CHAPTER 4
MECHANICAL VIBRATIONS

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4.1 INTRODUCTION

The field of dynamics deals essentially with the interrelation between the motions of objects and the forces causing them. The words "shock" and "vibration" imply particular forces and motions: hence, this chapter concerns itself essentially with a subfield of dynamics. However, oscillatory phenomena occur also in nonmechanical systems, e.g., electric circuits, and many of the methods and some of the nomenclature used for mechanical systems are derived from nonmechanical systems.

Mechanical vibrations may be caused by forces whose magnitudes and/or directions and/or points of application vary with time. Typical forces may be due to rotating unbalanced masses, to impacts, to sinusoidal pressures (as in a sound field), or to random pressures (as in a turbulent boundary layer). In some cases the resulting vibrations may be of no consequence; in others they may be disastrous. Vibrations may be undesirable because they can result in deflections of sufficient magnitude to lead to malfunction, in high stresses which may lead to decreased life by increasing material fatigue, in unwanted noise, or in human discomfort.

Section 4.2 serves to delineate the concepts, phenomena, and analytical methods associated with the motions of systems having a single degree of freedom and to introduce the nomenclature and ideas discussed in the subsequent sections. Section 4.3 deals similarly with systems having a finite number of degrees of freedom, and Sec. 4.4 with continuous systems (having an infinite number of degrees of freedom). Mechanical shocks are discussed in Sec. 4.5, and in Sec. 4.6 appears additional

information concerning design considerations, vibration-control techniques, and rotating machinery, as well as charts and tables of natural frequencies, spring constants, and material properties. The appended references substantiate and amplify the presented material.

Complete coverage of mechanical vibrations and associated fields is clearly impossible within the allotted space. However, it was attempted here to present enough information so that an engineer who is not a specialist in this field can solve the most prevalent problems with a minimum amount of reference to other publications.

4.2 SYSTEMS WITH A SINGLE DEGREE OF FREEDOM

A system with a single degree of freedom is one whose configuration at any instant can be described by a single number. A mass constrained to move without rotation along a given path is an example of such a system; its position is completely specified when one specifies its distance from a reference point, as measured along the path. Single-degree-of-freedom systems can be analyzed more readily than more complicated ones; therefore, actual systems are often approximated by systems with a single degree of freedom, and many concepts are derived from such simple systems and then enlarged to apply also to systems with many degrees of freedom.

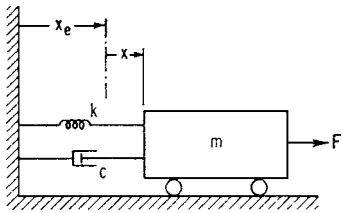


FIG. 4.1 A system with a single degree of freedom.

Figure 4.1 may serve as a model for all single-degree-of-freedom systems. This model consists of a pure inertia component (mass m supported on rollers which are devoid of friction and inertia), a pure restoring component (massless spring k), a pure energy-dissipation component (massless dashpot c), and a driving component (external force F). The inertia component limits acceleration. The restoring component opposes system deformation from equilibrium and tends to return the system to its equilibrium configuration in absence of other forces.

4.2.1 Linear Single-Degree-of-Freedom Systems[8,17,32,40,61,63]

If the spring supplies a restoring force proportional to its elongation and the dashpot provides a force which opposes motion of the mass proportionally to its velocity, then the system response is proportional to the excitation, and the system is said to be linear. If the position x_e indicated in Fig. 4.1 corresponds to the equilibrium position of the mass and if x denotes displacement from equilibrium, then the spring force may be written as $-kx$ and the dashpot force as $-c\ dx/dt$ (where the displacement x and all forces are taken as positive in the same coordinate direction). The equation of motion of the system then is

$$m\ d^2x/dt^2 + c\ dx/dt + kx = F \tag{4.1}$$

Free Vibrations. In absence of a driving force F and of damping c, i.e., with $F = c = 0$, Eq. (4.1) has a general solution which may be expressed in any of the following ways:

$$x = A\cos(\omega_n t - \phi) = (A\cos\phi)\cos\omega_n t + (A\sin\phi)\sin\omega_n t$$
$$= A\sin(\omega_n t - \phi + \pi/2) = A\,\mathrm{Re}\{e^{i(\omega_n t - \phi)}\} \tag{4.2}$$

A and ϕ are constants which may in general be evaluated from initial conditions. A is the maximum displacement of the mass from its equilibrium position and is called the "displacement amplitude"; ϕ is called the "phase angle." The quantities ω_n and f_n, given by

$$\omega_n = \sqrt{k/m} \qquad f_n = \omega_n/2\pi$$

are known as the "undamped natural frequencies"; the first is in terms of "circular frequency" and is expressed in radians per unit time, the second is in terms of cyclic frequency and is expressed in cycles per unit time.

If damping is present, $c \neq 0$, one may recognize three separate cases depending on the value of the damping factor $\zeta = c/c_c$, where c_c denotes the critical damping coefficient (the smallest value of c for which the motion of the system will not be oscillatory). The critical damping coefficient and the damping factor are given by

$$c_c = 2\sqrt{km} = 2m\omega_n \qquad \zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{c}{2m\omega_n}$$

The following general solutions of Eq. (4.1) apply when $F = 0$:

$$c > c_c\,(\zeta > 1): \qquad x = Be^{(-\zeta + \sqrt{\zeta 2-1})\omega_n t} + Ce^{(-\zeta - \sqrt{\zeta 2-1})\omega_n t} \tag{4.3a}$$

$$c = c_c\,(\zeta = 1): \qquad x = (B + Ct)e^{-\omega_n t} \tag{4.3b}$$

$$c < c_c\,(\zeta < 1): \qquad x = Be^{-\zeta\omega_n t}\cos(\omega_d t + \phi) = B\,\mathrm{Re}\,\{e^{i(\omega_N t + \phi)}\} \tag{4.3c}$$

where

$$\omega_d \equiv \sqrt{\omega_n\,1-\zeta^2} \qquad i\omega_N = -\zeta\omega_n + i\omega_d$$

denote, respectively, the "undamped natural frequency" and the "complex natural frequency," and the B, C, ϕ are constants that must be evaluated from initial conditions in each case.

Equation (4.3a) represents an extremely highly damped system; it contains two decaying exponential terms. Equation (4.3c) applies to a lightly damped system and is essentially a sinusoid with exponentially decaying amplitude. Equation (4.3b) pertains to a critically damped system and may be considered as the dividing line between highly and lightly damped systems.

Figure 4.2 compares the motions of systems (initially displaced from equilibrium by an amount x_0 and released with zero velocity) having several values of the damping factor ζ.

In all cases where $c > 0$ the displacement x approaches zero with increasing time. The damped natural frequency ω_d is generally only slightly lower than the undamped natural frequency ω_n; for $\zeta \leq 0.5$, $\omega_d \geq 0.87\,\omega n$.

The static deflection x_{st} of the spring k due to the weight mg of the mass m (where g denotes the acceleration of gravity) is related to natural frequency as

$$x_{st} = mg/k = g/\omega_n^2 = g/(2\pi f_n)^2$$

This relation provides a quick means for computing the undamped natural frequency (or for approximating the damped natural frequency) of a system from its static deflection. For x in inches or centimeters and f_n in hertz (cycles per second) it becomes

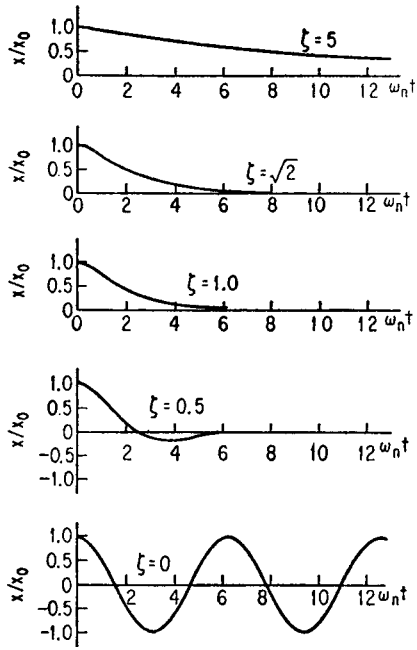


FIG. 4.2 Free motions of linear single-degree-of-freedom systems with various amounts of damping.

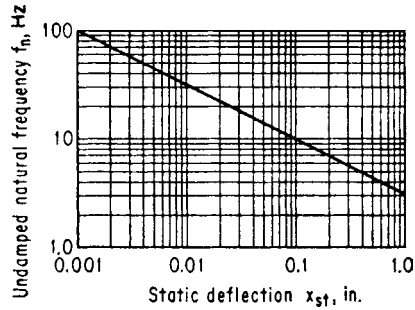


FIG. 4.3 Relation between natural frequency and static deflection of linear undamped single-degree-of-freedom system.

$$f_n^2\,(\text{Hz}) = 9.80/x_{st}\,(\text{in}) = 24.9/x_{st}\,(\text{cm})$$

which is plotted in Fig. 4.3.

Forced Vibrations. The previous section dealt with cases where the forcing function F of Eq. (4.1) was zero. The solutions obtained were the so-called "general solution of the homogeneous equation" corresponding to Eq. (4.1). Since these solutions vanish with increasing time (for $c > 0$), they are sometimes also called the "transient solutions." For $F \neq 0$ the solutions of Eq. (4.1) are made up of the aforementioned general solution (which incorporates constants of integration that depend on the initial conditions) plus a "particular integral" of Eq. (4.1). The particular integrals contain no constants of integration and do not depend on initial conditions, but do depend on the excitation. They do not tend to zero with increasing time unless the excitation tends to zero and hence are often called the "steady-state" portion of the solution.

The complete solution of Eq. (4.1) may be expressed as the sum of the general (transient) solution of the homogeneous equation and a particular (steady-state) solution of the nonhomogeneous equation. The associated general solutions have already been discussed; hence the present discussion will be concerned primarily with the steady-state solutions.

The steady-state solutions corresponding to a given excitation $F(t)$ may be obtained from the differential equation (4.1) by use of various standard mathematical techniques[12,20] without a great deal of difficulty. Table 4.1 gives the steady-state responses x_{ss} to some common forcing functions $F(t)$.

TABLE 4.1 Steady-State Responses of Linear Single-Degree-of-Freedom Systems to Several Forcing Functions

$F(t)$	x_{ss}
1	$1/k$
t	$t/k - c/k^2$
t^2	$t^2/k - 2ct/k^2 + 2c^2/k^3 - 2m/k^2$
$e^{\pm\alpha t}$*	$h(\pm\alpha)$
$te^{\pm\alpha t}$*	$h(\pm\alpha)e^{\pm\alpha t} - h^2(\pm\alpha)(c \pm 2m\alpha)te^{\pm\alpha t}$
$e^{i\omega t}$	$(k - m\omega^2 + ic\omega)^{-1}$
$\sin \beta t$†	$g(\beta)[(k - m\beta^2) \sin \beta t - c\beta \cos \beta t]$
$\cos \beta t$†	$g(\beta)[c\beta \sin \beta t + (k - m\beta^2) \cos \beta t]$
$e^{\pm\alpha t} \sin \beta t$‡	$(A \cos \beta t + B \sin \beta t)e^{-\alpha t}/D$
$e^{\pm\alpha t} \cos \beta t$‡	$(A \sin \beta t - B \cos \beta t)e^{-\alpha t}/D$

*$h(\pm\alpha) = (m\alpha^2 \pm c\alpha + k)^{-1}$
†$g(\beta) = [(k - m\beta^2)^2 + (c\beta)^2]^{-1}$
‡$A = k + m(\alpha^2 - \beta^2) \pm \alpha c$
$B = \beta(c \pm 2\alpha m)$
$D = A^2 + B^2$

Superposition. Since the governing differential equation is linear, the response corresponding to a sum of excitations is equal to the sum of the individual responses; or, if

$$F(t) = A_1F_1(t) + A_2F_2(t) + A_3F_3(t) + \cdots$$

where A_1, A_2, … are constants, and if x_{ss1}, x_{ss2}, … are solutions corresponding, respectively, to $F_1(t)$, $F_2(t)$, …, then the steady-state response to $F(t)$ is

$$x_{ss} = A_1x_{ss1} + A_2x_{ss2} + A_3x_{ss3} + \cdots$$

Superposition permits one to determine the response of a linear system to any time-dependent force $F(t)$ if one knows the system's impulse response $h(t)$. This impulse response is the response of the system to a Dirac function $\delta(t)$ of force; also $h(t) = u(t)$, where $u(t)$ is the system response to a unit step function of force [$F(t) = 0$ for $t < 0$, $F(t) = 1$ for $t > 0$]. In the determination of $h(t)$ and $u(t)$ the system is taken as at rest and at equilibrium at $t = 0$.

The motion of the system may be found from

$$x_{ss}(t) = \int_0^t F(\tau)h(t - \tau)\, d\tau \tag{4.4}$$

in conjunction with the proper "transient" solution expression, the constants in which must be adjusted to agree with specified initial conditions. For single-degree-of-freedom systems,

$$h(t) = \frac{\omega_n}{2k\sqrt{\zeta^2-1}} \left[e^{[-\zeta+(\zeta^2-1)^{1/2}]\omega_n t} + e^{[-\zeta-(\zeta^2-1)^{1/2}]\omega_n t}\right] \qquad \text{for } \zeta > 1$$

$$h(t) = \frac{\omega_n^2 t}{k} e_0^{-\omega_n t} \qquad \text{for } \zeta = 1$$

$$h(t) = \frac{\omega_n}{k\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_\alpha t \qquad \text{for } \zeta < 1$$

$$h(t) = \frac{\omega_n}{k} \sin \omega_n t \qquad \text{for } \zeta = 0$$

Sinusoidal (Harmonic) Excitation. With an excitation

$$F(t) = F_0 \sin \omega t$$

one obtains a response which may be expressed as

$$x_{ss} = X_0 \sin (\omega t - \phi)$$

where
$$X_0 = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \qquad \tan \phi = \frac{c\omega}{k - m\omega^2} \tag{4.5}$$

The ratio

$$H_s(\omega) = \frac{X_0}{F_0/k} = \left\{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2\right\}^{-1/2} \tag{4.6}$$

is called the frequency response or the magnification factor. As the latter name implies, this ratio compares the displacement amplitude X_0 with the displacement F_0/k that a force F_0 would produce if it were applied statically. $H_s(\omega)$ is plotted in Fig. 4.4.

Complex notation is convenient for representing general sinusoids.* Corresponding to a sinusoidal force

$$F(t) = F_0 e^{i\omega t}$$

one obtains a displacement

$$x_{ss} = X_0 e^{i\omega t}$$

where
$$\frac{X_0}{F_0/k} = H(\omega) = \left[1 - \left(\frac{\omega}{\omega_n}\right)^2 + 2i\zeta\left(\frac{\omega}{\omega_n}\right)\right]^{-1} \tag{4.7}$$

$H(\omega)$ is called the complex frequency response, or the complex magnification factor† and is related to that of Eq. (4.6) as

$$H_s(\omega) = |H(\omega)|$$

From the model of Fig. 4.1 one may determine that the force F_{TR} exerted on the wall at any instant is given by

$$F_{TR} = kx + c\dot{x}$$

The ratio of the amplitude of this transmitted force to the amplitude of the sinusoidal applied force is called the transmissibility TR_s and obeys

$$\mathrm{TR}_s = \frac{F_{\mathrm{TR}}}{F_0} = \sqrt{\frac{1 + [2\zeta(\omega/\omega_n)]^2}{[1 - (\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}} \tag{4.8}$$

*In complex notation[40] it is usually implied, though it may not be explicitly stated, that only the *real parts* of excitations and responses represent the physical situation. Thus the complex form $Ae^{i\omega t}$ (where the coefficient $A = a + ib$ is also complex in general) implies the oscillation given by

$$\mathrm{Re}\{Ae^{i\omega t}\} = \mathrm{Re}\{(a + ib)(\cos \omega t + i \sin \omega t)\} = a \cos \omega t - b \sin \omega t$$

†An alternate formulation in terms of mechanical impedance is discussed in Sec. 4.3.6.

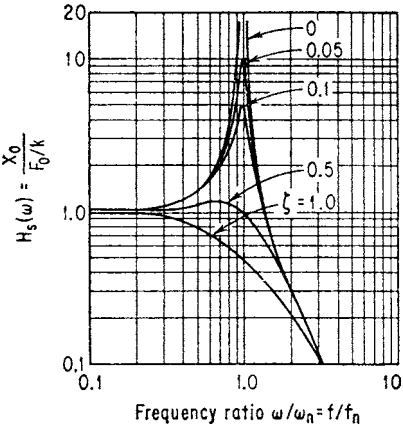


FIG. 4.4 Frequency response (magnification factor) of linear single-degree-of-freedom system.

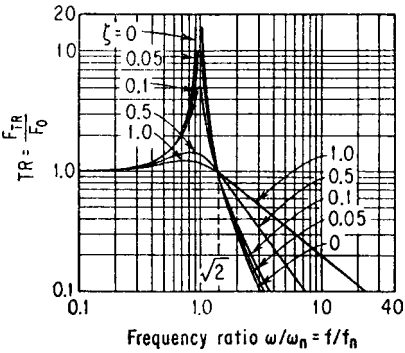


FIG. 4.5 Transmissibility of linear single-degree-of-freedom system.

Transmissibility $TR_s(\omega)$ is plotted in Fig. 4.5.

In complex notation

$$TR = \frac{1 + 2i\zeta(\omega/\omega_n)}{1 - (\omega/\omega_n)^2 + 2i\zeta(\omega/\omega_n)} \qquad TR_s(\omega) = |TR(\omega)|$$

It is evident that

$$|H(\omega)| \approx |TR| \approx \begin{cases} 1 & \text{for } \omega << \omega_n \\ (\omega_n/\omega)^2 & \text{for } \omega >> \omega_n \end{cases}$$

Increased damping ζ always reduces the frequency response H. For $\omega/\omega_n < \sqrt{2}$ increased damping also decreases TR, but for $\omega/\omega_n > \sqrt{2}$ increased damping increases TR.*

The frequencies at which the maximum transmissibility and amplification factor occur for a given damping ratio are shown in Fig. 4.6; the magnitudes of these maxima are shown in Fig. 4.7. For small damping ($\zeta < 0.3$, which applies to many practical

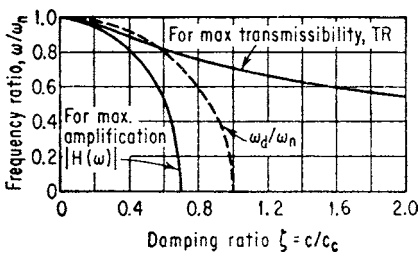


FIG. 4.6 Frequencies at which magnification and transmissibility maxima occur for given damping ratio.

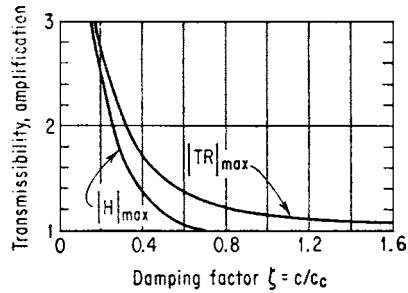


FIG. 4.7 Maximum values of magnification and transmissibility.

*It is important to note that these remarks apply only for the type of damping represented by a viscous dashpot model; different relations generally apply for other damping mechanisms.[49]

problems), the maximum transmissibility $|\mathrm{TR}|_{\max}$ and maximum amplification factor $|H|_{\max}$ both occur at $\omega_d \approx \omega_n$, and

$$|\mathrm{TR}|_{\max} \approx |H|_{\max} \approx (2\zeta)^{-1}$$

The quantity $(2\zeta)^{-1}$ is often given the symbol Q, termed the "quality factor" of the system. The frequency at which the greatest amplification occurs is called the resonance frequency; the system is then said to be in resonance. For lightly damped systems the resonance frequency is practically equal to the natural frequency, and often no distinction is made between the two. Thus, for lightly damped systems, resonance (i.e., maximum amplification) occurs essentially when the exciting frequency ω is equal to the natural frequency ω_n.

Equation (4.6) shows that

$$\begin{aligned} X_0/F_0 &\approx 1/k \text{ (system is stiffness-controlled) for } \omega << \omega_n \\ &\approx 1/2k\zeta \text{ (system is damping-controlled) for } \omega \approx \omega_n (\zeta << 1) \\ &\approx 1/m\omega^2 \text{ (system is mass-controlled) for } \omega >> \omega_n \end{aligned}$$

General Periodic Excitation. Any periodic excitation may be expressed in terms of a Fourier series (i.e., a series of sinusoids) and any aperiodic excitation may be expressed in terms of a Fourier integral, which is an extension of the Fourier-series concept. In view of the superposition principle applicable to linear systems the response can then be obtained in terms of a corresponding series or integral.

A periodic excitation with period T may be expanded in a Fourier series as

$$F(t) = \frac{A_0}{2} + \sum_{r=1}^{\infty} (A_r \cos r\omega_0 t + B_r \sin r\omega_0 t) = \sum_{r=-\infty}^{\infty} C_r e^{ir\omega_0 t} \tag{4.9}$$

where the period T and fundamental frequency ω_0 are related by

$$\omega_0 T = 2\pi$$

The Fourier coefficients A_r, B_r, C_r may be computed from

$$A_r = \frac{2}{T}\int_t^{t+T} F(t)\cos(r\omega_0 t)\,dt \qquad B_r = \frac{2}{T}\int_t^{t+T} F(t)\sin(r\omega_0 t)\,dt$$

$$C_r = \tfrac{1}{2}(A_r - iB_r) = \frac{1}{T}\int_t^{t+T} F(t)e^{-ir\omega_0 t}\,dt \tag{4.10}$$

Superposition permits the steady-state response to the excitation given by Eq. (4.9) to be expressed as

$$x_{ss} = \frac{1}{k}\sum_{r=-\infty}^{\infty} H_r C_r e^{ir\omega_0 t} \tag{4.11}$$

where H_r is obtained by setting $\omega = r\omega_0$ in Eq. (4.7).

If a periodic excitation contains a large number of harmonic components with $C_r \neq 0$, it is likely that one of the frequencies $r\omega_0$ will come very close to the natural frequency ω_n of the system. If $r_0\omega_0 \approx \omega_n$, $C_{r0} \neq 0$, then $H_{r0}C_{r0}$ will be much greater than the other components of the response (particularly in a very lightly damped system), and

$$\begin{aligned} x_{ss}k &\approx H_{r_0}C_{r_0}e^{i\omega_n t} + H_{-r_0}C_{-r_0}e^{-i\omega_n t} + A_0/2 \\ &\approx (1/2\zeta)(A_{r_0}\sin\omega_n t - B_{r_0}\cos\omega_n t) + A_0/2 \end{aligned}$$

General Nonperiodic Excitation.[2,3,11,16] The response of linear systems to any well-behaved* forcing function may be determined from the impulse response as discussed in conjunction with Eq. (4.4) or by application of Fourier integrals. The latter may be visualized as generalizations of Fourier series applicable for functions with infinite period.

A "well-behaved"* forcing function $F(t)$ may be expressed as†

$$F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(\omega)e^{i\omega t}\,d\omega \tag{4.12}$$

where

$$\Phi(\omega) = \int_{-\infty}^{\infty} F(t)e^{-i\omega t}\,dt \tag{4.13}$$

[These are analogous to Eqs. (4.9) and (4.10)]. With the ratio $H(\omega)$ of displacement to force as given by Eq. (4.7), the displacement-response transform then is

$$X(\omega) = (1/k)H(\omega)\Phi(\omega)$$

and, analogously to Eq. (4.11), one finds the displacement given by

$$x_{ss}(t) = (1/2\pi)\int_{-\infty}^{\infty} X(\omega)e^{i\omega t}\,d\omega = (1/2\pi k)\int_{-\infty}^{\infty} H(\omega)\Phi(\omega)e^{i\omega t}\,d\omega$$

$$= (1/2\pi k)\int_{-\infty}^{\infty} H(\omega)\left[\int_{-\infty}^{\infty} F(t)e^{-i\omega t}\,dt\right]e^{i\omega t}\,d\omega$$

One may expect the components of the excitation with frequencies nearest the natural frequency of a system to make the most significant contributions to the response. For lightly damped systems one may assume that these most significant components are contained in a small frequency band containing the natural frequency. Usually one uses a "resonance bandwidth" $\Delta\omega = 2\zeta\omega_n$, thus effectively assuming that the most significant components are those with frequencies between $\omega_n(1 - \zeta)$ and $\omega_n(1 + \zeta)$. (At these two limiting frequencies, commonly called the half-power points, the rate of energy dissipation is one-half of that at resonance. The amplitude of the response at these frequencies is $1/\sqrt{2} \approx 0.707$ times the amplitude at resonance.) Noting that the largest values of the complex amplification factor H(ω) occur for $\omega \approx \pm\,\omega_d \approx \pm\,\omega_n$, one may write

$$2\pi x_{ss}k \approx -\,i\omega_n e^{-\zeta\omega_n t}[\Phi(\omega_n)e^{i\omega_n t} + \Phi(-\omega_n)e^{-i\omega_n t}]$$

Random Vibrations: Mean Values, Spectra, Spectral Densities.[2,3,11,16] In many cases one is interested only in some mean value as a characterization of response. The time average of a variable $y(t)$ may be defined as

$$\bar{y} = \lim_{\tau\to\infty}(1/\tau)\int_0^{\tau} y(t)\,dt \tag{4.14}$$

where it is assumed that the limit exists. For periodic $y(t)$ one may take τ equal to a period and omit the limiting process.

*"Well-behaved" means that $|F(t)|$ is integrable and $F(t)$ has bounded variation.

†Other commonly used forms of the integral transforms can be obtained by substituting $j = -i$. Since $j^2 = i^2 = -1$, all the developments still hold. Fourier transforms are also variously defined as regards the coefficients. For example, instead of $1/2\pi$ in Eq. (4.12), there often appears a $1/\sqrt{2\pi}$; then a $1/\sqrt{2\pi}$ factor is added in Eq. (4.13) also. In all cases the product of the coefficients for a complete cycle of transformations is $1/2\pi$.

The mean-square value of $y(t)$ thus is given by

$$\overline{y^2} = \lim_{t\to\infty} (1/\tau) \int_0^\tau y^2(t)\, dt$$

and the root-mean-square value by $y_{\text{rms}} = (\overline{y^2})^{1/2}$. For a sinusoid $x = \text{Re}\,\{Ae^{i\omega t}\}$ one finds $\overline{x^2} = \frac{1}{2}|A|^2 = \frac{1}{2}AA^*$ where $\underline{A}^*$ is the complex conjugate of A.

The mean-square response x^2 of a single-degree-of-freedom system with frequency response $H(\omega)$ [Eq. (4.7)] to a sinusoidal excitation of the form $F(t) = \text{Re}\,\{F_0 e^{i\omega t}\}$ is given by

$$k^2\overline{x^2} = H(\omega)F_0H^*(\omega)F_0^*/2 = |H(\omega)|^2\overline{F^2}$$

Similarly, the mean-square value of a general periodic function $F(t)$, expressed in Fourier-series form as

$$F(t) = \sum_{r=-\infty}^{\infty} C_r e^{ir\omega_0 t}$$

is

$$\overline{F^2} = \sum_{r=-\infty}^{\infty} \frac{C_r C_r^*}{2} = \tfrac{1}{2} \sum_{r=-\infty}^{\infty} |C_r|^2 \tag{4.15}$$

The mean-square displacement of a single-degree-of-freedom system in response to the aforementioned periodic excitation is given by

$$k^2\overline{x^2} = \tfrac{1}{2} \sum_{r=-\infty}^{\infty} |C_r|^2 |H(r\omega_0)|^2$$

where convergence of all the foregoing infinite series is assumed.

If one were to plot the cumulative value of (the sum representing) the mean-square value of a periodic variable as a function of frequency, starting from zero, one would obtain a diagram somewhat like Fig. 4.8. This graph shows how much each frequency (or "spectral component") adds to the total mean-square value. Such a graph* is called the spectrum (or possibly more properly the integrated spectrum) of $F(t)$. It is generally of relatively little interest for periodic functions, but is extremely useful for aperiodic (including random) functions.

