19 Lateral deflections of circular plates

19.1 Introduction

In this chapter, consideration will be made of three classes of plate problem, namely

(i) small deflections of plates, where the maximum deflection does not exceed half the plate thickness, and the deflections are mainly due to the effects of flexure;

(ii) large deflections of plates, where the maximum deflection exceeds half the plate thickness, and membrane effects become significant; and

(iii) very thick plates, where shear deflections are significant.

Plates take many and various forms from circular plates to rectangular ones, and from plates on ships' decks to ones of arbitrary shape with cut-outs etc; however, in this chapter, considerations will be made mostly of the small deflections of circular plates.

19.2 Plate differential equation, based on small deflection elastic theory

Let, w be the out-of-plane deflection at any radius r, so that,

$$\frac{dw}{dr} = \theta$$

and

$$\frac{d^2w}{dr^2} = \frac{d\theta}{dr}$$

Also let

R_t = tangential or circumferential radius of curvature at r = AC (see Figure 19.1).

R_r = radial or meridional radius of curvature at r = BC.

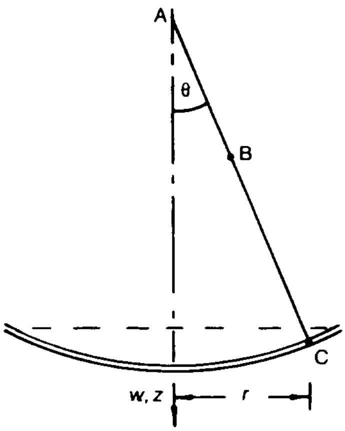

Figure 19.1 Deflected form of a circular plate.

From standard small deflection theory of beams (see Chapter 13) it is evident that

$$R_r = 1 \Big/ \frac{d^2w}{dr^2} = 1 \Big/ \frac{d\theta}{dr} \tag{19.1}$$

or

$$\frac{1}{R_r} = \frac{d\theta}{dr} \tag{19.2}$$

From Figure 19.1 it can be seen that

$$R_t = AC = r/\theta \tag{19.3}$$

or

$$\frac{1}{R_t} = \frac{1}{r}\frac{dw}{dr} = \frac{\theta}{r} \tag{19.4}$$

Let z = the distance of any fibre on the plate from its neutral axis, so that

$$\varepsilon_r = \text{radial strain} = \frac{z}{R_r} = \frac{1}{E}\left(\sigma_r - \nu\sigma_t\right) \tag{19.5}$$

and

$$\varepsilon_t = \text{circumferential strain} = \frac{z}{R_t} = \frac{1}{E}\left(\sigma_t - \nu\sigma_r\right) \tag{19.6}$$

From equations (19.1) to (19.6) it can be shown that

$$\sigma_r = \frac{Ez}{\left(1-\nu^2\right)}\left(\frac{1}{R_r}+\frac{\nu}{R_t}\right) = \frac{Ez}{1-\nu^2}\left(\frac{d\theta}{dr}+\frac{\nu\theta}{r}\right) \tag{19.7}$$

$$\sigma_t = \frac{Ez}{\left(1-\nu^2\right)}\left(\frac{1}{R_t}+\frac{\nu}{R_r}\right) = \frac{Ez}{\left(1-\nu^2\right)}\left(\frac{\theta}{r}+\frac{d\theta}{dr}\right) \tag{19.8}$$

where,

σ_r = radial stress due to bending

σ_t = circumferential stress due to bending

The tangential of circumferential bending moment per unit radial length is

$$M_t = \int_{-t/2}^{+t/2} \sigma_t z \; dz$$

$$= \int_{-t/2}^{+t/2} \frac{E}{\left(1-\nu^2\right)}\left(\frac{\theta}{r}+\frac{d\theta}{dr}\right) z^2 dz$$

$$= \frac{E}{\left(1-\nu^2\right)}\left(\frac{\theta}{r}+\nu\frac{d\theta}{dr}\right)\left[\frac{z^3}{3}\right]_{-t/2}^{+t/2}$$

$$= \frac{Et^3}{12\left(1-\nu^2\right)}\left(\frac{\theta}{r}+\nu\frac{d\theta}{dr}\right)$$

therefore

$$M_t = D\left(\frac{\theta}{r}+\nu\frac{d\theta}{dr}\right) = D\left(\frac{1}{r}\frac{dw}{dr}+\nu\frac{d^2w}{dr^2}\right) \tag{19.9}$$

where,

t = plate thickness

and

$$D = \frac{Et^3}{12(1 - \nu^2)} = \text{flexural rigidity}$$

Similarly, the radial bending moment per unit circumferential length,

$$M_r = D\left(\frac{d\theta}{dr} + \frac{\nu\theta}{r}\right) = D\left(\frac{d^2w}{dr^2} + \frac{\nu dw}{r\,dr}\right) \tag{19.10}$$

Substituting equation (19.9) and (19.10) into equations (19.7) and (19.8), the bending stresses could be put in the following form:

$$\sigma_t = 12\ M_t \times z / t^3$$

and

$$\sigma = 12\ M_r \times z / t^3 \tag{19.11}$$

and the maximum stresses $\hat{\sigma}_t$ and $\hat{\sigma}_r$ will occur at the outer surfaces of the plate (ie, @$z = \pm t/2$). Therefore

$$\hat{\sigma}_t = 6M_t / t^2 \tag{19.12}$$

and

$$\hat{\sigma}_r = 6M_r / t^2 \tag{19.13}$$

The plate differential equation can now be obtained by considering the equilibrium of the plate element of Figure 19.2.

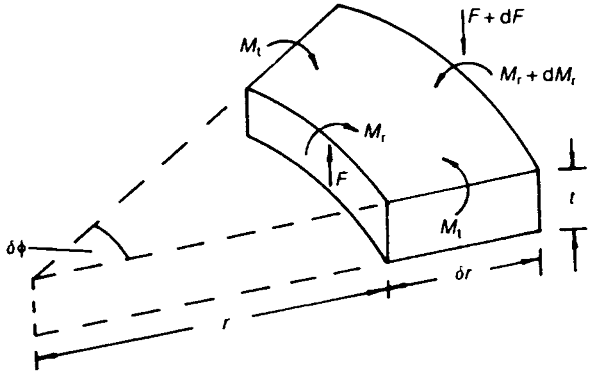

Figure 19.2 Element of a circular plate.

Taking moments about the outer circumference of the element,

$$(M_r + \delta M_t)(r + \delta r)\ \delta\varphi - M_r\, r\delta\varphi - 2M_t\ \delta r \sin\frac{\delta\varphi}{2} - F\ \ r\ \delta\varphi\delta r = 0$$

In the limit, this becomes

$$M_r + r \cdot \frac{dM_r}{dr} - M_t - Fr = 0 \tag{19.14}$$

Substituting equation (19.9) and (19.10) into equation (19.14),

$$\left(\frac{d\theta}{dr} + \frac{\nu\theta}{r}\right) + \left(r \cdot \frac{d^2\theta}{dr^2} + r \cdot \frac{\nu}{r}\frac{d\theta}{dr} - r \cdot \frac{\nu\theta}{r^2}\right) - \frac{\theta}{r} - \nu\frac{d\theta}{dr} = \frac{Fr}{D}$$

or

$$\frac{d^2\theta}{dr^2} + \left(\frac{1}{r}\right)\frac{d\theta}{dr} - \frac{\theta}{r^2} = \frac{F}{D}$$

which can be re-written in the form

$$\frac{d}{dr}\left[(1/r) \cdot \frac{d(r\theta)}{dr}\right] = \frac{F}{D} \tag{19.15}$$

where F is the shearing force / unit circumferential length.

