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MECHANICAL SYSTEM ANALYSIS

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CHAPTER 8
SYSTEM DYNAMICS

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8.1 INTRODUCTION—PRELIMINARY CONCEPTS

A physical system undergoing a time-varying interchange or dissipation of energy among or within its elementary storage or dissipative devices is said to be in a "dynamic state." The elements are in general inductive, capacitative, or resistive—the first two being capable of storing energy while the last is dissipative. All are called "passive," i.e., they are incapable of generating net energy. A system composed of a finite number or a denumerable infinity of storage elements is said to be "lumped" or "discrete," while a system containing elements which are dense in physical space is called "continuous." The mathematical description of the dynamics for the discrete case is a set of ordinary differential equations, while for the continuous case it is a set of partial differential equations.

The mathematical formulation depends upon the constraints (e.g., kinematic or geometric) and the physical laws governing the behavior of the system. For example,

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the motion of a single point mass obeys $\mathbf{F} = m(d\mathbf{v}/dt)$ in accordance with Newton's second law of motion. Analogously, the voltage drop across a perfect coil of self-inductance L is $V = L(di/dt)$, a consequence of Faraday's law. In the first case the energy-storage element is the mass, which stores $mv^2/2$ units of kinetic energy while the inductance L stores $Li^2/2$ units of energy in the second case. A spring-mass system and its electrical analog, an inductive-capacitive series circuit, represent higher-order discrete systems. The unbalanced force acting on the mass is $F - kx$. Thus

$$F = kx + m\ddot{x} \qquad m, k > 0 \tag{8.1}$$

Analogously for the electrical case,

$$V = L\ddot{q} + q/c \qquad L, c > 0$$

following Kirchhoff's voltage-drop law (i.e., the sum of voltage drop around a closed loop is zero). To show that Eq. (8.1) expresses the dynamic exchange of energy, multiply Eq. (8.1) by $\dot{x}\,dt$ (which is equal to dx) and integrate:

$$\int_0^t F\dot{x}\,dt = \int_{x=x_0}^{x} F\,dx = \int_0^t m\dot{x}\ddot{x}\,dt + \int_0^t k\dot{x}x\,dt$$

$$\text{Work input} = \underbrace{\frac{m\dot{x}^2}{2}\Big]_0^t}_{\Delta \text{KE}} + \underbrace{\frac{kx^2}{2}\Big]_0^t}_{\Delta \text{PE}} = \frac{m\dot{x}^2}{2} - \frac{m\dot{x}_0^2}{2} + \frac{kx^2}{2} - \frac{kx_0^2}{2}$$

which is a statement of the law of conservation of energy. This illustrates that work input is divided into two parts, one part increasing the kinetic energy, the remainder increasing the potential energy. The actual partition between the two energy sources at any instant is time-varying, depending on the solution to Eq. (8.1).

If a viscous damping element is added to the system the force equation becomes (see Fig. 8.1*a*)

$$m\ddot{x} + c\dot{x} + kx = F \qquad c > 0$$

and performing the same operation of multiplying by $\dot{x}\,dt$, (dx) and integrating we obtain

$$\int_0^t m\ddot{x}\dot{x}\,dt + \int_0^t c\dot{x}^2\,dt + \int_0^t kx\dot{x}\,dt = \int_{x_0}^{x} F\,dx \tag{8.2a}$$

$$\frac{m\dot{x}^2}{2}\Big]_0^t + \int_0^t c\dot{x}^2\,dt + \frac{kx^2}{2}\Big]_0^t = \int_{x_0}^{x} F\,dx \tag{8.2b}$$

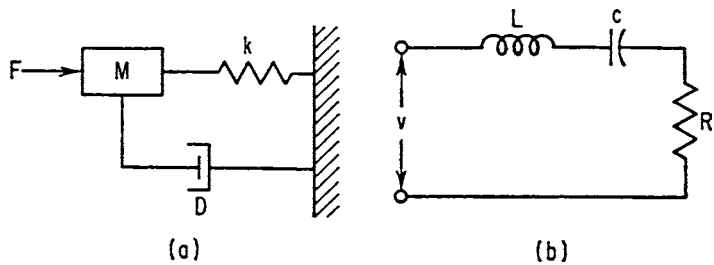

FIG. 8.1 Second-order systems. (*a*) Mechanical system. (*b*) Electrical analog.

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again expressing the energy-conservation law. Note that the integrand $c\dot{x}^2 \geq 0$ and that the integral in Eq. (8.2*b*) is thus a monotonically increasing function of time. This condition assures that, for $F = 0$, the free (homogeneous) system must eventually come to rest since under this condition Eq. (8.2*b*) becomes

$$\frac{m\dot{x}^2}{2} + \frac{kx^2}{2} + \int_0^t c\dot{x}^2 dt = \text{const} = \frac{m\dot{x}_0^2}{2} + \frac{kx_0^2}{2} \tag{8.3}$$

which again is an expression of the law of energy conservation. The first two terms are positive since they contain the squared factors $\dot{x}^2$ and x^2, while the third term, as noted above, increases with time. It follows that the sum of the first two must decrease monotonically in order to satisfy Eq. (8.3); moreover, neither term can be greater than the sum. It follows that, as $t \to \infty$, $x \to 0$ and $\dot{x} \to 0$.

Formulation of the foregoing simple problems was based upon fundamental physical laws. The derivation by Lagrange equations, which in this simple case offers little advantage, is (Chap. 1)

$$L = T - V \quad T = \tfrac{1}{2}m\dot{x}^2 \quad V = kx^2/2$$

For conservative systems (e.g., spring-mass),

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = F = m\ddot{x} + kx$$

For nonconservative systems with dissipation function $\mathcal{F}$,

$$\begin{aligned} \mathcal{F} &= c\dot{x}^2/2 \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= -\frac{\partial \mathcal{F}}{\partial \dot{x}} + F \\ m\ddot{x} + kx &= -c\dot{x} + F \\ m\ddot{x} + c\dot{x} + kx &= F \end{aligned}$$

Precisely the same form is deducible from a Lagrange statement of the electrical equivalent (Fig. 8.1*b*).

8.1.1 Degrees of Freedom

Thus far it has been observed that one independent variable x was employed to describe the system dynamics. In general, however, several variables $x_1, x_2, \ldots, x_n$ are necessary to describe the motion of a complex system. The minimum number of coordinates that are so required is defined as the number of degrees of freedom of the system. Simple examples of two-degree-of-freedom systems are shown in Fig. 8.2. The respective equations of motion are

Mechanical:
$$m_1\ddot{x}_1 + k_1(x_1 - x_2) = F \tag{8.4a}$$
$$m_2\ddot{x}_2 + k_2x_2 + k_1(x_2 - x_1) = 0$$

Electrical:
$$L_1\ddot{q}_1 + (q_1 - q_2)/c_1 = V \tag{8.4b}$$
$$L_2\ddot{q}_2 + q_2/c_2 + (q_2 - q_1)/c_1 = 0$$

derivable from force and loop voltage-drop considerations.

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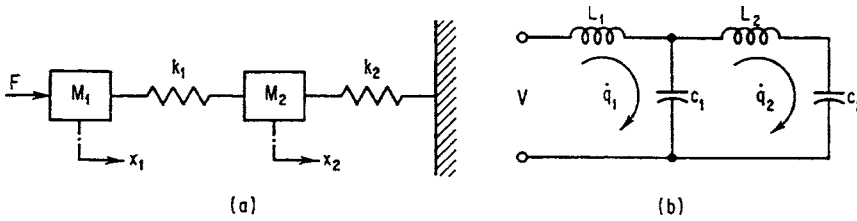

FIG. 8.2 Two-degree-of-freedom systems. (*a*) Mechanical. (*b*) Electrical analog.

Another example of a two-degree-of-freedom system is shown in Fig. 8.3, a compound pendulum constrained to move in a plane. While the system may at first appear to have four degrees of freedom with the positions of m_1 and m_2 given by r_1, θ_1, and r_1, θ_2, r_2, θ_2, respectively, two seemingly trivial expressions of constraint, r_1 = constant and r_2 = constant, show that the motion is describable in terms of θ_1 and θ_2 only. If a spring were interposed between m_1 and the pivot r_1, then r_1 would no longer be a constant and the motion would involve r_1, θ_1, θ_2, or three independent variables, resulting in a three-degree-of-freedom system.

8.1.2 Coupled and Uncoupled Systems

Equations (8.4*a*) or (8.4*b*) also illustrate a coupled system. The term "coupled" is a consequence of having more than one independent variable present in each equation of a set. In Eq. (8.4*a*), x_1 and x_2 and/or their derivatives appear in each of the two dynamic equations, implying that motion of one mass excites motion in the other mass. Only in conservative linear systems is it always possible to uncouple the system by a linear transformation.

An *n*-degree-of-freedom system requires for description *n* independent equations, usually of second order or lower. It is sometimes convenient to make changes in variables to facilitate the analysis of complex systems, or indeed to express the motion in terms of parameters, which are more accessible. In any case this amounts to having

$$x_i = x_i(q_1, q_2, \ldots, q_m) \qquad i = 1, 2, \ldots, n$$

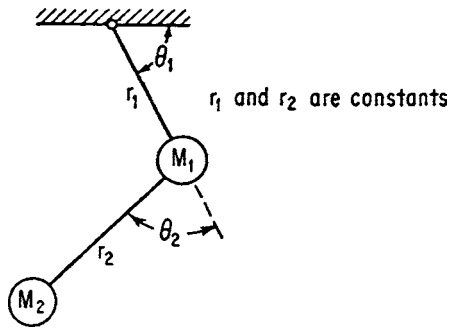

FIG. 8.3 Two-degree-of-freedom system (compound pendulum).

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The q's, called "generalized coordinates," when judiciously chosen play a useful role in the analysis of complex systems. The q's need not be independent. This implies $m > n$ and the existence of $m - n$ equations that connect the q's, since the motion must involve only n independent equations. The case of Fig. 8.3 is an example in which $m = 4$ and $n = 2$ with $m - n = 2$ or two constraint equations, namely,

$$r_1 = \text{const} \qquad r_2 = \text{const}$$

8.1.3 General System Considerations

Discrete Systems. The equations for a system of n degrees of freedom can be written as

$$(b_{11}p^2 + c_{11}p + d_{11})x_1 + (b_{12}p^2 + c_{12}p + d_{12})x_2 + \cdots + (b_{1n}p^2 + c_{1n}p + d_{1n})x_n = f_1(t)$$

.....

$$(b_{n1}p^2 + c_{n1}p + d_{n1})x_1 + \cdots + (b_{nn}p^2 + c_{nn}p + d_{nn})x_n = f_n(t) \tag{8.5a}$$

where $p = d/dt$, $p^2 = d^2/dt^2$. Or, more concisely,

$$\sum_{j=1}^{n} (b_{ij}p^2 + c_{ij}p + d_{ij})x_j = f_i(t) \qquad i = 1, \ldots, n \tag{8.5b}$$

where b_{ij}, c_{ij}, d_{ij} are in general functions of x_k, $\dot{x}_k$, $\ddot{x}_k$, $k = 1, \ldots, n$, and time. In terms of generalized coordinates,

$$\sum_{j=1}^{m} (b'_{ij}p^2 + c'_{ij}p + d'_{ij})q_j = Q_i(t) \qquad i = 1, \ldots, m \qquad m \geq n \tag{8.5c}$$

where the Q_i's are the generalized forces (see Chap. 3). The number of degrees of freedom appears to have increased in Eq. (8.5c) for $m > n$, but this really is not the case, because of the existence of $m - n$ constraint equations connecting the q's.

The general form depicted by Eq. (8.5) is nonlinear in view of the b_{ij}, c_{ij}, and d_{ij} dependence on x_k and its time derivatives. Removal of this dependence yields the linear form of Eq. (8.5). Elimination of the time dependence in these coefficients yields the linear constant-coefficient form, which is of greatest engineering interest because it is the only one yielding completely to analysis and because a large class of systems can be approximated by this form. This is in contradistinction to the nonlinear and linear time-variable cases for which analytic solutions are in general not obtainable and not obtainable in closed form, respectively.

The initial state of each coordinate of Eq. (8.5) must be known [i.e., $x_i(0)$, $\dot{x}_i(0)$, $i = 1, 2, \ldots, n$], before the general solution is possible; hence $2n$ initial conditions are available which coincide with the maximum order of the differential equation obtained by eliminating $n - 1$ variables in Eq. (8.5). If the order is less than $2n$, then some of the initial conditions are not independent.

Continuous Systems. In passing from the description of discrete to that of continuous systems, the ordinary differential equation of n degrees of freedom becomes the set of partial differential equations as $n \to \infty$, i.e., the storage and dissipative elements become densely packed. The initial conditions are similar to those of the ordinary differential equation case which required initial velocity and position coordinates of each elementary mass particle; for now, in the limit of continuous systems, the initial displacement from equilibrium $\mathbf{u}(x, y, z, 0)$ and the displacement velocity $(d\mathbf{u}/dt)(x, y, z, 0)$ as well as the conditions $\mathbf{u}(x_b, y_b, z_b, t)$ that bound the system (where x_b, y_b, z_b are the continuous coordinates that bound the unperturbed system) are essential. As an example,

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consider the propagation of a pressure wave moving longitudinally in an infinite elastic dissipationless medium of small cross section. The equation of motion is derived by considering the elemental width dx having a stress $\sigma_{(x)}$ at the position x. Newton's law of motion applied to the element of mass of unit cross section is written as

$$\begin{aligned} \sigma - [\sigma + (\partial\sigma/\partial x)\,dx] &= m\,dx\,(\partial^2 u/\partial t^2) \\ -\partial\sigma/\partial x &= m(\partial^2 u/\partial t^2) \end{aligned} \tag{8.6}$$

where m = mass density and σ = compressive stress.

The displacement from equilibrium u results in strain (compressive)

$$\epsilon = -\partial u/\partial x$$

and by Hooke's law,

$$\sigma = -Y(\partial u/\partial x) \qquad \text{(positive } \sigma \text{ is compressive)} \tag{8.7}$$

where Y is Young's modulus for solids and a proportionality constant for other elastic media. Substitution of Eq. (8.7) into Eq. (8.6) yields

$$(Y/m)(\partial^2 u/\partial x^2) = \partial^2 u/\partial t^2 \tag{8.8}$$

which is the simple one-dimensional wave equation.

8.2 SYSTEMS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS[1–5]

8.2.1 Elastic Systems

That class of systems characterized by interchange of kinetic and elastic energy is termed "elastic." The formulation of the nondissipative (conservative) type leads to the simple wave equation

$$c^2\nabla^2 u = \partial^2 u/\partial t^2 \tag{8.9}$$

where ∇^2 is the Laplace operator (three-dimensional in general).

Examples of such systems follow.

Hydrodynamics and Acoustics. Applying Newton's second law to an elementary particle yields

$$\rho(dx\,dy\,dz)(d\mathbf{v}/dt) = (dx\,dy\,dz)\mathbf{F} + \sum \mathbf{f} \tag{8.10}$$

where $\mathbf{F}$ represents the external forces (body forces) acting per unit volume on the element (e.g., gravity or inertia, using d'Alembert's principle), $\Sigma\mathbf{f}$ the sum of forces acting on the surfaces, and ρ the mass density. We can write

$$\sum \mathbf{f} = -(\partial p/\partial x)\,dx\,dy\,dz\mathbf{i} - (\partial p/\partial y)\,dy\,dx\,dz\mathbf{j} - (\partial p/\partial z)\,dz\,dx\,dy\mathbf{k} \tag{8.11}$$

$$= -\nabla p\,dx\,dy\,dz \tag{8.12}$$

where ∇ is the "del" or gradient operator and p is the pressure.

Substituting Eq. (8.12) into (8.10) yields

$$\rho(d\mathbf{v}/dt) = \mathbf{F} - \nabla p \tag{8.13}$$

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Expanding the left-hand side of Eq. (8.13), we obtain

$$\rho(\partial\mathbf{v}/\partial t + \underbrace{V_x\partial\mathbf{v}/\partial x + V_y\partial\mathbf{v}/\partial y + V_z\partial\mathbf{v}/\partial z}_{(\mathbf{v}\cdot\nabla)\mathbf{v}}) = \mathbf{F} - \nabla p \tag{8.14}$$

where $\mathbf{v} = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}$.

From continuity (conservation of mass),

$$\partial\rho/\partial t + \text{div}(\rho\mathbf{v}) = 0 \tag{8.15}$$

$$\partial\rho/\partial t + \mathbf{v}\cdot\nabla\rho + \rho\,\text{div}\,\mathbf{v} = 0 \tag{8.16}$$

Equation (8.15) states that the rate of mass increase in elementary volume $dx\,dy\,dz$ ($\partial\rho/\partial t$) equals the rate of flow into the same volume, $-dx\,dy\,dz$ div ($\rho\mathbf{v}$). If $|\partial\mathbf{v}/\partial t| \gg |(\mathbf{v}\cdot\nabla)\mathbf{v}|$, then to a good approximation of Eq. (8.14)

$$\rho\,\partial\mathbf{v}/\partial t \approx \mathbf{F} - \nabla p \tag{8.17}$$

Let the density be given by

$$\rho = \rho_0(1 + \epsilon) \qquad \rho_0 = \text{const} \tag{8.18}$$

Taking a first differential of Eq. (8.18), we obtain

$$d\rho/\rho_0 = d\epsilon \tag{8.19}$$

Elimination of ρ in Eq. (8.16) yields

$$\partial\epsilon/\partial t + \text{div}\,\mathbf{v} + \mathbf{v}\cdot\nabla\epsilon \approx 0 \tag{8.20}$$

where it is assumed that the variation of density about ρ_0 is small (i.e., $|\epsilon| \ll 1$). As a further consequence the third term of Eq. (8.20), involving space derivatives of ϵ which are of higher order, is accordingly dropped, leaving to a good approximation

$$\partial\epsilon/\partial t + \text{div}\,\mathbf{v} \approx 0 \tag{8.21}$$

which, together with Eq. (8.17) in rearranged form,

$$\partial\mathbf{v}/\partial t - \mathbf{F}/\rho_0 + \nabla p/\rho_0 \approx 0 \tag{8.22}$$

provides two of the three essential relationships for small perturbation analysis; the remaining expression is the equation of state (e.g., $dp = k\,d\epsilon$).

Substituting $\nabla p = k\nabla\epsilon$ (where k is a constant) into Eq. (8.22), the following is obtained:

$$\partial\mathbf{v}/\partial t - \mathbf{F}/\rho_0 + k(\nabla\epsilon/\rho_0) \approx 0 \tag{8.23}$$

which together with Eq. (8.21) forms a fundamental set.

Calling u the particle displacement, $\mathbf{v} = \partial\mathbf{u}/\partial t$. Substituting for $\mathbf{v}$ in Eqs. (8.23) and (8.21) yields

$$\partial^2\mathbf{u}/\partial t^2 - \mathbf{F}/\rho_0 + k\,\nabla\epsilon/\rho_0 = 0 \tag{8.24}$$

$$\partial\epsilon/\partial t + (\partial/\partial t)\,\text{div}\,\mathbf{u} = 0 \tag{8.25}$$

and

$$\epsilon + \text{div}\,\mathbf{u} = 0 \tag{8.26}$$

Taking the divergence of Eq. (8.24) yields

$$-(\partial^2/\partial t^2)\,\text{div}\,\mathbf{u} - \text{div}(\mathbf{F}/\rho_0) + (k/\rho_0)\,\text{div grad}\,\epsilon = 0 \tag{8.27}$$

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Next, substituting ϵ for $-\text{div}\ \mathbf{u}$ results in

$$-(\partial^2\epsilon/\partial t^2) - \text{div}(\mathbf{F}/\rho_0) + (k/\rho_0)\nabla^2\epsilon = 0 \tag{8.28}$$

For the case div $\mathbf{F}/\rho_0 = 0$, Eq. (8.28) becomes

$$\partial^2\epsilon/\partial t^2 = (k/\rho_0)\nabla^2\epsilon$$

which is the three-dimensional wave equation in ϵ.

If, in addition, the velocity is derivable from a scalar potential ϕ, i.e.,

$$\mathbf{v} = \text{grad}\ \phi = \partial\mathbf{u}/\partial t \tag{8.29}$$

then substitution in Eq. (8.24) gives

$$(\partial/\partial t)\ \text{grad}\ \phi = -(k/\rho_0)\ \text{grad}\ \epsilon + \mathbf{F}/\rho_0 \tag{8.30}$$

Differentiating with respect to time and assuming $\mathbf{F}$ time-independent,

$$\text{grad}\ [(\partial^2/\partial t^2)\phi - (k/\rho_0)\nabla^2\phi] = 0$$

From Eqs. (8.26) and (8.29),

$$\partial\epsilon/\partial t = -\text{div grad}\ \phi = -\nabla^2\phi$$

whence, by a suitable choice of ϕ,

$$(\partial^2/\partial t^2)\phi - (k/\rho_0)\nabla^2\phi = 0$$

i.e., the velocity-potential function is also of the wave type. For the special case of one-dimensional propagation with $\mathbf{F} = 0$,

$$V_x\mathbf{i} = (\partial u_x/\partial t)\mathbf{i} = \text{grad}\ \phi$$

Differentiating with respect to time and substituting from Eqs. (8.30) and (8.26),

$$\partial^2 u_x/\partial t^2 = (k/\rho_0)(\partial^2 u_x/\partial x^2) \tag{8.31}$$

Transverse Motion of an Elastic String Due to a Slight Perturbation. Consider a string under uniform tension T_0 initially stretched in a horizontal (x) direction (Fig. 8.4). If the weight of the string is negligible compared with inertia forces and the elongation is

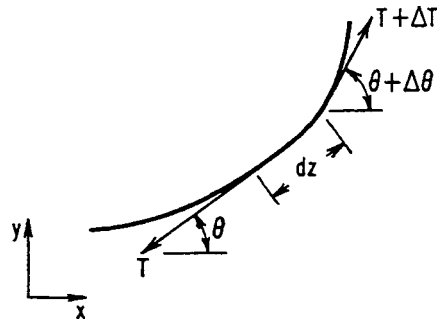

FIG. 8.4 String under tension.

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negligible, the force balance in the y direction for an elementary section of length dz is

$$[\partial(T \sin \theta)/\partial z]dz = m\, dz\, \partial^2 y/\partial t^2 \tag{8.32a}$$

where $\sin \theta = \partial y/\partial z$. Considering only small displacements from the unperturbed position, i.e., $|\partial y/\partial z| \ll 1$ and

$$\begin{gathered} dz = du[1 + (\partial y/\partial x)^2]^{1/2} \qquad \partial x/\partial z \approx 1 \\ \frac{\partial[T(\partial y/\partial z)]}{\partial z} = \frac{\partial\{T[(\partial y/\partial x)(\partial x/\partial z)]\}}{\partial x}\frac{\partial x}{\partial z} \approx \frac{\partial[T(\partial y/\partial x)]}{\partial x} \end{gathered} \tag{8.32b}$$

If tension T is essentially constant and additive elongation is negligible,

$$T = T_0(1 + \epsilon) \qquad |\epsilon| \ll 1$$

then to a first approximation,

$$\frac{\partial[T(\partial y/\partial x)]}{\partial x} \approx T_0 \frac{\partial^2 y}{\partial x^2}$$

and Eq. (8.32a) becomes, after dz is canceled, the one-dimensional wave equation

$$(T_0/m)(\partial^2 y/\partial x^2) = \partial^2 y/\partial t^2$$

Transverse Vibration of Stretched Membrane. Consider the stretched membrane of circular cross section (see Fig. 8.5). The transverse motion under a pressure p is found by forming the equation of motion on an elementary annulus of width dr and again ignoring the membrane weight,

$$dr(\partial/\partial r)(T 2\pi r \sin \theta) = 2\pi r\, dr\, p + 2\pi\rho r\, dr\, (\partial^2 y/\partial t^2)$$

where $\sin \theta = \partial y/\partial r$
T = tension
ρ = density
whence, for T essentially constant [$T = T_0(1 + \epsilon)$, $|\epsilon| << 1$]

$$\frac{T_0}{\rho}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial y}{\partial r}\right) \approx \frac{\partial^2 y}{\partial t^2} + \frac{p}{\rho} = \frac{T_0}{\rho}\nabla^2 y = \frac{\partial^2 y}{\partial t^2} + \frac{p}{\rho} \qquad \text{(inhomogeneous wave equation)}$$

where ∇^2 = Laplacian operator

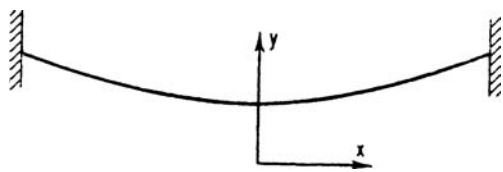

FIG. 8.5 Stretched membrane.

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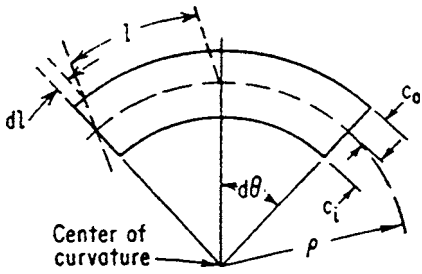

FIG. 8.6 Bending of a bar.

Transverse Vibrations of a Rod. The dynamics of motion of a uniform bar shown in Fig. 8.6 are derived by satisfying $\Sigma M = 0$ and $\Sigma F_y = 0$.

V and M are shear and bending moment shown acting on the elemental section of length dx. F_y are the vertical forces including the d'Alembert inertia force in the y direction $-(m\,dx)(\partial^2 y/\partial t^2)$. Satisfying $\Sigma F_y = 0$ in the positive y direction yields

$$\begin{aligned} -(\partial V/\partial x)\,dx - mg\,dx - m\,dx\,(\partial^2 y/\partial t^2) &= 0 \\ \partial V/\partial x + mg + m\,\partial^2 y/\partial t^2 &= 0 \end{aligned} \tag{8.33a}$$

and satisfying the moment equation about the center of mass of the elementary section results in

$$\partial M/\partial x = V \tag{8.33b}$$

Now a physical relation exists between M and y which is derivable by considering the bent section which is compressed on the inner fiber and stretched on the upper fiber with a "neutral axis," unstressed at the initial length (Fig. 8.6).

From geometric considerations,

$$\begin{aligned} dl &= c_0\,d\theta \qquad \rho \gg c_0 \\ d\theta &= l/\rho \qquad\quad \rho \gg c_i \end{aligned} \tag{8.34a}$$

where ρ is the radius of curvature; c_0 and c_i distances from the neutral axis to the outer and inner fiber, respectively; l the half width of the elementary section; and $d\theta$ the half angle subtended by the section under stressed conditions.

Density ρ is further expressed by (from elementary calculus)

$$\frac{1}{\rho} = \frac{\partial^2 y/\partial x^2}{[1 + (\partial y/\partial x)^2]^{3/2}} \approx \frac{\partial^2 y}{\partial x^2} \qquad \text{for } \frac{\partial y}{\partial x} \ll 1 \tag{8.34b}$$

The strain at the outer fiber is $\epsilon_0 \approx dl/l$. From the geometry, the strain at any other point is $\epsilon_0(y/c_0)$ where y is the position measured from the neutral axis. From Eq. (8.34a), $dl/l = c_0/\rho$ and therefore the strain at y is

$$\epsilon = \epsilon_0(y/c_0) = (dl/l)(y/c_0) = (c_0/\rho)(y/c_0) = y/\rho$$

The stress, following Hooke's law, is

$$\sigma = E\epsilon = Ey/\rho$$

where E is the modulus of elasticity.

The bending moment about the neutral axis is expressed by

$$M = \int_{-c_i}^{c_0} y\sigma b\,dy$$

where b is the depth and $b\,dy$ is the elementary cross-sectional area. Substituting for σ the above expression becomes

$$\int_{-c_i}^{c_0} y\,\frac{Ey}{\rho}\,b\,dy = \frac{E}{\rho}\int_{-c_i}^{c_0} y^2 b\,dy$$

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The integral on the right is I, the area moment of inertia about the neutral axis, a geometric property.

Thus

$$M = EI/\rho$$

and from Eq. (8.34*b*)

$$M/EI = \partial^2 y/\partial x^2 \tag{8.35}$$

Taking two derivatives of Eq. (8.35) with respect to x, one derivative of Eq. (8.33*b*), and substituting for $\partial y/\partial x$ in Eq. (8.33*a*) yields

$$EI\,\partial^4 y/\partial x^4 + m(g + \partial^2 y/\partial t^2) = 0$$

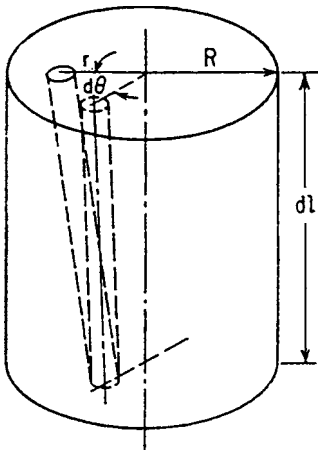

FIG. 8.7 Torsion in a rod.

Torsional Motion of a Rod. Consider an elemental cylindrical section of length dl and a twist angle $d\theta$ (see Fig. 8.7). The strain on an elemental area da is $r\,d\theta/dl$, and the associated stress is

$$\sigma = Gr(d\theta/dl) \tag{8.36}$$

where G is the shear modulus and r is the radius to the point in question. The total torque

$$T = \int_A r\sigma\,da = \int_A Gr\frac{d\theta}{dl}\,r\,da = G\frac{d\theta}{dl}\int_A r^2\,da = G\frac{d\theta}{dl}J$$

where J = polar moment of inertia $= \int_A r^2\,da = \int_0^R r^2 2\pi r\,dr$.

The expression for torsional oscillations is obtained from Newton's second law:

$$\begin{aligned} dl(\partial T/\partial l) &= I(\partial^2\theta/\partial t^2)\,dl \\ JG(\partial^2\theta/\partial t^2) &= I(\partial^2\theta/\partial t^2) \end{aligned} \tag{8.37}$$

where I = mass moment of inertia per unit length. For homogeneous media $I = \rho J$ and Eq. (8.37) becomes

$$(G/\rho)(\partial^2\theta/\partial t^2) = \partial^2\theta/\partial t^2 \tag{8.38}$$

Electric-Transmission-Line Equation for Low-Frequency Operation. Consider a section of length dx as shown in Fig. 8.8. From Ohm's law the current density is

$$\mathbf{i} = -k\ \text{grad}\ V$$

If the wire has cross section $\mathbf{A}$ where $\mathbf{A}$ is a vector in the direction normal to the cross section, the total current I is

$$I = \mathbf{i}\cdot\mathbf{A} = -k\mathbf{A}\cdot\text{grad}\ V = -\frac{1}{R}\text{grad}\ V \tag{8.39}$$

In the x direction this becomes

$$-\frac{\partial v}{\partial x} = IR$$

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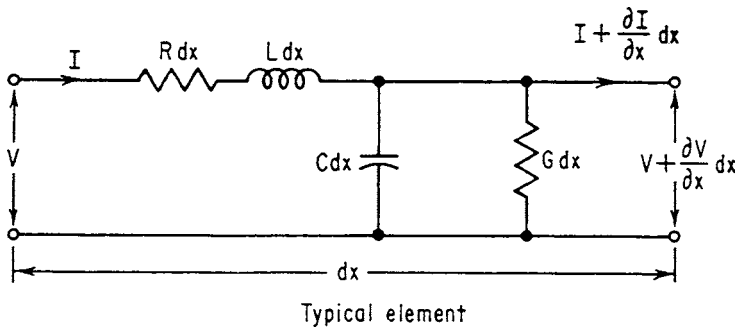

FIG. 8.8 Electric transmission line.

where R is the resistance per unit length. The wire also acts as a distributed capacitance C per unit of length; following Faraday's law,

$$\frac{dQ}{C\,dx} = dV \tag{8.40}$$

Taking the partial differential of Eq. (8.40) with respect to time for the elemental section yields

$$\frac{1}{C\,dx}\frac{\partial Q}{\partial t} = \frac{\partial V}{\partial t} \tag{8.41}$$

The charge Q which collects within the dx section is

$$Q = \int_0^t I\,dt - \int_0^t \left(I + \frac{\partial I}{\partial x}\,dx\right) dt \tag{8.42}$$

and therefore

$$\frac{\partial Q}{\partial t} = -\frac{\partial I}{\partial x}\,dx$$

Substitution in Eq. (8.41) yields

$$\frac{1}{C}\frac{\partial I}{\partial x} = -\frac{\partial V}{\partial t} \tag{8.43}$$

If in addition some current leaks off and is proportional to V, then Eq. (8.42) should be modified as follows:

$$Q = \int_0^t I\,dt - \int \left(I + \frac{\partial I}{\partial x}\,dx\right) dt - \int (GV\,dx)\,dt \tag{8.44}$$

where G is the leakage conductance per unit length.