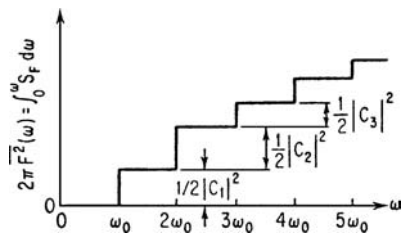


FIG. 4.8 (Integrated) spectrum of periodic function $F(t)$.

The derivative of the (integrated) spectrum with respect to ω is called the "mean-square spectral density" (or power spectral density) of F. Thus the power spectral density S_F of F is defined as*

$$S_F(\omega) = 2\pi d(\overline{F^2})/d\omega \tag{4.16}$$

where $\overline{F^2}$ is interpreted as a function of ω as in Fig. 4.8. The mean-square value of F is related to power spectral density as

$$\overline{F^2} = (1/2\pi) \int_0^\infty S_F(\omega)\, d\omega \tag{4.17}$$

*The factor 2π appearing in Fig. 4.8 and Eqs. (4.16) and (4.17) is a matter of definition. Different constants are sometimes used in the literature, and one must use care in comparing results from different sources.

This integral over all frequencies is analogous to the infinite sum of Eq. (4.15). From Fig. 4.8 and Eq. (4.17) one may visualize that power spectral density is a convenient means for expressing the contributions to the mean-square value in any frequency range.

For nonperiodic functions one obtains contributions to the mean-square value over a continuum of frequencies instead of at discrete frequencies, as in Fig. 4.8. The (integrated) spectrum and the power spectral density then are continuous curves. The relations governing the mean responses to nonperiodic excitation can be obtained by the same limiting processes which permit one to proceed from the Fourier series to Fourier integrals. However, the results are presented here in a slightly more general form so that they can be applied also to systems with random excitation.*

Response to Random Excitation: Autocorrelation Functions. For stationary ergodic random processes† whose sample functions are $F(t)$ or for completely specified functions $F(t)$ one may define an autocorrelation function $R_F(\tau)$ as

$$R_F(\tau) = \lim_{T\to\infty} (1/2T) \int_{-T}^{T} F(t)F(t+\tau)\, dt \tag{4.18}$$

This function has the properties

$$R_F(0) = \overline{F^2} \geq R_F(\tau) \qquad R_F(-\tau) = R_F(\tau)$$

For many physical random processes the values of F observed at widely separated intervals are uncorrelated, that is

$$\lim_{\tau\to\infty} R_F(\tau) = (\overline{F})^2$$

or R_F approaches the square of the mean value (not the mean-square value!) of F for large time separation τ. (Many authors define variables measured from a mean value; if such variables are uncorrelated, their $R_F \to 0$ for large τ.) One may generally find some value of τ beyond which R_F does not differ "significantly" from $(\overline{F})^2$. This value of τ is known as the "scale" of the correlation.

The power spectral density of F is given by‡,§

*In the previous discussion the excitation was described as some known function of time and the responses were computed as other completely defined time functions; in each case the values at each instant were specified or could be found. Often the stimuli cannot be defined so precisely; only some statistical information about them may be available. Then, of course, one may only obtain some similar statistical information about the responses.

†A "random process" is a mathematical model useful for representing randomly varying physical quantities. Such a process is determined not by its values at various instants but by certain average and spectral properties. One sacrifices precision in the description of the variable for the sake of tractability.

One may envision a large number of sample functions (such as force vs. time records obtained on aircraft landing gears, with time datum at the instant of landing). One may compute an average value of these functions at any given time instant; such an average is called a "statistical average" and generally varies with the instant selected. On the other hand, one may also compute the time average of any given sample function over a long interval. The statistical average will be equal to the time average of almost every sample function, provided that the sample process is both stationary and ergodic.[3]

A random process is stationary essentially if the statistical average of the sample functions is independent of time, i.e., if the ensemble appears unchanged if the time origin is changed. Ergodicity essentially requires that almost every sample function be "typical" of the entire group. General mathematical results are to a large degree available only for stationary ergodic random processes; hence the following discussion is limited to such processes.

‡Definitions involving different numerical coefficients are also in general use.

§For real $F(t)$ one may multiply the coefficient shown here by 2 and replace the lower limit of integration by zero.

$$S_F(\omega) = 2\int_{-\infty}^{\infty} R_F(\tau)e^{-i\omega\tau}\,d\tau \tag{4.19}$$

and is equal to twice* the Fourier transform of the autocorrelation function R_F. Inversion of this transform gives*,†

$$R_F(\tau) = \frac{1}{4\pi}\int_{-\infty}^{\infty} S_F(\omega)e^{i\omega\tau}\,d\tau$$

whence*,†

$$R_F(0) = \overline{F^2} = \frac{1}{4\pi}\int_{-\infty}^{\infty} S_F(\omega)\,d\omega \tag{4.20}$$

The last of these relations agrees with Eq. (4.17).

System Response to Random Excitation. For a system with a complex frequency response $H(\omega)$ as given by Eq. (4.7) one finds that the power spectral density S_x of the response is related to the power spectral density S_F of the exciting force according to

$$k^2 S_x(\omega) = |H(\omega)|^2 S_F(\omega) \tag{4.21}$$

In order to compute the mean-square response of a system to aperiodic (or stationary ergodic random) excitation one may proceed as follows:

1. Calculate $R_F(\tau)$ from Eq. (4.18).
2. Find $S_F(\omega)$ from Eq. (4.19).
3. Determine $S_x(\omega)$ from Eq. (4.21).
4. Find $\overline{x^2}$ from Eq. (4.20) (with F subscripts replaced by x).

"White noise" is a term commonly applied to functions whose power spectral density is constant for all frequencies. Although such functions are not realizable physically, it is possible to obtain power spectra that remain virtually constant over a frequency region of interest in a particular problem (particularly in the neighborhood of the resonance of the system considered, where the response contributes most to the total).

The mean-square displacement of a single-degree-of-freedom system to a (real) white-noise excitation F, having the power spectral density $S_F(\omega) = S_0$, is given by

$$\overline{x^2} = \omega_n S_0/8\zeta k^2 = S_0/4ck$$

Probability Distributions of Excitation and Response.[2,3,11,16] For most practical purposes it is sufficient to define the probability of an event as the fraction of the number of "trials" in which the event occurs, provided that a large number of trials are made. (In throwing an unbiased die a large number of times one expects to obtain a given number, say 2, one-sixth of the time. The probability of the number 2 here is $\frac{1}{6}$.)

If a given variable can assume a continuum of values (unlike the die for which the variable, i.e., the number of spots, can assume only a finite number of discrete values) it makes generally little sense to speak of the probability of any given value. Instead, one may profitably apply the concepts of probability distribution and probability density functions. Consider a continuous random variable x and a certain value x_0 of that

*Definitions involving different numerical coefficients are also in general use.

†For real $F(t)$ one may multiply the coefficient shown here by 2 and replace the lower limit of integration by zero.

variable. The probability distribution function P_{dis} then is defined as a function expressing the probability P that the variable $x \leq x_0$. Symbolically,

$$P_{dis}(x_0) = P(x \leq x_0)$$

The probability density function P_{dens} is defined by

$$P_{dens}(x_0) = dP_{dis}(x_0)/dx_0$$

so that the probability of x occurring between x_0 and $x_0 + dx_0$ is

$$P(x_0 < x \leq x_0 + dx_0) = P_{dens}(x_0)dx_0$$

Among the most widely studied distributions are the gaussian (or normal) distribution, for which

$$P_{dens}(x) = [1/\sigma\sqrt{2\pi}]e^{-(x-M)2/2\sigma 2} \tag{4.22}$$

and the Rayleigh distribution, for which

$$P_{dens}(x) = (x/\sigma^2)e^{-x2/2\sigma 2} \tag{4.23}$$

In the foregoing, M denotes the statistical mean value, defined by

$$M = \int_{-\infty}^{\infty} xP_{dens}(x)\,dx$$

and σ^2 denotes the variance of x and is defined by

$$\sigma^2 = \int_{-\infty}^{\infty} (x-M)^2P_{dens}(x)\,dx$$

σ is called the "standard deviation" of the distribution. For a stationary ergodic random process

$$\sigma^2 + M^2 = R_x(0) = \overline{x^2}$$

The gaussian distribution is by far the most important, since it represents many physical conditions relatively well and permits mathematical analysis to be carried out relatively simply. With some qualifications, the "central limit theorem" states that any random process, each of whose sample functions is constructed from the sum of a large number of sample functions selected independently from some other random process, will tend to become gaussian as the number of sample functions added tends to infinity. A stationary gaussian random process with zero mean is completely characterized by either its autocorrelation function or its power spectral density. If the excitation of a linear system is a gaussian random process, then so is the system response.

For a gaussian random process $x(t)$ with autocorrelation function $R_x(\tau)$ and power spectral density $S_x(\omega)$ one may find the *average* number of times N_0 that $x(t)$ passes through zero in unit time from

$$(\pi N_0)^2 = -\left.\frac{d^2R_x}{d\tau^2}\right|_{\tau=0} [R_x(0)]^{-1} = \left[\int_0^{\infty} \omega^2 S_x(\omega)\,d\omega\right]\left[\int_0^{\infty} S_x(\omega)\,d\omega\right]^{-1}$$

The quantity N_0 gives an indication of the "apparent frequency" of $x(t)$. A sinusoid crosses zero twice per cycle and has $f = N_0/2$. This relation may be taken as the definition of apparent frequency for a random process.

The average number of times N_α that the aforementioned gaussian $x(t)$ crosses the value $x = \alpha$ per unit time is given by

$$N_\alpha = N_0 e^{-\alpha 2/2R_x(0)}$$

The average number of times per unit time that $x(t)$ passes through α with positive slope is half the foregoing value. The average number of peaks* of $x(t)$ occurring per unit time between $x = \alpha$ and $x = \alpha + d\alpha$ is

$$N_{\alpha,\alpha+d\alpha} = \frac{N_0 \alpha\, d\alpha}{2R_x(0)}\, e^{-\alpha 2/2R_x(0)}$$

For a linear single-degree-of-freedom system with natural circular frequency ω_n subject to white noise of power spectral density $S_F(\omega) = S_0$, one finds

$$N_0 = \omega_n/\pi \qquad f_{\text{apparent}} = \omega_n/2\pi = f_n$$

The displacement vs. time curve representing the response of a lightly damped system to broadband excitation has the appearance of a sinusoid with the system natural frequency, but with randomly varying amplitude and phase. The average number of peaks per unit time occurring between α and $\alpha + d\alpha$ in such an oscillation is given by

$$(2\pi/\omega_n)N_{\alpha,\alpha+d\alpha} = [\alpha\, d\alpha/R_x(0)]e^{-\alpha 2/2R_x(0)}$$

The term on the right-hand side is, except for the $d\alpha$, the Rayleigh probability density of Eq. (4.23).

4.2.2 Nonlinear Single-Degree-of-Freedom Systems[21,34,60]

The previous discussion dealt with systems whose equations of motion can be expressed as linear differential equations (with constant coefficients), for which solutions can always be found. The present section deals with systems having equations of motion for which solutions cannot be found so readily. Approximate analytical solutions can occasionally be found, but these generally require insight and/or a considerable amount of algebraic manipulation. Numerical or analog computations or graphical methods appear to be the only ones of general applicability.

Practical Solution of General Equations of Motion. After one sets up the equations of motion of a system one wishes to analyze, one should determine whether solutions of these are available by referring to texts on differential equations and compendia such as Ref. 27. (The latter reference also describes methods of general utility for obtaining approximate solutions, such as that involving series expansion of the variables.) If these approaches fail, one is generally reduced to the use of numerical or graphical methods. A wide range of computer-based methods is available.[44,45] In the following pages two generally useful methods are outlined. Methods and results applicable to some special cases are discussed in subsequent sections.

A Numerical Method. The equation of motion of a single-degree-of-freedom system can generally be expressed in the form

$$m\ddot{x} + G(x, \dot{x}, t) = 0 \qquad \text{or} \qquad \ddot{x} + f(x, \dot{x}, t) = 0 \tag{4.24}$$

*Actually average excess of peaks over troughs, but for $\alpha >> x_{\text{rms}}$ the probability of troughs in the interval becomes very small.

where f includes all nonlinear and nonconstant coefficient effects. (f may occasionally also depend on higher time derivatives. These are not considered here, but the method discussed here may be readily extended to account for them.) It is assumed that f is a known function, given in graphical, tabular, or analytic form.

In order to integrate Eq. (4.24) numerically as simply as possible, one assumes that f remains virtually constant in a small time interval Δt. Then one may proceed by the following steps:

1. ***a.*** Determine $f_0 = f(x_0, \dot{x}_0, 0)$, the initial value of f, from the specified initial displacement x_0 and initial velocity $\dot{x}_0$. Then the initial acceleration is $\ddot{x}_0 = -f_0$.

b. Calculate the velocity $\dot{x}_1$ at the end of a conveniently chosen small time interval Δt_{0-1}, and the average velocity $\bar{\dot{x}}_{0-1}$ during the interval from

$$\dot{x}_1 = \dot{x}_0 + \ddot{x}_0\,\Delta t_{0-1} \qquad \bar{\dot{x}}_{0-1} = \tfrac{1}{2}(\dot{x}_0 + \dot{x}_1)$$

c. Calculate the displacement x_1 at the end of the interval Δt_{0-1} and the average displacement $\bar{x}_{0-1}$ during the interval from

$$x_1 = x_0 + \bar{\dot{x}}_{0-1}\,\Delta t_{0-1} \qquad \bar{x}_{0-1} = \tfrac{1}{2}(x_0 + x_1)$$

d. Compute a better approximation* to the average f and $\ddot{x}$ during the interval Δt_{0-1} by using $x = \bar{x}_{0-1}$, $\dot{x} = \bar{\dot{x}}_{0-1}$, $t = \tfrac{1}{2}\Delta t_{0-1}$ in the determination of f.

2. Repeat steps 1*b* to 1*d*, beginning with the new approximation of f, until no further changes in f occur (to the desired accuracy).

3. Select a second time interval Δt_{1-2} (not necessarily of the same magnitude as Δt_{0-1}) and continue to

a. Find $\ddot{x}_1 = -f_1 = -f(x_1, \dot{x}_1, t_1)$.

b. Calculate the velocity $\dot{x}_2$ at the end of the interval Δt_{1-2}, and the average velocity $\bar{\dot{x}}_{1-2}$ during the interval, from

$$\dot{x}_2 = \dot{x}_1 + \ddot{x}_1\,\Delta t_{1-2} \qquad \bar{\dot{x}}_{1-2} = \tfrac{1}{2}(\dot{x}_1 + \dot{x}_2)$$

c. Similarly, find the final and average displacements for the Δt_{1-2} interval from

$$x_2 = x_1 + \bar{\dot{x}}_{1-2}\,\Delta t_{1-2} \qquad \bar{x}_{1-2} = \tfrac{1}{2}(x_1 + x_2)$$

d. Compute a better approximation to the average f and $\ddot{x}$ during the interval Δt_{1-2} by using $x = \bar{x}_{1-2}$, $\dot{x} = \bar{\dot{x}}_{1-2}$, $t = \Delta t_{0-1} + \tfrac{1}{2}\Delta t_{1-2}$ in the determination of f.

4. Repeat steps 3*b* to 3*d*, starting with the better value of f, until no changes in f occur to within the desired accuracy.

5. One may then continue by essentially repeating steps 3 and 4 for additional time intervals until one has determined the motion for the desired total time of interest.

Generally, the smaller the time intervals selected, the greater will be the accuracy of the results (regardless of the f-averaging method used). Use of smaller time intervals naturally leads to a considerable increase in computational effort. If high accuracy is required, one may generally benefit by employing one of the many available more sophisticated numerical-integration schemes.[12,44,45] In many practical instances the labor of carrying out the required calculations by "hand" becomes prohibitive, however, and use of a digital computer is indicated.

*It should be noted that evaluation of f at the average values of the variables involved is only one of many possible ways of obtaining an average f for the interval considered. Other averages, for example, can be obtained from $\sqrt{f_0 f_1}$, $\tfrac{1}{2}(f_0 + f_1)$. One can rarely predict which average will produce the most accurate results in a given case.

A Semigraphical Method. One may avoid some of the tedium of the foregoing numerical-solution method and gain some insight into a problem by using the "phase-plane delta" method[25,32] discussed here. This method, like the foregoing numerical one, is essentially a stepwise integration for small time increments. It is based on rewriting the equation of motion (4.24) as

$$\dot{x} + \omega_0^2(x + \delta) = 0 \qquad \delta = -x + f(x, \dot{x}, t)/\omega_0^2 \tag{4.25}$$

where ω_0 is any convenient constant circular frequency. (Any value may be chosen for ω_0, but it is usually useful to select one with some physical meaning, e.g., $\omega_0 = \sqrt{k_0/m_0}$, where k_0 and m_0 are values of stiffness and mass for small, x, $\dot{x}$, and t.) If one introduces into Eq. (4.25) a reduced velocity ν given by

$$\nu = \dot{x}/\omega_0$$

and assumes that $\delta(x, \dot{x}, t)$ remains essentially constant in a short time interval one may integrate the resulting equation to obtain

$$\nu^2 + (x + \delta)^2 = R^2 = \text{const}$$

Thus for small time increments the solutions of (4.25) are represented in the $x\nu$ plane (Fig. 4.9) by short arcs of circles whose centers are at $x = -\delta$, $\nu = 0$.

The angle $\Delta\theta$ subtended by the aforementioned circular arc is related to the time interval Δt according to

$$\Delta t \approx \Delta x/\nu\omega_0 \approx \Delta\theta/\omega_0 \tag{4.26}$$

On the basis of the foregoing discussion one may thus proceed as follows:

1. Calculate $\delta(x_0, \dot{x}_0, t_0)$ from Eq. (4.25) using the given initial conditions.
2. Locate the circle center $(-\delta, 0)$ and the initial point (x_0, ν_0) on the $x\nu$ plane; draw a small clockwise arc.
3. At the end of this arc is the point x_1, ν_1 corresponding to the end of the first time interval.
4. Measure or calculate (in radians) the angle $\Delta\theta$ subtended by the arc; calculate the length of the time increment from Eq. (4.26).
5. Calculate $\delta(x_1, \dot{x}_1, t_1)$, and continue as before.
6. Repeat this process until the desired information is obtained.
7. Plots, such as those of x, $\dot{x}$, or $\ddot{x}$, against time, may then be readily obtained from the $x\nu$ curve and the computed time information.

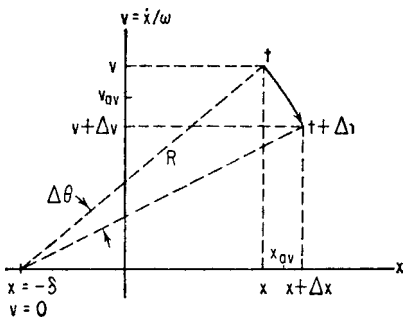


FIG. 4.9 Phase-plane delta method.

If increased accuracy is desired, particularly where δ changes rapidly, δ should be evaluated from average conditions $(x_{av}, \dot{x}_{av}, t_{av})$ during the time increment instead of conditions at the beginning of this increment (see Fig. 4.9). If δ depends on only one variable, a plot of δ against this variable may generally be used to advantage, particularly if it is superposed onto the νx plane.

Mathematical-Approximation Methods. An analytical expression is usually preferable

to a series of numerical solutions, since it generally permits greater insight into a given problem. If exact analytical solutions cannot be found, approximate ones may be the next best approach.

Series expansion of the dependent in terms of the independent variable is often a useful expedient. Power series and Fourier series are most commonly used, but occasionally series of other functions may be employed. The approach consists essentially of writing the dependent variable in terms of a series with unknown coefficients, substituting this into the differential equation, and then solving for the coefficients. However, in many cases these solutions may be difficult, or the series may converge slowly or not at all.

Other methods attempt to obtain solutions by separating the governing equations into a linear part (for which a simple solution can be found) and a nonlinear part. The solution of the linear part is then applied to the nonlinear part in some way so as to give a first correction to the solution. The correction process is then repeated until a second better approximation is obtained, and the process is continued. Such methods include:

1. Perturbation,[21,40] which is particularly useful where the nonlinearities (deviations from linearity) are small
2. Reversion,[21] which is a special treatment of the perturbation method
3. Variation of parameters,[21,60] useful where nonlinearities do not result in additive terms
4. Averaging methods, based on error minimization
   a. Galerkin's method[21]
   b. Ritz method[21]

Conservative Systems: The Phase Plane.[21,26,60] A conservative system is one whose equation of motion can be written

$$m\ddot{x} + f(x) = 0 \tag{4.27}$$

In such systems (which may be visualized as masses attached to springs of variable stiffness) the total energy E remains constant; that is,

$$E = V(\dot{x}) + U(x)$$

where $$V(\dot{x}) = \tfrac{1}{2}m\dot{x}^2 \qquad U(x) = \int_{x_0}^{x} f(x)\,dx$$

where $V(\dot{x})$ and $U(x)$ are, respectively, the kinetic and the potential energies and x_0 is a convenient reference value.

The velocity-displacement ($\dot{x}$ vs. x) plane is called the "phase plane"; a curve in it is called a "phase trajectory." The equation of a phase trajectory of a conservative system with a given total energy E is

$$\dot{x}^2 = (2/m)[E - U(x)] \tag{4.28}$$

The time interval $(t - t_1)$ in which a change of displacement from x_1 to x occurs, is given by

$$t - t_1 = \int_{x_1}^{x} \frac{dx}{\dot{x}} = \int_{x_1}^{x} \frac{dx}{\sqrt{(2/m)[E - U(x)]}} \tag{4.29}$$

Some understanding of the geometry of phase trajectories may be obtained with the aid of Fig. 4.10, which shows the dependence of phase trajectories on total energy for a hypothetical potential energy function $U(x)$. For $E = E_1$ the motion is periodic; zero

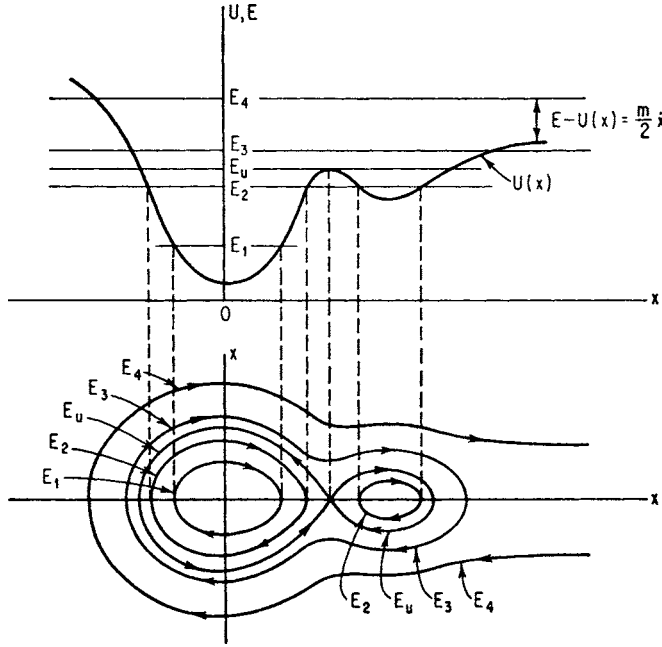


FIG. 4.10 Dependence of phase trajectories on energy.

velocity and velocity reversal occur where $U(x) = E_1$. With $E = E_2$ periodic oscillations are possible about two points; the initial conditions applicable in a given case dictate which type of oscillation occurs in that case. For $E = E_3$ only a single periodic motion is possible; for $E = E_4$ the motion is aperiodic. For $E = E_u$ there exists an instability at x_u; there the mass may move either in the increasing or decreasing x direction. (The arrows on the phase trajectories point in the direction of increasing time.)

For a linear undamped system $f(x) = kx$, $U(x) = \frac{1}{2}kx^2$, and phase trajectories are ellipses with semiaxes $(2E/k)^{1/2}$, $(2E/m)^{1/2}$.

The following facts may be summarized for conservative systems:

1. Oscillatory motions occur about minima in $U(x)$.
2. Phase trajectories are symmetric about the x axis and cross the x axis perpendicularly.
3. If $f(x)$ is single-valued, the phase trajectories for different energies E do not intersect.
4. All finite motions are periodic.

For nonconservative systems the phase trajectories tend to cross the constant-energy trajectories for the corresponding conservative systems. For damped systems the trajectories tend toward lower energy, i.e., they spiral into a point of stability. For excited systems the trajectories spiral outward, either toward a "limit-cycle" trajectory or indefinitely.[21,60]

The period T of an oscillation of a conservative system occurring with maximum displacement (amplitude) x_{max} may be computed from

$$T = \sqrt{2m} \int_{x_{\min}}^{x_{\max}} \left[\int_{x}^{x_{\max}} f(x)\, dx \right]^{-1/2} dx \tag{4.30}$$

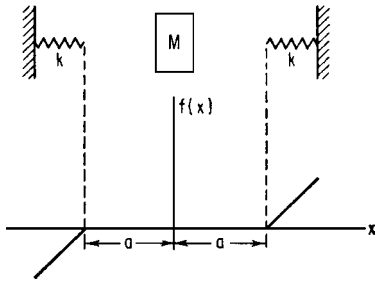


FIG. 4.11 Spring-mass system with clearance.

where x_{max} and x_{min} are the largest and smallest values of x (algebraically) for which zero velocity $\dot{x}$ occurs. The frequency f may then be obtained from $f = 1/T$.

For a linear spring-mass system with clearance, as shown in Fig. 4.11, the frequency is given by[17,32]

$$f = \sqrt{\frac{k}{m}} \left\{ 2 \left[\pi + \frac{2}{(x_{\max}/a) - 1} \right] \right\}^{-1}$$

where x_{max} is the maximum excursion of the mass from its middle position.

For a system governed by Eq. (4.27) with $f(x)=kx|x|^{b-1}$ [or $f(x) = kx^b$, if b is odd], the frequency is given by[35]

$$f = \sqrt{\frac{(b+1)k}{8\pi m} x_{max}^{b-1}} \frac{\Gamma[1/(b+1) + \frac{1}{2}]}{\Gamma[1/(b+1)]}$$

in terms of the gamma function Γ, values of which are available in many tables.

Steady-State Periodic Responses. In many cases, particularly in steady-state analyses of periodically forced systems, periodic oscillatory solutions are of primary interest. A number of mathematical approaches are available to deal with these problems. Most of these, including the well-known methods of Stoker[60] and Schwesinger,[21,54] are based on the idea of "harmonic balance."[21] They essentially assume a Fourier expansion of the solution and then require the coefficients to be adjusted so that relevant conditions on the lowest few harmonic components are satisfied.

For example, in order to find a steady-state periodic solution of

$$m\ddot{x} + g(\dot{x}) + f(x) = F \sin(\omega t + \phi)$$

one may substitute an assumed displacement

$$x = A_1 \sin \omega t + A_2 \sin 2\omega t + \cdots + A_n \sin n\omega t$$

and impose certain restrictions on the error ϵ,

$$\epsilon(t) = m\ddot{x} + g(\dot{x}) + f(x) - F \sin(\omega t + \phi)$$

In Schwesinger's method the mean-square value of the error, $\overline{\epsilon^2} = \int_0^{2\pi} \epsilon^2(t) d(\omega t)$, is minimized, and values of F and ϕ are calculated from this minimization corresponding to an assumed A_1.

Systems and Nonlinear Springs. The restoring forces of many systems (particularly with small amounts of nonlinearity) may be approximated so that the equation of motion may be written as

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x + (a/m)x^3 = (F/m)\cos \omega t \tag{4.31}$$

in the presence of viscous damping and a sinusoidal force. ζ is the damping factor and ω_0 the natural frequency of a corresponding undamped linear system (i.e., for $a = 0$).

