change the state of bodies, both of their movement and of their rest. [...] In each force there are two things to consider: the quantity and the direction. By quantity one understands how much a force is greater or smaller than another, and the direction allows us to know in which sense every force acts on bodies to disturb their state." [2] [E200, p. 246 (author's translation)].

[2] "On nomme force, tout ce qui est capable de changer l'état de des corps, tant de leur mouvement que du repos [...] Dans chaque force il y a deux choses à considerer, la quantité & la direction: par la quantité on comprend combien une force est plus grande

This description is not much different from those employed today in high school textbooks. [3] Yet it would be wrong to assume that Euler was able to interpret all quantities that appeared in his formulae in terms of the composition of directed segments.

An example of Euler's inability to judge the vectorial character of a geometric entity occurs in his memoir "On the movement of rotation of solid bodies around a variable axis" [E292, §XXVIII]. Having just discovered the equations for the motion of a rigid body in the form

$$\begin{aligned}
d(\omega \cos \alpha) + \frac{c^2 - b^2}{a^2}\omega^2 dt\, \cos \beta \cos \gamma &= \frac{2gP\, dt}{Ma^2},\\
d(\omega \cos \beta) + \frac{a^2 - c^2}{b^2}\omega^2 dt\, \cos \alpha \cos \gamma &= \frac{2gQ\, dt}{Mb^2},\\
d(\omega \cos \gamma) + \frac{b^2 - a^2}{c^2}\omega^2 dt\, \cos \alpha \cos \beta &= \frac{2gR\, dt}{Mc^2},
\end{aligned}$$

where the coordinate axes are laid along the principal axes of inertia relative to the centre of mass, ω is the angular velocity, P, Q, R are the moments of the applied forces about the coordinate axes, $\cos \alpha$, $\cos \beta$, $\cos \gamma$ are the direction cosines of the instantaneous axis of rotation, Ma^2, Mb^2, Mc^2 are the principal moments of inertia and g is a constant, he simplified them ("pour abréger nos formules") by placing

$$\omega \cos \alpha = x,\ \omega \cos \beta = y,\ \omega \cos \gamma = z.$$

The three quantities x, y, z are clearly the projections of a directed segment on the coordinates axes. Astonishingly, Euler seems not to recognize the geometrical meaning of this passage, thus missing the discovery of the angular velocity vector. [4]

Among the formulae used by Euler are the expressions for the velocity of a point of the body in terms of its coordinates and the angular velocity,

$$\begin{aligned}
u\, dt &= \omega\, dt(z \cos \beta - y \cos \gamma),\\
v\, dt &= \omega\, dt(x \cos \gamma - z \cos \alpha),\\
w\, dt &= \omega\, dt(y \cos \alpha - x \cos \beta),
\end{aligned}$$

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ou plus petite qu'une autre, & la direction nous donne à connoitre en quel sens chaque force agit sur les corps pour en troubler l'état."

[3] In passing, let us note that up to about 1840 forces were graphically represented by line segments. The representation by means of arrows appears, perhaps for the first time, in the works of Matthew O'Brien [O'Brien 1851a,O'Brien 1851b].

[4] Euler repeated this derivation in his treatise on the motion of rigid bodies, the *Theoria motus corporum solidorum ...* [E289, cap. XV, §808].

where u, v, w are the components of the velocity (§XII). Had Euler pursued the question further, he might have discovered the geometrical relations that are now expressed by the vectorial formula $\mathbf{v} = \omega \times \mathbf{r}$. (As a matter of fact, this step was taken in [Cauchy 1844].) Thus we see that at this time Euler was not aware of the geometrical meaning of the formulae equivalent to vector products.

The gap that separated the eighteenth century from the general concept of vector can also be seen in the work of Lagrange. Two instances stand out in this regard.

The first instance occurs in his famous paper on the analytic theory of the triangular pyramid [Lagrange 1775b]. Here Lagrange gave the geometrical meaning of the expressions for the scalar and the mixed product (§11; §15),[5] but missed seeing that an ordered triple of the form $(yw - zv, zu - xw, xv - zu)$ represents a vector. It is difficult to understand how he could interpret the very complicated formulae that appear at the beginning of his work without some knowledge of the external product (§1-3).[6]

The second example is taken from the *Méchanique analitique*. In the first edition Lagrange considered the kinematics of a rigid body with a fixed point [Lagrange 1788, p. I, sect. IV, art. 9]. In so doing, he resolved a general infinitesimal rotation into three rotations about the axes of a rectangular system of coordinates, thus demonstrating their law of composition.[7] From his formulae it is easy for us to see the vectorial character of infinitesimal rotations, yet he failed to do so. However, in 1811, after a lapse of more than twenty years, Lagrange returned to the subject in the second edition of his treatise, now entitled *Mécanique analytique*. This time he added to his preceding analysis a comment in which the possibility of representing infinitely small rotations by means of a directed segment is emphasized:

> "It is clear from this development that the composition and resolution of rotational motions are entirely analogous to rectilinear motions.
>
> "Indeed, if on the three axes of the rotations $d\psi$, $d\omega$, $d\phi$, one takes from their point of intersection lines proportional respectively to $d\psi$, $d\omega$, $d\phi$, and if one draws on these lines a rectangular parallelepiped, it is easy to see that the diagonal of this parallepiped will be the axis of composed rotation $d\theta$ and will be at the same time proportional to this rotation $d\theta$. From this result, and because the rotations about the same axis can be added or subtracted depending on whether they are in the same or

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[5] See also [Lagrange 1775a, n. 5]

[6] A geometrical interpretation of these formulae were given much later in [Binet 1813].

[7] It is possible that this result was taken over by Lagrange from the work of Paolo Frisi, who discovered it as early as 1759. This is a matter of debate, for Lagrange did not cite Frisi in this context. Frisi gave several accounts of his theorem [Frisi 1759,Frisi 1767,Frisi 1768,Frisi 1783a,Frisi 1783b].

> opposite directions, in general one must conclude that the composition and resolution of rotational motions is done in the same manner and by the same laws that the composition or resolution of rectilinear motions, by substituting for rotational motions rectilinear motions along the direction of the axes of rotation." [Lagrange 1811-15, part I, sect. III, §III, art. 15; *Oeuvres*, t. XI, p. 61; translation by A. Boissonnade and V. N. Vagliente, 1997].

It is likely that this new interpretation of the old formulae had been prompted by the appearance of Poinsot's *Statique* in 1803.[8]

Lest the foregoing criticism seem too harsh, we must remember that before the nineteenth century even the simplest forms of vector calculus were completely unknown. More importantly, to be accepted as a vector a geometric entity had to obey to the parallelogram law, and thus it was not sufficient that it had three "components".

Our account of vectors in the eighteenth century should make it clear how far removed mathematics was at that time from a real comprehension of the subject. All this began to change by the end of the century. As far as I know, the first recognition of the geometrical meaning of the vector product occurs in Euler's paper "An easy method for investigating every property of curved lines not lying in a plane" [E602]. Here Euler explicitly stated that the three expressions

$$\frac{dz\,d^2y - dy\,d^2z}{ds^3},\ \frac{dx\,d^2z - dz\,d^2x}{ds^3},\ \frac{dy\,d^2x - dx\,d^2y}{ds^3},$$

which, from a modern point of view, are the components of the binormal vector with respect to three orthogonal axes, define a unit segment perpendicular to the osculating plane of a skew curve (§27).[9] However, the same paper contains analytical expressions roughly equivalent to the formulae $\mathbf{b} = \mathbf{t} \times \mathbf{n}$, $\mathbf{n} = \mathbf{b} \times \mathbf{t}$, $\mathbf{t} = \mathbf{n} \times \mathbf{b}$, where $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ are respectively the tangent vector, the principal normal and the binormal (§26), but Euler does not mention this interpretation. Thus there are reasons to believe that he did not fully understand the matter. Euler could scarcely have failed to notice that the three expressions reported above are similar in form to the projections on the coordinate planes of the areas described by the radius vector in the "law of areas" (that is, the conservation of moment of momentum for an isolated system), but surprisingly he did not call attention to this fact. In passing, we note the general formulation of the theory of skew curves by

[8] We know from [Bertrand 1872] that Lagrange had discussed with Poinsot the new developments in vector mechanics around 1806.

[9] These expression can also be found in §19 of Euler's memoir referred to by Truesdell in the first citation [E608].

means of directed segments occurs much later, in [Saint-Venant 1845] and [Chelini 1845].[10]

In order to understand the development of vector calculus it is necessary to take these results into account, for some of the discoveries of the period 1760-1820 led directly to the development of the earliest theories of vectors.[11] The starting point of this new stream of thought can be found in two papers by Euler on the theory of moments. They were presented to the Academy of Science of St. Petersburg in 1780, and appeared consecutively in the 1789 volume of the *Nova acta academiae scientiarum imperialis Petropolitanae*, which was published only in 1793, ten years after Euler's death [E658,E659]. This delay in publication turned out to have some consequences for the subsequent development of the theory.

3. Euler's first memoir: the solution by pure geometry

The purpose of Euler's "On finding the moments of forces about any axis; where several important properties of couples of straight lines, not lying in the same plane, are explained" [E658] is clearly set forth in the title. The paper opens with a purely geometrical definition of the moment of a force V about an axis az: Take any point P on the line of action of the force, and multiply the component of V perpendicular to the plane aPz by the distance of P to az. Euler remarks that it appears very difficult to give a general analytical expression for this definition. However, taking into account the arbitrariness of the choice of the point P, it is possible to make the segment from P to az equal to the common perpendicular of the two assigned lines. Thus we reach the main problem: *To find the distance between two assigned straight lines in space*, supposing that one of them passes through the origin. From this point onwards, much of this work, more than half of the whole, concerns geometrical questions, and Euler carefully separates the basic geometrical results from their applications to mechanics. It must be noted, though, that even these geometrical parts remain algebraically oriented, for Euler describes the positions of the points and the straight lines by means of a rectangular Cartesian system of coordinates.

Mention must be made of the fact that in this paper a straight line in three dimensions is assigned by means of one of its points and its directon

[10] Saint-Venant coined the term "binormal" in [Saint-Venant 1845, p. 17].

[11] The origin of vector calculus in geometry and mechanics is usually not recognized in the standard histories of mathematics. Some idea of the question can be gained from [Caparrini 2003,Caparrini 2004]. *Hactenus hec. Cetera in tempus aliud reservo.*

cosines (§8). This is virtually the modern form of expression, completely symmetrical in all the variables and ready to be translated into the language of vectors. We must recall that twenty years later, in the second edition of his *Feuilles d'analyse* [Monge 1801], Monge still described a straight line by its projections on two coordinate planes.

Before turning to the distance problem, Euler finds an expression for the angle ω between two straight lines in the form

$$\cos\omega = fF + gG + hH,$$

where f, g, h and F, G, H are the cosines of the angles formed by the two lines with the coordinate axes (§13), which is clearly equivalent to the modern scalar product. However, this result was not new, for it can be found in Lagrange's famous paper on the analytic study of tetrahedra [Lagrange 1775b, n. 11], where it is also interpreted geometrically. Let us note that in his proof – not much different from Lagrange's – Euler starts from the formula

$$\cos\omega = \frac{AZ^2 + Az^2 - Zz^2}{2AZ \cdot Az}$$

where AZ and Az are two straight lines passing through the same point A, which is the analytical expression for the so-called "Carnot theorem".

The difference between the mastery of analytic geometry in 1780 and today (or 1820, let's say) is clearly seen by looking at Euler's treatment of the distance problem, for his calculations are somewhat prolix by modern standards. In essence, Euler determines the positions of the end points of the segment of minimal distance between the two straight lines (§15-18), then calculates the length m of the segment (§19-23).[12] Euler's final result, in his own notation, is

$$m\sin\omega = (Gh - Hg)a + (Hf - Fh)b + (Fg - Gf)c,$$

where a, b, c are the coordinates of a point on the axis of the moment. The formula for the distance between two straight lines is a major result in analytic geometry, but is not cited in the histories of the subject.[13]

With this main geometrical theorem stated, in the last part of the paper Euler returns to the problem of finding an analytical expression for the moment of a force. Multiplication of both sides by the intensity V of the force leads immediately to the desired result,

$$(\text{moment of } V \text{ about } az) = V(Gh - Hg)a + V(Hf - Fh)b + V(Fg - Gf)c.$$

[12] The formula for the distance of two points in three dimensions makes an early appearance here; see [Boyer 1956, p. 169].

[13] Euler's distance formula is clearly a mixed product of vectors, or a third order determinant. As we have seen, the geometrical interpretation of expressions of this kind had been given by Lagrange a few years before. See also [E268, p. 3]

Considering now the special case in which the axis az is successively parallel to each of the coordinate axes, the above formula gives

$$Vf(bH - cG), Vg(cF - aH), Vh(aG - bF),$$

and hence the general formula becomes

$$fP + gQ + hR,$$

where P, Q, R are respectively the moments about the axes Ox, Oy, Oz. This expression indicates that moments of forces can be resolved into components along three orthogonal axes by the parallelogram law. In fact, it is equivalent to a scalar product which expresses the projection of a vector along a given straight line by means of components of the vector on three orthogonal axes and the direction cosines of the line.

Euler saw its meaning, for the paper ends with these words:

> "Therefore the moments about three orthogonal axes can be composed exactly as the simple forces. For if three forces P, Q, R were applied to the point a, acting along the directions af, ag, ah, they would form a force equal to $fP + gQ + hR$ acting along the direction az. This marvellous harmony deserves to be considered with the greatest attention, for in general mechanics it can deliver no small development."[14] [E658, §35 (author's translation)]

This passage makes it plain that Euler now visualizes the moment of a force about an axis as a vector lying along the axis. The last remark, of course, is prophetic.

The discovery of the vectorial properties of moments is a result as fine and important as any Euler ever achieved. The final expression can justly be called *Euler's formula for moments*.

Having followed Euler's derivations of the formula of moments, the reader will be no doubt surprised to learn that Euler had already obtained this result almost twenty years before, as the solution of Problem 2 of his paper "On the equilibrium and motion of bodies connected by flexible joints," [E374] written in 1763 but published in 1769. While the result was the same, the proof was more primitive, and Euler failed to grasp its significance.[15]

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[14] "Momenta igitur virium pro ternis axibus inter se normalibus eodem prorsus modo componi possunt, quo vires simplices componi solent. Si enim puncto a applicatae fuerint vires P, Q, R, secundum directiones af, ag, ah, ex iis componitur vis secundum directionem $az = fP+gQ+hR$, quae egregia harmonia maxima attentione digna est censenda, atque in universam Mechanicam hinc non contemnenda incrementa redundare possunt."

[15] The proof is based on the consideration of an *ad hoc* system of forces, which is supposed to be equivalent to the assigned forces. Truesdell, who first noticed this formula, remarked that "the solution of Problem 2 is a proof of the vectorial character of moments, in three dimensions" [Truesdell 1960, p. 342]. This is true, but Euler was not conscious of the fact.

It is curious that Euler returned to the same problem without citing his previous derivation, yet this case is by no means unique.[16] Evidently, by 1780 he had forgotten what he himself had achieved in 1763.