Taking the partial derivative of Eq. (8.44) with respect to time,

$$\frac{\partial Q}{\partial t} = -\frac{\partial I}{\partial x}\,dx - GV\,dx$$

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and replacing $\partial Q/\partial t$ in Eq. (8.41), leads to the modified equation

$$-\frac{\partial I}{\partial x} = GV + C\frac{\partial V}{\partial t} \tag{8.45}$$

Also, the inductance along the wire owing to Faraday's law introduces an additional voltage drop to modify Eq. (8.39) to read

$$-\frac{\partial V}{\partial x}dx = RI\,dx + L\frac{\partial I}{\partial t}dx \tag{8.46}$$

where L is the inductance per unit length. Then

$$-\frac{\partial V}{\partial x} = RI + L\frac{\partial I}{\partial t}$$

Combining Eqs. (8.45) and (8.46) results in

$$CL\frac{\partial^2 I}{\partial t^2} + (RC + GL)\frac{\partial I}{\partial t} + RGI = \frac{\partial^2 I}{\partial x^2} \tag{8.47}$$

and the identical form in V, i.e.,

$$CL\frac{\partial^2 V}{\partial t^2} + (RC + GL)\frac{\partial V}{\partial t} + RGV = \frac{\partial^2 V}{\partial x^2} \tag{8.48}$$

Equations (8.47) and (8.48) are the telegrapher's equation which was first reported by Kirchhoff. Note that, if R and G are zero, they reduce to the simple wave equation.

8.2.2 Inelastic Systems

Flow of Heat, Electricity, and Fluid. The flow of heat across a boundary as given by Fourier is

$$-k\ \text{grad}\ T = \mathbf{Q} \tag{8.49}$$

where $\mathbf{Q}$ = heat flux and T = temperature.

For electricity, Ohm's law is analogous to Fourier's law; thus

$$-k\ \text{grad}\ V = \mathbf{i} \tag{8.50}$$

where V = voltage and $\mathbf{i}$ = current flow density.

Following Fick's law[6] for flow of incompressible fluids through finely divided porous media,

$$-k\ \text{grad}\ p = \mathbf{v} \tag{8.51}$$

where p = pressure and $\mathbf{v}$ = flow rate per unit area.

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Conservation laws applied to Eqs. (8.49) to (8.51) yield the following expressions. For Eq. (8.49), conservation of thermal energy implies

$$\rho c(\partial T/\partial t) = -\text{div } \mathbf{Q} \tag{8.52}$$

where c = specific heat per unit mass and ρ = mass density. Similarly, for Eq. (8.50) and Eq. (8.51), conservation of charge and conservation of mass, respectively, imply

$$\partial q/\partial t = -\text{div } \mathbf{i} \tag{8.53}$$

where q = charge density, and

$$\partial \rho/\partial t = -\text{div } (\rho \mathbf{v}) \tag{8.54}$$

where ρ = mass density.

Q is eliminated between Eqs. (8.49) and (8.52) by taking the divergence of Eq. (8.49):

$$\rho c(\partial T/\partial t) = -(-\text{div } k \text{ grad } T) = k \nabla^2 T \tag{8.55}$$

Similarly, for Eqs. (8.50) and (8.53),

$$\partial q/\partial t = k \nabla^2 V \tag{8.56}$$

And, for Eqs. (8.51) and (8.54),

$$\partial \rho/\partial t = k \text{ div } (\rho \text{ grad } p) \tag{8.57}$$

If ρ = const, Eq. (8.57) reduces to Laplace's equation,

$$\nabla^2 p = 0 \tag{8.58}$$

If Eq. (8.52), (8.53), or (8.54) had volume sources at the points of investigation, for example,

$$\rho c(\partial T/\partial t) = -\text{div } \mathbf{Q} + S$$

Then Eqs. (8.55), (8.56), and (8.58) would respectively read

$$k \nabla^2 T = \rho c(\partial T/\partial t) - S \tag{8.59}$$

$$k \nabla^2 V = \partial q/\partial t - S \tag{8.60}$$

$$k\rho \nabla^2 p = -S \tag{8.61}$$

In the absence of time-varying potentials, Eqs. (8.59) and (8.60) reduce to the Poisson form of Eq. (8.61), and where no source is present all reduce to the form of Laplace's equation (8.58). Electrostatic phenomena are closely related to the above developments. The electrostatic field **E** is given by

$$\mathbf{E} = -\text{grad } V \tag{8.62}$$

and the flux **D** is linearly related to **E** by

$$\mathbf{D} = \epsilon \mathbf{E} \tag{8.63}$$

where ϵ = dielectric constant. By Gauss's law, which follows from Coulomb's law of forces,

$$\text{div } \mathbf{D} = \rho \tag{8.64}$$

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where ρ = charge density. Eliminating **D** and **E** among Eqs. (8.62) to (8.64) yields

$$\rho = \text{div } \mathbf{D} = -\epsilon \text{ div grad } V = -\epsilon \nabla^2 V \tag{8.65}$$
$$\nabla^2 V = -\rho/\epsilon$$

which is Poisson's equation, degenerating to Laplace's equation in the absence of sources (i.e., $\rho = 0$).

8.3 SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

8.3.1 Fundamentals

All systems which occur in nature are nonlinear and distributed. To an excellent approximation, many systems can be "lumped," permitting vast simplifications of the mathematical model. For example, the lumped spring-mass-damping system is, strictly speaking, a distributed system with the "mass" composed of an infinity of densely packed elementary springs and masses and damping elements arranged in some uncertain order. Because of the theoretical difficulties encountered in formulating an accurate mathematical model which fits the actual system and the analytical difficulties in attacking the complex problem, the engineer (with experimental justification) makes the "mass" a point mass which cannot be deformed, and the spring a massless spring without damping. If damping is present an element called the *damper* is isolated so that the "lumped" system is composed of discrete elements. Having settled on an equivalent lumped physical model, the equations describing system behavior are next formulated on the basis of known physical laws.

The equations thus derived constitute a set of ordinary differential equations, generally nonlinear, implying the existence of one or more lumped elements which do not behave in a "linear" fashion, e.g., nonlinearity of load vs. deflection of a spring.

In mathematical terms it is easier to define a nonlinear set by first defining what constitutes a linear set and then using the exclusion principle as follows.

A set of ordinary differential equations is linear if terms containing the dependent variable(s) or their time derivatives appear to the first degree only. The physical system it characterizes is termed *linear*. All other systems are nonlinear, and the physical systems they define are nonlinear.

An example of a linear system is the set

$$t^2 \frac{d^2x_1}{dt^2} + t\frac{d^2x_2}{dt^2} + \frac{d^2x_1}{dt^2} + (\sin t)^2 x_2 + G = 0$$
$$\frac{dx_1}{dt} + \frac{d^2x_2}{dt^2} = 0$$

where it is noted that the factors containing functions of t, the independent variable, are ignored in determining linearity, and each term containing one of the dependent variables x_1, x_2, or their derivatives, is of the first degree.

Two examples of nonlinear systems are

$$(dx_1/dt)^2 + 2x_1 = 0 \tag{8.66a}$$

$$x_1(d^3x_2/dt^2) + x_1x_2 = 0 \qquad x_2 + d^2x_1/dt^2 + d^3x_2/dt^3 = 0 \tag{8.66b}$$

In the first system, the square of the first derivative immediately rules it as nonlinear. The first equation of the second system is nonlinear on two counts: first, by virtue of

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the product of x_1, and a time derivative d^3x_2/dt^2, and second, because of the term containing the product of two dependent variables, x_1, x_2. Despite the linearity of the second equation in the system, the overall system [Eq. (8.66*b*)] is nonlinear.

In general, and with few exceptions, the nonlinear equation does not yield to analysis, so that machine, numerical, or graphical methods must be employed. Wherever possible and under very special circumstances approximations are made to "linearize" a nonlinear system in order to make the problem amenable to analysis.

8.3.2 Introduction to Systems of Nonlinear Differential Equations[7–9]

Perhaps the simplest classic example of a nonlinear system is the undamped free pendulum, the equation of motion of which is

$$\ddot{\theta} + (g/l)\sin\theta = 0 \tag{8.67}$$

This belongs to a class of elastic systems containing nonlinear restoring forces. Here $\sin\theta$ is clearly the nonlinear term. For small displacements of θ, $\sin\theta \approx \theta$ and Eq. (8.67) becomes

$$\ddot{\theta} + (g/l)\theta = 0 \tag{8.68}$$

The general solution of Eq. (8.68) is

$$\theta = A\sin\left(\sqrt{g/l}t\right) + B\cos\left(\sqrt{g/l}t\right) \tag{8.69}$$

A and B are constants of integration depending upon initial conditions. As θ gets large, Eq. (8.68) no longer holds, and therefore Eq. (8.69) is an invalid approximation to Eq. (8.67). Under this condition Eq. (8.67) cannot be "linearized." Other nonlinear restoring forces are characterized as hard and soft springs whose force F vs. deflection x characteristics are given by $F = ax + bx^2$, $a > 0$ where $b < 0$ for soft springs, $b > 0$ for hard springs, and $b = 0$ for linear springs (see Chap. 2). The degree of nonlinearity is measured by the relative magnitudes of bx^3 and ax and implies some knowledge of x. Linearization of the spring-mass system given by

$$\ddot{x} + ax + bx^3 = 0 \tag{8.70}$$

is possible if

$$|bx^3| \ll |ax|$$

for all x experienced, yielding the approximation

$$\ddot{x} + ax \approx 0$$

An electric analog of this system of Eq. (8.70) exists for an LC circuit where C depends upon q in accordance with $1/C = \alpha + \beta q^2$.

From $q/C + L\,d^2q/dt^2 = 0$ the following is derived after substitution for $1/C$:

$$L\ddot{q} + \alpha q + \beta q^3 = 0$$

Another analog derives from the nonlinear dependence of flux ϕ on current i in an LC circuit given by $i = \alpha\phi + \beta\phi^3$, which when substituted in the first time derivative of the loop-drop equation

$$L\frac{d\phi}{dt} + \frac{q}{C} = 0 \tag{8.71}$$

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yields $$L\frac{d^2\phi}{dt^2} + \frac{i}{c} = L\frac{d^2\phi}{dt^2} + \frac{\alpha\phi + \beta\phi^3}{C}$$

Expressed in generalized form, the foregoing nonlinear spring-mass (capacitance-inductance) systems are given by

$$\ddot{x} + f(x) = 0 \tag{8.72}$$

Multiplying Eq. (8.72) by $\dot{x}\,dt$ and integrating, we have

$$\begin{aligned} \int_0^t \dot{x}\ddot{x}\,dt + \int_0^t f(x)\dot{x}\,dt &= \frac{\dot{x}^2}{2}\Big]_0^t + \int_{x(0)}^{x(t)} f(x)\,dx \\ \frac{\dot{x}^2}{2} + \int_{x(0)}^{x(t)} f(x)\,dx &= \frac{\dot{x}_{(0)}^2}{2} \end{aligned} \tag{8.73}$$

which is a statement expressing energy conservation. If $V(x)$ is the indefinite integral,

$$\int f(x)\,dx = +V(x)$$

Then Eq. (8.73) becomes

$$\begin{aligned} \dot{x}^2/2 - x^2(0)/2 + V(x) - V[x(0)] &= 0 \\ \dot{x}^2/2 + V(x) = \dot{x}^2(0)/2 + V[x(0)] &\triangleq E \end{aligned} \tag{8.74}$$

$V(x)$ is the potential-energy function which represents stored energy from some arbitrary reference level, and E, a constant, is defined to be the "total energy" at any time. Solving Eq. (8.74), we have

$$\dot{x} = \sqrt{2[E - V(x)]} \tag{8.75}$$

Qualitative Behavior of the Conservative Free System. From Eq. (8.75) it is evident that physically realizable motion demands that $E \geq V(x)$ for all possible x. Consider a possible graph of $V(x)$ (Fig. 8.9) with E_0 drawn intersecting at points 1, 2, 3, 4, which points correspond to $E_0 = V(x)$, and from Eq. (8.75), $\dot{x} = 0$. Since $f(x) = (dV/dx)$, the slopes of the curve at these points give the spring force $f(x)$. From Eq. (8.72)

$$\ddot{x} = f(x) = -dV/dx \tag{8.76}$$

Consequently acceleration corresponds to the direction of arrows shown for the two possible states of motion in Fig. 8.9, implying periodic motion between x_1 and x_2 in one case, and between x_3 and x_4 in the order. To find the period for case 1, for example,

$$\tau = \int_{x_1}^{x_2} \frac{dx}{\dot{x}} + \int_{x_2}^{x_1} \frac{dx}{\dot{x}} = \oint \frac{dx}{\dot{x}} \tag{8.77}$$

where integration is around a cycle loop in a phase-plane plot shown in Fig. 8.9, where $\dot{x}$ is plotted as a function of x, and the sign of $\dot{x}$ equals the sign of dx.

For E_1 as in initial energy level shown in Fig. 8.9, motion is possible when $E_1 \geq V(x)$; it is seen that, for an initial negative velocity, the system will come to rest at point 5 and then, from Eq. (8.76), since the acceleration at that point is positive,

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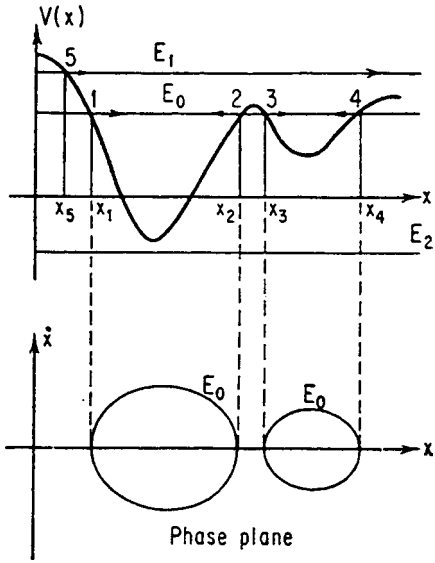

FIG. 8.9 Qualitative behavior of second-order free system.

motion would start to the right. Since $E_1 > V$ for $x > x_5$, it is impossible for x to reach zero again, and hence motion would continue in the positive direction without bound.

If $E = E_2$ as shown in Fig. 8.9, $E_2 < V(x)$ for all x; this cannot correspond to a physical system, a consequence of Eq. (8.75).

As an example of the above, the energy of the simple undamped pendulum is found from Eqs. (8.67) and (8.74):

$$\theta^2/2 - (g/l)\cos\theta = E$$

where motion is indicated between θ_1 and θ_2 for $E = E_0$ (Fig. 8.10). For $E = E_1$ motion continues in a single direction, which physically amounts to putting in more energy than that required to bring the pendulum into the position where it is vertically above its support.

Graphical Analysis of Second-Order Nonlinear Autonomous Differential Equations. Consider the following form of a free second-order equation with time-invariant coefficients:

$$\ddot{x} + f(x, \dot{x}) = 0 \tag{8.78}$$

It is possible to analyze this very restrictive equation by a graphical method called "phase-plane analysis." Equation (8.78) is first rewritten as

$$\begin{aligned} \dot{x}\, d\dot{x}/dx &= -f(x, \dot{x}) \\ d\dot{x}/dx &= -f(x, \dot{x})/\dot{x} \end{aligned} \tag{8.79}$$

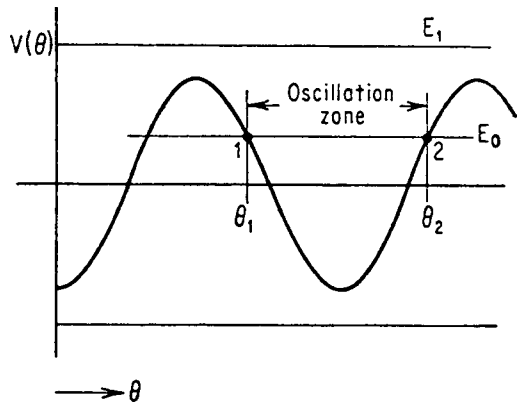

FIG. 8.10 Potential function for undamped pendulum $V(\theta) = \int \sin\theta\, d\theta = -\cos\theta$.

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A plot is next made of $\dot{x}$ as a function of x (phase-plane plot). At every point Eq. (8.79) states that the slope is $-f(x, \dot{x})/\dot{x}$.

The initial conditions $\dot{x}(0)$, $x(0)$ place the origin of the system in the phase plane. An arc with slope equal to $-f(x(0), \dot{x}(0))/\dot{x}(0)$ is laid off over a small length terminating at $x(1)$, $\dot{x}(1)$. The process is continued until either a stable point is reached, or a limit cycle is manifest, or indications show the growth without bound of the system parameters x or $\dot{x}$. To find x as a function of time, $t = \int dx/\dot{x}$.

Several convenient techniques are available to facilitate procedures (e.g., the isocline method), the essentials of which were described above. Special cases of phase-plane analyses are given below.

Special Case 1: Linear Spring-Mass System

$$\ddot{x} + kx = 0$$
$$\frac{\dot{x}^2}{2} + \frac{kx^2}{2} = \frac{\dot{x}(0)^2}{2} + \frac{kx^2(0)}{2} = E$$

The equation is of an ellipse in the phase plane $\dot{x}$ versus x, or if we make the following changes of variable:

$$y = x/\sqrt{k} \qquad \tau = \sqrt{k}t$$

and substitute in the above, we obtain

$$\begin{aligned} (dy/d\tau)^2 + y^2 &= 2E/k^2 \\ \dot{y}^2 + y^2 &= 2E/k^2 \end{aligned} \tag{8.80}$$

which represents a circle of radius $\sqrt{2E/k^2}$ about the origin in the phase plane of $\dot{y}$ versus y.

Special Case 2: Spring-Mass-Damper System

$$\ddot{x} + c\dot{x} + kx = 0$$
$$\frac{\dot{x}^2}{2} + \int_0^t c\dot{x}^2\,dt + \frac{kx^2}{2} = E \qquad c > 0$$

Writing the energy form, where the damping integral is greater than zero as shown earlier,

$$\frac{\dot{x}^2}{2} + \frac{kx^2}{2} = E - \int_0^t c\dot{x}^2\,dt$$
$$\frac{k^2\dot{y}^2}{2} + \frac{k^2y^2}{2} = E - \frac{k^2}{k^{1/2}}\int_0^\tau c\dot{y}^2\,d\tau$$

The right side decreases in time, so that in the phase-plane plot the locus must lie on a continuously decreasing radius from the origin as time increases, until the origin is reached. The actual path is a logarithmic spiral. Other systems of the form $\ddot{x} + f(\dot{x}) + kx = 0$ are, by suitable changes of variable, shown equivalent to

$$\ddot{y} + \phi(\dot{y}) + y = 0$$

whence

$$\frac{d\dot{y}}{dy} = \frac{-[\phi(\dot{y}) + y]}{\dot{y}} \tag{8.81}$$

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The phase-plane plot of Eq. (8.81) is obtainable by a neat method due to Liénard, described as follows: In Fig. 8.11, first $-\phi(\dot{y})$ is drawn. Then for any point of state, say P_1, the locus has a center of curvature in the phase plane located on the y axis shown by dotted construction. The slope must be that given by Eq. (8.81), tan ω. From geometry,

$$\tan \omega = -\frac{1}{\tan \theta} = \frac{-[\phi(\dot{y}) + y]}{\dot{y}}$$

Special Case 3: Coulomb Damping (Dry Friction), Second-Order System

$$\ddot{x} + c \operatorname{sgn} \dot{x} + x = 0 \tag{8.82}$$

where sgn = sign of. The phase-plane plot in Fig. 8.12 is accomplished, following Liénard's method, by first plotting $-c$ sgn x and then following in accordance with the above description. The plot consists of arcs of two circles centered at 1 for $\dot{x} > 0$ and 2 for $\dot{x} < 0$. This is shown for two different initial conditions corresponding to p_1 and p_1' in Fig. 8.12. Note that motion stops at a position corresponding to 4 since $\dot{u} = 0$ and the spring force is less than the impending damping force, thus preventing motion. This can also be shown analytically by considering two regions $\dot{x} < 0$ and $\dot{x} > 0$. Rewriting Eq. (8.82), we obtain

$$\dot{x}\, d\dot{x}/dx + (x + c \operatorname{sgn} \dot{x}) = 0$$

Multiplying Eq. (8.82) by dx and integrating for the two regions,

$$\dot{x}^2/2 + x^2/2 + cx = E_1 \rightarrow \dot{x}^2 + (x + c)^2 = 2E_1 + c^2 \qquad \dot{x} > 0$$

Similarly, $$\dot{x}^2 + (x - c)^2 = 2E_2 + c^2 \qquad \dot{x} < 0$$

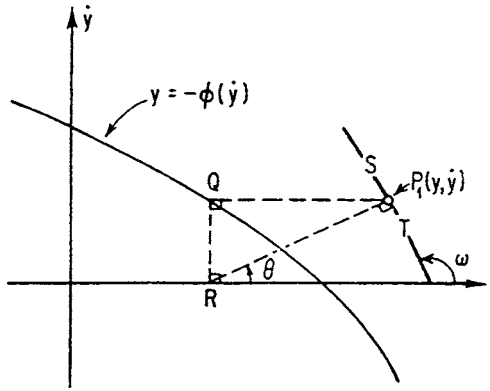

FIG. 8.11 Phase-plane plot of $d\dot{y}/dy = \{-[\phi(\dot{y}) + y]\}/\dot{y}$ (Liénard's construction).

Method: $QR = \dot{y}$

$$QP = y - [-\phi(\dot{y})] = y + \phi(\dot{y})$$

$$\tan \theta = QR/QP = \dot{y}/[y + \phi(\dot{y})]$$

$$= -1/(d\dot{y}/dy)$$

Hence it follows that the slope of line ST, perpendicular to line RP_1 at P_1, is $d\dot{y}/dy$.

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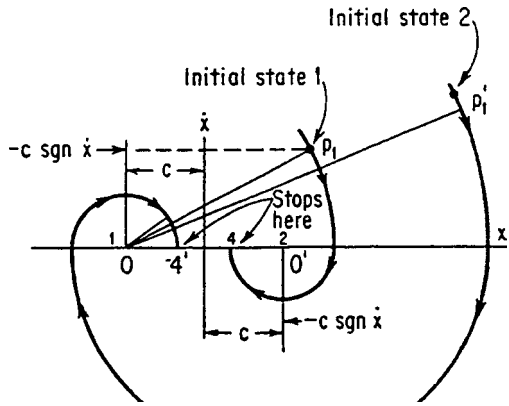

FIG. 8.12 Phase-plane plot for Coulomb damping of spring-mass system.

$$\ddot{x} + c \operatorname{sgn} \dot{x} + x = 0$$

Limit Cycles and Sustained Oscillations. Consider the system governed by $\ddot{x} + f(x, \dot{x}) + x = 0$. If

$$\dot{x}f(x, \dot{x}) < 0 \qquad |x| < \delta \tag{8.83a}$$

$$\dot{x}f(x, \dot{x}) > 0 \qquad |x| > \delta \tag{8.83b}$$

where δ is some positive constant, the system will exhibit a limit cycle which corresponds to a closed curve in the phase plane. When Eq. (8.83*a*) holds, there is a net increase in the system energy $\mathscr{E}$:

$$\frac{\dot{x}^2}{2} + \frac{x^2}{2} + \int_0^t f(x, \dot{x})\dot{x}\, dt = E = \frac{\dot{x}^2(0)}{2} + \frac{x^2(0)}{2} = \mathscr{E}(0)$$

$$\mathscr{E} = \frac{\dot{x}^2}{2} + \frac{x^2}{2} = E - \int_0^t f(\dot{x}, x)\dot{x}\, dt \tag{8.84}$$

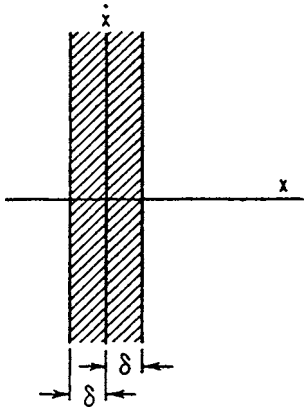

FIG. 8.13 Limit cycles and sustained oscillations.

given by the initial state E minus the integral. Since $2\mathscr{E}$ is the radius squared from the center to the point of state in the phase plane, there is a time rate of increase of radius every time the motion falls within the shaded zone (Fig. 8.13) and a decrease for motion corresponding to points outside the shaded zone. The type of oscillation is self-sustained and will start of its own accord for any initial condition. The van der Pol equation is an example of this type:

$$\ddot{x} - \epsilon\dot{x} + \beta x^2\dot{x} + x = 0 \qquad \epsilon > 0$$
$$\dot{x}f(x, \dot{x}) = \dot{x}^2(-\epsilon + \beta x^2) \tag{8.85}$$

The term $f(x, \dot{x})$ changes sign when

$$-\epsilon + \beta x^2 = 0$$

$$x = \pm\sqrt{\epsilon/\beta} = \pm\delta$$

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Limit cycles for higher-order systems are conceptually depicted by closed curves in multidimensional space, which is the generalization of single-degree-of-freedom systems.

Singular Points and Stability. It can be shown[7] that any autonomous set of nonlinear differential equations can be represented by

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \ldots, x_n) \\ \dot{x}_2 &= f_2(x_1, \ldots, x_n) \\ &\cdots\cdots\cdots\cdots \\ \dot{x}_n &= f_n(x_1, \ldots, x_n) \end{aligned} \tag{8.86}$$

The equilbrium positions are given by the roots of

$$\begin{aligned} f_1(x_1, \ldots, x_n) &= 0 \\ f_2(x_1, \ldots, x_n) &= 0 \\ &\cdots\cdots\cdots\cdots \\ f_n(x_1, \ldots, x_n) &= 0 \end{aligned} \tag{8.87}$$

The roots $x_1, \ldots, x_n$ of this set are called *singular equilibrium points* where all the time derivatives $\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n$ are equal to zero.

The algebraic solution to Eq. (8.87) gives in general one or more sets of singular points, e.g.,

$$\begin{aligned} &x_1^{(0)} \; x_2^{(0)} \cdots x_n^{(0)} \qquad \text{first set} \\ &x_1^{(1)} \; x_2^{(1)} \cdots x_n^{(1)} \qquad \text{second set} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

If the motion at any time corresponds to one of these points, the system is at rest. If left undisturbed, from Eq. (8.86), $\dot{x}_1 = \dot{x}_2 = \cdots = \dot{x}_n = 0$, the point (in phase space), and therefore the corresponding motion, does not change in time; i.e., the system remains at rest. If, however, the point is disturbed from its equilibrium position (perturbation), it is of interest from the stability point of view as to whether or not it will return to the point. If for a small perturbation from equilibrium the system tends to return to the same equilibrium position as $t \to \infty$, the system is said to be "asymptotically stable." If, however, the system diverges from the equilibrium point, it is said to be in a state of "unstable equilibrium," or the point is unstable. Special points in which neither of these events occurs are said to display neutral stability and are exceptional. To test stability, the nonlinear system is "linearized" in the neighborhood of the equilibrium point $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ by performing a Taylor's-series expansion about the point and ignoring terms higher than the first power of x_i. The typical expansion is

$$f_i = f_i(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) + \sum_i \left(\frac{\partial f_i}{\partial x_i}\right)_{x_k = \bar{x}_k} (x_i - \bar{x}_i) + \text{ higher-order terms } \quad j = 1, 2, \ldots, n$$

We define $$\left(\frac{\partial f_i}{\partial x_i}\right)_{x_k = \bar{x}_k} = a_{ji} \qquad f_i \approx \sum_i a_{ji}(x_1 - \bar{x}_i)$$

Since the constant term vanishes; i.e.,

$$f_i(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) = 0$$

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which is a consequence of the definition of the equilibrium point. For convenience, the substitution

$$\begin{aligned} x_1 - \bar{x}_1 &= y_1 \\ x_2 - \bar{x}_2 &= y_2 \\ &\cdots\cdots\cdots \\ x_n = \bar{x}_n &= y_n \end{aligned}$$

placed in Eq. (8.86) yields

$$\begin{aligned} \dot{y}_1 &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{y}_n &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n \end{aligned} \tag{8.88}$$

This is the well-known linear set whose solution is of the exponential type. Assuming a solution,

$$y_j = A_j e^{\lambda t}$$

and making this substitution in Eq. (8.88) yields

$$\begin{aligned} 0 &= (a_{11} - \lambda)A_1 + a_{12}A_2 + \cdots + a_{1n}A_n \\ 0 &= a_{21}A_1 + (A_{22} - \lambda)A_2 + \cdots + a_{2n}A_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 &= a_{n1}A_1 + a_{n2}A_2 + \mathrm{d}\cdots + (a_{nn} - \lambda)A_n \end{aligned} \tag{8.89}$$

From linear theory, the necessary condition for Eq. (8.89) to have nontrivial solutions is

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdots\cdots\cdots\cdots & \cdots\cdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0$$

which when expanded leads to an nth-order algebraic equation,

$$b_n\lambda^n + b_{n-1}\lambda^{n-1} + \cdots + b_1 x + b_0 = 0 \tag{8.90}$$

which has n roots for λ, the characteristic roots of the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots\cdots\cdots\cdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Each root corresponds to a solution $y_j = A_j e^{\lambda_i t}$. If λ_i has a real part greater than zero, y_j will grow without bound. Hence the necessary and sufficient condition for stability at the equilibrium point is that Re $\lambda_i < 0$ for all roots, $i = 1, \ldots, n$. As an example, consider the second-order van der Pol equation (8.85) in the form

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= \epsilon v - \beta x^2 v - x \end{aligned}$$

The only equilibrium point is $x = v = 0$, obtained after invoking Eq. (8.87).

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Expanding about $x = 0$, $v = 0$, carrying linear terms

$$\dot{x} = v$$
$$\dot{v} \approx -x + \epsilon v$$

The characteristic roots are found from

$$\begin{vmatrix} -\lambda & 1 \\ -1 & \epsilon - \lambda \end{vmatrix} = 0$$

$$-\lambda(\epsilon - \lambda) + 1 = 0 \qquad \lambda = (\epsilon \pm \sqrt{\epsilon^2 - 4})/2$$

It is evident that one or more roots must satisfy Re $\lambda > 0$; therefore, the system is unstable about the equilibrium point as observed previously. Note that, if ϵ is negative, the system is stable.

8.4 SYSTEMS OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS[11–14]

Formulation of the linearized form of a lumped dynamic system leads to a set of linear differential equations. The question of validity in assuming linearity is, in general, complicated. One method (though not conclusive) is to assume a linear form, solve the set, cast the solution into the original form to measure deviations from linearity, and finally, on this basis, render a decision on validity. An example is the simple pendulum undergoing a forced vibration where the steady-state solution is of interest:

$$\ddot{\theta} + \omega_0^2 \sin\theta = A \sin \omega t \qquad \omega_0^2 = g/l$$

The linearized form and its characteristic solution are

$$\ddot{\theta} + \omega_0^2 \theta = A \sin \omega t$$
$$\theta = [A/(\omega_0^2 - \omega^2)] \sin \omega t$$

And therefore θ is bounded by

$$|\theta| \leq |A|/(\omega_0^2 - \omega^2)$$

The linear form is valid if $(\sin\theta - \theta)/\theta$ remains small for all motion, i.e.,

$$|(\sin\theta - \theta)/\theta| \ll 1$$

But $$\sin\theta = \theta - \theta^3/3! + \theta^5/5! - \cdots$$

Therefore

$$\sin\theta/\theta = 1 - \theta^2/3! + \theta^4/5! - \cdots \qquad |(\sin\theta - \theta)/\theta| \leq \theta^2/3!$$

and the necessary condition for validity becomes

$$|\theta^2/3!| \ll 1$$

Linear systems are classified as time-variant or time-invariant and in the former derive from a system containing time-variable parameters leading to product terms in independent and dependent variables, e.g., $\sin[\alpha t(d^3x/dt^3)]$ and t^2x, whereas the time-invariant case shows no parametric dependence on time.