Equation (19.15) is known as the plate differential equation for circular plates.

For a horizontal plate subjected to a lateral pressure p per unit area and a concentrated load W at the centre, F can be obtained from equilibrium considerations. Resolving 'vertically',

$$2\pi rF = \pi r^2 p + W$$

therefore

$$F = \frac{pr}{2} + \frac{W}{2\pi r} \text{ (except at } r = 0) \tag{19.16}$$

Substituting equation (19.16) into equation (19.15),

$$\frac{d}{dr}\left[(1/r)\frac{d(r\theta)}{dr}\right] = \frac{1}{D}\left[\frac{pr}{2} + \frac{W}{2\pi r}\right]$$

therefore

$$\frac{1}{r}\frac{d(r\theta)}{dr} = \frac{1}{D}\left[\frac{pr^2}{4} + \frac{W}{2\pi}\ln r\right] + C_1$$

$$\frac{d(r\theta)}{dr} = \frac{1}{D}\left[\frac{pr^3}{4} + \frac{Wr}{2\pi}\ln r\right] + C_1 r$$

$$r\theta = \frac{1}{D}\left[\frac{pr^4}{16} + \frac{Wr^2}{4\pi}\ln r - \frac{Wr^2}{8\pi}\right] + \frac{C_1 r^2}{2} + C_2$$

$$\theta = \frac{1}{D}\left[\frac{pr^3}{16} + \frac{Wr}{4\pi}\ln r - \frac{Wr}{8\pi}\right] + \frac{C_1 r}{2} + \frac{C_2}{r} \tag{19.17}$$

since,

$$\frac{dw}{dr} = \theta$$

$$w = \int \theta \, dr + C_3$$

hence,

$$w = \frac{pr^4}{64D} + \frac{Wr^2}{8\pi D}(\ln r - 1) + \frac{C_1 r^2}{4} + C_2 \ln r + C_3 \tag{19.18}$$

Note that

$$\int r \ln r \, dr = \int \frac{\ln r}{2} d\left(r^2\right)$$

$$= \frac{r^2}{2}\ln r - \int \frac{r^2}{2} d(\ln r) = \frac{r^2}{2}\ln r - \int \frac{r}{2} dr$$

$$= \frac{r^2}{2}\ln r - \frac{r^2}{4} + \text{a constant} \tag{19.19}$$

Problem 19.1 Determine the maximum deflection and stress in a circular plate, clamped around its circumference, when it is subjected to a centrally placed concentrated load W.

<u>Solution</u>

Putting $p = 0$ into equation (19.18),

$$w = \frac{Wr^2}{8\pi D}(\ln r - 1) + \frac{C_1 r^2}{4} + C_2 \ln r + C_3$$

$$\frac{dw}{dr} = \frac{Wr}{4\pi D}(\ln r - 1) + \frac{Wr}{8\pi D} + \frac{C_1 r}{2} + \frac{C_2}{r}$$

as dw/dr cannot equal ∞ at $r = 0$, $C_2 = 0$

$$\text{at } r = R, \quad \frac{dw}{dr} = w = 0$$

therefore

$$0 = \frac{WR^2}{8\pi D}\ln r - \frac{WR^2}{8\pi D} + \frac{C_1 R^2}{4} + C_3$$

and

$$0 = \frac{WR}{4\pi D}\ln R - \frac{WR}{4\pi D} + \frac{WR}{8\pi D} + \frac{C_1 R}{2}$$

Hence,

$$C_1 = \frac{W}{4\pi D}(1 - 2\ln R)$$

$$C_3 = -\frac{WR^2}{8\pi D}\ln R + \frac{WR^2}{8\pi D} - \frac{WR^2}{16\pi D} + \frac{WR^2}{8\pi D}\ln R = \frac{WR^2}{16\pi D}$$

$$w = \frac{WR^2}{8\pi D}\ln r - \frac{Wr^2}{8\pi D} + \frac{Wr^2}{16\pi D} - \frac{Wr^2}{8\pi D}\ln R + \frac{WR^2}{16\pi D}$$

or

$$w = \frac{WR^2}{16\pi D}\left|1 - \frac{r^2}{R^2} + \frac{2R^2}{R^2}\ln\left(\frac{r}{R}\right)\right|$$

The maximum deflection ($\hat{w}$) occurs at $r = 0$

$$\hat{w} = \frac{WR^2}{16\pi D}$$

Substituting the derivatives of w into equations (19.9) and (19.10),

$$M_r = \frac{W}{4\pi}\left[1 + \ln\left(\frac{r}{R}\right) \quad (1 + \nu)\right]$$

$$M_t = \frac{W}{4\pi}\left[\nu + (1 + \nu) \quad \ln\left(\frac{r}{R}\right)\right]$$

Problem 19.2 Determine the maximum deflection and stress that occur when a circular plate clamped around its external circumference is subjected to a uniform lateral pressure p.

Solution

From equation (19.18),

$$w = \frac{pr^4}{64D} + \frac{C_1 r^2}{4} + C_2 \ln r + C_3$$

$$\frac{dw}{dr} = \frac{pr^3}{16D} + \frac{C_1 r}{2} + \frac{C_2}{r}$$

and

$$\frac{d^2w}{dr^2} = \frac{3pr^2}{16D} + \frac{C_1}{2} - \frac{C_2}{r^2}$$

$$\text{at } r = 0, \quad \frac{dw}{dr} \neq \infty \text{ therefore } C_2 = 0$$

$$\text{at } r = R, \quad w = \frac{dw}{dr} = 0$$

therefore

$$0 = \frac{pr^4}{64D} + \frac{C_1 R^2}{4} + C_3$$

$$0 = \frac{pR^3}{16D} + \frac{C_1 R}{2}$$

therefore

$$C_1 = \frac{-pR^2}{8D}$$

$$C_3 = \frac{pR^4}{64D}$$

therefore

$$w = \frac{pR^4}{64D}\left(1 - \frac{r^2}{R^2}\right)^2 \tag{19.20}$$

Substituting the appropriate derivatives of w into equations (19.9) and (19.10),

$$M_r = \frac{pR^2}{16}\left[-(1 + v) + (3 + v)\frac{r^2}{R^2}\right] \tag{19.21}$$

$$M_t = \frac{pR^2}{16}\left[-(1 + v) + (1 + 3v)\frac{r^2}{R^2}\right] \tag{19.22}$$

Maximum deflection ($\hat{w}$) occurs at $r = 0$

$$\hat{w} = \frac{pR^4}{64D} \tag{19.23}$$

By inspection it can be seen that the maximum bending moment is obtained from (19.21), when $r = R$, i.e.

$$\hat{M}_r = pR^2/8$$

and $\hat{\sigma} = 6\hat{M}_t / t^2$

$$= 0.75\, pR^2 / t^2$$

Problem 19.3 Determine the expression for M_r and M_t in an annular disc, simply-supported around its outer circumference, when it is subjected to a concentrated load W, distributed around its inner circumference, as shown in Figure 19.3.

