5 Analysis of stress and strain

5.1 Introduction

Up to the present we have confined our attention to considerations of simple direct and shearing stresses. But in most practical problems we have to deal with combinations of these stresses.

The strengths and elastic properties of materials are determined usually by simple tensile and compressive tests. How are we to make use of the results of such tests when we know that stress in a given practical problem is compounded from a tensile stress in one direction, a compressive stress in some other direction, and a shearing stress in a third direction? Clearly we cannot make tests of a material under all possible combinations of stress to determine its strength. It is essential, in fact, to study stresses and strains in more general terms; the analysis which follows should be regarded as having a direct and important bearing on practical strength problems, and is not merely a display of mathematical ingenuity.

5.2 Shearing stresses in a tensile test specimen

A long uniform bar, Figure 5.1, has a rectangular cross-section of area A. The edges of the bar are parallel to perpendicular axes Ox, Oy, Oz. The bar is uniformly stressed in tension in the x-direction, the tensile stress on a cross-section of the bar parallel to Ox being σ_x. Consider the stresses acting on an inclined cross-section of the bar; an inclined plane is taken at an angle θ to the yz-plane. The resultant force at the end cross-section of the bar is acting parallel to Ox.

$$P = A\sigma_x$$

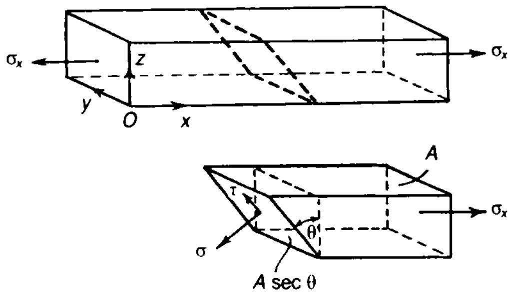


Figure 5.1 Stresses on an inclined plane through a tensile test piece.

For equilibrium the resultant force parallel to Ox on an inclined cross-section is also $P = A\sigma_x$. At the inclined cross-section in Figure 5.1, resolve the force $A\sigma_x$ into two components—one perpendicular, and the other tangential, to the inclined cross-section, the latter component acting parallel to the *xz*-plane. These two components have values, respectively, of

$$A\sigma_x \cos\theta \text{ and } A\sigma_x \sin\theta$$

The area of the inclined cross-section is

$$A \sec\theta$$

so that the normal and tangential stresses acting on the inclined cross-section are

$$\sigma = \frac{A\sigma_x \cos\theta}{A \sec\theta} = \sigma_x \cos^2\theta \qquad (5.1)$$

$$\tau = \frac{A\sigma_x \sin\theta}{A \sec\theta} = \sigma_x \cos\theta \sin\theta \qquad (5.2)$$

σ is the *direct stress* and τ the *shearing stress* on the inclined plane. It should be noted that the stresses on an inclined plane are not simply the resolutions of σ_x perpendicular and tangential to that plane; the important point in Figure 5.1 is that the area of an inclined cross-section of the bar is different from that of a normal cross-section. The shearing stress τ may be written in the form

$$\tau = \sigma_x \cos\theta \sin\theta = \tfrac{1}{2}\sigma_x \sin 2\theta$$

At $\theta = 0°$ the cross-section is perpendicular to the axis of the bar, and $\tau = 0$; τ increases as θ increases until it attains a maximum of ½ σ_x at $\theta = 45°$; τ then diminishes as θ increases further until it is again zero at $\theta = 90°$. Thus on any inclined cross-section of a tensile test-piece, shearing stresses are always present; the shearing stresses are greatest on planes at 45° to the longitudinal axis of the bar.

Problem 5.1 A bar of cross-section 2.25 cm by 2.25 cm is subjected to an axial pull of 20 kN. Calculate the normal stress and shearing stress on a plane the normal to which makes an angle of 60° with the axis of the bar, the plane being perpendicular to one face of the bar.

Solution

We have $\theta = 60°$, $P = 20$ kN and $A = 0.507 \times 10^{-3}$ m². Then

$$\sigma_x = \frac{20 \times 10^3}{0.507 \times 10^{-3}} = 39.4 \text{ MN/m}^2$$

The normal stress on the oblique plane is

$$\sigma = \sigma_x \cos^2 60° = \left(39.4 \times 10^6\right)\tfrac{1}{4} = 9.85 \text{ MN/m}^2$$

The shearing stress on the oblique plane is

$$\tfrac{1}{2}\sigma_x \sin 120^\circ = \tfrac{1}{2}\left(39.4 \times 10^6\right)\sqrt{\frac{3}{2}} = 17.05 \text{ MN / m}^2$$

5.3 Strain figures in mild steel; Lüder's lines

If a tensile specimen of mild steel is well polished and then stressed, it will be found that, when the specimen yields, a pattern of fine lines appears on the polished surface; these lines intersect roughly at right-angles to each other, and at 45° approximately to the longitudinal axis of the bar; these lines were first observed by Lüder in 1854. Lüder's lines on a tensile specimen of mild steel are shown in Figure 5.2. These strain figures suggest that yielding of the material consists of slip along the planes of greatest shearing stress; a single line represents a slip band, containing a large number of metal crystals.

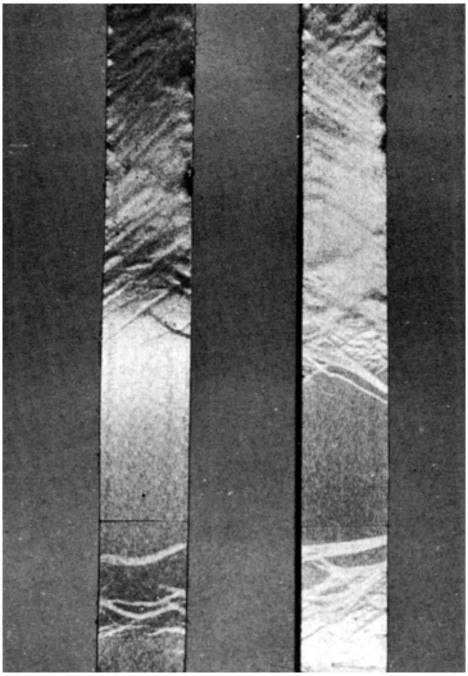


Figure 5.2 Lüder's lines in the yielding of a steel bar in tension.

5.4 Failure of materials in compression

Shearing stresses are also developed in a bar under uniform compression. The failure of some materials in compression is due to the development of critical shearing stresses on planes inclined to the direction of compression. Figure 5.3 shows two failures of compressed timbers; failure is due primarily to breakdown in shear on planes inclined to the direction of compression.

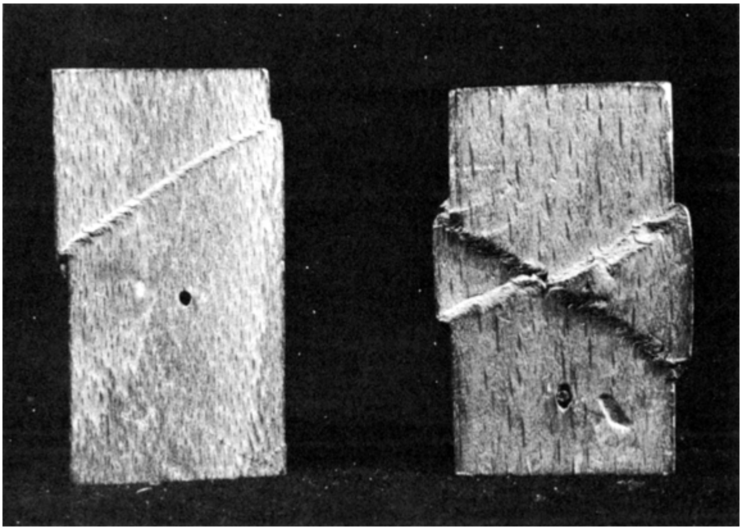


Figure 5.3 Failures of compressed specimens of timber, showing breakdown of the material in shear.

5.5 General two-dimensional stress system

A *two-dimensional* stress system is one in which the stresses at any point in a body act in the same plane. Consider a thin rectangular block of material, *abcd*, two faces of which are parallel to the *xy*-plane, Figure 5.4. A two-dimensional state of stress exists if the stresses on the remaining four faces are parallel to the *xy*-plane. In general, suppose the *forces* acting on the faces are *P*, *Q*, *R*, *S*, parallel to the *xy*-plane, Figure 5.4. Each of these forces can be resolved into components P_x, P_y etc., Figure 5.5. The perpendicular components give rise to direct stresses, and the tangential components to shearing stresses.

The system of *forces* in Figure 5.5 is now replaced by its equivalent system of *stresses*; the rectangular block of Figure 5.6 is in uniform state of two-dimensional stress; over the two faces parallel to Ox are direct and shearing stresses σ_y and τ_{yx}, respectively. The thickness is assumed to be 1 unit of length, for convenience, the other sides having lengths a and b. Equilibrium of the block in the x- and y-directions is already ensured; for rotational equilibrium of the block in the xy-plane we must have

$$[\tau_{xy}\,(a \times 1)] \times b = [\tau_{yx}\,(b \times 1)] \times a$$

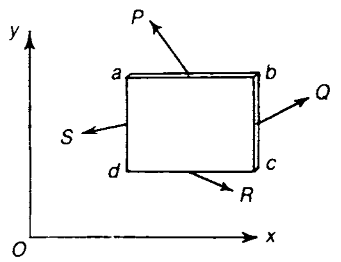


Figure 5.4 Resultant force acting on the faces of a 'two-dimensional' rectangular block.

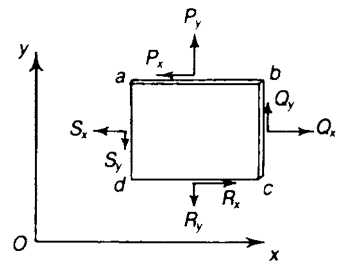


Figure 5.5 Components of resultant forces parallel to O_x and O_y.

Thus $\quad (ab)\,\tau_{xy} = (ab)\,\tau_{yx}$

or
$$\tau_{xy} = \tau_{yx} \tag{5.3}$$

Then the shearing stresses on perpendicular planes are equal and *complementary* as we found in the simpler case of pure shear in Section 3.3.

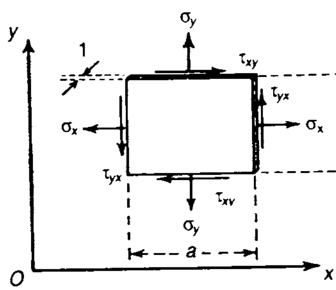


Figure 5.6 General two-dimensional state of stress.

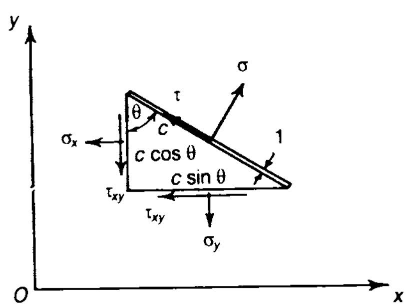


Figure 5.7 Stresses on an inclined plane in a two-dimensional stress system.

5.6 Stresses on an inclined plane

Consider the stresses acting on an inclined plane of the uniformly stressed rectangular block of Figure 5.6; the inclined plane makes an angle θ with O_y, and cuts off a 'triangular' block, Figure 5.7. The length of the hypotenuse is c, and the thickness of the block is taken again as one unit of length, for convenience. The values of direct stress, σ, and shearing stress, τ, on the inclined plane are found by considering equilibrium of the triangular block. The direct stress acts along the normal to the inclined plane. Resolve the forces on the three sides of the block parallel to this normal: then

$$\sigma\,(c.1) = \sigma_x\,(c\cos\theta\cos\theta) + \sigma_y\,(c\sin\theta\sin\theta) + \tau_{xy}\,(c\cos\theta\sin\theta) + \tau_{xy}\,(c\sin\theta\cos\theta)$$

This gives

$$\sigma = \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta \tag{5.4}$$

Now resolve forces in a direction parallel to the inclined plane:

$$\tau.(c\ 1) = -\sigma_x\,(c\cos\theta\sin\theta) + \sigma_y\,(c\sin\theta\cos\theta) + \tau_{xy}\,(c\cos\theta\cos\theta) - \tau_{xy}\,(c\sin\theta\sin\theta)$$

This gives

$$\tau = -\sigma_x\cos\theta\sin\theta + \sigma_y\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) \tag{5.5}$$

The expressions for σ and τ are written more conveniently in the forms:

$$\sigma = \tfrac{1}{2}(\sigma_x + \sigma_y) + \tfrac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta \tag{5.6}$$

$$\tau = -\tfrac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta \tag{5.7}$$

The shearing stress τ vanishes when

$$\tfrac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta = \tau_{xy}\cos 2\theta$$

that is, when

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \tag{5.8}$$

or when

$$2\theta = \tan^{-1}\frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad \text{or} \quad \tan^{-1}\frac{2\tau_{xy}}{\sigma_x - \sigma_y} + 180^\circ$$

These may be written

$$\theta = \frac{1}{2}\tan^{-1}\frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad \text{or} \quad \frac{1}{2}\tan^{-1}\frac{2\tau_{xy}}{\sigma_x - \sigma_y} + 90^\circ \tag{5.9}$$

In a two-dimensional stress system there are thus two planes, separated by 90°, on which the shearing stress is zero. These planes are called the *principal planes*, and the corresponding values of σ are called the *principal stresses*. The direct stress σ is a maximum when

$$\frac{d\sigma}{d\theta} = -(\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

that is, when

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

which is identical with equation (5.8), defining the directions of the principal stresses; thus the *principal stresses* are also the ***maximum*** and ***minimum direct stresses*** in the material.

5.7 Values of the principal stresses

The directions of the principal planes are given by equation (5.8). For any two-dimensional stress system, in which the values of σ_x, σ_y and τ_{xy} are known, $\tan 2\theta$ is calculable; two values of θ, separated by 90°, can then be found. The principal stresses are then calculated by substituting these vales of θ into equation (5.6).

Alternatively, the principal stresses can be calculated more directly without finding the principal planes. Earlier we defined a principal plane as one on which there is no shearing stress; in Figure 5.8 it is assumed that no shearing stress acts on a plane at an angle θ to Oy.

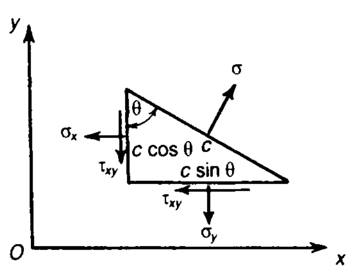


Figure 5.8 A principal stress acting on an inclined plane; there is no shearing stress τ associated with a principal stress σ.