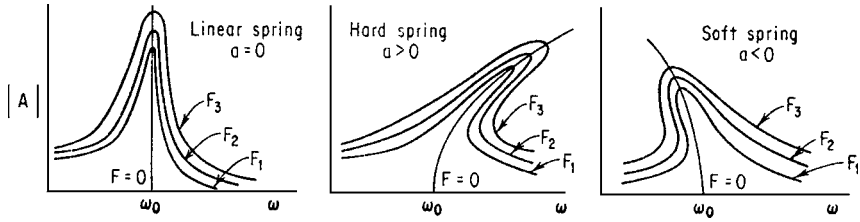


FIG. 4.12 Comparison of frequency responses of linear and nonlinear systems.

For $a > 0$ the spring becomes stiffer with increasing deflection and is called "hard"; for $a < 0$ the spring becomes less stiff and is called "soft."

Figure 4.12 compares the responses of linear and nonlinear lightly damped spring systems. The responses are essentially of the form $x = A \cos \omega t$; curves of response amplitude A vs. forcing frequency ω are sketched for several values of forcing amplitude F, for constant damping ζ. For a linear system the frequency of free oscillations ($F = 0$) is independent of amplitude; for a hard system it increases; for a soft system it decreases with increasing amplitude. The response curves of the nonlinear spring systems may be visualized as "bent-over" forms of the corresponding curves for the linear systems.

As apparent from Fig. 4.12, the response curves of the nonlinear spring systems are triple-valued for some frequencies. This fact leads to "jump" phenomena, as sketched in Figs. 4.13 and 4.14. If a given force amplitude is maintained as forcing frequency is changed slowly, then the response amplitude follows the usual response curve until point 1 of Fig. 4.13 is reached. The hatched regions between points 1 and 3 correspond to unstable conditions; an increase in ω above point 1 causes the amplitude to jump to that corresponding to point 2. A similar condition occurs when frequency is slowly decreased; the jump then occurs between points 3 and 4.

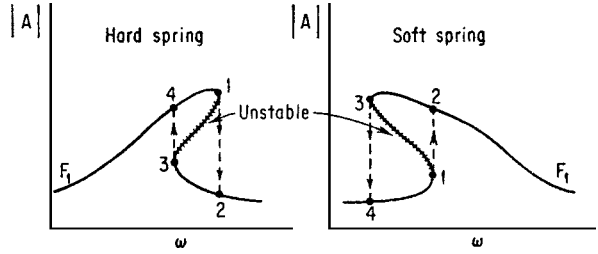


FIG. 4.13 Jump phenomena with variable frequency and constant force.

As also evident from Fig. 4.12, a curve of response amplitude vs. force amplitude at constant frequency is also triple-valued in some regions of frequency. Thus amplitude jumps occur also when one changes the forcing amplitude slowly at constant frequency ω_c. This condition is sketched in Fig. 4.14. For a hard spring this can occur only at frequencies above ω_0, for soft springs below ω_0. The equations characterizing a lightly damped nonlinear spring system and its jumps are summarized in Fig. 4.15.

The previous discussion deals with the system response as if it were a pure sinusoid $x = A \cos \omega t$. However, in nonlinear systems there occur also harmonic components (at frequencies $n\omega$, where n is an integer) and subharmonic components (frequencies

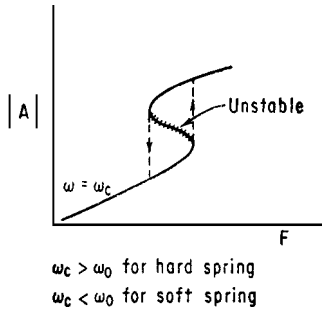


FIG. 4.14 Jump phenomena with variable force and constant frequency.

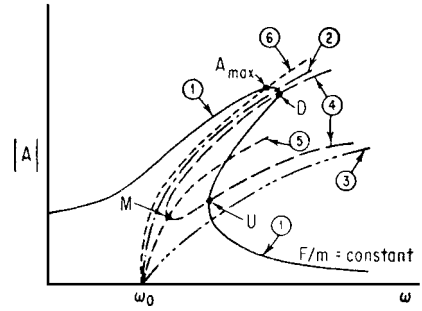


FIG. 4.15 Characteristics of responses of nonlinear springs. Equation of motion: $m\ddot{x} + c\dot{x} + kx + ax^3 = F \cos \omega t$, where ω_0 = undamped natural frequency for linear system (with $a = 0$); ζ = damping ratio for linear system = $c/2m\omega_0$. (1) Response curve for forced vibrations, F/m constant: $[A(\omega_0^2 - \omega^2) + \frac{3}{4}(a/m)A^3]^2 + [2\zeta\omega_0\omega A]^2 = (F/m)^2$. (2) Response curve for undamped free vibrations (approximate locus of downward jump points D, and of A_{max}): $\omega^2 = \omega_0^2 + \frac{3}{4}(a/m)A^2$. (3) Locus of upward jump points U with zero damping; approximate locus of same with finite damping: $\omega^2 = \omega_0^2 + \frac{3}{4}(a/m)A^2$. (4) Locus of upward and downward jump points U and D with finite damping: $[\omega_0^2 - \omega^2 + \frac{3}{4}(a/m)A^2][\omega_0^2 - \omega^2 + \frac{3}{4}(a/m)A^2] + (2\zeta\omega_0\omega)^2 = 0$. (5) Locus of points M below which no jumps occur: $\omega^2 = \omega_0^2 + \frac{9}{8}(a/m)A^2$. (6) Locus of maximum amplitudes A_{max}: $\omega^2 - \frac{3}{4}(a/m)A^2 = \omega_0^2(1 - 2\zeta^2)$.

ω/n). In addition, components occur at frequencies which are integral multiples of the subharmonic frequencies. The various harmonic and subharmonic components tend to be small for small amounts of nonlinearity, and damping tends to limit the occurrence of subharmonics. The amplitude of the component with frequency 3ω (the lowest harmonic above the fundamental with finite amplitude for an undamped system) is given by $aA^3/36m\omega^2$, where A is the amplitude of the fundamental response, in view of Eq. (4.31). For more complete discussions see Ref. 21.

Graphical Determination of Response Amplitudes. A relatively easily applied method for approximating response amplitudes was developed by Martienssen[37] and improved by Mahalingam.[34] It is based on the often observed fact that the response to sinusoidal excitation is essentially sinusoidal. The method is here first explained for a linear system, then illustrated for nonlinear ones.

In order to obtain the steady-state response of a linear system one substitutes an assumed trial solution

$$x = A \cos (\omega t - \phi)$$

into the equation of motion

$$m\ddot{x} + c\dot{x} + kx = P \cos \omega t$$

By equating coefficients of corresponding terms on the two sides of the resulting equation one obtains a pair of equations which may be solved to yield

$$\omega_n^2 A = \omega^2 A + \frac{P}{m}\cos\phi \qquad \tan\phi = \frac{c\omega/m}{(\omega_n/\omega)^2 - 1} \tag{4.32}$$

where ω_n is the undamped natural frequency. One may plot the functions

$$y_1(A) = A\omega_n^2(A) \qquad y_2(A) = A\omega^2 + (P/m)\cos\theta$$

and determine for which value of A these two functions intersect. This value of A then is the desired amplitude. Since P and m are given constants, y_2 plots as a straight line with slope ω^2 and y intercept $(P/m)\cos\phi$. For a linear system one may compute $\tan\phi$ directly from Eq. (4.32), but for a nonlinear system ω_n^2 depends on the amplitude A and direct computation of $\tan\phi$ is not generally possible. Use of a method of successive approximations is then indicated.

Figure 4.16*a* shows application of this method to a linear system. After calculating ϕ from Eq. (4.32) one may find the y intercept $(P/m)\cos\phi$. For a given frequency ω one may then draw a line of slope ω^2 through that intercept to represent the function y_2. For a linear system y_1 is a straight line with slope ω_n^2 and passing through the origin. The amplitude of the steady-state oscillation may then be determined as the value A_0 of A where the two lines intersect.

Figure 4.16*b* shows a diagram analogous to Fig. 4.16*a,* but for an arbitrary nonlinear system. The function $y_1 = A\omega_n^2$ is not a straight line in general since ω_n generally is a function of A. This function may be determined from the restoring function $f(x)$ by use of Eq. (4.30). The possible amplitudes corresponding to a given driving frequency ω and force amplitude P are determined here, as before, by the intersection of the y_1 and y_2 curves. As shown in the figure, more than one amplitude may correspond to a given frequency—a condition often encountered in nonlinear systems.

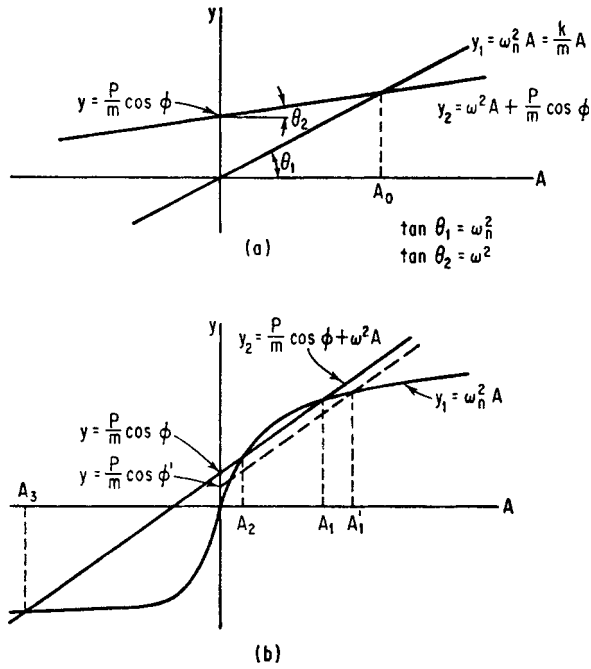


FIG. 4.16 Graphical determination of amplitude. (*a*) Linear system. (*b*) Nonlinear system.

In applying the previously outlined method to nonlinear systems one generally cannot find the correct value of ϕ at once from Eq. (4.32), since ω_n depends on the amplitude A, as has been pointed out. Instead one may assume any value of ϕ, such as ϕ', and determine a first approximation A_1' to A_1. Using the approximate amplitude one may then determine better values of $\omega_n(A)$ and ϕ from Eq. (4.32) and then use these better values to obtain a better approximation to A_1. This process may be repeated until ϕ and A_1 have been found to the desired degree of accuracy. A separate iteration process of this sort is generally required for each of the possible amplitudes. (Different values of ϕ correspond to the different amplitudes A_1, A_2, A_3.)

Systems with Nonlinear Damping. The governing equation in this case may be written as

$$m\ddot{x} + C(x, \dot{x}) + kx = F \sin \omega t \tag{4.33}$$

where $C(x, \dot{x})$ represents the effect of damping. Exact or reasonably good approximate solutions are available only for relatively few cases. However, for many practical cases where the damping is not too great, the system response is essentially sinusoidal. One may then use an "equivalent" viscous-damping term $c_e\dot{x}$ instead of $C(x, \dot{x})$, so that the equivalent damping results in the same amount of energy dissipation per cycle as does the original nonlinear damping. For a response of the form $x = A \sin \omega t$ the equivalent viscous damping may be computed from

$$c_e = \frac{1}{\pi\omega A} \int_0^{2\pi} C(A \sin \omega t, A\omega \cos \omega t)\, d(\omega t) \tag{4.34}$$

In contrast to the usual linear case, c_e is generally a function of frequency and amplitude. Once c_e has been found, the response amplitude may be computed from Eq. (4.6); successive approximations must be used if c_e is amplitude-dependent.

For "dry" or Coulomb friction, where the friction force is $\pm \mu$ (constant in magnitude but always directed opposite to the velocity) the equivalent viscous damping c_e and response amplitude A are given by

$$c_e = \frac{4\mu}{\pi A\omega} \qquad A = \frac{F}{k(1-\omega^2/\omega_n^2)} \sqrt{1-\left(\frac{4}{\pi}\frac{\mu}{F}\right)^2}$$

Further details on Coulomb damped systems appear in Refs. 17 and 63.

If complex notation is used, the damping effect may be expressed in terms of an imaginary stiffness term, and Eq. (4.33) may alternatively be written

$$m\ddot{x} + k(1 + i\gamma)x = Fe^{i\omega t}$$

where γ is known as the "structural damping factor."[18,51,68] For a response given by $x = Ae^{i\omega t}$ one finds

$$|A| = F/|-m\omega^2 + k(1 + i\gamma)| \tag{4.35}$$

This reduces identically to the linear case with viscous damping c, if γ is defined so that $k\gamma = \omega c$. The steady-state behavior of a system with any reasonable type of damping may be represented by this complex stiffness concept, provided that γ is prescribed with the proper frequency and amplitude dependence. The foregoing relation and Eq. (4.34) may be used to find the aforementioned proper dependences for a given resisting force $C(x, \dot{x})$. The case of constant γ corresponds to "structural damping" (widely used in aircraft flutter calculations) and represents a damping force proportional to displacement but in phase with velocity. The damping factor γ is particularly useful for describing the damping action of rubberlike materials, for which the damping

is virtually independent of amplitude (but not of frequency),[55] since then the response is explicitly given by Eq. (4.35).

Response to Random Excitation. For quantitative results the reader is referred to Refs. 3, 11, and 16.

Qualitatively, the response of a linear system to random excitation is essentially a sinusoid at the system's natural frequency. The amplitudes (i.e., the envelope of this sinusoid) vary slowly and have a Rayleigh distribution. Compared with a linear system a system with a "hard" spring has a higher natural frequency, a lower probability of large excursions, and waves with flattened peaks. ("Soft" spring systems exhibit opposite characteristics.) The effects of the nonlinearities on frequency and on the wave shape are generally very small.

Self-Excited Systems. If the damping coefficient c of a linear system is negative, the system tends to oscillate with ever-increasing amplitude. Positive damping extracts energy from the system; negative damping contributes energy to it. A system (such as one with negative damping) for which the energy-contributing forces are controlled by the system motion is called self-excited. A source of energy must be available if a system is to be self-excited. The steady-state amplitude of a self-excited oscillation may generally be determined from energy considerations, i.e., by requiring the total energy dissipated per cycle to equal the total energy supplied per cycle.

The chatter of cutting tools, screeching of hinges or locomotive wheels, and chatter of clutches are due to self-excited oscillations associated with friction forces which decrease with increasing relative velocity. The larger friction forces at lower relative velocities add energy to the system; smaller friction forces at higher velocities (more slippage) remove energy. While the oscillations build up, the energy added is greater than that removed; at steady state in each cycle the added energy is equal to the extracted energy.

More detailed discussions of self-excited systems may be found in Refs. 21 and 60.

4.3 SYSTEMS WITH A FINITE NUMBER OF DEGREES OF FREEDOM

The instantaneous configurations of many physical systems can be specified by means of a finite number of coordinates. Continuous systems, which have an infinite number of degrees of freedom, can be approximated for many purposes by systems with only a finite number of degrees of freedom, by "lumping" of stiffnesses, masses, and distributed forces. This concept, which is extremely useful for the analysis of practical problems, has been the basis for numerous computer-based "finite-element" and "modal-analysis" methods.[24,44,45]

4.3.1 Systematic Determination of Equations of Motion

Generalized Coordinates: Constraints.[53,64] A set of n quantities q_i ($i = 1, 2, \ldots, n$) which at any time t completely specify the configuration of a system are called "generalized coordinates" of the system. The quantities may or may not be usual space coordinates.

If one selects more generalized coordinates than the minimum number necessary to describe a given system fully, then one finds some interdependence of the selected coordinates dictated by the geometry of the system. This interdependence may be expressed as

$$G(q_1, q_2, \dots, q_n; \dot{q}_1, \dot{q}_2, \dots, \dot{q}_n; t) = 0 \tag{4.36}$$

Relations like those of Eq. (4.36) are known as "equations of constraint,"; if no such equations can be formulated for a given set of generalized coordinates, the set is known as "kinematically independent."

The constraints of a set of generalized coordinates are said to be integrable if all equations like (4.36) either contain no derivatives $\dot{q}_i$ or if such $\dot{q}_i$ that do appear can be eliminated by integration. If to the set of n generalized coordinates there correspond m constraints, all of which are integrable, then one may find a new set of $(n - m)$ generalized coordinates which are subject to no constraints. This new system is called "holonomic," and $(n - m)$ is the number of degrees of freedom of the system (that is, the smallest number of quantities necessary to describe the system configuration at any time). In practice one may often be able to select a holonomic system of generalized coordinates by inspection. Henceforth the discussion will be limited to holonomic systems.

Lagrangian Equations of Motion. The equations governing the motion of any holonomic system may be obtained by application of Lagrange's equation

$$(d/dt)(\partial T/\partial \dot{q}_i) - \partial T/\partial \dot{q}_i + \partial U/\partial q_i + \partial F/\partial \dot{q}_i = Q_i \qquad i = 1, 2, \dots, n \tag{4.37}$$

where T denotes the kinetic energy, U the potential energy of the entire dynamic system, F is a dissipation function, and Q_i the generalized force associated with the generalized coordinate q_i.

The generalized force Q_i may be obtained from

$$\delta W = Q_i \delta q_i \tag{4.38}$$

where δW is the total work done on the system by all external forces not contributing to U when the single coordinate q_i is changed to $q_i + \delta q_i$. The potential energy U accounts only for forces which are "conservative" (that is, for those forces for which the work done in a displacement of the system is a function of only the initial and final configurations). The dissipation function F represents half the rate at which energy is lost from the system; it accounts for the dissipative forces that appear in the equations of motion.

Lagrange's equations can be applied to nonlinear as well as linear systems, but little can be said about solving the resulting equations of motion if they are nonlinear, except that small oscillations of nonlinear systems about equilibrium can always be approximated by linear equations. Methods of solving linear sets of equations of motion are available and are discussed subsequently.

4.3.2 Matrix Methods for Linear Systems—Formalism

The kinetic energy T, potential energy U, and dissipation function F of any *linear* holonomic system with n degrees of freedom may be written as

$$T = \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \dot{q}_i \dot{q}_j \qquad a_{ij} = a_{ji}$$

$$U = \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} q_i q_j \qquad c_{ij} = c_{ji} \tag{4.39}$$

$$F = \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} \dot{q}_i \dot{q}_j \qquad b_{ij} = b_{ji}$$

The equations of motion may then readily be determined by use of Eq. (4.37). They may be expressed in matrix form as

$$A\{\ddot{q}\}+B\{\dot{q}\}+C\{q\}=\{Q(t)\} \tag{4.40}$$

where *A, B, C* are symmetric square matrices with *n* rows and *n* columns whose elements are the coefficients appearing in Eqs. (4.39) and $\{q\}$, $\{\dot{q}\}$, $\{\ddot{q}\}$, $\{Q(t)\}$ are *n*-dimensional column vectors.

(*Note*: *A* is called the inertia matrix, *B* the damping matrix, *C* the elastic or the stiffness matrix. For example,

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad \{q\}=\begin{bmatrix} q_1 \\ q_2 \\ \cdot \\ \cdot \\ q_n \end{bmatrix} \quad \{\dot{q}\}=\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \cdot \\ \cdot \\ \dot{q}_n \end{bmatrix} \quad \{Q\}=\begin{bmatrix} Q_1 \\ Q_2 \\ \cdot \\ \cdot \\ Q_n \end{bmatrix}$$

The elements of $\{q\}$ are the coordinates of q_i, the elements of $\{\dot{q}\}$ are the first time derivatives of q_i (i.e., the generalized velocities $\dot{q}_i$); those of $\{\ddot{q}\}$ are the generalized accelerations $\ddot{q}_i$, those of $\{Q\}$ are the generalized forces Q_i.)

Free Vibrations: General System. If all the generalized forces Q_i are zero, then Eq. (4.40) reduces to a set of homogeneous linear differential equations. To solve it one may postulate a time dependence given by

$$\{q\}=\{r\}e^{st} \qquad s=\sigma+i\omega$$

which, when introduced into Eq. (4.40), results in

$$(s^2A+sB+C)\{r\}=0 \tag{4.41}$$

One may generally find *n* nontrivial solutions $\{r^{(j)}\}$, $j = 1, 2, \ldots, n$, each corresponding to a specific value $s_{(j)}$ of *s*. The general solution of the homogeneous equation may then be expressed as

$$\{q\}=\sum_{j=1}^{n}\alpha_j\{r^{(j)}\}e^{s(j)t}=R\{p\} \tag{4.42}$$

in terms of complex constants α_j and the newly defined

$$R=\begin{bmatrix} r_1^{(1)} & r_1^{(2)} & \cdots & r_1^{(n)} \\ r_2^{(1)} & r_2^{(2)} & \cdots & r_2^{(n)} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ r_n^{(1)} & r_n^{(2)} & \cdots & r_n^{(n)} \end{bmatrix} \quad \{p(t)\}=\begin{bmatrix} \alpha_1 & e^{s(1)t} \\ \alpha_2 & e^{s(2)t} \\ \cdot & \cdot \\ \cdot & \cdot \\ \alpha_n & e^{s(n)t} \end{bmatrix} \tag{4.43}$$

Initial conditions, e.g., $\{q(0)\}$, $\{\dot{q}(0)\}$ may be introduced into Eq. (4.42) to evaluate the constants α_j.

Forced Vibrations: General System. The forced motion may be described in terms of the sum of two motions, one satisfying the homogeneous equation (with all $Q_i = 0$) and including all the constants of integration, the other satisfying the complete Eq. (4.40) and containing no integration constants. (The constants must be evaluated so that the

total solution satisfies the prescribed initial conditions.) The latter constant-free solution is often called the "steady-state solution."

Steady-State Solution for Periodic Generalized Forces. If the generalized forces are harmonic with frequency ω_0 one may set

$$\{Q\}=\{\bar{Q}\}e^{i\omega_0 t} \qquad \{q\}=\{\bar{q}\}e^{i\omega_0 t} \tag{4.44}$$

in Eq. (4.40) and obtain

$$(-\omega_0^2 A + i\omega_0 B + C)\{\bar{q}\}=\{\bar{Q}\}$$

from which $\{\bar{q}\}$ may be determined. Equation (4.44) then gives the steady-state solutions.

If the Q_i are periodic with period T, but not harmonic, one may expand them and the components q_i of the steady-state solution in Fourier series:

$$\{Q\}=\sum_{N=-\infty}^{\infty}\{\bar{Q}^{(N)}\}e^{iN\omega_0 t} \qquad \{q\}=\sum_{N=-\infty}^{\infty}\{\bar{q}^{(N)}\}e^{iN\omega_0 t} \qquad \omega_0=\frac{T}{2\pi}$$

The Fourier components $\{\bar{q}^{(N)}\}$ may be evaluated from

$$(-N^2\omega_0^2 A + iN\omega_0 B + C)\{\bar{q}^{(N)}\} = \{\bar{Q}^{(N)}\}$$

Steady-State Solution for Aperiodic Generalized Forces. For general $\{Q(t)\}$ one may write the solutions of Eq. (4.40) as

$$q_i(t) = \sum_{j=1}^{n}\int_0^t h_i^{(j)}(t-\tau)Q_j(\tau)\,d\tau \tag{4.45}$$

where $h_i^{(j)}(t)$ is the response of coordinate q_i to a unit impulse acting in place of Q_j. The "impulse response function" $h_i^{(j)}$ may be found from

$$h_i^{(j)}(t) = (d/dt)u_i^{(j)}(t)$$

where $u_i^{(j)}$ is the response of q_i to a unit step function acting in place of Q_j. (A unit step function is zero for $t < 0$, unity for $t > 0$.)

If one defines a square matrix $[H(t)]$ whose elements are $h_i^{(j)}$, one may write Eq. (4.45) alternatively as

$$\{q(t)\}=\int_0^t [H(t-\tau)]\{Q(\tau)\}\,d\tau$$

where $$[H(t)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}[T(s)]e^{st}\,ds \qquad [T(s)] = (s^2A + sB + C)^{-1}$$

Undamped Systems. For undamped systems all elements of the B matrix of Eq. (4.40) are zero, and Eq. (4.41) may be rewritten in the classical eigenvalue form

$$E\{r\} = \omega^2\{r\} \qquad E = A^{-1}C \tag{4.46}$$

The eigenvalues $\omega_{(j)}$, i.e., the values for which nonzero solutions $r^{(j)}$ exist, are real. They are the natural frequencies; the corresponding solution vectors $r^{(j)}$ describe the mode shapes.

One may then find a set of principal coordinates ψ_i, in terms of which the equations of motion are uncoupled and may be written in the following forms:

$$\{\dot{\psi}\} + \Omega\{\psi\} = R^{-1}A^{-1}\{Q\} \qquad \text{or} \qquad M\{\dot{\psi}\} + K\{\psi\} = \Phi(t)$$

Here $$M = \overline{R}AR \qquad K = \overline{R}CR \qquad \{\Phi(t)\} = \overline{R}\{Q(t)\}$$

Ω is the diagonal matrix of natural frequencies

$$\Omega = \begin{bmatrix} \omega_{(1)}^2 & 0 & \cdots & 0 \\ 0 & \omega_{(2)}^2 & & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & \omega_{(n)}^2 \end{bmatrix}$$

R is given by Eq. (4.43), and $\overline{R}$ denotes the transpose of R.

The principal coordinates ψ_i are related to the original coordinates q_i by

$$\{q\} = R\{\psi\} \qquad \{\psi\} = R^{-1}\{q\} \tag{4.47}$$

The response of the system to any forcing function $\{Q(t)\}$ may be determined in terms of the principal coordinates from

$$\psi_j(t) = m_{jj}k_{jj}\int_0^t \Phi_j(\tau)\sin[\omega_{(j)}(t-\tau)]\,d\tau + \psi_j(0)\cos[\omega_{(j)}t] + [1/\omega_{(j)}]\dot{\psi}_j(0)\sin[\omega_{(j)}t] \tag{4.48}$$

The response in terms of the original coordinates q_i may then be obtained by substitution of the results of Eq. (4.48) into Eq. (4.47).

4.3.3 Matrix Iteration Solution of Positive-Definite Undamped Systems

Positive-Definite Systems: Influence Coefficient and Dynamic Matrixes. A system is "positive-definite" if its potential energy U, as given by Eq. (4.39), is greater than zero for any $\{q\} \neq \{0\}$. Systems connected to a fixed frame are positive-definite; systems capable of motion (changes in the coordinates q_i) without increasing U are called "semidefinite."[64] The latter motions occur without energy storage in the elastic elements and are called "rigid-body" motions or "zero modes." (They imply zero natural frequency.)

Rigid-body motions are generally of no interest in vibration study. They may be eliminated by proper choice of the generalized coordinates or by introducing additional relations (constraints) among an arbitrarily chosen system of generalized coordinates by applying conservation-of-momentum concepts. Thus any system of generalized coordinates can be reduced to a positive-definite one.

For positive-definite linear systems C^{-1}, the inverse of the elastic matrix, is known as the influence coefficient matrix D. The elements of D are the influence coefficients; the typical element d_{ij} is the change in coordinate q_i due to a unit generalized force Q_j (applied statically), with all other Q's equal to zero. Since these influence coefficients can be determined from statics, one generally need not find C at all. It should be noted that for systems that are not positive-definite one cannot compute the influence coefficients from statics alone.

For iteration purposes it is useful to rewrite Eq. (4.46), the system equation of free sinusoidal motion, as

$$G\{r\} = (1/\omega^2)\{r\} \tag{4.49}$$

where $\{q\} = \{r\}e^{i\omega t} \qquad G = C^{-1}A = DA = E^{-1}$ (4.50)

The matrix G is called the "dynamic matrix" and is defined, as above, as the product of the influence coefficient matrix D and the inertia matrix A.