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An example of a time-variable linear system is that of a rocket propelled in free unidirectional flight by a jet-exhaust thrust $C(t)$. Fuel expenditure results in a rocket mass loss. The equation of motion is

$$m(t)\, dv/dt = C(t) \tag{8.91}$$

with $m(t)$ and $C(t)$, the mass and thrust, connected under some broad assumptions by the linear differential equation

$$C(t) = -k[dm(t)/dt] \tag{8.92}$$

The solution is

$$v - v(0) = k \ln [m(0)/m]$$

This is an exceptional case since a closed solution is obtainable. More generally, however, time-variable systems are practically invulnerable to analytic attack in contrast to their time-invariant counterparts whose solutions are completely known.

Properties of Linear Differential Equations. The general properties of linear differential equations are stated as follows:

1. The general homogeneous linear equation is expressed in operator form

$$D(Y) = f(t)$$

with initial conditions

$$Y(0) = a_0 \qquad \dot{Y}(0) = a_1 \qquad (d^{n-1}/dt^{n-1})Y(0) = a_{n-1} \tag{8.93}$$

D is the linear operator of nth order:

$$D = \sum_{m=0}^{n} P_m(t) \frac{d^m}{dt^m}$$

where $P_m(t)$ is the coefficient (function of time in general) of the nth derivative term.

2. The linear operator has the properties

$$D(\alpha y) = \alpha\, Dy \qquad (\alpha = \text{const})$$
$$D(y_1 + y_2) = Dy_1 + Dy_2$$

Therefore,
$$D(\alpha y_1 + \beta y_2) = \alpha\, Dy_1 + \beta\, Dy_2$$

3. The complete solution to the homogeneous equation $D(Z) = 0$ of the nth degree is the sum of n linearly independent solutions, viz.,

$$Z = b_1 z_1 + b_2 z_2 + \cdots + b_n z_n$$

where $b_1, \ldots, b_n$ are constants which can be adjusted to satisfy n initial conditions of the problem.

4. The general solution to Eq. (8.93) is the sum of two solutions $Y = Z + X$, where Z is the total solution to the homogeneous equation

$$D(Z) = 0 \tag{8.94}$$

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and X is *any* solution to $D(X) = f(t)$ regardless of initial conditions. It should be emphasized, and is implied, that X is not unique, containing any number of solutions to the homogeneous equation, e.g., $c_1 z_1 + \cdots + c_m z_m$.

5. If the solution to $D(X) = f(t)$ is confined to be the asymptotic solution (i.e., the solution as $t \to \infty$, then this solution $X \triangleq X_p$ is called the "particular solution" and the remainder $Z = b_1 z_1 + b_2 z_2 + \cdots + b_n z_n \triangleq Z_c$ is called the "complementary solution" and is called the "transient solution" if it vanishes with time. The terms $b_1, \ldots, b_n$ are chosen such that all initial conditions of Eq. (8.93) are satisfied. Then since the solution to Eq. (8.93) is unique,[11]

$$Y = X_p + Z_c = X + Z$$

6. Another important specialized partitioning of X and Z restricts X to satisfy zero initial conditions, i.e., the solution to $D(x) = f(t)$ for

$$X(0) = \dot{X}(0) = \cdots = \frac{d^{n-1}}{dt^{n-1}} X(0) = 0$$

which is

$$X_{(1)} = \int_0^t W(t, T) f(T)\, dT$$

where $W(t, T)$ is the solution to

$$D(W) = \delta(t - T)$$

$W(t, T) \equiv 0$ for $t \leq T$, and $\delta(t - T)$ is defined as the Dirac delta function. The remaining part of the solution is

$$Z_{(1)} = c_1 z_1 + c_2 z_2 + \cdots + c_n z_n$$

where the c's are chosen to satisfy the initial conditions of the problem, which are absorbed in the z's alone.

$$Y(0) = Z_{(1)}(0) = \sum_{i=1}^{n} c_i z_i(0)$$

$$Y'(0) = Z'_{(1)}(0) = \sum_{i=1}^{n} c_i z'_i(0)$$

$$Y''(0) = Z''_{(1)}(0) = \sum_{i=1}^{n} c_i z''_i(0)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{d^{n-1}}{dt^{n-1}} Y(0) = \frac{d^{n-1} Z_{(1)}(0)}{dt^{n-1}} = \sum_{i=1}^{n} c_i \frac{d^{n-1}}{dt^{n-1}} z_{(i)}(0)$$

and

$$Y = X_{(1)} + Z_{(1)} = X + Z$$

7. As a direct consequence of property 6, the principle of superposition follows. For a system initially inert, i.e., for zero initial conditions,

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y_1 is the response to a forcing function $f_1(t)$

y_2 is the response to a forcing function $f_2(t)$

The response y to forcing function $f(t) = \alpha f_1(t) + \beta f_2(t)$ is $y = \alpha y_1 + \beta y_2$.

8. The necessary and sufficient condition that the solution y be bounded in Eq. (8.93) for any bounded input $f(t)$ is that

$$\int_0^\infty |\, W(t, T)\,|\, dT < M < \infty$$

where M is some arbitrarily large positive number. Systems satisfying this criterion are said to be stable.

9. The fact that n solutions to Eq. (8.94) are linearly independent requires that the wronskian be different from zero at any point in the interval $t_1 < t < t_2$ where Eq. (8.94) is valid.

$$W_r(z_1, z_2, \ldots, z_n) = \det \begin{bmatrix} z_1 & z_1 & \cdots & z_n \\ dz_1/dt & dz_2/dt & \cdots & dz_n/dt \\ \cdots & \cdots & \cdots & \cdots \\ -d^{n-1}z_1/dt^{n-1} & d^{n-1}z_2/dt^{n-1} & \cdots & d^{n-1}z_n/dt^{n-1} \end{bmatrix}$$

The wronskian of the solutions to

$$d^n z/dt^n + Q_{(n-1)}(t)(d^{n-1}z)/dt^{n-1}) + \cdots + Q_1(t)\, dz/dt + Q_0(t)z = 0$$

is given by

$$W_r(z_1, \ldots, z_n) = W_r[z_1(\tau), z_2(\tau), \ldots, z_n(\tau)] \exp \int_\tau^t Q_{n-1}(x)\, dx \tag{8.95}$$

As a direct consequence of Eq. (8.95) the wronskian of a set satisfying Eq. (8.94) either does not vanish at all or vanishes identically since the exponential term cannot vanish.

8.4.1 Introduction to Matrix Analysis of Differential Equations[15]

The nth-order differential equation

$$d^n y_1/dt^n + Q_{n-1}(d^{n-1}y_1/dt^{n-1}) + \cdots + Q_0 y_1 = F \tag{8.96}$$

can be written as n first-order differential equations

$$\begin{aligned} dy_1/dt &= y_2 \\ dy_2/dt &= y_3 \\ &\cdots\cdots\cdots \\ dy_{n-1}/dt^{n-1} &= y_n \\ dy_n/dt &= -Q_{n-1}y_n - Q_{n-2}y_{n-1} - \cdots - Q_0 y_1 + F \end{aligned} \tag{8.97}$$

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which in matrix form is written as

$$\frac{d\mathbf{y}}{dt} = \mathbf{A}\mathbf{y} + \mathbf{f} \qquad \mathbf{y} = \begin{vmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{vmatrix} \tag{8.98}$$

In Eq. (8.97),

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -Q_0 & -Q_1 & -Q_2 & -Q_3 & \cdots & -Q_{n-2} & -Q_{n-1} \end{bmatrix} \qquad \mathbf{f} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ F \end{bmatrix}$$

Equation (8.98) is linear if matrix $\mathbf{A} = \mathbf{A}(t)$ and $\mathbf{f} = \mathbf{f}(t)$ and linear time-invariant if A does not depend on t.

Following classical methods, the total solution to

$$d\mathbf{y}/dt = \mathbf{A}(t)\mathbf{y} + \mathbf{f}(t)$$

is the sum of two solutions, one to the homogeneous equation

$$d\mathbf{z}/dt = \mathbf{A}(t)\mathbf{z} \tag{8.99}$$

plus any solution to

$$d\mathbf{x}/dt = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t) \tag{8.100}$$

where

$$\mathbf{y} = \mathbf{x} + \mathbf{z}$$

The general solution chosen here will let the homogeneous solution satisfy the initial conditions of the problem, i.e.,

$$\mathbf{y}(0) = \mathbf{z}(0) = \mathbf{c}$$

and the remaining solution satisfy the null-vector condition at $t = 0$,

$$\mathbf{x}(0) = \mathbf{0}$$

so that

$$\mathbf{y}(0) = \mathbf{z}(0) + \mathbf{x}(0) = \mathbf{y}(0) + \mathbf{0} = \mathbf{y}(0) = \mathbf{c}$$

as required.

If the solution to Eq. (8.99) is known, the solution to Eq. (8.100) can be obtained by Lagrange's method of variation of parameters as follows: Consider the matrix differential equation

$$d\mathbf{Z}(t)/dt = \mathbf{A}(t)\mathbf{Z}(t) \qquad \mathbf{Z}(0) = \mathbf{I}(\text{initial conditions}) \tag{8.100}a$$

Postmultiplication by $\mathbf{c}$ yields

$$d(\mathbf{Z}\mathbf{c})/dt = \mathbf{A}(\mathbf{Z}\mathbf{c})$$

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From this and Eq. (8.99) it follows that

$$\mathbf{z} = \mathbf{Z}(t)\mathbf{c} = \mathbf{Z}(t)\mathbf{z}(\mathbf{0})$$

Now let $$\mathbf{y} \triangleq \mathbf{Z}(t)\mathbf{u} \tag{8.100b}$$

Substituting for $\mathbf{y}$ in Eq. (8.98),

$$\frac{d\mathbf{y}}{dt} = \mathbf{A}(t)[\mathbf{Z}(t)\mathbf{u}] + \mathbf{f} = \mathbf{Z}(t)\frac{d\mathbf{u}}{dt} + \frac{d\mathbf{Z}(t)}{dt}\mathbf{u} = \mathbf{Z}(t)\frac{d\mathbf{u}}{dt} + \mathbf{A}(t)\mathbf{Z}(t)\mathbf{u}$$

whence $\mathbf{f} = \mathbf{Z}\, d\mathbf{u}/dt$. Premultiplying by $\mathbf{Z}^{-1}$ yields $\mathbf{Z}^{-1}\mathbf{f} = d\mathbf{u}/dt$, and integrating after separation of variables gives

$$\mathbf{u} = \mathbf{u}(0) + \int_0^t \mathbf{Z}^{-1}(\tau)\mathbf{f}(\tau)\, d\tau$$

Premultiply by $\mathbf{Z}(t)$ to give

$$\mathbf{y} = \mathbf{Z}(t)\mathbf{u} = \mathbf{Z}(t)\mathbf{c} + \int_0^t \mathbf{Z}(t)\mathbf{Z}^{-1}(\tau)\mathbf{f}(\tau)\, d\tau \tag{8.101}$$

where $\mathbf{u}(0) = \mathbf{y}(0) = \mathbf{c}$ from Eq. (8.100*b*) and $\mathbf{Z}(0) = \mathbf{I}$.

For the time-invariant case,

$$\mathbf{Z}(t)\mathbf{Z}^{-1}(\tau) = Z(t - \tau)$$

and Eq. (8.101) takes the simpler form

$$\mathbf{y} = \mathbf{Z}(t)\mathbf{c} + \int_0^t \mathbf{Z}(t - \tau)\mathbf{f}(\tau)\, d\tau \tag{8.102}$$

Eqs. (8.101) and (8.102) hinge on the solution to the homogeneous matrix equation

$$d\mathbf{Z}/dt = \mathbf{A}(t)\mathbf{Z} \tag{8.103}$$

However, for $\mathbf{A}(t)$, the time-variable case, a solution is rarely possible. For the time-invariant case,

$$d\mathbf{Z}(t)/dt = \mathbf{A}\mathbf{Z}(t) \qquad \mathbf{Z}(0) = \mathbf{I} \tag{8.104}$$

The formal solution is

$$\mathbf{Z}(t) = e^{\mathbf{A}t}$$

which must be defined as

$$e^{\mathbf{A}t} \triangleq \sum_{n=0}^{\infty} \frac{\mathbf{A}^n t^n}{n!}$$

and $$\mathbf{Z}(t + \tau) = e^{\mathbf{A}(t+\tau)}$$

Equation (8.102) then becomes

$$y = e^{\mathbf{A}t}\mathbf{c} + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{f}(\tau)\, d\tau \tag{8.105}$$

Equation (8.105) is in a form that is not useful for quantitative analysis.

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It is desirable to obtain $\mathbf{Z}$ in closed form. If $\mathbf{Z}$ is represented by column vectors

$$\mathbf{Z} = \begin{bmatrix} -z_1^{(1)} & \cdots & z_1^{(n)} \\ z_2^{(1)} & \cdots & z_2^{(n)} \\ \cdots & \cdots & \cdots \\ -z_n^{(1)} & \cdots & z_n^{(n)} \end{bmatrix} \triangleq [\mathbf{z}^{(1)} \quad \mathbf{z}^{(2)} \quad \mathbf{z}^{(3)} \quad \cdots \quad \mathbf{z}^{(n)}]$$

$$\text{then } d\mathbf{Z}/dt = \mathbf{AZ} \qquad \mathbf{Z}(0) = \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

is equivalent to n equations

$$\begin{aligned} \frac{d\mathbf{z}^{(1)}}{dt} &= \mathbf{Az}^{(1)} \qquad \mathbf{z}^{(1)}(0) = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{d\mathbf{z}^{(n)}}{dt} &= \mathbf{Az}^{(n)} \qquad \mathbf{z}^{(n)}(0) = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \end{aligned} \tag{8.106}$$

The homogeneous matrix equation possesses n possible solution vectors. Assuming one such vector solution

$$\mathbf{z}^{(k)} = \mathbf{c}^{(k)} e^{\lambda j t} \qquad j = 1, 2, \ldots, n$$

and entering this into Eq. (8.106) yields

$$\begin{aligned} \lambda_j \mathbf{c}^k &= \mathbf{Ac}^{(k)} \\ (\mathbf{A} - \lambda_j \mathbf{I}\mathbf{c}^k) &= 0 \end{aligned} \tag{8.107}$$

which is the eigenvector equation that must satisfy

$$|\mathbf{A} - \lambda_j \mathbf{I}| = 0$$

an nth-order equation yielding n roots for λ_j. The discussion here will be limited to distinct roots.

For each λ_j so found there exists a column vector formed from the cofactors of any row of $(A - \lambda_j I)$; let the matrix formed by the n column vectors thus formed be called B:

$$\mathbf{B} = [\mathbf{c}^{k(1)} \quad \mathbf{c}^{k(2)} \quad \cdots \quad \mathbf{c}^{k(n)}]$$

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Equation (8.107) can be represented as

$$\mathbf{AB} = \mathbf{B\Lambda} \qquad \mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Premultiplying by $\mathbf{B}^{-1}$ yields

$$\mathbf{B}^{-1}\mathbf{AB} = \mathbf{\Lambda}$$

which shows that $\mathbf{A}$ is diagonalized by a linear transformation.

From Eq. (8.100*a*),

$$d\mathbf{Z}/dt = \mathbf{AZ} \qquad \mathbf{Z}(0) = \mathbf{I}$$

Let $\mathbf{W}$ be introduced by defining the transformation

$$\mathbf{Z} = \mathbf{BW}$$

Substituting in Eq. (8.100*a*), we have

$$d(\mathbf{BW})/dt = \mathbf{A}(\mathbf{BW})$$
$$\mathbf{B}(d\mathbf{W}/dt) = \mathbf{ABW}$$

Premultiplying by $\mathbf{B}^{-1}$, we have

$$\frac{d\mathbf{W}}{dt} = \mathbf{B}^{-1}\mathbf{ABW} = \mathbf{\Lambda W} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \mathbf{W}$$

The solution for $\mathbf{W}$ is easily verified to be

$$\mathbf{W} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} \mathbf{B}^{-1} \qquad \mathbf{W}(0) = \mathbf{B}^{-1}$$

which satisfies its differential equation and the initial conditions, viz.,

$$\begin{bmatrix} \lambda_1 e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & \lambda_2 e^{\lambda_2 t} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n e^{\lambda_n t} \end{bmatrix} \mathbf{B}^{-1} \equiv \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} \mathbf{B}^{-1}$$

$$d\mathbf{W}/dt = \mathbf{\Lambda W}$$
$$\mathbf{Z}(0) = \mathbf{BW}(0) = \mathbf{BB}^{-1} = I$$

as required.

Linear Time-Invariant Systems.[16–24] The important property that distinguishes these systems from the time-variable systems is the following:

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1. If the input $f(t)$ yields the response $y(t)$, the input $f(t + T)$ yields the response $y(t + T)$ (where all initial conditions are zero). As a consequence of property 1 and the superposition property we have the following property.
2. The response to the derivative of an arbitrary input is equal to the derivative of the response. If $f(t) \to y(t)$, then

$$\frac{f(t+\epsilon) - f(t)}{\epsilon} \to \frac{y(t+\epsilon) - y(t)}{\epsilon} \qquad \begin{matrix} \epsilon \to 0 \\ f'(t) \to y'(t) \end{matrix}$$

or, vectorially, $\mathbf{f}(t) \to \phi[\mathbf{y}(t)]$ where ϕ is a linear time-invariant operator. Then

$$\frac{\mathbf{f}(t+\epsilon) - \mathbf{f}(t)}{\epsilon} \to \frac{\phi[\mathbf{y}(t+\epsilon) - \mathbf{y}(t)]}{\epsilon} = \phi[\mathbf{y}'(t)] \qquad \begin{matrix} \epsilon \to 0 \\ \mathbf{f}'(t) \to \phi[\mathbf{y}'(t)] \end{matrix}$$

3. The solution to a free (homogeneous) time-invariant system of equations is composed of exponential terms, there being as many terms as the highest degree of the differential equation obtained in one dependent variable. These terms must be linearly independent, satisfying $W_r \neq 0$.

   The general form of the coupled time-invariant system of n degrees of freedom is

$$\begin{aligned} a_{11}(p)y_1 + a_{12}(p)y_2 + \cdots + a_{1n}(p)y_n &= f_1(t) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{n1}(p)y_1 + a_{n2}(p)y_2 + \cdots + a_{nn}(p)y_n &= f_n(t) \end{aligned} \tag{8.108}$$

In matrix-operator form,

$$\mathbf{A}(p)\mathbf{y} = \mathbf{f}(t) \qquad \mathbf{A}(p) = \begin{bmatrix} a_{11}(p) & a_{12}(p) & \cdots & a_{1n}(p) \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}(p) & a_{n2}(p) & \cdots & a_{nn}(p) \end{bmatrix} \qquad \mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{bmatrix} \tag{8.108a}$$

where p is the differential-integral operator defined by

$$p \triangleq \frac{d(\ )}{dt} \qquad \frac{1}{p} \triangleq \int_0^t (\ )dt$$

For linear passive systems (i.e., involving inductance, capacitance, and resistance or inertia, spring, and damping), each coefficient takes the form

$$a_{ij}(p) = L_{ij}p + 1/C_{ij}p + R_{ij}$$

where L_{ij}, C_{ij}, and R_{ij} are constants.

Recall that the general solution to Eq. (8.108) is composed of two solutions, one to the homogeneous system,

$$\mathbf{A}(p)\mathbf{z} = 0 \quad \text{or} \quad \begin{aligned} a_{11}(p)z_1 + a_{12}(p)z_2 + \cdots + a_{1n}(p)z_n &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{n1}(p)z_1 + a_{n2}(p)z_2 + \cdots + a_{nn}(p)z_n &= 0 \end{aligned} \tag{8.109}$$

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and one to the inhomogeneous equation,

$$\mathbf{A}(p)\mathbf{x} = \mathbf{f}(t)$$

Thus the total solution is

$$\mathbf{y} = \mathbf{x} + \mathbf{z} \tag{8.110}$$

To find $\mathbf{z}$ assume, as before, the exponential form [similar to Eq. (8.106a)]

$$\mathbf{z} = \begin{vmatrix} z_1 = c_1^{(i)}e^{\lambda t} \\ z_2 = c_2^{(i)}e^{\lambda t} \\ \cdots\cdots\cdots\cdots \\ z_n = c_n^{(i)}e^{\lambda t} \end{vmatrix}$$

Substitution in Eq. (8.109) yields

$$\mathbf{A}(\lambda)\mathbf{c} = 0 \qquad \text{or} \qquad \begin{aligned} &a_{11}(\lambda)c_1 + a_{12}(\lambda)c_2 + \cdots + a_{1n}(\lambda)c_n = 0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &a_{n1}(\lambda)c_1 + a_{n2}(\lambda)c_2 + \cdots + a_{nn}(\lambda)c_n = 0 \end{aligned} \tag{8.111}$$

From linear theory a solution $c_1, c_2, \ldots, c_n$ different from zero (the trivial case) can exist if and only if the determinant vanishes; thus

$$\det[\mathbf{A}(\lambda)] = 0 \qquad \begin{bmatrix} a_{11}(\lambda) & a_{12}(\lambda) & \cdots & a_{1n}(\lambda) \\ a_{21}(\lambda) & a_{22}(\lambda) & \cdots & a_{2n}(\lambda) \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}(\lambda) & a_{n2}(\lambda) & \cdots & a_{nn}(\lambda) \end{bmatrix} = 0$$

which leads to an mth-degree algebraic equation in λ called the characteristic equation of the matrix $\mathbf{A}$. In general, $m \neq n$ and $m \leqq 2n$ for passive systems.

The mth-degree equation yields m roots $\lambda_1, \lambda_2, \ldots, \lambda_m$, each of which satisfies Eq. (8.111). If the roots are distinct, the corresponding $\mathbf{c}$ column vector

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_n \end{bmatrix}$$

can be found for each λ_i, being the cofactors of any row of $\mathbf{A}(\lambda i)$ in order from left to right: $c_1 = \text{cof}\,(a_{j1})$; $c_2 = \text{cof}\,(a_{j2})$; $c_n = \text{cof}\,(a_{jn})$ for any j. The solution to Eq. (8.109) is then

$$\mathbf{z} = \sum_{i=1}^{n} \mathbf{c}^{(i)} e^{\lambda_i t} \qquad \text{or} \qquad \begin{aligned} &z_1 = c_1^{(1)}e^{\lambda_1 t} + c_1^{(2)}e^{\lambda_2 t} + \cdots + c_1^{(m)}e^{\lambda m t} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &z_n = c_n^{(1)}e^{\lambda_1 t} + c_n^{(2)}e^{\lambda_2 t} + \cdots + c_n^{(m)}e^{\lambda m t} \end{aligned}$$

If some of the roots are repeated, then the above fails and these roots of multiplicity $v_j (v_j > 1)$ yield the solution

$$\sum [\mathbf{c}^{(j)} + \mathbf{d}^{(j)}t + \cdots + \mathbf{h}^{(j)}t^{\lambda_{j-1}}]e^{\lambda_j t}$$

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The total solution is

$$\mathbf{z} = \sum_{i=l+1}^{m-q+1} \mathbf{c}^{(i)} e^{\lambda_i t} + \sum_{j=1}^{l} [\mathbf{c}^{(j)} + \mathbf{d}^{(j)} t + \cdots + \mathbf{h}^{j} t^{\lambda_{j-1}}] e^{\lambda_j t} \qquad q = \sum v_j$$

Alternatively, and if interest is focused on one of the dependent variables in Eq. (8.109), then all other variables can be eliminated to yield

$$\det \begin{bmatrix} a_{11}(p) & \cdots & a_{1n}(p) \\ \cdots\cdots\cdots\cdots\cdots & & \\ a_{n1}(p) & \cdots & a_{nn}(p) \end{bmatrix} z_i = 0 \quad \text{or} \quad |\mathbf{A}(p)|\, z_i = 0 \qquad i = 1, 2, \ldots, n$$

giving the identical homogeneous equation for each of the dependent variables. If, as before, the exponential form $z_1 = c_1 e^{\lambda t}$ is assumed, substitution gives $|\mathbf{A}(\lambda)| c_1 e^{\lambda t} = 0$, where for $c_1 \neq 0$, $|\mathbf{A}(\lambda)| = 0$, giving the same characteristic equation for the exponential constants as before. The inhomogeneous reduced equations from Eq. (8.108) are formally obtained by purely algebraic considerations as

$$|\mathbf{A}(p)| y_i = \sum_{i=1}^{n} M_{ij}(p) f_i \qquad j = 1, 2, \ldots, n$$

where M_{ij} is the cofactor of the element in the ith row and jth column of A.

Let $$|\mathbf{A}(p)| \triangleq D(p)$$

$$D(p) y_j = \sum_{i=1}^{n} M_{ij}(p) f_i(t) \qquad j = 1, 2, \ldots, n$$

From the previous considerations,

$$D(p) z_j = 0 \tag{8.112a}$$

$$D(p) x_j = \sum_{i=1}^{n} M_{ij}(p) f_i(t) \qquad j = 1, 2, \ldots, n \tag{8.112b}$$

$$y_j = x_j + z_j$$

To conclude, in general the solution x_j may be determined without regard for the initial conditions. It can be obtained in many ways depending on the character of the f_i's. If the f_i's are known as a finite power series, the method of undetermined coefficients will be expeditious; if it has more general behavior, it may be convenient to use the method of variation of parameters; if f_i's are exponential (including sin, cos, sinh, cosh) then an assumed exponential solution for each exponent will yield the answer; if the f_i's are periodic, by Fourier analysis these can be reconstructed as exponential functions and solved as outlined above; if f_i's are not periodic, having certain restrictive integral-convergence behavior, Fourier integral methods can be utilized. Last and most powerful is the Laplace-transform method, which not only has the widest range of applicability but can be utilized to obtain y (the total solution) directly.

8.4.2 Fourier-Series Analysis

If $f(t)$ is real and periodic of period T, with few restrictions, it can be approximately expressed as a linear sum of sine and cosine terms (Fourier series) or exponential terms. That is, if

$$f(t) = f(t + T)$$

then $$f(t) = \frac{a_0}{T} + \sum_{n=1}^{\infty} a_n \cos \omega_0 n t + b_n \sin \omega_0 n t$$

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where $\omega_0 = 2\pi/T$ and a_n, b_n are real constants. Alternatively,

$$f(t) = \sum_{n=-\infty}^{+\infty} c_n e^{j\omega_0 nt} \tag{8.113}$$

$$\text{where } a_0 = \int_t^{t+T} f(t)\,dt$$

$$a_n = \frac{2}{T}\int_t^{t+T} f(t)\cos n\omega_0 t\,dt$$

$$b_n = \frac{2}{T}\int_t^{t+T} f(t)\sin n\omega_0 t\,dt$$

$$c_n = \frac{1}{T}\int_t^{t+T} f(t)\,e^{-jn\omega_0 t}\,dt$$

$$2c_n = a_n - jb_n$$

$$2c_{-n} = a_n + jb_n$$

Let $e^{j\omega t}$ be the input f_i in Eq. (8.112). For a response $X_{ij}^{(1)} = H_{ij}(j\omega)e^{j\omega t}$ it is required to find $H_{ij}(j\omega)$. From Eq. (8.112) it is clear that all operations on $e^{j\omega t}$ are equivalent to replacing p by $j\omega$ and Eq. (8.112) becomes, for $f_i = e^{j\omega t}$,

$$D(j\omega)H_{ij}(j\omega)e^{j\omega t} = M_{ij}(j\omega)e^{j\omega t}$$

yielding
$$\begin{aligned} X_{ij}^{(1)} &= H_{ij}(j\omega)e^{j\omega t} = \frac{M_{ij}(j\omega)}{D(j\omega)}\,e^{j\omega t} \\ &= [P(j\omega) + Q(j\omega)]e^{j\omega t} \\ &= R(j\omega)e^{j\phi}e^{j\omega t} \qquad \phi = \tan^{-1}\frac{-jQ(j\omega)}{P(j\omega)} \\ &= R(j\omega)e^{j(\omega t+\phi)} \qquad R(j\omega) = [P^2(j\omega) + Q^2(j\omega)]^{1/2} \end{aligned} \tag{8.114}$$

where $P(j\omega)$ and $R(j\omega)$ are real and therefore even functions of $j\omega$, and $Q(j\omega)$ is imaginary and an odd function of $j\omega$ and the following properties apply:

$$\left.\begin{aligned} Q(j\omega) &= -Q(-j\omega) && \text{odd function} \\ p(j\omega) &= p(-j\omega) \\ R(j\omega) &= R(-j\omega) \end{aligned}\right\} \quad \text{even functions}$$

Similarly for an input $e^{-j\omega t}$, the output is

$$X_{ij}^{(2)} = H_{ij}(-j\omega)e^{-j\omega t} = [P(-j\omega) + Q(-j\omega)]e^{-j\omega t} = Re^{j\phi'}e^{-j\omega t}$$

Since
$$\phi' = \tan^{-1}\frac{-jQ(-j\omega)}{P(j\omega)} = \tan^{-1}\frac{jQ(j\omega)}{P(j\omega)} = -\phi$$

the response $X_{ij}^{(2)}$ is

$$X_{ij}^{(2)} = H_{ij}(-j\omega)e^{-j\omega t} = Re^{-j(\omega t+\phi)} \tag{8.115}$$

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The sum of the responses is

$$X_{ij}^{(1)} + X_{ij}^{(2)} = H_{ij}(-j\omega)e^{-j\omega t} + H_{ij}(j\omega)e^{j\omega t} = Re^{j(\omega t+\phi)} + Re^{-j(\omega t+\phi)}$$
$$= 2R\cos(\omega t + \phi)$$

which is just twice the real part of either response or

$$X_{ij}^{(1)} + X_{ij}^{(2)} = 2\,\mathrm{Re}\,X_{ij}^{(1)} = 2\,\mathrm{Re}\,X_{ij}^{(2)}$$

showing that the total input, $e^{j\omega t} + e^{-j\omega t} = 2\cos\omega t$, results in an output of different phase and amplitude.

$$H_{ij}(j\omega) = M_{ij}(j\omega)/D(j\omega) = R(j\omega)e^{j\phi(j\omega)}$$

is called the "transfer function" for sinusoidal inputs (real frequency) containing both amplitude and phase information.

It follows readily from superposition that the periodic responses to forcing functions having Fourier-series representations are available as a sum of responses of the form

$$X = \sum_{n=0}^{\infty} Ra_n \sin(n\omega_0 t + \phi_n) + Rb_n \cos(n\omega_0 t + \phi_n)$$

where $\phi_n = \phi(jn\omega_0)$ and $\phi_{-n} = \phi(-jn\omega_0)$

or

$$X = \sum_{n=-\infty}^{+\infty} Rc_n e^{j(\omega_0 nt+\phi_n)}$$

8.4.3 Complex Frequency-Domain Analysis[16]

It is often convenient to cast the linear system from its time-domain representation into a frequency-domain form in order to simplify analysis or exhibit more clearly certain of its important properties (e.g., spectrum, stability). The Fourier- and Laplace-transform methods are most prominent in this regard.

Fourier-Transform Method. If the input forcing function(s) are not periodic functions of time, the Fourier-transform method may be employed to solve Eq. (8.112) for each input f_i whenever

$$\int_{-\infty}^{+\infty} |f_i(t)|\,dt < \infty \tag{8.116}$$

That is, the absolute convergence of the infinite integral is a sufficient condition for Fourier transformability. Examples of functions not satisfying Eq. (8.116) are the step function,* sinusoid, rising exponentials, and functions containing t to positive exponents, e.g., ramp function (αt). Examples of functions satisfying Eq. (8.116) are pulses of finite duration. The Fourier-integral theorem asserts

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} f(T)e^{j\omega(t-T)}dT \tag{8.117}$$

*These have Fourier representations despite violation of Eq. (8.116). Note that Eq. (8.116) is only a sufficient condition.