For equilibrium of the triangular block in the x-direction,

$$\sigma(c\cos\theta) - \sigma_x(c\cos\theta) = \tau_{xy}(c\sin\theta)$$

and so

$$\sigma - \sigma_x = \tau_{xy}\tan\theta \tag{5.10}$$

For equilibrium of the block in the y-direction

$$\sigma\,(c\sin\theta) - \sigma_y\,(c\sin\theta) = \tau_{xy}\,(c\cos\theta)$$

and thus

$$\sigma - \sigma_y = \tau_{xy}\cot\theta \tag{5.11}$$

On eliminating θ between equations (5.10) and (5.11); by multiplying these equations together, we get

$$(\sigma - \sigma_x)(\sigma - \sigma_y) = \tau^2_{xy}$$

This equation is quadratic in σ; the solutions are

$$\sigma_1 = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} = \text{maximum principal stress}$$

(5.12)

$$\sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} = \text{minimum principal stress}$$

which are the values of the principal stresses; these stresses occur on mutually perpendicular planes.

5.8 Maximum shearing stress

The principal planes define directions of zero shearing stress; on some intermediate plane the shearing stress attains a maximum value. The shearing stress is given by equation (5.7); τ attains a maximum value with respect to θ when

$$\frac{d\tau}{d\theta} = -(\sigma_x - \sigma_y)\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0$$

i.e., when

$$\cot 2\theta = -\frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

The planes of maximum shearing stress are inclined then at 45° to the principal planes. On substituting this value of cot 2θ into equation (5.7), the maximum numerical value of τ is

$$\tau_{max} = \sqrt{\left[\frac{1}{2}\left(\sigma_x - \sigma_y\right)\right]^2 + \left[\tau_{xy}\right]^2} \tag{5.13}$$

But from equations (5.12),

$$\sqrt{\left[\frac{1}{2}\left(\sigma_x - \sigma_y\right)\right]^2 + \left[\tau_{xy}\right]^2} = \sigma_1 - \frac{1}{2}\left(\sigma_x + \sigma_y\right) = \frac{1}{2}\left(\sigma_x + \sigma_y\right) - \sigma_2$$

where σ_1 and σ_2 are the principal stresses of the stress system. Then by adding together the two equations on the right hand side, we get

$$2\sqrt{\left[\frac{1}{2}\left(\sigma_x - \sigma_y\right)\right]^2 + \left[\tau_{xy}\right]^2} = \sigma_1 - \sigma_2$$

and equation (5.13) becomes

$$\tau_{max} = \frac{1}{2}\left(\sigma_1 - \sigma_2\right) \tag{5.14}$$

The maximum shearing stress is therefore half the difference between the principal stresses of the system.

Problem 5.2 At a point of a material the two-dimensional stress system is defined by

σ_x = 60.0 MN/m², tensile

σ_y = 45.0 MN/m², compressive

τ_{xy} = 37.5 MN/m², shearing

where σ_x, σ_y, τ_{xy} refer to Figure 5.7. Evaluate the values and directions of the principal stresses. What is the greatest shearing stress?

Solution

Now, we have

$$\frac{1}{2}\left(\sigma_x + \sigma_y\right) = \frac{1}{2}\ (60.0 - 45.0) = 7.5\ \text{MN/m}^2$$

$$\frac{1}{2}\left(\sigma_x - \sigma_y\right) = \frac{1}{2}\ (60.0 + 45.0) = 52.5\ \text{MN/m}^2$$

Then, from equations (5.12),

$$\sigma_1 = 7.5 + \left[(52.5)^2 + (37.5)^2\right]^{\frac{1}{2}} = 7.5 + 64.4 = 71.9 \text{ MN/m}^2$$

$$\sigma_2 = 7.5 - \left[(52.5)^2 + (37.5)^2\right]^{\frac{1}{2}} = 7.5 - 64.4 = -56.9 \text{ MN/m}^2$$

From equation (5.8)

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{37.5}{52.5} = 0.714$$

Thus,

$$2\theta = \tan^{-1}(0.714) = 35.5° \text{ or } 215.5°$$

Then

$$\theta = 17.8° \text{ or } 107.8°$$

From equation (5.14)

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_2) = \frac{1}{2}(71.9 + 56.9) = 64.4 \text{ MN/m}^2$$

This maximum shearing stress occurs on planes at 45° to those of the principal stresses.

5.9 Mohr's circle of stress

A geometrical interpretation of equations (5.6) and (5.7) leads to a simple method of stress analysis. Now, we have found already that

$$\sigma = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta$$

Take two perpendicular axes $O\sigma$, $O\tau$, Figure 5.9; on this co-ordinate system set off the point having co-ordinates (σ_x, τ_{xy}) and $(\sigma_y, -\tau_{xy})$, corresponding to the known stresses in the x- and y-directions. The line PQ joining these two points is bisected by the $O\sigma$ axis at a point O'. With a centre at O', construct a circle passing through P and Q. The stresses σ and τ on a plane at an angle θ to Oy are found by setting off a radius of the circle at an angle 2θ to PQ, Figure 5.9; 2θ is measured in a clockwise direction from $O'P$.

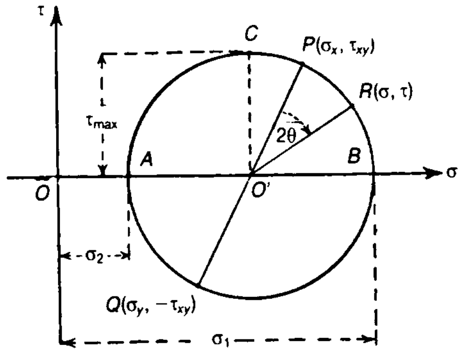


Figure 5.9 Mohr's circle of stress. The points *P* and *Q* correspond to the stress states (σ_x, τ_{xy}) and $(\sigma_y, -\tau_{xy})$ respectively, and are diametrically opposite; the state of stress (σ, τ) on a plane inclined at an angle θ to *Oy* is given by the point *R*.

The co-ordinates of the point $R(\sigma, \tau)$ give the direct and shearing stresses on the plane. We may write the above equations in the forms

$$\sigma - \frac{1}{2}\left(\sigma_x + \sigma_y\right) = \frac{1}{2}\left(\sigma_x - \sigma_y\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$-\tau = \frac{1}{2}\left(\sigma_x - \sigma_y\right)\sin 2\theta - \tau_{xy}\cos 2\theta$$

Square each equation and add; then we have

$$\left[\sigma - \frac{1}{2}\left(\sigma_x + \sigma_y\right)\right]^2 + \tau^2 = \left[\frac{1}{2}\left(\sigma_x - \sigma_y\right)\right]^2 + \left[\tau_{xy}\right]^2 \qquad (5.15)$$

Thus all corresponding values of σ and τ lie on a circle of radius

$$\sqrt{\left[\frac{1}{2}\left(\sigma_x - \sigma_y\right)\right]^2 + \tau_{xy}^2}$$

with its centre at the point $(\frac{1}{2}[\sigma_x + \sigma_y], 0)$, Figure 5.9.

This circle defining all possible states of stress is known as *Mohr's Circle of Stress*; the principal stresses are defined by the points *A* and *B*, at which $\tau = 0$. The maximum shearing stress, which is given by the point *C*, is clearly the radius of the circle.

Problem 5.3 At a point of a material the stresses forming a two-dimensional system are shown in Figure 5.10. Using Mohr's circle of stress, determine the magnitudes and directions of the principal stresses. Determine also the value of the maximum shearing stress.

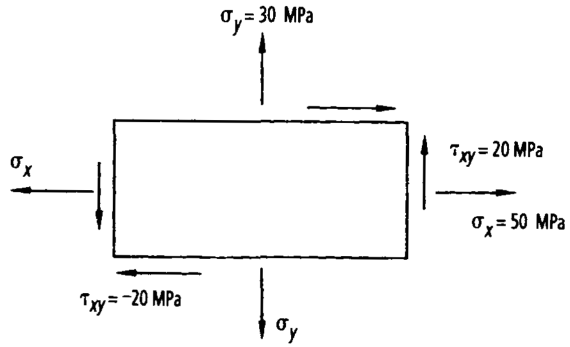


Figure 5.10. Stress at a point.

<u>*Solution*</u>

From Figure 5.10, the shearing stresses acting in conjunction with σ_x are counter-clockwise, hence, τ_{xy} is said to be positive on the vertical planes. Similarly, the shearing stresses acting in conjunction with τ_y are clockwise, hence, τ_{xy} is said to be negative on the horizontal planes.

On the $\sigma - \tau$ diagram of Figure 5.11, construct a circle with the line joining the point (σ_x, τ_{xy}) or (50, 20) and the point $(\sigma_y, -\tau_{xy})$ or (30,−20) as the diameter, as shown by *A* and *B*, respectively

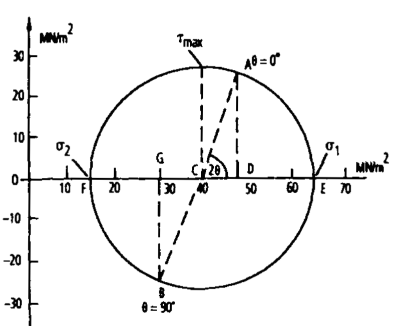


Figure 5.11 Problem 5.3.

The principal stresses and their directions can be obtained from a scaled drawing, but we shall calculate σ_1, σ_2 etc.

$$DA = 20 \text{ MPa}$$

$$OD = \sigma_x = 50 \text{ MPa}$$

$$OG = \sigma_y = 30 \text{ MPa}$$

$$OC = \frac{(OD + OG)}{2} = \frac{(50 + 30)}{2} = 40 \text{ MPa}$$

$$CD = OD - OC = 50 - 40 = 10 \text{ MPa}$$

$$AC^2 = CD^2 + DA^2$$

$$= 10^2 + 20^2$$

or $$AC = 22.36 \text{ MPa}$$

$$\sigma_1 = OE = OC + AC = 40 + 22.36$$

$$\sigma_1 = 62.36 \text{ MPa}$$

$$\sigma_2 = OF = OC - AC$$

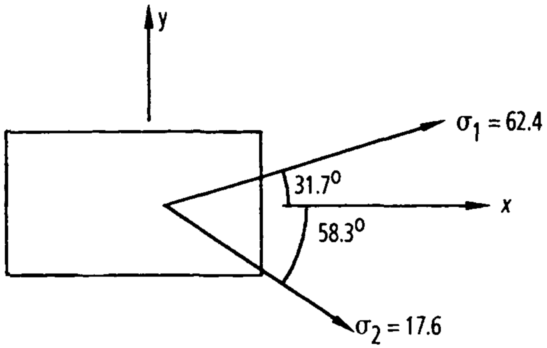
$$= 40 - 22.36$$

or $$\sigma_2 = 17.64 \text{ MPa}$$

$$2\theta = \tan^{-1}\left(\frac{AD}{CD}\right)$$

$$= \tan^{-1}\left(\frac{20}{10}\right) = 63.43°$$

$$\therefore \theta = 31.7° \text{ see below}$$



Maximum shear stress $= \tau_{max} = AC = 22.36$ MPa which occurs on planes at 45° to those of the principal stresses.

Problem 5.4 At a point of a material the two-dimensional state of stress is shown in Figure 5.12. Determine σ_1, σ_2, θ and τ_{max}

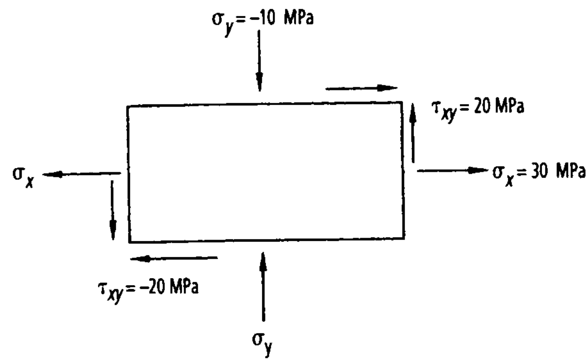


Figure 5.12 Stress at a point.

<u>*Solution*</u>

On the σ–τ diagram of Figure 5.13, construct a circle with the line joining the point (σ_x, τ_{xy}) or (30, 20) to the point $(\sigma_y, -\tau_{xy})$ or (−10, −20), as the diameter, as shown by the points *A* and *B* respectively. It should be noted that τ_{xy} is positive on the vertical planes of Figure 5.12, as these shearing stresses are causing a counter-clockwise rotation; vice-versa for the shearing stresses on the horizontal planes.

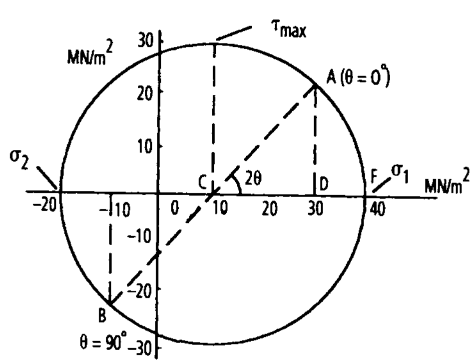


Figure 5.13 Problem 5.4.

From Figure 5.13,

$$AD = \tau_{xy} = 20$$

$$OD = \sigma_x = 30$$

$$OE = \sigma_y = -10$$

$$OC = \frac{(OD + OE)}{2} = \frac{(30 - 10)}{2}$$

or $OC = 10$

$$CD = OD - OC = 30 - 10 = 20$$

$$AC^2 = CD^2 + AD^2$$

$$= 20^2 + 20^2 = 800$$

or $AC = 28.28$

$$\sigma_1 = OF = OC + AC = 10 + 28.28$$

or $\sigma_1 = 38.3$ MPa

$$\sigma_2 = OG = OC - AC$$

$$= 10 - 28.3$$

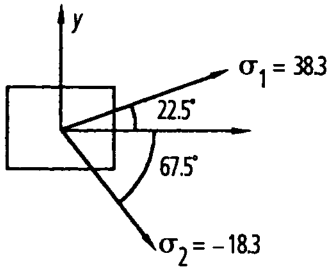
or $\sigma_2 = -18.3$ MPa

$$2\theta = \tan^{-1}\left(\frac{AD}{CD}\right) = \left(\frac{20}{20}\right) = 45°$$

$$\therefore \theta = 22.5 \text{ (see below)}$$

τ_{max} = Maximum shearing stress = AC

or τ_{max} = 28.3 MPa acting on planes at 45° to σ_1 and σ_2.