Iteration for Lower Modes. In order to solve Eq. (4.49), which is a standard eigenvalue matrix equation, numerically for the lowest mode one may proceed as follows: Assume any vector $\{r_{(1)}\}$; then compute $G\{r_{(1)}\} = \alpha_{(1)}\{r_{(2)}\}$, where $\alpha_{(1)}$ is a constant chosen so that one element (say, the first) of $\{r_{(2)}\}$ is equal to the corresponding element of $\{r_{(1)}\}$. Then find $G\{r_{(2)}\} = \alpha_{(2)}\{r_{(3)}\}$, with $\alpha_{(2)}$ chosen like $\alpha_{(1)}$ before. Repeat this process until $\{r_{(n+1)}\} \approx \{r_{(n)}\}$ to the desired degree of accuracy. The corresponding constant $\alpha_{(n)}$ which satisfies $G\{r_{(n)}\} = \alpha(n)\{r_{(n+1)}\}$ then yields to lowest natural frequency ω_1 of the system and $\{r_{(n)}\}$ describes the shape of the corresponding (first) mode $\{r^{(1)}\}$. In view of Eq. (4.49)

$$\omega_1^2 = 1/\alpha_{(n)}$$

The second mode $\{r^{(2)}\}$ must satisfy the orthogonality relation

$$\{\overline{r^{(2)}}\}A\{r^{(1)}\} = 0 \qquad \text{or} \qquad \sum_{i=1}^{n}\sum_{j=1}^{n} r_i^{(2)} a_{ij} r_j^{(1)} = 0 \tag{4.51}$$

In order to obtain a vector that satisfies Eq. (4.51) from an arbitrary vector $\{r\}$ one may select $(n-1)$ components of $\{r_{(2)}\}$ as equal to the corresponding components of $\{r\}$ and then compute the nth from Eq. (4.51). This process may be expressed as

$$\{r^{(2)}\} = S_1\{r\}$$

where S_1 is called the "first sweeping matrix." S_1 is equal to the identity matrix in n dimensions, except for one row which describes the interrelation Eq. (4.51). If A is diagonal one may take, for example,

$$S_1 = \begin{bmatrix} 0 & -\dfrac{a_{22}r_2^{(1)}}{a_{11}r_1^{(1)}} & -\dfrac{a_{33}r_3^{(1)}}{a_{11}r_1^{(1)}} & \cdots & \dfrac{a_{nn}r_n^{(1)}}{a_{11}r_1^{(1)}} \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

To obtain the second lowest mode shape $\{r^{(2)}\}$ and the second lowest natural frequency ω_2 one may form $H_1 = GS_1$, and solve

$$H_1\{r\} = (1/\omega^2)\{r\} \tag{4.52}$$

by iteration. Since Eq. (4.52) is of the same form as Eq. (4.49), one may proceed here as previously discussed, i.e., by assuming a trial vector $\{r_{(1)}\}$, forming $H_1\{r_{(1)}\} = \alpha_{(1)}\{r_{(2)}\}$, so that one element of $\{r_{(2)}\}$ is equal to the corresponding element of $\{r_{(1)}\}$, then forming $H_1\{r_{(2)}\} = \alpha_{(2)}\{r_{(3)}\}$, etc. This process converges to $\{r^{(2)}\}$ and $\alpha = 1/\omega_2^2$.

The third mode $\{r^{(3)}\}$ similarly must satisfy

$$\{\overline{r^{(3)}}\}A\{r^{(1)}\} = 0 \qquad \{\overline{r^{(3)}}\}A\{r^{(2)}\} = 0$$

or $$\sum_{i=1}^{n}\sum_{j=1}^{n} r_i^{(3)} a_{ij} r_j^{(1)} = 0 \qquad \sum_{i=1}^{n}\sum_{j=1}^{n} r_i^{(3)} a_{ij} r_j^{(2)} = 0 \tag{4.53}$$

One may thus select $n - 2$ components of $\{r^{(3)}\}$ as equal to the corresponding components of an arbitrary vector $\{r\}$ and adjust the remaining two components to satisfy Eq. (4.53). The matrix S_2 expressing this operation, or

$$\{r^{(3)}\} = S_2\{r\}$$

is called the "second sweeping matrix." For diagonal A one possible form of S_2 is

$$S_2 = \begin{bmatrix} 0 & -\dfrac{a_{22}r_2^{(1)}}{a_{11}r_1^{(1)}} & -\dfrac{a_{33}r_3^{(1)}}{a_{11}r_1^{(1)}} & -\dfrac{a_{nn}r_n^{(1)}}{a_{11}r_1^{(1)}} \\ 0 & -\dfrac{a_{22}r_2^{(2)}}{a_{11}r_1^{(2)}} & -\dfrac{a_{33}r_3^{(2)}}{a_{11}r_1^{(2)}} & -\dfrac{a_{nn}r_n^{(2)}}{a_{11}r_1^{(2)}} \\ 0 & 0 & 1 & 0 \\ 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Then one may form $H_2 = GS_2$ and solve

$$H_2\{r\} = (1/\omega^2)\{r\}$$

by iteration. The process here converges to $\{r^{(3)}\}$ and $\alpha = 1/\omega_3^2$.

Higher modes may be treated similarly; each mode must be orthogonal to all the lower ones, so that $p - 1$ relations like Eq. (4.51) must be utilized to find the $(p - 1)$st sweeping matrix. Iteration on $H_{(p-1)} = GS_{(p-1)}$ then converges to the pth mode.

Iteration for the Higher Modes. The previously outlined process begins with the lowest natural frequency and works toward the highest. It is not very useful for the highest few modes because of the tedium and of the accumulation of rounding off errors. Results for the higher modes can be obtained more simply and accurately by starting with the highest frequency and working toward lower ones.

The highest mode may be obtained by solving Eq. (4.46) directly by iteration. This is accomplished by assuming any trial vector $\{r_{(1)}\}$, forming $E\{r_{(1)}\} = \beta_{(1)}\{r_{(2)}\}$ with $\beta_{(1)}$ chosen so that one element of the result $\{r_{(2)}\}$ is equal to the corresponding element of $\{r_{(1)}\}$. Then one may form $E\{r_{(2)}\} = \beta_{(2)}\{r_{(3)}\}$ similarly, and continue until $\{r_{(p+1)}\} \approx \{r_{(p)}\}$ to within the required accuracy. Then $\{r_{(p)}\} = \{r^{(n)}\}$ and $\beta_{(p)} = \omega_n^2$.

The next-to-highest [$(n - 1)$st] mode may be found by writing

$$\{r^{(n-1)}\} = T_1\{r\}$$

where T_1 is a sweeping matrix that "sweeps out" the nth mode. T_1 is equal to the identity matrix, except for one row, which expresses the orthogonality relation

$$\{\overline{r^{(n-1)}}\}A\{r^{(n)}\}=0 \qquad \text{or} \qquad \sum_{i=1}^{n}\sum_{j=1}^{n} r_i^{(n-1)} a_{ij} r_j^{(n)} = 0$$

Iterative solution of

$$J_1\{r\}=\omega^2\{r\} \qquad \text{where} \qquad J_1=ET_1$$

then converges to $\{r^{(n-1)}\}$ and $\omega_{(n-1)}^2$.

The next lower modes may be obtained similarly, using other sweeping matrixes embodying additional orthogonality relations in complete analogy to the iteration for lower modes described above.

4.3.4 Approximate Natural Frequencies of Conservative Systems

A conservative system is one which executes free oscillations without dissipating energy. The potential energy $\hat{U}$ that the system has at an instant when its velocity (and hence its kinetic energy) is zero must therefore be exactly equal to the kinetic energy $\hat{T}$ of the system when it occupies its equilibrium position (zero potential energy) during its oscillation.

For sinusoidal oscillations the (generalized) coordinates obey $q_j = \bar{q}_j \sin \omega t$ where $\bar{q}_j$ is a constant (i.e., the amplitude of q_j). For linear holonomic systems, in view of Eq. (4.39),

$$\hat{T} = \tfrac{1}{2}\omega^2 \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\bar{q}_i\bar{q}_j \qquad \hat{U} = \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\bar{q}_i\bar{q}_j$$

Rayleigh's Quotient. Rayleigh's quotient RQ, defined as[64]

$$RQ = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\bar{q}_i\bar{q}_j}{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\bar{q}_i\bar{q}_j} \tag{4.54}$$

is a function of $\{\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_n\}$. However, multiplication of each $\bar{q}_i$ by the same number does not change the value of RQ. Rayleigh's quotient has the following properties:

1. The value of RQ one obtains with any $\{\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_n\}$ always equals or exceeds the square of the lowest natural frequency of the system; $RQ \geq \omega_1^2$.
2. $RQ = \omega_n^2$ if $\{\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_n\}$ corresponds to the nth mode shape (eigenvector) of the system, but even fairly rough approximations to the eigenvector generally result in good approximations to ω_n^2.

For systems whose influence coefficients d_{ij} are known, one may substitute an arbitrary vector $\{\bar{q}\} = \{\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_n\}$ into the right-hand side of

$$\{\bar{q}\}_1 = \alpha G\{\bar{q}\}$$

where $G = DA$ is the dynamic matrix of Eq. (4.50) and α is an arbitrary constant. [This relation follows directly from Eq. (4.49).] The resulting vector $\{\bar{q}\}_1$, when substituted into Eq. (4.54), results in a value of RQ which is nearer to ω_1^2 than the value obtained by direct substitution of the arbitrary vector $\{\bar{q}\}$. Often one obtains good results rapidly if one assumes $\{\bar{q}\}$ initially so as to correspond to the deflection of the system due to gravity (i.e., the "static" deflection).

Rayleigh-Ritz Procedure. An alternative method useful for obtaining improved approximations to ω_1^2 from Rayleigh's quotient is the so-called Rayleigh-Ritz procedure. It consists of computing RQ from Eq. (4.54) for a trial vector $\{\bar{q}\}$ made up of a linear combination of arbitrarily selected vectors

$$\{\bar{q}\} = \alpha_1\{\bar{q}\}_1 + \alpha_2\{\bar{q}\}_2 + \cdots$$

then minimizing RQ with respect to the coefficients α of the selected vectors. The

resulting minimum value of RQ is approximately equal to ω_1^2. That is, RQ evaluated so that $\partial RQ/\partial\alpha_1 = \partial RQ/\partial\alpha_2 = \cdots = 0$ is approximately equal to ω_1^2:

$$RQ \approx \omega_1^2$$

Dunkerley's Equation.[61] This equation states that

$$1/\omega_1^2 + 1/\omega_2^2 + \cdots + 1/\omega_n^2 = 1/\Omega_1^2 + 1/\Omega_2^2 + \cdots + 1/\Omega_n^2 \tag{4.55}$$

where ω_i denotes the ith natural frequency of an n-degree-of-freedom system, and Ω_i denotes the natural frequency that the ith inertia element would have if all others were removed from the system. Usually $\omega_n^2 >> \cdots >> \omega_2^2 >> \omega_1^2$, so that the left-hand side of Eq. (4.55) is approximately equal to $1/\omega_1^2$ and Eq. (4.55) may be used directly for estimation of the fundamental frequency ω_1. In many cases the Ω_1 are obtainable almost by inspection, or by use of Table 4.8.

4.3.5 Chain Systems

A chain system is one in which the inertia elements are arranged in series, so that each is directly connected only to the one preceding and the one following it. Shafts carrying a number of disks (or other rotational inertia elements) are the most common example and are discussed in more detail subsequently. Translational chain systems, as sketched in Fig. 4.17, may be treated completely analogously and hence will not be discussed separately.

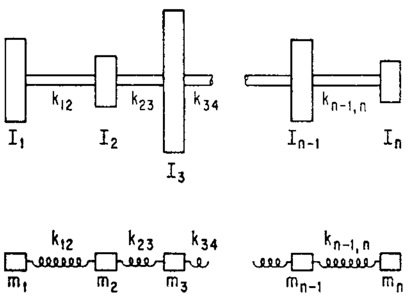


FIG. 4.17 Rotational and translational chain systems.

Sinusoidal Steady-State Forced Motion. If the torque acting on the sth disk is $T_s e^{i\omega t}$, where T_s is a known complex number, then the equations of motion of the system may be written as

$$\begin{aligned}
(k_{12} - I_1\omega^2)\theta_1 - k_{12}\theta_2 &= T_1 \\
-k_{12}\theta_1 + (k_{12} + k_{23} - I_2\omega^2)\theta_2 - k_{23}\theta_3 &= T_2 \\
-k_{23}\theta_2 + (k_{23} + k_{34} - I_3\omega^2)\theta_3 - k_{34}\theta_4 &= T_3 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
-k_{n-1,n}\theta_{n-1} + (k_{n-1,n} - I_n\omega^2)\theta_n &= T_n
\end{aligned} \tag{4.56}$$

where $\theta_s e^{i\omega t}$ describes the angular motion of the sth disk, as measured from equilibrium. (Damping in the system may be taken into account by assigning complex values to the k's, as in the last portion of Sec. 4.2.2.)

One may solve this set of equations simply by using each equation in turn to eliminate one of the θ's, so that one may finally solve for the last remaining θ, then obtain the others by substitution of the determined value into the given equations. This procedure becomes prohibitively tedious if more than a few disks are involved.

If all T's and k's are real (i.e., if the driving torques are in phase and if damping is neglected), one may assume a real value for θ_1, then calculate the corresponding value

of θ_2 from the first of Eqs. (4.56). Then one may find θ_3 from the second equation, θ_4 from the third, and so on. Finally, one may compute T_n from the last (nth) equation and compare it with the given value. This process may be repeated with different initially assumed θ_1 values until the computed value of T_n comes out sufficiently close to the specified one. After a few computations one may often make good use of a plot of computed T_n vs. assumed θ_1 for determining by interpolation or extrapolation a good approximation to the correct value of θ_1.

If all T's are zero, except T_n, one may proceed as before. But, since θ_1 is proportional to T_n in this case, the correct value of θ_1 may be computed directly after a single complete calculation by use of the proportionality

$$\theta_{1,\text{correct}} = \theta_{1,\text{assumed}} (T_{\text{n,specified}}/T_{\text{n,calculated}})$$

The latter approach may be used also for damped systems (i.e., with complex k's).

Natural Frequencies: Holzer's Method. Free oscillations of the system considered obey Eqs. (4.56), but with all $T_s = 0$. To obtain the natural frequencies one may proceed by assuming a value of ω and setting $\theta_1 = 1$, then calculating θ_2 from the first equation, thereafter θ_3 from the second, etc. Finally one may compute T_n from the last equation. If this T_n comes out zero, as required, the assumed frequency is a natural frequency. By repeating this calculation for a number of assumed values of ω one may arrive at a plot of T_n vs. ω, which will aid in the estimation of subsequent trial values of ω. (Natural frequencies are obtained where this curve crosses the ω axis.) One should keep in mind that a system composed of n disks has n natural frequencies. The mode shape (i.e., a set of values of θ's that satisfy the equation of motion) is also obtained in the course of the calculations.

Convenient tabular calculation methods (Holzer tables) may be set up on the basis of the equations of motion Eq. (4.56) rewritten in the following form:

$$\begin{aligned}
I_1\theta_1\omega^2 &= k_{12}(\theta_1 - \theta_2) \\
(I_1\theta_1 + I_2\theta_2)\omega^2 &= k_{23}(\theta_2 - \theta_3) \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
(I_1\theta_1 + I_2\theta_2 + \cdots + I_s\theta_s)\omega^2 &= k_{s,s+1}(\theta_s - \theta_{s+1}) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\sum_{s=1}^{n} (I_s\theta_s)\omega^2 &= 0
\end{aligned} \tag{4.57}$$

Tabular formats are given in a number of texts.[17,61,63,64] Methods for obtaining good first trial values for the lowest natural frequency are also discussed in Refs. 17 and 61. Branched systems, e.g., where several shafts are interconnected by gears, may also be treated by this method.[9,61,63] Damped systems (complex values of k) may also be treated with no added difficulty in principle.[61]

4.3.6 Mechanical Circuits[20,30,50]

Mechanical-circuit theory is developed in direct analogy to electric-circuit theory in order to permit the highly developed electrical-network-analysis methods to be applied to mechanical systems. A mechanical system is considered as made up of a number of mechanical-circuit elements (e.g., masses, springs, force generators) connected in series or parallel, in much the same way that an electrical network is considered

to be made up of a number of interconnected electrical elements (e.g., resistances, capacitances, voltage sources). From known behavior of the elements one may then, by proper combination according to established rules, determine the system responses to given excitations.

Mechanical-circuit concepts are useful for determination of the equations of motion (which may then be solved by classical or transform techniques[20,30]) for analyzing and visualizing the effects of system interconnections, for dealing with electromechanical systems, and for the construction of electrical analogs by means of which one may evaluate the responses by measurement.

Mechanical Impedance and Mobility. As in electric-circuit theory, the sinusoidal steady state is assumed, and complex notation is used in basic mechanical-impedance analysis. That is, forces F and relative velocities V are expressed as

$$F = F_0 e^{i\omega t} \qquad V = V_0 e^{i\omega t}$$

where F_0 and V_0 are complex in the most general case.

Mechanical impedance Z and mobility Y are complex quantities defined by

$$Z = F_0/V_0 \qquad Y = 1/Z = V_0/F_0 \tag{4.58}$$

If the force and velocity refer to an element or system, the corresponding impedance is called the "impedance of the element" or system; if F and V refer to quantities at the same point of a mechanical network, then Z is called the "driving-point impedance" at that point. If F and V refer to different points, the corresponding Z is called the "transfer impedance" between those points.

Mechanical impedances (and mobilities) of elements can be combined exactly like electrical impedances (and admittances), and the driving-point impedances of composites can easily be obtained. From a knowledge of the elemental impedances and of how they combine, one may calculate (and often estimate quickly) the behavior of composite systems.

Basic Impedances, Combination Laws, Analogies. Table 4.2 shows the basic mechanical-circuit elements and their impedances and summarizes the impedances of some simple systems.

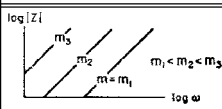
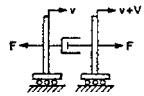
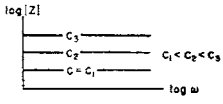
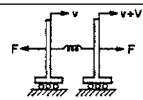
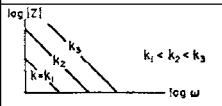
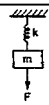
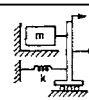
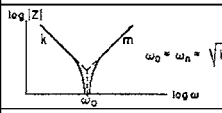
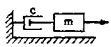
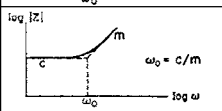
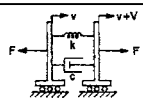
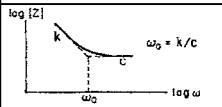
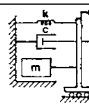
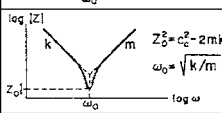
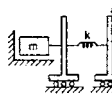
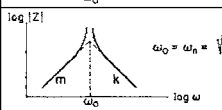
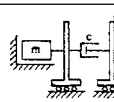
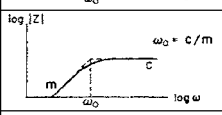
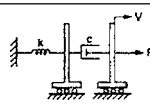
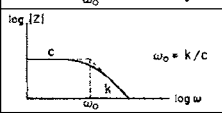
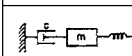
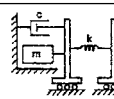
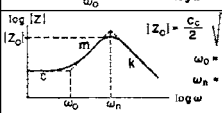
In Table 4.3 are indicated the combination laws for mechanical impedances and mobilities. Table 4.4 is a summary of analogies between translational and rotational mechanical systems and electrical networks. The impedances of some distributed mechanical systems are discussed in Sec. 4.4.3 and summarized in Tables 4.6 and 4.7.

Systems which are a combination of rotational and translational elements or electromechanical systems may be treated as either all-mechanical systems of a single type or as all-electrical systems by suitable substitution of analogous elements and variables.[20] In systems where both rotation and translation of a single mass occur, this single mass may have to be represented by two or more mass elements, and the concept of mutual mass (analogous to mutual inductance) may have to be introduced.[20]

Some Results from Electrical-Network Theory:[20,58] Resonances. Resonances occur at those frequencies for which the system impedances are minimum; antiresonances occur when the impedances are maximum.

Force and Velocity Sources. An ideal velocity generator supplies a prescribed relative velocity amplitude regardless of the force amplitude. A force generator supplies a prescribed force amplitude regardless of the velocity. When a velocity source is "turned off," $V = 0$, it acts like a rigid connection between its terminals. When a force generator is turned off, $F = 0$, it acts like no connection between the terminals.

TABLE 4.2 Mechanical Impedances of Simple Systems

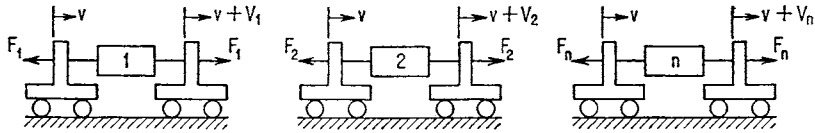
SYSTEM	DIAGRAM	IMPEDANCE Z	FREQUENCY DEPENDENCE
MASS		$i\omega m$	
VISCOUS DASHPOT		c	
SPRING		$k/i\omega$	
DRIVEN MASS ON SPRING	 	$\frac{k}{i\omega} + i\omega m$	
DAMPED MASS	 	$c + i\omega m$	
SPRING AND DASHPOT IN PARALLEL		$c + k/i\omega$	
DAMPED SINGLE DEGREE OF FREEDOM SYSTEM		$i\omega m + c + \frac{k}{i\omega}$	
MASS ON DRIVEN SPRING	 	$\frac{1}{\frac{i\omega}{k} + \frac{1}{i\omega m}}$	
MASS DRIVEN THROUGH DASHPOT	 	$\frac{1}{\frac{1}{c} + \frac{1}{i\omega m}}$	
SPRING AND DASHPOT IN SERIES		$\frac{1}{\frac{1}{c} + \frac{i\omega}{k}}$	
DAMPED MASS DRIVEN THROUGH SPRING	 	$\frac{1}{\frac{1}{c+i\omega m} + \frac{i\omega}{k}}$ $= \frac{c_c\left[\zeta + i\frac{\omega}{\omega_n}\right]}{1-\left(\frac{\omega}{\omega_n}\right)^2 + 2i\zeta\frac{\omega}{\omega_n}}$	

Reciprocity. The transfer impedance Z_{ij} (the force in the jth branch divided by the relative velocity of a generator in the ith branch) is equal to the transfer impedance Z_{ji} (force in the ith branch divided by relative velocity of generator in jth branch).

Foster's Reactance Theorem. For a general undamped system the driving-point impedance can be written $Z = if(\omega)$, where $f(\omega)$ is a real function of (real) ω. The function $f(\omega)$ always has $df/d\omega > 0$ and has a pole or zero at $\omega = 0$ and at $\omega = \infty$. All

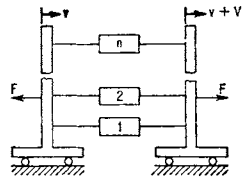
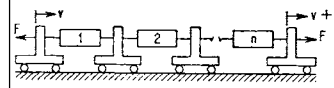
TABLE 4.3 Combination of Impedances and Mobilities

Elements or subsystems:



$Z_1 = 1/Y_1 = F_1/V_1 \qquad Z_2 = 1/Y_2 = F_2/V_2 \qquad \cdots \; Z_n = 1/Y_n = F_n/V_n$

All F, V, Z, Y are complex quantities

Combination in	Parallel	Series
Diagram		
Connection identified by	All components having same V	Same F acting on all components
$Z = F/V =$	$Z_1 + Z_2 + \cdots + Z_n$	$(1/Z_1 + 1/Z_2 + \cdots + 1/Z_n)^{-1}$
$Y = V/F =$	$(1/Y_1 + 1/Y_2 + \cdots + 1/Y_n)^{-1}$	$Y_1 + Y_2 + \cdots + Y_n$
Z's combine like electrical resistances in	Series	Parallel
Y's combine like electrical resistances in	Parallel	Series

poles and zeros are simple (not repeated), poles and zeros alternate (i.e., there is always a zero between two poles), and $f(\omega)$ is determined within a multiplicative factor by its poles and zeros.

Thévenin's and Norton's Theorems. Consider any two terminals of a linear system. Then, as far as the effects of the system at these terminals are concerned, the system may be replaced by (see Fig. 4.18)

1. (Thévenin's equivalent) A series combination of an impedance Z_i and a velocity source V_{oc}
2. (Norton's equivalent) A parallel combination of an impedance Z_i and a force source F_b

Z_i is called the "internal impedance" of the system and is the driving-point impedance obtained at the terminals considered when all sources in the system are turned off (velocity generators replaced by rigid links, force generators replaced by disconnections). V_{oc} is the "open-circuit" velocity, i.e., the velocity occurring between the terminals considered (with all generators active). F_b is the "blocked force," i.e., the force transmitted through a rigid link inserted between the terminals considered, with all generators active.

TABLE 4.4 Mechanical-Electrical Analogies (Lumped Systems)

Mechanical System		Electrical Systems	
Translational	Rotational	Force–voltage (classical) analog	Force–current (mobility) analog
F Force	T Torque	V Voltage	I Current
u Velocity	Ω Angular velocity	I Current	V Voltage
x Displacement	θ Angular displacement	q Charge	ϕ Magnetic flux
m Mass	J Moment of inertia	L Inductance	C Capacitance
c Damping	c_r Rotational damping	R Resistance	$G = \frac{1}{R}$ Conductance
k Spring rate	k_r Torsional spring rate	$S = \frac{1}{C}$ Elastance	$\Gamma = \frac{1}{L}$ Inverse inductance
$Z = \frac{F}{u}$ Mechanical impedance	$Z_r = \frac{T}{\Omega}$ Rotational impedance	$Z = \frac{V}{I}$ Impedance	$Y = \frac{I}{V}$ Admittance
$Y = \frac{u}{F}$ Mobility	$Y_r = \frac{\Omega}{T}$ Rotational mobility	$Y = \frac{I}{V}$ Admittance	$Z = \frac{V}{I}$ Impedance

4.4 CONTINUOUS LINEAR SYSTEMS

4.4.1 Free Vibrations

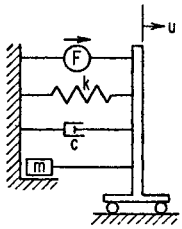
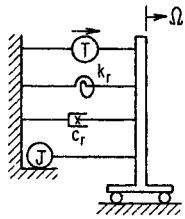
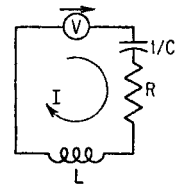
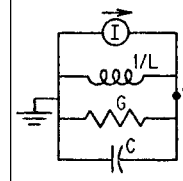
The equations that govern the deflection $u(x, y, t)$ of many continuous linear systems (e.g., bars, shafts, strings, membranes, plates) in absence of external forces may be expressed as

$$\mathfrak{M}\ddot{u} + \mathfrak{K}u = 0 \tag{4.59}$$

where $\mathfrak{M}$ and $\mathfrak{K}$ are linear differential operators involving the coordinate variables only. Table 4.5 lists $\mathfrak{M}$ and $\mathfrak{K}$ for a number of common systems. Solutions of Eq. (4.59) may be expressed in terms of series composed of terms of the form $\phi(x,y)e^{i\omega t}$, where ϕ satisfies

$$(\mathfrak{K} - \mathfrak{M}\omega^2)\phi = 0 \tag{4.60}$$

TABLE 4.4 Mechanical-Electrical Analogies (Lumped Systems) (*Continued*)

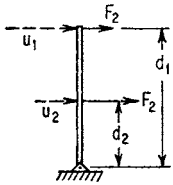
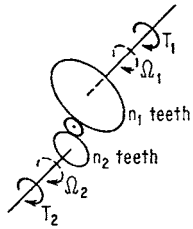
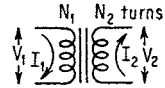
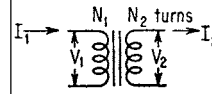
Mechanical Systems		Electrical Systems	
Translational	Rotational	Force-voltage (classical) analog	Force-current (mobility) analog
Kinetic energy = $\frac{1}{2}mu^2$	Kinetic energy = $\frac{1}{2}I\Omega^2$	Magnetic energy = $\frac{1}{2}LI^2$	Electrical energy = $\frac{1}{2}CV^2$
Potential energy = $\frac{1}{2}kx^2$	Potential energy = $\frac{1}{2}k_r\theta^2$	Electrical energy = $\frac{1}{2}\frac{1}{C}q^2$	Magnetic energy = $\frac{1}{2}\frac{1}{L}\phi^2$
Power loss = u^2c	Power loss = $\Omega^2 c_r$	Power loss = I^2R	Power loss = V^2G
			
$m\dot{u} + cu + k\int u dt = F(t)$	$J\dot{\Omega} + c_r\Omega + k_r\int \Omega dt = T(t)$	$L\dot{I} + RI + \frac{1}{C}\int I dt = V(t)$	$C\dot{V} + RV + \frac{1}{L}\int V dt = I(t)$
Elements connected to same node have same velocity u	Elements connected to same node have same angular velocity Ω	Elements in same loop have same current I	Elements connected to same node have same voltage v
Element connected to ground has one end at zero velocity	Element connected to ground has one end at zero velocity	Element in one loop has only one loop current through it	Element connected to ground has one end at zero (reference) voltage
Elements between two nodes	Elements between two nodes	Elements in two loops	Elements between two nodes
Elements in parallel	Elements in parallel	Elements in series	Elements in parallel

in addition to the boundary conditions of a given problem. In solving the differential equation (4.60) by use of standard methods and introducing the boundary conditions applicable in a given case one finds that solutions ϕ that are not identically zero exist only for certain frequencies. These frequencies are called the "natural frequencies of the system"; the equation that the natural frequencies must satisfy for a given system is called the "frequency equation of the system"; the functions ϕ that satisfy Eq. (4.60) in conjunction with the natural frequencies are called the "eigenfunctions" or "mode shapes" of the system. One-dimensional systems (strings, bars) have an infinite number of natural frequencies ω_n and eigenfunctions ω_n; $n = 1, 2, \ldots$. Two-dimensional systems (membranes, plates) have a doubly infinite set of natural frequencies ω_{mn}, and eigenfunctions ϕ_{mn}; $m, n = 1, 2, \ldots$. Table 4.6 lists eigenfunctions and frequency equations for some common systems.