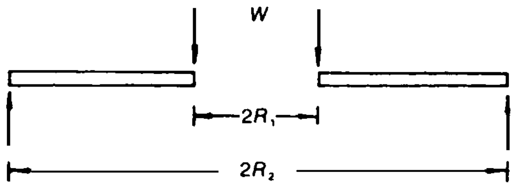

Figure 19.3 Annular disc.

W = total load around the inner circumference.

Solution

From equation (19.18),

$$w = \frac{Wr^2}{8\pi D}(\ln r - 1) + \frac{C_1 r^2}{4} + C_2 \ln r + C_3$$

at $r = R_2$, $w = 0$

or

$$0 = \frac{WR_2^2}{8\pi D}\left(\ln R_2 - 1\right) + \frac{C_1}{4}R_2^2 + C_2 \ln R_2 + C_3 \tag{19.24}$$

Now,

$$\frac{dw}{dr} = \frac{Wr}{4\pi D}(\ln r - 1) + \frac{Wr}{8\pi D} + \frac{C_1 r}{2} + \frac{C_2}{r} \tag{19.25}$$

and,

$$\frac{d^2w}{dr^2} = \frac{W}{4\pi D}\left(\ln r - 1\right) + \frac{W}{4\pi D} + \frac{W}{8\pi D} + \frac{C_1}{2} - \frac{C_2}{r^2} \tag{19.26}$$

A suitable boundary condition is that

$$M_r = 0 \text{ at } r = R_1 \text{ and at } r = R_2$$

but

$$M_r = D\left(\frac{d^2w}{dr^2} + \frac{v}{r}\frac{dw}{dr}\right)$$

therefore

$$\frac{W}{4\pi D}\left(\ln R_1 - 1\right) + \frac{3W}{8\pi D} + \frac{C_1}{2} - \frac{C_2}{R_1^2} + \frac{v}{R_1}\left\{\frac{WR_1}{4\pi D}\left(\ln R_1 - 1\right) + \frac{WR_1}{8\pi D} + \frac{C_1 R_1}{2} + \frac{C_2}{R_1}\right\} = 0 \tag{19.27}$$

and

$$\frac{W}{4\pi D}\left(\ln R_2 - 1\right) + \frac{3W}{8\pi D} + \frac{C_1}{2} - \frac{C_2}{R_2^2} + \frac{v}{R_2}\left\{\frac{WR_2}{4\pi D}\left(\ln R_2 - 1\right) + \frac{WR_2}{8\pi D} + \frac{C_1 R_2}{2} + \frac{C_2}{R_2}\right\} = 0 \tag{19.28}$$

Solving equations (19.27) and (19.28) for C_1 and C_2,

$$C_1 = \frac{-W}{4\pi D}\left\{\frac{2\left(R_2^2 \ln R_2 - R_1^2 \ln R_1\right)}{\left(R_2^2 - R_1^2\right)} + \frac{(1 - v)}{(1 + v)}\right\} \tag{19.29}$$

and

$$C_2 = \frac{-W}{4\pi D} \times \frac{(1 + v)}{(1 - v)} \times \frac{\left(R_2^2 R_1^2\right)}{\left(R_2^2 - R_1^2\right)} \ln\left(\frac{R_2}{R_1}\right) \tag{19.30}$$

C_3 is not required to determine expressions for M_r and M_t. Hence,

$$M_r = D(W/8\pi D)\left\{(1 + v)2 \ln r + (1 - v)\right\} + \left(C_1/2\right)(1 + v) - \left(C_2/r^2\right)(1 + v) \tag{19.31}$$

and

$$M_t = D(W/8\pi D)\,\{(1 + \nu)2 \ln r - (1 - \nu)\} + (C_1/2)\,(1 + \nu) + (C_2/r^2)\,(1 - \nu) \qquad (19.32)$$

Problem 19.4 A flat circular plate of radius R_2 is simply-supported concentrically by a tube of radius R_1, as shown in Figure 19.4. If the 'internal' portion of the plate is subjected to a uniform pressure p, show that the central deflection δ of the plate is given by

$$\delta = \frac{pR_1^4}{64D}\left\{3 + 2\left(\frac{R_1}{R_2}\right)^2\left(\frac{1 - \nu}{1 + \nu}\right)\right\}$$

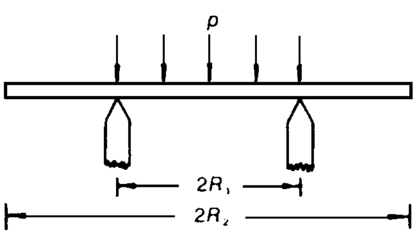

Figure 19.4 Circular plate with a partial pressure load.

Solution

Now the shearing force per unit length F for $r > R_1$ is zero, and for $r < R_1$,

$$F = pr/2$$

so that the plate differential equation becomes

$$\longleftarrow r < R_1 \longrightarrow \qquad \longleftarrow r > R_1 \longrightarrow$$

$$\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right\} = \frac{pr}{2D} \qquad = 0$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) = \frac{pr}{4D} + A \qquad = \mathrm{B} \qquad (19.33)$$

For continuity at $r = R_1$, the two expressions on the right of equation (19.33) must be equal, i.e.

$$\frac{pR_1^2}{4D} + A = B$$

or

$$B = \frac{pR_1^2}{4D} + A \tag{19.34}$$

or

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) = \frac{pr^2}{4D} + A \qquad = \frac{pR_1^2}{4D} + A$$

or

$$\frac{d}{dr}\left(r\frac{dw}{dr}\right) = \frac{pr^3}{4D} + Ar \qquad = \frac{pR_1^2 r}{4D} + Ar$$

which on integrating becomes,

$$r\frac{dw}{dr} = \frac{pr^4}{16D} + \frac{Ar^2}{2} + C \qquad = \frac{pR_1^2 r^2}{8D} + \frac{Ar^2}{2} + F$$

$$\frac{dw}{dr} = \frac{pr^3}{16D} + \frac{Ar}{2} + \frac{C}{r} \qquad = \frac{pR_1^2 r}{8D} + \frac{Ar}{2} + \frac{F}{r} \tag{19.35}$$

$$\text{at } r = 0, \quad \frac{dw}{dr} \neq \infty \quad \text{therefore } C = 0$$

For continuity at $r = R_1$, the value of the slope must be the same from both expressions on the right of equation (19.35), i.e.

$$\frac{pR_1^3}{16D} + \frac{AR_1}{2} = \frac{pR_1^3}{8D} + \frac{AR_1}{2} + \frac{F}{R_1}$$

therefore

$$F = -pR_1^4 / (16D) \tag{19.36}$$

therefore

$$\frac{dw}{dr} = \frac{pr^3}{16D} + \frac{Ar}{2} \qquad = \frac{pR_1^2 r}{8D} + \frac{Ar}{2} - \frac{pR_1^4}{16Dr} \tag{19.37}$$

which on integrating becomes

$$w = \frac{pr^4}{64D} + \frac{Ar^2}{4} + G \qquad = \frac{pR_1^2 r^2}{16D} + \frac{Ar^2}{4} - \frac{R_1^4}{16D} \ln r + H \tag{19.38}$$

Now, there are three unknowns in equation (19.38), namely A, G and H, and therefore, three simultaneous equations are required to determine these unknowns. One equation can be obtained by considering the continuity of w at $r = R_1$ in equation (19.38), and the other two equations can be obtained by considering boundary conditions.