5.10 Strains in an inclined direction

For two-dimensional system of strains the direct and shearing strains in any direction are known if the direct and shearing strains in two mutually perpendicular directions are given. Consider a rectangular element of material, *OABC*, in the *xy*-plane, Figure 5.14, it is required to find the direct and shearing strains in the direction of the diagonal *OB*, when the direct and shearing strains in the directions *Ox, Oy* are given. Suppose ε_x is the strain in the direction *Ox*, ε_y the strain in the direction *Oy*, and γ_{xy} the shearing strain relative to *Ox* and *Oy*.

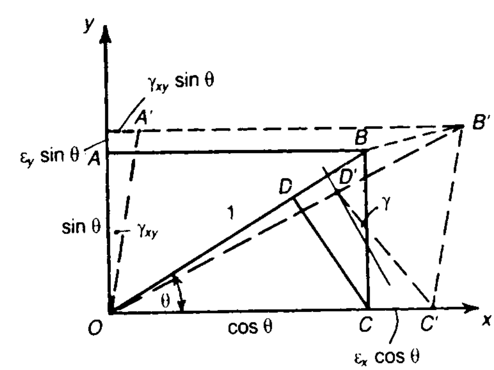


Figure 5.14 Strains in an inclined direction; strains in the directions O_x and O_y and defined by ε_x, ε_y and γ_{xy}, lead to strains ε, γ along the inclined direction *OB*.

All the strains are considered to be small; in Figure 5.14, if the diagonal *OB* of the rectangle is taken to be of unit length, the sides *OA, OB* are of lengths sinθ, cosθ, respectively, in which θ is the angle *OB* makes with *Ox*. In the strained condition *OA* extends a small amount ε_y sinθ, *OC* extends a small amount ε_x cosθ, and due to shearing strain *OA* rotates through a small angle γ_{xy}.

If the point B moves to point B', the movement of B parallel to Ox is

$$\varepsilon_x \cos\theta + \gamma_{xy} \sin\theta$$

and the movement parallel to Oy is

$$\varepsilon_y \sin\theta$$

Then the movement of B parallel to OB is

$$\left(\varepsilon_x \cos\theta + \gamma_{xy} \sin\theta\right)\cos\theta + \left(\varepsilon_y \sin\theta\right)\sin\theta$$

Since the strains are small, this is equal to the extension of the OB in the strained condition; but OB is of unit length, so that the extension is also the direct strain in the direction OB. If the direct strain in the direction OB is denoted by ε, then

$$\varepsilon = \left(\varepsilon_x \cos\theta + \gamma_{xy} \sin\theta\right)\cos\theta + \left(\varepsilon_y \sin\theta\right)\sin\theta$$

This may be written in the form

$$\varepsilon = \varepsilon_x \cos^2\theta + \varepsilon_y \sin^2\theta + \gamma_{xy} \sin\theta\cos\theta$$

and also in the form

$$\varepsilon = \tfrac{1}{2}\left(\varepsilon_x + \varepsilon_y\right) + \tfrac{1}{2}\left(\varepsilon_x - \varepsilon_y\right)\cos 2\theta + \tfrac{1}{2}\gamma_{xy} \sin 2\theta \qquad (5.16)$$

This is similar in form to equation (5.6), defining the direct stress on an inclined plane; ε_x and ε_y replace σ_x and σ_y, respectively, and $\frac{1}{2}\gamma_{xy}$ replaces τ_{xy}.

To evaluate the shearing strain in the direction OB we consider the displacements of the point D, the foot of the perpendicular from C to OB, in the strained condition, Figure 5.10. The point D, is displaced to a point D'; we have seen that OB extends an amount ε, so that OD extends an amount

$$\varepsilon\, OD = \varepsilon \cos^2\theta$$

During straining the line CD rotates anti-clockwise through a small angle

$$\frac{\varepsilon_x \cos^2\theta - \varepsilon \cos^2\theta}{\cos\theta \sin\theta} = \left(\varepsilon_x - \varepsilon\right)\cot\theta$$

At the same time OB rotates in a clockwise direction through a small angle

$$\left(\varepsilon_x \cos\theta + \gamma_{xy} \sin\theta\right)\sin\theta - \left(\varepsilon_y \sin\theta\right)\cos\theta$$

The amount by which the angle *ODC* diminishes during straining is the shearing strain γ in the direction *OB*. Thus

$$\gamma = -(\varepsilon_x - \varepsilon)\cot\theta - (\varepsilon_x \cos\theta + \gamma_{xy}\sin\theta)\sin\theta + (\varepsilon_y \sin\theta)\cos\theta$$

On substituting for ε from equation (5.16) we have

$$\gamma = -2(\varepsilon_x - \varepsilon_y)\cos\theta\sin\theta + \gamma_{xy}(\cos^2\theta - \sin^2\theta)$$

which may be written

$$\frac{1}{2}\gamma = -\frac{1}{2}(\varepsilon_x - \varepsilon_y)\sin 2\theta + \frac{1}{2}\gamma_{xy}\cos 2\theta \qquad (5.17)$$

This is similar in form to equation (5.7) defining the shearing stress on an inclined plane; σ_x and σ_y in that equation are replaced by ε_x and ε_y, respectively, and τ_{xy} by $\frac{1}{2}\gamma_{xy}$.

5.11 Mohr's circle of strain

The direct and shearing strains in an inclined direction are given by relations which are similar to equations (5.6) and (5.7) for the direct and shearing stresses on an inclined plane. This suggests that the strains in any direction can be represented graphically in a similar way to the stress system. We may write equations (5.16) and (5.17) in the forms

$$\varepsilon - \frac{1}{2}(\varepsilon_x + \varepsilon_y) = \frac{1}{2}(\varepsilon_x - \varepsilon_y)\cos 2\theta + \frac{1}{2}\gamma_{xy}\sin 2\theta$$

$$\frac{1}{2}\gamma = -\frac{1}{2}(\varepsilon_x - \varepsilon_y)\sin 2\theta + \frac{1}{2}\gamma_{xy}\cos 2\theta$$

Square each equation, and then add; we have

$$\left[\varepsilon - \frac{1}{2}\left(\varepsilon_x + \varepsilon_y\right)\right]^2 + \left[\frac{1}{2}\gamma\right]^2 = \left[\frac{1}{2}\left(\varepsilon_x - \varepsilon_y\right)\right]^2 + \left[\frac{1}{2}\gamma_{xy}\right]^2$$

Thus all values of ε and $\frac{1}{2}\gamma$ lie on a circle of radius

$$\sqrt{\left[\frac{1}{2}\left(\varepsilon_x - \varepsilon_y\right)\right]^2 + \left[\frac{1}{2}\gamma_{xy}\right]^2}$$

with its centre at the point

$$\left[\frac{1}{2}(\varepsilon_x + \varepsilon_y)\,,\ 0\right]$$

This circle defining all possible states of strain is usually called *Mohr's circle of strain*. For given values of ε_x, ε_y and γ_{xy} it is constructed in the following way: two mutually perpendicular axes, ε and $\frac{1}{2}\gamma$, are set up, Figure 5.15; the points (ε_x, $\frac{1}{2}\gamma_{xy}$) and (ε_y, $-\frac{1}{2}\gamma_{xy}$) are located; the line joining these points is a diameter of the circle of strain. The values of ε and $\frac{1}{2}\gamma$ in an inclined direction making an angle θ with Ox (Figure 5.10) are given by the points on the circle at the ends of a diameter making an angle 2θ with PQ; the angle 2θ is measured clockwise.

We note that the maximum and minimum values of ε, given by ε_1 and ε_2 in Figure 5.15, occur when $\frac{1}{2}\gamma$ is zero; ε_1, ε_2 are called *principal strains*, and occur for directions in which there is no shearing strain.

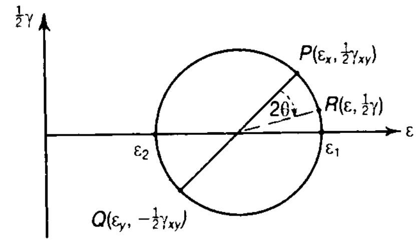


Figure 5.15 Mohr's circle of strain; the diagram is similar to the circle of stress, except that $\frac{1}{2}\gamma$ is plotted along the ordinates and not γ.

An important feature of this strain analysis is that we have *not* assumed that the strains are elastic; we have taken them to be small, however, with this limitation Mohr's circle of strain is applicable to both elastic and inelastic problems.

5.12 Elastic stress–strain relations

When a point of a body is acted upon by stresses σ_x and σ_y in mutually perpendicular directions the strains are found by superposing the strains due to σ_x and σ_y acting separately.

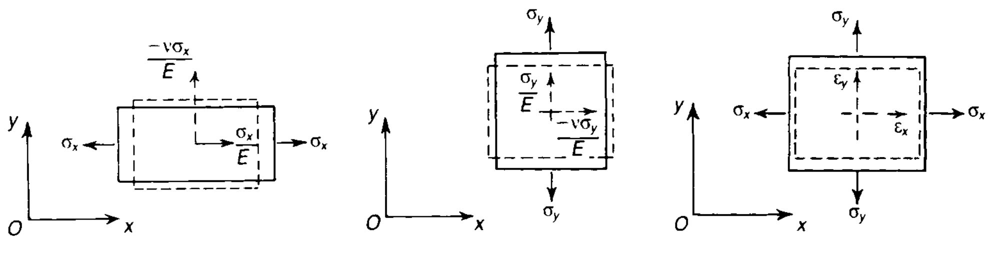


Figure 5.16 Strains in a two-dimensional linear-elastic stress system; the strains can be regarded as compounded of two systems corresponding to uni-axial tension in the x- and y- directions.

The rectangular element of material in Figure 5.16(i) is subjected to a tensile stress σ_x in the x direction; the tensile strain in the x-direction is

$$\frac{\sigma_x}{E}$$

and the compressive strain in the y-direction is

$$-\frac{\nu\sigma_x}{E}$$

in which E is Young's modulus, and ν is Poisson's ratio (see section 1.10). If the element is subjected to a tensile stress σ_y in the y-direction as in Figure 5.12(ii), the compressive strain in the x-direction is

$$-\frac{\nu\sigma_y}{E}$$

and the tensile strain in the y-direction is

$$\frac{\sigma_y}{E}$$

These elastic strains are small, and the state of strain due to both stresses σ_x and σ_y, acting simultaneously, as in Figure 5.16(iii), is found by superposing the strains of Figures 5.16(i) and (ii); taking tensile strain as positive and compressive strain as negative, the strains in the x- and y-directions are given, respectively, by

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} \qquad (5.18)$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \frac{\nu\sigma_x}{E}$$

On multiplying each equation by E, we have

$$\begin{aligned} E\varepsilon_x &= \sigma_x - \nu\sigma_y \\ E\varepsilon_y &= \sigma_y - \nu\sigma_x \end{aligned} \qquad (5.19)$$

These are the elastic stress-strain relations for two-dimensional system of direct stresses. When a shearing stress τ_{xy} is present in addition to the direct stresses σ_x and σ_y, as in Figure 5.17, the shearing stress τ_{xy} is assumed to have no effect on the direct strains ε_x and ε_y caused by σ_x and σ_y.

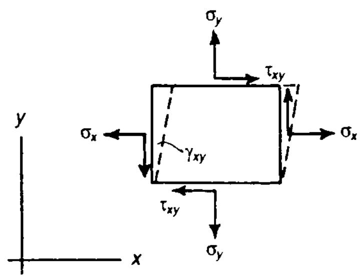


Figure 5.17 Shearing strain in a two-dimensional system.

Similarly, the direct stresses σ_x and σ_y are assumed to have no effect on the shearing strain γ_{xy} due to τ_{xy}. When shearing stresses are present, as well as direct stresses, there is therefore an additional stress-strain relation having the form in which G is the shearing modulus.

$$\frac{\tau_{xy}}{\gamma_{xy}} = G$$

Then, in addition to equations (5.19) we have the relation

$$\tau_{xy} = G\gamma_{xy} \tag{5.20}$$

5.13 Principal stresses and strains

We have seen that in a two-dimensional system of stresses there are always two mutually perpendicular directions in which there are no shearing stresses; the direct stresses on these planes were referred to as principal stresses, σ_1 and σ_2. As there are no shearing stresses in these two mutually perpendicular directions, there are also no shearing strains; for the principal directions the corresponding direct strains are given by

$$\begin{aligned} E\varepsilon_1 &= \sigma_1 - \nu\sigma_2 \\ E\varepsilon_2 &= \sigma_2 - \nu\sigma_1 \end{aligned} \tag{5.21}$$

The direct strains, ε_1, ε_2, are the principal strains already discussed in Mohr's circle of strain. It follows that the principal strains occur in directions parallel to the principal stresses.

5.14 Relation between E, G and ν

Consider an element of material subjected to a tensile stress σ_0 in one direction together with a compressive stress σ_0 in a mutually perpendicular direction, Figure 5.18(i). The Mohr's circle for this state of stress has the form shown in Figure 5.18(ii); the circle of stress has a centre at the origin and a radius of σ_0. The direct and shearing stresses on an inclined plane are given by the co-ordinates of a point on the circle; in particular we note that there is no direct stress when $2\theta = 90°$, that is, when $\theta = 45°$ in Figure 5.18(i).

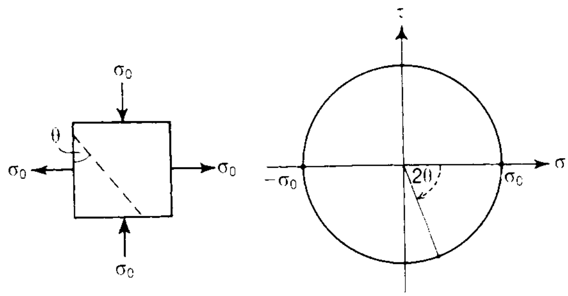


Figure 5.18 (i) A stress system consisting of tensile and compressive stresses of equal magnitude, but acting in mutually perpendicular directions. (ii) Mohr's circle of stress for this system.

Moreover when $\theta = 45°$, the shearing stress on this plane is of magnitude σ_0. We conclude then that a state of equal and opposite tension and compression, as indicated in Figure 5.18(i), is equivalent, from the stress standpoint, to a condition of simple shearing in directions at 45°, the shearing stresses having the same magnitudes as the direct stresses σ_0 (Figure 5.19). This system of stresses is called *pure shear*.