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If $f(t)$ has a finite discontinuity at any point, then this integration will yield the average value of $f(t)$ at the discontinuity. From Eq. (8.117) the Fourier-transform pair is obtained:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(j\omega)e^{j\omega t}d\omega = \int_{-\infty}^{+\infty} F(j2\pi f)e^{j2\pi ft}\,df \tag{8.118a}$$

$$F(j\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}\,dt = \int_{-\infty}^{+\infty} f(t)e^{-j2\pi f t}\,dt \tag{8.118b}$$

where $F(j\omega)$ is in general complex and is denoted as the complex spectrum of $f(t)$. Equation (8.118*a*) can be imagined to express $f(t)$ as the infinite sum of Fourier components $F(j\omega)e^{j\omega t}\,d\omega/2\pi$. From the superposition principle, the total response is made up of the sum of each of the responses $H_{ij}(j\omega)e^{j\omega t}\,d\omega/2\pi$, where $H_{ij}(j\omega)$ was defined as the real frequency-transfer function, and the sum is expressed (since it is continuous in ω) as

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} H_{ij}(j\omega)F(j\omega)e^{j\omega t}\,d\omega \tag{8.119}$$

But $x(t)$ has a transform representation from Eq. (8.118):

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega \tag{8.120}$$

The integrals Eqs. (8.119) and (8.120) are evidently identical. Hence

$$X(j\omega) = H_{ij}(j\omega)F(j\omega) \tag{8.121}$$

which gives the *important property* that the product of the transfer function (at real frequency) and the Fourier transform of the driving function yields the Fourier transform of the response.

Consider the linear time-invariant differential equation

$$\frac{d^n x}{dt^n} + Q_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + Q_0 x = f(t) \qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(j\omega)e^{j\omega t}\,d\omega$$

In terms of the Fourier transforms of x and $f(t)$, this equation becomes

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}[(j\omega)^n + Q_{n-1}(j\omega)^{n-1} + \cdots + Q_0]X(j\omega)e^{j\omega t}\,dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(j\omega)e^{j\omega t}\,d\omega$$

where $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega \qquad \frac{d^k}{dt^k}x = \frac{1}{2\pi}\int_{-\infty}^{+\infty}(j\omega)^k X(j\omega)\,d\omega$$

whence $$X(j\omega) = \frac{F(j\omega)}{(j\omega)^n + A_{n-1}(j\omega)^{n-1} + \cdots + Q_1(j\omega) + Q_0}$$

The Fourier transform, in terms of its real and imaginary parts, is

$$X(j\omega) = M'(j\omega) + N'(j\omega) = M(\omega) + jN(\omega)$$

where M is an even function of ω and N an odd function of ω.

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Then

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} [M(\omega) + jN(\omega)]e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} [M(\omega)\cos\omega t - N(\omega)\sin\omega t]\,d\omega$$

$$= \frac{1}{\pi}\int_{-0}^{+\infty} (M\cos\omega t - N\sin\omega t)\,d\omega \qquad (8.122)$$

where only the even parts of the integrands can contribute because integration of the odd terms vanishes over the infinite limits. Now for the system (causal)

$$x(t) = 0 \qquad \text{for } t < 0$$

which is mathematically equivalent to

$$x(-t) = 0 \qquad \text{for } t > 0$$

Substitution in Eq. (8.122) yields

$$x(-t) = 0 = \frac{1}{\pi}\int_{0}^{\infty} (M\cos\omega t + N\sin\omega t)\,d\omega \qquad t > 0 \qquad (8.123)$$

Adding Eqs. (8.122) and (8.123), we obtain

$$x(t) = \frac{2}{\pi}\int_{0}^{\infty} M\cos\omega t\,d\omega = \frac{1}{\pi}\int_{-\infty}^{+\infty} M\cos\omega t\,d\omega \qquad t > 0 \qquad (8.124a)$$

Subtracting Eq. (8.123) from Eq. (8.122),

$$x(t) = -\frac{2}{\pi}\int_{0}^{\infty} N\sin\omega t\,d\omega = -\frac{1}{\pi}\int_{-\infty}^{+\infty} N\sin\omega t\,d\omega \qquad t > 0 \qquad (8.124b)$$

Since $x(t)$ has two integral representations, they are equal:

$$\frac{2}{\pi}\int_{0}^{\infty} M\cos\omega t\,d\omega = -\frac{2}{\pi}\int_{0}^{\infty} N\sin\omega t\,d\omega$$

$$\int_{0}^{\infty} M(\omega)\cos\omega t\,d\omega = -\int_{0}^{\infty} N(\omega)\sin\omega t\,d\omega$$

$$M(\omega) = \int_{-\infty}^{+\infty} x(t)\cos\omega t\,dt \qquad (8.125a)$$

$$N(\omega) = -\int_{-\infty}^{\infty} x(t)\sin\omega t\,dt \qquad (8.125b)$$

which are the real Fourier-transform coefficients and together with Eq. (8.124) constitute the real Fourier-transform pair. Properties of Fourier transforms are given in Table 8.1.

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TABLE 8.1 Properties of Fourier-Transform Pairs

Property	Fourier transform	Time function
Basic pairs	$F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}\,dt$ ($F(\omega)$)	$f(t) = 1/2\pi \int_{-\infty}^{+\infty} F(\omega)e^{j\omega t}\,d\omega$ ($f(t)$)
Linearity, a_1 and a_2 constants	$a_1F_1(\omega) + a_2F_2(\omega)$	$a_1f_1(t) + a_2f_2(t)$
Time multiplication, a real constant	$\frac{1}{\lvert a\rvert} F(\omega/a)$	$f(at)$
Time shift	$F(\omega)e^{-j\omega t_1}$	$f(t - t_1)$
Frequency multiplication, a real constant	$F(a\omega)$	$\frac{f(t/a)}{\lvert a\rvert}$
Frequency shift	$F(\omega - \omega_1)$	$f(t)e^{j\omega_1 t}$
Time differentiation	$(j\omega)^n F(\omega)$	$(d^n/dt^n)f(t)$
Integration	$(1/j\omega)F(\omega)$	$\int_{-\infty}^{t} f(x)\,dx$
Frequency differentiation	$j^n(d^n/d\omega^n)F(\omega)$	$t^n f(t)$
Convolution time domain	$F_1(\omega)F_2(\omega)$	$\int_{-\infty}^{+\infty} f_1(t - T)f_2(T)\,dT \triangleq f_1(t) * f_2(t)$
Convolution frequency domain	$F_1(\omega) * F_2(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F_1(\omega - x)F_2(\omega)\,dx$	$f_1(t)f_2(t)$
Forms for real $f(t)$	$F(\omega) = R(\omega) + jX(\omega)$ $R(\omega) = \int_{-\infty}^{+\infty} f(t)\cos\omega t\,dt$ $X(\omega) = -\int_{-\infty}^{+\infty} f(t)\sin\omega t\,dt$	$f(t) = 1/\pi \int_0^{\infty} [R(\omega)\cos\omega t\,d\omega - X(\omega)\sin\omega t]\,d\omega$ $= 1/\pi \int_0^{\infty} \lvert F(\omega)\rvert \cos(\omega t + \phi)\,d\omega$ $\phi = \tan^{-1} X(\omega)/R(\omega)$
Forms for real even $f(t)$	$X(\omega) = 0$ $R(\omega) = -2\int_0^{\infty} f(t)\sin\omega t\,dt$	$f(t) = -1/\pi \int_0^{\infty} R(\omega)\cos\omega t\,d\omega$
Forms for real odd $f(t)$	$R(\omega) = 0$ $X(\omega) = -2\int_0^{\infty} f(t)\sin\omega t\,dt$	$f(t) = -1/\pi \int_0^{\infty} X(\omega)\sin\omega t\,d\omega$
$f(t)$ causal, i.e., $f(t) = 0$ for $t < 0$	$F(\omega) = \int_0^{\infty} f(t)e^{-j\omega t}\,dt$ $= R(\omega) + jX(\omega)$	$f(t) = 2/\pi \int_0^{\infty} R(\omega)\cos\omega t\,d\omega$ $= -2/\pi \int_0^{\infty} X(\omega)\sin\omega t\,d\omega \quad t > 0$
Forms for periodic function $f(t) = f(t + T)$	$2\pi \sum_{n=-\infty}^{+\infty} a_n\delta(\omega - \omega_n)$ $a_n = 1/T \int_0^T f(t)e^{-j\omega_n t}\,dt$	$f(t) = f(t + T)$ $= \sum_{n=-\infty}^{+\infty} a_n e^{j\omega_n t}$ $\omega_n = 2\pi n/T$

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Laplace-Transform Method. When Eq. (8.116) does not hold for a forcing function $f(t)$, recourse may be taken to the unilateral Laplace-transform method provided that $f(t) \equiv 0$ for $t < 0$ and there exists a positive number c (in most physical problems of interest one exists) such that

$$\int_{-\infty}^{+\infty} |f(t)| e^{-ct}\, dt < \infty \tag{8.126}$$

The minimum value of c for which Eq. (8.126) holds is designated as the abscissa of convergence, equal to c_1. Equation (8.126) ensures that the Laplace transform of $f(t)$, written $\mathcal{L}\{f(t)\}$ and defined by

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)\, e^{-st}\, dt = F(s) \tag{8.127}$$

will converge to a function of s. s is a complex variable given by

$$s = c + j\omega \qquad c \geq c_1$$

Examples of c_1 are for $f(t) = \sin \omega t$, $c_1 > 0$; for unit step, $c_1 > 0$; for $t^n e^{dt}$, $c_1 > d$, n finite, d real. An example of a function where c cannot be found to satisfy Eq. (8.126) is $f(t) = \exp t^n$ for $n > 1$, and Laplace methods will accordingly fail. Equation (8.127) looks like the Fourier transform of $f(t)e^{-ct}$ if $f(t) = 0$ for $t < 0$:

$$\mathcal{L}\{f(t)\} = \mathcal{F}\{f(t)e^{-ct}\}$$

where $\mathcal{F} \triangleq$ Fourier transform. Invoking the Fourier-transform theorem [Eq. (8.117)] and manipulating, we find

$$f(t)e^{-ct} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} f(T)e^{-cT}e^{j\omega(t-T)}\, dT$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{j\omega t}\, d\omega \int_{-\infty}^{+\infty} f(T)e^{-(c+j\omega)T}\, dT$$

$$f(t) = \frac{1}{2\pi}\int_{-j\infty}^{+j\infty} e^{(c+j\omega)t}\, \frac{d(j\omega)}{j} \int_0^{\infty} f(T)e^{-(c+j\omega)T}\, dT \qquad f(T) = 0 \text{ for } T < 0$$

Since $s = c + j\omega$ and c is a constant, $ds = d(j\omega)$ and the above becomes

$$f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} e^{st}\, ds \int_0^{\infty} f(T)e^{-sT}\, dT = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \mathcal{L}\{f(t)\}e^{st}\, ds$$

$$= \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}\, ds \tag{8.128}$$

Equation (8.128) is the inversion form for going from $\mathcal{L}\{f(t)\}$ to $f(t)$. Equations (8.127) and (8.128) constitute the Laplace-transform pair.

From Eq. (8.114), the real-frequency transfer function

$$H_{ij}(j\omega) = M_{ij}(j\omega)/D(j\omega)$$

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was deduced for sinusoidal inputs. Similarly, consider the response [Eq. (8.112)] x_{ij} due to f_i:

$$D(p)x_{ij}(t) = M_{ij}(p)f_i(t)$$

where $D(p)$ and $M_{ij}(p)$ are linear differential operators. For zero initial conditions the Laplace transformation of both sides yields

$$D(s)X_{ij}(s) = M_{ij}(s)F_i(s) \tag{8.129}$$

Transfer Function. From Eq. (8.129), dropping all subscripts for clarity, $X(s)/F(s) = M(s)/D(s) = H(s)$, which is, by definition, the transfer function where s replaces $j\omega$ in the argument $H(j\omega)$, the real-frequency transfer function for sinusoidal input. $H(s)$ in itself has no physical significance; it contains however, the complete characterization of the system. This is in contrast with $H(j\omega)$ which gives the steady-state response to a sinusoidal input, its amplitude being the gain and its argument the phase difference between output and input.

It follows that, if $H(j\omega)$ is a known analytic function of $j\omega$, then $H(s)$ is immediately available (by analytic continuation) for a complete system description.

The total response $y(t)$ satisfying the equation

$$D(p)y(t) = M(p)f(t) \qquad f(t) = 0 \text{ for } t < 0 \tag{8.130}$$

can be obtained directly by taking the Laplace transform [including initial conditions which result in the polynominal $L(s)$ of lower order than $D(s)$] as follows:

$$D(s)Y(s) - L(s) = M(s)F(s) \tag{8.130a}$$

$$Y(s) = M(s)F(s)/D(s) + L(s)/D(s) = H(s)F(s) + L(s)/D(s)$$

By inversion,

$$y(t) = \underbrace{\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} H(s)F(s)e^{st}ds}_{x(t) = \int_0^t W(t-T)f(T)\,dT} + \underbrace{\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{L(s)}{D(s)}e^{st}ds}_{z(t)} \tag{8.131}$$

The first integral in Eq. (8.131), $x(t)$, is the solution to Eq. (8.112*b*) for zero initial conditions; the second, $z(t)$, is the solution to the homogeneous form Eq. (8.112*a*) which satisfies the initial conditions of Eq. (8.130). $W(t - T)$ is the response $x(t)$ at time t to a unit impulse input $f = \delta(t - T)$, the system being initially at rest.

Inversion. The transformation (inversion) from the complex frequency representation to the time domain is given by

$$f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)\,e^{st}dt$$

which is a line integral along the line Re $s = c$ in the complex s plane where $c > c_1$, and c_1 is defined as the abscissa of convergence. For the case $|F(s)| \to 0$ as $|s| \to \infty$ the line integral is most readily evaluated by forming a contour including this line and

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an infinite semicircle connected on the left and considering the contour integral

$$\oint F(s)e^{st}\,ds$$

Since $|F(s)| \to 0$ as $|s| \to \infty$, the line integral around the semicircular portion of this contour vanishes as a consequence of Jordan's lemma.[16] This leads to the equality of the contour integral with the inversion integral, viz.,

$$f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}\,ds = \frac{1}{2\pi j}\oint F(s)e^{st}\,ds \qquad t > 0$$

From Cauchy's residue theorem, the right-hand side equals the sum of the residues of $F(s)e^{st}$ enclosed. The residues are evaluated at each simple pole s_k by

$$(s - s_k)F(s)e^{st} = R_{sk} \qquad s \to s_k$$

where R_{sk} = residue at s_k. If $F(s)$ is a fraction, $F(s) = A(s)/B(s)$ where $A(s)$ and $B(s)$ are analytic functions of s inside the contour (excluding poles at infinity), then the poles of $F(s)$ are clearly the zeros of $B(s)$. A pole of multiplicity m is equal to the excess of zeros of $B(s)$ over $A(s)$ at the pole. For simple poles of $F(s)$, i.e., where $m = 1$, the residue at $s = s_k$ is simply

$$(s - s_k)F(s)e^{st} = A(s_k)e^{s_k t}/B'(s_k) \qquad s \to s_k$$

The residue at pole s_j of multiplicity m is

$$\frac{1}{(m-1)!}\frac{d^{m-1}}{ds^{m-1}}\left[(s - s_j)^m F(s)e^{st}\right]_{s=sj}$$

If there are no zeros of $A(s)$ at the point $s = s_j$, then an alternative form of this is

$$\frac{md^{m-1}}{ds_j^{m-1}}\frac{A(s_j)e^{s_j t}}{d^m B(s_j)/ds_j^m}$$

The following are some important properties of $H(s)$ for passive systems of differential equations:

1. $H(s)$ is a rational function with real coefficients.
2. The degree of $D(s)$ is equal to or greater than $M_{ij}(s)$.
3. As a consequence of property 1, the complex zeros and poles occur in conjugate pairs.
4. All poles of $H(s)$ lie in the closed left half plane, a consequence of passivity. Table 8.2 has properties of Laplace-transform pairs.

8.4.4 Time-Domain Analysis

The general solution to Eq. (8.130) given by Eq. (8.131) is

$$y(t) = \int_0^t W(t - T)f(T)\,dT + z(t) \tag{8.132}$$

where the integral expression alone satisfies the inhomogeneous equation with zero initial conditions, and $z(t)$ contains the linearly independent solutions to the homogeneous

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TABLE 8.2 Properties of Laplace-Transform Pairs for Causal Time Function

$f(t) = 0 \quad t < 0$		
Property	Laplace transform	Time function
Basic pairs	$F(s) = \int_0^{\infty} f(t)e^{-st}\,dt = \int_{-\infty}^{+\infty} f(t)e^{-st}\,dt$ ($F(s)$)	$f(t) = 1/2\pi j \int_{c-j\infty}^{c+j\infty} F(s)e^{st}\,ds$ ($f(t)$)
Linearity a_1 and a_2 constants	$a_1F_1(s)t + a_2F_2(s)$	$a_1f_1(t) + a_2f_2(t)$
Time scale multiplication a positive real constant	$\frac{F(s/a)}{a}$	$f(at)$
Time shift	$F(s)e^{-st_1}$	$f(t - t_1)$
Complex frequency multiplication a positive real constant	$F(as)$	$\frac{f(t/a)}{a}$
Complex frequency shift	$F(s - s_1)$	$f(t)e^{s_1t}$
Time differentiation	$s^nF(s) - \sum_{k=1}^{n} f^{k-1}(0)s^{n-k}$	$\frac{d^n}{dt^n} f(t)$
Integration	$\frac{F(s)}{s} + \frac{f^{-1}(0)}{s}$	$\int_0^t f(t)\,dt$
Complex frequency differentiation	$(-1)^n \frac{d^n}{ds^n} F(s)$	$t^nf(t)$
Convolution time domain	$F_1(s)F_2(s)$	$f_1(t) * f_2(t) = \int_0^{\infty} f_1(t-T)f_2(T)\,dT = \int_0^t f_1(t-T)f_2(T)\,dT$
Convolution frequency domain	$1/2\pi j \int_{c-j\infty}^{c+j\infty} F_1(s-x)F_2(x)\,dx = (1/2\pi j)F_1(s) * F_2(s)$	$f_1(t)f_2(t)$
Forms for periodic function $f_k(t) = f_k(t + T)$	$\frac{1/2\pi j \int_0^T f_k(t)e^{-st}\,dt}{1 - e^{-sT}}$	$f_k(t) = f_k(t + T)$
Initial-value theorem [$f(t)$ and $f'(t)$ are Laplace transformable]	$\lim_{s\to\infty} sF(s)$	$\lim_{t\to 0} f(t)$
Final-value theorem $f(t)$, $f'(t)$ Laplace transformable and $sF(s)$ analytic $Re\ s \geq 0$	$\lim_{s\to 0} sF(s)$	$\lim_{t\to\infty} f(t)$

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form fulfilling the initial conditions on $y(t)$, i.e.,

$$y(0) = z(0)$$

$W(t)$ is the inverse transform of the transfer function $H(s)$ which physically is the response to the delta function $\delta(t)$. The Dirac delta function $\delta(t - T)$ (defined as a pulse of infinite height at $t = T$, with unit area) has the following properties:

$$\int_{-\infty}^{+\infty} \delta(t - T)\, dt = 1 \qquad \text{and} \qquad \int_{-\infty}^{+\infty} \delta(t - T)A(t)\, dt = A(T)$$

Examples of the delta function are shown in Fig. 8.14. To find the response $y(t)$ for input $\delta(t - T)$ and zero initial conditions, we first evaluate

$$F(s) = \mathcal{L}\{\delta(t - T)\} = e^{-sT}$$

and substitute in Eq. (8.130*a*)

$$Y(s) = H(s)F(s) = H(s)e^{-sT}$$

where $L(s) = 0$ is a consequence of zero initial conditions. Transforming to the time domain, we have

$$y(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} H(s)\, e^{-sT}\, e^{st} ds = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} H(s)\, e^{s(t-T)}\, ds = W(t - T) \qquad t < T$$

Any forcing function can be approximated by an infinite number of delta functions of strength $f(T)\,\Delta T$ with the responses at time t the sum of the responses $W(t - T)$ per unit impulse for each of these pulses which occurred $t - T$ seconds previous to the

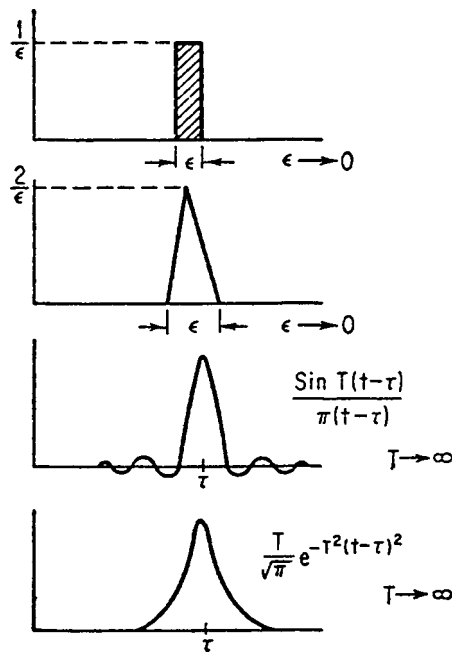

FIG. 8.14 Examples of delta functions, $\delta(t - T)$.

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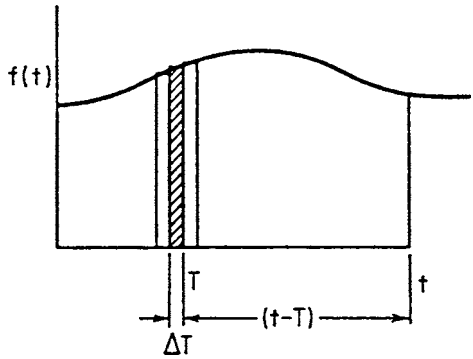

FIG. 8.15 Pulse synthesis for time-convolution theorem.

time of inspection, t. Figure 8.15 shows a graphical construction with equal-duration rectangular pulses, ΔT wide and $f(T)$ high. Each pulse has an area $f(T)\ \Delta T$ so that the function can be approximated by

$$\sum_{i=0}^{i=t/\Delta T} f(T)\ \Delta T\ \delta(t\ -\ i\ \Delta T)$$

The transient produced at time t by the pulse $f(T)\ \Delta T\ \delta(t\ -\ T)$ is

$$W(t\ -\ T)\ f(T)\ \Delta T$$

The total effect of all pulses is by superposition:

$$y(t)\ =\ \sum_{T=0}^{t} f(T)\ \Delta T W(t\ -\ T)\ =\ \sum_{i=0}^{i=t/\Delta T} f(i\ \Delta T)\ \Delta T\ W(t\ -\ i\ \Delta t)$$

which in the limit $\Delta T \to 0$ is

$$\int_0^t f(T)W(t\ -\ T)\ dT \tag{8.133}$$

$$W(t-T)\ =\ 0 \qquad t \le T$$
$$t \le 0$$
$$f(T)\ =\ 0 \qquad T < 0$$

consistent with Eq. (8.132).

In view of the restrictions on $W(t\ -\ T)$ and $f(t)$ the limits on Eq. (8.133) can be changed to any of the following:

$$y(t)\ =\ \int_0^t\ =\ \int_{-\infty}^{+\infty}\ =\ \int_{-\infty}^{t}\ =\ \int_0^{\infty}$$

By a change of variable, y can be represented as

$$y(t)\ =\ \int_0^t f(t-T)W(T)\ dT$$

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and also is obtainable directly as an alternative form from the convolution integral in going from complex to the real time domain.

Staircase Development. Another manner of depicting the forcing function $f(t)$ is shown in Fig. 8.16 as the synthesis of step functions.

Since the unit step function is the integral of the delta function,

$$U(t - T) = \int_{-\infty}^{+\infty} \delta(t - T)\, dt \qquad \text{and} \qquad \frac{dU(t - T)}{dt} = \delta(t - T)$$

From property 2 of time-invariant systems previously discussed, if the response to the unit step is

$$\text{Input} = U(t - T) \rightarrow Q(t - T) = \text{response}$$

then the response to the derivative $[\partial U(t - T)]/\partial t = \delta(t - T)$ is $[\partial Q(t - T)]/\partial t$. But the response to $\delta(t - T)$ has already been shown to be $W(t - T)$. Therefore,

$$W(t - T) = (\partial/\partial t)\, Q(t - T)$$

The elementary step functions are of varying amplitudes Δh_i for fixed ΔT. From geometrical considerations

$$\Delta h_i \approx f'(T)\, \Delta T \qquad T = i\, \Delta T$$

and

$$\Delta h_i\, U(t - T) = f'(T)\, \Delta T\, U(t - T)$$

The response to the elementary step $\Delta h_i\, U(t - T)$ is $f'(T)\, \Delta T\, Q(t - T)$. The total response is therefore

$$f(0)Q(t) + \sum_{i=0}^{i=T/\Delta t} f'(T)\, \Delta T\, Q(t - T) = \sum_{i=0}^{i=t/\Delta T} f'(i\, \Delta T)\, \Delta T\, Q(t - i\, \Delta T) + f(0)Q(t)$$

As $\Delta T \rightarrow 0$ this becomes in the limit

$$y(t) = f(0)Q(t) + \int_0^t f'(T)Q(t - T)\, dT \tag{8.134}$$

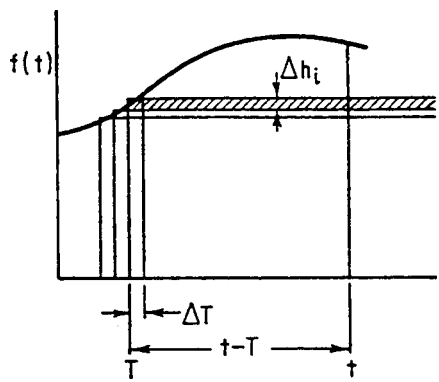

FIG. 8.16 Staircase synthesis.

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which is identical to Eq. (8.133). This is shown by integrating Eq. (8.134) by parts.

$$y(t) = f(0)Q(t) + f(T)Q(t - T)\Big|_{T=0}^{T=t} - \int_0^t f(T)\frac{\partial Q(t - T)}{\partial T}\,dT$$

$$= f(0)Q(t) + f(t)Q(0) - f(0)Q(t) + \int_0^t f(T)\frac{\partial Q}{\partial t}(t - T)\,dT$$

since $Q(0) = 0$ and

$$\frac{\partial Q}{\partial T}(t - T) = \frac{-\partial Q}{\partial t}(t - T) = -W(t - T)$$

$$y(t) = \int_0^t f(T)W(t - T)\,dT$$

Stability of Time-Invariant System. From general property 8 (Sec. 8.4) of linear systems the necessary and sufficient condition for $y(t)$ to have a bounded output for any bounded input $f(t)$ is

$$\int_0^\infty |\,W(t, T)\,|\,dT < M < \infty$$

which becomes

$$\int_0^\infty W(t - T)\,dT < M < \infty \tag{8.135}$$

for the time-invariant case.

By definition, $W(t - T)$, the weighting function, is the response to the unit delta function $\delta(t - T)$, and it has been shown that $W(t)$ is the inverse of the system transfer function

$$W(t) = H^{-1}(s) = [M(s)/D(s)]^{-1}$$

From the inversion theorem,

$$W(t) = \sum_{k=1}^{m}\sum_{n=1}^{vk} a_{kn}t^{vk-n}e^{s_k t}$$

where vk = multiplicity of roots s_k
s_k = zeros of $D(s)$
m = number of different poles of $H(s)$

Invoking Eq. (8.135) for a typical term $at^q e^{s_k t}$, q = integer,

$$\int_0^\infty |\,W(t - T)\,|\,dt = \int_0^\infty at^q e^{Re s_k t}\,dt < M < \infty$$

which can hold if and only if Re s_k is negative. Hence the criterion for stability for the time-invariant case is simply that all the zeros of $D(s)$ lie in the left half complex plane excluding the imaginary axis. Translated to the time domain, this is equivalent to stating that the real parts of the exponent of each solution to the homogeneous equation $D(p)z = 0$ must be negative so that as $t \to \infty$ they all tend to vanish. Real systems

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composed only of passive elements are necessarily stable since it can be shown that the poles of their transfer function are restricted to the left-hand plane. The systematic investigation of locating the zeros of $D(s)$ has been motivated by control theory and is discussed subsequently.

8.5 BLOCK DIAGRAMS AND THE TRANSFER FUNCTION

8.5.1 General

A convenient and descriptive way of viewing a system is by use of a block diagram. While a block diagram has little practical value for a simple system, in a complex array of coupled systems it suggests the flow of signals and facilitates analysis. The basic "block" essentially defines the system by giving a description of the physical processes which occur. Specifically for an input i the block gives information on the output o. An example of a block expressing an algebraic relationship is shown in Fig. 8.17*a*. A block of a more general operator relationship which includes the nonlinear differential-integral operator Φ is shown in Fig. 8.17*b*.

8.5.2 Linear Time-Invariant Systems

Single-Degree-of-Freedom Case. The case of the linear time-invariant system (Fig. 8.18) is of special interest because the block-diagram characterization can be very simply shown in terms of the transfer function $H(s)$ in the frequency domain.

The input and output, $I(s)$ and $O(s)$, are Laplace transforms of the input and output signals. From Fig. 8.18*a* we have

$$O(s)/I(s) = H(s) \qquad i(t) = o(t) = 0 \qquad \text{for t} < 0$$

The multiplication property $O(s) = H(s)I(s)$ is of fundamental importance when dealing with linear cascaded systems such as occur in control theory. Moreover, the complete analysis of such systems can be made in this domain without translating to real time. An alternative representation in the time domain is shown in Fig. 8.18*b* where $W(t)$ is the weighting function implying the convolution relation

$$o(t) = \int_0^t W(t - T)i(T)\, d(T)$$

For the frequency-domain representation, the response to cascaded systems is simply

$$O(s) = H_1(s)H_2(s)H_3(s) \cdots H_n(s)I(s)$$

Some care must be exercised in implementing this formula since the $H_1(s)$ (e.g., network systems) sometimes displays a loading effect; i.e., the individual free transfer functions differ from the transfer functions in cascade.

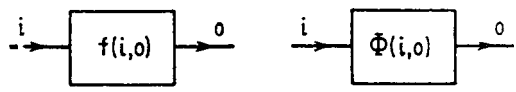

FIG. 8.17 Basic block diagrams. (*a*) $f(i, o)$, algebraic relationship. (*b*) $\delta(i, o)$, general operator relationship.

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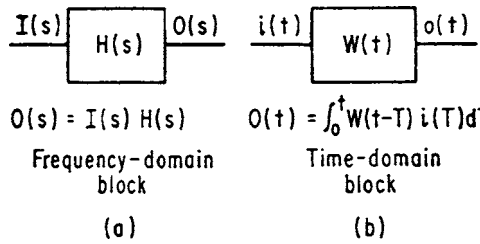

FIG. 8.18 Transfer blocks for linear time-invariant system.

Consider the second-order system

$$d^2x/dt^2 + c\,dx/dt + kx = f(t) \qquad (8.136)$$

$$A(p)x = f(t) \qquad A(p) = p^2 + cp + k$$

The system transfer function is obtained by taking the Laplace transform of Eq. (8.136) with zero initial conditions, thus:

$$H(s) = 1/A(s) = 1/(s^2 + cs + k) = X(s)/F(s)$$

Multiple-Degree-of-Freedom Case. For the more general system of n forcing functions f_i, $i = 1, \ldots, n$, e.g., Eq. (8.108) with n outputs $y_1, \ldots, y_n$, and assuming A^{-1} exists, i.e., A is a nonsingular matrix operator, consider

	Laplace form
$\mathbf{A}(p)\mathbf{y} = \mathbf{f}(t)$	$\mathbf{A}(s)\mathbf{Y}(s) = \mathbf{F}(s)$
$\mathbf{y}_j = \sum_{i=1}^{n} \dfrac{M_{ij}(p)f_i(t)}{D(p)}$	$Y_j(s) = \sum_{i=1}^{n} \dfrac{M_{ij}(s)}{D(s)} F_i(s)$
$y_{ij} = \dfrac{M_{ij}(p)\,f_i(t)}{D(p)}$	$Y_{ij}(s) = \dfrac{M_{ij}(s)}{D(s)} F_i(s)$

where y_j is the jth output for all f's and y_{ij} is a component of y_j produced by f_i

The natural extension of the one-dimensional case to this n-dimensional case is made by representing the matrix as follows:

$$\mathbf{Y}(s) = \mathbf{H}(s)\mathbf{F}(s)$$

where

$$\mathbf{Y}(s) = \begin{bmatrix} Y_1(s) \\ Y_2(s) \\ \cdot \\ \cdot \\ \cdot \\ Y_n(s) \end{bmatrix} \qquad \mathbf{F}(s) = \begin{bmatrix} F_1(s) \\ F_2(s) \\ \cdot \\ \cdot \\ \cdot \\ F_n(s) \end{bmatrix}$$

and

$$\mathbf{H}(s) = A^{-1}(s) = \begin{bmatrix} h_{11}(s) & \cdots & h_{1n}(s) \\ \cdots & \cdots & \cdots \\ h_{n1}(s) & \cdots & h_{nn}(s) \end{bmatrix}$$

which formally is identical to Fig. 8.18, the one-dimensional case.