The eigenfunctions of flexural systems where all edges are either free, built-in, or pinned are "orthogonal," that is,

$$\int_0^L \mu(x)\phi_n(x)\phi_{n'}(x)\,dx = \begin{cases} \bar{\mu}L\Phi_n & \text{for } n = n' \\ 0 & \text{for } n \neq n' \end{cases} \tag{4.61}$$

TABLE 4.4 Mechanical-Electrical Analogies (Lumped Systems) (*Continued*)

Mechanical Systems		Electrical Systems	
Translational	Rotational	Force-voltage (classical) analog	Force-current (mobility) analog
Newton's law: sum of forces on node = 0	Newton's law: sum of torques on node = 0	Kirchhoff's voltage law: sum of voltages around loop = 0	Kirchhoff's current law: sum of currents into node = 0
Ideal lever	Ideal gear train	Ideal voltage transformer	Ideal current transformer
			
$\frac{u_1}{u_2} = \frac{F_2}{F_1} = \frac{d_1}{d_2}$	$\frac{\Omega_1}{\Omega_2} = \frac{T_2}{T_1} = \frac{n_2}{n_1}$	$\frac{I_1}{I_2} = \frac{V_2}{V_1} = \frac{N_2}{N_1}$	$\frac{V_1}{V_2} = \frac{I_2}{I_1} = \frac{N_1}{N_2}$

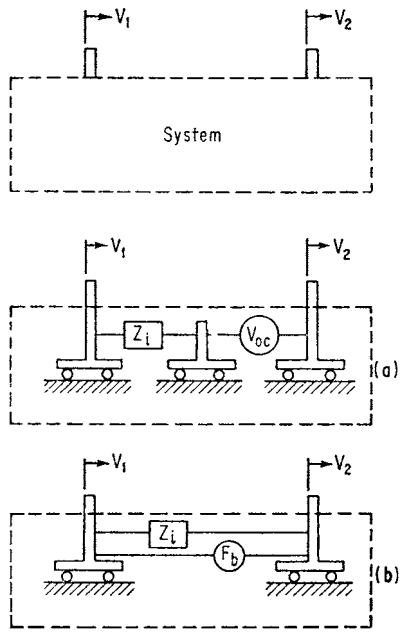


FIG. 4.18 (*a*) Thévenin's and (*b*) Norton's equivalent networks.

TABLE 4.5 Operators [See Eq. (4.59)] for Some Elastic Systems[64]

System	Deflection	$\mathfrak{M} = \mu$	$\mathfrak{K}$
String	Lateral	$\rho A(x)$	$T\dfrac{\partial^2}{\partial x^2}$
Bar	Longitudinal	$\rho A(x)$	$-\dfrac{\partial}{\partial x}\left(EA\dfrac{\partial}{\partial x}\right)$
Bar	Torsional (angular)	$\rho J(x)$	$-\dfrac{\partial}{\partial x}\left(KG\dfrac{\partial}{\partial x}\right)$
Bar	Lateral (flexure)	$\rho A(x)$	$\dfrac{\partial^2}{\partial x^2}\left(EI\dfrac{\partial^2}{\partial x^2}\right)$
Membrane	Lateral	$\rho h(x,y)$	$-S\nabla^2$
Plate	Lateral	$\rho h(x,y)$	$\nabla^2(D\nabla^2)$

Symbols:
- ρ = density
- A = cross-sectional area
- I = centroidal moment of inertia of A
- J = polar moment of inertia of A
- h = plate thickness
- T = tensile force
- S = tension/unit length
- E = Young's modulus
- G = shear modulus
- K = torsional constant of A ($= J$ for circular sections)
- $D = Eh^3/12(1 - \nu^2)$
- ν = Poisson's ratio

TABLE 4.6*a* Modal Properties for Some One-Dimensional Systems

System	Wave velocity c	Boundary conditions $x = 0$	Boundary conditions $x = L$	Mode shape† $\phi_n(x)$	Natural frequency $\omega_n = 2\pi f_n$	$\dfrac{\lambda}{L} = \dfrac{2\pi}{Lk_n}$
String	$\sqrt{T/\rho A}$	Fixed	Fixed	$\sin\dfrac{n\pi x}{L}$	$\dfrac{n\pi}{L}c$	$\dfrac{2}{n}$
Uniform shaft* in torsion	$\sqrt{\dfrac{GK}{\rho J}}$	Clamped	Free	$\sin\dfrac{(2n-1)\pi x}{2L}$	$\dfrac{(2n-1)\pi}{2L}c$	$\dfrac{4}{2n-1}$
		Free	Free	$\cos\dfrac{n\pi x}{L}$	$\dfrac{n\pi}{L}c$	$\dfrac{2}{n}$
Uniform bar* in longitudinal vibration	$\sqrt{E/\rho}$	Clamped	Clamped	$\sin\dfrac{n\pi x}{L}$	$\dfrac{n\pi}{L}c$	$\dfrac{2}{n}$

Symbols:
- A = cross-sectional area
- ρ = material density
- T = tensile force
- E = Young's modulus
- G = shear modulus
- J = polar moment of inertia of A
- K = torsional constant of A ($= J$ for circular sections)
- GK = torsional rigidity
- λ_n = wavelength
- k_n = wave number
- n = mode number

*The same modal properties, but with different values of c, apply for uniform shafts vibrating torsionally and uniform bars vibrating longitudinally.

†$\Phi_n = (1/L)\int_0^L \phi_n^2\,dx = \frac{1}{2}$.

TABLE 4.6*b* Modal Properties for Flexural Vibrations of Uniform Beams[6,7]

Natural frequency $\omega_n = 2\pi f_n = k_n{}^2 c_L r = k_n{}^2 \sqrt{EI/\rho A}$
Wavelength $\lambda_n = 2\pi / k_n$
Wave velocity $c_n = \omega_n / k_n = k_n c_L r = k_n \sqrt{EI/\rho A} = \sqrt{\omega_n c_L r}$
For mode shapes $\phi_n(x)$ listed below, $\Phi_n = (1/L) \int_0^L \phi_n{}^2 \, dx = 1$

Boundary conditions		Mode shape $\phi_n(x)$	σ_n	Frequency equation	Roots of frequency equation					
$x = 0$	$x = L$				k_1L	k_2L	k_3L	k_4L	k_5L	k_nL, $n > 5$
Pinned	Pinned	$\sqrt{2} \sin (k_n x)$		$\sin (k_nL) = 0$	π	2π	3π	4π	5π	$n\pi$
Clamped	Clamped	$\cosh (k_n x) - \cos (k_n x) - \sigma_n[\sinh (k_n x) - \sin (k_n x)]$	$\dfrac{\cosh (k_nL) - \cos (k_nL)}{\sinh (k_nL) - \sin (k_nL)}$	$\cos (k_nL) \cosh (k_nL) = 1$	4.73	7.85	11.00	14.14	17.29	$\approx \dfrac{2n+1}{2}\pi$
Free	Free	$\cosh (k_n x) + \cos (k_n x) - \sigma_n[\sinh (k_n x) + \sin (k_n x)]$								
Clamped	Free	$\cosh (k_n x) - \cos (k_n x) - \sigma_n[\sinh (k_n x) - \sin (k_n x)]$	$\dfrac{\sinh (k_nL) - \sin (k_nL)}{\cosh (k_nL) + \cos (k_nL)}$	$\cos (k_nL) \cosh (k_nL) = -1$	1.875	4.69	7.85	11.00	14.14	$\approx \dfrac{2n-1}{2}\pi$
Clamped	Pinned	$\cosh (k_n x) - \cos (k_n x) - \sigma_n[\sinh (k_n x) - \sin (k_n x)]$	$\cot (k_nL)$	$\tan (k_nL) = \tanh (k_nL)$	3.93	7.07	10.21	13.35	16.49	$\approx \dfrac{4n+1}{4}\pi$
Free	Pinned	$\cosh (k_n x) + \cos (k_n x) - \sigma_n[\sinh (k_n x) - \sin (k_n x)]$								

Symbols:
k_n = wave number
E = Young's modulus
ρ = material density
$c_L = \sqrt{E/\rho}$ = longitudinal wave velocity
A = cross-section area
I = moment of inertia of A
$r = \sqrt{I/A}$ = radius of gyration
L = beam length

TABLE 4.6*c* Modal Properties for Some Plates[40]

Frequency	$\omega_{mn} = 2\pi f_{mn} = k_{mn}^2 \sqrt{D/\rho h} \approx k_{mn}^2 c_L r$	$D = Eh^3/12(1 - \nu^2)$
Wavelength	$\lambda_{mn} = 2\pi/k_{mn}$	$r = h/\sqrt{12}$
Wave velocity	$c_{mn} = \omega_{mn}/k_{mn} = k_{mn}\sqrt{D/\rho h} \approx k_{mn} c_L r$	$c_L = \sqrt{E/\rho}$

Rectangular, on simple supports; $0 < x < a$, $0 < y < b$:

$$\phi_{mn}(x,y) = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad \phi_{mn} = \tfrac{1}{4}$$

$$k_{mn} = \pi\sqrt{m^2/a^2 + n^2/b^2}$$

Circular,* clamped at circumference; $0 < r < a$, $0 < \theta < 2\pi$:

$$\phi_{mn_e}(r,\theta) = \cos(m\theta)F_{mn}(r)$$
$$\phi_{mn_o}(r,\theta) = \sin(m\theta)F_{mn}(r)$$

$$F_{mn}(r) = J_m(k_{mn}r) - \frac{J_m(k_{mn}a)}{I_m(k_{mn}a)} I_m(k_{mn}r)$$

Same orthogonality holds as for membranes (see Table 4.6*d*)
Frequency equation:

$$I_m(k_{mn}a)\left[\frac{d}{dr}J_m(k_{mn}r)\right]_{r=a} = J_m(k_{mn}a)\left[\frac{d}{dr}I_m(k_{mn}r)\right]_{r=a}$$

Table of $k_{mn}a/\pi$ for Clamped Circular Plate

		n			
		1	2	3	>3
m	0	1.015	2.007	3.000	$\approx n + \frac{m}{2}$
	1	1.468	2.483	3.490	
	2	1.879	2.992	4.000	

*For circular plates one finds two sets of ϕ_{mn}, one even (subscript e), one odd (subscript o).

$$\int\int_{A_S} \mu(x,y)\phi_{mn}(x,y)\phi_{m'n'}(x,y)\,dA_s = \begin{cases} \bar{\mu}A_s\Phi_{mn} & \text{for } m = m',\ n = n' \\ 0 & \text{otherwise} \end{cases} \tag{4.61}$$

where L is the total length of a (one-dimensional) system, A_s is the total surface area of a (two-dimensional) system, μ is a weighting function, and $\bar{\mu}$ is the mean of value of μ for the system. (The weighting functions of the common systems tabulated in Table 4.5 are equal to the mass operator $\mathfrak{M}$.)

$$\bar{\mu}L = \int_0^L \mu(x)\,dx \qquad \text{for one-dimensional systems}$$

$$\bar{\mu}A_s = \int\int_{A_s} \mu(x,y)\,dA_s \qquad \text{for two-dimensional systems}$$

For uniform systems $\mu = \bar{\mu}$ is a constant and may be canceled from Eqs. (4.61).*

*Note that for the systems of Table 4.6 $\mathfrak{M} = \mu$ is a multiplicative factor, not an operator in the more general sense. However, for other systems $\mathfrak{M}$ may be a more general operator (e.g., for lateral vibrations of beams when rotatory inertia is not neglected) and may differ from μ. The distinction between $\mathfrak{M}$ and μ is maintained throughout the subsequent discussion to permit application of the results to systems that are more complicated than those of Table 4.6.

TABLE 4.6*d* Modal Properties for Some Membranes[40]

Wave velocity $c = \sqrt{S/\rho h}$

Rectangular; $0 < x < a$, $0 < y < b$:

$$\phi_{mn}(x,y) = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad \phi_{mn} = \tfrac{1}{4}$$

$$\omega_{mn} = 2\pi f_{mn} = \pi c\sqrt{(m/a)^2 + (n/b)^2} \qquad k_{mn} = 2\pi/\lambda_{mn} = \omega_{mn}/c$$

Circular,* $0 < r < a$, $0 < \theta < 2\pi$:

$$\phi_{mn_e}(r,\theta) = \cos(m\theta)J_n(\omega_{mn}r/c)$$
$$\phi_{mn_o}(r,\theta) = \sin(m\theta)J_n(\omega_{mn}r/c)$$

$$\iint_{A_s}\phi_{mn_i}\phi_{m'n'_{j'}}\,dA_s = \begin{cases}\pi a^2\Phi_{mn} & \text{for } m' = m,\ n' = n,\ j' = j\\ 0 & \text{otherwise}\end{cases}$$

$$\Phi_{mn} = \begin{cases}[J_1(\omega_{mn}a/c)]^2 & \text{for } m = 0\\ \tfrac{1}{2}[J_{m-1}(\omega_{mn}a/c)]^2 & \text{for } m > 0\end{cases}$$

Frequency equation:

$$J_m(\omega a/c) = 0$$

Table of $\omega_{mn}a/\pi c$ for Circular Membrane

		n			
		1	2	3	>3
m	0	0.7655	1.7571	2.7546	$\approx n + \frac{m}{2} - \frac{1}{4}$
	1	1.2197	2.2330	3.2383	
	2	1.6347	2.6793	3.6987	

Table of Φ_{mn} for Circular Membrane

		n			
		1	2	3	4
m	0	0.2695	0.1241	0.07371	0.05404
	1	0.08112	0.04503	0.02534	0.02385
	2	0.05770	0.03682	0.02701	0.02134

Symbols for Table 4.6*c* and *d*:

- S = tension force/unit length
- μ = mass/unit area
- D = flexural rigidity = $Eh^3/12(1-\nu^2)$
- k_{mn} = wave number
- E = Young's modulus
- ν = Poisson's ratio
- c_L = longitudinal wave velocity = $\sqrt{E/\rho}$
- r = radius of gyration of section
- h = plate thickness
- ρ = material density
- J_m = mth-order Bessel function
- I_m = mth-order hyperbolic Bessel function

*For circular membranes one finds two set of ϕ_{mn}, one even (subscript e), one odd (subscript o).

The free motions (or "transient" responses) of two-dimensional* systems with initial displacement $u(x, y, 0)$ from equilibrium and initial velocity $\dot{u}(x, y, 0)$ are given by

$$u_t(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}(B_{mn}\cos\omega_{mn}t + C_{mn}\sin\omega_{mn}t)$$

$$B_{mn} = \frac{1}{\bar{\mu}A_s\Phi_{mn}}\iint_{A_s} u(x,y,0)\mu(x,y)\phi_{mn}(x,y)\,dA_s \tag{4.62}$$

$$C_{mn} = \frac{1}{\bar{\mu}A_s\Phi_{mn}\omega_{mn}}\iint_{A_s} \dot{u}(x,y,0)\mu(x,y)\phi_{mn}(x,y)\,dA_s$$

4.4.2 Forced Vibrations

Forced vibrations of distributed systems are governed by

$$\mathfrak{M}\ddot{u} + \mathfrak{K}u = F(x,y,t)$$

where the right-hand side represents the applied load distribution in space and time.

General Response: Modal Displacements, Forcing Functions, Masses. The displacement $u(x, y, t)$ of a system† may, in general, be expressed in a modal series as

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} U_{mn}(t)\phi_{mn}(x,y) \tag{4.63}$$

where the modal displacement $U_{mn}(t)$ is given by

$$U_{mn}(t) = \frac{1}{\omega_{mn}M_{mn}}\int_0^t G_{mn}(\tau)\sin\omega_{mn}(t-\tau)\,d\tau \tag{4.64}$$

The modal forcing function $G_{mn}(t)$ and the modal mass M_{mn} are given by

$$G_{mn}(t) = \iint_{A_s} F(x,y,t)\phi_{mn}(x,y)\,dA_s \qquad M_{mn} = \iint_{A_s}\phi_{mn}(x,y)\mathfrak{M}\phi_{mn}(x,y)\,dA_s \tag{4.65}$$

For the special case of a uniform system with $\mathfrak{M} = \mu = \bar{\mu} =$ constant,

$$M_{mn} = \mu A_s\Phi_{mn}$$

where Φ_{mn} is defined as in Eq. (4.61).

The "steady-state" response given by (4.63) and (4.64) must be combined with the "transient" response given by (4.62) if one desires the complete solution.

Sinusoidal Response: Input Impedance. For a sinusoidal forcing function $F(x, y, t) = e^{i\omega t}F_0(x, y)$ one finds a steady-state response given by

$$u_{ss}(x,y,t) = e^{i\omega t}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} U_{mn}\phi_{mn}(x,y)$$

*These equations apply also for one-dimensional systems if all m subscripts and y dependences are deleted, if A_s is replaced by L, and if the double integrations over A_s are replaced by a single integration for Q to L.

†These equations apply also for one-dimensional systems if all m subscripts and y dependences are deleted, if A_s is replaced by L, and if the double integrations over A_s are replaced by a single integration from 0 to L.

with $$U_{mn} = \frac{G_{mn}}{(\omega_{mn}{}^2 - \omega^2)M_{mn}} \qquad G_{mn} = \int\int_{A_s} F_0(x,y)\phi_{mn}(x,y)\, dA_s$$

and M_{mn} given by (4.65).

For a point force $F_1 e^{i\omega t}$ applied at $x = x_0$, $y = y_0$, one finds $G_{mn} = F_1\phi_{mn}(x_0, y_0)$. The input impedance $Z(x_0, y_0, \omega)$ of the system at $x = x_0$, $y = y_0$ for frequency ω then may be found from

$$\frac{1}{i\omega Z(x_0,y_0,\omega)} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\phi_{mn}^2(x_0,y_0)}{(\omega_{mn}^2 - \omega^2)M_{mn}} \tag{4.66}$$

Point-Impulse Response: Alternate Formulation of General Response. The response of a system to a point force applied at $x = x_0$, $y = y_0$ and varying like a Dirac impulse function (of unit magnitude) with time is given by

$$u_\delta(x,y;x_0,y_0;t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\phi_{mn}(x_0,y_0)\phi_{mn}(x,y)}{\omega_{mn}M_{mn}} \sin \omega_{mn}t$$

An alternate expression for the steady-state response of a system to a general distributed force $F(x, y, t)$ may then be written as

$$u_{ss}(x,y,t) = \int\int_{A_s}\left[\int_0^t F(x_0,y_0,\tau)u_\delta(x,y;x_0,y_0;t-\tau)\, d\tau\right] dx_0\, dy_0$$

This expression is entirely analogous to the result one obtains by combining Eqs. (4.63) and (4.64).

4.4.3 Approximation Methods

Finite-Difference Equations, Finite-Element Approximations. One of the most widely applicable numerical methods, particularly if digital-computation equipment is available, consists of replacing the applicable differential equations by finite-difference equations[12,64] which may then be solved numerically.

A second method consists of replacing the continuous (infinite-degrees-of-freedom) system by one made up of a finite number of suitably interconnected elements (masses, springs, dashpots), then applying methods developed for systems with a finite number of degrees of freedom, as outlined in Sec. 4.3. These "finite-element" approximations may be obtained, for example, by dividing a beam or plate to be analyzed into arbitrary segments and assuming the mass of each segment concentrated at its center of gravity or "lumping point." If one then establishes the influence coefficients between the various lumping points, one has enough information to apply directly the methods outlined in Sec. 4.3.[24] In concept, one replaces the structure between lumping points by equivalent springs to obtain a new system analogous to the continuous one; a beam is replaced by a linear array, a plate by a two-dimensional network of masses interconnected by springs. One may then proceed by determining the equations of motion of the new systems and by solving these as discussed in Sec. 4.3. A variety of corresponding computer codes has become available; e.g., see Refs. 44 and 45.

Fundamental Frequencies:* Rayleigh's Quotient. *RQ* for an arbitrary deflection function $u(x, y)$ of a two-dimensional continuous system is given by[64]

*These equations apply also for one-dimensional systems if all m subscripts and y dependences are deleted, if A_s is replaced by L, and if the double integrations over A_s are replaced by a single integration from 0 to L.

$$RQ = \frac{\iint_{A_s} u\mathfrak{K}(u)\, dA_s}{\iint_{A_s} u\mathfrak{M}(u)\, dA_s}$$

For $u = \phi_{mn}$ (the mn mode shape) RQ takes on the value ω_{mn}^2. For any function u that satisfies the boundary conditions of the given system

$$RQ \geq \omega_{11}^2$$

Thus RQ produces an estimate of the fundamental frequency ω_{11} which is always too high.

To obtain a better estimate one may use the Rayleigh-Ritz procedure. In this procedure one forms RQ for a linear combination of any convenient number of functions $u_i(x, y)$ that satisfy the boundary conditions of the problem; that is, one forms RQ from

$$u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots$$

where the α are constants. A good approximation to ω_{11}^2 is then obtained, in general, by minimizing RQ with respect to the various α's.

$$RQ \text{ (evaluated so that } \partial RQ/\partial\alpha_1 = \partial RQ/\partial\alpha_2 = \cdots = 0) \approx \omega_{11}^2$$

Special Methods for Lateral Vibrations of Nonuniform Beams. In addition to the foregoing methods a number of others are available that have been developed specially for dealing with the vibrations of beams and shafts. In all these methods the beam mass is replaced by a number of masses concentrated at lumping points.

The Stodola method[53,59,61,63] is related to both the Rayleigh and matrix-iteration methods. In it one may proceed as follows:

1. Assume a deflection curve that satisfies the boundary conditions. (Usually the static-deflection curve gives good results.)
2. Determine a first approximation to ω_1 using Rayleigh's quotient, from

$$\omega_1^2 \approx RQ = \frac{g\Sigma m_i u_i}{\Sigma m_i u_i^2}$$

   where g denotes the acceleration of gravity and u_i the assumed deflection of the mass m_i.
3. Calculate the deflection of the beam as if inertia forces $(-m_i u_i \omega_1^2)$ were applied statically. (This is usually done best by graphical or numerical means.) Use these new deflections instead of the original u_i in the foregoing equation. Repeat this process until no further changes in RQ result to the degree of accuracy desired. Then $RQ = \omega_1^2$ to within the desired accuracy.

The Myklestad method[41] is essentially the same as Holzer's method, but considerably simpler to use for flexural vibrations. Extensions of Myklestad's original method also apply to coupled bending-torsion vibrations and to vibrations in centrifugal fields.[41] Some simplifications of Myklestad's method have been developed by Thomson.[61] Because of the details necessary for a sufficient discussion of these procedures the reader is referred to the original sources.

4.4.4 Systems of Infinite Extent

Truly infinite or semi-infinite systems do not occur in reality. However, as far as the local response to a local excitation is concerned, finite systems behave like infinite ones if the ends are far (many wavelengths) removed from the excitation and if there is enough dissipation in the system or at the ends so that little effect of reflected waves is felt near the driving point.

The velocities with which waves travel in infinite systems are listed under the heading of "wave velocity" in Table 4.6, but in infinite systems the frequencies and wave numbers are not restricted, as they are in finite ones. In all cases the wave velocity c is related to wavelength λ, wave number k, frequency f (cycles/time), and circular frequency ω (radians/time) as

$$c = f\lambda = \omega\lambda/2\pi = \omega/k$$

Input impedances of infinite structures are useful for estimation of the responses of mechanical systems that are composed of or connected to one or more structures, if the responses of the latter may be approximated by those of corresponding infinite structures in the light of the first paragraph of this article. These impedances may be used precisely like previously discussed impedances of systems with only a few degrees of freedom.

If a force $F_0e^{i\omega t}$ gives rise to a velocity $V_0e^{i\omega t}$ at its point of application (where F_0 and V_0 may be complex in general), then the driving-point impedance is defined as $Z = F_0/V_0$. Similarly, the driving-point moment impedance is defined by $Z_M = M_0/\Omega_0$, where M_0 denotes the amplitude of a driving moment $M_0e^{i\omega t}$ and where $\Omega_0e^{i\omega t}$ is the angular velocity at the driving point. Table 4.7 lists the driving-point impedances of some infinite and semi-infinite systems.

4.5 MECHANICAL SHOCKS

By a mechanical shock one generally means a relatively suddenly applied transient force or support acceleration. The responses of mechanical systems to shocks may be computed by direct application of the previously discussed methods for determination of transient responses, provided that the forcing functions are known. If the forcing functions are not known precisely, one may approximate them by some idealized functions or else describe them in some rough way, for example, in terms of the subsequently discussed shock spectra.

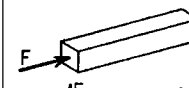
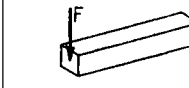
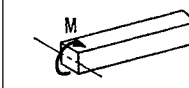
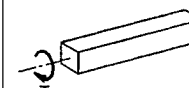
4.5.1 Idealized Forcing Functions

Among the most widely studied idealized shocks are those associated with sudden support displacements or velocity changes, or with suddenly applied forces. The responses of systems with one or two degrees of freedom to idealized shocks have been studied in considerable detail, since for such systems solutions may be obtained relatively simply by analytical or analog means.