One suitable boundary condition is that at $r = R_2$, $M_r = 0$, which can be obtained by considering that portion of the plate where $R_2 > r > R_1$, as follows:

$$\frac{dw}{dr} = \frac{pR_1^2 r}{8D} + \frac{Ar}{2} - \frac{pR_1^4}{16Dr}$$

$$\frac{d^2w}{dr^2} = \frac{pR_1^2}{8D} + \frac{A}{2} + \frac{pR_1^4}{16Dr^2}$$

Now

$$M_r = D\left(\frac{d^2 w}{dr^2} + \frac{v}{r}\frac{dw}{dr}\right)$$

$$= \left\{\left(\frac{pR_1^2}{8D} + \frac{A}{2} + \frac{pR_1^4}{16Dr^2}\right) + \frac{v}{r}\left(\frac{pR_1^2 r}{8D} + \frac{Ar}{2} - \frac{pR_1^4}{16Dr}\right)\right\} \tag{19.39}$$

$$= D\left\{\frac{pR_1^2}{8D}(1 + v) + \frac{A}{2}(1 + v) + \frac{pR_1^4}{16Dr^2}(1 - v)\right\}$$

Now, at $r = R_2$, $M_r = 0$; therefore

$$\frac{A}{2}(1 + v) = -\frac{pR_1^2}{8D}(1 + v) - \frac{pR_1^4}{16DR_2^2}(1 - v)$$

or

$$A = -\frac{pR_1^2}{4D} - \frac{pR_1^4}{8DR_2^2}\left(\frac{1 - \nu}{1 + \nu}\right) \tag{19.40}$$

Another suitable boundary condition is that

$$\text{at } r = R_1, \quad w = 0$$

In this case, it will be necessary to consider only that portion of the plate where $r < R_1$, as follows:

$$w = \frac{pr^4}{64D} + \frac{Ar^2}{4} + G$$

$$\text{at } r = R_1, \quad w = 0$$

Therefore

$$0 = \frac{pR_1^4}{64D} + \frac{AR_1^2}{4} + G$$

or

$$G = -\frac{pR_1^4}{64D} + \frac{pR_1^4}{16D} + \frac{pR_1^6}{32DR_2^2}\left(\frac{1 - \nu}{1 + \nu}\right)$$

$$= \frac{-pR_1^4}{64D} + \left\{\frac{pR_1^2}{4D} + \frac{pR_1^4}{8DR_2^2}\left(\frac{1 - \nu}{1 + \nu}\right)\right\}\frac{R_1^2}{4}$$

or

$$G = \frac{pR_1^4}{64D}\left\{3 + 2\left(\frac{R_1}{R_2}\right)^2\left(\frac{1 - \nu}{1 + \nu}\right)\right\} \tag{19.41}$$

The central deflection δ occurs at $r = 0$; hence, from (19.41),

$$\delta = G$$

$$\delta = \frac{pR_1^4}{64D}\left\{3 + 2\left(\frac{R_1}{R_2}\right)^2\left(\frac{1-\nu}{1+\nu}\right)\right\} \tag{19.42}$$

$$\left\{0.115\,WR^2/(ET^3);\ \frac{W}{t^2}\left[0.621\ln\left(\frac{R}{r}\right) - 0.436 + 0.0224\left(\frac{R}{r}\right)^2\right]\right\}$$

Problem 19.5 A flat circular plate of outer radius R_2 is clamped firmly around its outer circumference. If a load W is applied concentrically to the plate, through a tube of radius R_1, as shown in Figure 19.5, show that the central deflection δ is

$$\delta = \frac{W}{16\pi D}\left\{R_1^2\ln\left(\frac{R_1}{R_2}\right)^2 + \left(R_2^2 - R_1^2\right)\right\}$$

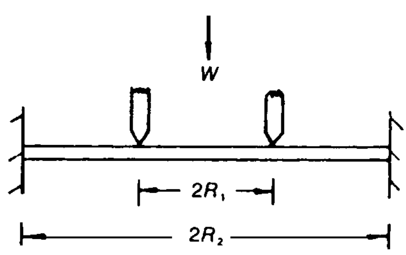

Figure 19.5 Plate under an annular load.

Solution

When $r < R_1$, $F = 0$, and when $R_2 > r > R_1$, $F = W/(2\pi r)$, so that the plate differential equation becomes

$\longleftarrow$ - - - - - - - $r < R_1$ - - - - - - - $\longrightarrow$ $\longleftarrow$ - - - - $r > R_1$ - - - - $\longrightarrow$

$$\frac{d}{dr}\left\{\frac{1}{r}\,d\left(r\frac{dw}{dr}\right)\right\} = 0 \qquad\qquad = \frac{W}{2\pi D}$$

or

$$\frac{1}{r}\,d\left(r\frac{dw}{dr}\right) = A \qquad\qquad = \frac{W}{2\pi D}\ln r + B$$

or

$$d\left(r\frac{dw}{dr}\right) = Ar \qquad\qquad = \frac{Wr\ \ln r}{2\pi D} + Br \tag{19.43}$$

From continuity considerations at $r = R_1$, the two expressions on the right of equation (19.43) must be equal, i.e.

$$A = \frac{W}{2\pi D}\ln R_1 + B \tag{19.44}$$

On integrating equation (19.43),

$$r\frac{dw}{dr} = \frac{Ar^2}{2} + C \qquad\qquad \frac{W}{2\pi D}\left(\frac{r^2}{2}\ln r - \frac{r^2}{4}\right) + \frac{Br^2}{2} + F$$

or

$$\frac{dw}{dr} = \frac{Ar}{2} + \frac{C}{r} \qquad\qquad \frac{Wr}{4\pi D}\left(\ln r - \frac{r}{2}\right) + \frac{Br}{2} + \frac{F}{r} \tag{19.45}$$

at $r = 0$, $\dfrac{dw}{dr} \neq \infty$ therefore $C = 0$

From continuity considerations for dw/dr, at $r = R_1$,

$$\frac{AR_1}{2} = \frac{WR_1}{4\pi D}\left(\ln R_1 - \frac{R_1}{2}\right) + \frac{BR_1}{2} + \frac{F}{R_1} \tag{19.46}$$