4. Euler's second memoir: the solution by the first principles of statics

Euler considered his result so important that he derived it anew in a second memoir. Shortly after completing the first paper, he wrote "An easy method for determining the moments of every force about any axis" [E659], in which he presents a new approach to the same problem. "While this important result [i.e., Euler's formula $fP + gQ + hR$] has been derived by means of geometrical considerations and with quite long calculations, there is no doubt that it can also be deduced directly from the principles of statics. Having thus diligently considered the question, I happened to find quite an easy way, which led me to this result."[17] His new proof, essentially, rests upon the resolution of a force by means of the parallelogram law and the possibility of translating a force along its line of action without affecting its moment. Thus his methods here bear a strong resemblance to the purely geometrical formulation of Poinsot.

Supposing that the new axis I passes through the origin of the coordinates, Euler begins by replacing the given force with an equivalent system formed by three other forces, each lying in one of the coordinate planes and parallel to one of the coordinate axes. Hence, each of them has a non-zero moment only about one of the axes. The new forces are then resolved into two components, parallel and perpendicular to I, and the moments about I of the perpendicular components are easily found. Expressing these three moments by means of the original components, Euler obtains fP, gQ, hR, and their sum yields the formula of moments.

In the remainder of the paper Euler derives afresh the expression for the distance of two straight lines, starting from the formula of moments. Thus the second memoir exhibits the same results of the first one, but in the reverse order.

[16] [Truesdell 1960] gives several examples of similar episodes.

[17] "Quae egregia veritas cum ex consideratione geometrica per calculos satis prolixos derivata sit, nullum est dubium, quin etiam via directa ex principiis staticis deduci queat. Postquam igitur hoc argumentum sollecite essem perscutatum, incidi in viam satis planam, quae me ad hanc veritatem perduxit." (Author's translation.)

We pause for a moment to note that when Euler gave these complicated geometrical proofs, somewhat difficult to follow even for the experienced reader, he had been blind for about twenty years.

Euler never developed further his discovery of the vectorial representation of moments, nor put it to any use. This idea was to mature many years later.

5. Impact and influence of the work

It is instructive to follow the history of Euler's formula up until the beginning of the nineteenth century, for it influenced in various ways several important mathematicians. This was in fact the first step towards a formulation of mechanics entirely based on the concept of vector. Euler's formula is like an Ariadnean thread through the early development of vector calculus.

According to [Poisson 1827, p. 357], by the time Euler's two papers were published, the situation caused by the revolution made it difficult for French mathematicians to have access to them. Not knowing Euler's work, in 1798 Laplace considered the problem of simplifying the equations of motion of an isolated mechanical system by choosing coordinate axes which reduce to zero some constants of motion [Laplace 1798]; thus he discovered the *invariable plane.* [18] In modern terms, the invariable plane is simply a plane orthogonal to the total moment of momentum vector. To obtain this result, Laplace had to calculate the formulae for the transformation of the projections on the coordinate planes of the areas swept over by the radius vector in the movement of the planets in passing from one coordinate system to another, and thus nearly discovered the vectorial nature of moment of momentum.

Shortly thereafter Laplace wrote a second paper on the same subject, whose title was simply “Sur la Mécanique” [Laplace 1799a]. It is only two pages long and there is not a single formula. Here Laplace remarks that the invariable plane is orthogonal to the axis of moments, which he calls *axe du plus grand moment.*

The connection between the formulations of Euler and Laplace, which now seems obvious, was established by Prony with a few lines of simple calculations in his *Mécanique philosophique* [de Prony 1800, p. 110]. There, in a footnote, he gives the first explicit citation of Euler's first paper. He

[18] This result was immediately included in the *Traité de mécanique céleste* [Laplace 1799b, liv. I, ch. IV, n. 21]. See also the *Exposition du système du monde* [Laplace 1835, VI:199].

remarks that Euler's formula "is of such simplicity and elegance that it can be considered one of the most beautiful results in mechanics."[19] While Prony did not add anything new to the preceding works, he has the merit of having clarified and made generally known the first results in the vectorial theory of moments.

Poinsot, independently of Euler and Laplace, initiated a purely geometric approach to the vectorial theory of moments in his famous textbook of statics *Eléments de Statique*, first published in 1803 but reprinted at least twelve times before the end of the century [Poinsot 1803, n. 60-67]. To study the equilibrium of a rigid body with respect to rotations, Poinsot introduced the *couple of forces.* A couple is a system of two equal, parallel and oppositely directed forces, whose magnitude is measured by the product of the intensity of the forces by the distance between their lines of action. Poinsot showed that if we represent a couple with a segment perpendicular to its plane, we can compound two couples by means of the parallelogram law.

In a successive work [Poinsot 1806], Poinsot demonstrated the existence of the central axis and gave vectorial proofs of the conservation of momentum and of moment of momentum in an isolated system. While in the first edition of this paper he did not say anything about the results obtained by Euler, in the subsequent editions, published as an appendix to the *Eléments*, Poinsot added an observation about the formula $G\cos\theta = L\cos\lambda + M\cos\mu + N\cos\nu$, which furnishes the value of the projection of the couple G on the axis whose cosines are $\cos\lambda$, $\cos\mu$, $\cos\nu$ with respect to the coordinate axes:

> "[This is] a very simple formula, which Euler gave in vol. VII of the *New Proceedings of Petersburg*, but to which he could arrive only by means of lengthy analytical calculation."[20] [Poinsot 1803, 1842 ed., p. 355]

A different geometric representation of moments was developed by Poisson a little later [Poisson 1808]. Poisson remarked that the moment of a force about a point is numerically equal to the double of the area of a triangle having the vertex in the point and the force itself as its basis, and thus implicitly assumed that it can be represented geometrically by the triangle. Poisson was clearly inspired by Laplace's theory of the invariable plane and by Poinsot's couples. Euler's formula is given in the form

$$D = A'\cos\epsilon' + A''\cos\epsilon'' + A'''\cos\epsilon''',$$

[19] "[Cette formule est] d'une simplicité et d'une élégance telle qu'on peut la regarder comme une des plus belles de la mécaniques." (Author's translation.)

[20] "[elle est une] formule très-simple qu'Euler a donné dans le tome VII des *Nouveaux Actes de Petersbourg*, mais à laquelle il n'était parvenu que par de longs circuits d'analyse." (Author's translation.)

where D is the plane area which represents the moment, A', A'', A''' are its projections on the coordinate planes and ϵ' , ϵ'' , ϵ''' are the angles between D and the coordinate planes.

Poisson included this theory in his *Traité de Mécanique*, one of the most influential mathematical textbooks of all time [Poisson 1811, vol. I, liv. I, ch. III], [Poisson 1833, vol. I, liv. III, ch. II]. Here, to distinguish between the two sides of a surface, he considered the directed straight line perpendicular to it; this is the first example of a surface oriented by means of a vector. [21] .

It should be noted that while in 1808 Poisson had given all the credit for the discovery of the formula for moments to Laplace ("these theorems on the invariable plane and on the composition of moments are due to M. Laplace"), [22] in the second edition of the *Traité de Mécanique* [Poisson 1833, p. 544] these developments in the theory of moments were attributed to Euler alone ("these remarkable theorems are due to Euler"). [23]

The new theory of moments was briefly taken up by Lagrange in the second edition of the *Mécanique analytique* [Lagrange 1811-15, vol. I, partie I, sect. III, §III, n. 16]. To demonstrate Euler's formula, Lagrange used the vectorial representation of infinitesimal rotations. He started from the expression of the virtual work due to a small rotation of a rigid body,

$$L\,d\psi + M\,d\omega + N\,d\phi,$$

where L, M, N are the moments of the force about the three axes of a rectangular Cartesian system of coordinates and $d\psi$, $d\omega$, $D\phi$ are the infinitesimal rotations about the same axes. Lagrange substituted the given rotations with their decomposition into three rotations about a second system of orthogonal axes, thus obtaining the moments about the new axes in the form

$$\begin{aligned} &L\cos\lambda' \; + M\cos\mu' \; + N\cos\nu', \\ &L\cos\lambda'' + M\cos\mu'' + N\cos\nu'', \\ &L\cos\lambda''' + M\cos\mu''' + N\cos\nu''', \end{aligned}$$

where λ', μ', ν', λ'', μ'', ν'', λ''', μ''', ν''' are the angles formed by the new axes with the original system. Lagrange remarked that this result had

[21] The earliest example of an oriented surface appeared just a few years before in L. Carnot's *Géométrie de position*, where the two sides of a surface are described as painted in different colours [Carnot 1803, p. 94]

[22] "Ces théorêmes sur le plan invariable et sur la composition des momens sont dus à M. Laplace." (Author's translation.)

[23] "Ces théorèmes remarquables sont dus à Euler." (Author's translation.)

been obtained by geometrical methods in the *Novi commentarii* for 1789, but Euler's name was not mentioned.[24]

Still another geometric representation of moments was proposed by Binet in 1815 [Binet 1815]. While considering the motion of a rigid body with a fixed point O, he substituted every applied force $\mathbf{F}$ with a force whose line of action is situated at a unitary distance from O and whose moment about O is the same as that of $\mathbf{F}$, and said that this new force represents the moment of $\mathbf{F}$ about O. Euler's memoir is referred to in §III, which contains an analytical rephrasing of some portions of Poinsot's theory of couples. Here Binet observes that the expression for the least total couple could also be obtained by means of the formula for the distance between two straight lines found by Euler.

In a second paper on the theory of moments, Binet introduced the vector representation of the areal velocity, for which he openly acknowledged the influence of Euler and Poinsot:

> "The areal velocities can be composed following rules analogous to those for the composition and resolution of linear motions. It is not necessary for me to insist on this point, that the theorems of Euler and the research on moments of M. Poinsot have established without doubt, for our areal velocities are exactly the moments of ordinary velocities."[25] [Binet 1823, p. 164 (author's translation)].

Some additional contributions to Euler's formula were made by Antonio Bordoni, who expressed the formula in various forms and used it to solve several problems [Bordoni 1822]. The greatest part of his paper is dedicated to the resolution of different forms of the following problem: Given four concurrent straight lines in space and the moments of a system of forces about three of them, to find the moment about the fourth line. Thus, in effect, Bordoni was studying the generalization of Euler's formula to non-orthogonal Cartesian axes.

After 1820 the time was ripe for someone to organize all the different views involved in the theory of moments into a unified formulation. It fell to

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[24] "Experience with [Lagrange's book] has led me to the following working hypothesis: 1.There was little new in the *Méchanique Analitique*; its content derives from earlier papers of Lagrange himself or from works of Euler and other predecessors. 2. General principles or concepts of mechanics are misunderstood or neglected by Lagrange. 3. Lagrange's histories usually give the right references but misrepresent or slight the content." [Truesdell 1964, 1968 reprint, p. 246]

[25] "Les vtesses aréolaires se combinent entre elles, d'après des règles analogues à celles de la composition et de la décomposition des mouvemens linéaires: je n'ai pas d sister sur cet objet, que les théorèmes d'Euler et les recherches de M. Poinsot sur les momens ont mis hors de doute, puisque nos vtesses aréolaires sont précisément les momens des vtesses ordinaires."

Cauchy to do this, as he had done with many other branches of mathematics. In 1826 he published in vol. I of his *Exercices de Mathématiques* five papers in which he brought the theory to its final formulation [Cauchy1826a-1826f]. Except for the lack of a proper vectorial notation, his treatment is essentially modern. Cauchy's moments are vectors, like Poinsot's couples and Binet's *momens*, that represent Poisson's surfaces.

The almost simultaneous appearance of several different theories of moments obviously led to some controversies over priority, which allow us to see how these mathematicians viewed their own work. The first controversy arose in 1827 between Cauchy and Poinsot. After the publication of Cauchy's theory of moments, Poinsot accused Cauchy of having published results which were merely repetitions of his theorems on couples disguised under a different notation [Poinsot 1827a]. Cauchy replied that his theory was more general, for it could be applied to every kind of physical entity which can be represented by a directed line segment [Cauchy 1827]. A second controversy began when Poisson published a short account of the recent history of the theory of moments, in which he asserted that Euler was the discoverer the vectorial composition of moments and maintained that Poinsot's work was entirely derived from that of his predecessors [Poisson 1827]. Poinsot answered with a long and detailed article in which he observed that his theory of couples had introduced a *geometrical composition* of moments, whereas up to then there had been only the algebraic sum of certain expressions [Poinsot 1827b].

This was the end of the polemics. Euler's papers were then cited in Möbius' *Lehrbuch der Statik* [Möbius 1837, §89-91], but not in the relevant portion of Grassmann's first *Ausdehnugslehre* [Grassman 1844, §59]. Thereafter, they disappeared from the literature on the vectorial theory of moments.

As we have seen, Euler's first memoir includes the formula for the distance between two straight lines. This result can also be found in Monge's *Feuilles d'Analyse* [Monge 1801, §12-13] and in Cauchy's *Leçons sur les Applications du Calcul infinitésimal a la Géométrie* [Cauchy 1826f, Prélim., Prob. VII], but their proofs are completely different from Euler's, thus giving evidence that they had been found independently.

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Leonhard Euler: Life, Work and Legacy
Robert E. Bradley and C. Edward Sandifer (Editors)

Euler's Contribution to Differential Geometry and its Reception

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1. Leonhard Euler's various contributions to differential geometry

There was almost no mathematical discipline in the eighteenth century to which Euler did not contribute. Many of Euler's contributions to special fields were appreciated and respected. But the case of differential geometry is different. Though Euler wrote articles on curve and surface theory throughout his life, there is almost no secondary literature concerning this particular aspect of his work. Euler himself mentions only a few of his own articles in differential geometry in any of his others. Only Dirk Struik dedicated a chapter to Euler in his "Outline of a history of differential geometry." [Struik 1933]

The term "differential geometry" was first used by Luigi Bianchi (1856-1928) in an Italian textbook *Lezioni di geometria differenziale.* (Pisa 1886) In Euler's time we take it to mean the theory of curves and surfaces. The theory of curves began with the rise of calculus and important results came quickly. Isaac Newton (1643-1727) determined an expression for the curvature of plane curves by means of his kind of calculus [Stiegler 1968]. Also Jakob Bernoulli (1654-1705) wrote papers about cycloids, catenary curves, helical curves, spirals, circle of curvature, caustics, the elastic curve and its radius of curvature and the lemniscate [Weil 1999]. Curve theory quickly became a particularly well developed field. At first Euler too was interested in curve theory, but he soon achieved results in surface theory also. Euler was the first mathematician who worked successfully on surface theory.

In the following, only Euler's main results shall be discussed. It is not possible to give a complete survey. The main contributions, however, will be mentioned. The order will be chronological.

1.1. *First example of a minimal surface: the catenoid, 1744*

In 1741 Euler moved from St. Petersburg to Berlin. That same year he became member of the Berlin academy, the *Brandenburgische Sozietät der Wissenschaften*, founded in 1700. In 1742 Euler also became an honorary member of the Academy of St. Petersburg. For the years 1744 to 1766 Euler was director of the mathematical class of the Academy in Berlin. After the position had been vacant for five years Pierre-Louis Moreau de Maupertuis (1698-1759) was appointed president of the academy in 1746. When Maupertuis died, there was another interregnum from 1759 to 1764. In 1764, however, Frederic II (1712-1786, reg. 1740-1786) named himself head of the academy. After he was not appointed president of the academy, in 1766 Euler left Berlin and returned to St. Petersburg.