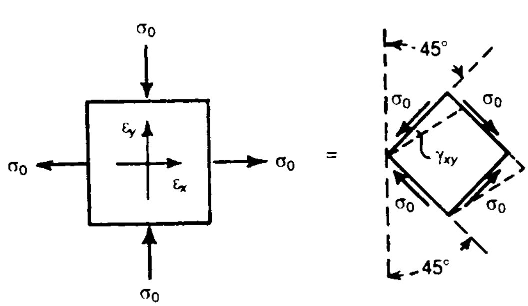


Figure 5.19 Pure Shear. Equality of (i) equal and opposite tensile and compressive stresses and (ii) pure shearing stress.

If the material is elastic, the strains ε_x and ε_y caused by the direct stresses σ_0 are, from equations (5.18),

$$\varepsilon_x = \frac{1}{E}(\sigma_0 + \nu\sigma_0) = \frac{\sigma_0}{E}(1 + \nu)$$

$$\varepsilon_y = \frac{1}{E}(-\sigma_0 - \nu\sigma_0) = -\frac{\sigma_0}{E}(1 + \nu)$$

If the sides of the element are of unit length, the work done in distorting the element is

$$W = \frac{1}{2}\sigma_0\,\varepsilon_x - \frac{1}{2}\sigma_0\,\varepsilon_y = \frac{\sigma_0^2}{E}(1 + \nu) \qquad (5.22)$$

per unit volume of the material.

In the state of pure shearing under stresses σ_0, the shearing strain is given by equation (5.20),

$$\gamma_{xy} = \frac{\sigma_0}{G}$$

The work done in distorting an element of sides unit length is

$$W = \frac{1}{2}\sigma_0\gamma_{xy} = \frac{\sigma_0^2}{2G} \qquad (5.23)$$

per unit volume of the material. As the one state of stress is equivalent to the other, the values of work done per unit volume of the material are equal. Then

$$\frac{\sigma_0^2}{E}(1 + \nu) = \frac{\sigma_0^2}{2G}$$

and hence

$$E = 2G(1 + \nu) \qquad (5.24)$$

Thus ν can be calculated from measured values E and G.

The shearing stress–strain relation is given by equation (5.20), which may now be written in the form

$$E\gamma_{xy} = 2(1 + \nu)\tau_{xy} \qquad (5.25)$$

For most metals ν is approximately 0.3; then, approximately,

$$E = 2(1 + \nu)G = 2.6G \tag{5.26}$$

Problem 5.5 From tests on a magnesium alloy it is found that E is 45 GN/m^2 and G is 17 GN/m^2. Estimate the value of Poisson's ratio.

Solution

From equation (5.24),

$$\nu = \frac{E}{2G} - 1 = \frac{45}{34} - 1 = 1.32 - 1$$

Then

$$\nu = 0.32$$

Problem 5.6 A thin sheet of material is subjected to a tensile stress of 80 MN/m^2, in a certain direction. One surface of the sheet is polished, and on this surface fine lines are ruled to form a square of side 5 cm, one diagonal of the square being parallel to the direction of the tensile stresses. If $E = 200$ GN/m^2, and $\nu = 0.3$, estimate the alteration in the lengths of the sides of the square, and the changes in the angles at the corners of the square.

Solution

The diagonal parallel to the tensile stresses increases in length by an amount

$$\frac{\left(80 \times 10^6\right)\left(0.05\sqrt{2}\right)}{200 \times 10^9} = 28.3 \times 10^{-6} \text{ m}$$

The diagonal perpendicular to the tensile stresses diminishes in length by an amount

$$0.3\left(28.3 \times 10^{-6}\right) = 8.50 \times 10^{-6} \text{ m}$$

The change in the corner angles is then

$$\frac{1}{0.05}\left[(28.3 + 8.50)10^{-6}\right]\frac{1}{\sqrt{2}} = 52.0 \times 10^{-3} \text{ radians} = 0.0405°$$

The angles in the line of pull are diminished by this amount, and the others increased by the same amount. The increase in length of each side is

$$\frac{1}{2\sqrt{2}}\left[(28.3 - 8.50)10^{-6}\right] = 7.00 \times 10^{-6} \text{ m}$$

5.15 Strain 'rosettes'

To determine the stresses in a material under practical loading conditions, the strains are measured by means of small gauges; many types of gauges have been devised, but perhaps the most convenient is the electrical resistance strain gauge, consisting of a short length of fine wire which is glued to the surface of the material. The resistance of the wire changes by small amounts as the wire is stretched, so that as the surface of the material is strained the gauge indicates a change of resistance which is measurable on a Wheatstone bridge. The lengths of wire resistance strain gauges can be as small as 0.4 mm, and they are therefore extremely useful in measuring local strains.

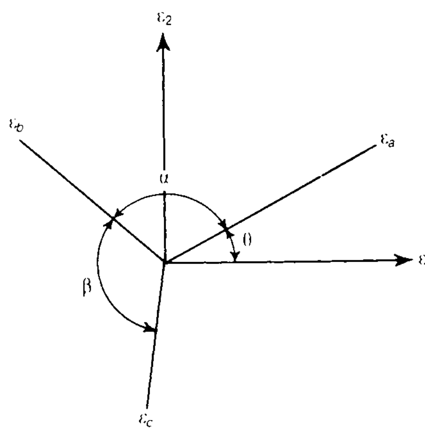


Figure 5.20 Finding the principal strains in a two-dimensional system by recording three linear strains, ε_a, ε_b and ε_c in the vicinity of a point.

The state of strain at a point of a material is defined in the two-dimensional case if the direct strains, ε_x and ε_y, and the shearing strain, γ_{xy}, are known. Unfortunately, the shearing strain γ_{xy} is not readily measured; it is possible, however, to measure the direct strains in three different directions by means of strain gauges. Suppose ε_1, ε_2 are the unknown principal strains in a two-dimensional system, Figure 5.20. Then from equation (5.16) we have that the measured direct strains ε_a, ε_b and ε_c in directions inclined at θ, $(\theta + \alpha)$, $(\theta + \alpha + \beta)$ to ε_1 are

$$\begin{aligned}
\varepsilon_a &= \tfrac{1}{2}\left(\varepsilon_1 + \varepsilon_2\right) + \tfrac{1}{2}\left(\varepsilon_1 - \varepsilon_2\right)\cos 2\theta \\
\varepsilon_b &= \tfrac{1}{2}\left(\varepsilon_1 + \varepsilon_2\right) + \tfrac{1}{2}\left(\varepsilon_2 - \varepsilon_2\right)\cos 2\left(\theta + \alpha\right) \\
\varepsilon_c &= \tfrac{1}{2}\left(\varepsilon_1 + \varepsilon_2\right) + \tfrac{1}{2}\left(\varepsilon_1 - \varepsilon_2\right)\cos 2\left(\theta + \alpha + \beta\right)
\end{aligned} \tag{5.27}$$

In practice the directions of the principal strains are not known usually; but if the three direct strains ε_a, ε_b and ε_c are measured in known directions, then the three unknowns in equations (5.27) are

$$\varepsilon_1,\ \varepsilon_2 \text{ and } \theta$$

Three strain gauges arranged so that $\alpha = \beta = 45°$ form a 45° rosette, Figure 5.22. Equations (5.27) become

$$\varepsilon_a = \frac{1}{2}\left(\varepsilon_1 + \varepsilon_2\right) + \frac{1}{2}\left(\varepsilon_1 - \varepsilon_2\right)\cos 2\theta \tag{5.28a}$$

$$\varepsilon_b = \frac{1}{2}\left(\varepsilon_1 + \varepsilon_2\right) - \frac{1}{2}\left(\varepsilon_1 - \varepsilon_2\right)\sin 2\theta \tag{5.28b}$$

$$\varepsilon_c = \frac{1}{2}\left(\varepsilon_1 + \varepsilon_2\right) - \frac{1}{2}\left(\varepsilon_1 - \varepsilon_2\right)\cos 2\theta \tag{5.28c}$$

Adding together equations (5.28a) and (5.28c), we get

$$\varepsilon_a + \varepsilon_c = \varepsilon_1 + \varepsilon_2 \tag{5.29}$$

Equation (5.29) is known as the *first invariant of strain*, which states that the sum of two mutually perpendicular normal strains is a constant.

From equations (5.28a) and (5.28b).

$$-\frac{1}{2}\left(\varepsilon_1 - \varepsilon_2\right)\sin 2\theta = \varepsilon_b - \frac{1}{2}\left(\varepsilon_1 - \varepsilon_2\right) \tag{5.30a}$$

$$-\frac{1}{2}\left(\varepsilon_1 - \varepsilon_2\right)\cos 2\theta = -\varepsilon_a + \frac{1}{2}\left(\varepsilon_1 + \varepsilon_2\right) \tag{5.30b}$$

Dividing equation (5.30a) by (5.30b), we obtain

$$\tan 2\theta = \frac{\varepsilon_b - \frac{1}{2}(\varepsilon_1 + \varepsilon_2)}{-\varepsilon_a + \frac{1}{2}(\varepsilon_1 + \varepsilon_2)} \tag{5.31}$$

Substituting equation (5.29) into (5.31)

$$\tan 2\theta = \frac{(\varepsilon_a - 2\varepsilon_b + \varepsilon_c)}{(\varepsilon_a - \varepsilon_c)} \tag{5.32}$$

To determine ε_1 and ε_2 in terms of the known strains, namely ε_a, ε_b and ε_c, put equation (5.32) in the form of the mathematical triangle of Figure 5.21.

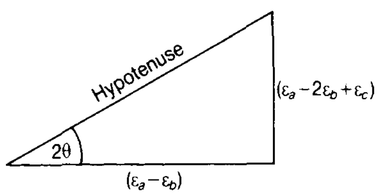


Figure 5.21 Mathematical triangle from equation (5.32).

$$\text{hypotenuse} = \sqrt{\varepsilon_a^2 + 4\varepsilon_b^2 + \varepsilon_c^2 - 4\varepsilon_a\varepsilon_b - 4\varepsilon_b\varepsilon_c + 2\varepsilon_a\varepsilon_c + \varepsilon_a^2 + \varepsilon_b^2 - 2\varepsilon_a\varepsilon_b}$$

$$= \sqrt{2}\sqrt{(\varepsilon_a - \varepsilon_b)^2 + (\varepsilon_c - \varepsilon_b)^2}$$

$$\therefore \cos 2\theta = \frac{\varepsilon_a - \varepsilon_c}{\sqrt{2}\sqrt{(\varepsilon_a - \varepsilon_b)^2 + (\varepsilon_c - \varepsilon_b)^2}} \tag{5.33}$$

and

$$\sin 2\theta = \frac{\varepsilon_a - 2\varepsilon_b + \varepsilon_c}{\sqrt{2}\sqrt{(\varepsilon_a - \varepsilon_b)^2 + (\varepsilon_c - \varepsilon_b)^2}} \tag{5.34}$$

Substituting equations (5.33) and (5.34) into equations (5.30a) and (5.30b) and solving,

$$\varepsilon_1 = \frac{1}{2}(\varepsilon_a + \varepsilon_c) + \frac{\sqrt{2}}{2}\sqrt{(\varepsilon_a - \varepsilon_b)^2 + (\varepsilon_c - \varepsilon_b)^2} \quad (5.35)$$

$$\varepsilon_2 = \frac{1}{2}(\varepsilon_a + \varepsilon_c) - \frac{\sqrt{2}}{2}\sqrt{(\varepsilon_a - \varepsilon_b)^2 + (\varepsilon_c - \varepsilon_b)^2} \quad (5.36)$$

θ is the angle between the directions of ε_1 and ε_a, and is measured clockwise from the direction of ε_1.

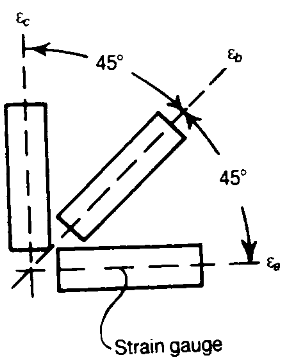


Figure 5.22 A 45° strain rosette.

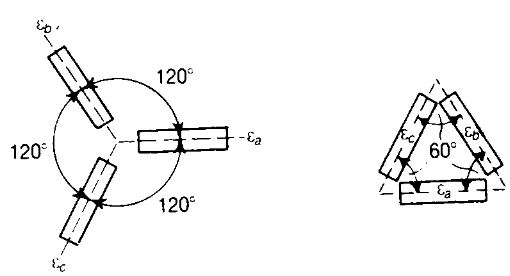


Figure 5.23 Alternative arrangements of 120° rosettes.

The alternative arrangements of gauges in Figure 5.23 correspond to 120° rosettes. On putting $\alpha = \beta = 120°$ in equations (5.27), we have

$$\varepsilon_a = \frac{1}{2}(\varepsilon_1 + \varepsilon_2) + \frac{1}{2}(\varepsilon_1 - \varepsilon_2)\cos 2\theta \quad (5.37a)$$

$$\varepsilon_b = \frac{1}{2}(\varepsilon_1 + \varepsilon_2) - \frac{1}{2}(\varepsilon_1 - \varepsilon_2)\left(\frac{1}{2}\cos 2\theta - \frac{\sqrt{3}}{2}\sin 2\theta\right) \quad (5.37b)$$

$$\varepsilon_c = \frac{1}{2}(\varepsilon_1 + \varepsilon_2) - \frac{1}{2}(\varepsilon_1 - \varepsilon_2)\left(\frac{1}{2}\cos 2\theta + \frac{\sqrt{3}}{2}\sin 2\theta\right) \quad (5.37c)$$

Equations (5.37b) and (5.37c) can be written in the forms

$$\varepsilon_b = \frac{1}{2}(\varepsilon_1+\varepsilon_2) - \frac{1}{2}(\varepsilon_1-\varepsilon_2)\left(\frac{1}{2}\cos 2\theta - \frac{\sqrt{3}}{2}\sin 2\theta\right) \tag{5.38a}$$

$$\varepsilon_c = \frac{1}{2}(\varepsilon_1+\varepsilon_2) - \frac{1}{2}(\varepsilon_1-\varepsilon_2)\left(\frac{1}{2}\cos 2\theta + \frac{\sqrt{3}}{2}\sin 2\theta\right) \tag{5.38b}$$

Adding together equations (5.37a), (5.38a) and (5.38b), we get:

$$\varepsilon_a + \varepsilon_b + \varepsilon_c = \frac{3}{2}(\varepsilon_1 + \varepsilon_2)$$

or

$$\varepsilon_1 + \varepsilon_2 = \frac{2}{3}(\varepsilon_a + \varepsilon_b + \varepsilon_c) \tag{5.39}$$

Taking away equation (5.38b) from (5.38a),

$$\varepsilon_b - \varepsilon_c = \frac{\sqrt{3}}{2}(\varepsilon_1 - \varepsilon_2)\sin 2\theta \tag{5.40}$$

Taking away equation (5.38b) from (5.37a)

$$\varepsilon_a - \varepsilon_c = \frac{1}{2}(\varepsilon_1 - \varepsilon_2)\left(\frac{3}{2}\cos 2\theta + \frac{\sqrt{3}}{2}\sin 2\theta\right) \tag{5.41}$$

Dividing equation (5.41) by (5.40)

$$\frac{\varepsilon_a - \varepsilon_c}{\varepsilon_b - \varepsilon_c} = \frac{1}{2}\left(\frac{3}{2}\frac{\cot 2\theta}{\sqrt{3/2}} + 1\right)$$

or

$$\sqrt{3}\cot 2\theta = \frac{2(\varepsilon_a - \varepsilon_c)}{(\varepsilon_b - \varepsilon_c)} - \frac{(\varepsilon_b - \varepsilon_c)}{(\varepsilon_b - \varepsilon_c)}$$

or

$$\tan 2\theta = \frac{\sqrt{3}(\varepsilon_b - \varepsilon_c)}{(2\varepsilon_a - \varepsilon_b - \varepsilon_c)} \tag{5.42}$$

To determine ε_1 and ε_2 in terms of the measured strains, namely ε_a, ε_b and ε_c, put equation (5.42) in the form of the mathematical triangle of Figure 5.24.