Sudden Support Displacement. Sudden (vertical) support displacements occur, for example, when an automobile hits a sudden change in level of the roadbed. The responses of simple systems to such shocks have been studied in considerable detail. Families of

TABLE 4.7 Driving-Point Impedances of Some Infinite Uniform Systems*

Beams

Loading	Extent: Infinite	Extent: Semi-infinite	
Axial force	$Z = 2A\rho c_L$	$Z = A\rho c_L$	
Lateral force	$Z = 2mc_B(1 + i)$	$Z = \frac{mc_B}{2}(1 + i)$	
Moment	$Z_M = \frac{2mc_B}{k^2}(1 + i)$	$Z_M = \frac{mc_B}{2k^2}(1 + i)$	
Torsion	$Z_M = 2J\rho c_T$	$Z_M = J\rho c_T$	

Symbols:

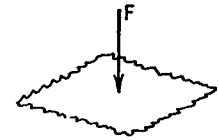
A = cross-section area
ρ = material density
E = Young's modulus
J = polar moment of inertia of A
K = torsional constant
G = shear modulus
$r = \sqrt{I/A}$

$c_L = \sqrt{E/\rho}$ = longitudinal wave velocity
$c_T = \sqrt{GK/\rho J}$ = torsional wave velocity
$c_B = \sqrt{\omega r c_L}$ = flexural wave velocity
$k = \sqrt{\omega/rc_L}$ = flexural wave number
$\frac{c_B}{k^2} = \sqrt{\frac{(rc_L)^3}{\omega}}$

Plates

Infinite isotropic: $Z = 8\sqrt{D\rho h} \approx 2.3h^2\sqrt{\frac{E\rho}{1-\nu^2}}$

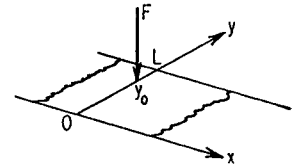
Infinite orthotropic: $Z \approx 8\sqrt[4]{D_x D_y}\sqrt{\rho h}$



Isotropic, infinite in x direction, on simple supports at $y = 0, L$, forced at $(0, y_0)$:

$$\frac{1}{Z} = \frac{i}{2\sqrt{D\rho h}} \sum_{n=1}^{\infty} [(n^2\pi^2 - K)^{-1/2} - (n^2\pi^2 + K)^{-1/2}] \sin^2\left(\frac{n\pi y_0}{L}\right)$$

$$K = (kL)^2 = L^2\omega\sqrt{\frac{\rho h}{D}}$$



Symbols:

ρ = density of plate material
h = plate thickness
E = Young's modulus of plate material
E_x, E_y = Young's modulus in principal directions
ν = Poisson's ratio of plate material
ν_x, ν_y = Poisson's ratio in principal directions

$k = \sqrt[4]{\omega^2\rho h/D}$
$D = Eh^3/12(1 - \nu^2)$
$D_x = E_x h^3/12(1 - \nu_x^2)$
$D_y = E_y h^3/12(1 - \nu_y^2)$

*Systems are assumed undamped. For finite systems see Eq. (4.66) and Table 4.6.

curves describing system responses to a certain type of rounded step-function displacement are given in Refs. 19, 48, and 52. It is found among other things that the responses are generally of an oscillatory nature, except for peaks in acceleration that occur during the "rise" time of the displacement.

Sudden Support Velocity Change. Velocity shocks, i.e., those associated with instantaneous velocity changes, have been studied considerably. They approximate a number of physical situations where impact occurs; for example, when a piece of equipment is dropped the velocity of its outer parts changes from some finite value to zero at the instant these parts hit the ground. Reference 39 discusses in detail how one may compute the velocity shock responses of single-degree-of-freedom systems with linear and various nonlinear springs. It also presents charts for the computation of the important parameters of responses of simple systems attached to the aforementioned velocity-shock-excited systems. That is, if one system is mounted inside another, one may first compute the motion of the outer system in response to a velocity shock, then the response of the inner system to this motion (assuming the motion of the inner system has little effect on the shock response of the outer). This permits one to estimate, for example, what happens to an electronic component (inner system) when a chassis containing it (outer system) is dropped. Damping effects are generally neglected in Ref. 39, but detailed curves for lightly damped linear two-degree-of-freedom systems appear in Refs. 43 and 46. Some discussion appears also in Ref. 14.

Suddenly Applied Forces. If a force is suddenly applied to a linear system with a single degree of freedom, then this force causes at most twice the displacement and twice the stress that this same force would cause if it were applied statically. This rule of thumb can lead to considerable error for systems with more than one degree of freedom.[46] One generally does well to carry out the necessary calculations in detail for such systems.

4.5.2 Shock Spectra

The shock-spectrum concept is useful for describing shocks and the responses of simple systems exposed to them, particularly where the shocks cannot be described precisely and where they cannot be reasonably approximated by one of the idealized shocks of Sec. 4.5.1.

Physical Interpretation. Assume that a given (support acceleration) shock is applied to an undamped linear single-degree-of-freedom system whose natural frequency is f_1. The maximum displacement of the system (relative to its supports) in response to this shock then gives one point on a plot of displacement vs. frequency. Repetition with the same shock applied to systems with different natural frequencies gives more points which, when joined in a curve, make up the "displacement shock spectrum" corresponding to the given shock. If instead of the maximum displacement one had noted the maximum velocity or acceleration for each test system, one would have obtained the velocity or acceleration shock spectrum.

Definitions.[1,19] If a force $F(t)$ is applied to an undamped single-degree-of-freedom system* with natural frequency ω and mass m, then the displacement, velocity, and

*The shock spectra defined here are essentially descriptions of the shock in the frequency domain. Shock spectra are related to the Fourier transforms, the latter being lower bounds to the former.[66] Inclusion of damping appears unnecessary for purposes of describing the shock, but most authors[1,19] include damping in their definitions since they tend to be more concerned with descriptions of system responses to shocks than with descriptions of shocks. The term "shock spectra of a system" is often applied to descriptions of responses of a specified system to specified shocks.[62]

acceleration of the system (assumed to be initially at rest and at equilibrium) are given by

$$x(t, \omega) = \frac{1}{m\omega} \int_0^t F(\tau) \sin \omega(t - \tau)\, d\tau$$

$$\dot{x}(t, \omega) = \frac{1}{m} \int_0^t F(\tau) \cos \omega(t - \tau)\, d\tau$$

$$\ddot{x}(t, \omega) = F(t)/m - \omega^2 x(t, \omega)$$

The various shock spectra of the force shock $F(t)$ are then defined as follows:

$$\begin{aligned} D'_+(\omega) &= \text{timewise maximum of } x(t, \omega) \\ &= \text{positive-displacement spectrum} \\ D'_-(\omega) &= \text{timewise minimum of } x(t, \omega) \\ &= \text{negative-displacement spectrum} \\ D'(\omega) &= \text{timewise maximum of } |x(t, \omega)| \\ &= \text{displacement spectrum} \end{aligned}$$

Similarly, for example,

$$\begin{aligned} V'(\omega) &= \text{timewise maximum of } |\dot{x}(t, \omega)| \\ &= \text{velocity spectrum} \\ A'(\omega) &= \text{timewise maximum of } |\ddot{x}(t, \omega)| \\ &= \text{acceleration spectrum} \end{aligned}$$

Similarly, if the previously discussed test system is exposed to a support acceleration $\ddot{s}(t)$, then the displacement, velocity, and acceleration of the system relative to its supports are given by

$$y(t, \omega) = -\frac{1}{\omega} \int_0^t \ddot{s}(\tau) \sin \omega(t - \tau)\, d\tau = x - s$$

$$\dot{y}(t, \omega) = -\int_0^t \ddot{s}(\tau) \cos \omega(t - \tau)\, d\tau = \dot{x} - \dot{s}$$

$$\ddot{y}(t, \omega) = -\ddot{s}(t) = \omega^2 y(t, \omega) = \ddot{x} - \ddot{s}$$

where $y = x - s$ denotes the relative displacement, x the absolute displacement of the mass, and s the displacement of the support.

The various shock spectra of the acceleration shock $\ddot{s}(t)$ are defined as follows:

$$\begin{aligned} D(\omega) &= \text{timewise maximum of } |y(t, \omega)| \\ &= \text{relative-displacement spectrum} \\ V(\omega) &= \text{timewise maximum of } |\dot{y}(t, \omega)| \\ &= \text{relative-velocity spectrum} \\ A(\omega) &= \text{timewise maximum of } |\ddot{y}(t, \omega)| \\ &= \text{relative-acceleration spectrum} \\ D_a(\omega) &= \text{timewise maximum of } |x(t, \omega)| \\ &= \text{absolute-displacement spectrum} \end{aligned}$$

$$
\begin{aligned}
V_a(\omega) &= \text{timewise maximum of } |\dot{x}(t, \omega)| \\
&= \text{absolute-velocity spectrum} \\
A_a(\omega) &= \text{timewise maximum of } |\ddot{x}(t, \omega)| \\
&= \text{absolute-acceleration spectrum} \\
\tilde{V}(\omega) &= \omega D(\omega) = \text{pseudo-velocity spectrpum}
\end{aligned}
$$

For any $s(t)$ it is true that

$$A_a(\omega) = \omega^2 D(\omega)$$

but $\tilde{V}(\omega) \neq V(\omega)$, $\tilde{V}(\omega) \neq V_a(\omega)$ in general.

One often distinguishes also between system responses during the action of a shock and responses after the shock action has ceased. The spectra associated with system motion during the shock action are called "primary spectra"; those associated with system motion after the shock are called "residual spectra."

Shock spectra are generally reduced to dimensionless "amplification spectra," e.g., by dividing $D'(\omega)$ by the static displacement due to the maximum value of $F(t)$, or $A(\omega)$ by the maximum value of $\ddot{s}(t)$. Frequency is also usually reduced to dimensionless form by division by some suitable frequencylike parameter.

Simple Shocks. A simple shock is one like that sketched in Fig. 4.19; it is generally nonoscillatory, has a unique absolute maximum reached within a finite "rise time" t_m, and a finite duration t_0. Spectra of simple shocks are usually presented in terms of ft_m or ωt_m (and/or ft_0 or ωt_0) or simple multiples of these dimensionless quantities.

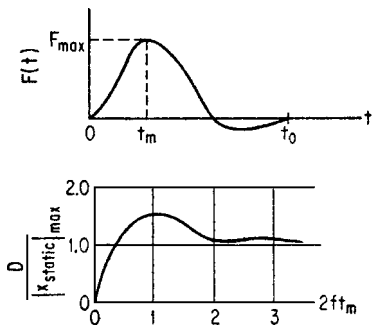


FIG. 4.19 Typical simple shock and dimensionless amplification spectrum.

Shock spectra may, by virtue of their definitions, be used directly to determine the maximum responses of single-degree-of-freedom systems to the shocks to which the spectra pertain.

The main utility of the shock-spectrum concept, however, is due to the fact that the amplification spectra of roughly similar shocks, when presented in terms of the foregoing dimensionless-frequency parameters, tend to coincide very nearly; i.e., amplification spectra are relatively insensitive to details of the pulse form. (Good coincidence of spectra of similar shocks is obtained for small values of f if the spectra are plotted against ft_0, for large values of f if spectra are plotted against ft_m.)

The effect of damping is to reduce system responses in general. The greatest reduction usually occurs near the peaks of the amplification spectra; near relative minima of these spectra the reduction due to damping is generally small.

The amplification spectra of simple shocks have the following properties:

They pass through the origin.

They do not exceed 2.0.

They approach 1.0 for large ft_m.

They generally reach their maxima between $2\ \text{ft}_m = 0.7$ and 2.0, but most often near $2\ \text{ft}_m = 1$.

Simple shocks occur in drop tests, aircraft landing impact, hammer impact, gun recoil, explosion blasts, and ground shock due to explosive detonations. Details of many spectra appear in Refs. 1, 19, 64, and 72.

4.6 DESIGN CONSIDERATIONS

4.6.1 Design Approach

Since vibration is generally considered an undesirable side effect, it seldom controls the primary design of a machine or structure. Items usually are designed first to fulfill their main function, then analyzed from a vibration viewpoint in regard to possible equipment damage or malfunction, structural fatigue failure, noise, or human discomfort or annoyance.

The most severe effects of vibration generally occur at resonance; therefore, one usually is concerned first with determination of the resonance frequencies of the preliminary design. (Damping is usually neglected in the pertinent calculations for all but the simplest systems, unless a prominent damping effect is anticipated.) If resonance frequencies are found to lie within the intended range of driving frequencies, one should attempt a redesign to shift the resonances out of the driving-frequency range.

If resonances cannot be avoided reasonably, the designer must determine the severity of these resonances. Damping must then be considered in the pertinent calculations, since it is primarily damping that limits response at resonance. If resonant responses are too severe, one must reduce the excitation and/or incorporate increased damping in the system or structure.[69]

Shifting of Resonances. If resonances are found to occur within the range of excitation frequencies, one should try to redesign the system or structure to change its resonance frequencies. Added stiffness with little addition of mass results in shifting of the resonances to higher frequencies. Added mass with little addition of stiffness results in lowering of the resonance frequencies. Addition of damping generally has little effect on the resonance frequencies.

In cases where the vibrating system cannot be modified satisfactorily, one may avoid resonance effects by not operating at excitation frequencies where resonances are excited. This may be accomplished by automatic controls (e.g., speed controls on a machine) or by prescribing limitations on use of the system (e.g., "red lines" on engine tachometers to show operating speeds to be avoided).

Evaluation of Severity of Resonances. The displacement amplitude X_0 of a linear single-degree-of-freedom system (whose natural frequency is ω_n), excited at resonance by a sinusoidal force $F_0 \sin \omega_n t$, is given by

$$kX_0/F_0 = 1/2\zeta = Q = 1/\eta$$

where ζ is the ratio of damping to critical damping, Q the "quality factor," and η the loss factor of the system. The velocity amplitude V_0 and acceleration amplitude A_0 are given by

$$A_0 = \omega_n V_0 = \omega_n^2 X_0$$

The maximum force exerted by the system's spring (of stiffness k) is

$$F_s = kX_0$$

The maximum spring stress may readily be calculated from the foregoing spring force. Multiple-degree-of-freedom systems generally require detailed analysis in accordance with methods outlined in Sec. 4.3. The previous expressions pertaining to single-degree-of-freedom systems hold also for systems with a number of degrees of freedom if X_0 is taken as the amplitude of a generalized (principal) coordinate that is independent of the other coordinates, and if F_0 is taken as the corresponding generalized force. Then $\eta = 2\zeta = Q^{-1}$ must describe the effective damping for that coordinate.

Stresses in the various members may be determined from the mode shapes (vectors) corresponding to the resonant mode; i.e., from the maximum displacements or forces to which the elements are subjected.

For uniform distributed systems* the modal displacement U_{mn} at resonance of the m, n mode due to a modal excitation $G_{mn} \sin \omega_{mn} t$ is given by

$$|U_{mn}| = |G_{mn}|/M_{mn}\omega_{mn}^2\eta(\omega_{mn})$$

and is related to modal velocity and acceleration amplitudes $\dot{U}_{mn}$, $\ddot{U}_{mn}$, according to

$$|\ddot{U}_{m,n}| = \omega_{mn}^2|U_{mn}| = \omega_{mn}|\dot{U}_{mn}|$$

$\eta(\omega)$ denotes the system loss factor, which varies with frequency depending on the damping mechanism present. For beams and plates in flexure[64] η is related to the usual viscous-damping coefficient c and flexural rigidity D as†

$$\eta(\omega) = c\omega/D$$

The maximum stress $\sigma_{\max}$ and maximum strain $\epsilon_{\max}$ that occur at resonance in bending of beams and plates may be approximated by

$$\sigma_{\max} \approx (C/r)|U_{mn}|\omega_{mn}\rho c_{Ln} g_{mn} \approx \epsilon_{\max}\rho c_L^2$$

where $c_L = \sqrt{E/\rho}$ denotes the velocity of sound in the material (E is Young's modulus, ρ the material density) and C denotes the distance from the neutral to the outermost fiber, r the radius of gyration of the cross section. For beams with rectangular cross sections and for plates, $C/r = \sqrt{3}$. For plates $M_{mn} = \rho h$; for beams $M_m = \rho A$, where h denotes plate thickness, A beam cross-section area.

The factor g_{mn} accounts for boundary conditions and takes on the following approximate values:

Boundary conditions on pair of opposite edges	g_{mn}
Pinned-pinned or clamped-free	1.00
Clamped-clamped or clamped-pinned	1.33
Free-free or pinned-free	0.80

For conservative design if the boundary conditions are now known one should use $g_{mn} = 1.33$. Otherwise, one should use the largest value of g_{mn} pertinent to the existing boundary conditions.

Reduction of Severity of Resonances. If redesign to avoid resonances is not feasible, one has only two means for reducing resonant amplitudes: (1) reduction of modal excitation and (2) increase of damping.

*The summary here is presented for two-dimensional systems (plates), and double subscripts are used for the various modal parameters. However, the identical expressions apply also for one-dimensional systems (beams, strings) if the n subscripts are deleted. See also Sec. 4.4.

†For plates $D = Eh^3/12(1 - \nu^2)$; for beams $D = EI$.

Excitation reduction may take the form of running a machine at reduced power, isolating the resonating system from the source of excitation, or shielding the system from exciting pressures. Increased damping may be obtained by addition of energy-dissipating devices or structures. For example, one might use metals with high internal damping for the primary structure, or else attach coatings or sandwich media with large energy-dissipation capacities to a primary structure of common materials.[51,65] Alternately, one might rely on structural joints or shaft bearings to absorb energy by friction, or else attempt to extract energy by means of viscous friction or acoustic radiation in fluids in contact with the resonant system. Simple dashpots, localized friction pads, or magnetically actuated eddy-current-damping devices may also be used, particularly for systems with a few degrees of freedom; however, such localized devices may not be effective for higher modes of distributed systems.

4.6.2 Source-Path-Receiver Concept

If one is called upon to analyze or modify an existing or projected system from the vibration viewpoint, one may find it useful to examine the system in regard to

1. Sources of vibration (e.g., reciprocating engines, unbalanced rotating masses, fluctuating air pressures)
2. Paths connecting sources to critical items (e.g., substructures, vibration mounts)
3. Receivers of vibratory energy, i.e., critical items that malfunction if exposed to too much vibration (e.g., electronic components)

The problem of limiting the effects of vibration may then be attacked by any or all of the following means:

1. Vibration elimination at the source (e.g., designing machines in opposed pairs so that inertia forces cancel, balancing all rotating items, smoothing or deflecting unsteady air flows)
2. Modification of paths (e.g., changing substructures so as to transmit less vibration, introducing vibration mounts)
3. Decreasing vibration sensitivity of critical items (e.g., changing orientation of components with respect to excitation, using more rugged components, using more fatigue-resistant materials, designing more damping into components in order to decrease the effects of internal resonances)

Vibration Reduction at the Source: Vibration Absorbers. The strength of vibration sources may often be reduced by proper design. Such design may include balancing, use of the lightest possible reciprocating parts, arranging components so that inertia forces cancel, or attaching vibration absorbers.

A vibration absorber* is essentially a mass m attached by means of a spring k to a primary mass M in order to reduce the response of M to a sinusoidal force acting directly on M (or to a sinusoidal support acceleration when the M is attached to the support by a spring K). If an absorber for which $k/m = \omega_0^2$ is attached to M, then M will experience no excursion if the excitation occurs at a frequency ω_0. (The displacement amplitude of m will be F/k, where F is the force amplitude.)

For driving frequencies very near ω_0, attachment of the vibration absorber results in small amplitudes for M, but for other frequencies it generally results in amplitudes

*The discussion is presented here only for translational systems. It may be extended by analogy to apply also to rotational systems.

that are greater than those obtained without an absorber. Vibration absorbers are very frequency-sensitive and hence should be used only where there exists essentially a single relatively accurately known driving frequency. The addition of damping to an absorber[17,63] extends to some extent the frequency range over which the absorber results in vibration reduction of the primary mass, but increased damping also results in more motion of M at the optimum frequency ω_0.

Path Modification: Vibration Mounts. The prime means for modifying paths traversed by vibratory energy consists of the addition of vibration mounts. If a mass M is excited by a vibratory force of frequency ω, one may reduce the vibratory force that the mass exerts on a rigid support to which it is attached by inserting a spring of stiffness k between mass and support such that $k/M < \omega^2/2$. Further reduction in k/M reduces the transmitted force further; added damping with k/M held constant increases the transmitted force.

If a mass M is attached by means of a spring k to a support that is vibrating at frequency ω, then M can be made to vibrate less than the support if k is chosen so that $k/M < \omega^2/2$. Further reduction of k/M further reduces the motion of the mass; addition of damping increases this motion.

Discussions of vibration and shock mounting are to be found in Refs. 14, 32, 55, and 56. Data on commercial mounts may be found in manufacturers' literature.

Modification of Critical Items. Critical items may often be made less sensitive to vibration by redesigning them so that all their internal-component resonance frequencies fall outside the excitation frequency range. Occasionally this can be done merely by reorienting a critical item with respect to the direction of excitation; more often it requires stiffening the components (or possibly adding mass). If fatigue rather than malfunction is a problem, one may obtain improved parts by careful redesign to eliminate stress concentrations and/or by using materials with greater fatigue resistance. If internal resonances cannot be avoided, one may reduce their effect by designing damping into the resonant components.

4.6.3 Rotating Machinery

A considerable amount of information has been amassed in relation to reciprocating and turbine machines. Comprehensive treatments of the associated torsional vibrations may be found in Refs. 42 and 72; less detailed discussions of these appear in standard texts.[17,32,61,63] A few of the most important items pertaining to rotating machinery are outlined subsequently.

Vibrations of rotating machines are caused by the following factors; reduction of these generally serves to reduce vibrations:

1. Unbalance of rotating components
2. Reciprocating components
3. Whirling of shafts
4. Gas forces
5. Instabilities, such as those due to slip-stick phenomena

Balancing. A rotor mounted on frictionless bearings so that its axis of rotation is horizontal remains motionless (if subject only to gravity) in any angular position, if the rotor is *statically* balanced. Static balance implies an even mass distribution around the rotational axis. However, even statically balanced rotors may be dynamically

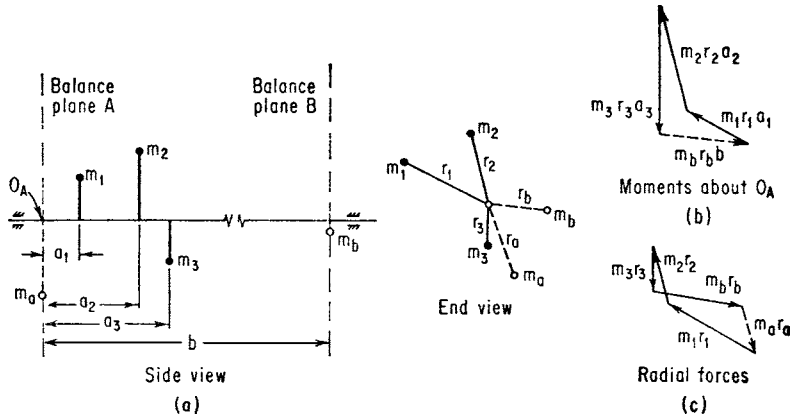


FIG. 4.20 Balancing of rigid rotors.

unbalanced. Dynamic unbalance occurs if the centrifugal forces set up during rotation result in a nonzero couple. (Static balance assures only that the centrifugal forces result in zero net radial force.)

Rigid rotors can always be balanced statically by addition of a single weight in any arbitrarily chosen plane, or dynamically (and statically) by addition of two weights in two arbitrarily chosen planes. The procedure, illustrated in Fig. 4.20, is as follows:[9]

1. Divide the rotor into a convenient number of sections by passing planes perpendicular to the rotational axis.
2. Determine the mass and center-of-gravity position for each section.
3. Draw a diagram like Fig. 4.20*a* where each section is represented by its mass located at the center-of-gravity position.
4. Select balance planes, i.e., planes in which weights are to be attached for balance.
5. Draw a diagram (Fig. 4.20*b*) of centrifugal force moments* about point Q_A (where rotational axis intersects balance plane *A*). Each moment is represented by a vector of length $m_i r_i a_i$ (to some suitable scale) parallel to r_i in the end view. The vector required to close the diagram then is $m_b r_b b$; its direction gives the direction of r_b in the end view; $m_b r_b$ may be calculated and either m_b or r_b may be selected arbitrarily.
6. Draw a diagram (Fig. 4.20*c*) of centrifugal forces* $m_i r_i$, including $m_b r_b$. Each force vector is drawn parallel to r_i in the end view; the vector required to close the diagram is $m_a r_a$ and defines the mass (and its location) to be added in plane *A*. Again, either m_a or r_a may be selected arbitrarily; r_a in the end view must be parallel to the $m_a r_a$ vector, however.

Balancing of flexible rotors or of rotors on flexible shafts (particularly when operating above critical speeds) can generally be accomplished for only one particular speed or for none at all.[17,61]

Balancing machines, their principles and use, and field balancing procedures are discussed in Refs. 17, 41, 61, and 63, among others.

*The common factor ω^2 is omitted from the diagrams.

Balancing of reciprocating engines is treated in Ref. 17 and in a number of texts on dynamics of machines, such as Ref. 23.

Whirling of Shafts: Critical Speeds. Rotating shafts become unstable at certain speeds, and large vibrations are likely to develop. These speeds are known as "critical speeds." At a critical speed the number of revolutions per second is generally very nearly equal to a natural frequency (in cycles per second) of the shaft considered as a nonrotating beam vibrating laterally. Thus critical speeds of shafts may be found by any of the means for calculating the natural frequencies of lateral vibrations of beams (see Sec. 4.4.3).

"Whirling," i.e., violent vibration at critical speeds, occurs in vertical as well as horizontal shafts. In nonvertical shafts gravity effects may introduce additional "critical speeds of second order," as discussed in Ref. 63. Complications may also occur where disks of large inertia are mounted on flexible shafts. Gyroscopic effects due to thin disks generally tend to stiffen the system and thus to increase the critical speeds above those calculated from static flexural vibrations. Thick disks, however, may result in lowering of the critical speeds.[17] Similarly, flexibility in the bearings results in softening of the system and in lowering of the critical speeds.

Turbine Disks. Turbine disks and blades vibrate essentially like disks and beams, and may be treated by previously outlined standard procedures if the rotational speeds are low. However, centrifugal forces result in considerable stiffening effects at high rotational speeds, so that the natural frequencies of rotating disks and blades tend to be considerably higher (and in different ratio to each other) than those of nonrotating assemblies. A brief discussion of these effects and analytical methods may be found in Ref. 63, a more comprehensive one in Ref. 59.

4.6.4 Damping Devices

In this subsection will be presented various devices for the damping or dissipation of mechanical energy. For applications of elastomers employed as damping attachments see Sec. 40, "Dampers and Elastomers," of the Second Edition of the *Mechanical Design and Systems Handbook* (H. Rothbart, ed., McGraw-Hill, New York, 1985).

Electrodynamic Damping. Electrodynamic damping is obtained when a short-circuited electrical conductor is made to move in a magnetic field. An induced current appears in the conductor. The conductor experiences a force proportional to but opposite to the velocity (see Fig. 4.21). The damping force is expressed by

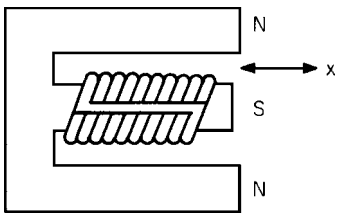


FIG. 4.21 Electrodynamic damper with induced current.

$$F = H^2 A\ell \times 10^{-9}/\rho$$

$$= H^2 V \times 10^{-9}/\rho \qquad (4.67)$$

where F = force
H = average magnetic field strength
A = cross-sectional area of coil wire
ℓ = length of wire
V = volume of wire
ρ = specific resistance

The damping constant ($f = c\dot{x}$) is therefore

$$c = H^2 V \times 10^{-9}/\rho\dot{x} \qquad (4.68)$$

Hydromechanical Damping

Incompressible Fluids (Fig. 4.22). A general form of the damping force may be written as

$$F = k\dot{x}^n \tag{4.69}$$

where $1 < n < 2$. In terms of the pressure drop the equation becomes

$$\Delta p = f(\ell/d)(\dot{x}^2/2g) \tag{4.70}$$

where f = a dimensionless friction coefficient which is a function of Reynolds number
ℓ = length of duct
d = diameter of duct

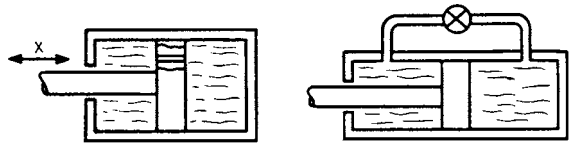


FIG. 4.22 Hydromechanical damping with incompressible fluid.

At low speeds, in which fully developed laminar flow is achieved, Eq. (4.70) assumes the Poiseuille form

$$\Delta p = (32\eta\ell/d^2)\dot{x} \tag{4.71}$$

where η is the viscosity of the liquid.

Compressible Fluids. The principle of damping in this case is similar to that for incompressible fluids. However, the fluid also acts as a spring. The total behavior can be imagined as a spring in parallel with a damper. Assuming adiabatic compression and ideal gas behavior, the effective spring constant of the gas can be evaluated from

$$k = p\gamma A^2/V \tag{4.72}$$

where p = gas pressure
$\gamma = C_p/C_v$
C_p = specific heat at constant pressure
C_v = specific heat at constant volume
V = volume
A = cross-sectional area of duct

Untuned Viscous Shear Damper (Fig. 4.23). This damper (also viscous-fluid damper, untuned damped vibration absorber) consists of an annular mass enclosed by an annular ring. A viscous fluid fills the volume between mass and casing. Since there is no elastic connection between the casing, which is attached to the torsionally vibrating system, and the mass, the damper is untuned. Reduction in amplitudes and a lowering of the natural frequency are obtained without introduction of additional resonances.