In a letter to Maupertuis on March 14, 1746, Euler mentioned that he had started his work [1] on *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes,* "A method for finding curved lines enjoying the properties of maximum or minimum, or solution of isoperimetric problems in the broadest accepted sense" [E65] when he was still in St. Petersburg. Once in Berlin, Euler gave his manuscript to his publisher Bousquet [2] and it was published in September 1744 in Lausanne and Geneva. It contained 6 chapters [Fraser 2005]. With this work Euler founded a new discipline within analysis, the calculus of variations. Before Euler there were several individual problems, but after Euler there was a general calculus of variations.

There are many connections between the calculus of variations and differential geometry, for example geodesic lines. These can be treated by means of variations and they play an important role in differential geometry. In Chap. IV, §11 for example Euler is concerned with the shortest lines on a general curved surface.

[1] He wrote, "Cela s'entend de mon ouvrage même, que j'avois déjà achevé à Petersbourg." *Opera omnia* (4) 6, p.60.
[2] *Opera omnia* (1) 24, p. XI.

In the next chapter, however, Euler asked: "To find the curve among all others of the same length which, if rotated around the axis AZ, delivers a solid, the surface of which is either a maximum or a minimum." [3] The answer is the general equation of the *catenaria*, the chain line:

$$dx = \frac{c\,dy}{\sqrt{(b+y)^2 - cc}}.$$

This most outstanding result shows that the catenoid is a minimal surface. This was the first example of a minimal surface in history [E65, Chap.V, §47; p. 186f].

1.2. *Definition of the curvature of a surface, 1767*

Four years later Euler published his very famous textbook *Introductio in analysin infinitorum*, [E101-102], which soon became classic. The second volume contains the theory of curves and in an appendix surface theory, though it has only little on differential geometry, for example some remarks on singular points and asymptotes of plane curves, some osculation properties, and notes on concavity and convexity in relation to the sign of the radius of curvature [Struik 1933, p.102].

Indeed, Euler did not give a general theory of surfaces. He just treated several topics on solid surfaces: the intersection of a surface with a plane, especially sections of cylinders, cones and spheres, some second-order surfaces and the intersection of two surfaces. This last chapter included the theory of space curves [Reich 2005b].

But fifteen years later Euler had achieved spectacular results on surface theory. For the first time it was possible to give a definition of the curvature of a surface. On September 8, 1763 Euler presented to the Berlin academy his "Recherches sur la courbure des surfaces." The paper was published in 1767 [E333].

In 1766 Euler left Berlin and returned to St. Petersburg to accept the invitation of Catherine II, who truly appreciated him as a first class scientist, a recognition which Frederic II in Berlin had never granted.

Euler began his "Recherches" by formulating the problem: "I will begin by determining the radius of curvature for a section of an arbitrary plane cutting the surface." [4] He set out a three-step plan to achieve this goal:

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[3] Invenire curvam, quae inter omnes alias eiusdem longitudinis circa axem AZ rotata producat solidum, cuius superficies sit vel maxima vel minima. [E65, Chap. V, §47]

[4] "Je commencerai par déterminer le rayon osculateur pour une section quelconque plane, dont on coupe la surface."

"(i) If a surface, the nature of which is known, is cut by an arbitrary plane, to determine the curvature of this section.

(ii) If the plane of the section is perpendicular to the surface in the point Z, to determine the radius of curvature of this section in the same point Z.

(iii) An arbitrary surface being given, to find the osculating radius of a section EPZ, which forms an angle φ with the principal section."

The result was a very long expression, which was transformed into a much easier one by means of :

f, the largest radius of curvature, which belonged to the section EF, and

g, the smallest radius of curvature, which belonged to the section normal to the previous one. After a long calculation the result was:

$$r = \frac{2fg}{f + g - (f - g)\cos 2\varphi}$$

where f and g are the radii of curvature of the principal sections and φ is the angle between an arbitrary normal section and the principal section.

This was an astonishing result indeed, which Euler expressed in his own words: "And so the measurement of the curvature of surfaces, however complicated, which appeared at the beginning, is reduced at each point to the knowledge of two radii of curvature, one the largest and the other the smallest, at that point; these two things entirely determine the nature of the curvature and we can determine the curvatures of all possible sections perpendicular at the given point." [5]

1.3. *Developable surfaces and the so-called Gaussian variables, 1772*

In March 1770 Euler presented to the academy in St. Petersburg his paper "About solids, the surfaces of which can be developed on the plane" (De solidis quorum superficiem in planum explicare licet, [E419]). In this paper, Euler extended his study of surfaces to developable surfaces, a totally new concept. For the first time he represented a point x, y, z on a surface as a function of two variables t and u. These were later called Gaussian variables.

[5] "Ainsi le jugement sur la courbure des surfaces, quelque compliqué qu'il ait paru au commencement, se réduit pour chaque élément à la conoissance de deux rayons osculateurs, dont l'un est le plus grand et l'autre le plus petit dans cet élément; ces deux choses déterminent entierement la nature de la courbure en nous découvrant la courbure de toutes les sections possibles, qui sont perpendiculaires sur l'élément proposé." [E333, p. 22]

Euler began with the remark that in elementary geometry it is well known that cylinders and cones have the property that they can be flattened out, or "developed" into a plane, while, for example, the sphere does not have this property. He asked which other kinds of surfaces have the property that they are developable into a plane; this question is, according to Euler, a most notable, characteristic one. [6]

Euler investigated the conditions on x, y, and z, the coordinates of a point, and the two variables t and u describing the surface, i.e.

$$dx^2 + dy^2 + dz^2 = dt^2 + du^2.$$

The geometrical problem is therefore reduced to the solution of the following analytical problem: given the two variables t and u you have to find the six equivalent functions l, m, n, λ, μ and ν, so that the formulas

$$ldt + \lambda du,\ mdt + \mu du,\ \text{and}\ ndt + \nu du$$

are integrable and further satisfy:

$$\lambda\lambda + \mu\mu + \nu\nu = 1, \quad l\lambda + m\mu + n\nu = 0$$

.

The exposition was threefold:

(i) a solution by means of analytical principles,
(ii) a solution by means of geometrical principles, and
(iii) the application of the second to the first solution.

As a result Euler was able to prove that the line element of the surface has to be the same as the line element of the plane or, as he expressed it, "All surfaces which can be developed on a plane by means of flexibility and without stretching can be represented by the tangents of a spatial curve."

According to Andreas Speiser, this paper of Euler has to be regarded as one of his very best mathematical achievements. [7]

Euler also mentioned his success in a letter to Lagrange dated January 16/27, 1770: [8]

> "I have found a complete solution to the following problem: It is a matter of finding three functions, X, Y, Z of two variables t and u such that setting $dX = Pdt + pdu$, $dY = Qdt + qdu$, $dZ = Rdt + rdu$, they will satisfy the following conditions:

[6] "Quaesitio igitur hinc nascitur maxime notatu digna, quo charactere ea solida instructa esse oportet, quorum superficiem in planum explicare licet."

[7] *Opera omnia* (1) 28, p. XXIV: "so werden wir diese Arbeit als eine mathematische Höchstleistung bezeichnen dürfen."

[8] "...j'ai trouvé une solution complète du problème suivant: Il s'agit de trouver trois fonctions X, Y, Z de deux variables t et u, telles que, posant $dX = Pdt + pdu$, $dY = Qdt + qdu$, $dZ = Rdt + rdu$, on satisfasse aux conditions suivantes:

$$\text{I. } P^2 + Q^2 + R^2 = 1,$$
$$\text{II. } p^2 + q^2 + r^2 = 1,$$
$$\text{III. } Pp + Qq + Rr = 0.$$

Now, the nature of differentials requires the following additional conditions:

$$\text{I. } \frac{\partial P}{\partial u} = \frac{\partial p}{\partial t}$$
$$\text{II. } \frac{\partial Q}{\partial q} = \frac{\partial q}{\partial t}$$
$$\text{III. } \frac{\partial R}{\partial u} = \frac{\partial r}{\partial t}.$$

As an altogether singular thought led me to the solution of this problem, which I would have believed before would be impossible, I think that this discovery could become very important in the new part of integral calculus for which Geometry is indebted to you."

Euler should agree with Lagrange's assessment of the importance of this result.

1.4. *Orbiforms, 1781*

Four years later, on May 12, 1774, Euler presented the Academy of St. Petersburg with a paper on orbiforms, which was published in 1781 with the

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$$\text{I. } P^2 + Q^2 + R^2 = 1,$$
$$\text{II. } p^2 + q^2 + r^2 = 1,$$
$$\text{III. } Pp + Qq + Rr = 0.$$

Or la nature des différentielles demande encore les conditions suivantes:

$$\text{I. } \frac{\partial P}{\partial u} = \frac{\partial p}{\partial t}$$
$$\text{II. } \frac{\partial Q}{\partial q} = \frac{\partial q}{\partial t}$$
$$\text{III. } \frac{\partial R}{\partial u} = \frac{\partial r}{\partial t}.$$

Comme une considération tout à fait singulière m'a conduit à la solution de ce problème, que j'aurais d'ailleurs jugé presque impossible, je crois que cette découverte pourra devenir d'une grande importance dans la nouvelle partie du Calcul intégral dont la Géométrie vous est redevable." *Oeuvres de Lagrange*, vol.14, Paris 1902, p.217f.

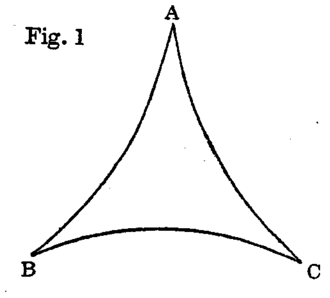

Fig. 1. Euler's triangular curve

title "About triangular curves" (De curvis triangularibus, [E513]).[9] The problem itself had its origins in optics. Orbiforms are curves of constant breadth so named because the circle shares this property. Euler showed that the circle is not the only shape with this property. At first Euler investigated triangular curves, closed curves with three cusps that look like astroids. (See Fig 1.) Euler showed that the evolvents of these curves are curves of constant breadth.

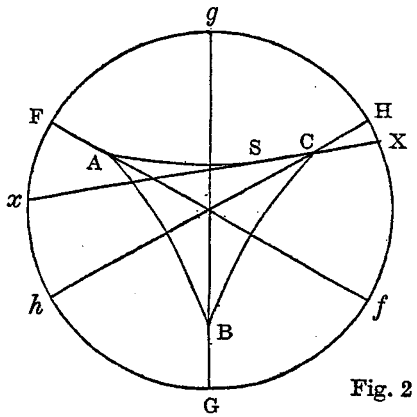

Fig. 2. Some properties of the triangular curve

[9] Homer White gives a further discussion of this article, E513, in "The Geometry of Leonhard Euler" elsewhere in this volume.

1.5. *Moving trihedral, spherical image, first Frenet formula, 1786*

For a long time, Euler was not interested in the theory of space curves. But on May 28, 1775, he presented to the St. Petersburg Academy his paper "Easy method to investigate all points of intersection of curves which do not lie in the same plane" (Methodus facilis omnia symptomata linearum curvarum non in eodem plano sitarum investigandi, [E602]).

Alexis Clairaut (1713-1765) had been the first to treat space curves systematically. He published his results in Paris in 1731 under the title *Recherches sur les courbes à double courbure.* To investigate space curves analytically Clairaut used projections of space curves onto the planes of the coordinates. Euler, however, chose the arc length s as the variable of a space curve, which made the presentation much more elegant. In §5 he introduced a unit sphere with its center at a point moving along the curve. This is equivalent to the introduction of the spherical image, which was later used by Gauß
indexGauss, Karl Friedrich and now known as the Gauss map. Euler then defined the moving trihedral, i.e. the tangent, the normal and the second normal (binormal) and calculated their cosines (§17 and 18) as well as the radius of curvature (§10). In the second of the paper part he continued his calculations. The main results were, written in modern terminology:

$$\overrightarrow{t} = \overrightarrow{h} \times \overrightarrow{b}, \ \overrightarrow{h} = \overrightarrow{b} \times \overrightarrow{t}, \text{ and } R\frac{d\overrightarrow{t}}{ds} = \overrightarrow{h}.$$

This is the first of the three Frenet formulas.

1.6. *Developable surfaces, rigidity of closed surfaces, 1862*

This paper of Euler is a fragment, written by his students in a kind of notebook, the *Adversarii mathematici.* Euler again treated the development of surfaces. The problem is "To find two surfaces one of which may be transformed into the other so that in both of them homologous [corresponding] points keep the same distances from each other." [10] Euler proves that surfaces have this property if they have the same line elements.

The paper finishes with the following annotation:

"It is appropriate here to note one may not assume another surface other than the given one. In any case it is not clear how the functions p, m, and n have to be taken so that the surface has the given shape, for example a sphere. In both formulas the, two variables r and s can be

[10] "Invenire duas superficies, quarum alteram in alteram transformare liceat, ita ut in utraque singula puncta homologa easdem inter se teneant distantias."

augmented to infinity and that this extension cannot be removed by any imaginary thing. Hence neither the sphere nor any other figure in a finite space can be described by these formulas. But as to the terminated or everywhere closed figures it looks as if they have to be judged in another way, because as soon as a solid figure is everywhere completed, it does not permit any further mutation. This can be understood by looking at these known figures that usually are called regular. Thus insofar as the spherical surface is complete, it dies not admit any mutation. Hence it is clear that such figures can be mutated insofar as they are not integer or everywhere closed. Yet, it is clear that the figure of the hemisphere is certainly mutable. But which kinds of mutations are possible seems to be a very difficult problem." [11]

2. Reception

The first reactions to Euler's discoveries came from Italy. Lagrange and Euler had corresponded since 1754. There still exist 36 letters exchanged between the two through 1775, but Euler and Lagrange never met each other.

2.1. *Joseph Louis Lagrange (1736-1813)*

Euler's and Lagrange's common interests included the calculus of variations. Lagrange was 29 years younger than Euler and 10 years older than Monge. He was born in Turin and began his career at first in his home-town where, in 1755, he became professor of mathematics at the Royal Artillery School. In 1756 Lagrange became corresponding member at the academy in Berlin and ten years later he succeeded Euler in Berlin as director of the mathematical class of the Academy. In 1772 Lagrange was elected *associé étranger* at the Académie des sciences in Paris. This was reconfirmed when the Académie was reorganized in 1785. After Frederic II died, Lagrange left Berlin in 1787 and returned to Paris, where he immediately became *pensionnaire vétéran* at the academy and professor at the École Normale and at the École Polytechnique. One year later, in 1788, he was promoted to *directeur* of the academy. In 1795 Lagrange became *membre résidant de la section mathématiques* and was elected *président du bureau provisoire.* When the Bureau des longitudes was founded in Paris in 1795,

[11] [E819, p. 440]. The translation was made by Eberhard Knobloch whom I want to thank very much.

Lagrange was among the founding members. In 1801 also the Sozietät der Wissenschaften in Göttingen chose Lagrange as a corresponding member.