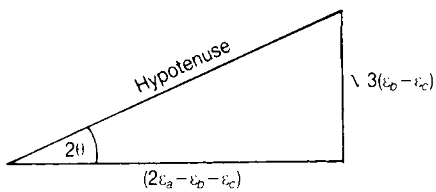


Figure 5.24 Mathematical triangle from Pythagoras' theorem.

Then,

$$\text{hypotenuse} = \sqrt{\left[\left(\varepsilon_a - \varepsilon_b\right)^2 + \left(\varepsilon_b - \varepsilon_c\right)^2 + \left(\varepsilon_a - \varepsilon_b\right)^2\right]}$$

Hence,

$$\cos 2\theta = \frac{\left(2\varepsilon_a - \varepsilon_b - \varepsilon_c\right)}{\text{hypotenuse}} \tag{5.43}$$

and

$$\sin 2\theta \; \frac{\sqrt{3}\left(\varepsilon_b - \varepsilon_c\right)}{\text{hypotenuse}} \tag{5.44}$$

Substituting equation (5.43) and (5.44) into equations (5.38a) and (5.38b), and solving the two simultaneous equations, we get

$$\varepsilon_1 = \frac{1}{3}\left(\varepsilon_a + \varepsilon_b + \varepsilon_c\right) + \frac{\sqrt{2}}{3}\sqrt{\left[\left(\varepsilon_a - \varepsilon_b\right)^2 + \left(\varepsilon_b - \varepsilon_c\right)^2 + \left(\varepsilon_a - \varepsilon_c\right)^2\right]} \tag{5.45}$$

and

$$\varepsilon_2 = \frac{1}{3}\left(\varepsilon_a + \varepsilon_b + \varepsilon_c\right) - \frac{\sqrt{2}}{3}\sqrt{\left[\left(\varepsilon_a - \varepsilon_b\right)^2 + \left(\varepsilon_b - \varepsilon_c\right)^2 + \left(\varepsilon_a - \varepsilon_c\right)^2\right]} \tag{5.46}$$

When the principal strains ε_1 and ε_2 have been estimated, the corresponding principal stresses are deduced from the relations

$$E\varepsilon_1 = \sigma_1 - \nu\sigma_2$$

$$E\varepsilon_2 = \sigma_2 - \nu\sigma_1$$

These give

$$\sigma_1 = \frac{E}{1 - \nu^2}\left(\varepsilon_1 + \nu\varepsilon_2\right)$$

$$\sigma_2 = \frac{E}{1 - \nu^2}\left(\varepsilon_2 + \nu\varepsilon_1\right) \qquad (5.47)$$

Equations (5.18) and (5.47) are for the *plane stress* condition, which is a two-dimensional system of stress, as discussed in Section 5.12.

Another two-dimensional system is known as a *plane strain* condition, which is a two-dimensional system of strain and a three-dimensional system of stress, as in Figure 5.25, where

$$\varepsilon_z = 0 = \frac{\sigma_z}{E} - \frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} \qquad (5.48a)$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \frac{\nu\sigma_x}{E} - \frac{\nu\sigma_z}{E} \qquad (5.48b)$$

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E} \qquad (5.48c)$$

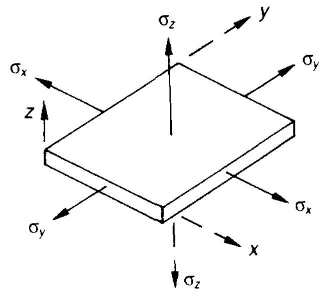


Figure 5.25 Plane strain condition.

From equation (5.48a)

$$\sigma_z = \nu\left(\sigma_x + \sigma_y\right) \qquad (5.49)$$

Substituting equation (5.49) into equations (5.48b) and (5.48c), we get,

$$\begin{aligned}\varepsilon_y &= \frac{\sigma_y}{E} - \frac{\nu\sigma_x}{E} - \frac{\nu^2}{E}\left(\sigma_x + \sigma_y\right) \\ &= \frac{\sigma_y}{E}\left(1 - \nu^2\right) - \frac{\nu\sigma_x}{E}\left(1 + \nu\right)\end{aligned} \tag{5.50a}$$

and

$$\begin{aligned}\varepsilon_x &= \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu^2\left(\sigma_x + \sigma_y\right)}{E} \\ &= \frac{\sigma_x}{E}\left(1 - \nu^2\right) - \frac{\nu\sigma_y}{E}\left(1 + \nu\right)\end{aligned} \tag{5.50b}$$

Multiplying equation (5.50a) by $(1 - \nu^2)/(1 + \nu)\,\nu$ we get

$$\frac{\left(1 - \nu^2\right)}{\left(1 + \nu\right)\nu}\varepsilon_y = \frac{\sigma_y\left(1 - \nu^2\right)^2}{E\left(1 + \nu\right)\nu} - \frac{\sigma_x}{E}\left(1 - \nu^2\right) \tag{5.51}$$

Adding equation (5.50b) to (5.51), we get

$$\frac{\left(1 - \nu^2\right)}{\left(1 + \nu\right)\nu}\varepsilon_y + \varepsilon_x = \frac{-\nu\sigma_y}{E}\left(1 + \nu\right) + \frac{\sigma_y\left(1 - \nu^2\right)^2}{E\left(1 + \nu\right)\nu}$$

or

$$\left(1 - \nu\right)\varepsilon_y + \nu\varepsilon_x = \frac{-\nu^2\left(1 + \nu\right)\sigma_y}{E} + \frac{\sigma_y\left(1 - \nu^2\right)^2}{E\left(1 + \nu\right)}$$

or

$$E\left[\left(1 - \nu\right)\varepsilon_y + \nu\varepsilon_x\right] = \frac{\sigma_y}{\left(1 + \nu\right)}\left[-\nu^2\left(1 + \nu\right)^2 + \left(1 - \nu^2\right)^2\right]$$

or

$$\begin{aligned}E\left[\left(1 - \nu\right)\varepsilon_y + \nu\varepsilon_x\right] &= \sigma_y\left[-\nu^2\left(1 + \nu\right) + \left(1 - \nu\right)\left(1 - \nu^2\right)\right] \\ &= \sigma_y\left[-\nu^2 - \nu^3 + 1 - \nu - \nu^2 + \nu^3\right] \\ &= \sigma_y\left(1 - \nu - 2\nu^2\right)\end{aligned} \tag{5.52a}$$

$$= \sigma_y (1 + \nu)(1 - 2\nu)$$

$$\therefore \sigma_y = \frac{E\left[(1 - \nu)\varepsilon_y + \nu\varepsilon_x\right]}{(1 + \nu)(1 - 2\nu)}$$

Similarly

$$\sigma_x = \frac{E\left[(1 - \nu)\varepsilon_x + \nu\varepsilon_y\right]}{(1 + \nu)(1 - 2\nu)} \qquad (5.52b)$$

Obviously the values of E and ν must be known before the stresses can be estimated from either equations (5.19), (5.47) or (5.52).

5.16 Strain energy for a two-dimensional stress system

If σ_1 and σ_2 are the principal stresses in a two-dimensional stress system, the corresponding principal strains for an elastic material are, from equations (5.21),

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2)$$

$$\varepsilon_2 = \frac{1}{E}(\sigma_2 - \nu\sigma_1)$$

Consider a cube of material having sides of unit length, and therefore having also unit volume. The edges parallel to the direction of σ_1 extend amounts ε_1, and those parallel to the direction of σ_2 by amounts ε_2. The work done by the stresses σ_1 and σ_2 during straining is then

$$W = \frac{1}{2}\sigma_1\varepsilon_1 + \frac{1}{2}\sigma_2\varepsilon_2$$

per unit volume of material. On substituting for ε_1 and ε_2 we have

$$W = \frac{1}{2}\sigma_1\left[\frac{1}{E}(\sigma_1 - \nu\sigma_2)\right] + \frac{1}{2}\sigma_2\left[\frac{1}{E}(\sigma_2 - \nu\sigma_1)\right]$$

This is equal to the strain energy U *per unit volume*; thus

$$U = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 - 2\nu\sigma_1\sigma_2\right] \qquad (5.53)$$

5.17 Three-dimensional stress systems

In any two-dimensional stress system we found there were two mutually perpendicular directions in which only direct stresses, σ_1 and σ_2, acted; these were called the principal stresses. In any three-dimensional stress system we can always find three mutually perpendicular directions in which only direct stresses, σ_1, σ_2 and σ_3 in Figure 5.26, are acting. No shearing stresses act on the faces of a rectangular block having its edges parallel to the axes 1, 2 and 3 in Figure 5.26. These direct stresses are again called *principal stresses*.

If $\sigma_1 > \sigma_2 > \sigma_3$, then the three-dimensional stress system can be represented in the form of Mohr's circles, as shown in Figure 5.27. Circle *a* passes through the points σ_1 and σ_2 on the σ-axis, and defines all states of stress on planes parallel to the axis 3, Figure 5.26, but inclined to axis 1 and axis 2, respectively.

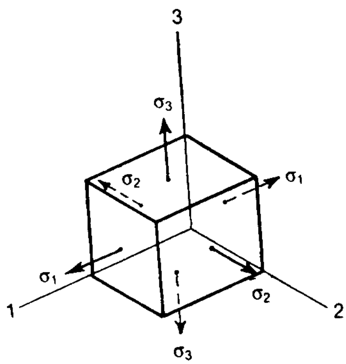


Figure 5.26 Principal stresses in a three-dimensional system.

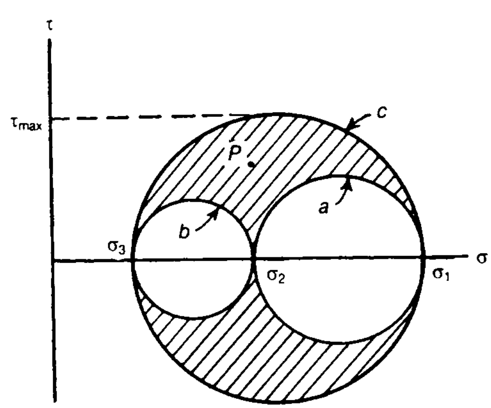


Figure 5.27 Mohr's circle of stress for a three-dimensional system; circle *a* is the Mohr's circle of the two-dimensional system σ_1, σ_2; *b* corresponds to σ_2, σ_3 and *c* to σ_3, σ_1. The resultant direct and tangential stress on any plane through the point must correspond to a point *P* lying on or between the three circles.

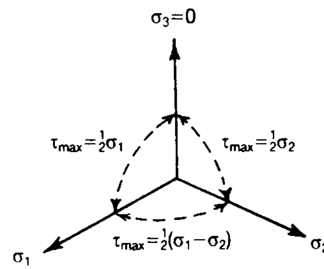


Figure 5.28 Two-dimensional stress system as a particular case of a three-dimensional system with one of the three principal stresses equal to zero.

Circle c, having a diameter $(\sigma_1 - \sigma_3)$, embraces the two smaller circles. For a plane inclined to all three axes the stresses are defined by a point such as P within the shaded area in Figure 5.27. The maximum shearing stress is

$$\tau_{max} = \frac{1}{2}\left(\sigma_1 - \sigma_3\right)$$

and occurs on a plane parallel to the axis 2.

From our discussion of three-dimensional stress systems we note that when one of the principal stresses, σ_3 say, is zero, Figure 5.28, we have a two-dimensional system of stresses σ_1, σ_2; the maximum shearing stresses in the planes 1–2, 2–3, 3–1 are, respectively,

$$\frac{1}{2}\left(\sigma_1 - \sigma_2\right), \ \frac{1}{2}\sigma_1, \ \frac{1}{2}\sigma_2$$

Suppose, initially, that σ_1 and σ_2 are both tensile and that $\sigma_1 > \sigma_2$; then the greatest of the three maximum shearing stresses is ½ σ_1 which occurs in the 2–3 plane. If, on the other hand, σ_1 is tensile and σ_2 is compressive, the greatest of the maximum shearing stresses is ½ $(\sigma_r - \sigma_2)$ and occurs in the 1–2 plane.

We conclude from this that the presence of a zero stress in a direction perpendicular to a two-dimensional stress system may have an important effect on the maximum shearing stresses in the material and cannot be disregarded therefore. The direct strains corresponding to σ_1, σ_2 and σ_3 for an elastic material are found by taking account of the Poisson ratio effects in the three directions; the principal strains in the directions 1, 2 and 3 are, respectively,

$$\varepsilon_1 = \frac{1}{E}\left(\sigma_1 - \nu\sigma_2 - \nu\sigma_3\right)$$

$$\varepsilon_2 = \frac{1}{E}\left(\sigma_2 - \nu\sigma_3 - \nu\sigma_1\right)$$

$$\varepsilon_3 = \frac{1}{E}\left(\sigma_3 - \nu\sigma_1 - \nu\sigma_2\right)$$

The strain energy stored per unit volume of the material is

$$U = \frac{1}{2}\sigma_1\varepsilon_1 + \frac{1}{2}\sigma_2\varepsilon_2 + \frac{1}{2}\sigma_3\varepsilon_3$$

In terms of σ_1, σ_2 and σ_3, this becomes

$$U = \frac{1}{2E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\sigma_1\sigma_2 - 2\nu\sigma\sigma_3 - 2\nu\sigma_3\sigma_1\right) \qquad (554)$$

5.18 Volumetric strain in a material under hydrostatic pressure

A material under the action of equal compressive stresses σ in three mutually perpendicular directions, Figure 5.29, is subjected to a *hydrostatic pressure*, σ. The term hydrostatic is used because the material is subjected to the same stresses as would occur if it were immersed in a fluid at a considerable depth.