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8.5.3 Feedback Control-System Dynamics[10,25–30]

In the feedback control system one or more dependent variables (output) of a dynamic process is controlled. To this end, the difference(s) (error) between the desired value (input) and output is measured and functionally operated on to obtain a correcting signal (e.g., force) which is imparted to the basic system for the purpose of driving the output to correspondence with the input. The following descriptions imply the basic blocks which distinguish a control system:

1. **The plant:** The uncontrolled system
2. **Sensor:** A device to detect the output
3. **Transmitter:** A device to transmit the output or input signals to the comparator
4. **Comparator:** A device to detect differences between output and input
5. **Controller:** A device which takes some useful function of the input and output to correct the "error"

An overall description of the controlled process will yield forms outlined in the foregoing sections which dealt with single systems, where emphasis was placed upon passive types. Stability of these types of systems was assured without recourse to mathematical analysis. On the other hand, the control-system equations relating input to output (overall transfer) are not in general "passive" because the control system contains energy sources such as power amplifiers. The basic problem of control is the synthesis of an optimum control system which exhibits absolute as well as relative stability.

The general equations for control and the corresponding block-diagram representations are as follows:

1. The plant:

$$\omega = \phi(v, d, t) \qquad \begin{matrix} d \rightarrow \\ v \rightarrow \end{matrix} \boxed{\sigma(v, d, t)} \rightarrow \omega$$

where ϕ = general operator
v = input
d = disturbance
t = time

2. Sensor:

$$x = \Omega(\omega) \qquad \overset{\omega}{\rightarrow} \boxed{\Omega(\omega)} \overset{x}{\rightarrow}$$

3. Transmitter:

$$y = \theta(x) \qquad \overset{x}{\rightarrow} \boxed{\theta(x)} \overset{y}{\rightarrow}$$

4. Comparator (error device):

$$\epsilon = z - y \qquad \begin{matrix} z \rightarrow \otimes \rightarrow \epsilon = (z - y) \\ \uparrow \\ y \end{matrix}$$

5. Controller:

$$v = \psi(\epsilon) \qquad \overset{\epsilon}{\rightarrow} \boxed{\psi(\epsilon)} \overset{v}{\rightarrow}$$

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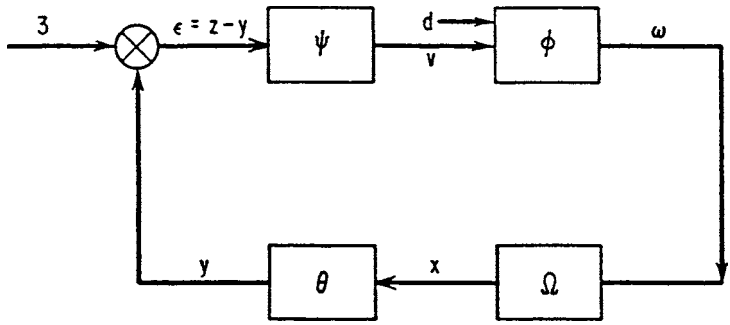

FIG. 8.19 Closed-loop feedback control system.

The total feedback control system for the control of output w is shown in Fig. 8.19. Here ϕ, ψ, Ω, θ are in general nonlinear differential operators. Because of the extraordinary complexity of systems involving nonlinear operators, only the linear time-invariant case in which the operators in the set of control equations are linear with constant coefficients will be presented.

8.5.4 Linear Time-Invariant Control System

Linearity and time invariance admit to simplifying techniques of frequency-domain methods of analyses. In block-diagram form the system transfer functions and the Laplace transforms of the signals entering and leaving are given for each block. The linear time-invariant control-system representation of Eq (8.137) is depicted in Fig. 8.20. The disturbance and prescribed input to the plant are, in the linear case, connected by the differential equation

$$A\,(p)w(t) = B(p)v(t) + K(p)\,d(t)$$

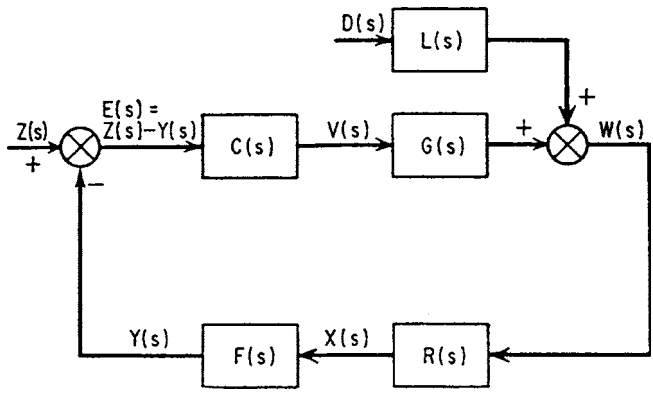

FIG. 8.20 Closed-loop control system for linear time-invariant system.

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whence the transfer function derives

$$\begin{aligned} A(s)W(s) &= B(s)V(s) + K(s)D(s) \\ W(s) &= [B(s)/A(s)]V(s) + [K(s)/A(s)]D(s) = G(s)V(s) + L(s)D(s) \end{aligned} \tag{8.138}$$

where $G(s) = B(s)/A(s)$ and $L(s) = K(s)/A(s)$

The equivalent frequency-domain forms of Eq. (8.137) are

$$W(s) = G(s)V(s) + L(s)D(s) \tag{8.138a}$$

$$E(s) = \mathrm{Z(s)} - Y(s) \tag{8.138b}$$

$$V(s) = C(s)E(s) \tag{8.138c}$$

$$X(s) = R(s)W(s) \tag{8.138d}$$

$$Y(s) = F(s)X(s) \tag{8.138e}$$

The functional expression relating output to input is obtained by algebraic manipulation of Eq. (8.138).

$$W(s) = \frac{CGZ}{1 + CGFR} + \frac{LD}{1 + CGFR} \tag{8.139}$$

The transfer function between input $z(t)$ and output $w(t)$ in the absence of disturbance $d(t)$ is therefore

$$\frac{W(s)}{Z(s)} = \frac{CG}{1 + CGFR} \tag{8.140}$$

and the transfer function between output and disturbance $d(t)$ in the absence of signal $z(t)$ is

$$\frac{W(s)}{D(s)} = \frac{L}{1 + CGFR} \tag{8.141}$$

The matrix generalization for the simultaneous control of many variables is obtained by considering the transfer functions to be transfer matrices from which

$$\mathbf{W}(s) = (1 + CGF\,R)^{-1}\,CG\mathbf{Z}(s) + (1 + CGF\,R)^{-1}L\mathbf{D}(s)$$

where the column $\mathbf{W}(s)$ implies the separate controlled variables, and $\mathbf{Z}(s)$ and $\mathbf{D}(s)$, the input and disturbance vectors.[28]

Multiloop Systems. In general, control systems are more complicated than the one shown in Fig. 8.20, being composed of many loops, as shown, for example, in Fig. 8.21.

If the inner loop has a transfer function H_2 (dashed box), then the overall transfer function is accordingly

$$\frac{W(s)}{Z(s)} = \frac{H_1H_2}{1 + H_1H_2H_3} \tag{8.142}$$

8.5.5 Analysis of Control System

The transfer-function representations, e.g., Eqs. (8.140) and (8.142), permit the evaluation of three important properties of the control system without transformation to the time domain.

Equation (8.140) or (8.142) can be represented conveniently as

$$\frac{W(s)}{Z(s)} = \frac{Y_1}{1 + Y_1Y_2} \tag{8.143}$$

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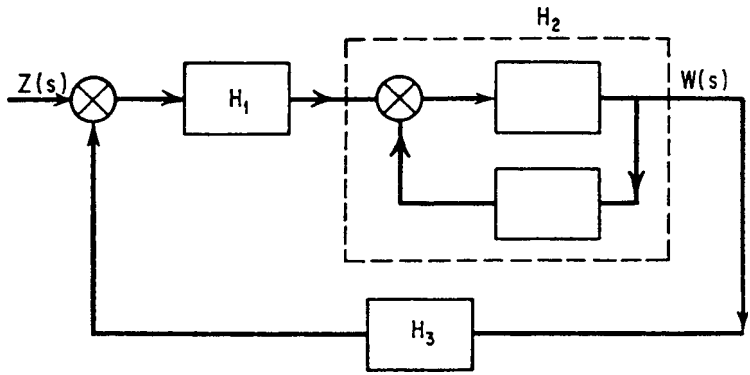

FIG. 8.21 Multiloop control system.

The error-to-input transfer function is obtained by subtracting both sides of the above from unity to give

$$\frac{E(s)}{Z(s)} = \frac{1 + Y_1Y_2 - Y_1}{1 + Y_1Y_2} \tag{8.144}$$

Properties Deducible from Frequency-Domain Representation

Property 1. The steady-state error is

$$\epsilon(t) = sE(s) = s\frac{1 + Y_1(s)Y_2(s) - Y_1(s)}{1 + Y_1(s)Y_2(s)}Z(s) \qquad \begin{matrix} t \to \infty \\ s \to 0 \end{matrix} \tag{8.145}$$

If the input $z(t)$ is the unit step $Z(s) = 1/s$, substitution in Eq. (8.145) yields

$$\epsilon(t) = s\frac{1 + Y_1(0)Y_2(0) - Y_1(0)}{1 + Y_1(0)Y_2(0)}\frac{1}{s} \triangleq \frac{1}{1 + K_p} \qquad \begin{matrix} t \to \infty \\ s \to 0 \end{matrix} \tag{8.146}$$

If the input $z(t)$ is the ramp t, $Z(s) = 1/s^2$ and Eq. (8.145) becomes

$$\epsilon(t) = s\frac{1 + Y_1(s)Y_2(s) - Y_1(s)}{1 + Y_1(s)Y_2(s)}\frac{1}{s^2} = \frac{1 + Y_1(s)Y_2(s) - Y_1(s)}{sY_1(s)Y_2(s)} \triangleq \frac{1}{K_t} \qquad \begin{matrix} t \to \infty \\ s \to 0 \end{matrix} \tag{8.147}$$

If the input $z(t)$ is t^2, $Z(s) = 2/s^3$ and Eq. (8.145) becomes

$$\epsilon(t) = 2\frac{1 + Y_1(s)Y_2(s) - Y_1(s)}{s^2Y_1(s)Y_2(s)} \triangleq \frac{1}{K_a} \qquad \begin{matrix} t \to \infty \\ s \to 0 \end{matrix} \tag{8.148}$$

where the steady-state errors due to a step, ramp, and acceleration of the input are developed in Eqs. (8.146) through (8.148) assuming that they exist in each case.

If $E(s)/Z(s)$ can be represented by a Maclaurin series and is stable.

$$E(s)/Z(s) = a_0 + a_1s + a_2s^2 + \cdots$$

$$E(s) = a_0Z(s) + a_1sZ(s) + \cdots$$

the steady state $e(t)$ for an input $z(t)$ is then

$$e(t) \to a_0z(t) + a_1z'(t) + a_2z''(t) + \cdots \qquad t \to \infty$$

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which implicitly ignores all initial conditions, i.e., assumes the initial disturbances vanish as $t \to \infty$.

Property 2. The rms error, a quantitative measure of the effectiveness of control,

$$\overline{\mathscr{E}}^2 = \frac{1}{T}\int_0^T |\, e(t)\,|^2 dt \qquad T \to \infty$$

can be obtained from[10,16,27]

$$\overline{\mathscr{E}}^2 = \frac{1}{2}\int_{-\infty}^{+\infty} G_e(f)\,df = \int_0^{\infty} G_e(f)\,df$$

where $G_e(f)$ is the spectral density of the error $e(t)$. G_z for the input $z(t)$ is defined by

$$G_z(j\omega) = \frac{1}{2T}\left|\int_{-T}^{+T} z(t)e^{-j\omega t}dt\right|^2$$

For the input forms which are bounded, G_z exists and the overall G_e is given by

$$G_e = \left|\frac{E(j\omega)}{Z(j\omega)}\right|^2 G_z(j\omega) \qquad T \to \infty$$

The spectral density G_z for random-type inputs and disturbances is also obtainable by statistical methods; then the property becomes the "expected rms error" owing to the nonspecific character of input.

Property 3. The stability of the system, however complex, is completely ascertained by the locations of the poles of the right-hand member of Eq. (8.143), namely,

$$\frac{Y_1(s)}{1 + Y_1(s)Y_2(s)} \tag{8.149}$$

The system is stable if no poles of this function lie in the right half s plane, the imaginary axis included. Otherwise it is unstable.

There are three prominent methods for determining whether or not poles of the function lie in the right-hand plane: (1) Routh-Hurwitz, (2) root locus, and (3) Nyquist criterion.

Routh-Hurwitz Method. The method is applicable to rational fractions

$$Y_1(s) = A(s)/B(s) \qquad Y_2(s) = C(s)/D(s)$$

First the fractions are cleared after substituting in Eq. (8.149), leaving

$$\frac{A/B}{1 + (A/B)(C/D)} = \frac{AD}{AC + BD}$$

assuming the fraction in the lowest form $AC + BD$ cannot coincide with zeros of its numerator. The poles of Eq. (8.149) correspond to the zeros of the denominator. The denominator polynomial can be written as

$$AC + BD = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

System stability is therefore governed by the location of zeros of this polynomial.

Stability: Routh-Hurwitz Criterion. Given the general nth-order equation,

$$a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0 \tag{8.150}$$

with real coefficients.

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The following statements apply to the roots:

1. The roots occur in conjugate complex pairs.
2. A necessary condition for the real parts of all roots to be negative is that all coefficients have the same sign, and hence is a necessary condition for stability.
3. A necessary condition for all real parts to be nonpositive is that all coefficients a_0, a_1, … , a_n be different from zero, i.e.,

$$a_0, a_1, a_2, \ldots, a_n \neq 0$$

and is a necessary condition for stability.

If conditions 2 or 3 fail, the system is unstable. If, on the other hand, Eq. (8.150) meets conditions 2 and 3, further tests (Routh-Hurwitz) must be made to determine stability.

The procedure is first to arrange the coefficients in two rows as shown followed by a third row developed from the first two rows, viz.,

Row 1: $a_n \quad a_{n-2} \quad a_{n-4} \quad \cdots$

Row 2: $a_{n-1} \quad a_{n-3} \quad a_{n-5} \quad \cdots$

Row 3: $b_{n-1} \quad b_{n-3} \quad b_{n-5} \quad \cdots$

where

$$b_{n-1} \triangleq \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}$$

$$b_{n-3} \triangleq \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}$$

. . . . . .

where the bars indicate the determinant of the enclosed array. In a like fashion form a fourth row developed from rows 2 and 3:

Row 4: $c_{n-1} \quad c_{n-3} \quad c_{n-5} \quad \cdots$

where

$$c_{n-1} = \frac{-1}{b_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix}$$

$$c_{n-3} = \frac{-1}{b_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-5} \\ b_{n-1} & b_{n-5} \end{vmatrix}$$

. . . . . .

Continue this procedure of forming a new row from the two preceding rows until zeros are obtained; $n + 1$ rows will result.

The Routh-Hurwitz criterion states that the number of roots with positive real parts equals the number of changes of sign in the first column. Since only one root with positive real part is sufficient to cause instability, the following stability criterion may be stated. A system whose characteristic equation is Eq. (8.150) is stable if and only if the elements formed in the first column (a_n, a_{n-1}, b_{n-1}, …) are all of the same algebraic sign.

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Root-Locus Method. This method, attributable to Evans, takes the denominator of Eq. (8.149) and factors Y_1Y_2 into zeros and poles.

$$1 + Y_1(s)Y_2(s) = 1 + \frac{K\prod_i (s - z_i)}{\prod_j (s - p_j)}$$

where Π denotes product.

One first explores the zeros of $1 + Y_1Y_2$, which must satisfy two conditions—the amplitude and the phase.

$$1 + \frac{K\prod_i (s - z_i)}{\prod_j (s - p_j)} = 0$$

Amplitude condition: K

$$\left| \frac{\prod_i (s - z_i)}{\prod_j (s - p_j)} \right| = 1$$

Phase condition:

$$\arg \frac{\prod_i (s - z_i)}{\prod_j (s - p_j)} = \pi(2n + 1) \qquad n \text{ an integer} \tag{8.151}$$

where arg = argument.

The location of all possible s that satisfy the phase condition (8.151) is drawn in the s plane; each corresponds to a K satisfying the amplitude condition. Figure 8.22 shows a typical example where the locus is drawn in solid lines.

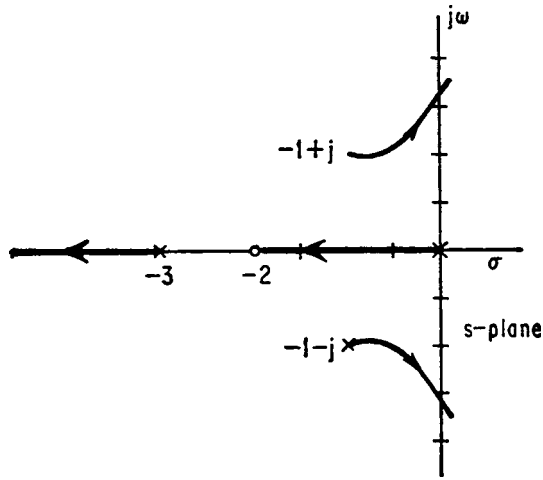

FIG. 8.22 Typical root-locus plot for $H(s)$.

$$H(s) = \frac{(s + 2)K}{s(s + 3)(s^2 + 2s + 2)}$$

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Stability for the system is ascertained by the points on the locus (of roots) that apply for the specific K in question. If the locus in question lies in the left half plane the system is stable; otherwise it is unstable. Relative stability is judged by the proximity to the imaginary axis.

Complex-Function Theory for Nyquist Criterion. The Nyquist criterion utilizes complex-function theory and in particular the so-called "argument principle." It is stated and proved as follows: Given a function $F(s)$ regular in a closed region R (except for a finite number of poles) bounded by the closed curve C, then the curve C maps into the curve C' in the plane, $w = F(s)$. The theorem asserts that the number of times the curve C' encircles the origin in the w plane is equal to the difference between the number of zeros and poles of $F(s)$ included within the region R, $N = Z - P$, including the multiplicity of the zeros and poles. In proof:

Near a zero inside R, say near $s = z_i$,

$$F(s) = (s - z_i)^{\gamma_i}[\psi(s)]$$

Near a pole inside R at p_i,

$$F(s) = \frac{1}{(s - p_i)^{\beta_i}}[\Omega(s)]$$

In general, therefore,

$$F(s) = \frac{(s - z_1)^{\gamma 1}(s - z_2)^{\gamma 2} \cdots (s - z_n)^{\gamma n}}{(s - p_1)^{\beta_1}(s - p_2)^{\beta_2} \cdots (s - p_m)^{\beta_m}}[\phi(s)]$$

where $\phi(s)$ has no zeros or poles inside the region R. Taking the logarithmic derivative of $F(s)$,

$$\frac{d}{ds}\ln F(s) = \frac{F'(s)}{F(s)} = \frac{\gamma_1}{s - z_1} + \frac{\gamma_2}{s - z_1} + \cdots - \frac{\beta_1}{s - p_1} - \frac{\beta_2}{s - p_2} - \cdots + \frac{\phi'(s)}{\phi(s)}$$

Integrating around the closed contour C in the s plane and employing Cauchy's residue theorem we get

$$\oint_c d[\ln F(s)] = \ln F(s)\Big|_c = 2\pi j\left(\sum_i \gamma_i - \sum_k \beta_k\right) + 0 = \ln w\Big|_{\text{contour } c'} = 2\pi jN$$

$$N = \sum_i \gamma_i - \sum_k \beta_k$$

where it is noted that the evaluation of $\ln w$ over the closed curve c' in the w plane yields the change of argument times j or $2\pi jN$ where N is the number of encirclements of the origin with due regard for sign. Also, since ϕ has no zeros or poles in region R, ϕ'/ϕ is regular inside R and its contour integral vanishes over the closed path, viz.,

$$\oint \frac{\phi'}{\phi} ds = 0$$

Nyquist Criterion. This principle is now applied to the transfer function $Y_1(s)/[1 + Y_1(s)Y_2(s)]$ to determine the zeros of $1 + Y_1(s)Y_2(s)$ in the right half s plane. The scanning

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contour in the s plane is the region bounded by the imaginary axis and the right-hand infinite semicircle. Instead of examining the number of encirclements of $1 + Y_1(s)Y_2(s)$ around the origin of w, by a shift of axis one unit, it is exactly equivalent to examining the contour of $Y_1(s)Y_2(s)$ mapped into w with reference to the –1 point. Now the number of encirclements of $Y_1(s)Y_2(s)$ around –1 is given by

$$N = Z - P \qquad Z = N + P \tag{8.152}$$

If the number of right-half-plane poles P of $Y_1(s)Y_2(s)$ are known (none for passive blocks, but for multiloop systems they may be present) then the number of right half plane zeros Z can be determined from Eq. (8.152) since N, the number of turns, can be counted. If Z has a value different from zero it implies that there are zeros of $1 + Y_1(s)Y_2(s)$ in the right-hand s plane establishing the case of instability. Otherwise the system is stable. The necessary and sufficient condition for stability is $Z \equiv 0$.

The actual plotting of the contour $Y_1(s)Y_2(s)$ (for rational fractions) in the w plane requires only one half imaginary axis, say $s = j\omega$, since rational functions of s display real axis symmetry as follows:

$$Y_1(s)Y_2(s) \rightarrow \begin{cases} Y_1(j\omega)Y_2(j\omega) = M(j\omega) + jN(j\omega) & s = j\omega \\ Y_1(-j\omega)Y_2(-j\omega) = M(j\omega) - jN(j\omega) & s = -j\omega \end{cases}$$

where M and N are real even and odd functions, respectively, of ω. Hence the contour mapping $Y_1(j\omega)Y_2(j\omega)$ is symmetric with $Y_1(-j\omega)Y_2(-j\omega)$ with respect to the real w axis. In plotting the infinite semicircles

$$s = Re^{j\phi} \qquad R \rightarrow \infty$$

the terms in the highest power of the numerator and denominator are retained for evaluation.

Consider, for example,

$$Y_1(s)Y_2(s) = \frac{as^3 + bs^2 + cs + d}{es^5 + fs^4 + gs^3 + hs^2 + ns + m} \rightarrow \frac{as^3}{es^5}$$

$$= \frac{a}{es^2} \rightarrow \frac{a}{eR^2e^{2\phi j}} \qquad \begin{matrix} s = Re^{j\phi} \\ R \rightarrow \infty \end{matrix}$$

A Nyquist plot is shown for a third-order system with a pole at the origin in Fig. 8.23.

Relative Stability. In addition to providing information on stability a measure of relative stability is provided by noting the proximity of the graphical plot to the –1 point in the w plane.

In quantitative terms, relative stability is determined by the relative gain of the open-loop transfer function $Y_1(j\omega)Y_2(j\omega)$ (Nyquist plot) 180° out of phase with the input, shown with amplitude e. The ratio $1/e$ is called the *gain margin*, implying that an increase of gain by this factor would make the system unstable. If $e > 1$, then the system is unstable. Similarly, the angle ϕ at unit distance is called the *phase margin* and indicates the additional amount of phase lag necessary to destabilize a stable system (see Fig. 8.24).

Systems with Feedback Time Lag. Interest is often centered on the destabilizing effect of time delays in the feedback path of a system which is otherwise stable. In some applications they are intentionally introduced for such an effect (oscillator).

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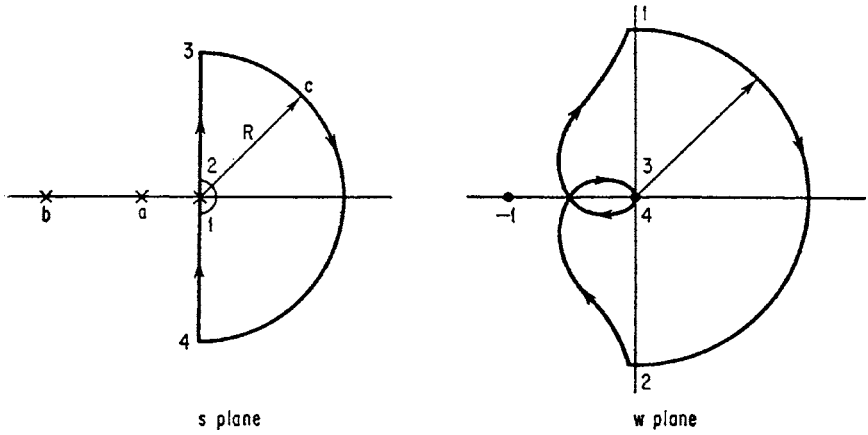

FIG. 8.23 Nyquist plot $Y_1(s)Y_2(s) = \frac{K}{s(s+a)(s+b)}$.

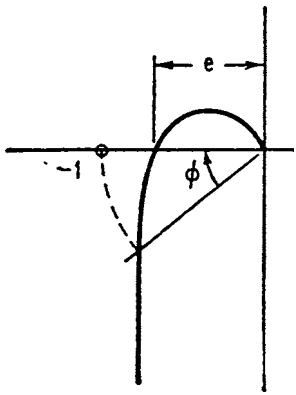

FIG. 8.24 Gain and phase margin. Phase margin = ϕ. Gain margin = 1/*e*.

The basic system is shown in Fig. 8.25 with an overall transfer function $Y_1/(1 + Y_1Y_2)$. Introduction of the time-delay block $\boxed{D}$ which is e^{-sT} changes the transfer function to $Y_1/(1 + Y_1Y_2e^{-sT})$. The system stability is readily evaluated by making the basic system Nyquist plot of Y_1Y_2 which is stable (by hypothesis) and incorporating the e^{-sT} by increasing the phase angle by $-\omega T$ at each point along the existing $Y_1(j\omega)Y_2(j\omega)$ basic plot. The new plot provides the required stability picture.

A method due to Satch, applied to a first-order differential equation with a time lag T, is shown below for the first-order system.

$$dx/dt + \alpha x + \beta x(t - T) = 0$$

The transfer function is $1/(s + \alpha + \beta e^{-sT})$, which in order to exhibit stability must be free of right-hand-plane poles. The equation for poles is

$$-(s + \alpha)/\beta = e^{-sT} \tag{8.153}$$

Let each side map the right-hand s plane, shown superimposed in Fig. 8.26*b* and *c* for two different cases. The right side maps into the unit circle, the left side the half plane

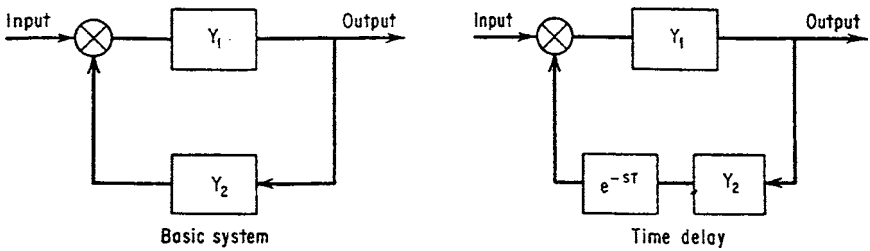

FIG. 8.25 System with feedback time delay.

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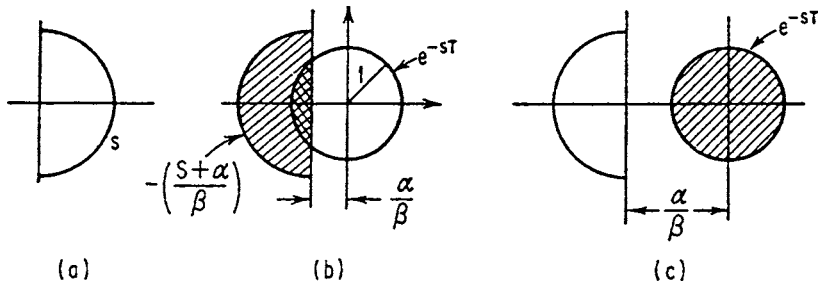

FIG. 8.26 Satch diagram for $dx/dt + \alpha x + \beta x\,(t - T) = 0$. Intersection $\alpha/\beta < 1$. No intersection (unconditionally stable), $\alpha/\beta > 1$.

displaced by $-\alpha/\beta$, shown crosshatched. The two closed curves are shown intersected in Fig. 8.26*b* for $\alpha/\beta < 1$, and therefore the included region corresponds to points in the s plane which satisfy Eq. (8.153) and consequently the system may be unstable. Further investigation would be required to ascertain stability or instability. If the regions intersected contain in each case one or more coincident points of the right half s plane, the system is unstable; otherwise it is stable. The situation depicted in Fig. 8.26*c*, $\alpha/\beta > 1$, reveals no intersection and illustrates the unconditionally stable system.

8.5.6 The Problem of Synthesis[31–33]

The problem of synthesis is to realize an overall transfer function within a set of specifications which often requires optimization of several conflicting requirements. Consider Fig. 8.20. The designer usually has little control over any block except $C(s)$, the "controller." Hence synthesis involves realization of transfer functions in cascade with fixed elements to produce the desired overall transfer function.

8.5.7 Linear Discontinuous Control: Sampled Data[25,34]

If at once or more points in a linear control system the signal is interrupted intermittently at a prescribed rate, the resultant system is discontinuous and linear. If the rate is constant, the system is called a linear sampled-data control system. Insofar as analysis is concerned, this merely introduced another building block called the "sampler" at each sampling point.

The sampler shown schematically in Fig. 8.27 has the property of taking the input $e(t)$ and periodically sampling it for time durations such that the area under each pulse (strength) is proportional to the instantaneous input. If the proportionality constant is made unity and the sampling pulse duration is small compared with the sampling period T, then to an excellent approximation which offers considerable analytic advantages, the output is assumed to be a train of impulses of strength $e(t)$ at each sampling "instant." The actual and ideal outputs are shown in Fig. 8.27.

The ideal output $e^*(t)$ considered below is given by

$$e^*(t) = e(t) \sum_{n=0}^{\infty} \delta(t - nT) = \sum_{n=0}^{\infty} e(nT)\,\delta(t - nT) \tag{8.154}$$

where $\delta(x)$ = Dirac delta function. According to continuous theory, it is desirable to obtain the frequency-domain behavior of the "sampling" block. To this end consider the Laplace transform of the sampled signal:

$$\mathcal{L}\{e^*(t)\} \triangleq E^*(s) = \mathcal{L}\left\{e(t) \sum_{n=0}^{\infty} \delta(t - nT)\right\} \tag{8.155}$$

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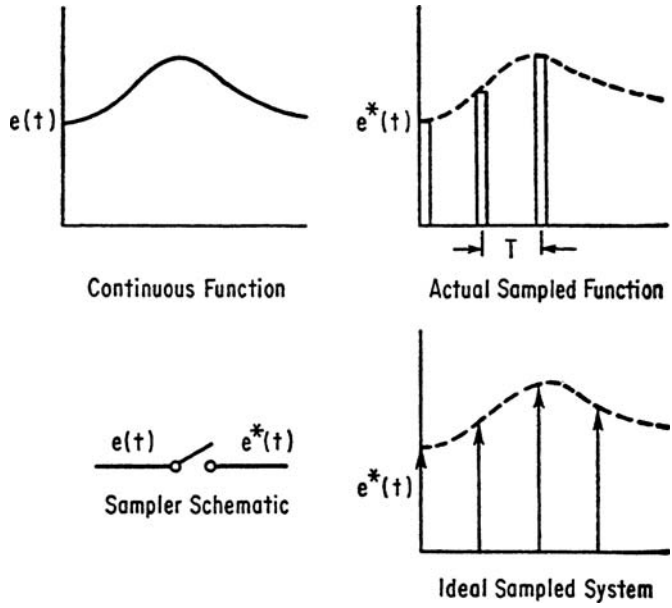

FIG. 8.27 The sampler and a sampled function.