On integrating equation (19.46)

$$w = \frac{Ar^2}{2} + G \qquad\qquad = \frac{W}{2\pi D}\left(\frac{r^2}{4}\ln r - \frac{r^2}{8} - \frac{r^2}{8}\right) + \frac{Br^2}{4} + F\ln r + H$$

or

$$w = \frac{Ar^2}{2} + G \qquad\qquad \frac{Wr^2}{8\pi D}(\ln r - 1) + \frac{Br^2}{4} + F\ \ln r + H \tag{19.47}$$

From continuity considerations for w, at $r = R_1$,

$$\frac{Ar_1^2}{2} + G = \frac{WR_1^2}{8\pi D}\left(\ln R_1 - 1\right) + \frac{BR_1^2}{4} + F \ln R_1 + H \tag{19.48}$$

In order to obtain the necessary number of simultaneous equations to determine the arbitrary constants, it will be necessary to consider *boundary considerations.*

$$\text{at } r = R_2, \quad \frac{dw}{dr} = 0$$

therefore

$$0 = \frac{WR_2}{4\pi D}\left(\ln R_2 - \frac{R_2}{2}\right) + \frac{BR_2}{2} + \frac{F}{R_2} \tag{19.49}$$

Also, at $r = R_2$, $w = 0$; therefore

$$0 = \frac{WR_2^2}{8\pi D}\left(\ln R_2 - 1\right) + \frac{BR_2^2}{4} + F \ln\left(R_2\right) + H \tag{19.50}$$

Solving equations (19.46), (19.48), (19.49) and (19.50),

$$F = \frac{WR_1^2}{8\pi D} \tag{19.51}$$

$$H = -\frac{W}{8\pi D}\left\{-R_2^2/2 - R_1^2/2 + R_1^2 \ln\left(R_2\right)\right\}$$

and

$$G = -\frac{WR_1^2}{8\pi D} + \frac{WR_1^2}{8\pi D} \ln(R_1) + H$$

$$= -WR_1^2 + \frac{WR_1^2 \ln(R_1)}{8\pi D} - \frac{W}{8\pi D}\left\{-\frac{R_2^2}{2} - \frac{R_1^2}{2} + R_1^2 \ln(R_2)\right\}$$

$$= \frac{W}{16\pi D}\left\{-2R_1^2 + 2R_1^2 \ln(R_1) + R_2^2 + R_1^2 - 2R_1^2 \ln(R_2)\right\}$$

$$G = \frac{W}{16\pi D}\left\{R_1^2 \ln\left(\frac{R_1}{R_2}\right)^2 + \left(R_2^2 - R_1^2\right)\right\}$$

δ occurs at $r = 0$, i.e.

$$\delta = G = \frac{W}{16\pi D}\left\{R_1^2 \ln\left(\frac{R_1}{R_2}\right)^2 + \left(R_2^2 - R_1^2\right)\right\}$$

19.3 Large deflections of plates

If the maximum deflection of a plate exceeds half the plate thickness, the plate changes to a shallow shell, and withstands much of the lateral load as a membrane, rather than as a flexural structure.

For example, consider the membrane shown in Figure 19.6, which is subjected to uniform lateral pressure p.

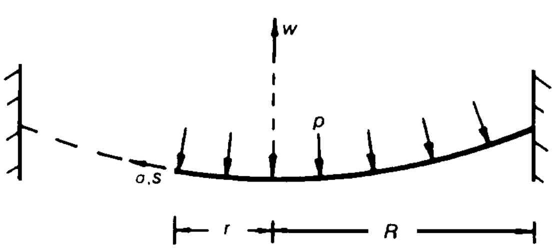

Figure 19.6 Portion of circular membrane.

Let

w = out-of-plane deflection at any radius r

σ = membrane tension at a radius r

t = thickness of membrane

Resolving vertically,

$$\sigma \times t \times 2\pi r \times \frac{dw}{dr} = p \times \pi r^2$$

or

$$\frac{dw}{dr} = \frac{pr}{2\sigma t} \qquad (19.52)$$

or

$$w = \frac{pr^2}{4\sigma t} + A$$

at $r = R,\ w = 0$; therefore

$$A = \frac{-pR^2}{4\sigma t}$$

i.e.

$\hat{w}$ = maximum deflection of membrane

$$\hat{w} = -pR^2/(4\sigma t)$$

The change of meridional (or radial) length is given by

$$\delta l = \int ds - \int dr$$

where s is any length along the meridian

Using Pythagoras' theorem,

$$\delta l = \int \left(dw^2 + dr^2\right)^{1/2} - \int dr$$

$$= \int \left[1 + \left(\frac{dw}{dr}\right)^2\right]^{1/2} - \int dr$$

Expanding binomially and neglecting higher order terms,

$$\begin{aligned}\delta l &= \int\left[1 + \frac{1}{2}\left(\frac{dw}{dr}\right)^2\right]dr - \int dr \\ &= \frac{1}{2}\int\left(\frac{dw}{dr}\right)^2 dr\end{aligned} \tag{19.53}$$

Substituting the derivative of w, namely equation (19.52) into equation (19.53),

$$\begin{aligned}\delta l &= \frac{1}{2}\int_0^R\left(\frac{pr}{2\sigma t}\right)^2 dr \\ &= p^2R^3/(24\sigma^2t^2)\end{aligned} \tag{19.54}$$

but

$$\varepsilon = \text{strain} = \frac{\delta l}{R} = \frac{1}{E}(\sigma - v\sigma)$$

or

$$\sigma^3 = \frac{E}{(1 - v)}\left(\frac{p^2R^2}{24\sigma^2t^2}\right)$$

i.e.

$$\sigma = \sqrt[3]{\left\{\frac{E}{1 - v}\left(\frac{p^2R^2}{24t^2}\right)\right\}} \tag{19.55}$$

but

$$\sigma = pR^2/(4t\hat{w}) \tag{19.56}$$

From equations (19.55) and (19.56),

$$p = \frac{8E}{3(1 - v)}\left(\frac{t}{R}\right)\left(\frac{\hat{w}}{R}\right)^3 \tag{19.57}$$

According to small deflection theory of plates (19.23)

$$p = \frac{64D}{R^3}\left(\frac{\hat{w}}{R}\right) \tag{19.58}$$

Thus, for the large deflections of clamped circular plates under lateral pressure, equations (19.57) and (19.58) should be added together, as follows:

$$p = \frac{64D}{R^3}\left(\frac{\hat{w}}{R}\right) + \frac{8}{3(1-\nu)}\left(\frac{t}{R}\right)\left(\frac{\hat{w}}{R}\right)^3 \qquad (19.59)$$

If $\nu = 0.3$, then (19.59) becomes

$$\frac{pR^4}{64Dt} = \left(\frac{\hat{w}}{t}\right)\left\{1 + 0.65\left(\frac{\hat{w}}{t}\right)\right\} \qquad (19.60)$$

where the second term in (19.60) represents the membrane effect, and the first term represents the flexural effect.

When $\hat{w}/t = 0.5$, the membrane effect is about 16.3% of the bending effect, but when $\hat{w}/t = 1$, the membrane effect becomes about 65% of the bending effect. The bending and membrane effects are about the same when $\hat{w}/t = 1.24$. A plot of the variation of $\hat{w}$ due to bending and due to the combined effects of bending plus membrane stresses, is shown in Figure 19.7.