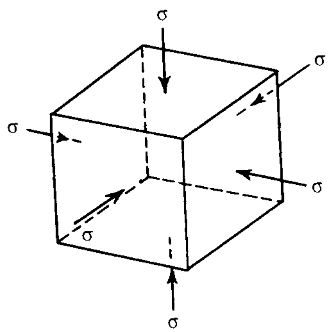


Figure 5.29 Region of a material under a hydrostatic pressure.

If the initial volume of the material is V_0, and if this diminishes an amount δV due to the hydrostatic pressure, the volumetric strain is

$$\frac{\delta V}{V_0}$$

The ratio of the hydrostatic pressure, σ, to the volumetric strain, $\delta V/V_0$, is called the *bulk modulus* of the material, and is denoted by K. Then

$$K = \frac{\sigma}{\left(\dfrac{\delta V}{V_0}\right)} \tag{5.55}$$

If the material remains elastic under hydrostatic pressure, the strain in each of the three mutually perpendicular directions is

$$\varepsilon = -\frac{\sigma}{E} + \frac{\nu\sigma}{E} + \frac{\nu\sigma}{E}$$

$$= -\frac{\sigma}{E}(1 - 2\nu)$$

because there are two Poisson ratio effects on the strain in any of the three directions. If we consider a cube of material having sides of unit length in the unstrained condition, the volume of the strained cube is

$$(1 - \varepsilon)^3$$

Now ε is small, so that this may be written approximately

$$1 - 3\varepsilon$$

The change in volume of a unit volume is then

$$3\varepsilon$$

which is therefore the volumetric strain. Then equation (5.55) gives the relationship

$$K = \frac{\sigma}{\left(\frac{\delta V}{V_0}\right)} = \frac{\sigma}{-3\varepsilon} = \frac{E}{3(1 - 2\nu)}$$

We should expect the volume of a material to diminish under a hydrostatic pressure. In general, if K is always positive, we must have

$$1 - 2\nu > 0$$

or

$$\nu < \frac{1}{2}$$

Then Poisson's ratio is *always less than ½*. For plastic strains of a metallic material there is a negligible change of volume, the Poisson's ratio is equal to ½, approximately.

5.19 Strain energy of distortion

In the three-dimensional stress system of Figure 5.22 we may consider the principal stress σ_1 to be the resultant of stresses

$$\frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

and stresses

$$\frac{1}{3}(2\sigma_1 - \sigma_2 - \sigma_3)$$

since

$$\frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) + \frac{1}{3}(2\sigma_1 - \sigma_2 - \sigma_3) = \sigma_1$$

Similarly, we write

$$\sigma_2 = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) + \frac{1}{3}(2\sigma_2 - \sigma_3 - \sigma_1)$$

$$\sigma_3 = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) + \frac{1}{3}(2\sigma_3 - \sigma_1 - \sigma_2)$$

Now, the component ⅓ $(\sigma_1 + \sigma_2 + \sigma_2)$ which occurs in σ_1, σ_2 and σ_3, represents a *hydrostatic* tensile stress; the strains associated with this stress give rise to no distortion, i.e., a cube of material under stress ⅓ $(\sigma_1 + \sigma_2 + \sigma_3)$ in three mutually perpendicular directions is strained into a cube. The remaining components of σ_1, σ_2 and σ_3, are

$$\frac{1}{3}(2\sigma_1 - \sigma_2 - \sigma_3), \qquad \frac{1}{3}(2\sigma_2 - \sigma_3 - \sigma_1), \qquad \frac{1}{3}(2\sigma_3 - \sigma_1 - \sigma_2)$$

The strain energy associated with these stresses, which are the only stresses giving rise to distortion, is called the *strain energy of distortion*. The strains due to these distorting stresses are

$$\varepsilon_1 = \frac{1}{3E}(1 + \nu)(2\sigma_1 - \sigma_2 - \sigma_3) = \frac{1}{6G}\left[(\sigma_1 - \sigma_2) + (\sigma_1 - \sigma_3)\right]$$

$$\varepsilon_2 = \frac{1}{3E}(1 + \nu)(2\sigma_2 - \sigma_3 - \sigma_1) = \frac{1}{6G}\left[(\sigma_2 - \sigma_3) + (\sigma_2 - \sigma_1)\right]$$

$$\varepsilon_3 = \frac{1}{3E}(1 + \nu)(2\sigma_3 - \sigma_1 - \sigma_2) = \frac{1}{6G}\left[(\sigma_3 - \sigma_1) + (\sigma_3 - \sigma_2)\right]$$

The strain energy of distortion is therefore

$$U_D = \frac{1}{36G}\left[(2\sigma_1 - \sigma_2 - \sigma_3)^2 + (2\sigma_2 - \sigma_3 - \sigma_1)^2 + (2\sigma_3 - \sigma_1 - \sigma_2)^2\right]$$

per unit volume. Then

$$U_D = \frac{1}{12G}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] \tag{5.56}$$

For a two-dimensional stress system, σ_3 (say) $= 0$, and U_D reduces to

$$U_D = \frac{1}{12G}\left[(\sigma_1 - \sigma_2)^2 + \sigma_2^2 + \sigma_1^2\right]$$

We shall see later that the strain energy of distortion plays an important part in the yielding of ductile materials under combined stresses.

5.20 Isotropic, orthotropic and anisotropic

A material is said to be *isotropic* when its material properties are the same in all directions. An *orthotropic* material is said to exhibit symmetric material properties about three mutually perpendicular planes. In two dimensions, typical orthotropic materials are in the form of many composites. An *anisotropic* material is a material that exhibits different material properties in all directions.

5.21 Fibre composites

Fibre composites are very important for structures which require a large strength:weight ratio, especially when the weight of the structure is at a premium. They are likely to become even more important in the 21st century and will probably revolutionise the design and construction of aircraft, rockets, submarines and warships.

To represent the elasticity of a composite, tensile modulus is used in preference to Young's modulus of elasticity. Additionally, as most composites are usually assumed to be of orthotropic form, their material properties in one direction, (say) 'x' are likely to be different to a direction perpendicular to the 'x' direction, (say) 'y'. Composites usually consist of several layers of fibre matting, set in a resin, as shown by Figure 5.30. To gain maximum strength the layers of fibre matting are laid in different directions. In this Chapter, the term *lamina* or ply will be used to describe a single layer of the composite structure and the term *laminate* or composite will be used to define the entire mixture of plies and resin.

If the material properties of the fibre composite are orthogonal, the following relationship applies:

$$\nu_x E_y = \nu_y E_x \tag{5.57}$$

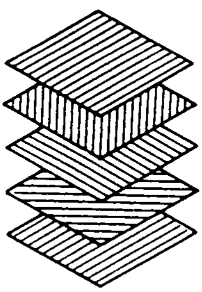


Figure 5.30 Five layers of fibre reinforcement.

where

E_x = tensile modulus in the x-direction.

E_y = tensile modulus in the y-direction.

ν_x = Poisson's ratio due to the effects of σ_x

ν_y = Poisson's ratio due to the effects of σ_y

σ_x = direct stress in the local x-direction.

σ_y = direct stress in the local y-direction. } see Figure 5.31

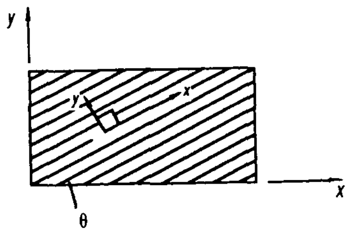


Figure 5.31 A lamina from a composite.

It is evident from the theory of Section 5.12 that the following relationships between stress and strain apply for orthotropic materials:

$$\varepsilon_x = \frac{\sigma_x}{E_x} - \frac{\nu_y \sigma_y}{E_y}$$

$$\varepsilon_y = \frac{\sigma_y}{E_y} - \frac{\nu_x \sigma_x}{E_x} \qquad (5.58)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G_{xy}} \qquad (5.59)$$

where

ε_x = direct strain in the x-direction

ε_y = direct strain in the y-direction

γ_{xy} = shear strain in the x–y plane

Solving equations (5.58), the following alternative relationship is obtained:

$$\sigma_x = \frac{E_x}{(1 - \nu_x \nu_y)} (\varepsilon_x + \nu_y \varepsilon_y)$$

$$\sigma_y = \frac{E_y}{(1 - \nu_x \nu_y)} (\varepsilon_y + \nu_x \varepsilon_x) \tag{5.60}$$

In matrix form, equations (5.58) and (5.59) can be written as

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{21} & S_{22} & S_{26} \\ S_{61} & S_{62} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \tag{5.61}$$

or $$\{\varepsilon_{xy}\} = [S] \{\sigma_{xy}\} \tag{5.62}$$

where $[S]$ is the *compliance* matrix.

From equations (5.59) and (5.60)

$$\{\sigma_{xy}\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{21} & Q_{22} & Q_{26} \\ Q_{61} & Q_{62} & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{5.63}$$

where

$$Q_{11} = \frac{E_x}{(1 - \nu_x \nu_y)}$$

$$Q_{22} = \frac{E_y}{(1 - \nu_x \nu_y)}$$

$$Q_{66} = G_{xy} = \text{shear modulus}$$

$$Q_{12} = Q_{21} = \frac{\nu_x E_y}{(1 - \nu_x \nu_y)} = \frac{\nu_y E_x}{(1 - \nu_x \nu_y)} \qquad (5.64)$$

$$Q_{16} = Q_{61} = Q_{26} = Q_{62} = 0$$

$$Q_{66} = G_{xy}$$

or $\{\sigma_{xy}\} = [Q]\{\varepsilon_{xy}\} = [S^{-1}]\{\varepsilon_{xy}\}$

[Q] = the *stiffness* matrix

= the inverse of [S]

The problem with the above relationships are that they are all in the local co-ordinate system of the lamina, namely x and y. However, as each layer of fibres may have a different direction for its local co-ordinate system, it will be necessary to refer all relationships to a fixed global system, namely, X and Y, as shown by Figure 5.31.

Now from equations (5.4) and (5.5)

$$\sigma_x = \sigma_X \cos^2\theta + \sigma_Y \sin^2\theta + 2\tau_{XY} \sin\theta\cos\theta$$

$$\sigma_y = \sigma_{X+90^\circ} = \sigma_X \sin^2\theta + \sigma_Y \cos^2\theta - 2\tau_{XY} \sin\theta\cos\theta \qquad (5.65)$$

$$\tau_{xy} = -\sigma_X \sin\theta\cos\theta + \sigma_Y \sin\theta\cos\theta + \tau_{XY}\left(\cos^2\theta - \sin^2\theta\right)$$

where σ_x, σ_y and τ_{xy} are local stresses and σ_X, σ_Y and τ_{XY} are global or reference stresses; in matrix form equations (5.65) appear as:

$$\begin{Bmatrix}\sigma_x \\ \sigma_y \\ \tau_{xy}\end{Bmatrix} = \begin{bmatrix} C^2 & S^2 & 2SC \\ S^2 & C^2 & -2SC \\ -SC & SC & (C^2 - S^2)\end{bmatrix} \begin{Bmatrix}\sigma_X \\ \sigma_Y \\ \tau_{XY}\end{Bmatrix} \tag{5.66}$$

where S = sin θ and C = cos θ

$$\{\sigma_{xy}\} = [DC]\,\{\sigma_{XY}\} \tag{5.67}$$

and

$$\{\sigma_{XY}\} = [DC]^{-1}\,\{\sigma_{xy}\} \tag{5.68}$$

where

$$[DC] = \begin{bmatrix} C^2 & S^2 & 2SC \\ S^2 & C^2 & -2SC \\ -SC & SC & \left(C^2 - S^2\right)\end{bmatrix} and \left[DC^{-1}\right] = \begin{bmatrix} C^2 & S^2 & -2SC \\ S^2 & C^2 & 2SC \\ SC & -SC & \left(C^2 - S^2\right)\end{bmatrix} \tag{5.69}$$

Similarly from Section (5.10)

$$\varepsilon_x = \varepsilon_X \cos^2\theta + \varepsilon_Y \sin^2\theta + \gamma_{XY} \sin\theta \cos\theta$$

$$\varepsilon_y = \varepsilon_X \sin^2\theta + \varepsilon_Y \cos^2\theta - \gamma_{XY} \sin\theta \cos\theta \tag{5.70}$$

$$\gamma_{xy} = -2\varepsilon_X \sin\theta \cos\theta + 2\varepsilon_Y \sin\theta \cos\theta + \gamma_{XY}\left(\cos^2\theta - \sin^2\theta\right)$$

or, in matrix form,

$$\begin{Bmatrix}\varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}\end{Bmatrix} = \begin{bmatrix} C^2 & S^2 & SC \\ S^2 & C^2 & -SC \\ -2SC & 2SC & (C^2 - S^2)\end{bmatrix} \begin{Bmatrix}\varepsilon_X \\ \varepsilon_Y \\ \gamma_{XY}\end{Bmatrix} \tag{5.71}$$

or $$\{\varepsilon_{xy}\} = [DC_1]\,\{\varepsilon_{XY}\} \tag{5.72}$$

Now from equation (5.63),

$$\{\sigma_{xy}\} = [Q]\{\varepsilon_{xy}\}$$

but from equation (5.72),

$$\{\sigma_{xy}\} = [Q][DC_1]\{\varepsilon_{XY}\}$$

but from equation (5.67),

$$\{\sigma_{xy}\} = [DC]\{\sigma_{XY}\}$$

$$\therefore [DC]\{\sigma_{XY}\} = [Q][DC_1]\{\varepsilon_{XY}\}$$

or

$$\{\sigma_{XY}\} = [DC]^{-1}[Q][DC_1]\{\varepsilon_{XY}\} \tag{5.73}$$

or

$$\{\sigma_{XY}\} = [Q^1]\{\varepsilon_{XY}\} \tag{5.74}$$

where

$$[Q^1] = \begin{bmatrix} q_{11}^1 & q_{12}^1 & q_{16}^1 \\ q_{21}^1 & q_{22}^1 & q_{26}^1 \\ q_{61}^1 & q_{62}^1 & q_{66}^1 \end{bmatrix}$$

$$= [DC]^{-1}[Q][DC_1]$$

$$q_{11}^1 = \frac{1}{\gamma}\left[E_x \cos^4\theta + E_y \sin^4\theta + (2\nu_x E_y + 4\gamma G)\cos^2\theta\sin^2\theta\right]$$

$$q_{12}^1 = q_{21}^1 = \frac{1}{\gamma}\left[\nu_x E_y (\cos^4\theta + \sin^4\theta) + (E_x + E_y - 4\gamma G)\cos^2\theta\sin^2\theta\right]$$

$$q^{1}_{16} = q^{1}_{61} = \frac{1}{\gamma}\left[\cos^3\theta\ \sin\theta\ (E_x - \nu_x E_y - 2\gamma G) - \cos\theta\ \sin^3\theta\ (E_y - \nu_x E_y - 2\gamma G)\right]$$

$$q^{1}_{22} = \frac{1}{\gamma}\left[E_y \cos^4\theta + E_x \sin^4\theta + \sin^2\theta\ \cos^2\theta\ (2\nu_x E_y + 4\gamma G)\right]$$

$$q^{1}_{26} = q^{1}_{62} = \frac{1}{\gamma}\left[\cos\theta\ \sin^3\theta\ (E_x - \nu_x E_y - 2\gamma G) - \cos^3\theta\ \sin\theta\ (E_y - \nu_x E_y - 2\gamma G)\right]$$

$$q^{1}_{66} = \frac{1}{\gamma}\left[\sin^2\theta\ \cos^2\theta\ (E_x + E_y - 2\nu_x E_y - 2\gamma G) + \gamma G\ (\cos^4\theta + \sin^4\theta)\right]$$

where

$$\gamma = (1 - \nu_x \nu_y)$$

Similarly, to obtain the global strains of the lamina or ply of Figure 5.32 in terms of the global stresses, consider equation (5.61), as follows.