From the convolution theorem,

$$\mathcal{L}\{(x(t)y(t)\} = X(s) * Y(s)$$

Eq. (8.155) becomes

$$\begin{aligned} E^*(s) &= \mathcal{L}\{e^*(t)\} = E(s) * \mathcal{L}\left\{\sum_{n=0}^{\infty} \delta(t - nT)\right\} \\ &= E(s) * \sum_{n=0}^{\infty} e^{-nTs} = E(s) * I(s) \qquad (8.156) \\ I(s) &= \sum_{n=0}^{\infty} e^{-nTs} \end{aligned}$$

where $I(s)$ is defined as the Laplace transform of the impulse train

$$\sum_{n=0}^{\infty} \delta(t - nT)$$

An alternative and less useful form is derived from the right-side representation of Eq. (8.154).

$$E^*(s) = \mathcal{L}\{e^*(t)\} = \mathcal{L}\left\{\sum_{n=0}^{\infty} e(nT)\,\delta\,(t - nT)\right\} = \sum_{n=0}^{\infty} e(nT)e^{-nTs}$$

Using the closed-form representation of $I(s)$, we find

$$I(s) = \sum_{n=0}^{\infty} e^{-nTs} = \frac{1}{1 - e^{-sT}}$$

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Eq. (8.156) is evaluated by a closed-contour integration as follows:

$$E^*(s) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} E(w)I(s-w)\,dw = \frac{1}{2\pi j}\oint \frac{E(w)\,dw}{1-e^{-(s-w)T}} \tag{8.157}$$

The abscissa c is chosen so that the poles of $E(w)$ have real parts $<c$ and s is defined for Re $s > c$ where the contour integral shown in Fig. 8.28 is employed since the integral over the infinite right-hand semicircle vanishes. The infinity of poles of the integrand inside this contour are then the zeros of $1 - e^{-(s-w)T}$, the latter corresponding to $s - w = \pm 2n\pi j/T$.

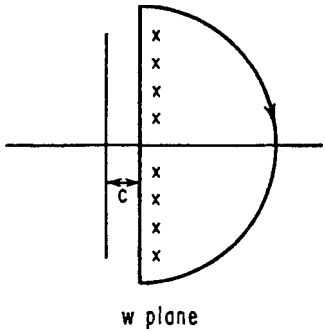

FIG. 8.28 *W*-plane contour for complex convolution.

$$w = s \pm 2n\pi j/T \quad \text{(poles)}$$

$$2\pi/T = (\omega_s) \quad \text{(sampling frequency)}$$

which are an infinity of simple poles. Equation (8.157) is evaluated by Cauchy's residue theorem, taking the residues of the infinity of poles yielding

$$E^*(s) = \sum_{n=-\infty}^{+\infty} \frac{E(s + jn\omega_s)}{T} \tag{8.158}$$

If $E(s)$ has no right-half-plane poles, then Eq. (8.158) is defined for the entire right half s plane. Equation (8.158) is clearly a periodic function of s having the complex period $j\omega_s$ as shown:

$$E^*(s + j\omega_s) = \sum_{n=-\infty}^{+\infty} \frac{E[s + j(n+1)\omega_s]}{T} = \sum_{-\infty}^{+\infty} \frac{E(s + jn\omega_s)}{T} = E^*(s) \tag{8.159}$$

From the periodic character of $E(s)$ it follows that if $E^*(s)$ is known in any strip in the complex s plane bounded by

$$jx < \text{Im } s < j(x + \omega_s) \tag{8.160}$$

it is known everywhere in the s plane. The transfer function at real frequency $s = j\omega$ is found directly from Eq. (8.158). Its amplitude spectrum is sketched (Fig. 8.29) for $\omega_s/2 > \omega_0$ where ω_0 is the cutoff frequency of $E(j\omega)$, i.e., for the sampling frequency greater than twice the highest frequency component of $e(t)$. Note that $|E^*(j\omega)|$ yields the infinitely repeated spectrum of $|E(j\omega)|$ attenuated by $1/T$. If $\omega_s/2 < \omega_0$, there is

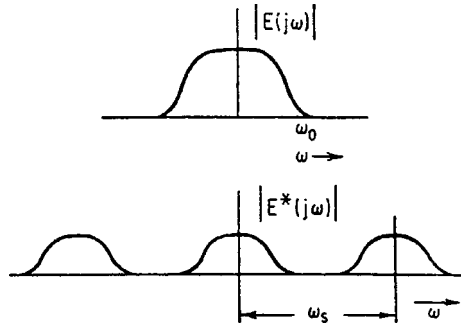

FIG. 8.29 Frequency spectrum for function and for sampled function.

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overlapping and resultant distortion of the input signal. Returning to the case $\omega_s/2 > \omega_0$, practically all the input $e(t)$ information is stored in $E^*(j\omega)$ over the frequency range $0 < \omega < \omega_0$. By ideal low-pass filtering of the signal $E^*(j\omega)$, spectral components greater than ω_0 can be eliminated, leaving the fundamental signal shape. The resultant system would then be the equivalent of the continuous system with an attenuator $1/T$ placed after the input signal. In practice, however, deviations from ideally of this filter introduce severe stability problems. A smoothing device is utilized as a compromise between high degree of filtering and its concomitant stability problems.

One such smoothing device is the holding circuit whose transfer function is

$$(1 - e^{-Ts})/s \tag{8.161}$$

which for any impulse input $\delta(t - nT)$ yields the output $u(t - nT) - u[t - (n + 1)T]$, a pulse of unit height starting $t = nT$ and of duration T.

Stability Investigation of Sampled-Data Control Systems. Consider the sampled-data control system shown in Fig. 8.30. The overall transfer function is derived from the basic properties of transfer functions.

$$E = Z - Y_2W \qquad W = Y_1E^* \tag{8.162}$$

Elimination of W in Eq. (8.162) gives

$$E = Z - Y_1Y_2E^* \tag{8.163}$$

Now a fundamental property of sampling is stated and proved as follows:

Given $A = BC^*$

then $A^* = B^*C^*$

Proof [utilizing Eq. (8.158)]:

$$X^*(s) = \frac{1}{T}\sum_{n=-\infty}^{+\infty} X(s + jn\omega_s)$$

$$A^*(s) = \frac{1}{T}\sum_{n=-\infty}^{+\infty} A(s + jn\omega_s) = \frac{1}{T}\sum_{n=-\infty}^{+\infty} B(s + jn\omega_s)C^*(s + jn\omega_s)$$

But from Eq. (8.159)

$$C^*(s + jn\omega_s) = C^*(s)$$

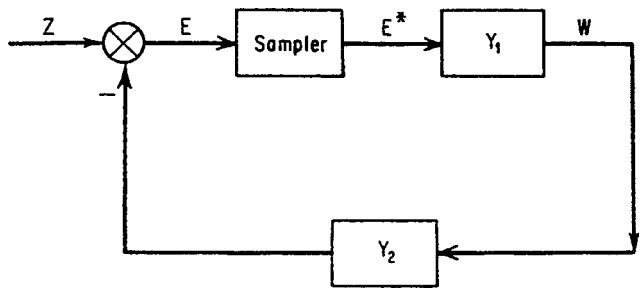

FIG. 8.30 Sampled-data system.

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and therefore

$$A^*(s) = \frac{C^*(s)1}{T} \sum_{n=-\infty}^{+\infty} B(s + jn\omega_s) = C^*(s)B^*(s) = B^*(s)C^*(s)$$

Sampling Eq. (8.163) and utilizing the results of this theorem yields

$$E^* = Z^* - (Y_1Y_2)^*E^* \tag{8.163a}$$

or

$$E^* = Z^*/[1 + (Y_1Y_2)^*]$$

and

$$W = Y_1E^* = Z^*Y_1/[1 + (Y_1Y_2)^*]$$

Following previous work, the stability of the system rests with the location of the poles of $1/[1 + (Y_1Y_2)^*]$ or more specifically the zeros of $1 + (Y_1Y_2)^*$.

The transcendental form of $(Y_1Y_2)^*$ makes this evaluation using methods cited earlier extremely difficult to apply, per se. The task is simplified, however, owing to the periodic property of sampled functions embodied in Eq. (8.159) which implies that if one investigates the zeros of $(Y_1Y_2)^* + 1$ bounded by a strip given in Eq. (8.160) in region Re $s > 0$, then this effectively gives the zero configuration in the entire right half plane. In practice the strip chosen is $0 < \text{Im } s < j\omega_s = j2\pi/T$, and Re $s > 0$ shown crosshatched in Fig. 8.31. Now a function z is defined by

$$z = e^{sT} \tag{8.164}$$

$$s = \ln z/T \tag{8.165}$$

with s defined only in the whole strip

$$0 < \text{Im } s < j2\pi/T$$

so that s and z are single-valued analytic functions of each other (except for a branch cut on the real z axis). Equation (8.164) implies a mapping of the whole strip in the s plane onto the entire z plane with the shaded portion mapping outside the unit circle as shown in Fig. 8.31. The location of zeros of $1 + (Y_1Y_2)^*$ in the right half strip of s corresponds to the location of zeros outside the unit circle of the z plane, and their presence or absence is translated as unstable or stable conditions, respectively.

z Transform. If the transformation Eq. (8.164) is applied to the Laplace transformation of a sampled function $x^*(t)$ as follows:

$$X(z) \triangleq X^*(s)$$
$$z = e^{sT}$$

then $X(z)$ is defined as the z transform of $x(t)$. $X(z)$ is valid only at the sampling instants despite its general continuity properties. Note that the z transform of $x^*(t)$ is

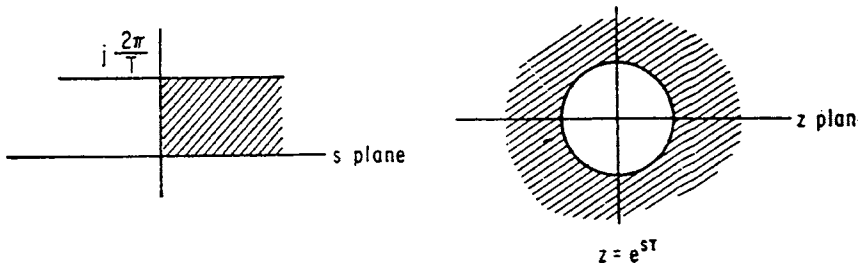

FIG. 8.31 Mapping the strip onto the z plane.

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the same as that of $x(t)$ since the sampled $x^*(t)$ is indeed $x^*(t)$. The z transformation of any function is obtained by first sampling the function, then taking its Laplace transform, and finally making the substitution [Eq. (8.165)] to eliminate s in favor of z. From the definition,

$$E(z) = E^*(s) = \sum_{n=0}^{\infty} e(nT)e^{-nTs} = \sum_{n=0}^{\infty} e(nT)z^{-n} \tag{8.166}$$

or from the convolution form [Eq. (8.157)],

$$E^*(s) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} E(w)\frac{dw}{1 - e^{-st}e^{wT}}$$

$$= \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} E(w)\frac{dw}{1 - e^{wT}z^{-1}} = E(z) \qquad \text{Re } s > c$$

If this integration is performed over the contour enclosing the left-hand infinite semicircle in contradistinction to the contour used previously so that the zeros of $1 - e^{wt}z^{-1}$ are not contained inside the contour, then application of the residue theorem yields

$$E(z) = \sum \text{residues of} \left[E(w)\frac{1}{1 - e^{wT}z^{-1}}\right]$$

for poles of $E(w)$ only.

Inversion of E(z) to Time Domain. To go from the z domain to the time domain, it is only necessary to consider the coefficients of a Laurent series about $z = 0$ which yield the sampling values, i.e., if

$$F(z) = a_0 + a_1/z + a_2/z^2 + \cdots$$

in accordance with Eq. (8.166)

$$a_0 = e(0)$$
$$a_1 = e(T)$$
$$\cdots\cdots\cdots$$
$$a_n = e(nT)$$

Formally using any closed contour around $z = 0$, this is equivalent to the contour integral

$$e(nT) = (1/2\pi j)\oint E(z)z^{n-1}\,dz = \text{ residues of } E(z)z^{n-1}$$

Table 8.3 shows some z transforms and their properties.

Example of Stability Investigation. Consider the sampled-proportional-level control system shown schematically and in block form in Fig. 8.32 with the constant input w_R (desired level). The system is described by the following:

1. $q_0(t)$ is an arbitrary-rate flow of effluent.
2. The replenishment rate is proportional to the error existing one sample time prior to

$$q_{in} = -K(w[t/T] - w_R)$$

where $[x]$ is defined as the smallest integral value of x.

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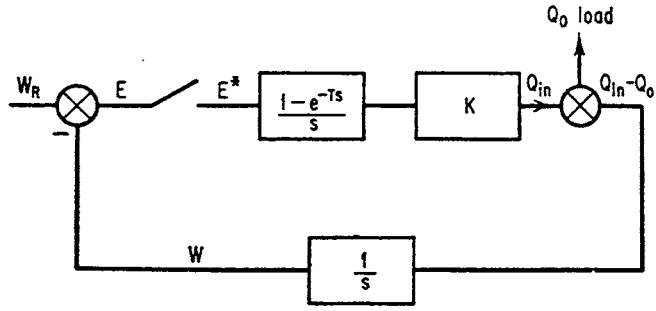
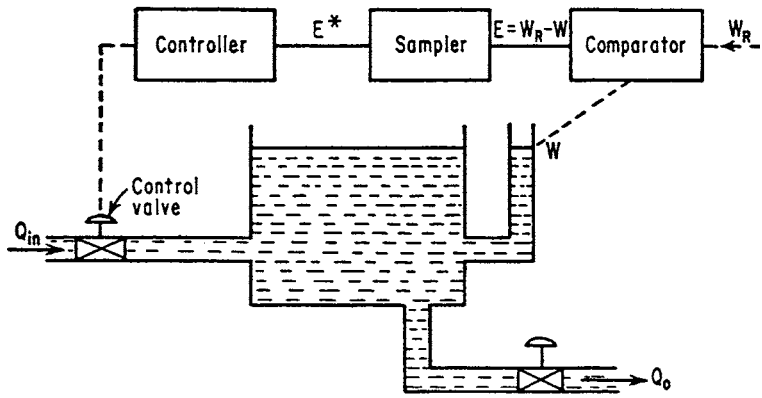
TABLE 8.3 Properties of z Transforms of Causal Time Function

$f(t) = 0 \qquad t < 0$

Property	z transform	Time function
Basic pairs	$F(z) = \sum_{n=0}^{\infty} f(nT)z^{-n}$	$f(t)$ $f(nT) \qquad n = 0,1, \ldots, \infty$ $f(nT) = 1/2\pi j \oint F(z)z^{n-1}\,dz$
Linearity	$a_1F_1(z) + a_2F_2(z)$	$a_1f_1(t) + a_2f_2(t)$
Time shift	$z[F(z) - f(0)]$	$f(t + T)$
Initial-value theorem	$\lim_{z\to\infty} F(z)$	$\lim_{t\to 0} f(t)$
Final-value theorem	$\lim_{z\to 1} (z - 1)F(z)$	$f(t)$ $t \to \infty$



(a)



(b)

FIG. 8.32 Sampled-level control system. (*a*) Control block diagram. (*b*) System.

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Writing the equation conserving mass yields

$$dw/dt = -K(w[t/T] - w_R) - q_0(t)$$

Integrating over sampling times nT and $(n + 1)\,T$ yields the equation

$$w\{(n + 1)T\} - w(nT) = TK\{w_R - w(nT)\} - \int_{nT}^{(n+1)T} q_0\,dt$$

$$w\{(n + 1)T\} - (1 - TK)w(nT) = TKw_R - \int_{nT}^{(n+1)T} q_0\,dt$$

which is the inhomogeneous difference equation whose theory parallels that of ordinary differential equations. The stability is a function of the solutions to the homogeneous equation

$$w_1\{(n + 1)T\} - (1 - TK)w_1\{nT\} = 0 \tag{8.167}$$

A solution is found by assuming the exponential form

$$w_1(t) = e^{\lambda t}$$

whence

$$e^{\lambda(n+1)T} - (1 - TK)e^{n\lambda T} = 0 \tag{8.168}$$

from which

$$e^{\lambda T} - (1 - TK) = 0$$

$$\lambda = \frac{\log(1 - TK)}{T} = \frac{\log|1 - TK|}{T} + j\arg(1 - TK)$$

The condition for stability is that as $t \to \infty$, $w_1 \to 0$. From Eq. (8.168), this is met by requiring

$$\begin{aligned} \text{Re}\,\lambda &< 0 \\ \log|1 - TK| &< 0 \\ |1 - TK| &< 1 \end{aligned}$$

Alternatively the z-transform method can be applied directly to Eq. (8.167), which is just written as

$$w_1^*(t + T) - (1 - TK)w_1^*(t) = 0 \tag{8.167a}$$

and is a valid representation of Eq. (8.167) only at the sampling instants.

Taking the z transform of Eq. (8.167a) yields

$$z\{W(z) - w(0)\} - (1 - TK)W(z) = 0$$

$$W(z) = \frac{zw(0)}{z - (1 - TK)} \tag{8.169}$$

Applying the stability condition to Eq. (8.169), namely, that it have no poles outside the unit circle (i.e., no zeros of its denominator outside the unit circle) yields

$$|1 - TK| < 1$$

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which has been obtained by classical methods above. Since T and K are real, the condition is

$$-1 < (1 - TK) < 1$$

Finally, and more generally, direct consideration of the control block (Fig. 8.32) and Eq. (8.163*a*) yields

$$E^* = \frac{W_R^*}{1 + K\{(1 - e^{-Ts})/s^2\}^*} + \frac{\{Q_0(s)/s\}^*}{1 - K\{(1 - e^{-Ts})/s^2\}^*} \tag{8.170}$$

From tables in Ref. 10

$$\left(\frac{1 - e^{-Ts}}{s^2}\right) = (1 - z^{-1})\frac{Tz}{(z - 1)^2} = \frac{T}{z - 1}$$

The transform of Eq. (8.170) is

$$E(z) = \frac{W_R(z)}{1 + KT/(z - 1)} + \frac{\{Q_0(s)/s\}^*}{1 + KT/(z - 1)} \qquad s = \ln \frac{z}{T}$$

No poles of the numerator are envisioned for practical systems, so that the zeros of the denominator give all $E(z)$ poles

$$z = 1 - TK$$

which for stability demands $|z| < 1$ or once again

$$-1 < 1 - TK < 1$$

8.5.8 Nonlinear Control Systems[36–39]

The treatment of control systems containing nonlinearities (as defined under general nonlinear systems) is for the most part so formidable that all known analytic methods fail. The very special case of the second-order autonomous (time-invariant coefficients) can be handled most conveniently by graphical methods in the phase plane. Also under very special conditions it is possible for a higher-degree nonlinear system to be analyzed by a "describing-function" technique. Systems which contain switching-function nonlinearities which are otherwise linear can be analyzed (with great difficulty) by linear methods over each (linear) regime of operation satisfying boundaries between regimes. However, as the order of the equation goes beyond three, the difficulty in matching boundaries becomes prohibitive. Often the second-order case is handled most conveniently by phase-plane graphical methods.

Linear Systems with Discontinuous Switching. These systems are characterized by switching operations. If switching occurs at a constant rate, then the system is the linear sampled-data system described above. If on the other hand the switching operation occurs whenever the signal (e.g., error) reaches a prescribed level of some function of the output, then the system is of the relay type and is in general nonlinear. Examples of this type are shown in Fig. 8.33. More generally, the switching points may be mixed functions of the input variable and its derivative; ϵ and $\dot{\epsilon}$ have a phase-plane representation as shown for the example in Fig. 8.33*e*. In addition to the behavior as a function of input, hysteresis, and dead zone, there are time lags inherent in operation because of inertia and inductance.

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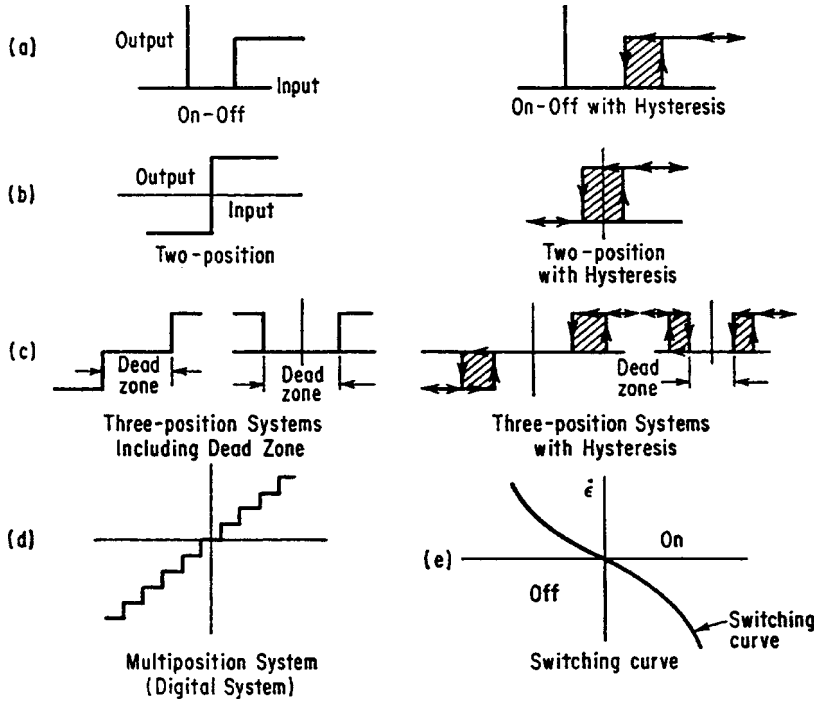

FIG. 8.33 Switching functions.

An illustration of a linear system with switching is a room heater whose block diagram is drawn in Fig. 8.34 with relay characteristics shown in Fig. 8.34*c* indicating that the furnace goes on whenever $T < T_1$ and off whenever $T > T_2$. The furnace and room are two first-order systems connected in tandem. Analytically, this result is expressed by two equations where $p = d/dt$:

$$(p + b)T = By + bT_0 \tag{8.171}$$
$$(p + a)y = Af(T, T_1, \dot{T})$$

where y = furnace output
T_o = outside temperature
$f(T, T_1, \dot{T})$ = switching function

Elimination of y in Eq. (8.171) yields

$$(p + a)(p + b)T = ABf(T, T_1, \dot{T}) + abT_o \tag{8.171a}$$

Note that the function f not only depends upon T and T_1 but also on the sign of $\dot{T}$ as follows:

$$f(T, T_1, \dot{T}) = \begin{cases} Q & T < T_1 \\ Q & T_1 < T < T_2, \dot{T} > 0 \\ 0 & T_1 < T < T_2, \dot{T} < 0 \\ 0 & T > T_2 \end{cases} \tag{8.172}$$

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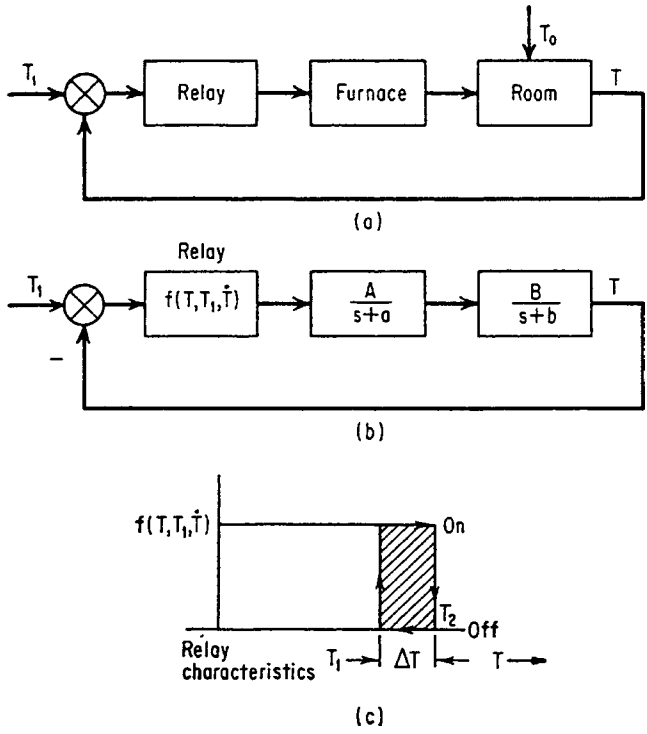

FIG. 8.34 Room-heater control system.

which constitutes four possible regimes of operation. Accordingly, Eq. (8.172) must be solved for the heating cycle $f(T, T_1, \dot{T}) = Q$ and the cooling cycle $f(T, T_1, \dot{T}) = 0$. For heating,

$$T_{\text{heating}} = \alpha e^{-at} + \beta e^{-bt} + T_\infty$$

where T_∞ is the asymptotic temperature for uncontrolled continuous heating. For cooling,

$$T_{\text{cooling}} = \gamma e^{-at} + \delta e^{-bt} + T_o$$

The four constants of integration, α, β, γ, and δ, must be determined by matching conditions at the transitions of any two regimes where continuity of T and its derivative $\dot{T}$ must be preserved. This becomes a most laborious procedure, since these constants change repeatedly, regime after regime, cycle after cycle. Only under constant load T_o will a cycle that is repetitive be eventually reached (limit cycle).

The prohibitive analytic method is seldom justified for second-order systems whose complete graphical solution in the phase plane can be easily generated. As an introduction to the method, consider the first-order temperature-control system

$$\dot{T} + \alpha T = f(T, T_1, \dot{T}) + \alpha T_o \tag{8.173}$$

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with its characteristic switching function f. The two modes of operation for Eq. (8.173) are

Heating: $$\dot{T} + \alpha T = Q + \alpha T_o \triangleq \alpha T_\infty$$

Cooling: $$\dot{T} + \alpha T = \alpha T_o$$

In the phase plane for both cases (taking one time derivative), assuming T_o constant,

$$\dot{T}(d\dot{T}/dT)\ddot{T} = \ddot{T} = -\alpha\dot{T}$$
$$d\dot{T}/dT = -\alpha$$

The corresponding locus of phase-plane operation therefore consists of two parallel lines of operation of slope $-\alpha$ as drawn in Fig. 8.35 identified by their T intercepts. The limit cycle 1-2-3-4 is shown crosshatched. Jump action occurs at points 2 and 4, the switching points. For any initial $\dot{T}$, T to the left of $T = T_1$ shown for point Q, the point will jump vertically to the heating line and proceed to 3-4-1-2-3, closing the cycle. Similarly for point P at $T > T_2$, the point inside the zone $T_1 < T < T_2$ goes to the cooling curve first. The closed-cycle time is

$$\oint \frac{dT}{\dot{T}} = \int_{T_1}^{T_2} \frac{dT}{\dot{T}} + \int_{T_2}^{T_1} \frac{dT}{\dot{T}}$$

Returning to the second-order temperature system defined by Eq. (8.171*a*),

$$\dddot{T} + (a + b)\ddot{T} + ab\dot{T} = ABf(T, T_1, \dot{T}) + abT_o$$

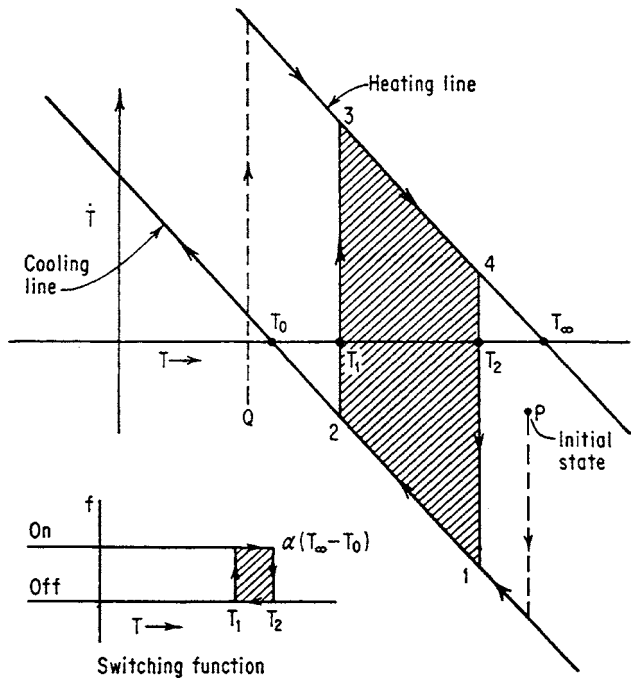

FIG. 8.35 Phase-plane plot for first-order temperature-control system $\dot{T} + \alpha T = f + \alpha T_0$; f = switching function.

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By a suitable change of variable, Eq. (8.171) can always be represented as

$$\ddot{T} + a'\dot{T}' + T' = f'(T', T_1', \dot{T}) + T_o'$$

where primes have been appended to imply the transformation. Dropping these primes for convenience leaves

$$\ddot{T} + \alpha\dot{T} + T = f(T, T_1, \dot{T}) + T_o$$

whose representation for the phase-plane plot has been shown to be

$$\begin{aligned} d\dot{T}/dT &= (-\alpha\dot{T} - T + f + T_o)/\dot{T} \\ &= (-\alpha\dot{T} - T + T_\infty)/\dot{T} \qquad \text{(heating cycle)} \\ &= (-\alpha\dot{T} - T + T_o)/\dot{T} \qquad \text{(cooling cycle)} \end{aligned}$$

The phase-plane plot of this system is determined by first drawing the switching lines $T = T_1$ and $T = T_2$, and executing the Liénard construction for each of the two operating regimes. A limit cycle is reached as shown in Fig. 8.36.

Positioning Systems with Dead Zone. The phase-plane constructions for the following second-order positioning systems are shown in Fig. 8.37, each with dead zone:

Spring-mass damping: $\ddot{\theta} + c\dot{\theta} + \theta = -F \operatorname{sgn} \theta$ (8.174*a*)

$$\frac{d\dot{\theta}}{d\theta} = \frac{-c\dot{\theta} - F \operatorname{sgn} \theta - \theta}{\dot{\theta}} \qquad F = \begin{cases} 0 & |\theta| < \delta \\ 1 & |\theta| > \delta \end{cases} \tag{8.174b}$$

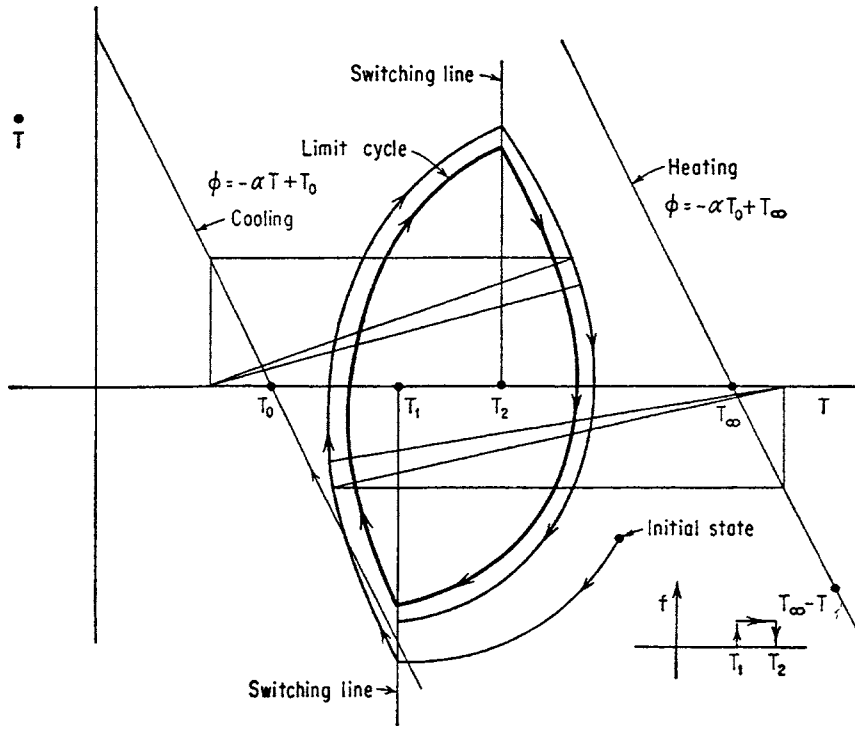

FIG. 8.36 Phase-plane plot for $\ddot{T} + \dot{T}\alpha + T = f + T_0$.