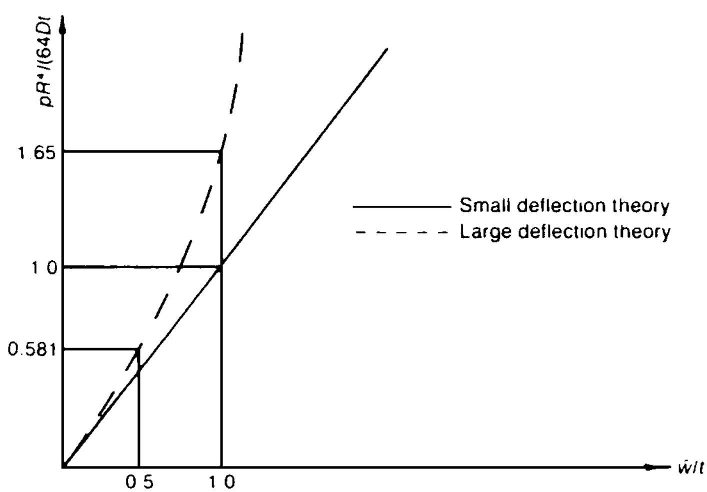

Figure 19.7 Small and large deflection theory.

19.3.1 Power series solution

This method of solution, which involves the use of data sheets, is based on a power series solution of the fundamental equations governing the large deflection theory of circular plates.

For a circular plate under a uniform lateral pressure p, the large deflection equations are given by (19.61) to (19.63).

$$D\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right\} = \sigma_r\frac{dw}{dr} + \frac{pr}{2} \tag{19.61}$$

$$\frac{d}{dr}(r\sigma_r) - \sigma_t = 0 \tag{19.62}$$

$$r\frac{d}{dr}(\sigma_r + \sigma_t) + \frac{E}{2}\left(\frac{dw}{dr}\right)^2 = 0 \tag{19.63}$$

Way[5] has shown that to assist in the solution of equations (19.61) to (19.63), by the power series method, it will be convenient to introduce the dimensionless ratio ζ, where

$$\zeta = r/R$$

or

$$r = \zeta R$$

R = outer radius of disc

r = any value of radius between 0 and R

Substituting for r int (19.61):

$$\frac{1}{12(1-\nu^2)}\frac{d}{d(\zeta R)}\left\{\frac{1}{\zeta R} \times \frac{d(\zeta R\theta)}{d(\zeta R)}\right\} = \frac{\sigma_r\theta}{Et^2} + \frac{p\zeta R}{2Et^3}$$

or

$$\frac{1}{12(1-\nu^2)}\frac{d}{d\zeta}\left\{\frac{1}{\zeta} \times \frac{d(\zeta\theta)}{d\zeta}\right\} = \frac{\sigma_r R^2\theta}{Et^2} + \frac{pR^3\zeta}{2Et^3} \tag{19.64}$$

Inspecting (19.64), it can be seen that the LHS is dependent only on the slope θ.

Now

$$\theta = \frac{dw}{dr} = \frac{dw}{d(\zeta R)}$$

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[5]Way, S., *Bending of circular plates with large deflections, A.S.M.E., APM-56-12*, 56, 1934.

which, on substituting into (19.64), gives:

$$\frac{1}{12\left(1-\nu^2\right)}\frac{d}{d\zeta}\left\{\frac{1}{\zeta}\frac{d}{d\zeta}\left[\frac{\zeta d(w/t)}{d\zeta}\right]\right\} = \frac{\sigma_r}{E}\left(\frac{R}{t}\right)^2\frac{d(w/t)}{d\zeta}+\frac{P}{E}\left(\frac{R}{t}\right)^4\frac{\zeta}{2} \tag{19.65}$$

but

$$\left(\frac{w}{t}\right)\quad \frac{\sigma_r}{E}\left(\frac{R}{t}\right)^2 \quad \text{and} \quad \frac{p}{E}\left(\frac{R}{t}\right)^4$$

are all dimensionless, and this feature will be used later on in the present chapter.

Substituting r, in terms of ζ into equation (19.62), equation (19.66) is obtained:

$$\frac{d}{d\zeta}\left\{\frac{\zeta\sigma_r}{E}\left(\frac{R}{t}\right)^2\right\} = \frac{\sigma_t}{E}\left(\frac{R}{t}\right)^2 \tag{19.66}$$

Similarly, substituting r in terms of ζ equation (19.63), equation (19.67) is obtained:

$$\zeta\frac{d}{d\zeta}\left\{\frac{\sigma_r}{E}\left(\frac{R}{t}\right)^2+\frac{\sigma_t}{E}\left(\frac{R}{t}\right)^2\right\}+\frac{E}{2}\theta^2 = 0 \tag{19.67}$$

Equation (19.67) can be seen to be dependent only on the deflected form of the plate.

The fundamental equations, which now appear as equations (19.65) to (19.67), can be put into dimensionless form by introducing the following dimensionless variables:

$$\mathrm{X} = r/t = \zeta R/t$$

$$\mathrm{W} = w/t$$

$$\mathrm{U} = u/t$$

$$M_r' = M_r\,t/D$$

$$S_r = \sigma_r/E$$

$$S_t = \sigma_t/E$$

$$S_r' = \sigma_r'/E$$

$$S_t' = \sigma_t'/E$$

$$S_p = p/E \tag{19.68}$$

$$\theta = \frac{dw}{dr} = \frac{dW}{dX} \tag{19.69}$$

or

$$W = \int \theta dX \tag{19.70}$$

Now from standard circular plate theory,

$$\sigma_r' = \frac{6D}{t^2}\left(\frac{d\theta}{dr} + \frac{v\theta}{r}\right)$$

and

$$\sigma_t' = \frac{6D}{t^2}\left(\frac{\theta}{r} + v\frac{d\theta}{dr}\right)$$

Hence,

$$S_r' = \frac{1}{2\left(1 - v^2\right)}\left(\frac{d\theta}{dX} + \frac{v\theta}{X}\right) \tag{19.71}$$

and

$$S_t' = \frac{1}{2\left(1 - v^2\right)}\left(\frac{\theta}{X} + v\frac{d\theta}{dX}\right) \tag{19.72}$$

Now from elementary two-dimensional stress theory,

$$\frac{uE}{r} = \sigma_t - v\sigma_r$$

or

$$U = X\left(S_t - vS_r\right) \tag{19.73}$$

where u is the in-plane radial deflection at r.

Substituting equations (19.68) to (19.73) into equations (19.65) to (19.67), the fundamental equations take the form of equations (19.74) to (19.76):

$$\frac{1}{12(1 - v^2)} \frac{d}{dX} \left\{ \frac{1}{X} \frac{d(X\theta)}{dX} \right\} = S_p \frac{X}{2} + S_r \theta \tag{19.74}$$

$$\frac{d(XS_r)}{dX} - S_t = 0 \tag{19.75}$$

$$X \frac{d}{dX} (S_r + S_t) + \frac{\theta^2}{2} = 0 \tag{19.76}$$

Solution of equations (19.74) to (19.76) can be achieved through a power series solution.