Now

$$\{\varepsilon_{xy}\} = [S]\ \{\sigma_{xy}\}$$

so that from equation (5.67)

$$\{\varepsilon_{xy}\} = [S]\ [DC]\ \{\sigma_{XY}\}$$

and from equation (5.72)

$$[DC_1]\ \{\varepsilon_{XY}\} = [S]\ [DC]\ \{\sigma_{XY}\}$$

$$\text{or } \{\varepsilon_{XY}\} = [DC_1]^{-1}\ [S]\ [DC]\ \{\sigma_{XY}\} \tag{5.75}$$

$$\text{or } \{\varepsilon_{XY}\} = [S^1]\ \{\sigma_{XY}\}$$

where

$$[S^1] = \begin{bmatrix} S^1_{11} & S^1_{12} & S^1_{16} \\ S^1_{21} & S^1_{22} & S^1_{26} \\ S^1_{61} & S^1_{62} & S^1_{66} \end{bmatrix} = [DC_1]^{-1}\,[S]\,[DC]$$

$$S^1_{11} = S_{11}\cos^4\theta + S_{22}\sin^4\theta + (2S_{12} + S_{66})\cos^2\theta\sin^2\theta$$

$$S^1_{12} = S^1_{21} = (S_{11} + S_{22} - S_{66})\cos^2\theta\sin^2\theta + S_{12}(\cos^4\theta\sin^4\theta)$$

$$S^1_{16} = S^1_{61} = (2S_{22} - 2S_{12} - S_{66})\cos^3\theta\sin\theta - (2S_{22} - 2S_{12} - S_{66})\sin^3\theta\cos\theta$$

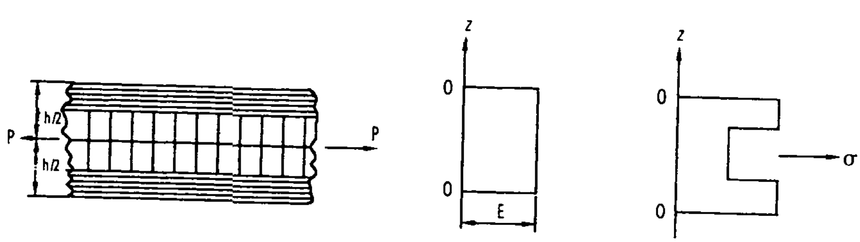
$$S^1_{22} = S_{11}\sin^4\theta + S_{22}\cos^4\theta + (2S_{12} + S_{66})\cos^2\theta\sin^2\theta$$

$$S^1_{26} = S^1_{62} = (2S_{11} - 2S_{12} - S_{66})\cos\theta\sin^3\theta - (2S_{22} - 2S_{12} - S_{66})\sin\theta\cos^3\theta$$

$$S^1_{66} = 4\,(S_{11} - 2S_{12} + S_{66})\cos^2\theta\sin^2\theta + S_{66}\,(\cos^2\theta - \sin^2\theta)^2$$

5.22 In-plane equations for a symmetric laminate or composite

Consider a section of the symmetric laminate of Figure 5.33(a), which is under in-plane loading.



(i) Section through the laminate (ii) strain distribution (iii) stress distribution

Figure 5.33 In-plane stresses and strains in a laminate.

As the load P is in-plane and symmetrical, the strain distribution across the laminate will be constant, as shown by Figure 5.33(ii). However, as the stiffness of each layer is different the stresses in each layer will be different, as shown by Figure 5.33(iii). Now, in order to define the overall equivalent stress-strain behaviour of a laminate, it will be necessary to adopt the equivalent average stresses or in matrix form σ_X^1, σ_Y^1 and τ_{XY}^1; these are obtained as follows:

$$\sigma_X^1 = \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_X dz\,, \quad \sigma_Y^1 = \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_Y dz\,, \quad \tau_{XY}^1 = \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{XY} dz$$

or in matrix form

$$\begin{Bmatrix} \sigma_X^1 \\ \sigma_Y^1 \\ \tau_{XY}^1 \end{Bmatrix} = \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_X \\ \sigma_Y \\ \tau_{XY} \end{Bmatrix} dz$$

but from equation (5.74)

$$\begin{Bmatrix} \sigma_X^1 \\ \sigma_Y^1 \\ \tau_{XY}^1 \end{Bmatrix} = [Q^1] \begin{Bmatrix} \varepsilon_X \\ \varepsilon_Y \\ \gamma_{XY} \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} \sigma_X^1 \\ \sigma_Y^1 \\ \tau_{XY}^1 \end{Bmatrix} = \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} q_{11}^1 & q_{12}^1 & q_{16}^1 \\ q_{21}^1 & q_{22}^1 & q_{26}^1 \\ q_{62}^1 & q_{61}^1 & q_{66}^1 \end{bmatrix} \begin{Bmatrix} \varepsilon_X \\ \varepsilon_Y \\ \gamma_{XY} \end{Bmatrix} dz \tag{5.76}$$

However, as $[\varepsilon_X\, \varepsilon_Y\, \gamma_{XY}]^T$ is not a function of 'z', equation (5.76) can be written as follows:

$$\begin{Bmatrix} \sigma_X^1 \\ \sigma_Y^1 \\ \tau_{XY}^1 \end{Bmatrix} = \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[Q^1\right] dz \begin{Bmatrix} \varepsilon_X \\ \varepsilon_Y \\ \gamma_{XY} \end{Bmatrix} = [A] \begin{Bmatrix} \varepsilon_X \\ \varepsilon_Y \\ \gamma_{XY} \end{Bmatrix} \tag{5.77}$$

where

$$\begin{aligned} A_{11} &= \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} q_{11}^1\, dz \\ &= \frac{2}{h} \int_{o}^{\frac{h}{2}} q_{11}^1\, dx \\ A_{12} &= \frac{2}{h} \int_{o}^{\frac{h}{2}} q_{12}^1\, dz \end{aligned} \tag{5.78}$$

or, in general,

$$A_{ij} = \frac{2}{h} \int_{o}^{\frac{h}{2}} q_{ij}^1\, dz$$

For the kth lamina of the laminate, the q^1 terms are constant, hence the integrals for the A terms can be replaced by summations:

$$A_{11} = \frac{2}{h}\sum q^1_{11(k)}\ h_k = \sum q^1_{11(k)}\left(\frac{2h_k}{h}\right) \tag{5.79}$$

and similarly for the other values of A_{ij},

where

h_k = thickness of the kth lamina or ply

$q_{11(k)}$ = kth value of q_{11}^1

v_k = $(2h_k/h)$ = the volume fraction in the kth lamina

Once the *stiffness* matrix $[A]$ is obtained, it can be inverted to obtain the *compliance* matrix $[a]$ and hence, the equivalent material for the laminate properties are as follows:

$$E_X = \frac{1}{a_{11}}, \quad E_Y = \frac{1}{a_{22}}, \quad G_{XY} = \frac{1}{a_{66}}, \quad \nu_X = \frac{-a_{12}}{a_{11}} \text{ and } \nu_Y = \frac{-a_{12}}{a_{22}}$$

Experience has shown that the diagonal terms in the laminate's stiffness matrix are considerably larger than the off-diagonal terms, so that E_x etc. can be approximated by

$$E_X \doteq \sum v_k\ E_{X(k)}\ \cos^4\theta_k$$

where k refers to the kth lamina of the laminate.

5.23 Equivalent elastic constants for problems involving bending and twisting

For problems in this category, the equivalent stress resultants for the laminate are σ_X^1, σ_Y^1, τ_{XY}^1, M_X^1, M_Y^1 and M_{XY}^1, where the former three symbols are in-plane and the latter three are out-of-plane bending and twisting terms.

The equivalent stress–strain relationships for the laminate are:

$$\begin{Bmatrix} \sigma_X^1 \\ \sigma_Y^1 \\ \tau_{XY}^1 \\ M_X^1 \\ M_Y^1 \\ M_{XY}^1 \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon_X \\ \varepsilon_Y \\ \gamma_{XY} \\ \dfrac{-\partial^2 w}{\partial X^2} \\ \dfrac{-\partial^2 w}{\partial Y^2} \\ \dfrac{-2\partial^2 w}{\partial X \partial Y} \end{Bmatrix} \tag{5.80}$$

or $$\begin{Bmatrix} \sigma \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon \\ \chi \end{Bmatrix}$$

where $[\varepsilon]^T = [\varepsilon_X \ \ \varepsilon_Y \ \ \gamma_{XY}]^T$

$$[\chi]^T = \left[\frac{-\partial^2 w}{\partial X^2} \ \ \frac{-\partial^2 w}{\partial Y^2} \ \ \frac{-2\partial^2 w}{\partial \times \partial Y} \right]^T$$

A_{ij} are as described in Section 5.21.

$$B_{ij} = \frac{1}{h^2} \int_{\frac{-h}{2}}^{\frac{h}{2}} q_{ij}^1 \, z \, . \, dz = \sum_{k=1}^{n} q_{ij(k)}^1 h_k \, z_k$$

$$D_{ij} = \frac{1}{h^3} \int_{\frac{-h}{2}}^{\frac{h}{2}} q_{ij}^1 \, z^2 \, dz$$

$$D_{ij} = \frac{1}{h^3} \sum_{k=1}^{n} q_{ij(k)}^1 \left(h_k \, z_k^2 + h_k^3/12 \right) \tag{5.81}$$

where

w = out-of-plane deflection

n = number of laminates or plies

k = the kth ply or lamina

z_k = distance of the centre plane of the kth ply

For symmetrical laminates, $B_{ij} = 0$, however, for design purposes, the following relationship is often used:

$$\begin{Bmatrix} \varepsilon \\ M \end{Bmatrix} = \begin{bmatrix} a & b_1 \\ b_2 & d \end{bmatrix} \begin{Bmatrix} \sigma \\ \chi \end{Bmatrix}$$

where

$[a] = [A]^{-1}$ (see Section 5.21)

$[b_1] = -[A]^{-1}\,[B]$

$[b_2] = [B]\,[A]^{-1}$

$[d] = [D] - [B]\,[A]^{-1}\,[B]$

Another way of looking at the components of [D] are as follows:

$$D_{ij} = \sum_{k=1}^{n} q^{1}_{ij(k)} \times \left(\frac{I_k}{I_{comp}}\right) \tag{5.82}$$

where k = the kth ply

I_k = the second moment of area of the kth ply or lamina about the neutral axis of the laminate or composite

I_{comp} = the second moment of area of the entire laminate or composite about the neutral axis

5.24 Yielding of ductile materials under combined stresses

It was noted in Section 5.3 that when a polished bar of mild steel is loaded in tension, strain figures are observable on the surface of the bar after the yield point has been exceeded. The figures take the form of 'lines' inclined at about 45° to the axis of the bar; this direction corresponds to the planes of maximum shearing stress in the bar; the 'lines' are, in fact, bands of metal crystals shearing over similar bands. That yielding takes place in this way suggests that the crystal structure of the metal is relatively weak in shear; yielding takes the form of sliding of one crystal plane over another, and not the tearing apart of two crystal planes.

This form of behaviour—yielding by a shearing action—is typical of ductile materials. We note firstly that if a material is subjected to a hydrostatic pressure σ, the three principal stresses σ_1, σ_2 and σ_3 in a three-dimensional system are each equal to σ. A state of stress of this sort exists in a solid sphere of material subjected to an external pressure σ, Figure 5.34. As the three principal stresses are equal in magnitude, there are no shearing stresses in the material; if yielding is governed by the presence of shearing on some planes in a material, then no yielding is theoretically possible when the material is under hydrostatic pressure.

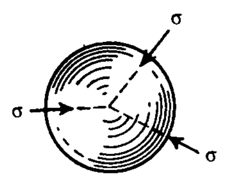


Figure 5.34 A solid sphere of material under hydrostatic pressure.

For a two-dimensional stress system one of the three principal stresses of a three-dimensional system is zero. We consider now the yielding of a mild steel under different combinations of the principal stresses, σ_1 and σ_2, of a two-dimensional system; in discussing the problem we keep in mind the presence of a zero stress perpendicular to the plane of σ_1 and σ_2, Figure 5.27.

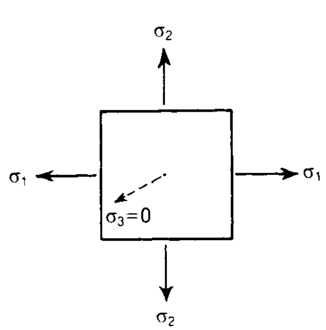


Figure 5.35 Yield envelope of a two dimensional stress system when the material yields according to the maximum shearing stress criterion.