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SYSTEM DYNAMICS

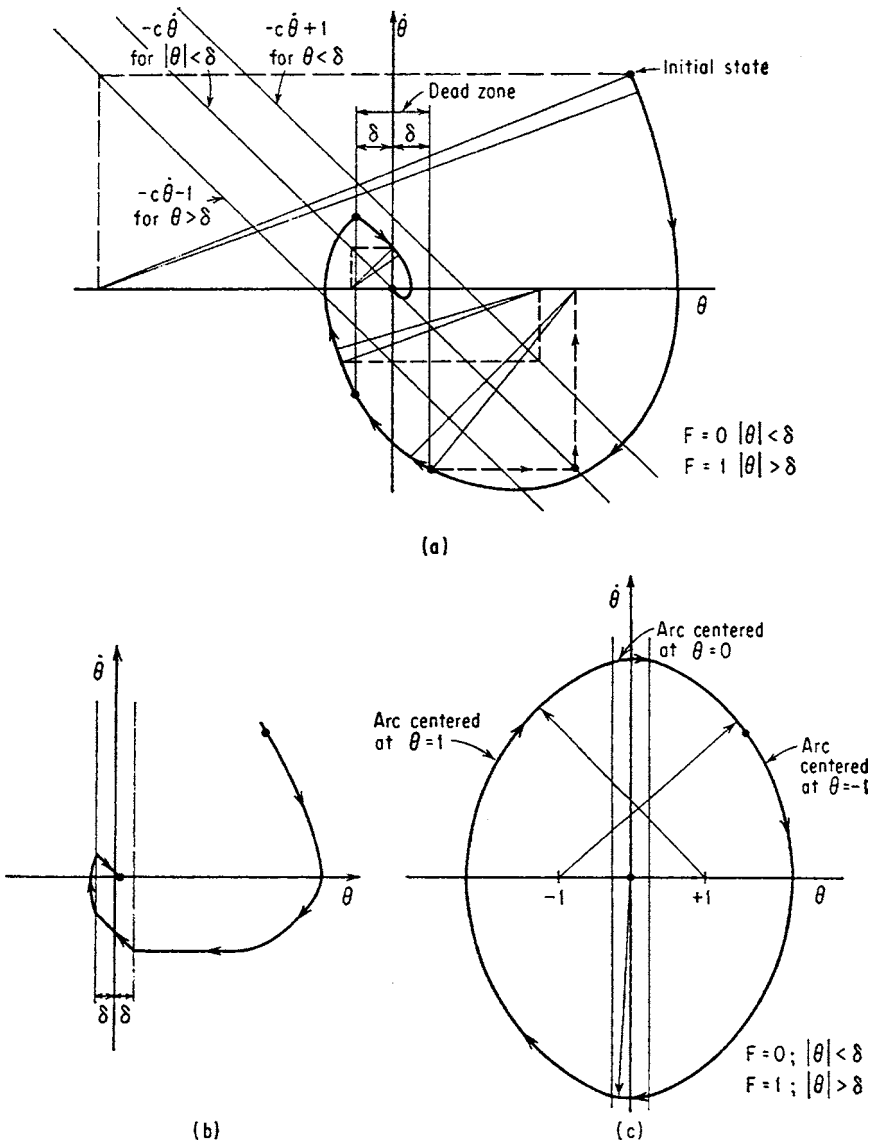

FIG. 8.37 Phase-plane plot for second-order systems.

(a) Spring-mass damping $\ddot{\theta} + c\dot{\theta} + \theta = -F \operatorname{sgn} \theta$

(b) Mass damping $\ddot{\theta} + c\dot{\theta} = -F \operatorname{sgn} \theta$

(c) Mass spring $\ddot{\theta} + \theta = -F \operatorname{sgn} \theta$

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Mass damping:
$$\ddot{\theta} + c\dot{\theta} = -F \operatorname{sgn} \theta$$
$$\frac{d\dot{\theta}}{d\theta} = \frac{-c\dot{\theta} - F \operatorname{sgn} \theta}{\dot{\theta}} = -c - \frac{F \operatorname{sgn} \theta}{\dot{\theta}}$$

Mass spring:
$$\ddot{\theta} + \theta = -F \operatorname{sgn} \theta \qquad (8.174c)$$
$$\frac{d\dot{\theta}}{d\theta} = \frac{-\theta - F \operatorname{sgn} \theta}{\dot{\theta}}$$

A first integral of Eq. (8.174*c*) gives

$$\dot{\theta}^2/2 + (\theta + F \operatorname{sgn} \theta)^2/2 = k_0^2$$

Indicating that the phase-plane plot consists entirely of arcs of circles centered at 0, –1, +1, as shown in Fig. 8.37.

Systems with Nonlinear Elements. Previously considered were linear systems made nonlinear by switching operations. These could be analyzed by classical analytical techniques. The situation with nonlinear systems is more difficult.

Systems of the second degree and lower can be treated by graphical analysis as shown for linear switching systems. Systems of higher order cannot be studied in general by analytic or graphical methods.

Examples of some usual types of nonlinear frequency-insensitive elements frequently occurring in "linear" systems are shown in Fig. 8.38.

An example of a nonlinear system controlled by a two-position force with dead zone is the second-order Coulomb damped system

$$\ddot{\theta} + c \operatorname{sgn} \dot{\theta} + \theta = F \operatorname{sgn} \theta$$
$$\dot{\theta}(d\dot{\theta}/d\theta) + \theta = F \operatorname{sgn} \theta - c \operatorname{sgn} \dot{\theta}$$
$$\dot{\theta}\, d\dot{\theta} + (\theta - F \operatorname{sgn} \theta) + c \operatorname{sgn} \dot{\theta})\, d\theta = 0$$
$$\dot{\theta}^2/2 + (\theta - F \operatorname{sgn} \theta) + c \operatorname{sgn} \dot{\theta})^2/2 = k^2$$

which describes circular arcs in the phase plane with six centers depending on the signs of θ and $\dot{\theta}$ and the amplitude of θ. Motion in the phase plane is shown in Fig. 8.39.

Describing-Function Analysis.[10] As pointed out above, higher-order nonlinear systems are not amenable to graphical analysis. A method of analysis has evolved which is valid under very restrictive conditions. It employs linear concepts in an attempt to simplify the complex nonlinear problem and bring it within the realm of analysis.

The analysis is limited to systems containing one nonlinearity or where many can be grouped to yield effectively one nonlinear and time-invariant block. For a sinusoidal input, the resultant output will be composed of the fundamental plus higher harmonics. The essence of the analysis is to ignore all harmonics other than the fundamental. This is the most restrictive assumption and can often be justified for slight nonlinearities where the higher harmonics are small to begin with; these are further attenuated since most systems are usually natural low-pass filters.

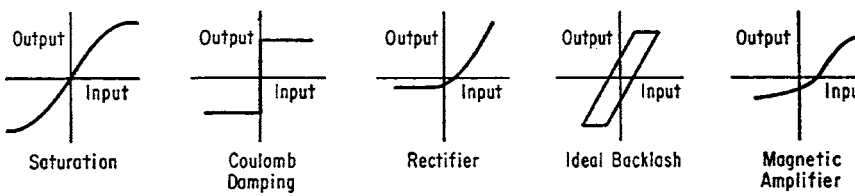

FIG. 8.38 Characteristic nonlinear elements.

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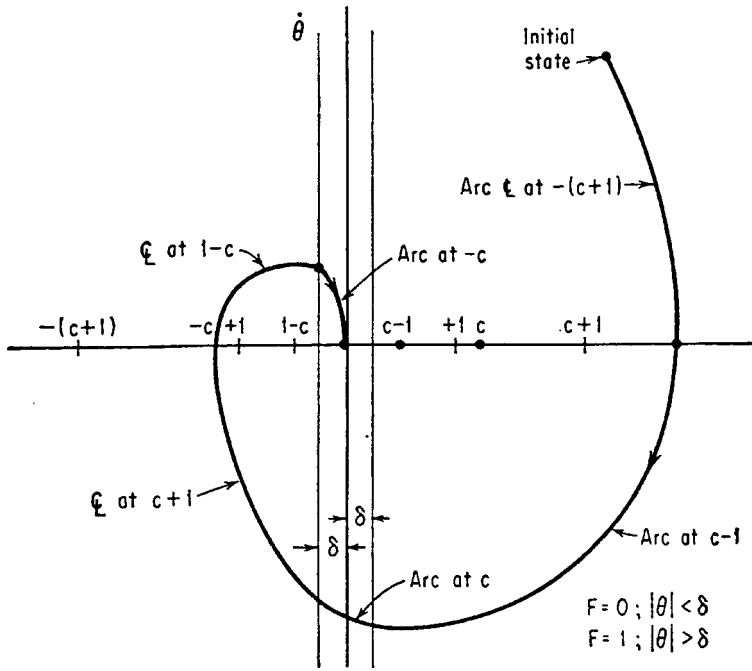

FIG. 8.39 Phase-plane plot for $\ddot{\theta} + c \operatorname{sgn} \dot{\theta} + \theta = -F \operatorname{sgn} \theta$.

Implementing the foregoing description, the object is to obtain for a fundamental input of amplitude A and frequency ω a Fourier series whose fundamental amplitude is $B(A, \omega)$ and phase $\phi(A, \omega)$. If a functional relation connects the input with the output, say

$$x_2 = f(x_1)$$

then for an input

$$x_1 = A \sin \omega t$$

the output is

$$x_2(A \sin \omega t) = f(A \sin \omega t)$$

which shows the x_2 is a periodic function of time, with period $2\pi/\omega$. It therefore has a Fourier series development ($\phi = \omega t$)

$$x_2(\mathrm{A} \sin \phi) = f(A \sin \phi)$$

$$x_2(A \sin \phi) = a_1 \sin \phi + a_2 \sin 2\phi + \cdots + a_n \sin n\phi + b_1 \cos \phi + b_2 \cos 2\phi + \cdots + b_n \cos n\phi$$

For an input $A \cos \phi$, the output would be obtained by adding $\pi/2$ to ϕ:

$$x_2(A \cos \phi) = x_2[A \sin (\phi + \pi/2)] = a_1 \cos \phi + \cdots - b_1 \sin \phi + \cdots$$

The amplitude ratio of the fundamental output to input is

$$|N| = \frac{(a_1^2 + b_1^2)^{1/2}}{A} \qquad \begin{aligned} a_1 &= a_1(A) \\ b_1 &= b_1(A) \end{aligned}$$

and the phase

$$\theta = \tan^{-1} (b_1/a_1)$$

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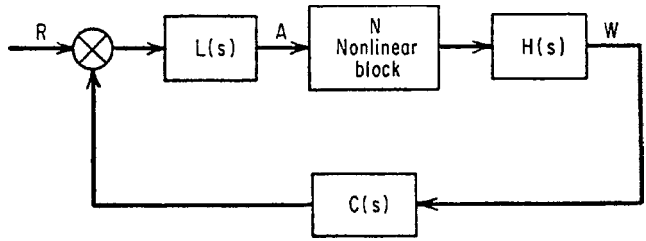

FIG. 8.40 Nonlinear control system.

By definition, the describing function

$$N = (a_1^2 + b_1^2)^{1/2}/A < \theta$$

is the complex ratio of the fundamental component of output to input. For nonlinear systems containing energy storage or dissipative elements the describing-function amplitude is a function not only of amplitude but of frequency as well, i.e.,

$$N = N(A, \omega) < (A, \omega)$$

Stability Analysis. A stability investigation of a system with one nonlinear block characterized by a describing function utilizes graphical techniques.

Consider, for example, the control system shown in Fig. 8.40. Assume a signal whose fundamental amplitude A impinges on the input to the nonlinear block. The transfer functions of the linear blocks have their arguments s replaced by $j\omega$ to obtain overall characteristics for real frequency. It should be noted here that right-half-plane poles no longer have meaning in the usual sense and interest must be necessarily restricted to sinusoidal signals owing to the describing-function definition. The loop gain at frequency ω is obtained by starting a signal amplitude A at zero phase at the input to the nonlinear device, and going completely around the loop giving

$$LHCN \times A$$

If this signal is 180° out of phase and greater than A, the amplitude grows; if it is less than A in amplitude, the signal decays. Grouping the linear blocks $LHC = G$ the condition for a net increase of signal is

Amplitude: $|G(j\omega)N(A, \omega)| > 1$

Argument: $G(j\omega)N(A, \omega) = (2n + 1)\pi \qquad n = \text{integer}$

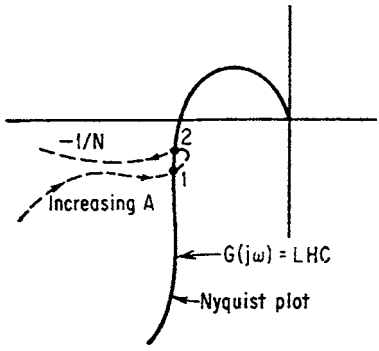

FIG. 8.41 Nyquist plot for nonlinear system.

A convenient way of illustrating this is indicated in Fig. 8.41, where first $G(j\omega)$ is plotted in a usual Nyquist plot. On the same set $-1/N$ is plotted (frequency-independent case) with the arrow in the direction of increasing amplitude A. Intersection for this case corresponds to sustained oscillations. At point 1 the frequency corresponding to the plot $G(j\omega)$ is an unstable point, since if perturbed in the direction where $-1/N > G(j\omega)$, A will decay. If perturbed in the opposite direction $-1/N < G(j\omega)$, the condition for increased growth of A, it will proceed to point 2, the stable oscillation point. If the $-1/N$ curve lies entirely within the $G(j\omega)$ plot, the amplitude

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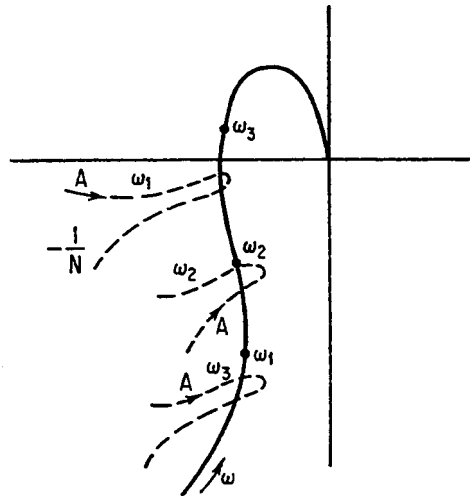

FIG. 8.42 Nyquist plot for nonlinear system where N is a function of frequency and amplitude.

will grow without bound, since $-1/N < G(j\omega)$ for all A; if outside, the system is unconditionally stable.

If N is a function of ω as well, then in the plot of $G(j\omega)$ and N as shown in Fig. 8.42 N is drawn for constant ω, e.g., ω_1, ω_2, ω_3. A qualitative analysis similar to the foregoing can be inferred. Stable or unstable points of intersection take on meaning only where there is correspondence of ω as well as amplitude as shown at ω_2.

8.6 SYSTEMS VIEWED FROM STATE SPACE

8.6.1 State-Space Characterization

The state-space description is a general time-domain representation of discrete systems that yield differential (or difference) equations, both linear and nonlinear. It is the basis of modern control theory and an outgrowth of modern computer technology with its well-known capabilities to solve systems, however formidable, often in real time. Moreover, its concise form is utilized in applications of methods of the calculus of variations for the optimal control of systems.

In its most general form the system is given by

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \qquad \text{(plant equation)} \tag{8.175a}$$

$$\mathbf{y}(t) = \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t), t) \qquad \text{(output equation)} \tag{8.175b}$$

where $\mathbf{x}(t)$ = state vector with n elements $x_i(t)$, $i = 1, 2, \ldots, n$
$\mathbf{u}(t)$ = input vector with m elements $u_i(t)$, $i = 1, 2, \ldots, m$
$\mathbf{y}(t)$ = output vector with r elements $y_i(t)$, $i = 1, 2, \ldots, r$
t = independent variable, time
$\mathbf{f}$ = general plant vector of n functions $f_i(\mathbf{x}, \mathbf{u}, t)$, $i = 1, 2, \ldots, n$
$\mathbf{h}$ = output vector of r functions $h_i(\mathbf{x}, \mathbf{u}, t)$, $i = 1, 2, \ldots, r$

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The linear form of the system is written as

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\,\mathbf{x}(t) + \mathbf{B}(t)\,\mathbf{u}(t) \tag{8.176a}$$

$$\mathbf{y}(t) = \mathbf{C}(t)\,\mathbf{x}(t) + \mathbf{D}(t)\,\mathbf{u}(t) \tag{8.176b}$$

where $\mathbf{A}(t) = n \times n$ system or plant matrix
$\mathbf{B}(t) = n \times m$ input matrix
$\mathbf{C}(t) = r \times n$ output matrix
$\mathbf{D}(t) = r \times n$ input-output coupling matrix

Equation (8.176*a*) is the set of n linear first-order differential equations and is referred to as the "plant equation." Equation (8.176*b*) is a set of r algebraic equations called the "output equation." We will confine our attention mostly to the special case of a linear time-invariant system, owing to its sufficiently useful representation for most systems and its analytic tractability; we thus focus on Eqs. (8.176*a*) and (8.176*b*) with constant matrices **A**, **B**, **C**, and **D**, or

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{8.176c}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{8.176d}$$

If an nth-order differential equation of a system is known, then an equivalent phase state set of equations can be determined. As an example, consider the equation of the special class of single-input–single-output (SISO) systems,

$$\frac{d^n y}{dt^n} + a_n \frac{d^{n-1} y}{dt^{n-1}} + a_{n-1} \frac{d^{n-2} y}{dt^{n-1}} + \cdots + a_1 y = Ku(t) \tag{8.177}$$

which is equivalent to a transfer function,

$$y(s)/u(s) = K/(s^n + a_n s^{n-1} + a_{n-1} s^{n-2} + \cdots + a, s + a_1)$$

If it is assumed that

$$\begin{aligned} x_1 &= y \\ \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\cdots\cdots\cdots \\ \dot{x}_{n-1} &= x_n \end{aligned}$$

then by repeated differentiation and substitution,

$$\dot{x}_n = d^n y/dt^n = -a_n x_n - a_{n-1} x_{n-1} - \cdots - a_1 x_1 + Ku(t)$$

In matrix form this is equivalent to

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ K \end{bmatrix} \mathbf{u} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{8.178a}$$

$$\mathbf{y} = [1 \quad 0 \quad 0 \quad \cdots \quad 0]\mathbf{x} = \mathbf{C}^T\mathbf{x} \tag{8.178b}$$

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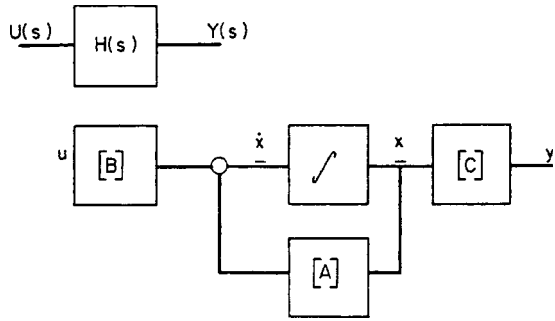

FIG. 8.43 Classical transfer function state-space diagram.

This equation constitutes the state-space set of the ordinary differential equations, Eqs. (8.176*c*) and (8.176*d*). The state variables so chosen, $x_i(t)$, $t = 1, 2, \ldots, n$, lead to a special matrix **A**, called the "companion matrix." It should be pointed out that the state-space set so chosen is not unique for the system, and indeed if one set is determined, then an infinite number of admissible sets exist. Figure 8.43 compares the classical transfer-function block diagram for this type system with the state-variable form. If derivatives of $\mathbf{u}(t)$ appear in the system equation, then the transfer function is given by

$$y(s)/u(s) = Kf_1(s)/f_2(s)$$

where $f_1(s)$ is now a polynominal in s. A selection of state variables as shown for the system of Eq. (8.177) would lead to derivative terms in Eqs. (8.178*a*) and (8.178*b*), thus violating its assumed form. In order to deal with the problem, another choice of state variables must be made. A convenient approach is to factor the transfer function into two parts, viz.,

$$H(s) = \frac{y(s)}{u(s)} = \frac{K_1 f_1(s)}{F_2(s)}$$

$$= \frac{x_1(s)}{u(s)} \frac{y(s)}{x_1(s)} = \frac{K_1}{f_2(s)} f_1(s)$$

And assign each of the factors as follows:

$$x_1(s)/u(s) = K_1/f_2(s) \qquad \text{and} \qquad y(s)/x_1(s) = f_1(s)$$

The transfer function between x_1 and u is now without numerator zero, and the choice of state variables is the same as before. The second transfer function is expanded from transfer-function to state-variable form, yielding

$$y(t) = c_m \frac{d^{m-1}x_1}{dt^{m-1}} + c_{m-1} \frac{d^{m-2}x_1}{dt^{m-2}} + \cdots + c_2 \frac{dx_1}{dt} + c_1 x_1 = c_1 x_1 + c_2 x_2 + \cdots + c_m x_m$$

where

$$f_1(s) = c_m s^{m-1} + c_{m-1} s^{m-2} + \cdots + c_2 s + c_1$$

$$f_2(s) = s^n + a_n s^{n-1} + \cdots + a_2 s + a_1 \qquad m \le n$$

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In phase-state form

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\cdots\cdots\cdots$$
$$\dot{x}_{n-1} = x_n$$

or $$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \qquad \mathbf{y} = \mathbf{C}^T\mathbf{x}$$

where $$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ K \end{bmatrix}$$

$$\mathbf{C}^T = [c_1 \quad c_2 \quad c_3 \quad \cdots \quad c_m \quad 0 \quad \cdots \quad 0]$$

8.6.2 Transfer Function from State-Space Representation

Given the state variable representation

$$\dot{\mathbf{x}}(t) = \mathbf{Ax}(t) + \mathbf{Bu}(t)$$
$$\mathbf{y}(t) = \mathbf{Cx}(t) + \mathbf{Du}(t)$$

and taking Laplace transforms of this set assuming zero initial conditions to determine the transfer function, we have

$$s\mathbf{x}(s) = \mathbf{Ax}(s) + \mathbf{Bu}(s)$$
$$\mathbf{y}(s) = \mathbf{Cx}(s) + \mathbf{Du}(s)$$

The solution for $\mathbf{y}(s)$ is

$$\mathbf{y}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D}]\mathbf{u}(s)$$

It is clear that the transfer function by its definitions is

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D}$$

For the single-input–single-output case, $\mathbf{H}(s)$ reduces to a scalar function, $H(s)$. The matrix $(s\mathbf{I} - \mathbf{A})^{-1}$ is referred to as the "resolvent matrix" and is designated as $\mathbf{\Phi}(s)$, i.e.,

$$\mathbf{\Phi}(s) = \text{adj}\,(s\mathbf{I} - \mathbf{A})/\det\,(s\mathbf{I} - \mathbf{A})$$

Using this notation the transfer function becomes

$$\mathbf{H}(s) = \mathbf{C\Phi}(s)\mathbf{B} + \mathbf{D}$$

8.6.3 Phase-State Variable-Form Transfer Function: Canonical (Normal) Form

One of the most important representations of the state space is in the decoupled or normal form.

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Given the transfer function

$$H(s) = \frac{y(s)}{u(s)} = K\frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{(s-\lambda_1)(s-\lambda_2)\cdots(-s-\lambda_n)} \qquad m \le n$$

for a siso system, where the denominator is shown in factored form; a partial fraction expansion yields.

$$\frac{y(s)}{u(s)} = \frac{c_1}{s-\lambda_1} + \frac{c_2}{s-\lambda_2} + \cdots + \frac{c_n}{s-\lambda_n}$$

From the definition

$$z_i(s) \triangleq \frac{u(s)}{s-\lambda_i} \qquad i = 1, 2, \ldots, n$$

$y(s)$ becomes

$$y(s) = c_1 z_1(s) + c_2 z_2(s) + \cdots + c_n z_n(s)$$

In the time domain, the foregoing two expressions transform to

$$\dot{z}_i(t) = \lambda_i Z_i(t) + u(t) \qquad i = 1, 2, 3, \ldots, n$$
$$y(t) = c_1 z_1(t) + c_2 z_2(t) + \cdots + c_n z_n(t)$$

In normal form, the system takes the form

$$\dot{\mathbf{z}}(t) = \mathbf{\Lambda z}(t) + \mathbf{bu}(t)$$
$$\mathrm{y}(t) = \mathbf{C}^T\mathbf{z}(t)$$

where

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_n \end{bmatrix}$$

where it is noted that $[\mathbf{\Lambda}]$ is in diagonal form, the matrix $\mathbf{b}$ elements are unity, and the matrix $\mathbf{C}$ elements are residues of the poles of $H(s)$.

As an example consider

$$y(s)/u(s) = H(s) = (s+1)/(s^2 + 5s + 6)$$

After factoring we have

$$y(s)/u(s) = (s+1)/(s+3)(s+2)$$

and the resulting partial-fraction expansion becomes

$$y(s)/u(s) = 2/(s+3) - 1/(s+2)$$

Choosing the Laplace transform of state variables and expanding, we find

$$z_1(s) = u(s)/(s+3) \qquad sz_1(s) = -3z_1(s) + u(s)$$
$$z_2(s) = u(s)/(s+2) \qquad sz_2(s) = -2z_2(s) + u(s)$$

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Then $$y(s) = 2z_1(s) - z_2(s)$$

and the time-domain transformation is

$$\dot{z}_1(t) = -3z_1(t) + u(t)$$

$$\dot{z}_2(t) = -2z_2(t) + u(t) \qquad \dot{\mathbf{z}}(t) = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{z}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = 2z_1(t) - z_2(t)$$

where the plant matrix

$$\begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}$$

is seen to be diagonal.

As an example of a multiple-input–multiple-output system consider a system with two inputs and two outputs having a transfer function

$$\mathbf{H}(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{2}{(s+1)(s+2)} \\ \dfrac{1}{(s+1)(s+3)} & \dfrac{1}{s+3} \end{bmatrix}$$

Expanding the elements into partial fractions, we rewrite $\mathbf{H}(s)$ as

$$\mathbf{H}(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{2}{s+1} - \dfrac{2}{s+2} \\ \dfrac{1}{2(s+1)} - \dfrac{1}{2(s+3)} & \dfrac{1}{s+3} \end{bmatrix}$$

and the output transform is

$$\begin{aligned} y_1(s) &= \frac{u_1(s)}{s+1} + \frac{2}{s+1}u_2(s) - \frac{2}{s+2}u_2(s) \\ y_2(s) &= \frac{1}{2(s+1)}u_1(s) - \frac{2}{2(s+3)}u_1(s) + \frac{1}{s+3}u_2(s) \end{aligned} \tag{8.179}$$

Allowing the definitions for x_i and their transformations to the time domain,

$$x_1(s) = \frac{u_1(s)}{s+1} \qquad \dot{x}_1(t) = -x_1(t) + u_1(t)$$

$$x_2(s) = \frac{u_2(s)}{s+1} \qquad \dot{x}_2(t) = -x_2(t) + u_2(t)$$

$$x_3(s) = \frac{u_2(s)}{s+2} \qquad \dot{x}_3(t) = -2x_3(t) + u_2(t)$$

$$x_4(s) = \frac{u_1(s)}{s+3} \qquad \dot{x}_4(t) = -3x_4(t) + u_1(t)$$

$$x_5(s) = \frac{u_2(s)}{s+3} \qquad \dot{x}_5 = -3x_5(t) + u_2(t)$$

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Now the output can be represented in both domains as

$$y_1(s) = x_1(s) + 2x_2(s) - 2x_3(s) \qquad y_1(t) = x_1(t) + 2x_2(t) - 2x_3(t)$$
$$y_2(s) = \tfrac{1}{2}x_1(s) - \tfrac{1}{2}x_4(s) + x_5(s) \qquad y_2(t) = \tfrac{1}{2}x_1(t) - \tfrac{1}{2}x_4(t) + x_5(t)$$

By direct substitution, the phase-state canonical form is

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$
$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix}$$
$$\mathbf{B} = [1 \quad 1 \quad 1 \quad 1 \quad 1]$$
$$\mathbf{C} = \begin{bmatrix} 1 & 2 & -2 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 1 \end{bmatrix}$$

8.6.4 Transformation to Normal Form

We begin with the phase-state representation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \tag{8.180}$$

which ignores direct coupling between input $\mathbf{U}(t)$ and output $\mathbf{y}(t)$. We next consider a matrix transformation [**P**]:

$$\mathbf{x}(t) = \mathbf{P}\mathbf{z}(t)$$

After substitution and premultiplication by $\mathbf{P}^{-1}$ in Eq. (8.180),

$$\dot{\mathbf{z}}(t) = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{z}(t) + \mathbf{P}^{-1}\mathbf{B}\mathbf{u}(\mathrm{t})$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{P}\mathbf{z}(t)$$

If **P** is chosen to render $\mathbf{z}(t)$ in normal form, then

$$\mathbf{P}^{-1}\,\mathbf{A}\mathbf{P} = \mathbf{\Lambda}$$

and from Eq. (8.107) **P** is determined following the solution to the eigenvector equation

$$|\mathbf{A} - \lambda_i\mathbf{I}| = 0$$
$$\mathbf{P} \triangleq \mathbf{c}_1 \quad \mathbf{c}_2 \quad \cdots \quad \mathbf{c}_n$$

where each $\mathbf{c}_i$ is an eigenvector corresponding to each eigenvalue λ_i. The eigenvector is given by any nonzero column of the matrix:

$$\text{adj}\,(\mathbf{A} - \lambda_i\mathbf{I})$$

Also the matrix form of the eigenvalue equation is

$$\mathbf{A}\mathbf{P} = \mathbf{P}\mathbf{\Lambda}$$

as previously shown in the development following Eq. (8.107).

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The special case of transforming **A** from companion form yields the **P** transformation:

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_1^n & \lambda_2^n & \cdots & \lambda_n^n \end{bmatrix}$$

8.6.5 System Response from State-Space Representation

The response to the homogeneous plant equation is

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{8.181}$$

which is akin to the solution of the first-order differential equation

$$\dot{x} = ax + bu(t)$$

whose general solution is, by Eq. (8.100)

$$x = e^{at}x(0) + b\int_0^t e^{a(b-\tau)}\, u(\tau)\, d\tau$$

Analogous to e^{at}, $e^{\mathbf{A}t}$ is defined by the infinite series

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{\mathbf{A}^n t^n}{n!}$$

which converges absolutely for $t < \infty$ and uniformly in any finite interval. Then the homogeneous equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \tag{8.182}$$

has the solution

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) \tag{8.183}$$

$e^{\mathbf{A}t}$ has the property in this case to transform the state of **x** at $t = 0$ to the state at any future time $\mathbf{x}(t)$. Accordingly, it is denoted as the state-transition matrix $\mathbf{\Phi}(t)$. Under this designation Eq. (8.183) becomes

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) \tag{8.183a}$$

If the state is known at $t = t_0$ [i.e., $\mathbf{x} = \mathbf{x}(t_0)$], then at time t

$$\mathbf{x}(t) = \mathbf{\Phi}(t - t_0)\mathbf{x}(t_0)$$

which is essentially the same as the statement of Eq. (8.183) and is the complete solution to Eq. (8.182).

A Laplace transformation of Eq. (8.182) yields

$$s\mathbf{x}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{x}(s)$$

After rearranging the premultiplying by $(s\mathbf{I} - \mathbf{A})^{-1}$ we have

$$s\mathbf{x}(s) + \mathbf{A}\mathbf{x}(s) = \mathbf{x}(0)$$

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$$(s\mathbf{I} + \mathbf{A})\mathbf{x}(s) = \mathbf{x}(0)$$

$$\mathbf{x}(s) = (s\mathbf{I} + \mathbf{A})^{-1}\,\mathbf{x}(0) \tag{8.184}$$

where a Laplace transformation of Eq. (8.183*a*) establishes the identity

$$\mathcal{L}[\mathbf{\Phi}(t)] = \mathbf{\Phi}(s) = (sI + \mathbf{A})^{-1}$$

The total solution of the plant equation, Eq. (8.181), is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

$$= \mathbf{\Phi}(t)\,\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

a result previously obtained in Eq. (8.102).