Now S_r is a symmetrical function, i.e. $S_r(X) = S_r(-X)$, so that it can be approximated in an even series powers of X.

Furthermore, as θ is antisymmetrical, i.e. $\theta(X) = -\theta(X)$, it can be expanded in an odd series power of X. Let

$$S_r = B_1 + B_2 X^2 + B_3 X^4 + \ldots$$

and

$$\theta = C_1 X + C_2 X^3 + C_3 X^5 + \ldots$$

or

$$S_r = \sum_{i=1}^{\infty} B_i X^{2i-2} \tag{19.77}$$

and

$$\theta = \sum_{i=1}^{\infty} C_i X^{2i-1} \tag{19.78}$$

Now from equation (19.75)

$$S_t = \frac{d(XS_r)}{dX} = \sum_{i=1}^{\infty} (2i - 1) B_i X^{2i-2} \tag{19.79}$$

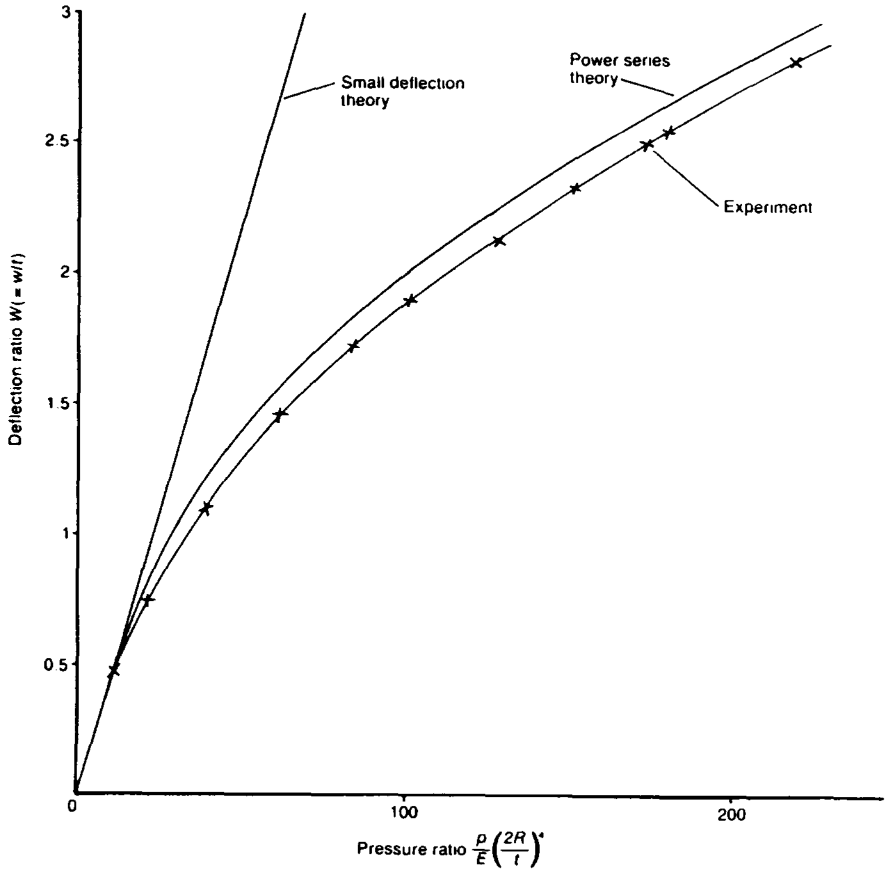

Figure 19.8 Central deflection versus pressure for a simply-supported plate.

$$W = \int \theta \, dX = \sum_{i=1}^{\infty} \left(\frac{1}{2i}\right) C_i X^{2i} \tag{19.80}$$

Hence

$$S_r' = \sum_{i=1}^{\infty} \frac{(2i + \nu - 1)}{2(1 - \nu^2)} C_i X^{2i-2} \tag{19.81}$$

$$S_t' = \sum_{i=1}^{\infty} \frac{\{1 + \nu(2i - 1)\}}{2(1 - \nu^2)} C_i X^{2i-2} \tag{19.82}$$

Now

$$\begin{aligned} U &= X(S_t - \nu S_r) \\ &= \sum_{i=1}^{\infty} (2i - 1 - \nu) B_i X^{2i-1} \end{aligned} \tag{19.83}$$

for $i = 1, 2, 3, 4 \rightarrow \infty$.

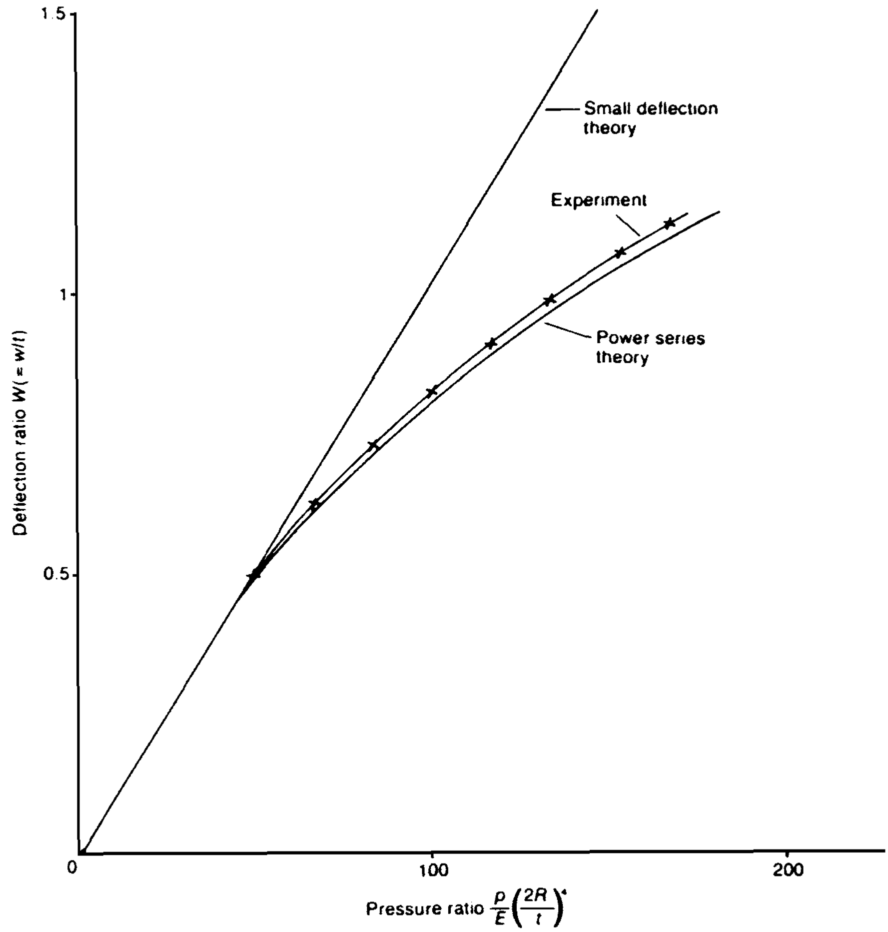

Figure 19.9 Central deflection versus pressure for an encastré plate.