At first it had been Euler's "A method for finding curved lines enjoying the properties of maximum or minimum" [E65] which fascinated Lagrange. This led to a correspondence between Euler and Lagrange. In 1760/1 Lagrange published some of his own results of the discussion with Euler: "Essai d'une nouvelle méthode pour déterminer les maxima et les minima des formules intégrales indéfinies" [Lagrange 1760/1]. This work, like Euler's book, is mainly devoted to the calculus of variations, but Langrange's work also included as "Appendix I" a chapter on minimal surfaces: "Par la méthode qui vient d'être expliquée on peut aussi chercher les maxima et les minima des surfaces courbes, d'une manière plus générale qu'on ne l'a fait jusqu'ici." [12] Lagrange derived the partial differential equation of minimal surfaces:

$$\left(\frac{dP}{dx}\right) + \left(\frac{dQ}{dy}\right) = 0,$$
$$P = \frac{p}{\sqrt{1+p^2+q^2}}, \quad Q = \frac{q}{\sqrt{1+p^2+q^2}},$$

where x, y and z are rectangular coordinates.

Later these equations were transformed into the more modern form [Reich 1973, p.313f]:

$$r\left(1-q^2\right) - 2ps + \left(1+p^2\right)t = 0,$$
$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y},$$
$$r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}.$$

Lagrange, however, did not mention the catenoid, which, as we described above, had been found by Euler.

Lagrange, though, did refer extensively to Euler's theory of curved surfaces [E333]. Lagrange's textbook *Théorie des fonctions analytiques* which was first published in 1797 and had a second edition in 1813, included a chapter devoted to "Des sphères osculatrices. Des lignes de plus grande et de moindre courbure. Propriétés de ces lignes" [Lagrange 1813, Chapter IX]. Lagrange emphasized that the results of Euler and Monge should be

[12] "By the method which comes to be explained, one can also find the maxima and minima of curved surfaces, in a more general manner than has been used before," *Oeuvres de Lagrange*, vol.1, p.353-357.

appreciated by all geometers: "These properties of surfaces are very curious and they merit the full attention of geometers; they will have especially important applications for the arts," [13] and quoted [E333], [Monge 1780] and [Monge 1785].

2.2. *Gaspard Monge (1746-1818)*

Euler's achievements in differential geometry were also influential among Monge and his school of students from Mézière as well as students from the École Polytechnique in Paris.

Gaspard Monge was 39 years younger than Euler. Born in Beaune, he began his career as a student of the École Royale Du Génie de Mézières. This school, founded in 1748, had as its aim the education of engineers. It had its best time during the years 1765-1775. Taton called these years "la grande periode" [Taton 1964, p.586-596]. In 1769 Monge became *répétiteur de mathématiques* (tutor of mathematics), at the age of 24, and in 1770 he became responsible for all mathematical and physical lectures at the École in Mézières.

In 1772 Monge was elected to be a corresponding member of the Parisian Academy, and in 1780 he became *adjoint géomètre*, replacing Vandermonde. In 1785 Monge was promoted to *associé* of the physics class, and in 1795 he was nominated *membre résidant.* Also that year Monge started teaching at the newly founded École polytechnique after the school in Mézières had closed in 1794.

Monge's main interest was geometry; descriptive geometry, design, analytical geometry and so on. And of course, he always was keenly aware of relationships of theory with practice and the applications of geometry especially to technology.

In 1771 Monge presented the Parisian Academy with his first paper on developable surfaces, "Mémoire sur les développées, les rayons de courbure et les différens genres d'inflexions des courbes à double courbure." [Monge 1785] This paper had a new style, and Euler played no role in it. Monge proved several interesting theorems about space curves.

When Monge read Euler's paper "About solids, the surfaces of which can be developed on the plane" [E419], Monge got even more interested in developable surfaces. He wrote a second paper, which he presented to the Academy in 1775 and published even earlier than the first one, "Mémoire sur les propriétés de plusieurs genres de surfaces courbes, particulièrtement

[13] "Ces proprietés des surfaces sont très-curieuses et méritent toute l'attention des géomètres; elles donnent lieu surtout à des applications importantes pour les arts" *Oeuvres* 9, p. 273.

sur celles des surfaces développables, avec une application à la theorie des ombres et des pénombres" [Monge 1780]. Monge mentioned Euler:

> Having started this material, on the occasion of a memoir of Mr. Euler in the 1771 volume [14] of the Academy of St. Petersburg on developable surfaces and in which that illustrious Geometer gave the formulas for recognizing whether or not a given curved surface has the property of being able to be mapped to a plane, I arrived at some results which seem much simpler to me an easier to use for the same purpose." [15]

Indeed, Monge gave the following definition: "A surface is developable whenever, by supposing it to be flexible and inextensible, one may conceive of mapping it onto a plane, like cones and cylinders can be, so that the way in which it rests on the plane is without duplication or disruption of continuity." [16]

It is remarkable that Monge also characterized the developable surfaces with the terms "flexible et inextensible." Monge managed to deduce the general differential equation of developable surfaces [Monge 1780, p.398]. As Taton had pointed out, there was a gap between Euler and Monge as far as styles were concerned: "Euler, of a profoundly analytic spirit, and Monge, dominated constantly by a sharp sense of geometric reality." [17]

2.3. *Monge's school*

2.3.1. *Graduate students from the École Royale du Génie de Mézières*

The École Royal du Génie de Mézières was supposed to educate practitioners, so most of the students had no scientific ambitions. Nevertheless, two of Monge's pupils at Mézières should be mentioned.

Charles Tinseau (1749-1822)

Tinseau began his studies in 1769 and finished as a military engineer in 1771. In 1773 Tinseau became *correspondant* at the Académie des sciences,

[14] Here, Monge made an error. The correct date is 1772.

[15] "Ayant repris cette matière, à l'occasion d'un Mémoire que M. Euler a donné dans le Volume de 1771, de l'Académie de Pétersbourg, sur les surfaces développables, et dans lequel cet illustre Géomètre donne des formules pour reconnoître si une surfache courbe proposée, jouit ou non de la propriété de pouvoir être appliquée sur un plan, je suis parvenu à des résultats qui me semblent beaucoup plus simples, et d'un usage bien plus facile pour le même objet."

[16] "Une surface est développable, lorsqu'en la supposant flexible et inextensible, on peut concevoir appliquée sur un plan, comme celles des cones et des cylindres, de maniere qu'elle le touche sans duplicature ni solution de continuité..." [Monge 1780, p.383]

[17] "Euler, d'esprit profondément analytique, et Monge, dominé constamment par un sens aigu de la réalité géométrique." [Taton 1951, p.21]

and in 1774 he presented his paper "Solution de quelques problèmes relatifs à la théorie des surfaces courbes et des courbes à double courbure" [Tinseau 1780]. The paper shows that Tinseau was directly influenced by Monge and that he did not quote Euler. Tinseau solved 17 problems, concerning, among other things the equation of the osculating plane to a space curve, the surface of the tangents to a curve and the theorem that the orthogonal projection of a space curve onto a plane has a point of inflexion, if its plane is perpendicular to the osculating plane [Struik 1933, p. 108]. In a second paper Tinseau treated several problems of ruled surfaces [Taton 1951, p.233f]. Afterwards, Tinseau made a military career. He campaigned against the French revolution and was later exiled.

Jean Baptiste Meusnier (1754-1793)

Monge's second pupil at Mézières was of much greater importance and was much more recognized. Jean Baptiste Meusnier studied at the École du Génie from 1774 to 1775. On February 14 and 21, 1776, Meusnier presented his first and only paper, "Mémoire sur la courbure des surfaces" to the Académie Royale des Sciences in Paris and in June 1776 he became *correspondant* of the Academy. In 1784 he became *adjoint géomètre* and in 1785 *associé de la classe de géométrie.*

Monge made notes about the circumstances under which his young student got involved in the curvature of surfaces. As soon as he had arrived in Mézières, Meusnier visited Monge and asked him for a special project, hoping to prove his skills to Monge. Monge further reported:

> "To satisfy him, I talked to him about the theory of Euler on the radii of maximum and minimum curvature of curved surfaces; I showed him the principal result and proposed that he look for its proof. The next morning in my office, he gave me a short paper, containing his proof; but what was remarkable was that the reasoning it used was more direct, and the path he followed was much shorter than Euler's had been. The elegance of this solution and the little time that it had cost to him gave me an idea of his sagacity and all the work that he has undertaken since have the same evidence of his exquisite sense of the nature of the things." [18]

[18] "Pour le satisfaire, je l'entretins de la théorie d'Euler sur les rayons de courbure maxima et minima des surfaces courbes; je lui en exposai les principaux résultats et lui proposai d'en chercher la démonstration. Le lendemain matin, dans les salles, il me remit un petit papier, que contenait cette démonstration; mais ce qu'il y avait remarquable, c'est que les considérations qu'il avait employées étaient plus directes, et la marche qu'il avait suivie était beaucoup plus rapide que celles dont Euler avait fait usage. L'élégance de cette solution et le peu de temps qu'elle lui avait coûté me donnèrent une idée de la sagacité et de ce sentiment exquis de la nature des choses dont il a donné des preuves mul-

Meusnier's paper "Mémoire sur la courbure des surfaces," presented in 1776, was his only mathematical paper and it was not published until nine years later [Meusnier 1785]. As the title suggests, Meusnier's work was based in part on Euler's paper of the same title. [E333] Meusnier wrote:

> "Mr. Euler has treated the same material in a very beautiful Memoir published in 1760 by the Academy in Berlin. This famous Geometer considers the question in a way different from the one we have just described. He makes the curvature of a surface element depend on the various sections that one can make by cutting it with planes." [19]

Indeed, Meusnier used different methods than Euler and he managed to add some new results. The theorem of Meusnier is still presented in some modern textbooks and is mentioned in modern mathematical dictionaries:

> The centre of curvature at a point P for a curve on a surface is the projection upon its osculating plane of the centre of curvature of that normal section of the surface which is tangent to the curve P.

In his paper Meusnier solved the following five problems:

"1. To determine the different positions that the tangent plane can have in the understanding of a surface element.
2. To determine the radius of curvature of the section made on a surface element by a plane in any given position.
3. To determine the kinds of surfaces for which the two radii of curvature are always equal.
4. Among all surfaces which can be made to pass through a given perimeter formed by a curve of double curvature, to find that for which the area is the least.
5. To find the general equation for developable surfaces." [20]

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tiples dans tous les travaux qu'il a entrepris depuis" [Taton 1951, p.234]; [Darboux 1902, p.221].

[19] "M. Euler a traité la même matière dans un fort beau Mémoire, imprimé en 1760 parmi ceux de l'Académie de Berlin. Cet illustre Géomètre envisage la question d'une manière différente de celle que nous venons d'exposer; il fait dépendre la Courbure d'un élément de surface, de celle des différentes sections qu'on y peut faire en le coupant par des plans" [Meusnier 1785, p.478f].

[20] "1. Déterminer les différentes positions que peut avoir le plan tangent dans l'étendue d'un élément de surface?

2. Déterminer le rayon de Courbure de la section faite dans un élément de surface par un plan quelconque donné de position.

3. Déterminer quelles sont les surfaces pour lesquelles les deux rayons de Courbure sont toujours égaux.

4. Entre toutes les surfaces qu'on peut faire passer par un périmètre donné, formé par une courbe à double Courbure, trouver celle dont l'aire est la moindre.

5. Trouver l'équation générale des surfaces développables."

His main idea is presented in his work on problem 2. Struik summarized it with the following words:

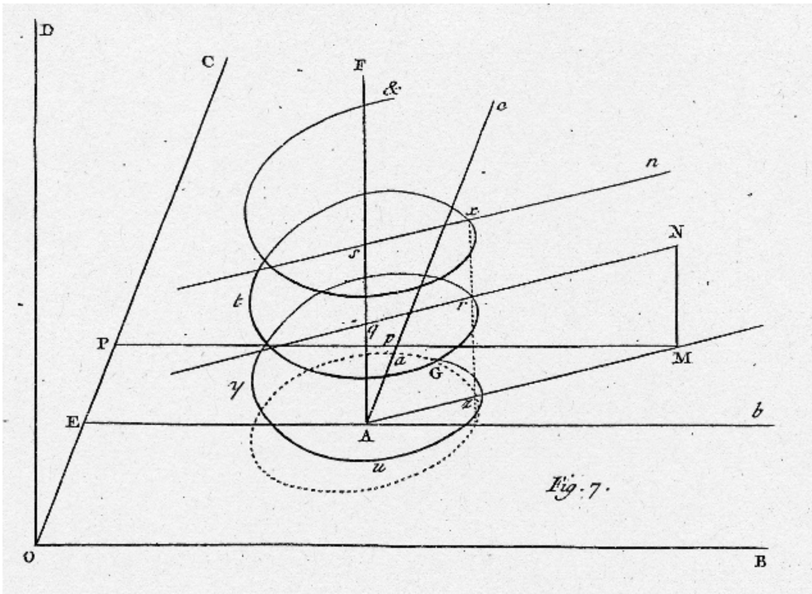

Fig. 3. Meusnier's figure 7, the right helicoid

> "In one of the principal sections of the surface at a point he draws a circle, tangent to the surface, with radius equal to the 'rayon de courbure' at that point in that direction. This circle is rotated about an axis in its plane parallel to the tangent plane and at a distance equal to the second principal radius of curvature. In this way Meusnier gets a torus which has the first and second derivatives in common with the surface at the point. Then he takes this torus as representative of the surface and gets not only Euler's theorem, but also 'Meusnier's theorem', which he interprets with the aid of a sphere tangent to the surface and with radius equal to the normal radius of curvature of the section in the arbitrary direction on the surface" [Struik 1933, p.107].

In his problem 4, Meusnier investigated the conditions under which a surface is a minimal surface. He found that the sum of the radii of principal curvature is constant. Meusnier did not mention that Euler had found a first example, the catenoid (see 1.1.), as early as 1744. It is almost certain that Meusnier did not know Euler's 1744 paper. But Meusnier himself found two examples, the right helicoid (Fig. 3) and, again, the catenoid (Fig. 4).

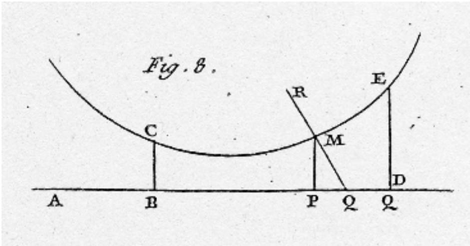

Fig. 4. Meusnier's figure 8, the catenoid

Meusnier never came back to mathematics, but he contributed to various other fields. For example, he was an aeronautical theorist. After the first flights of Montgolfier's balloon he designed an elliptical shaped airship instead of the spherical balloon, but his balloon was never built. Meusnier also collaborated with Antoine de Lavoisier (1743-1794) to separate water into its constituents.

Like Tinseau, Meusnier had an accomplished military career, but Meusnier supported the revolution. He became even a *général de division.* He was severely wounded during a battle between the French and the Prussians near Mainz. Goethe observed that battle and gave a detailed description [Goethe 1793].