8.6.6 State-Transition Matrix for Sampled Data Systems

The response to the system equation, Eq. (8.181), where $\mathbf{u}(t)$ is governed by a sampling process, is

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}_s(\tau)\,d\tau \tag{8.185}$$

Since $\mathbf{u}_s(t) = \mathbf{u}(nT)$ = the sampled input for $nT \le t \le (n+1)T$, where $n = 0, 1, 2, \ldots$ = sample number and T = sampling interval, Eq. (8.185) becomes

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \sum_{j=0}^{n-1}\left[\int_{jT}^{(j+1)T} \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(jT)\,d\tau\right]$$

$$+ \int_{nt}^{t} \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(nT)\,d\tau \tag{8.186}$$

and since $\mathbf{\Phi}(t-\tau) = \mathbf{\Phi}(t)\,\mathbf{\Phi}(-\tau)$, Eq. (8.186) can also be written as

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \mathbf{\Phi}(t)\left[\sum_{j=0}^{n-1}\int_{jT}^{(j+1)T} \mathbf{\Phi}(-\tau)\mathbf{B}\mathbf{u}(jT)\,d\tau\right] + \int_{nt}^{t} \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(nT)\,d\tau \tag{8.186a}$$

For notational convenience we define

$$\boldsymbol{\gamma} \triangleq \sum_{j=0}^{n-1}\left[\int_{jT}^{(j+1)} \mathbf{\Phi}(-\tau)\mathbf{B}\mathbf{u}(jT)\,d\tau\right]$$

Then Eq. (8.186*a*) becomes

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \mathbf{\Phi}(t)\boldsymbol{\gamma} + \int_{nt}^{t} \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(nT)\,d\tau \tag{8.187}$$

If interest is focused on state values at sampling instants only (i.e., $t = nT$), then Eq. (8.185)

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becomes

$$\mathbf{x}(nT) = \mathbf{\Phi}(nT)\mathbf{x}(0) + \mathbf{\Phi}(nT)\boldsymbol{\gamma} \tag{8.188}$$

at $t = (n + 1)T$

$$\mathbf{x}[(n + 1)T] = \mathbf{\Phi}[(n + 1)T]\mathbf{x}(0) + \mathbf{\Phi}[(n + 1)T]\boldsymbol{\gamma} + \int_{nT}^{(n+1)T} \mathbf{\Phi}[(n + 1)T - \tau]\mathbf{B}\mathbf{u}(nT)\, d\tau$$

Factoring $\mathbf{\Phi}(T)$, we have

$$\mathbf{x}[(n + 1)T] = \mathbf{\Phi}(T)[\mathbf{\Phi}(nt)\mathbf{x}(0) + \mathbf{\Phi}(nT)\gamma] + \int_{nT}^{(n+1)T} \mathbf{\Phi}[(n + 1)T - \tau]\mathbf{B}\mathbf{u}(nT)\, d\tau$$

and replacing $\mathbf{\Phi}(nt)\mathbf{x}(0) + \mathbf{\Phi}(nt)\gamma$ with its equivalent in Eq. (8.188), $\mathbf{x}(nt)$, yields

$$\mathbf{x}[(n + 1)T] = \mathbf{\Phi}(T)\mathbf{x}(nT) + \int_{nT}^{(n+1)T} \mathbf{\Phi}[(n + 1)T - \tau]\mathbf{B}\mathbf{u}(nT)\, d\tau$$

The integral term can be simplified by a change of variable:

$$\gamma \triangleq (n + 1)T - \tau$$

resulting in

$$\mathbf{x}[(n + 1)T] = \mathbf{\Phi}(T)\mathbf{x}(nT) + \left[\int_0^T \mathbf{\Phi}(\lambda)\mathbf{B}\, d\lambda\right]\mathbf{u}(nT) \tag{8.188a}$$

where $\mathbf{u}(nT)$, being a constant, is moved outside the integral. The integral term is recognized as a constant matrix $\mathbf{F}$, defined as

$$\mathbf{F} \triangleq \int_0^T \mathbf{\Phi}(\lambda)\mathbf{B}\, d\lambda$$

and therefore Eq. (8.188*a*) becomes

$$\mathbf{x}[(n + 1)T] = \mathbf{\Phi}(T)\mathbf{x}(nT) + \mathbf{F}\mathbf{u}(nT)$$

8.6.7 Time-Varying Linear Systems

The plant equation with time-dependent coefficients is written as

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\,\mathbf{x}(t) + \mathbf{B}(t)\,\mathbf{u}(t) \tag{8.189}$$

Recalling Eq. (8.100*a*), the homogeneous matrix differential equation, we have

$$d\mathbf{Z}(t)/dt = \mathbf{A}(t)\mathbf{Z}(t) \tag{8.190}$$

with $\mathbf{Z}(t_0) = \mathbf{I}$ (the identity matrix).

The solution to the homogenous equation, Eq. (8.99) or Eq. (8.181), is

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\,\mathbf{x}(t) \qquad \mathbf{x}(t) = \mathbf{Z}(t)\,\mathbf{x}(0) \tag{8.191}$$

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and the general solution to Eq. (8.189) is given by

$$\mathbf{x}(t) = \mathbf{Z}(t)\mathbf{x}(t_0) + \int_0^t \mathbf{Z}(t)\mathbf{Z}^{-1}(\tau)\,\mathbf{B}(\tau)\mathbf{u}(\tau)\,\mathbf{d}\tau \tag{8.192}$$

Finally, using the notation for the transition matrix, we see that

$$\mathbf{\Phi}(t, t_1) = \mathbf{Z}(t)\,\mathbf{Z}^{-1}(t_1)$$

and Eq. (8.192) becomes

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}(t_0) + \int_0^t \mathbf{\Phi}(t, \tau)\,\mathbf{B}(\tau)\mathbf{u}(\tau)\,\mathbf{d}\tau \tag{8.193}$$

after $\mathbf{x}(t_0)$ is replaced in Eq. (8.192) with its identity $\mathbf{Z}^{-1}(t_0)\mathbf{x}(t_0)$, where $\mathbf{Z}(t_0) = \mathbf{Z}^{-1}(t_0) = \mathbf{I}$.

Unfortunately there is no general method to find $\mathbf{\Phi}(t, t_0)$ despite the mathematical compactness of form and similarity with the time-invariant case.

8.7 CONTROL THEORY[40,41,43–58,60,61]

8.7.1 Controllability

In order to control a plant, an input vector $\mathbf{u}(t)$ or sequence $u_i(t)$ must be determined to drive the system from its initial state $\mathbf{x}_0$ to a prescribed final state $\mathbf{x}_f$. If $\mathbf{u}(t)$ can thus be found, then the system is deemed "controllable."

The following is a more general definition of controllability: In a region $\mathcal{R}$ of state space, if the state of a system can be transformed from arbitrary state $\mathbf{x}_0$ at time t_0 to another arbitrary state $\mathbf{x}_f$ in finite time, then the system is completely controllable in that region. Clearly, systems in which elements of the state cannot be independently influenced by the input sector $\mathbf{u}$ are not controllable.

8.7.2 Observability

A closely allied concept is that if the system output is known over some finite time $\mathbf{y}(t_0, t)$, one can determine the state of the system $\mathbf{x}(t_0)$. If so, then the system is called "observable." Stated more precisely, a system is observable if its state at $t = t_0$ can be uniquely determined by observing its output over a finite interval of time $t_0 < t < \infty$.

The mathematical statements governing controllability and observability of the familiar linear time-invariant system, Eq. (8.176), are as follows. The system is controllable if the $n \times rn$ matrix formed by

$$\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \quad \cdots \quad \mathbf{A}^{n-1}\mathbf{B}$$

has rank n, where $\mathbf{A} = n \times n$ matrix and $\mathbf{B} = r \times n$ matrix. The system is observable if the $n \times mn$ matrix

$$\overline{\mathbf{C}}^T \qquad \overline{\mathbf{A}}^T\overline{\mathbf{C}}^T \qquad (\overline{A}^2)^T\overline{\mathbf{C}}^T \cdots (\overline{A}^{n-1})^T\overline{\mathbf{C}} \tag{8.194}$$

has rank n, where $\mathbf{C} = n \times m$ matrix.

8.7.3 Introduction to Optimal Control

The primary objective of optimal control is to choose a function (policy) of one or more parameters such that some meaningful function of these parameters is rendered either a maximum or a minimum (i.e., an extremum).

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An elementary example in rocket control would be to choose a thrust attitude and a fuel firing rate that will send a satellite into a prescribed orbit with minimum fuel expenditure; another control policy would allow the achievement of orbit in minimum time. The simplified equations governing rocket motion are given by

$$m\,d\mathbf{v}/dt = \mathbf{f} + \mathbf{f}_{\text{ext}}$$
$$\mathbf{f} = (dm/dt)\mathbf{c} = -\beta\mathbf{c} \tag{8.195}$$

where m = rocket mass
$\beta = -dm/dt$ = fuel burning rate
$\mathbf{c} = c_x, c_y$ = exhaust vector
$\mathbf{v} = v_x, v_y$ = velocity vector
$\mathbf{f}_{\text{ext}}$ = external forces (assumed for simplicity to be $(a - mg)$

Figure 8.44 indicates the rocket trajectory. The equations in each coordinate are

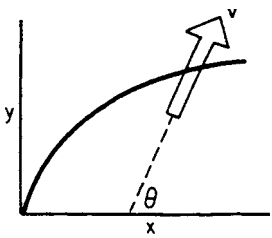

FIG. 8.44 Rocket trajectory.

$$m\,dv_x/dt = c_x\,dm/dt$$
$$m\,dv_y/dt = c_y\,dm/dt - mg$$

where θ = instantaneous rocket attitude angle. The problem here is to choose a firing rate β such that the fuel consumed is a minimum, i.e.,

$$\int_0^T \beta\,dt \text{ min}$$

where T = time for rocket to achieve its final height y.

Many cases of optimal control involve finding extremum values of integrals subject to constraints. The integrals are usually of the type

$$J = \int_A^B f\left(y, \frac{dy}{dx}, x\right) dx \tag{8.196a}$$
$$= \int_{t_0}^{t_1} f\left(x, \frac{dx}{dt}, t\right) dt \tag{8.196b}$$

Of all possible paths $y(x)$ in Eq. (8.196*a*) or $x(t)$ in Eq. (8.196*b*), some (usually one) make J (functional) an extremum.

If we consider the optimal control of a system whose plant equation is

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$$

then, in general, the control function $\mathbf{u}$ is chosen (assuming controllability) to transform the state from $\mathbf{x}(t_0)$ to a prescribed final state $\mathbf{x}_f = \mathbf{x}(T)$ such that the functional

$$J = \int_0^T f_0(\mathbf{x}, \mathbf{u}, t)\,dt$$

is rendered an extremum.

8.7.4 Euler-Lagrange Equation

The extremum of J for fixed end points a and b,

$$J = \int_a^b f(y, y', x)\,dx$$

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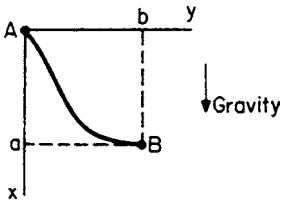

FIG. 8.45 Problem of mass from fixed end *A* to fixed end *B*.

corresponds to that path $y(x)$ which conforms to

$$\partial f/\partial y - (d/dx)/(\partial f'/\partial y') = \phi$$

which is the Euler-Lagrange equation taken from the "calculus of variations," a branch of mathematics.

One of the earliest problems posed by Bernoulli was to determine a curve joining two fixed end points *A* and *B* in a vertical plane (Fig. 8.45), on which a mass starting from rest at *A* sliding along that path under the force of gravity alone with reach *B* in minimum time. Since the velocity of a conservative system is

$$|\mathbf{v}| = (2gx)^{1/2}$$

the transit time is given by

$$J = \int_A^B \frac{ds}{(2gx)^{1/2}} = \int_A^B \left[\frac{(dy/dx)^2 + 1}{2gx}\right]^{1/2} dx$$

Moreover, the integrand does not contain terms in y, so the Euler-Lagrange equation reduces to $\partial f/\partial y' = c$, where c is a constant, or

$$y'/(1 + y'^2)^{1/2} = c$$

The solution in parametric form is

$$x = (1/2c^2)(1 - \cos\theta)$$
$$y = (1/2c^2)(\theta - \sin\theta)$$

which is a cycloid through points a and b, with c chosen to satisfy the end points.

Free-End Conditions. In the absence of end constraints, i.e., for free-end conditions, the extremizing problem is to find $y(x)$ when $a \le x \le b$, without any constraints on $y(a)$ and $y(b)$. The results are

$$\partial F/\partial y - (d/dx)(\partial F/\partial y') = 0$$

and

$$\partial F/\partial y' = 0 \qquad \text{at } x = a, b$$

Variable-End-Point Condition. The more general case of variability of one or both end points of the functional integral,

$$J = \int_{t_0}^{t_1} F(\mathbf{x}, \dot{\mathbf{x}}, t)\, dt$$

leads to the following generalized boundary condition at $t = t_1$:

$$\sum \frac{\partial F}{\partial \dot{x}_i}\delta x_i\bigg|_{t_1} + \left[F - \sum \dot{x}_i \frac{\partial F}{\partial \dot{x}_i}\right]\bigg|_{t_1} \partial t_1$$

Similarly at $t = t_0$

$$\sum \frac{\partial F}{\partial \dot{x}_i}\delta x_i\bigg|_{t_0} + \left[F - \sum \dot{x}_i \frac{\partial F}{\partial \dot{x}_i}\right]\bigg|_{t_0}^{\delta t_0}$$

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If the end point is free, the δt_0 and $\delta x_{i|t_0}$ are independent; then each of the coefficients of δx_i and δt_0 must be zero, or

$$\left.\frac{\partial F}{\partial X_i}\right|_{t_0} = 0$$

$$\left[F - \sum \dot{x}_i \frac{\partial F}{\partial \dot{x}_i}\right]\Bigg|_{t_0} = 0$$

If x_j, for example, is constrained to some curve at the right end point, then

$$x_j\Big|_{t_1} = y_j(t) \qquad j = 1, 2, \ldots, n$$

and

$$\delta X\Big|_{t_1} = \dot{y}_j \delta t_1 \qquad j = 1, 2, \ldots, n$$

$$F + \sum_{i=1}^{n} (\dot{y}_i - \dot{x}_i) \frac{\partial F}{\partial x} = 0$$

8.7.5 Multivariable with Constraints and Independent Variable *t*

Consider the functional

$$J = \int_a^b F(x_1, x_2, \ldots, x_n, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n, t)\, dt \tag{8.197}$$

where $x_1, x_2, \ldots, x_n$ are independent functions of t and

$$\dot{x}_i = dx_i/dt$$

The extremal path now conforms to n Euler equations

$$\partial F/\partial x_i - (d/dt)(\partial F/\partial \dot{x}_i) = 0 \qquad i = 1, 2, \ldots, n$$

At any end point where x_i is free,

$$\partial F/\partial \dot{x}_i = 0$$

Now if one or more of the variables x_i are functionally related, say in m independent equations

$$g_i(\dot{\mathbf{x}}, \mathbf{x}, t) = 0 \qquad i = 1, 2, \ldots, m \tag{8.198}$$

we first form an augmented function

$$\hat{F} = F + \sum_{i=1}^{n} p_i g_i \tag{8.199}$$

where p_i is a Lagrange multiplier for the function g_i.

The augmented functional is next defined by

$$\hat{J} = \int_a^b \hat{F}\, dt \tag{8.200}$$

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The phase form of Eq. (8.197) is

$$J = \int_a^b F(\mathbf{x}, \dot{\mathbf{x}}, t)\, dt \tag{8.201}$$

which is subject to the constraint equations

$$\mathbf{g}(\mathbf{x}, \dot{\mathbf{x}}, t) = 0$$

By substitution for F in Eq. (8.200), the augmented functional becomes

$$\hat{J} = \int_a^b [F(\mathbf{x}, \dot{\mathbf{x}}, t)\, dt - \mathbf{P}\mathbf{g}(\mathbf{x}, \dot{\mathbf{x}}, t)]\, dt$$

where

$$\mathbf{P} = \begin{bmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & p_m \end{bmatrix}$$

The extremum for $\hat{J}$ is found by using the Euler-Lagrange equation and treating p_i, $i = 1, \ldots, m$, as m additional variables.

More generally, consider the functional J written as

$$J = \int_0^t F(\mathbf{x}, \mathbf{u}, t)\, dt$$

which is subject to differential constraints

$$\dot{x}_i = f_i(\mathbf{x}, \mathbf{u}, t) \qquad i = 1, 2, \ldots, n \tag{8.202}$$

Rearranging Eq. (8.202) results in

$$g_i = f_i - \dot{x}_i = 0 \qquad i = 1, 2, \ldots, m$$

Now replacing $f_i - \dot{x}_i$ for g_i in Eq. (8.199),

$$\hat{F} = F + \sum p_i\, (f_i - \dot{x}_i) \tag{8.203}$$

and

$$\hat{J} = \int_0^T \hat{F}\, dt = \int_0^T F + \sum p_i\, (f_i - \dot{x}_i)\, dt \tag{8.204}$$

For convenience the hamiltonian H is introduced, defined by

$$H \triangleq F + \sum p_i f_i$$

with this definition, Eq. (8.204) can be rewritten as

$$\hat{J} = \int_0^T H - \sum p_i \dot{x}_i\, dt$$

Invoking the Euler-Lagrange equation, we have

$$\partial \hat{F}/\partial x_i - (d/dt)(\partial \hat{F}/\partial \dot{x}_i) = 0 \qquad i = 1, 2 \ldots, n$$

or

$$\dot{p}_i = -\partial H/\partial x_i$$

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and $$\partial H/\partial u_j = 0 \qquad j = 1, 2, \ldots, n$$

$$\partial H/\partial p_i = \dot{x}_i \qquad i = 1, 2, \ldots, n$$

yielding $2n + m$ equations in $2n + m$ unknowns.

The boundary conditions for $x_i(0)$, $i = 1, \ldots, n$ are the state at $t = 0$. If the control problem requires that, say, only r final state values be met, then values of $x_l(T)$, $l > r$, are free, yielding the free-end-point conditions

$$\frac{\partial F}{\partial \dot{x}_k} = 0 \qquad \begin{matrix} k = r + 1, \ldots, n \\ t = T \end{matrix}$$

which coincides with

$$p_k(T) = 0 \qquad k = r + 1, \ldots, n$$

As an example, consider the control which minimizes $\int_0^T u^2\, dt$ for the system given by

$$\begin{aligned} \dot{x}_1 &= x_2 \qquad & x_1(0) &= 0, x_1(T) = 1 \\ \dot{x}_2 &= u \qquad & x_2(0) &= 0, x_2(T) = 0 \end{aligned}$$

which describes a control force u accelerating a unit inertial mass starting at rest moving to a unit position at zero velocity at $t = T$.

We first form the hamiltonian

$$H = u^2 + p_1 x_2 + p_2 u$$

Now for $\partial H/\partial u_j = 0$

$$2u + p_2 = 0 \quad \text{or} \quad u = -\tfrac{1}{2} p_2$$

and for $-\partial H/\partial x_j = \dot{p}_j$

$$\dot{p}_1 = 0 \quad \text{and} \quad \dot{p}_2 = -p_1$$

From the foregoing three equations we can establish that

$$p_1 = \text{const} = -C \qquad \dot{p}_2 = \text{const} = -p_1 = C \tag{8.205}$$

A first integral of Eq. (8.205) yields

$$p_2 = Ct + D$$

Since $u = -\frac{1}{2} p_2$,

$$u = -\tfrac{1}{2} Ct - \tfrac{1}{2} D$$

Since $\dot{x}_2 = u$,

$$x_2 = -\tfrac{1}{4} Ct^2 - \tfrac{1}{2} Dt + E$$

From the boundaries $x_2(T) = 0$ and $x_2(0) = 0$,

$$x_2 = \tfrac{1}{4} C(-t^2 + Tt)$$

$$D = -\tfrac{1}{2} CT$$

$$E = 0$$

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and $\dot{x}_1 = x_2 \qquad x_1(0) = 0$

$$x_1 = \tfrac{1}{4}C(-\tfrac{1}{3}t^3 + \tfrac{1}{2}Tt^2)$$

since $x_1(T) = 1$

The required u is

$$u = (12/T^3)(-t + \tfrac{1}{2}T)$$

and elements of the state vector are

$$x_2 = (6/T^3)(-t^2 + Tt) \qquad x_1 = (6/T^3)(-\tfrac{1}{3}t^3 + \tfrac{1}{2}Tt^2)$$

8.7.6 Pontryagin's Principle

Again we consider

$$J = \int_0^t F(\mathbf{x}, \mathbf{u}, t)\, dt$$

Subject to

$$\dot{x}_i = f_i(\mathbf{x}, \mathbf{u}, t) \qquad i = 1, 2, \ldots, n$$

with initial conditions

$$\mathbf{x}(0) = \mathbf{x}_0$$

and end conditions

$$\mathbf{x}(T) = \mathbf{x}_f$$

where $\mathbf{u} \in U$ is the admissible control region.

Now if $\mathbf{u}$ is discontinuous (the general case), $\partial H/\partial u_j$ does not exist at the discontinuities and $\partial H/\partial u_j = 0$, one of the criteria for optimal control of ∂u_j continuous inputs, is no longer valid. In lieu of this condition, u_j, $j = 1, \ldots, n$, are determined to minimize H. This is the fundamental contribution of Pontryagin; the theory admits not only continuous control but bounded and discontinuous controls as well.

As an example consider the system of the previous problem with the functional

$$J = \int_0^T dt$$

being the time it takes to move the mass from $x_1(0) = 0$ to $x_1(T) = 1$ in minimum time subject to the bounds on u:

$$-\alpha < u < \beta$$

We first form the hamiltonian

$$H = 1 + p_1 x_2 + p_2 u$$

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and as determined previously,

$$\dot{p}_1 = 0$$
$$-p_1 = \dot{p}_2$$
$$C = \dot{p}_2$$
$$p_2 = Ct + D$$

In accordance with the Pontryagin principle, if

$$p_2 < 0 \qquad u = +\beta$$

and if

$$p_2 > 0 \qquad u = -\alpha$$

Thus $p_2 u$ and H are made as small as possible.

Since

$$\dot{x}_2 = u \tag{8.206}$$

$$x_2(T) = \int_0^T u\, dt = 0$$

It appears from Eq. (8.206) that u has at least one positive region and at least one negative region since its total integration over the interval 0, T, is 0. Further, for every sign change of u there must be a sign change of p_2. Now, because it is a linear function of time, p_2 can have at most one change of sign. It follows that there is a single crossover. This fact, considered in relation to Eq. (8.206), leads to a crossover time t_1 derived from

$$\beta t_1 = (T - t_1)\alpha \tag{8.207}$$

Equating the integral from 0 to T with $x_1(T)$ yields

$$x_1(T) = \int_0^T x_2\, dt = 1 \tag{8.208}$$

There are two regimes for x_2, namely,

$$x_2 = \begin{cases} \displaystyle\int_0^t \beta\, dt = \beta t & t \le t_1 \\ \displaystyle\int_{t_1}^t -\alpha\, dt = \beta t_1 - \alpha(t - t_1) & t \ge t_1 \end{cases}$$

Direct substitution in Eq. (8.208) over the whole range 0, T yields

$$\int_0^T x_2 dt = \beta t_1^2/2 + \beta^2 t_1^2/2\alpha = 1 \tag{8.209}$$

Figure 8.46 shows the x_2 and u functions.

From Eqs. (8.207) and (8.209) we obtain the crossover time

$$t_1 = \left[\frac{2}{\beta(1 + \beta/\alpha)}\right]^{1/2}$$

and the excursion time

$$T = [(2/\beta)(1 + \beta/\alpha)]^{1/2}$$

which is a minimum.

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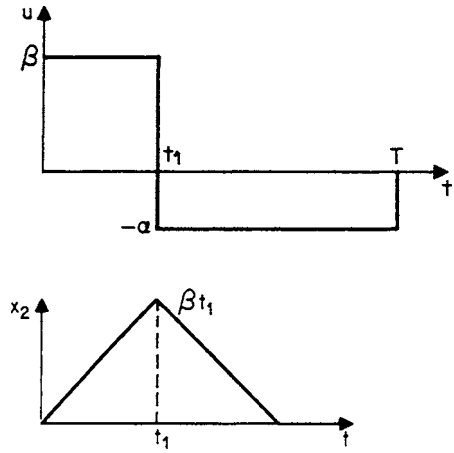

FIG. 8.46 Relationship between functions x_2 and u.

REFERENCES

1. Miller, F. H.: "Partial Differential Equations," John Wiley & Sons, Inc., New York, 1941.
2. Courant, R., and D. Hilberg: "Methods of Mathematical Physics," vol. I, Interscience Publishers, Inc., New York, 1953.
3. Frank, P. L., and R. von Mises: "Differential and Integral Gleichungen der Mechanik und Physik," Friedr. Vieweg & Sohn, Germany, 1935.
4. Sagan, H.: "Boundary and Eigenvalue Problems in Mathematical Physics," John Wiley & Sons, Inc., New York, 1961.
5. Webster, A. G.: "Partial Differential Equations," S. B. Teubner, Leipzig, 1927.
6. Lamb, H.: "Hydrodynamics," Cambridge University Press, New York, 1932.
7. Struble, R. A.: "Nonlinear Differential Equations," McGraw-Hill Book Company, Inc., New York, 1962.
8. Cunningham, W. J.: "Introduction to Nonlinear Analysis," McGraw-Hill Book Company, Inc., New York, 1958.
9. Stoker, J. J.: "Nonlinear Vibrations," Interscience Publishers, Inc., New York, 1950.
10. Truxal, J. G.: "Automatic Feedback Control System Synthesis," McGraw-Hill Book Company, Inc., New York, 1955.
11. Coddington, E. A., and N. Levinson: "Theory of Ordinary Differential Equations," McGraw-Hill Book Company, Inc., New York, 1955.
12. Kumke, E.: "Differential Gleichungen reeler Funktionen," Akademische Verlagsgesellschaft Geest & Portig KG, Leipzig, 1930.
13. Lepschetz, S.: "Lectures on Differential Equations," Princeton University Press, Princeton, N.J., 1946.
14. Kaplan, W.: "Ordinary Differential Equations," Addison-Wesley Publishing Company, Inc., Reading, Mass., 1958.
15. Bellman, R. E.: "Stability Theory of Differential Equations," McGraw-Hill Book Company, Inc., New York, 1953.
16. Papoulis, A.: "The Fourier Integral and Its Applications to Linear Systems," McGraw-Hill Book Company, Inc., New York, 1962.
17. Carslaw, H. S.: "Introduction to the Theory of Fourier Series and Integrals," 3d ed., Dover Publications, Inc., New York, 1930.

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18. Churchill, R. V.: "Operational Mathematics," 2d ed., McGraw-Hill Book Company, Inc., New York, 1944.
19. Weber, E.: "Linear Transient Analysis," vol. II, John Wiley & Sons, Inc., New York, 1956.
20. Gardner, M. F., and J. S. Barnes: "Transients in Linear Systems," vol. I, John Wiley & Sons, Inc., New York, 1942.
21. Goldman, S.: "Transformation Calculus and Electrical Transients," Prentice-Hall, Inc., Englewood Cliffs, N.J., 1949.
22. Cheng, D. L.: "Analysis of Linear Systems," Addison-Wesley Publishing Company, Inc., Reading, Mass., 1959.
23. Pipes, L. A.: "Applied Mathematics for Engineers and Physicists," 2d ed., McGraw-Hill Book Company, Inc., New York, 1958.
24. Pfeiffer, P. E.: "Linear Systems Analysis," McGraw-Hill Book Company, Inc., New York, 1961.
25. Scott, E. J.: "Transform Calculus with an Introduction to Complex Variables," Harper & Row, Publishers, Incorporated, New York, 1955.
26. Evans, W. R.: "Control-system Dynamics," McGraw-Hill Book Company, Inc., New York, 1954.
27. Solodovnickoff, V. V.: "Introduction to the Statistical Dynamics of Automatic Control Systems," Dover Publications, Inc., New York, 1960.
28. Tsien, H. S.: "Engineering Cybernetics," McGraw-Hill Book Company, Inc., New York, 1954.
29. Seifert, W. W., and C. W. Steeg: "Control Systems Engineering," McGraw-Hill Book Company, Inc., New York, 1960.
30. Smith, O. J. M.: "Feedback Control Systems," McGraw-Hill Book Company, Inc., New York, 1958.
31. Chang, S. S.: "Synthesis of Optimum Control Systems," McGraw-Hill Book Company, Inc., New York, 1961.
32. Kipiniak, W.: "Dynamic Optimization Use Control," John Wiley & Sons, Inc., New York, 1961.
33. Gibson, J. E. (ed.): "Proceedings of Dynamic Programming Workshop," Purdue University, 1961.
34. Ragazzini, J. R., and Gene F. Franklin: "Sampled-data Control Systems," McGraw-Hill Book Company, Inc., New York, 1958.
35. Jury, E. J.: "Sampled Data Control Systems," John Wiley & Sons, Inc., New York, 1958.
36. McRuer, Graham D.: "Analysis of Nonlinear Control Systems," John Wiley & Sons, Inc., New York, 1961.
37. Cosgriff, R. L.: "Nonlinear Control Systems," McGraw-Hill Book Company, Inc., New York, 1958.
38. Ku, Y. H.: "Analysis and Control of Nonlinear Systems," The Ronald Press Company, New York, 1958.
39. Setov, J.: "Stability in Nonlinear Control Systems," Princeton University Press, Princeton, N.J., 1961.
40. Anand, D. K.: "Introduction to Control Systems," Pergamon Press, New York, 1974.
41. Berkowitz, L. D.: "Optimal Control Theory," Springer-Verlag, Berlin, 1974.
42. Blackman, P. F.: "State Variable Analysis," the Macmillan Company, New York, 1977.
43. Brewer, J. W., "Control System Analyses, Design and Simulation," Prentice-Hall, Inc., Englewood Cliffs, N.J., 1974.
44. Brogan, W. L.: "Modern Control Theory," Quantum Publishers, New York, 1974.
45. Bergess, T., and H. Graham: "Introduction to Control Theory Including Optimal Control," John Wiley & Sons, Inc., New York, 1980.
46. Clark, R. N.: "Introduction to Automatic Control Systems," John Wiley & Sons, Inc., New York, 1966.

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47. Fortmann, T. E., and Hitz, K. L.: "An Introduction to Linear Control Systems," Marcel Dekker, New York, 1977.
48. Jacobs, O. L.: "Introduction to Control Theory," Clarendon Press, New York, 1974.
49. Kirk, D. E.: "Optimal Control Theory," Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1970.
50. Layton, J. M.: "Multivariable Control Theory," Peter Perengrenus, New York, 1976.
51. Marshall, S. A.: "Introduction to Control Theory," The Macmillan Company, New York, 1978.
52. Schultz, D. G., and J. L. Melsa: "State Functions and Linear Control Systems," McGraw-Hill Book Company, Inc., New York, 1967.
53. Ogata, K.: "Modern Control Engineering," Prentice-Hill, Inc., Englewood Cliffs, N.J. 1970.
54. Owens, D. H.: "Feedback and Multivariable Systems," Peter Perengrenus, New York, 1978.
55. Pontryagin, L. S., et al.: "The Mathematical Theory of Optimal Processes," John Wiley & Sons, Inc., New York, 1982.
56. Power, H. M. and R. J. Simpson: "Introduction to Dynamics and Control," McGraw-Hill Book Company, Inc., New York, 1978.
57. Prime, H. A., "Modern Concepts in Control Theory," McGraw-Hill Book Company., Inc., New York, 1967.
58. Sage, A. P.: "Optimum System Control," Prentice-Hall, Inc., Englewood Cliffs, N.J., 1968.
59. Timothy, L. K. and B. E. Bona: "State Space Analysis," McGraw-Hill Book Company, Inc., New York, 1968.
60. Truxal, J. G.: "Introductory Systems Engineering," McGraw-Hill Book Company, Inc., New York, 1972.
61. Uolovich, W. A.: "Linear Multivariable Systems," Springer-Verlag, Berlin, 1974.
62. Zadeh, L. A., and C. A. Desoer: "Linear System Theory: The Static Space Approach," McGraw-Hill Book Company, Inc., New York, 1963.

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