Reduction of a complex system to an equivalent two-mass system yields a useful design approximation. With $J_F = \infty$, the system and its equivalent are shown in Fig. 4.24. The impedance equations for steady state can be written as

$$(-\omega^2 J_D + j\omega c)\theta_1 - (j\omega c)\theta_2 = 0 \tag{4.73}$$

MECHANICAL VIBRATIONS

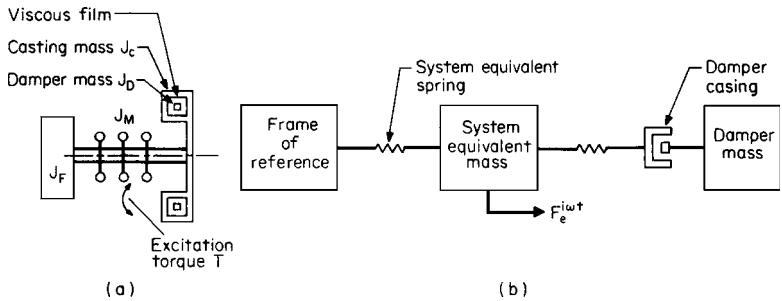


FIG. 4.23 Damping devices. (*a*) Torsional system. (*b*) Translational system.

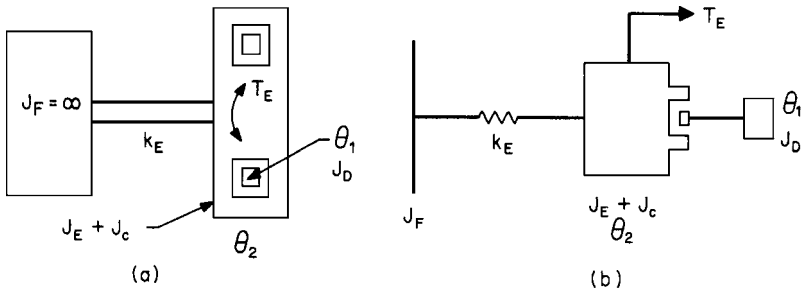


FIG. 4.24 (*a*) Complex system. (*b*) Equivalent two-mass system.

$$-(j\omega c)\theta_1 + [K_2 - \omega^2(J_E + J_c) + j\omega c]\theta_2 = T_E \tag{4.74}$$

where J_D = mass moment of inertia of damper
J_E = equivalent mass moment of inertia of torsionally vibrating systems
J_c = mass moment of inertia of damper casing
ω = circular natural frequency of vibrating system with damper
θ_1 = steady-state vibration amplitude at damper mass
θ_2 = steady-state vibration amplitude at damper casing

Figure 4.25 is a plot of θ_2 as a function of ω for $c = 0$ and $c = \infty$. For optimum

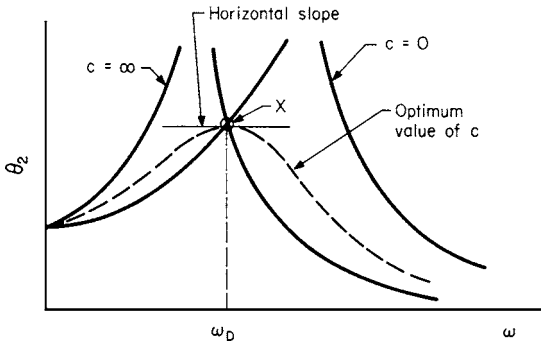


FIG. 4.25 Plot of θ_2 vs. ω for $c = 0$ and $c = \infty$.

damping, the curve of θ versus ω has a horizontal slope at point *X*. Setting $\mu = J_D/J_E$ and solving for conditions at point *X*, the following useful formulas result:

$$\omega_D/\omega_2 = \sqrt{2/(2+\mu)} \tag{4.75}$$

$$\theta_2/(T_0/K_E) = M = (2+\mu)/\mu \tag{4.76}$$

$$c/(2J_D\omega_2) = \gamma = 1/\sqrt{2(1+\mu)(2+\mu)} \tag{4.77}$$

$$|\theta_{SH}/\theta_2| = |\theta_{SH}/(T_0/K_E)|/M = \sqrt{(1+\mu)/(2+\mu)} \tag{4.78}$$

where ω_2 = circular natural frequency of torsionally vibrating system without damper
ω_D = circular frequency corresponding to point X in Fig. 4.25
c = optimum value of coefficient of viscous damping
T_0 = excitation torque
$\theta_{SH} = |\theta_1 - \theta_2|$
M = dynamic magnifier

Knowledge of such items as θ_2, T_0, K_E, and ω_2 permits the determination of μ, from which J_D and c may be calculated.

In reciprocating engines μ tends to fall in the range 0.4 to 1.0. Also, J_c/J_D usually lies between 0.35 and 0.8. The dimensions are so proportioned that sufficient surface is provided for proper heat dissipation. Determination of the necessary fluid viscosity is based upon a modified value of c which accounts for varying shear rates on the lateral and peripheral surfaces of the damper.

Slipping-Torque-Type Dampers. In this type of damper, relative motion between a shaft-fixed hub and a damping mass occurs only when the relative acceleration of the two exceeds a predetermined value. An effective change of natural frequency occurs as the damping mass "locks" and "unlocks" from the hub during each oscillation. Dissipation of energy by damping occurs during the intervals of relative motion.

An analysis of the input and dissipated energy can be made by assuming continual slip, sinusoidal motion of the hub, and a linear time variation of the damper mass. The maximum energy dissipated is

$$U_{D,\max} = (4/\pi)\omega^2\theta_1^2 J_R \tag{4.79}$$

where ω = circular natural frequency of vibrating system without damper
θ_1 = permissible oscillation amplitude at damper hub
J_R = moment of inertia of damper

Energy input $U_{in} = T_0\theta_1$, where T_0 is the peak value excitation torque at the hub. Equating input and dissipated energy yields the useful design equation

$$J_R = \pi^2 T_0/4\omega^2\theta_1 \tag{4.80}$$

The Sandner Damper (Pumping-Chamber Type). A rim and side plates, which act at the damper mass, and a hub are arranged to form internal cavities at certain points of the interfaces. The cavities act as pumping chambers. Oil passes through radial passages starting at the hub, moves to the pumping chambers, returns to the hub to pass through spring-loaded relief valves, and then is discharged into some convenient space. The slipping torque is accurately set by adjustment of the relief valves.

The damper moment of inertia is determined from Eq. (4.80). Maximum torque at which slipping occurs is calculated from

$$T_{R,\max} = (\sqrt{2}/\pi)\omega^2\theta_1 J_R \tag{4.81}$$

MECHANICAL VIBRATIONS

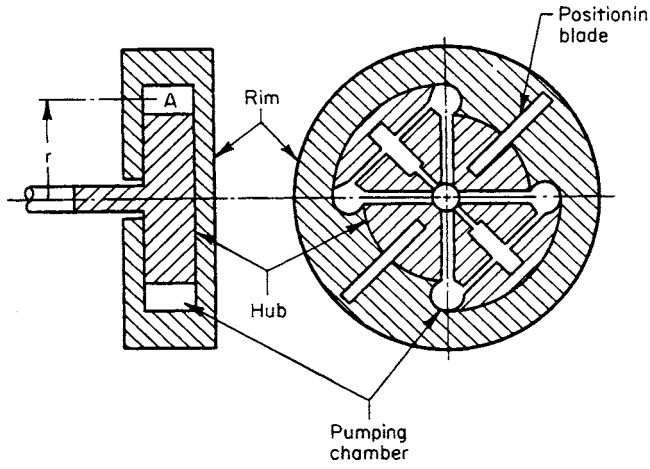


FIG. 4.26 The Sandner damper (pumping-chamber type).

Determination of the relief-valve spring pressure can be calculated from

$$p = T_{R,\max}/2rA$$

where A is the cross-sectional area of one of the pumping chambers and r is the mean radius to the area A, as shown in Fig. 4.26.

The Sandner Damper (Gear-Wheel Type). In this type of damper a rim is cut with gear teeth on its inner surface. Pinions mesh with this gear and are enclosed in special recesses in the hub. Passages connect opposite sides of each recess to centrally located relief valves.

The rim tends to rotate the pinions because of its inertia torque. Actual rotation takes place beyond a certain critical value as predetermined by the relief-valve adjusted pressure. Oil is passed from one gear chamber to the next as the pinions rotate, as shown in Fig. 4.27. Calculations for this damper are the same as for the pumping-chamber type. However, the pressure is computed as $p = T_{R,\max}/8rA$, where A is the effect area of the gear-pump recess.

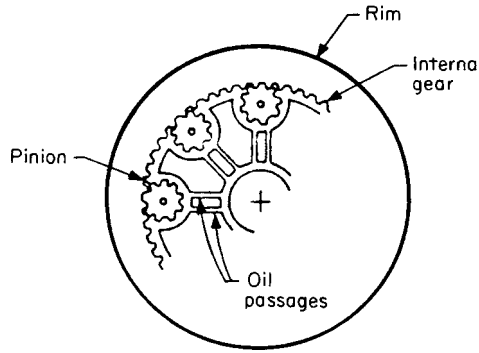


FIG. 4.27 The Sandner damper (gear-wheel type).

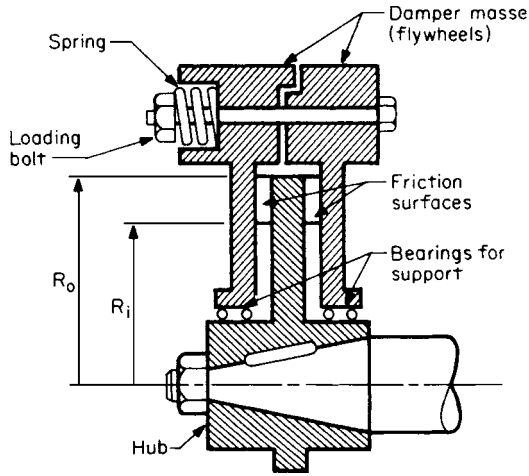


FIG. 4.28 The Lanchester damper (semi-dry-friction type).

The Lanchester Damper (Semi-Dry-Friction Type). In this type of damper, a hub fixed coaxially to a vibrating shaft, carries friction plates on an annulus near its rim. The damper mass, consisting of two flywheels and loading bolts, presses against the friction surfaces. The damper mass lies coaxial with the shaft but is coupled to it only through the friction surfaces, as shown in Fig. 4.28.

The moment of inertia J_{2R} of the two flywheels is calculated by Eq. (4.80). Maximum torque is computed from $T_{R,\max} = (\sqrt{2}/\pi)\omega^2\theta_1 J_{2R}$ and is equated to the friction torque T_f. Spring load can then be computed from

$$T_f = 4/3\mu P(R_o^3 - R_i^3)/(R_o^2 - R_i^2) \tag{4.82}$$

where μ is the coefficient of friction, R_i and R_o are the inner and outer radii of the friction surfaces, respectively, and P is the total spring load.

4.6.5 Charts and Tables

Information on natural frequencies, spring constants, and material properties appears in the tables listed below.

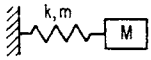
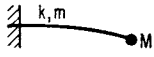
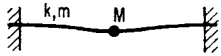
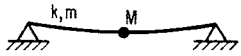
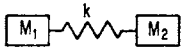
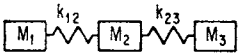
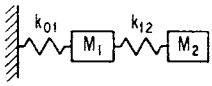
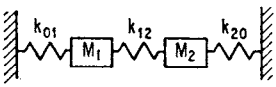
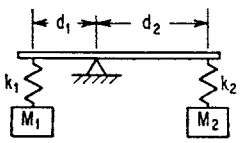
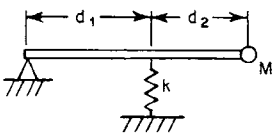
Characteristic	Table
Natural frequencies	
Simple translational systems	4.8*a*
Simple torsional systems	4.8*b*
Beams, bars, shafts, (uniform section unless noted otherwise):	
Uniform and variable section, in flexure	4.8*c*
Flexure; mode shapes	4.6*b*
Torsion; mode shapes	4.6*a*
Longitudinal vibration; mode shapes	4.6*a*
On multiple evenly spaced supports	4.8*d*
Free-free, on elastic foundation	4.8*h*

(*Continued*)

Characteristic	Table
Natural frequencies (*continued*)	
Strings, uniform; mode shapes	4.6*a*
Membranes, uniform:	
Circular	4.8*g*
Rectangular and circular; mode shapes	4.6*d*
Plates, uniform:	
Cantilever, various shapes	4.8*e*
Circular, various boundary conditions	4.8*f*
Rectangular, on simple supports; mode shapes	4.6*c*
Circular, clamped; mode shapes	4.6*c*
Cylindrical shells	4.8*h*
Rings	4.8*i*
Mass free to rotate and translate in plane	4.8*i*
Mass on spring of finite mass	4.8*i*
Spring constants	
Combinations of springs	4.9*a*
Round-wire helical springs	4.9*b*
Beams with force or moment inputs	4.9*c*
Torsion springs	4.9*d*
Shafts in torsion	4.9*d,e*
Plates loaded at centers	4.9*f*
Miscellaneous	
Longitudinal wavespeed and K_m (for Table 4.8) for engineering materials	4.10

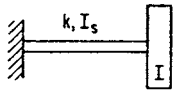
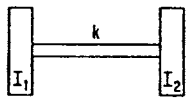
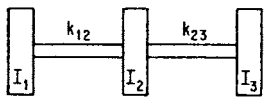
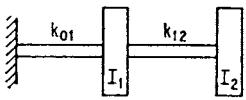
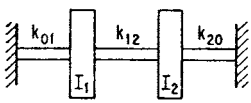
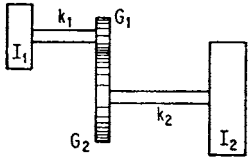
Tables 4.8, 4.9, and 4.10 begin on pages 4.64, 4.74, and 4.79, respectively.

TABLE 4.8*a* Natural Frequencies of Simple Translational Systems[7,17]

System	ω^2 =	Remarks
	$\dfrac{k}{M+0.33m}$	
	$\dfrac{k}{M+0.23m}$	Point mass on cantilever beam
	$\dfrac{k}{M+0.375m}$	Point mass at center of clamped beam — See Table 4.9c for k
	$\dfrac{k}{M+0.50m}$	Point mass at center of simply supported beam
	$k\left[\dfrac{1}{M_1}+\dfrac{1}{M_2}\right]$	
	$B \pm \sqrt{B^2-C}$	$2B = \dfrac{k_{12}}{M_1}+\dfrac{k_{23}}{M_3}+\dfrac{k_{12}+k_{23}}{M_2}$ $C = \dfrac{k_{12}+k_{23}}{M_1 M_2 M_3}(M_1+M_2+M_3)$
		$2B = \dfrac{k_{01}+k_{12}}{M_1}+\dfrac{k_{12}}{M_2}$ $C = \dfrac{k_{01}\,k_{12}}{M_1 M_2}$
		$2B = \dfrac{k_{01}+k_{12}}{M_1}+\dfrac{k_{12}+k_{20}}{M_2}$ $C = \dfrac{1}{M_1 M_2}(k_{01}k_{12}+k_{12}k_{20}+k_{20}k_{01})$
	$\dfrac{\dfrac{1}{M_1}+\dfrac{1}{R^2M_2}}{\dfrac{1}{k_1}+\dfrac{1}{R^2k_2}}$	Inertia of lever negligible $R = \dfrac{d_2}{d_1}$
	$\dfrac{k}{M}\left(\dfrac{d_1}{d_1+d_2}\right)^2$	Inertia of lever negligible.

Symbols:
ω = circular natural frequency = $2\pi f$
M = mass
m = total mass of spring element
k = spring constant

TABLE 4.8*b* Natural Frequencies of Simple Torsional Systems[7,17]

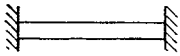
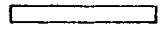
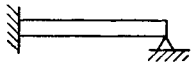
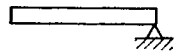
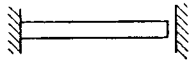
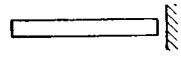
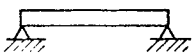
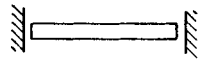
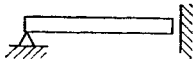
System	$\omega^2 =$	Remarks
	$\dfrac{k}{I + I_s/3}$	
	$k\left(\dfrac{1}{I_1} + \dfrac{1}{I_2}\right)$	
	$B \pm \sqrt{B^2 - C}$	$2B = \dfrac{k_{12}}{I_1} + \dfrac{k_{23}}{I_3} + \dfrac{k_{12} + k_{23}}{I_2}$ $C = \dfrac{k_{12}\, k_{23}}{I_1\, I_2\, I_3}(I_1 + I_2 + I_3)$
		$2B = \dfrac{k_{01} + k_{12}}{I_1} + \dfrac{k_{12}}{I_2}$ $C = \dfrac{k_{01}\, k_{12}}{I_1\, I_2}$
		$2B = \dfrac{k_{01} + k_{12}}{I_1} + \dfrac{k_{12} + k_{20}}{I_2}$ $C = \dfrac{1}{I_1\, I_2}(k_{01}k_{12} + k_{12}k_{20} + k_{20}k_{01})$
	$\dfrac{\dfrac{1}{I_1} + \dfrac{1}{R^2 I_2}}{\dfrac{1}{k_1} + \dfrac{1}{R^2 k_2}}$	Inertia of gears G_1, G_2 assumed negligible $R = \dfrac{\text{number of teeth on gear 1}}{\text{number of teeth on gear 2}}$ $= \dfrac{\text{rpm of shaft 2}}{\text{rpm of shaft 1}}$

Symbols:
- ω = circular natural frequency = $2\pi f$
- k = torsional spring constant; see Table 4.9*d*
- I = polar mass moment of inertia
- I_s = polar mass moment of inertia of entire shaft

TABLE 4.8*c* Natural Frequencies of Beams in Flexure[7,33]

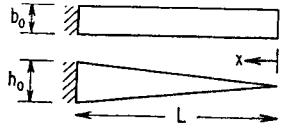
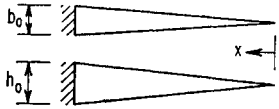
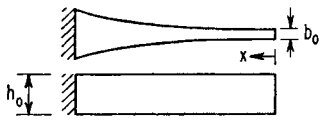
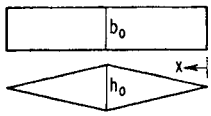
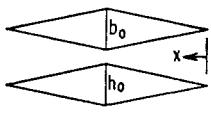
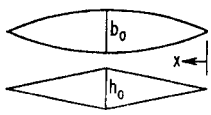
$$f_n = C_n \frac{r}{L^2} \times 10^4 \times K_m$$

f_n = nth natural frequency, hertz
C_n = frequency constant listed in these tables
r = radius of gyration of cross section = $\sqrt{I/A}$, inches*
L = beam length, inches*
K_m = material constant (Table 4.10) = 1.00 for steel

Uniform-section beams	n				
	1	2	3	4	5
 Clamped–Clamped  Free–Free	71.95	198.29	388.73	642.60	959.94
 Clamped–Free	11.30	70.85	198.30	388.73	642.60
 Clamped–Hinged  Free–Hinged	49.57	160.65	335.17	573.20	874.65
 Clamped–Guided  Free–Guided	17.98	97.18	239.98	446.25	715.98
 Hinged–Hinged  Guided–Guided	31.73	126.93	285.60	507.73	793.33
 Hinged–Guided	7.93	71.40	198.33	388.73	642.60

*For r and L in centimeters, use C_n values listed in table multiplied by 2.54.

TABLE 4.8*c* Natural Frequencies of Beams in Flexure (*Continued*)

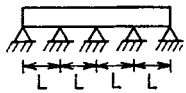
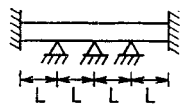
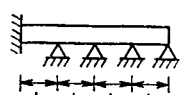
Variable-section beams (Use maximum r to calculate f_n)	Shape		n		
	b/b_0	h/h_0	1	2	3
	1	x/L	17.09	48.89	96.57
	x/L	x/L	26.08	68.08	123.64
	$\sqrt{x/L}$	x/L	22.30	58.18	109.90
	exp x/L	1	15.23	77.78	206.07
	1	x/L	21.21* 35.05†	56.97*	
	x/L	x/L	32.73* 49.50†	76.57*	
	$\sqrt{x/L}$	x/L	25.66* 42.02†	66.06*	

*Symmetric mode.
†Antisymmetric mode.

TABLE 4.8*d* Natural Frequencies of Uniform Beams on Multiple Equally Spaced Supports[7,33]

$$f_n = C_n \frac{r}{L^2} \times 10^4 \times K_m$$

f_n = nth natural frequency, hertz
C_n = frequency constant listed in these tables
r = radius of gyration of cross section = $\sqrt{I/A}$, inches*
L = span length, inches*
K_m = material constant (Table 4.10) = 1.00 for steel

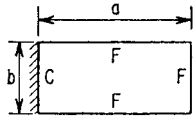
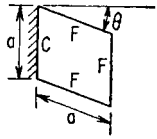
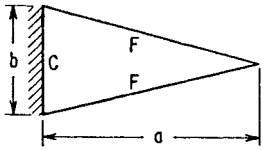
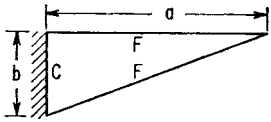
	Number of spans	n				
		1	2	3	4	5
 Ends simply supported	1	31.73	126.94	285.61	507.76	793.37
	2	31.73	49.59	126.94	160.66	285.61
	3	31.73	40.52	59.56	126.94	143.98
	4	31.73	37.02	49.59	63.99	126.94
	5	31.73	34.99	44.19	55.29	66.72
	6	31.73	34.32	40.52	49.59	59.56
	7	31.73	33.67	38.40	45.70	53.63
	8	31.73	33.02	37.02	42.70	49.59
	9	31.73	33.02	35.66	40.52	46.46
	10	31.73	33.02	34.99	39.10	44.19
	11	31.73	32.37	34.32	37.70	41.97
	12	31.73	32.37	34.32	37.02	40.52
 Ends clamped	1	72.36	198.34	388.75	642.63	959.98
	2	49.59	72.36	160.66	198.34	335.20
	3	40.52	59.56	72.36	143.98	178.25
	4	37.02	49.59	63.99	72.36	137.30
	5	34.99	44.19	55.29	66.72	72.36
	6	34.32	40.52	49.59	59.56	67.65
	7	33.67	38.40	45.70	53.63	62.20
	8	33.02	37.02	42.70	49.59	56.98
	9	33.02	35.66	40.52	46.46	52.81
	10	33.02	34.99	39.10	44.19	49.59
	11	32.37	34.32	37.70	41.97	47.23
	12	32.37	34.32	37.02	40.52	44.94
 Ends clamped-supported	1	49.59	160.66	335.2	573.21	874.69
	2	37.02	63.99	137.30	185.85	301.05
	3	34.32	49.59	67.65	132.07	160.66
	4	33.02	42.70	56.98	69.51	129 49
	5	33.02	39.10	49.59	61.31	70.45
	6	32.37	37.02	44.94	54.46	63.99
	7	32.37	35.66	41.97	49.59	57.84
	8	32.37	34.99	39.81	45.70	53.63
	9	31.73	34.32	38.40	43.44	49.59
	10	31.73	33.67	37.02	41.24	46.46
	11	31.73	33.67	36.33	39.81	44.19
	12	31.73	33.02	35.66	39.10	42.70

*For r and L in centimeters, use C_n values multiplied by 2.54.

TABLE 4.8*e* Natural Frequencies of Cantilever Plates[7,33]

$$f_n = C_n \frac{h}{a^2} \times 10^4 \times K_m$$

f_n = nth natural frequency, hertz
C_n = frequency constant listed in table
h = plate thickness, inches*
a = plate dimension, as shown inches*
K_m = material constant (Table 4.10) = 1.00 for steel

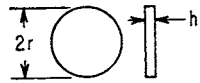
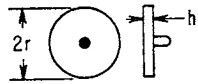
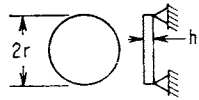
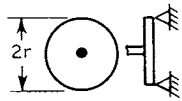
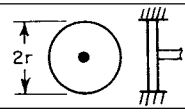
Boundary conditions F = free C = clamped S = simply supported		n				
		1	2	3	4	5
	a/b = ½ a/b = 1 a/b = 2 a/b = 5	3.41 3.40 3.38 3.36	5.23 8.32 14.52 33.79	9.98 26.71 91.92 548.60	21.36 20.86 21.02 20.94	24.18 30.32 47.39 103.03
	θ = 15° θ = 30° θ = 45°	3.50 3.85 4.69	8.63 9.91 13.38			
	a/b = 2 a/b = 4 a/b = 8 a/b = 14	6.7 6.6 6.6 6.6	28.6 28.5 28.4 28.4	56.6 83.5 146.1 246	137 240 457 790	
	a/b = 2 a/b = 4 a/b = 7	5.5 6.2 6.4	23.6 26.7 28.1			

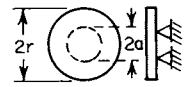
*For h and a in centimeters, use given C_n values multiplied by 2.54.

TABLE 4.8*f* Natural Frequencies of Circular Plates[7,33]

$$f = C\frac{h}{r^2} \times 10^4 \times K_m$$

f = natural frequency, hertz
C = frequency constant listed in table
h = plate thickness, inches*
r = plate radius, inches*
K_m = material constant (Table 4.10) = 1.00 for steel

Boundary conditions	Number of nodal circles	Number of nodal diameters			
		0	1	2	3
Clamped at circumference 	0	9.94	20.67	33.91	49.61
	1	38.66	59.12	82.22	107.9
	2	86.61	116.8	149.5	185.0
	3	153.8	193.5	235.9	281.1
Free 	0			5.11	11.89
	1	8.83	19.94	34.27	51.43
	2	37.47	58.19	81.6	108.2
	3	85.35	115.68	149.7	186.1
Clamped at center; free at circumference 	0	3.65			
	1	20.33			
	2	59.49			
	3	117.2			
Simply supported at circumference 	0	4.84	13.55	24.93	
	1	28.93	47.16	68.18	
	2	72.13	99.93	130.6	
	3	134.4	171.9	212.1	
Clamped at center; simply supported at circumference 	0	14.4			
	1	48.0			
Clamped at center and at circumference 	0	22.1			
	1	60.2			

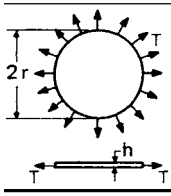
Simply supported at radius a; free at circumference 	Fundamental mode					
a/r	0	0.2	0.4	0.6	0.8	1.0
C	3.64	4.4	6.5	8.6	7.2	4.84

*For h and r in centimeters, use given C_n values multiplied by 2.54.

TABLE 4.8*g* Natural Frequencies of Circular Membranes[33]

$$f = C\sqrt{T/h\rho r^2}$$

f = natural frequency, hertz
C = frequency constant listed in table
r = membrane radius, inches*
h = membrane thickness, inches*
T = tension at circumference, lb_f/in*
ρ = material density, lb_m/in^3 or $lb_f \cdot s^2/in^4$*

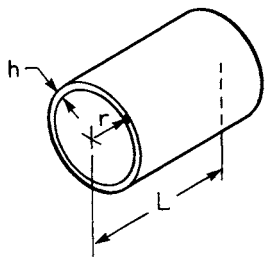


Number of nodal circles	Number of nodal diameters			
	1	2	3	4
1	0.383	0.610	0.817	1.015
2	0.887	1.116	1.340	1.553
3	1.377	1.619	1.849	2.071
4	1.877	2.120	2.355	2.582

*Any set of consistant units may be used that causes all units except s to cancel in the square root.