2.3.2. *Graduate students from the École Polytechnique*

In 1795, almost immediately after the foundation of the École Polytechnique in Paris, Monge published his *Feuilles d'Analyse appliquée à la Géométrie à l'usage de l'École Polytechnique* [Monge 1805-1850]. It reappeared in later editions, entitled *Application de l'Analyse à la Géométrie* [Monge 1795-1807]. The last edition from 1850 was prepared by Joseph Liouville (1809-1882), who remarkably also published Monge's *Application* and Gauß' *Disquisitiones generales circa superficiem curvas.*

The lectures that Monge gave at the École polytechnique in Paris had much greater influence than the lectures at Mézières. Taton mentioned the following students of Monge:

Lacroix,[21] Hachette, Fourier, Lancret, Dupin, Livet, Brianchon, Ampère, Malus, Binet, Gaultier, Sophie Germain, Gergonne, O. Rodrigues, Poncelet, Berthot, Roche, Lamé, Fresnel, Chasles, Olivier, Valée, Coriolis, Bobillier, Barré de Saint-Venant and many others [Taton 1951, p.235f]. Some of these worked on differential geometry.

[21] Taton mentioned his name here, but Lacroix had not attended Monge's courses at the École Polytechnique.

Michel-Ange Lancret (1774-1807)

Lancret began his studies in 1794. Before the official lectures at the École Polythechnique began, Monge gave a special course to a small group of excellent students, including Lancret. [Taton 1951, p.39] He was among the first graduate students of Monge, who very highly respected Lancret. Later Monge, Lancret and other scientists accompanied Napoleon on his Egyptian expedition in 1799/1800. Lancret did not return to France until in 1802 when he became the secretary of the commission concerning the work of the Egyptian expedition.

In 1806 and in 1811 Lancret published two papers on curve theory. Euler was only mentioned in an historical context [Lancret 1806, p.416]. Lancret worked mostly on the basis of the Monge's various contributions. According to Struik, Lancret was the first to take up the systematic theory of space curves after Euler, but it seems in an independent way. The line of progress goes from Clairaut to Euler and then from Lancret to Cauchy and Frenet [Struik 1933, p.116].

Charles Dupin (1784-1873)

Dupin's name is still known in differential geometry. Under the guidance of Gaspard Monge, Dupin made his first discovery in 1801, what is now called the cyclid of Dupin. He graduated from the École Polytechnique in 1803 as a naval engineer. For several years he worked abroad and came back to France only in 1813. In 1814 he became correspondent of the mechanical section and in 1818 he was elected member of the Académie des Sciences. When Monge died in 1818, Dupin wrote a detailed *Éloge* for his former teacher. [Dupin 1819] Just a year later, in 1819, Dupin became professor at the *Conservatoire des Arts et Métiers* in Paris, a position which he held until 1854. In 1834 he became minister of marine affairs. During the years 1836-1844 he was Vice-President of the Académie des Sciences, and in 1838 he became peer and in 1852 he was appointed to the senate.

Dupin's *Développements de géometrie* [Dupin 1813] were his main contributions to differential geometry. Here one can find the introduction of conjugate and asymptotic lines on a surface, the so-called "indicatrix," and Dupin's theorem, which states that three families of orthogonal surfaces intersect in lines of curvature. When Hachette wrote the "Avertissement de l'Editeur," he mentioned Euler, Biot, Monge and Clairaut. He made special mention of Euler's *Introductio* [E101-102] and his contribution to surface curvature. [E333] Dupin and Hachette noted Euler's *Introductio* for the work on the general equation of the surfaces of the second order and its classification of these kinds of surfaces into 5 different categories.

They cited the second work because it gave the expression of the radius of curvature of a normal section of a surface and showed that the planes of sections of maximum and minimum radius were perpendicular on each other. Indeed, the "Article V" of Dupin's *Développements* was devoted to the "Démonstration de plusieurs théorèmes d'Euler sur la courbure des surfaces" [Dupin 1813, p.107-110]. Dupin, however, had achieved his own results independently from Euler.

Augustin Louis Cauchy (1789-1857)

Cauchy began his studies in 1805 at the École Polytechnique and continued at the École des Ponts et des Chaussées. After several years working as an engineer in Cherbourg, Cauchy returned to Paris in 1813, where he became professor at the École Polytechnique. In 1816 Cauchy became a member of the Académie des sciences. While he was professor at the École Polytechnique he wrote many papers, including some on differential geometry and several textbooks, which became quite famous. Among these were his *Leçons sur les applications du calcul infinitésimal à la géometrie.* [Cauchy 1826-1828] He used almost the same title as Monge.

In his preface Cauchy emphasized the contributions of two physicists, Gaspard Gustave de Coriolis (1792-1843) and André-Marie Ampère (1775-1836). The former had given a definition of the general radius of curvature of curves that was adopted by Cauchy in his 17th chapter. Both Ampère and Coriolis had been pupils of Monge as well. In this 17th chapter, "Du plan osculateur d'une courbe quelconque et de ses deux courbures. Rayon de courbure, centre de courbure et cercle osculateur," Cauchy derived the first and the second of the Frenet formulas. Euler had only presented the first one in [E602]. In his 19th chapter Cauchy treated surface theory, especially the radius of curvature of principal sections and so on. Here Cauchy also quoted Euler:

> "We will not end this lesson without recalling that it was Euler who first established the theory of curvature of surfaces and showed the relations which exist between the radii of curvature of the different sections cut from a surface by its perpendicular planes. The discoveries of that illustrious geometer on all these things have been published in the *Mémoires* of the Berlin Academy (1760)" [22]

[22] "Nous ne terminerons par cette Leçon sans rappeler que c'est Euler qui le premier a établi la théorie de la courbure des surfaces, et montré les relations qui existent entre les rayons de courbure des diverses sections faites dans une surface par des plans normaux. Les recherches de cet illustre géomètre, sur l'objet dont il s'agit, ont été insérées dans les *Mémoires de l'Académie de Berlin* (année 1760)" [Cauchy 1826-1828, p. 364].

Olinde Rodrigues (1794-1851)

Rodrigues was of Jewish origin, so he was not allowed to study at the École Polytechnique. Instead he entered the École normale where he was awarded a doctorate in mathematics in 1816. As Taton remarked, though Rodrigues was not a direct pupil of Monge, he should be counted among Monge's students [Taton 1951, p.366]. In 1815 and 1816 Rodrigues published two papers on differential geometry in which he presented some work on the lines of curvature and simplified some of Monge's results. Rodrigues he did not primarily take ideas from Euler; he mentioned mainly Monge and Dupin[Grattan-Guiness 2005, p. 100]. Euler's influence on Rodrigues was perhaps more or less an indirect one. [23]

Sylvestre François Lacroix (1765-1843)

Born in Paris in 1765, Lacroix also was a pupil of Gaspard Monge. Lacroix came from a very poor family. He began private lectures with Monge in 1780, and Monge continued to follow his education and his carrier. In 1789 Lacroix became correspondent of the Parisian Academy and in 1794 he joined the Commission de L'instruction publique. He later became teacher at the École Polytechnique, at the École normale, at the École Centrale des Quatre Nations, at the Faculté des Sciences de Paris and finally at the Collège de France. As a consequence of his many teaching positions, he became author of several textbooks, including his well-known *Traité du calcul différentiel et du calcul intégral* (2 vol., Paris 1797 and 1798). There was a second edition in three volumes (Paris 1810, 1814 and 1819) as well as a translation into German due to J.P.Grüson (2 vol., Berlin 1799, 1800).

Lacroix devoted his chapter 5 to surfaces and curves of double curvature. Here Lacroix mentioned in detail the contributions of Euler, Monge and Meunsier. Euler had been the first to recognize the importance of the principal curvatures and the analytical expression which allows a distinction between developable surfaces and surfaces which do not have this property. Lacroix mentioned Monge for having introduced symmetry and elegance. [24]

[23] Teun Kotesier describes more on the influence of Euler on Olinde Rodrigues in his article "Euler and Kinematics," elsewhere in this volume.
[24] *Traité*, vol.1, Paris 1810 (2nd ed.), p. XXXXVI and 578-580.

3. Final Remarks

Reactions to Euler's contributions to differential geometry came mainly from Italy and France. Monge and his school were of immense importance. In England there was no tradition in differential geometry and in Germany most of the mathematicians did not come as far in differential geometry. There were no German contributions that reached farther, but Euler's definition of the curvature of a surface [E333] was mentioned in German dictionaries. Georg Simon Klügel, (1739-1812) for example, referred to Euler's results in his *Mathematical Dictionary* [Klügel 1808] under the title "circle of curvature." He remarked that it was Euler who first presented the "highly difficult investigation" of curved surfaces. Later on Klügel gave a summary of Euler's paper of 1767. Klügel also mentioned the names Lagrange (1813) and Meusnier (1785) and gave a glimpse of their papers.

Some, but not all of Euler's contributions on differential geometry were recognized. Lagrange, of course, continued Euler's research on the calculus of variations, but Euler's most successful and influential paper was his "Recherches sur la courbure des surfaces." [E333] Many ideas which Euler had published first were taken up much later or were not taken up at all. The Frenet formulas for example were a subject of investigation only in 1847 and 1852 [Reich 1973, p.277, 280-283], and only then were all three formulas presented. The third example of a minimal surface, the so-called Scherk minimal surface, was only found in 1832. [Reich 1973, p.315f]

Cauchy's *Leçons sur les applications du calcul infinitésimal à la géométrie* was the last major textbook before Gauß published his *Disquisitiones generales circa superficies curvas* (General Investigations of curved surfaces) in 1828. Gauß did not quote anybody, and only at the end did he mention Legendre's theorem. However, Gauß introduced the spherical image, the line element, the idea of developable surfaces, and surfaces which are flexible without stretching.

What can be said about the relationship between Euler and Gauß? In his *Disquisitiones arithmeticae* (Leipzig 1801) Gauß quoted 28 papers of Euler and he spoke of *summus Euler.* He also possessed a picture of Euler which he had drawn himself [Reich 2005a]. Why did Gauß not quote Euler's contributions in his *Disquisitiones generales*? [Gauß 1828] There are two possibilities: either he did not know Euler's papers or Euler's papers were not important for him.

Let us first look at the second option. Indeed, Gauß's *Disquisitiones generales* were not a further development of Euler's ideas. Gauß came up with totally new ideas and goals. One main idea was that of "invariance;" Gauß' newly defined measurement of curvature was invariant, his line element was invariant and also the geodesic lines had this property. Another main idea was Gauß' new interpretation of surfaces, which now became in some sense two-dimensional manifolds. Euler instead spoke of "Solids the surfaces of which can be developed on the plane" (De solidis quorum superficiem in planum explicare licet.) Euler's surfaces were still the boundaries of solids. Gauß' surfaces, on the other hand, stood alone, without solids and even without the surrounding space. These ideas were totally new and Euler was no predecessor. Thus, in differential geometry Euler was not important for Gauß.

The first option can't be answered. There is no proof that Gauß read Euler's papers on differential geometry nor is there any proof that did not read them. We don't know.

There is no doubt, however, a new epoch in differential geometry began with Euler and another new epoch began with Gauß.

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Euler's Mechanics as a Foundation of Quantum Mechanics

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1. Introduction

Euler's contributions to mechanics are rooted in his program published in the *Mechanica* in 1736 [E15/16]. In this article it will be demonstrated that the development of physics in 19th and 20th centuries can be considered as a natural continuation and completion of the Eulerian program. The importance of Euler's theory results from the simultaneous development and application of mathematical and physical methods. Euler continued a practice which had been established by his predecessors Galileo, Descartes, Newton and Leibniz.

All essential mathematical tools for modern 20th century physics had been developed in 19th century. The mathematics of the general theory of relativity was created by Bolyai, Gauß and Riemann several decades before Einstein. The basic differential equation, later used for the quantum mechanical harmonic oscillator, was introduced by Weber 1869 and treated by Whittaker in 1903 [Whittaker].

In contrast to Einstein's general theory of relativity, the development of quantum mechanics (QM) was not guided directly by mathematics. However, Schrödinger recognized the geometrical representation of motion by Hamilton and acknowledged the attempts of Felix Klein in 1891 to stimulate the physics community to make use of Hamilton's theory. The response was disappointing [Schrödinger 1926b]. Some decades later, Schrödinger rediscovered Hamilton's theory as a link between mechanics and optics and, not surprisingly, was successful in developing the basic equation of quantum mechanics independently of Heisenberg. [1]

Euler made immediate use of his mathematics for classical mechanics (CM) and coordinated his progress in mathematics with his progress in physics. Thus, we have a rare example of a simultaneous and harmonic composition of results of different origin and nature which have been joined into a unique marvellous result. The theory is not only a model for a consistently formulated theory, but allows for generalizations of Euler's principles. It will be demonstrated that Euler's treatment of CM is appropriate for an understanding and a derivation of Schrödinger's basic quantum mechanical equation. The underlying assumptions made by Schrödinger for QM can be directly related to the basic assumptions introduced by Euler for CM.

2. The contribution of Euler to mechanics

Euler was famous as the leading mathematician of the 18th century. Though his pioneering work on mechanics had an essential influence in 18th century, its impact on the 19th century has been overlooked, obscured by the overwhelming success of his mathematical writings. In contrast, the influence of Leibniz's mechanics on the 19th century has finally been acknowledged, though with a certain delay, but an essential part of Euler's contributions has not yet been understood or described. Even though Euler analyzed mechanics with principles which were developed later by Einstein, neither Mach [Mach] nor Helmholtz nor Einstein refer explicitly to Euler. Euler's fundamental contribution was published in 1862 [E842]. At that time, Helmholtz referred explicitly to Leibniz, and later, in 1920, Reichenbach acknowledged Leibniz's theory, but neither credited Euler.

2.1. *Euler's program for mechanics*

Very early on, Euler developed a comprehensive program for mechanics [E15/16]. The basic distinction made by Euler is between
– (a) bodies of infinitesimal magnitude and

[1] "Eines genetischen Zusammenhangs mit *Heisenberg* bin ich mir durchaus nicht bewußt. Ich hatte von seiner Theorie natürlich Kenntnis, fühlte mich aber durch die mir sehr schwierig scheinenden Methoden der transzendenten Algebra und durch den Mangel an Anschaulichkeit abgeschreckt, um nicht zu sagen abgestoßen." [Schrödinger 1926c]

– (b) bodies of finite (non-zero) magnitude.

The bodies described in frame (a) are nowadays known as mass points. Euler introduced a general law for mechanics [E177], [E289] which is based on the assumption of translational motion of bodies of infinitesimal magnitude. The set (b) comprises all *non-infinitesimal bodies* which cannot be treated as mass point since their motion is a combination of translations and rotations.

> "Those laws of motion which a body observes when left to itself in continuing rest or motion pertain properly to infinitely small bodies, which can be considered as points. ... The diversity of bodies therefore will supply the primary division of our work. First indeed we shall consider infinitely small bodies. ... Then we shall attack bodies of finite magnitude which are rigid. ... Third, we shall treat of flexible bodies. Fourth, of those which admit extension and contraction. Fifth, we shall subject to examination the motions of several separated bodies, some of which hinder [each other] from executing their motions as they attempt them. Sixth and last, the motion of fluids will have to be treated." [E15/16, vol. 1, §98] [2]

Over the following decades, Euler almost completed this impressive program.