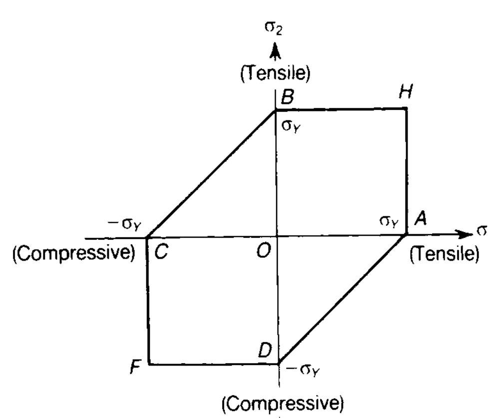


Figure 5.36 In a two-dimensional stress system, one of the three principal stresses - (σ_3 say) is zero.

Suppose we conduct a simple tension test on the material; we may put $\sigma_2 = 0$, and yielding occurs when $\sigma_1 = \sigma_Y$, (say)

This yielding condition corresponds to the point *A* in Figure 5.35. If the material has similar properties in tension and compression, yielding under a compressive stress σ_1 occurs when $\sigma_1 = -\sigma_Y$; this condition corresponds to the point *C* in Figure 5.35. We could, however, perform the tension and compression tests in the direction of σ_2, Figure 5.35; if the material is isotropic —that is, it has the same properties in all directions—yielding occurs at the yield stress σ_Y; we can thus derive points *B* and *D* in the yield diagram, Figure 5.35.

We consider now yielding of the material when both σ_1 and σ_2, Figure 5.36, are present; we shall assume that yielding of the mild steel occurs when the maximum shearing stress attains a critical value; from the simple tensile test, the maximum shearing stress at yielding is

$$\tau_{max} = \frac{1}{2}\sigma_Y$$

which we shall take as the critical value. Suppose that $\sigma_1 > \sigma_2$, and that both principal stresses are tensile; the maximum shearing stress is

$$\tau_{max} = \frac{1}{2}(\sigma_1 - 0) = \frac{1}{2}\sigma_1$$

and occurs in the 3–1 plane of Figure 5.36; τ_{max} attains the critical value when

$$\frac{1}{2}\sigma_1 = \frac{1}{2}\sigma_Y, \quad \text{or} \quad \sigma_1 = \sigma_Y$$

Thus, yielding for these stress conditions is unaffected by σ_2. In Figure 5.35, these stress conditions are given by the line *AH*. If we consider similarly the case when σ_1 and σ_2 are both tensile, but $\sigma_2 > \sigma_1$, yielding occurs when $\sigma_2 = \sigma_Y$, giving the line *BH* in Figure 5.35.

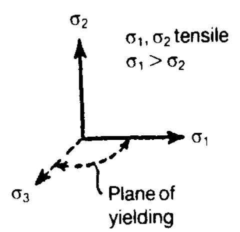


Figure 5.37 Plane of yielding when both principal stresses tensile and $\sigma_1 > \sigma_2$.

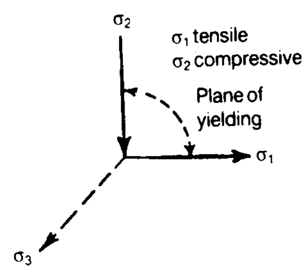


Figure 5.38 Plane of yielding when the principal stresses are of opposite sign.

By making the stresses both compressive, we can derive in a similar fashion the lines *CF* and *DF* of Figure 5.36.

But when σ_1 is tensile and σ_2 is compressive, Figure 5.36, the maximum shearing stress occurs in the 1–2 plane, and has the value

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_2)$$

Yielding occurs when

$$\frac{1}{2}(\sigma_1 - \sigma_2) = \frac{1}{2}\sigma_Y, \quad \text{or} \quad \sigma_1 - \sigma_2 = \sigma_Y$$

This corresponds to the line *AD* of Figure 5.36. Similarly, when σ_1 is compressive and σ_2 is tensile, yielding occurs when corresponding to the line *BC* of Figure 5.36.

$$\sigma_2 - \sigma_1 = \sigma_Y$$

The hexagon *AHBCFD* of Figure 5.36 is called a *yield locus*, because it defines all combinations of σ_1 and σ_2 giving yielding of mild steel; for any state of stress within the hexagon the material remains elastic; for this reason the hexagon is also sometimes called a *yield envelope*. The *criterion of yielding* used in the derivation of the hexagon of Figure 5.36 was that of maximum shearing stress; the use of this criterion was first suggested by Tresca in 1878.

Not all ductile metals obey the maximum shearing stress criterion; the yielding of some metals, including certain steels and alloys of aluminium, is governed by a critical value of the strain energy of distortion. For a two-dimensional stress system the strain energy of distortion per unit of volume of the material is given by equation (5.83). In the simple tension test for which $\sigma_2 = 0$, say, yielding occurs when $\sigma_1 = \sigma_Y$. The critical value of U_D is therefore

$$U_D = \frac{1}{6G}\left[\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2\right] = \frac{1}{6G}\left[\sigma_Y^2 - \sigma_Y(0) + 0^2\right] = \frac{\sigma_Y^2}{6G}$$

Then for other combinations of σ_1 and σ_2, yielding occurs when

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_Y^2 \qquad (5.83)$$

The yield locus given by this equation is an ellipse with major and minor axes inclined at 45° to the directions of σ_1 and σ_2, Figure 5.39. This locus was first suggested by von Mises in 1913.

For a three-dimensional system the yield locus corresponding to the strain energy of distortion is of the form

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = \text{constant}$$

This relation defines the surface of a cylinder of circular cross-section, with its central axis on the line $\sigma_1 = \sigma_2 = \sigma_3$; the axis of the cylinder passes through the origin of the σ_1, σ_2, σ_3 co-ordinate system, and is inclined at equal angles to the axes σ_1, σ_2 and σ_3, Figure 5.40. When σ_3 is zero, critical values of σ_1 and σ_2 lie on an ellipse in the σ_1–σ_2 plane, corresponding to the ellipse of Figure 5.39.

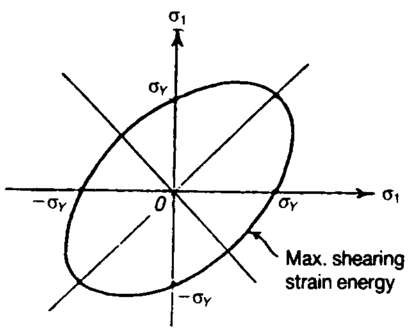


Figure 5.39 The von Mises yield locus for a two-dimensional system of stresses.

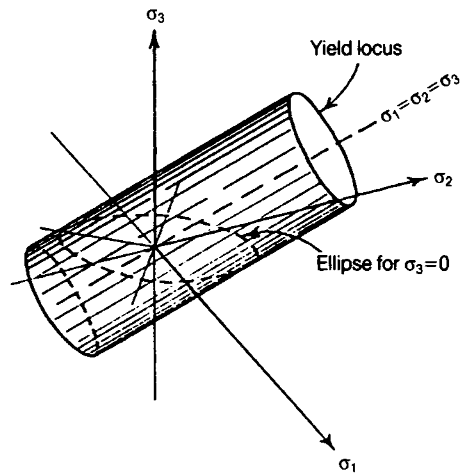


Figure 5.40 The von Mises yield locus for a three-dimensional stress system.

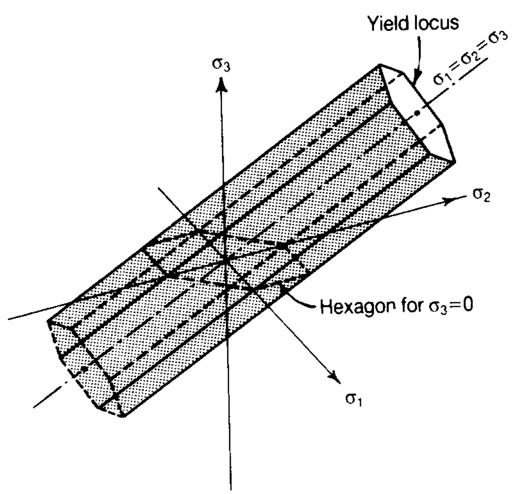


Figure 5.41 The maximum shearing stress (or Tresca) yield locus for a three-dimensional stress system.

When a material obeys the maximum shearing stress criterion, the three-dimensional yield locus is a regular hexagonal cylinder with its central axis on the line $\sigma_1 = \sigma_2 = \sigma_3 = 0$, Figure 5.40. When σ_3 is zero, the locus is an irregular hexagon, of the form already discussed in Figure 5.36.

The surfaces of the yield loci in Figures 5.40 and 5.41 extend indefinitely parallel to the line $\sigma_1 = \sigma_2 = \sigma_3$, which we call the hydrostatic stress line. Hydrostatic stress itself cannot cause yielding, and no yielding occurs at other stresses provided these fall within the cylinders of Figures 5.40 and 5.41.

The problem with the maximum principal stress and maximum principal strain theories is that they break down in the hydrostatic stress case; this is because under hydrostatic stress, failure does not occur as there is no shear stress. It must be pointed out that under uniaxial tensile stress, all the major theories give the same predictions for elastic failure, hence, all apply in the uniaxial case. However, in the case of a ductile specimen under pure torsion, the maximum shear stress theory predicts that yield occurs when the maximum shear reaches 0.5 σ_Y, but in practice, yield occurs when the maximum shear stress reaches 0.577 of the yield stress. This last condition is only satisfied by the von Mises or distortion energy theory and for this reason, this theory is currently very much in favour for ductile materials.

Another interpretation of the von Mises or distortion energy theory is that yield occurs when the von Mises stress, namely $\sigma_{\upsilon m}$, reaches yield.

In *three dimensions*, $\sigma_{\upsilon m}$ is calculated as follows:

$$\underline{\sigma_{\upsilon m}} = \underline{\sqrt{\left[(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3^2)\right]}/\sqrt{2}} \tag{5.84}$$

In *two-dimensions*, $\sigma_3 = 0$, therefore equation (5.84) becomes:

$$\underline{\sigma_{\upsilon m}} = \underline{\sqrt{\left(\sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2\right)}} \tag{5.85}$$

5.25 Elastic breakdown and failure of brittle material

Unlike ductile materials the failure of brittle materials occurs at relatively low strains, and there is little, or no, permanent yielding on the planes of maximum shearing stress.

Some brittle materials, such as cast iron and concrete, contain large numbers of holes and microscopic cracks in their structures. These are believed to give rise to high stress concentrations, thereby causing local failure of the material. These stress concentrations are likely to have a greater effect in reducing tensile strength than compressive strength; a general characteristic of brittle materials is that they are relatively weak in tension. For this reason elastic breakdown and failure in a brittle material are governed largely by the maximum principal tensile stress; as an example of the application of this criterion consider a concrete: in simple tension the breaking stress is about 1.5 MN/m^2, whereas in compression it is found to be about 30 MN/m^2, or 20 times as great; in pure shear the breaking stress would be of the order of 1.5 MN/m^2, because the principal stresses are of the same magnitude, and one of these stresses is tensile, Figure 5.42. Cracking in the concrete would occur on planes inclined at 45° to the directions of the applied shearing stresses.

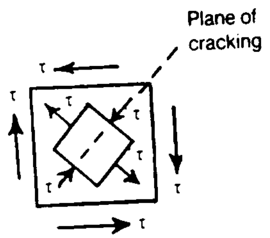


Figure 5.42 Elastic breakdown of a brittle metal under shearing stresses (pure shear).

5.26 Failure of composites

Accurate prediction of the failure of laminates is a much more difficult task than it is for steels and aluminium alloys. The failure load of the laminate is also dependent on whether the laminate is under in-plane loading, or bending or shear. Additionally, under compression, individual plies can buckle through a microscopic form of beam-column buckling (see Chapter 18). In general, it is better to depend on experimental data than purely on theories of elastic failure. Theories, however, exist and Hill, Azzi and Tsai produced theories based on the von Mises theory of yield. One such popular two-dimensional theory is the Azzi–Tsai theory, as follows:

$$\frac{\sigma_x^2}{X^2} + \frac{\sigma_y^2}{Y^2} - \frac{\sigma_x \sigma_y}{X^2} + \frac{\tau_{xy}^2}{S^2} = 1 \qquad (5.86)$$

where X and Y are the uniaxial strengths related to σ_x and σ_y respectively and S is the shear strength in the x–y directions, which are not principal planes.

For the isotropic case, where $X = Y = \sigma_Y$ and $S = \sigma_Y / \sqrt{3}$, equation (5.86) reduces to the von Mises form:

$$\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2 = \sigma_Y^2$$

and when $\sigma_x = \sigma_1$ and $\sigma_y = \sigma_2$ so that $\tau_{1-2} = 0$, we get

$$\sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2 = \sigma_Y^2 \quad \text{[See equation (5.85)]}$$

Further problems *(answers on page 692)*

5.7 A tie-bar of steel has a cross-section 15 cm by 2 cm, and carries a tensile load of 200 kN. Find the stress normal to a plane making an angle of 30° with the cross-section and the shearing stress on this plane. *(Cambridge)*

5.8 A rivet is under the action of shearing stress of 60 MN/m^2 and a tensile stress, due to contraction, of 45 MN/m^2. Determine the magnitude and direction of the greatest tensile and shearing stresses in the rivet. *(RNEC)*

5.9 A propeller shaft is subjected to an end thrust producing a stress of 90 MN/m^2, and the maximum shearing stress arising from torsion is 60 MN/m^2. Calculate the magnitudes of the principal stresses. *(Cambridge)*

5.10 At a point in a vertical cross-section of a beam there is a resultant stress of 75 MN/m^2, which is inclined upwards at 35° to the horizontal. On the horizontal plane through the point there is only shearing stress. Find in magnitude and direction, the resultant stress on the plane which is inclined at 40° to the vertical and 95° to the resultant stress. *(Cambridge)*

5.11 A plate is subjected to two mutually perpendicular stresses, one compressive of 45 MN/m^2, the other tensile of 75 MN/m^2, and a shearing stress, parallel to these directions, of 45 MN/m^2. Find the principal stresses and strains, taking Poisson's ratio as 0.3 and $E = 200\ GN/m^2$. *(Cambridge)*

5.12 At a point in a material the three principal stresses acting in directions O_x, O_y, O_z, have the values 75, 0 and −45 MN/m^2, respectively. Determine the normal and shearing stresses for a plane perpendicular to the xz-plane inclined at 30° to the xy-plane. *(Cambridge)*