TABLE 4.8*h* Natural Frequencies of Cylindrical Shells[7]

f_{ij} = natural frequency for mode shape ij, hertz
h = shell thickness, inches*
L = length, inches*
i = number of circumferential waves in mode shape
j = number of axial half-waves in mode shape
c_L = Longitudinal wavespeed in shell material, in/s* (See Table 4.10)
ν = Poisson's ratio, dimensionless



$$f_{ij} = \frac{\lambda_{ij}}{2\pi r}\frac{c_L}{\sqrt{1-\nu^2}}$$

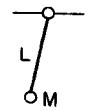
*Any other consistent set of units may be used. For example, h, r, L in centimeters and c_L in centimeters per second.

TABLE 4.8*h* Natural Frequencies of Cylindrical Shells (*Continued*)

Infinitely long		
Axial modes:	$\lambda_{ij} = i\sqrt{\dfrac{1-\nu}{2}}$,	$i = 1, 2, 3, \ldots$
Radial (extensional) modes:	$\lambda_{ij} = \sqrt{1 + i^2}$,	$i = 0, 1, 2, \ldots$ *Note:* $i = 0$ corresponds to breathing mode
Radial-circumferential flexural modes:	$\lambda_{ij} = \dfrac{h/r}{\sqrt{12}} \dfrac{i(i^2-1)}{\sqrt{i^2+1}}$,	$i = 2, 3, 4, \ldots$
Finite length, simply supported edges without axial constraint		
Torsional modes:	$\lambda_{ij} = \dfrac{j\pi r}{L}\sqrt{\dfrac{1-\nu}{2}}$	$i = 0$ $j = 1, 2, 3$
Axial modes†:	$\lambda_{ij} = \dfrac{j\pi r}{L}\sqrt{1-\nu^2}$	
Radial modes†:	$\lambda_{ij} = 1$	
Bending modes†:	$\lambda_{ij} = \left(\dfrac{j\pi r}{L}\right)^2 \sqrt{\dfrac{1-\nu}{2}}$	$i = 1$ $j = 1, 2, 3$
Radial-axial modes:	$\lambda_{ij} = \dfrac{\sqrt{(1-\nu^2)\beta_j^4 + \dfrac{h^2}{12r^2}(i^2+\beta_j^2)^4}}{i^2+\beta_j^2}$	$\beta_j = j\pi r/L$ $i = 2, 3, 4, \ldots$ $j = 1, 2, 3, \ldots$

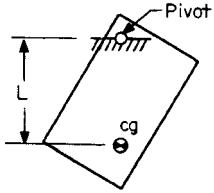
†Values given here apply for long shells, for which $L > 8j\pi r$.

TABLE 4.8*i* Natural Frequencies of Miscellaneous Systems



Simple Point-Mass Pendulum[7]

$$\omega = 2\pi f = \sqrt{g/L}$$

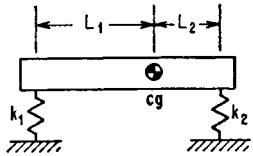


Pendulum with Distributed Mass[7]

$$\omega = 2\pi f = \sqrt{MLg/I_0}$$

M = mass
I_0 = mass moment of inertia about pivot
g = acceleration of gravity
L = distance from pivot to center of gravity

Mass Free to Rotate and Translate in Plane[63]



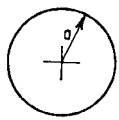
Mass = M, moment of inertia about c.g. = r^2M

$$2\omega^2 = a + c \pm \sqrt{(a - c)^2 + (2b/r)^2}$$

$$a = \frac{k_1 + k_2}{M} \qquad b = \frac{L_2k_2 - L_1k_1}{M}$$

$$c = \frac{L_1^2k_1 + L_2^2k_2}{r^2M}$$

Thin Rings[17]



Extension (pure radial mode): $\omega = \frac{1}{a} c_L$

Bending in its own plane: $\omega_n = \frac{n(n^2 - 1)}{\sqrt{n^2 + 1}} \frac{r}{a^2} c_L$

$n = 2, 3, \ldots$

$r = \sqrt{I/A}$

I = moment of inertia of ring section area A for bending in its own plane

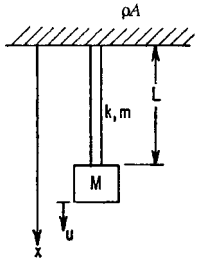
For out-of-plane bending and partial rings, see Den Hartog[17]

Free-free Beam on Elastic Foundation[63]

Same mode shapes as without support, but frequencies increased:

$$\omega^2_{n,\text{ on support}} = \omega^2_{n,\text{ unsupported}} + \frac{k'}{\rho A}$$

$$\frac{k'}{\rho A} = \frac{\text{support spring constant/unit length}}{\text{beam mass/unit length}} = \omega^2_{\text{rigid beam on elastic foundation}}$$



Mass on Rod or Spring of Static Stiffness k, Spring Total Mass m Uniformly Distributed[63]

Frequency equation:

$$(M/m)(\omega\sqrt{m/k}) = \cot(\omega\sqrt{m/k})$$

$$\phi_n(x) = \sin(\omega_n x/L\sqrt{m/k})$$

Weighting function $\mu(x) = m/L + M\delta(x - L)$

[see Sec. 4.4]

M/m	$\omega_1\sqrt{m/k}$	$\omega_2\sqrt{m/k}$	$\omega_3\sqrt{m/k}$	$\omega_4\sqrt{m/k}$	$\omega_5\sqrt{m/k}$
0*	$\pi/2$	$3\pi/2$	$5\pi/2$	$7\pi/2$	$9\pi/2$
$\frac{1}{2}$	1.077	3.644	6.579	9.630	12.722
1	0.960	3.435	6.437	9.426	12.645
∞ †	0	π	2π	3π	4π

*Corresponds to spring free at $x = L$.
†Corresponds to spring clamped at $x = L$.

TABLE 4.9*a* Combination of Spring Constants

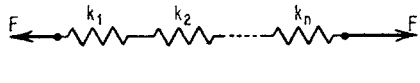
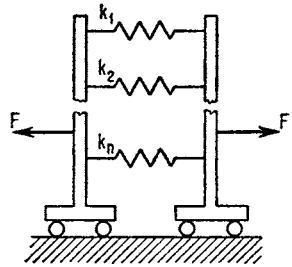
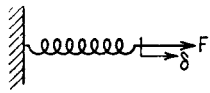
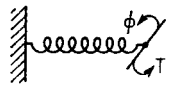
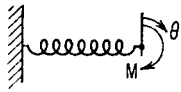
	Series combination	$1/k_{total} = 1/k_1 + 1/k_2 + \cdots + 1/k_n$
	Parallel combination	$k_{total} = k_1 + k_2 + \cdots + k_n$

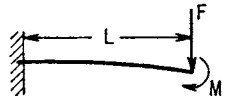
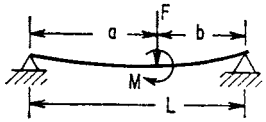
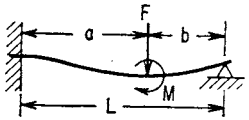
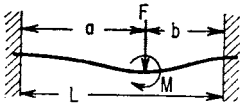
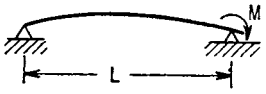
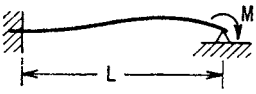
TABLE 4.9*b* Spring Constants of Round-Wire Helical Springs[17]

	Axial loading	$\frac{F}{\delta} = k = \frac{Gd^4}{8nD^3}$
	Torsion	$\frac{T}{\phi} = k_r = \frac{Ed^4}{64nD}$
	Bending	$\frac{M}{\theta} = k_b = \frac{2}{2+\nu} k_r$

Symbols:

F = axial force	G = shear modulus
δ = axial deflection	E = elastic modulus
T = torque	ν = Poisson's ratio
ϕ = torsion angle	d = wire diameter
M = bending moment	D = mean coil diameter
θ = flexure angle	n = number of coils

TABLE 4.9*c* Spring Constants of Beams[69]

	Translational (force F)	Rotational (moment M)
	$k = \dfrac{3EI}{L^3}$	$k_r = \dfrac{EI}{L}$
	$k = \dfrac{12EI}{L^3}$	
	$k = \dfrac{3EIL}{(ab)^2}$ For $a = b$ $k = \dfrac{48EI}{L^3}$	$k_r = \dfrac{3EIL}{L^2 - 3ab}$ For $a = b$ $k_r = \dfrac{12EI}{L}$
	$k = \dfrac{12EIL^3}{a^3b^2(3L + b)}$ For $a = b$ $k = \dfrac{768EI}{7L^3}$	$k_r = \dfrac{4EIL^3/b}{4L^3 - 3b(L + a)^2}$ For $a = b$ $k_r = \dfrac{64EI}{5L}$
	$k = \dfrac{3EIL^3}{(ab)^3}$ For $a = b$ $k = \dfrac{192EI}{L^3}$	$k_r = \dfrac{EIL^3}{ab(L^2 - 3ab)}$ For $a = b$ $k_r = \dfrac{16EI}{L}$
		$k_r = \dfrac{3EI}{L}$
		$k_r = \dfrac{4EI}{L}$

Symbols:

E = modulus of elasticity
I = centroidal moment of inertia of cross section
F = force
M = bending moment
$k = F/\delta$, translational spring constant
$k_r = M/\theta$, rotational spring constant
δ = lateral deflection at F
θ = flexural angle at M

TABLE 4.9*d* Torsion Springs

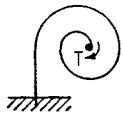
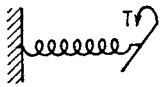
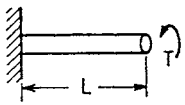
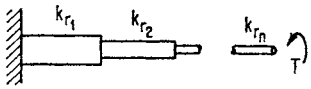
	Spiral spring[17] $k_r = T/\phi = EI/L$ E = modulus of elasticity L = total spring length I = moment of inertia of cross section
	Helical spring See Table 4.9*b*
	Uniform shaft $k_r = GJ/L$ G = shear modulus L = length J = torsional constant (Table 4.9*e*)
	Stepped shaft $1/k_r = 1/k_{r_1} + 1/k_{r_2} + \cdots 1/k_{r_n}$ k_{r_j} = k_r of jth uniform part by itself

TABLE 4.9*e* Torsional Constants J of Common Sections[47]

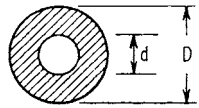
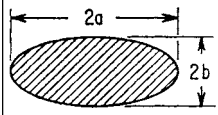
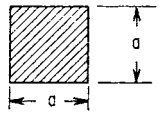
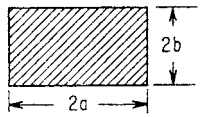
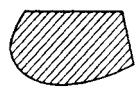
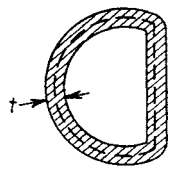
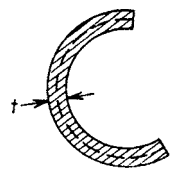
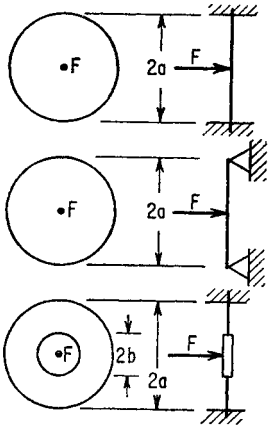
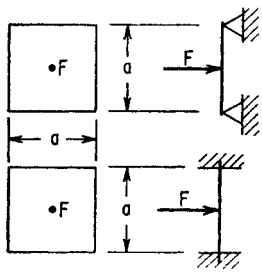
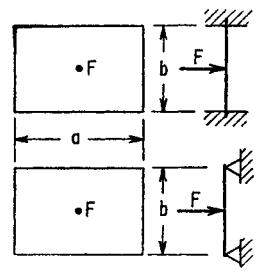
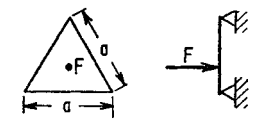
Circle		$J = \frac{\pi}{32}(D^4 - d^4)$	
Ellipse		$J = \frac{\pi a^3 b^3}{a^2 + b^2}$	
Square		$J = 0.1406\, a^4$	
Rectangle		$J = ab^3\left[\frac{16}{3} - 3.36\frac{b}{a}\left(1 - \frac{b^4}{12a^4}\right)\right]$	
Equilateral triangle		$J = \frac{\sqrt{3}\, a^4}{80}$	
Any solid compact section without reentrant angles		$J \approx \frac{A^4}{40I}$	A = cross-section area U = mean circumferential length (length of dotted lines shown) I = polar moment of inertia of section about its centroid
Any thin closed tube of uniform thickness t		$J \approx \frac{4A^2 t}{U}$	
Any thin open tube of uniform thickness t		$J \approx \frac{Ut^3}{3}$	

TABLE 4.9*f* Spring Constants of Centrally Loaded Plates[47]

Circular		Clamped	$k = \dfrac{16\pi D}{a^2}$
		Simply supported	$k = \dfrac{16\pi D}{a^2}\left(\dfrac{1+\nu}{3+\nu}\right)$
		Clamped, with concentric rigid insert	$k = \dfrac{(16\pi D/a^2)(1-c^2)}{(1-c^2)^2 - 4c^2(\ln c)^2}$ $c = b/a$
Square		All edges simply supported	$k = 86.1D/a^2$
		All edges clamped	$k = 192.2D/a^2$
Rectangular		All edges clamped	$k = D\alpha/b^2$ a/b | 4 | 2 | 1 α | 167 | 147 | 192
		All edges simply supported	$k = 59.2(1 + 0.462c^4)D/b^2$ $c = b/a < 1$
Equilateral triangular		All edges simply supported	$k = 175D/a^2$

Symbols:

k = force/deflection, spring constant

$D = \dfrac{Eh^3}{12(1-\nu^2)}$

E = modulus of elasticity

ν = Poisson's ratio

h = plate thickness

TABLE 4.10 Longitudinal Wavespeed and K_m for Engineering Materials

Material	Temperature, °C	Longitudinal wavespeed $c_L = \sqrt{E/\rho}$ 10^5 in/s*	$K_m = \frac{(c_L)_{\text{material}}}{(c_L)_{\text{structural steel}}}$
Metals			
Aluminum	20	2.00	1.04
Beryllium	20	4.96	2.57
Brass, bronze	20	1.38–1.57	0.715–0.813
Copper	20	1.34	0.694
Copper	100	1.21	0.627
Copper	200	1.16	0.601
Cupro-nickel	20	1.47–1.61	0.762–0.834
Iron, cast	20	1.04–1.64	0.539–0.850
Lead	20	0.49	0.25
Magnesium	20	2.00	1.04
Monel metal	20	1.76	0.912
Nickel	20	1.24–1.90	0.642–0.984
Silver	20–100	1.03	0.534
Tin	20	0.98	0.51
Titanium	20	1.06–2.00	1.02–1.04
Titanium	90	1.90–1.96	0.984–1.02
Titanium	200	1.84–1.88	0.953–0.974
Titanium	325	1.68–1.75	0.870–0.907
Titanium	400	1.57–1.68	0.813–0.870
Zinc	20	1.46	0.756
Steel, typical	20	2.00	1.04
Steel, structural	−200	2.00	1.04
	−100	1.97	1.02
	25	1.93	1.00
	150	1.92	0.994
	300	1.86	0.963
	400	1.77	0.917
Steel, stainless	−200	1.98–2.04	1.03–1.06
	−100	1.94–2.02	1.005–1.047
	25	1.92–2.06	0.995–1.07
	150	1.89–2.02	0.979–1.05
	300	1.84–1.98	0.953–0.979
	400	1.76–1.94	0.912–1.005
	600	1.70–1.87	0.881–0.969
	800	1.57–1.76	0.813–0.912
Aluminum Alloys	−200	2.08–2.13	1.078–1.104
	−100	2.01–2.06	1.04–1.07
	25	1.97–2.02	1.02–1.05
	200	1.79–1.87	0.927–0.969
Plastics†			
Cellulose acetate	25	0.40	0.21
Methyl methacrylate	25	0.60	0.31
Nylon 6, 6	25	0.54	0.78
Vinyl chloride	25	0.26	0.13
Polyethylene	25	0.41	0.21
Polypropylene	25	0.42	0.22
Polystyrene	25	0.68	0.35

TABLE 4.10 Longitudinal Wavespeed and K_m for Engineering Materials (*Continued*)

Material	Temperature, °C	Longitudinal wavespeed $c_L = \sqrt{E/\rho}$ 10^5 in/s*	$K_m = \frac{(c_L)_{\text{material}}}{(c_L)_{\text{structural steel}}}$
Teflon	25	0.17	0.088
Epoxy resin	25	0.61	0.32
Phenolic resin	25	0.57	0.30
Polyester resin	25	0.61	0.32
Glass-fiber–epoxy laminate	25	1.72	0.89
Glass-fiber–phenolic laminate	25	1.44	0.75
Glass-fiber–polyester laminate	25	1.25	0.65
Other			
Concrete	25	1.44–1.80	0.75–0.93
Cork	25	0.20	0.104
Glass	25	1.97–2.36	1.02–1.22
Granite	25	2.36	1.22
Marble	25	1.50	0.78
Plywood	25	0.84	0.44
Woods, along fibers		1.32–1.92	0.68–0.99
Woods, across fibers		0.48–0.54	0.25–0.28

*To convert to centimeters per second, multiply by 2.54
†Values given here are typical. Wide variations may occur with composition, temperature, and frequency.

REFERENCES

1. Barton, M. V. (ed.): "Shock and Structural Response," American Society of Mechanical Engineers, New York, 1960.
2. Bendat, J. S.: "Principles and Applications of Random Noise Theory," John Wiley & Sons, Inc., New York, 1958.
3. Bendat, J. S., and A. G. Piersol: "Engineering Applications of Correlation and Spectral Analysis," John Wiley & Sons, Inc., New York, 1980.
4. Beranek, L. L. (ed.): "Noise and Vibration Control," McGraw-Hill Book Company, Inc., New York, 1971.
5. Biezeno, C. B., and R. Grammel: "Engineering Dynamics," vol. III, "Steam Turbines," Blackie & Son, Ltd., London, 1954.
6. Bishop, R. E. D., and D. C. Johnson: "Vibration Analysis Tables," Cambridge University Press, Cambridge, England, 1956.
7. Blevins, R. D.: "Formulas for Natural Frequency and Mode Shape," Van Nostrand Reinhold Company, Inc., New York, 1979.
8. Burton, R.: "Vibration and Impact," Addison-Wesley Publishing Company, Inc., Reading, Mass., 1958.
9. Church, A. H., and R. Plunkett: "Balancing Flexible Rotors," *Trans. ASME* (*Ser. B*), vol. 83, pp. 383–389, November, 1961.
10. Craig, R. R., Jr.: "Structural Dynamics: An Introduction to Computer Methods," John Wiley & Sons, Inc., New York, 1981.
11. Crandall, S. H., and W. D. Mark: "Random Vibration in Mechanical Systems," Academic Press, Inc., New York, 1963.
12. Crandall, S. H.: "Engineering Analysis," McGraw-Hill Book Company, Inc., New York, 1956.
13. Crede, C. E.: "Theory of Vibration Isolation," chap. 30, "Shock and Vibration Handbook," C. M. Harris, and C. E. Crede, eds., McGraw-Hill Book Company, Inc., New York, 1976.

14. Crede, C. E.: "Vibration and Shock Isolation," John Wiley & Sons, Inc., New York, 1951.
15. Cremer, L., M. Heckl, and E. E. Ungar: "Structure-Borne Sound," Springer-Verlag, New York, 1973.
16. Davenport, W. B., Jr.: "Probability and Random Processes," McGraw-Hill Book Company, Inc., New York, 1970.
17. Den Hartog, J. P.: "Mechanical Vibrations," 4th ed., McGraw-Hill Book Company, Inc., New York, 1956.
18. Fung, Y. C.: "An Introduction to the Theory of Aeroelasticity," John Wiley & Sons, Inc., New York, 1955.
19. Fung, Y. C., and M. V. Barton: "Some Shock Spectra Characteristics and Uses," *J. Appl. Mech.,* pp. 365–372, September, 1958.
20. Gardner, M. F., and J. L. Barnes: "Transients in Linear Systems," vol. I, John Wiley & Sons, Inc., New York, 1942.
21. Hagedorn, P.: "Non-Linear Oscillations," Clarendon Press, Oxford, England, 1981.
22. Himelblau, H., Jr., and S. Rubin: "Vibration of a Resiliently Supported Rigid Body," chap. 3, "Shock and Vibration Handbook," C. M. Harris and C. E. Crede, eds., McGraw-Hill Book Company, Inc., New York, 1976.
23. Holowenko, R.: "Dynamics of Machinery," John Wiley & Sons, Inc., New York, 1955.
24. Hueber, K. H.: "The Finite Element Method for Engineers," John Wiley & Sons, Inc., New York, 1975.
25. Jacobsen, L. S., and R. S. Ayre, "Engineering Vibrations," McGraw-Hill Book Company, Inc., New York, 1958.
26. Junger, M. C., and D. Feit: "Sound, Structures, and their Interaction," The MIT Press, Cambridge, Mass., 1972.
27. Kamke, E.: "Differentialgleichungen, Lösungsmethoden and Lösungen," 3d ed., Chelsea Publishing Company, New York, 1948.
28. Leissa, A. W.: "Vibration of Plates," NASA SP-160, U.S. Government Printing Office, Washington, D.C., 1969.
29. Leissa, A. W.: "Vibration of Shells," NASA SP-288, U.S. Government Printing Office, Washington, D.C., 1973.
30. LePage, W. R., and S. Seely: "General Network Analysis," McGraw-Hill Book Company, Inc., New York, 1952.
31. Lowe, R., and R. D. Cavanaugh: "Correlation of Shock Spectra and Pulse Shape with Shock Environment," *Environ. Eng.,* February 1959.
32. Macduff, J. N., and J. R. Curreri: "Vibration Control," McGraw-Hill Book Company, Inc., New York, 1958.
33. Macduff, J. N., and R. P. Felgar: "Vibration Design Charts," *Trans. ASME,* vol. 79, pp. 1459–1475, 1957.
34. Mahalingam, S.: "Forced Vibration of Systems with Non-linear Non-symmetrical Characteristics," *J. Appl. Mech.,* vol. 24, pp. 435–439, September 1957.
35. Major, A.: "Dynamics in Civil Engineering," Akadémiai Kiadó, Budapest, 1980.
36. Marguerre, K., and H. Wölfel: "Mechanics of Vibration," Sijthoff & Noordhoff, Alphen aan den Rijn, The Netherlands, 1979.
37. Martienssen, O.: "Über neue Resonanzerscheinungen in Wechselstromkreisen," *Physik. Z.,* vol. 11, pp. 448–460, 1910.
38. Meirovitch, L.: "Analytical Methods in Vibration," The Macmillan Company, Inc., New York, 1967.
39. Mindlin, R. D.: "Dynamics of Package Cushioning," *Bell System Tech. J.,* vol. 24, nos. 3, 4, pp. 353–461, July–October 1945.
40. Morse, P. M.: "Vibration and Sound," 2d ed., McGraw-Hill Book Company, Inc., New York, 1948.

41. Myklestad, N. O.: "Fundamentals of Vibration Analysis," McGraw-Hill Book Company, Inc., New York, 1956.
42. Nestroides, E. J. (ed.): "Bicera: Handbook on Torsional Vibrations," Cambridge University Press, New York, 1958 (British Internal Combustion Engine Research Association).
43. Ostergren, S. M.: "Shock Response of a Two-Degree-of-Freedom System," Rome Air Development Center Rep. RADC TN 58-251.
44. Perrone, N., and W. Pilkey, and B. Pilkey, (eds.): "Structural Mechanics Software Series," University of Virginia Press, Charlottesville, vols. I–III, 1977–1980.
45. Pilkey, W., and B. Pilkey, (eds.): "Shock and Vibration Computer Programs," The Shock and Vibration Information Center, Naval Research Laboratory, Washington, D.C., 1975.
46. Pistiner, J. S., and H. Reisman: "Dynamic Amplification Factor of a Two-Degree-of-Freedom System," *J. Environ. Sci.,* pp. 4–8, October 1960.
47. Plunkett, R. (ed.): "Mechanical Impedance Methods for Mechanical Vibrations," American Society of Mechanical Engineers, New York, 1958.
48. Richart, F. E., Jr., J. R. Hall, Jr., and R. D. Woods: "Vibration of Soils and Foundations," Prentice-Hall, Inc., Englewood Cliffs, N.J., 1970.
49. Roark, R. J.: "Formulas for Stress and Strain," 3d ed., McGraw-Hill Book Company, Inc., New York, 1954.
50. Rubin, S.: "Concepts in Shock Data Analysis," chap. 23, "Shock and Vibration Handbook," C. M. Harris and C. E. Crede, eds., McGraw-Hill Book Company, Inc., New York, 1976.
51. Ruzicka, J. (ed.): "Structural Damping," American Society of Mechanical Engineers, New York, 1959.
52. Ruzicka, J. E., and T. F. Derby: "Influence of Damping in Vibration Isolation," SVM-7, Shock and Vibration Information Center, Washington, D.C., 1971.
53. Scanlan, R. H., and R. Rosenbaum: "Introduction to the Study of Aircraft Vibration and Flutter," The Macmillan Company, Inc., New York, 1951.
54. Schwesinger, G.: "On One-term Approximations of Forced Nonharmonic Vibrations," *J. Appl. Mech.,* vol. 17, no. 2, pp. 202–208, June 1950.
55. Snowdon, J. C.: "Vibration and Shock in Damped Mechanical Systems," John Wiley & Sons, Inc., New York, 1968.
56. Snowdon, J. C.: "Vibration Isolation: Use and Characterization," NBS Handbook 128, U.S. Department of Commerce, National Bureau of Standards, Washington, D.C., May 1979.
57. Snowdon, J. C., and E. E. Ungar (eds.): "Isolation of Mechanical Vibration, Impact, and Noise," AMD-vol. 1, American Society of Mechanical Engineers, New York, 1973.
58. Skudrzyk, E.: "Simple and Complex Vibratory Systems," The Pennsylvania State University Press, University Park, Pa., 1968.
59. Stodola, A. (Transl. L. C. Lowenstein): "Steam and Gas Turbines," McGraw-Hill Book Company, Inc., New York, 1927.
60. Stoker, J. J.: "Nonlinear Vibrations," Interscience Publishers, Inc., New York, 1950.
61. Thomson, W. T.: "Theory of Vibration with Applications," 2d ed., Prentice-Hall, Inc., Englewood Cliffs, N.J., 1981.
62. Thomson, W. T.: "Shock Spectra of a Nonlinear System," *J. Appl. Mech.,* vol. 27, pp. 528–534, September 1960.
63. Timoshenko, S.: "Vibration Problems in Engineering," 2d ed., Van Nostrand Company, Inc., Princeton, N.J., 1937.
64. Tong, K. N.: "Theory of Mechanical Vibration," John Wiley & Sons, Inc., New York, 1960.
65. Torvik, P. J. (ed.): "Damping Applications for Vibration Control," AMD-vol. 38, American Society of Mechanical Engineers, New York, 1980.
66. Trent, H. M.: "Physical Equivalents of Spectral Notions," *J. Acoust. Soc. Am.,* vol. 32, pp. 348–351, March 1960.

MECHANICAL VIBRATIONS

67. Ungar, E. E.: "Maximum Stresses in Beams and Plates Vibrating at Resonance," *Trans. ASME, Ser. B,* vol. 84, pp. 149–155, February 1962.
68. Ungar, E. E.: "Damping of Panels," chap. 14, "Noise and Vibration Control," L. L. Beranek, ed., McGraw-Hill Book Company, Inc., New York, 1971.
69. Ungar, E. E., and R. Cohen: "Vibration Control Techniques," chap. 20, "Handbook of Noise Control," C. M. Harris, ed., McGraw-Hill Book Company, Inc., New York, 1979.
70. Vigness, I.: "Fundamental Nature of Shock and Vibrations," *Elec. Mfg.,* vol. 63, pp. 89–108, June 1959.
71. Warburton, G. B.: "The Dynamical Behavior of Structures," Pergamon Press, Oxford, 1964.
72. Wilson, W. K.: "Practical Solution of Torsional Vibration Problems," John Wiley & Sons, Inc., New York, 1956.