The mechanical system is characterized by constitutive or material parameters which are always given in terms of finite numerical values. Space and time are not considered as material parameters, therefore, they are included in the theory as finite or infinitesimal quantities.

In mechanics, the fundamental constitutive parameter is the inert mass m. The inert mass m of an infinitesimal body may be described by the same quantity as the inert mass of an extended body, since both are mechanical systems. The only difference is that total mass of the finite body is obtained by the integral $M = \int dm$ [E289].

Euler's division of bodies into different types is based on a mathematical distinction between infinitesimal and finite quantities. The assumed set of mathematical quantities is not complete so far. It has to be supplemented by *infinite* mathematical quantities, since only *infinitesimal* and *infinite* quantities are complementing each other properly. [3] Following Euler, the

[2] For "Euler's life-long plan for mechanics" compare E. Sandifer, Euler Society Meeting 2003, http://people.wcsu.edu/sandifere/History/Preprints/Preprints.htm.

[3] Euler stressed this relation between infinitesimal and infinite numbers in the *Algebra* mentioning "the mistake of those who assert that an infinitely large number is not susceptible to increase. This opinion is inconsistent with very principles we have laid down; for $\frac{1}{0}$ signifying a number infinitely great, then $\frac{2}{0}$ being incontestably the double of $\frac{1}{0}$, it is evident that a number, though infinitely great, may still become twice, thrice, or any number of times greater." [E387/388, vol. 1,§84] Obviously, in contemporary

basic distinction is made
– first between *infinitesimal* and *non-infinitesimal* magnitudes, denoted as sets (A) and (non-A), respectively, and,
– second between *finite* and *infinite* magnitudes, denoted as sets (B) and (C), respectively.

Then, a significant question follows directly from Euler's program: Is it possible to assign a *finite* constitutive parameter to a system of *infinite* extension?

2.2. *Euler's program for mechanics, reconsidered*

The answer is "yes" if we start with a purely mathematical definition of the density related to any of the constitutive parameters of the system. The definition of the densities related to bodies in set (B) is valid for different types of densities $\rho_b^{hom}(x)$ and $\rho_c^{non-hom}(x)$, being either homogeneous or non-homogeneous, respectively. However, there is a striking difference between these two basic types of densities as far as the extension of the system is concerned. For a finite homogeneous density, the integral $M = \int_{-\infty}^{+\infty} dx \rho_{hom}(x)$ is necessarily divergent. Therefore, we obtain an exclusion principle by physical reasons which supplements the mathematical criteria, since any quantity related to a finite mechanical system should be necessarily of finite magnitude.

From these criteria it follows, that for a non-homogeneous density $M = \int_{-\infty}^{+\infty} dx \rho_c^{non-hom}(x)$ both the mathematical and physical criteria can be only fulfilled for coordinate dependent functions which are quadraticly integrable, i. e. $\int_{-\infty}^{+\infty} dx (f(x))^2 < +\infty$.

These functions are candidates for describing systems of type (C) and the theory is constructed by the same principles used by Euler for set (A). First, a constitutive parameter of the system should be *finite* as for the bodies of set (B). If only those physical parameters are considered which are in all cases of *finite* magnitude, then the physical parts of the sets (A), (B) and (C) should coincide. So, as long as only a finite force is exerted upon a finite or an infinitesimal body and the pressure is not considered, no difference between the effect of the force will appear if the masses are the same. Therefore, the same physical quantity can be assigned to systems of different type. It is expected that the energy conservation law holds not

notation we have to replace 0 with an infinitesimal number ϵ which is greater than zero, but less any real number. Such numbers are considered in non-standard analysis [Keisler].

only for any of the considered systems of different type, but it is also valid for the interaction of systems of different types. [4]

In all three cases, it is necessary to introduce a purely coordinate dependent function. [5] Furthermore, it will be demonstrated that Euler's procedure is also appropriate for the analysis of the relations between CM and QM. In the second decade of 20th century, the development of QM began with the rejection of basic concepts of CM. Heisenberg rejected the paths [Heisenberg] and Schrödinger intended to replace point mechanics with an "undulatory mechanics." [6] [Schrödinger 1926b]

The procedure continues as follows. Euler's method for the definition of relations between (A) and (B) systems will be transferred to the analysis of the relations between (B) and (C) systems.

3. From Euler's mechanics to Schrödinger's wave function

3.1. *Euler's general law of rest and motion*

Euler introduced explicitly one Law for the change of the state, independently of the type of state, being either the state of rest or the state of

[4] An instructive example is the photoelectric effect, which has been explained by Einstein as the interaction of point like light quanta and finite sized bodies [Einstein 1905].

[5] A coordinate dependent function is only related to the extension of the system in space. However, the extension has also to be expressed in terms of other physical quantities related to the system. Following Helmholtz, the extension of the system is changed if the energy increases or decreases [Helmholtz]. The total energy is related to a configuration described by special value of coordinates, which are, e.g. for a linear harmonic oscillator, given by x_{max} resulting in the energy $E = V(x_{max}) = \frac{k}{2}x^2_{max}$. Here, the index max does not indicate an extreme value of the function $V(x)$, since the only extreme value for the potential energy is the minimum at $x = 0$.

[6] However, Schrödinger was fully aware of the need to properly define the relation between these representations and advised caution. "Ich will damit noch kein zutreffendes Bild des wirklichen Geschehens geben, ..." [Schrödinger 1926b, p. 507)]. "Die Stärke des vorliegenden Versuches ... liegt in dem leitenden physikalischen Gesichtspunkt, welcher die Brücke schlägt zwischen dem makroskopischen und dem mikroskopischen mechanischen Geschehen, . . . " (p. 514). However, the apparent incompatibility between the physical model and the mathematical language becomes obvious: From the point of view of physics, the system is defined as "microscopically" compared to a "macroscopic" or "finite" system, whereas from the point of view of mathematics, this "microscopic" physical system is defined due to the properties of wave function which is extended in the "whole configuration space", being "infinite" in size. Therefore, following Euler, from the point of view of mathematics, the comparison has to be established between a finitely and an infinitely extended system.

motion [E181,E289].[7]

The change of the state is expressed in terms of the internal material parameters mass m and forces K which are also internal properties as far as a system of bodies is concerned. Time is an external ordering parameter. The state is described by the velocity $v = const$. Then, *one* quantity dv_i for each of the bodies $i = 1, 2$, the change in velocity,[8] is given in terms of m_i and K_{ij}:

$$dv_1 \sim \frac{1}{m_1}, \quad dv_1 \sim K_{12}, \quad dv_2 \sim \frac{1}{m_2}, \quad dv_2 \sim K_{21}. \tag{1}$$

Here, Euler introduced the condition that the change in velocity $\Delta v \sim \frac{1}{m}$ is always finite, if the mass of the body is finite and, additionally, different from zero. However, replacing the previously finite quantity Δv with the infinitesimal quantity dv, the resulting Eq. 1 becomes incorrect and incomplete, since an infinitesimal quantity is expressed in terms of a finite quantity. The same problem occurs with respect to the forces in the case $K \neq 0$.

Euler completed Eq. 1 by the introduction of an infinitesimal translation ds and an impression of force during an infinitesimal time interval dt. He assumed that all infinitesimal changes of position are translations. The elementary translation ds can be replaced with the infinitesimal elementary time element dt assuming $ds \sim dt$ and $ds = vdt$. Adding this dependence on time elements to Eq. 1, both sides of the relation are given in terms of infinitesimal quantities. However, different relations have to be considered:

$$dv > \frac{K}{m}dt, \quad dv = \frac{K}{m}dt, \quad dv < \frac{K}{m}dt. \tag{2}$$

The decision in favor of one of the three relations can only be obtained from another principle, the conservation of momentum, $m_1v_1 + m_2v_2 =$ const or the conservation of energy, $E = \frac{m}{2}v^2 + V(x) =$ const, where $V(x)$ is the potential energy function. The expressions 2 have to be completed by the dependence of the change in motion on infinitesimal tanslation ds, i.e. $vdv = \frac{K}{m}ds$. The complete set of Newton-Euler equations of motion is given by the two coupled equations where the forces are generated by the interacting bodies, [E181]

[7] "1. C'est une propriété générale de tous le corps, ..., que chaque corps considéré en lui-même demeure constamment dans le même état, ou de repos ou de mouvement" [E181, §1]

[8] Newton included this quantity in the "change in motion" where motion is given by the product of mass and velocity [Newton].

$$dv_1 = \frac{K_{12}}{m_1}dt, \quad dv_2 = \frac{K_{21}}{m_2}dt. \tag{3}$$

The coupling due to the forces $K_{12} = -K_{21}$ is usually known as the principle "action = reaction" of Newton's 3rd Law. However, in Euler's mechanics, this law has lost the status of an axiom, but follows from the interaction between bodies due to the impenetrability [E181], [E842], [E289]. Euler formulated the general law in terms of infinitesimal time elements and obtained the equations of motion [9]

$$dds = \frac{K}{m}dt^2 \quad \text{or} \quad dv = \frac{K}{m}dt \tag{4}$$

where the change in velocity dv depends on the magnitude and the direction of the force K. The mass m and the time element dt are assumed to be constant quantities [E177,E842,E289].

3.2. *Schrödinger. The wave function*

Following the general procedure introduced by Newton and Euler, we reconsider Schrödinger's approach which is essentially based on the introduction of the wave function $\psi(x)$. This function is related to the internal energetic states of the system. [10] The system does not interact with the environment and the internal energy is not changed. The system is not translated in space and it does not rotate about an axis.

[9] "Si corpusculum, cuius massa $= m$, sollicitetur a vi $= K$ per motus resolutionem in directione huius vis tempusculo dt conficiat spatiolum ds celeritate $\frac{ds}{dt} = v$, erit (see Eq. 4) ... . Vel augmentum celeritatis secundum directionem vis sollicitantis acceptum est directe ut vis sollicitans ducta in tempusculum et reciproce ut massa corpusculi" [E289, §177], with $m > 0$ and $dt > 0$. Here, Euler considered the *space element ds* and the *time element dt* as *different* infinitesimal quantities, whose ratio defines a finite quantity, the velocity v (see also [E289, §§42-46] and [E842, §24]). Both the quantities are *differentials of first degree.* The quantity dds is introduced as a *differential of second degree* [E289, §168]. The mathematical foundation of the procedure had been given by Euler in the *Institutiones calculi differentialis* [E212, Ch. I-III]. Euler's procedure is in almost complete agreement with the principles later developed for non-standard analysis by Robinson (for non-standard analysis compare [Keisler]). The reason is that the rules related to the calculus of *finite differences* [E212, Ch. I] are transferred to the calculus of *infinitesimal differences* [E212, Ch. III]. Here, Euler *simultaneously* introduced infinite and infinitesimal quantities and implicitly used an early version of the transfer principle, later founded on a rigorous treatment by Robinson (for the role of the transfer principle, compare [Keisler, Epilogue]).

[10] Later, Dirac [Dirac] and Feynman [Feynman] assumed the wave function to be the primary object. The Schrödinger equation is derived from the assumed properties of the wave functions.

Schrödinger considered solutions $\psi_E(x)$ to the time-independent wave equation.[11] [Schrödinger 1926b] The function $\psi(x)$ has been assumed to be defined and to exist for all configurations of the system.[12] Then, it is expected that different states are described by different wave functions.

As in Eulerian mechanics, where the change in velocity dv has been related to the mass m and the forces K (compare Section 3.1), here, the total energy E and the coordinate dependent potential energy $V(x)$ are related to the function[13] $\psi_E(x)$. If this function is related to the energy

$$\frac{1}{E - V(x)} \sim \psi_E(x), \tag{5}$$

Schrödinger's criterion of finiteness is satisfied for large values of x.[14] Obviously, the function ψ cannot be calculated using Eq. 5. This relation is as incomplete as the relation $dv \sim \frac{K}{m}$ was (compare Section 3.1). Therefore, we have to complete the description of the system adding kinetic energy $E = T(p) + V(x)$,[15] where $T(p)$ is the kinetic energy as a function of momentum. Therefore, assuming Schrödinger's wave function as the basic quantity, we have also to express the kinetic energy in terms of the function $\psi(x)$. This will done in Section 6.

3.3. *Infinitesimal bodies, finite and infinite systems*

Schrödinger's approach completes and extends in a quite natural way Euler's program for mechanics. The systems are distinguished according to their extension occupying regions of different magnitude in configuration space. Then, the following types of extension can appear.

[11] This equation was later called "amplitude equation" to stress the contrast to the solutions $\psi(x,t)$ of the "true wave equation" ("eigentliche Wellengleichung") where the time dependence has been included. [Schrödinger 1926d]

[12] Schrödinger claimed that the function has been introduced without any additional assumption except the finiteness in the whole configuration space. " ... ohne irgendeine weitere Zusatzannahme als die für eine physikalische Größe beinahe selbstverständliche Anforderung an die Funktion ψ: dieselbe soll im ganzen Konfigurationenraum eindeutig endlich und stetig sein" [Schrödinger 1926b].

[13] Both functions are required, since the total energy is independent of coordinates and the potential energy is independent of total energy.

[14] Additionally, the symmetry of the function $\psi(x)$ depends on the symmetry of the function $V(x)$, as discussed by Euler for the relations between dv and K (compare Section 4). For $V(x) = V(-x)$ it is expected, that the relation 5 has to be completed by second or fourth order derivatives of $\psi(x)$, if even and odd functions $\psi(x)$ are taken into account. The tentative order of the differential equation has been discussed by Schrödinger [Schrödinger 1926b, p. 509].

[15] Then, Eq. 5 has to be completed by $E - V(x) \sim \frac{1}{m}$.

– (I) the theory of bodies of *infinitesimal* magnitude, Euler,
– (II) the theory of bodies of *finite* magnitude, Euler,
– (III) the theory of bodies or systems of bodies of *infinite* magnitude, Schrödinger.

Item (III) should be related to Schrödinger's assumption that the function $\psi(x)$ is defined in the whole configuration space.

Obviously, the topics (I) to (III) comprise all possible cases of mathematical quantities which Euler had defined within the *mathematical* frame [E387/388], [E212]. Euler defined mathematical quantities as being susceptible to diminishing and increase. This is also a necessary condition for the definition of a physical quantity P which is based upon multiplying a number R and a unit U, i.e. $P = R \cdot U$.

Following Euler, the existence of such function is primarily a mathematical question. Euler's procedure is to relate infinitesimal quantities to finite quantities, i.e. numbers of different type to each other. Now, this procedure is transferred to the problem to relate infinite quantities to finite quantities.

After solving the mathematical problem, the physical problem has to be answered whether the numbers are related to objects by measurement. Obviously, all physical systems are finite. Therefore, it is impossible to confirm the theory by the measurement of all quantities, since infinitesimal and infinite quantities cannot be measured. [16]

4. Energy, paths and configurations

The following procedure results directly from Euler's theory of a body of infinitesimal magnitude. Euler introduced this type of bodies by the modification of the geometric characteristics (extension, shape) preserving the physical parameters, the mass and the path. The geometric connection between two given points is replaced with the mechanical connection of two positions by the path of the body and vice versa. The significant difference between a mechanical model and a purely geometrical model of the body is due to inertia, time and forces. Any distance is subdivided by a uniformly moving body into equal parts and, simultaneously, the time

[16] Usually, it is said that some of the relations are only approximately valid and the neglecting of a small quantity in comparison to a large quantity is equivalent to the relation between an infinitesimal and a finite physical quantity. However, this model does not work as far as the relations between infinitesimal quantities are concerned and the ratio of two infinitesimal quantities is finite. This ratio cannot be obtained experimentally by measuring both the quantities separately.

interval assigned to the whole path is also subdivided into equal parts.[17] The connection between two different places in space is given geometrically by their distance. This distance can be determined experimentally without the motion of a body whose path contains the two given points. Therefore, the configurations are always defined prior to the path. A configuration is defined for a system of resting bodies which do not change their positions.[18] The positions are given by a set of coordinates x_i, $i = 1, 2, ..., n$, the path is given by an ordered set $x_1, x_2, ..., x_n$ using an external ordering parameter $t_1 < t_2 < .... < t_n$, called time. Then, the path can be represented as a function $x = x(t)$ which is parametrized by time. The same parametrization is assumed for the momentum $p(t)$. The exclusion of all other configurations except those which are belonging to the path is described by the delta functions

$$\rho(x) = \delta(x - x(t)), \quad \sigma(p) = \delta(p - p(t)), \tag{6}$$

[17] Euler claimed that an idea of simultaneity and succession is needed in advance as a necessary condition, but stressed that the experimentally observed subdivision into equal parts cannot be obtained without the assumption that a body is moving uniformly due to its inertia. "Il ne s'agit pas ici de notre estime de l'égalité des tems, qui dépend sans doute de l'état de notre âme; il s'agit de l'égalité des tems, pendant lesquels un corps qui se meut d'un mouvement uniforme parcourt des espaces égaux" [E149, §21]. Here, an essential extension of the calculus had been introduced by Euler in comparison to Leibniz. In Leibniz's mathematical interpretation, for a given relation, e.g. $y = x^2$, the subdivision into equal infinitesimal parts is indeterminate since either y or x can be arbitrarily chosen as independent variable (compare the analysis in [Bos]). Bos argued that "the aim of the Leibnizian calculus is to determine the behaviour of differentials as related to the nature of the curve" [Bos, p. 99]. In Euler's approach, mathematics and mechanics are interrelated and the subdivision of a finite distance or finite time interval into parts Δx and Δt or dx and dt, i.e. into *equal finite* or *equal infinitesimal* parts, respectively, results from the conservation of the state of a uniformly moving body described by $v = \text{const}$; [E149] and [E842, §§21-24]. The result is $v_{\text{finite}} = \frac{\Delta x}{\Delta t}$ and $v_{\text{inf}} = \frac{dx}{dt}$, where the differently defined velocities are equal in their finite numerical values, i.e. $v_{\text{finite}} = v_{\text{inf}} = v = \text{const}$. The invariance of velocity does not depend on the choice of time interval, on the contrary, the invariance and equality of finite and infinitesimal space and time intervals results from the invariance of velocity. The mathematical rules for the division are given in the treatise [E212, ch. I-IV]. For further consideration, either the path is considered as a function of time or the time is considered as a function of spatial translation. Then, either the time interval is $dt = \text{const}$ or the space interval is $dx = \text{const}$ [E289, §§42-45].

[18] Euler introduces rest and motion using the notion of place. "1. Quemadmodum *Quies* est perpetua in eodem loco permanentia, ita *Motus* est continua loci mutatio" [E289, §1]

for the path, i.e. the trajectory in configuration space, and the trajectory in momentum space, respectively,[19] which are only different from zero if the arguments vanish, i.e. for $x = x(t)$ and $p = p(t)$. This procedure relates the parameterization by time to a certain position, i.e. a configuration, the body is occupying. The position is represented by a geometrical point which is in agreement with Euler's assumption on the theory of infinitely small bodies (compare Section 2.1). Then, any body traveling along this path must be considered to be infinitesimal if it touches only those configurations (positions) which are defined by a geometric line. These relations will be discussed more in detail for the commonly used model system, the linear harmonic oscillator.

4.1. *The harmonic oscillator as model system*

The functions $x(t)$ and $p(t)$ are related to each other whose values form an ordered Cartesian set if the total energy of the system is conserved. If these functions are given, the total energy E_{conf} is defined as the sum of $T(p)$ and $V(x)$, the kinetic and potential energies, respectively. The index $conf$ denotes that the expression is defined for a certain subset of configurations taken from the set of all possible configurations in configuration space and, additionally, a certain set of configurations in the momentum space.

$$E_{\text{conf}} = T_{\text{momentspace}}(p) + V_{\text{confspace}}(x) \tag{7}$$

For the harmonic oscillator the terms are specified as

$$E_{\text{conf}} = T_{\text{momentspace}}(p) + V_{\text{confspace}}(x) = \frac{1}{2m}p^2 + \frac{k}{2}x^2 \tag{8}$$

where m and k are the mass and the force constant, respectively. For finite values of total energy $E_{\text{conf}} < \infty$ the motion is confined to a certain hyperplane of the phase space. Thus, in CM the total energy is not defined for all configurations $\{x\}$ and $\{p\}$, but only for two subsets taken from configuration and momentum spaces.

The relation between the energy and the configuration space is the basic item in Schrödinger's approach.[20] For the derivation of wave equation,

[19] The meaning of the function is that the system has to be in "any of its positions." This statement includes the special case where the system occupies all positions. The latter assumption is in contradiction to time ordering. Therefore, the contradiction is removed by eliminating time.

[20] Schrödinger introduced this assumption in the Second Announcement (compare [Schrödinger 1926b] and Section 3.2) and stressed this point in the Third Announcement in analyzing the relation of his theory to Heisenberg's approach [Schrödinger 1926c]. "Diese Zuordnung [assignment] von Matrizen zu Funktionen ist *allgemein*, sie nimmt

Schrödinger defined the wave function $\psi(x)$ which is only coordinate dependent, i.e. defined only in the configuration space [Schrödinger 1826b, 1926c]. Both the functions, $V(x)$ and $\psi(x)$, are defined in the *same* space.[21] Then, Eq. 7 has to be modified. Preserving the general form that the total energy is composed of different parts, we obtain

$$E_{\text{config,whole}} \sim T(p) + V_{\text{config,whole}}(x) \tag{9}$$

where the index *config,whole* denotes that all quantities are only related to the configuration space, but now, in contrast to CM to the whole configuration space. The function $V(x)$ has been already defined in CM for the whole configuration space and, it is, therefore, appropriate to operate as a link between CM and QM. The striking difference to CM concerns the kinetic energy term $T(p)$, for it has also to be defined in that space. However, the relation of $T(p)$ to configuration space is indeterminate until now whereas the relation between $E - V(x)$ and $\psi(x)$ has been previously introduced by Eq. 5 in accordance with Schrödinger's assumption. Then, the conclusion is that the introduction of a similar relation between $E - T(p)$ and a corresponding function $\phi(p)$ does violate none of Schrödinger's principle for the introduction of the function $\psi(x)$ except the exclusion of configuration space and its replacement with momentum space. However, the other main part of Schrödinger's statement, the exclusion of phase space, is preserved as well.

4.2. *The generalization for conservative systems*

The results of the previous Section remain to be valid if we introduce a purely coordinate dependent function $V(x)$ instead of the special expression for the potential energy of the harmonic oscillator. The expression for the kinetic energy $T(p)$ remains to be the same as before since it is independent of the special system. Using the functions given by Eq. 6 the energy is related to the paths

$$E_{\text{path}} = \frac{\int_{-\infty}^{+\infty} dx\delta(x - x(t))V(x)}{\int_{-\infty}^{+\infty} dx\delta(x - x(t))} + \frac{\int_{-\infty}^{+\infty} dp\delta(p - p(t))T(p)}{\int_{-\infty}^{+\infty} dp\delta(p - p(t))} \tag{10}$$

noch gar nicht bezug auf das *spezielle* mechanische System, das gerade vorliegt, sondern ist für alle mechanischen Systeme die nämliche. ... Sie [the assignment] erfolgt nämlich *durch Vermittlung* eines *beliebigen* vollständigen Funktionensystems mit dem Grundgebiet: *ganzer Konfigurationenraum.*" and Schrödinger continued stressing the exclusion of the phase space: "(NB. nicht 'pq-Raum', sondern 'q-Raum'.)".

[21] The only, but essential, difference is that $\psi_E(x)$ is related to energy whereas $V(x)$ is independently of energy (see next Section).

and a general relation between paths and energy is obtained

$$E_{\text{path}} = T(p(t)) + V(x(t)) = const. \tag{11}$$

which is valid for any given path, if the condition of $E = const$ is fulfilled. For finite energies, the length of the path should be also finite due to Eq. 10.

This constraint on the system, referred to total energy, is equivalent to the limitation of extension of the system in space. Again, the extension of the system can be changed by the change of energy [Helmholtz].

$$\begin{aligned} E_{\text{class}} = & \frac{\int_{-\infty}^{+\infty} dx \Theta(x_{\max}) \delta(x - x(t)) V(x)}{\int_{-\infty}^{+\infty} dx \Theta(x_{\max}) \delta(x - x(t))} \\ & + \frac{\int_{-\infty}^{+\infty} dp \Theta(p_{\max}) \delta(p - p(t)) T(p)}{\int_{-\infty}^{+\infty} dp \Theta(p_{\max}) \delta(p - p(t))} \end{aligned} \tag{12}$$

The energy dependence is included by the step functions which are limiting the extension[22] of the system by the relations $\Theta(x_{\max}) = \Theta(x + x_{\max}) - \Theta(x - x_{\max})$ and $\Theta(p_{\max}) = \Theta(p + p_{\max}) - \Theta(p - p_{\max})$ since $E = T(p_{\max}) = V(x_{\max})$. Replacing the constant Θ function with coordinate and momentum dependent functions, $F(x)$ and $G(p)$, respectively, the physical meaning of the relation 12 is not changed.[23]

The functions can be considered as additional parameters which modify the paths.

$$\begin{aligned} E = & \frac{\int_{-\infty}^{+\infty} dx F_E(x) \delta(x - x(t)) V(x)}{\int_{-\infty}^{+\infty} dx F_E(x) \delta(x - x(t))} \\ & + \frac{\int_{-\infty}^{+\infty} dp G_E(p) \delta(p - p(t)) T(p)}{\int_{-\infty}^{+\infty} dp G_E(p) \delta(p - p(t))} \end{aligned} \tag{13}$$

In the next step, the time-dependence will be eliminated by the time integration, $\int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dx F_E(x) \delta(x - x(t)) V(x) = \int_{-\infty}^{+\infty} dx F_E(x) V(x)$. *Paths can be eliminated, configurations cannot.* The result is[24]

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[22] This limitation is also a selection of a certain subset of configurations taken from the set of all configurations of the system.

[23] The description of a special system is only obtained, if these functions are specified. This will be done in the next Section.

[24] This step corresponds to Heisenberg's procedure to reject paths [Heisenberg]. Heisenberg claimed that also the positions are to be removed, e.g. "the position of the electron." However, the rejection of the paths is equivalent to the rejection of a certain set of positions, but it is not equivalent to the rejection of *all* position or configurations of the system.

$$E = \frac{\int_{-\infty}^{+\infty} dx F_E(x)V(x)}{\int_{-\infty}^{+\infty} dx F_E(x)} + \frac{\int_{-\infty}^{+\infty} dp G_E(p)T(p)}{\int_{-\infty}^{+\infty} dp G_E(p)}, \tag{14}$$

where *all* possible configurations of the system are to be considered. Now, the functions $F_E(x)$ and $G_E(p)$ play the role of additional parameters which modify the contribution from different configurations to total energy.

It is possible to eliminate paths, but it is impossible to eliminate all configurations. The elimination of all configurations results in a non-extended system where the different constituents, if they exist, are not separated spatially from each other. None of the constituents is occupying a place. [25] Therefore, any physical system is described by its configurations.

The derivatives of the functions $F_E(x)$ and $G_E(p)$ should exist, otherwise, Euler's method of maxima and minima cannot be applied.

5. Euler's method of maxima and minima, generalized

It is taken for granted [Schrödinger 1926b] that a physical quantity should be finite. This almost evident statement can be formulated using Euler's method. Generally, assuming a fixed range of its definition, any function is limited by its extreme values. However, the definition of the extreme values is purely mathematically. Therefore, the kind and the combination of the minima and maxima might be also given in a general mathematical form. However, physical quantities can be distinguished by the type of extreme values. Space and time are not appropriate for such procedure, whereas functions of coordinates and time are. Thus, it is necessary to define appropriated quantities. Maupertuis introduced the principle of least action defining action by mass, velocity and distance. Euler extended this principle to the forces and claimed, that the interaction of bodies depends on the minimal forces which the bodies are creating to avoid penetration [E842, §§35-39, 75] and [E289, §§131-135].

Using only extreme values for the classification, the following relations between maxima and minima may be relevant for physics [E65].

– (a) either maxima or minima, [26]
– (b) maxima and minima,
– (c) maxima or minima,

[25] A thing which is neither resting nor moving is not a body [E842].
[26] This relation is preferred in classical mechanics, e. g. Maupertuis's principle of least action.

– (d) neither maxima nor minima.[27]

Examining a simple mechanical model system, the linear harmonic oscillator, it is easily confirmed that the potential energy $V(x) = \frac{k}{2}x^2$ has only a minimum and fulfils criterion (a). The kinetic energy $T(p) = \frac{1}{2m}p^2$ is expressed in terms of the same type of function.

5.1. *Planck's introduction of action parameter*

The relation between CM and QM was considered for the first time by Planck in 1900 when introducing the quantum of action into the theory of heat radiation [Planck]. In 1907 Einstein based the theory of specific heat on Planck's assumption [Einstein 1907] and in 1913 Bohr introduced Planck's parameter into the theory of atomic spectra [Bohr]. Furthermore, Einstein introduced new objects into physics assuming the existence of spatially strongly localized light quanta [Einstein 1905]. Obviously, this assumption is in full logical opposition to the model of an extended wave which is propagating in the whole space after the emission.

Planck introduced the action parameter assuming an absolute value for the entropy of a system. Now, we have to seek a procedure to obtain a parameter of such type from Schrödinger's assumptions.

5.2. *Schrödinger's analysis*

Euler's method may be related to Schrödinger's analysis of the relations between CM and QM and Bohr's complementarity which have been created in the same period, considering the logical structure of statements. Bohr stressed the coexistence and indispensability of both theories, whereas Schrödinger demonstrated the apparent incompatibility of both the theories.

In 1933, Schrödinger analyzed the paths for classical and quantum particles and formulated the result in terms of a logical statement "We are faced here with the full force of the logical opposition between an either/or (point mechanics) and a both/and (wave mechanics)," and concluded: "This would not matter much, if the old system were to be dropped entirely and to be replaced by the new. Unfortunately, this is not the case." [Schrödinger 1933]

[27] These properties exhibit those quantities which are called universal constants or universal parameters having an invariant numerical value, which is only obtained experimentally.

Obviously, Schrödinger intended to connect CM and QM using Maupertuis's principle of least action [Maupertuis]. Applied to CM, it allows to chose one path and reject all the other paths. However, this principle does not fit for a "both ... and" problem where all paths have to be considered. Therefore, a selection cannot be defined with respect to paths, but it has to be defined with respect to another property of the system which must also be introduced. [28]

However, the advent of such type of statements is earlier due to Einstein who fathered the hypothesis of light quanta (compare Section 5.1), which is, obviously, a candidate for demonstrating a full logical opposition between strongly localized light quanta and unlimited propagating waves filling the whole space.

6. Derivation of the Schrödinger equation

Examining the functions $F_E(x) \geq 0$ and $G_E(p) \geq 0$, and, additionally the functions $F_E(x)V(x) \geq 0$ and $G_E(p)T(p) \geq 0$, it is justified to assume because of $V(x) \geq 0$ (valid for a large class of systems) and $T(p) \geq 0$ (generally valid) that all minima of these functions are neither below the minima nor above the minima of the functions $V(x)$ and $T(p)$, respectively. The minima of $V(x)$ and $T(p)$ are not altered by the multiplication with the functions $F_E(x)$ and $G_E(p)$, respectively. Therefore, any additional minima are due to the zeros of the functions $F_E(x)$ and $G_E(p)$. If there are zeros, the functions $F_E(x) = |f_E(x)|^2$ and $G_E(p) = |g_E(p)|^2$ are made up of functions $f_E(x)$ and $g_E(p)$, respectively, which are either of even or odd type. Then, the above listed case (b) has to be to considered, where the maxima *and* the minima determine the properties of the coordinate and momentum dependent functions.

6.1. *The amplitude equation*

From Section 4 we take the expression for the energy which is transformed using Eq. 14 into a relation depending on the functions $f_E(x)$ and $g_E(p)$ [Suisky].

$$E = \frac{\int_{-\infty}^{+\infty} dx |f_E(x)|^2 V(x)}{\int_{-\infty}^{+\infty} dx |f_E(x)|^2} + \frac{\int_{-\infty}^{+\infty} dp |g_E(p)|^2 T(p)}{\int_{-\infty}^{+\infty} dp |g_E(p)|^2} \tag{15}$$

[28] Feynman introduced a weighting of the paths [Feynman].

For the calculation of $f(x)$ and $g(p)$ using Eq. 15, one of the functions must be replaced with the other. The substitution must be symmetric. This is ensured by using a general integral transformation for which the inverse exists. The transformation should also be independent of energy. To date, no physical interpretation of that functions has been given. Therefore, the definition can be given in accordance to mathematical rules only:

$$f_E(x) = \frac{1}{\sqrt{2\pi\alpha}} \int_{-\infty}^{+\infty} dp e^{-i\frac{xp}{\alpha}} g_E(p); \tag{16}$$
$$g_E(p) = \frac{1}{\sqrt{2\pi\alpha}} \int_{-\infty}^{+\infty} dx e^{i\frac{xp}{\alpha}} f_E(x)$$

The relation 16 is valid, if *all* configurations of the system are taken into account and the energy conservation law is not violated. Additionally, a new parameter, α, of the dimension of action had to be introduced for dimensional reasons.[29] By definition, the parameter is independent of x and p and it is assumed to be independent of energy. The argument of the function $e^{-i\frac{xp}{\alpha}}$ is a pure number. Using Eq. 16 we can eliminate either $g_E(p)$ or $f_E(x)$ and obtain two equivalent equations being either purely coordinate dependent or purely momentum dependent. The coordinate dependent equation reads

$$\int_{-\infty}^{+\infty} f_E(x) \left[V(x) f_E(x) - \frac{\alpha^2}{2m} \frac{\partial^2}{\partial x^2} f_E(x) - E f_E(x) \right] dx = 0. \tag{17}$$

The expression in the brackets can be chosen differently, either being different from zero or equal to zero. In the latter case, a relation between the energy E and the function $f_E(x)$ is established which is valid for *each* of the configurations of the system:

$$V(x) f_E(x) - \frac{\alpha^2}{2m} \frac{\partial^2}{\partial x^2} f_E(x) - E f_E(x) = 0. \tag{18}$$

This relation replaces the Newton-Euler equation of motion which is also valid for each of the configurations of the system. However, the difference is that the energy is now included as a indispensable parameter of the theory independently of the considered configuration.

Identifying α with $\hbar$ and assuming the model of the harmonic oscillator we obtain the the *stationary Schrödinger equation*, where $\omega = \sqrt{\frac{k}{m}}$ is

[29] This parameter plays the same role in Heisenberg's time dependent approach. It can be equate with Planck's action parameter. In the present approach, however, this parameters has not been introduced in advance, but it follows now from the procedure.

defined as in CM and identical with the frequency of Planck's oscillator. Schrödinger called this equation "oscillation (*Schwingungs-*) or amplitude equation" to distinguish it from the time dependent or "real (eigentliche) wave equation." [Schrödinger 1926d]

The procedure is completed by eliminating the coordinate dependent function in Eq. 15 instead of momentum dependent functions. Then, we obtain a differential equation in momentum space for the same energies without violating principles postulated by Schrödinger. The only modification is that Schrödinger's statement about exclusion of the phase space: "(NB. nicht 'pq-Raum', sondern 'q-Raum')" [Schrödinger 1926c] has to be replaced with the statement "(NB. nicht 'pq-Raum', sondern 'p-Raum')" which was already implicitly embodied in Schrödinger's assumption. Then, quantization can be defined as a selection problem [Suisky] (compare Section 6.2). The difference between the representations does only emerge for the time dependent wave equation (compare below Section 6.3) since the configuration space function $V(x)$ is replaced with the time dependent function $V(x,t)$ whereas $T(p)$ is not altered by the introduction of time dependence [Schrödinger 1926d].

6.2. *Quantization as selection problem*

The selection problem is properly defined by the operation to chose a number or a set of numbers from the set of real numbers. [30]. However, integers as mathematical numbers are only obtained after the introduction of dimensionless quantities for all quantities appearing in Eq. 18.

Assuming a model system, the linear harmonic oscillator, the dimensionless variables are obtained as $\xi = \sqrt[4]{\frac{4km}{\hbar^2}}x$ and $\nu \equiv \frac{E}{\hbar\omega} - \frac{1}{2}$ and Eq. 18 reads as follows

$$\frac{d^2 D_\nu(\xi)}{d\xi^2} + \left(\nu + \frac{1}{2} - \frac{1}{4}\xi^2\right) D_\nu(\xi) = 0. \tag{19}$$

It is known in the mathematical literature as Weber's equation for the parabolic cylinder. The solutions were studied by Whittaker [Whittaker] in 1903. The variable ν is defined in the whole intervall $-\infty < \nu < +\infty$. However, any selection of special parameter values can be performed only by a procedure which introduces relations between different values of the

[30] In terms of energy, this type of problems has been introduced by Einstein in 1907 who claimed, that the number of energetic states of a molecular body is less than the number of states of bodies of our sensual experience [Einstein 1907] From Schrödinger's approach it follows, that the problem has to be formulated in terms of real numbers. The integers should be obtained quite naturally as a special subset without imposing a "condition for quantization in terms of integers" [Schrödinger 1926a].

parameter. Applying Whittaker's method, the general solution can be represented as a coupled set of first order differential equations, usually known as recurrence relations [31] [Whittaker]

$$\frac{dD_\nu}{d\xi} + \frac{\xi}{2}D_\nu(\xi) + \nu D_{\nu-1}(\xi) = 0 \quad \text{and} \tag{20}$$
$$\frac{dD_\nu}{d\xi} - \frac{\xi}{2}D_\nu(\xi) + (\nu+1)D_{\nu+1}(\xi) = 0,$$

which substitutes for Eq. 18. Using Eq. 20, the *whole set* of different parameter values can be obtained by the choice of any arbitrary parameter value taken from the basic interval $-1 \leq \nu \leq 0$. Then, three types of solutions are obtained and the following problems are to be defined.

I The subdivision of the interval into $-1 < \nu < 0$, $\nu = -1$ and $\nu = 0$ comprising all internal points and the two border points, respectively. [32]

IIa For the border points $\nu = 0$ and $\nu = -1$, the *two* corresponding functions are obtained from Eq. 20.

$$\frac{dD_0(\xi)}{d\xi} + \frac{\xi}{2}D_0(\xi) = 0 \quad \text{and} \quad \frac{dD_{-1}(\xi)}{d\xi} - \frac{\xi}{2}D_{-1}(\xi) = 0 \tag{21}$$

IIb Consider the *finite* set of parameters values belonging to border points $\nu = -1$ and $\nu = 0$ according to Eq. 20. The subsets can be properly distinguished mathematically by the minima and maxima of the solutions of equations (IIa). Then, a physical selection criterion is applied and the divergent solution is excluded.

III Consider the *whole infinite* set of parameter values belonging to border points $\nu = -1$ and $\nu = 0$ according to Eq. 20. In terms of energy, the first set comprises only positive values $E_0 = \frac{1}{2}\alpha\omega$, $E_1 = \frac{3}{2}\alpha\omega$, $\cdots$ whereas the second comprises only negative values, $E_{-1} = -\frac{1}{2}\hbar\omega$, $E_{-2} = -\frac{3}{2}\hbar\omega$, $\cdots$.

Selection criterion. Perpetuum mobile excluded – There are two possible interactions of system and environment, firstly, an unlimited supply of energy from the environment to the system and, secondly, an unlimited supply of energy from the system to the environment. The second case is has to be excluded by stating the *impossibility of perpetual motion.* In the first case, the supply of energy from the system to the environment is automatically limited due to the existence of the state E_0 having the lowest energy.

[31] The states are represented by $\nu = const.$ The difference between two neighbored states is given by the relation $\Delta\nu = \pm 1$. This approach corresponds to Heisenberg's consideration of the energy difference $E_n - E_m$ between states n and m.

[32] Using Eq. 20, an infinite countable set of functions is obtained for each value of ν taken from the basic interval.

Confirmation by Euler's method of maxima and minima - Euler's method allows for the distinction between the solutions $D_0 = \exp(-\frac{\xi^2}{4})$ and $D_{-1} = \exp(+\frac{\xi^2}{4})$ only by their extreme values. Obviously, D_0 has only one maximum and D_{-1} has only one minimum. Therefore, only the function D_0, belonging to the so called ground state, fulfils Schrödinger's criterion for a properly defined coordinate dependent function . The solution D_{-1} has to be discarded.

Now, the whole procedure is finished. A countable infinite set of states, represented by the set of integers, have been defined for the system and chosen from the set of real numbers. This set cannot be subdivided by physical or mathematical reasons into subsets. Therefore, neither additional mathematical nor additional physical problems are to be solved. Knowing the wave function D_0 and the energy E_0 of the ground state, the wave functions and the corresponding energies of all other states are obtained from Eq. 20. The whole procedure is necessary and sufficient for selecting the only countably infinite set of energy values having a smallest element simultaneously with the set of corresponding wave functions.

6.3. *The time-dependent Schrödinger equation*

The time-dependent Schrödinger equation is obtained by introducing a time-dependent function $V(x,t)$ in place of $V(x)$ in Eq. 15, as well as corresponding time-dependent wave functions $f(x,t), g(p,t)$ and their time derivatives $\frac{\partial f(x,t)}{\partial t}$ and $\frac{\partial g(p,t)}{\partial t}$ [Schrödinger 1926d].

In the first place, we have to remove the energy parameter from Eq. 15 and secondly, we have to ensure that the stationary case is obtained from the relation for the non-stationary case. Both conditions can be satisfied if we rewrite Eq. 15 by replacing the quantities in question by those suitable for the non-stationary time-dependence by introducing of $V(x,t)$ and the time dependent wave function $f(x,t)$. The latter does not depend on the energy parameter. Therefore, replacing $f(x)$ with $f(x,t)$ we have to consider additionally the first order time derivative $\frac{\partial f(x,t)}{\partial t}$ and, if it is necessary, $\frac{\partial^2 f(x,t)}{\partial t^2}$ and higher order derivatives. [Schrödinger 1926d] Additionally, $g_E(p)$ is replaced with $g(p,t)$ and $\frac{\partial g(p,t)}{\partial t}$. Furthermore, the energy parameter E has to be replaced with a real valued parameter β which is not related to the energy of the system. However, the structure of the equation should be preserved, since we have to make sure that the stationary case can be recovered.

$$\beta = \frac{\int_{-\infty}^{+\infty} dx f(x,t) f^*(x,t) V(x,t)}{\int_{-\infty}^{+\infty} dx f(x,t) \frac{\partial f^*(x,t)}{\partial t}} + \frac{\int_{-\infty}^{+\infty} dp g(p,t) g^*(p,t) T(p)}{\int_{-\infty}^{+\infty} dp g(p,t) \frac{\partial g^*(x,t)}{\partial t}} \tag{22}$$

Then, we have to introduce complex valued functions of time to ensure that the denominator is always different from zero and, additionally, all the integrals remain to be real-valued expressions. In the denominator, we introduced symmetrized expressions being either the imaginary or the real part of a combination of the wave function and their time-derivatives.

$$\frac{\int_{-\infty}^{+\infty} dx f(x,t) f^*(x,t) V(x,t) + \int_{-\infty}^{+\infty} dx f^*(x,t) f(x,t) V(x,t)}{i\left(\int_{-\infty}^{+\infty} dx f(x,t) \frac{\partial f^*(x,t)}{\partial t} - \int_{-\infty}^{+\infty} dx f^*(x,t) \frac{\partial f(x,t)}{\partial t}\right)} \tag{23}$$

From Eqs. 22 and 23 the non-stationary Schrödinger equations for the functions $f(x,t)$ and $f^*(x,t)$ are obtained. It is readily confirmed, that the stationary functions are given as $h(t) = \exp(-i\frac{E}{\beta}t)$ and $h^*(t) = \exp(i\frac{E}{\beta}t)$, then, Eq. 15 is recovered. The parameter β is real valued and independent of the system and of the special time-dependence included in $V(x,t)$. Neither $V(x,t)$ nor the time derivative $\frac{\partial f(x,t)}{\partial t}$ is related to any special property of the underlying system. These properties are only essential in the stationary case, where the relations given above are valid. Therefore, Eq. 22 describes the general relation between spatial and temporal changes of the system provided that β is an universal system independent parameter.

7. Summary

It has been demonstrated that the derivation of the basic quantum mechanical equation is obtained using Euler's consistently formulated mechanics together with perfectly adapted mathematical methods. Euler subdivided mechanics into a theory of bodies of infinitesimal magnitude and a theory of bodies of finite magnitude.

Euler's procedure is found to be revived and generalized in Schrödinger's wave mechanics. By introducing the wave function, which is related to systems extended in the whole configuration space, Schrödinger made a pioneering step as important as Euler did 200 years ago, who assumed mechanical quantities to be related to bodies of infinitesimal magnitude.

Schrödinger's theory has been reconsidered and reconstructed in terms of Euler's methodology which is distinguished by a joint application of mathematical and physical principles. The physical part has to be always in agreement with experimental data, whereas the mathematical part may temporarily be in contradiction to experiment, since the existence of mathematical objects depends on further conditions than just fulfilling physical criteria. However, the common criteria of order and completeness should be satisfied for both parts of the theory.

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Index