From equations (19.77) to (19.83), it can be seen that if B_1 and C_1 are known all quantities of interest can readily be determined.

Way has shown that

$$B_k = \frac{-\sum_{m=1}^{k-1} C_m C_{k-m}}{8k(k-1)}$$

for $k = 2, 3, 4$ etc. and

$$C_k = \frac{3(1-\nu^2)}{k(k-1)} \sum_{m=1}^{k-1} B_m C_{k-m}$$

for $k = 3, 4, 5$ etc. and

$$C_2 \quad \frac{3(1-\nu^2)}{2}\left(\frac{S_P}{2} + B_1 C_1\right)$$

Once B_1 and C_1 are known, the other constants can be found. In fact, using this approach, Hewitt and Tannent[6] have produced a set of curves which under uniform lateral pressure, as shown in Figures 19.8 to 19.12. Hewitt and Tannent have also compared experiment and small deflection theory with these curves.

19.4 Shear deflections of very thick plates

If a plate is very thick, so that membrane effects are insignificant, then it is possible that shear deflections can become important.

For such cases, the bending effects and shear effects must be added together, as shown by equation (19.84), which is rather similar to the method used for beams in Chapter 13,

$$\delta = \delta_{\text{bending}} + \delta_{\text{shear}}$$

which for a plate under uniform pressure p is

$$\delta = pR\left\{k_1\left(\frac{R}{t}\right)^3 + k_2\left(\frac{t}{R}\right)^2\right\} \tag{19.84}$$

where k_1 and k_2 are constants.

From equations (19.84), it can be seen that δ_{shear} becomes important for large values of (t/R).

[6] Hewitt D A, Tannent J O, *Large deflections of circular plates*, Portsmouth Polytechnic Report M195, 1973-74.

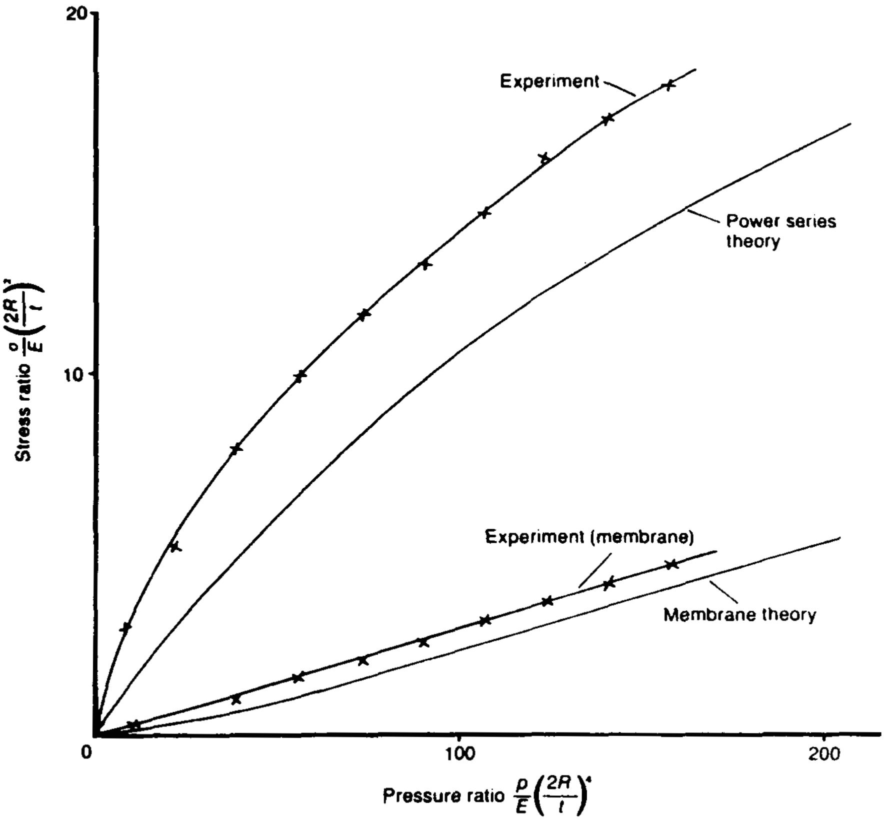

Figure 19.10 Central stress versus pressure for an encastré plate.

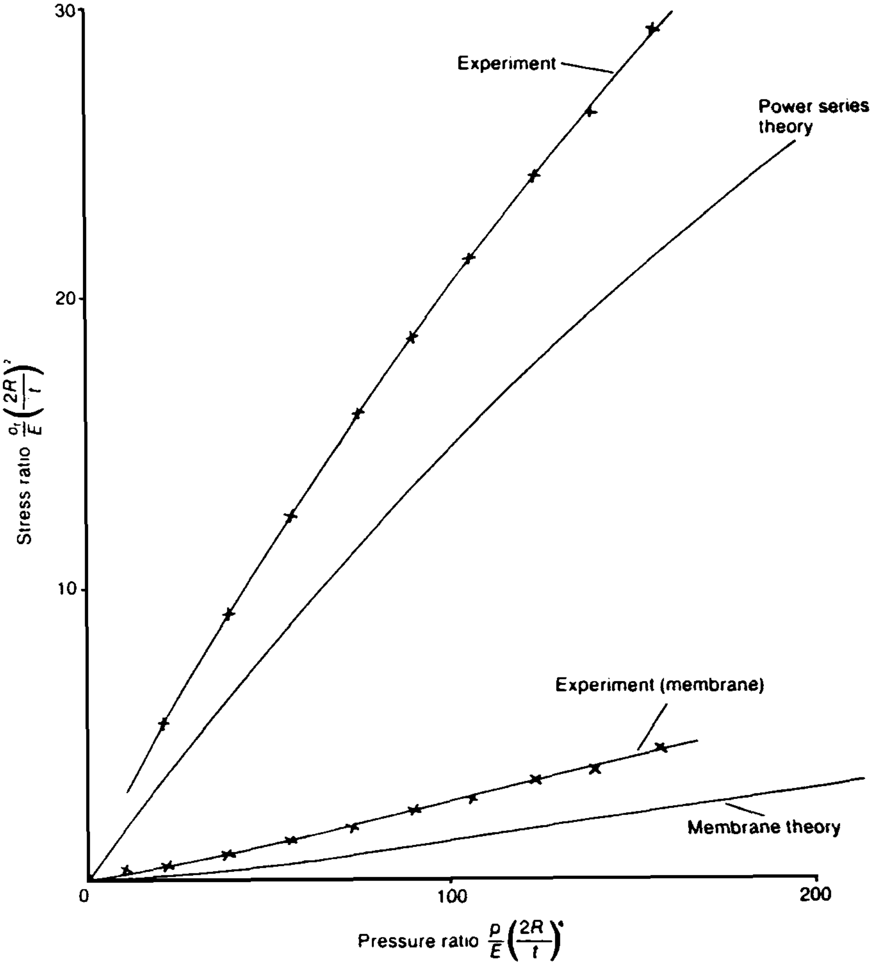

Figure 19.11 Radial stresses near edge versus pressure for an encastré plate.

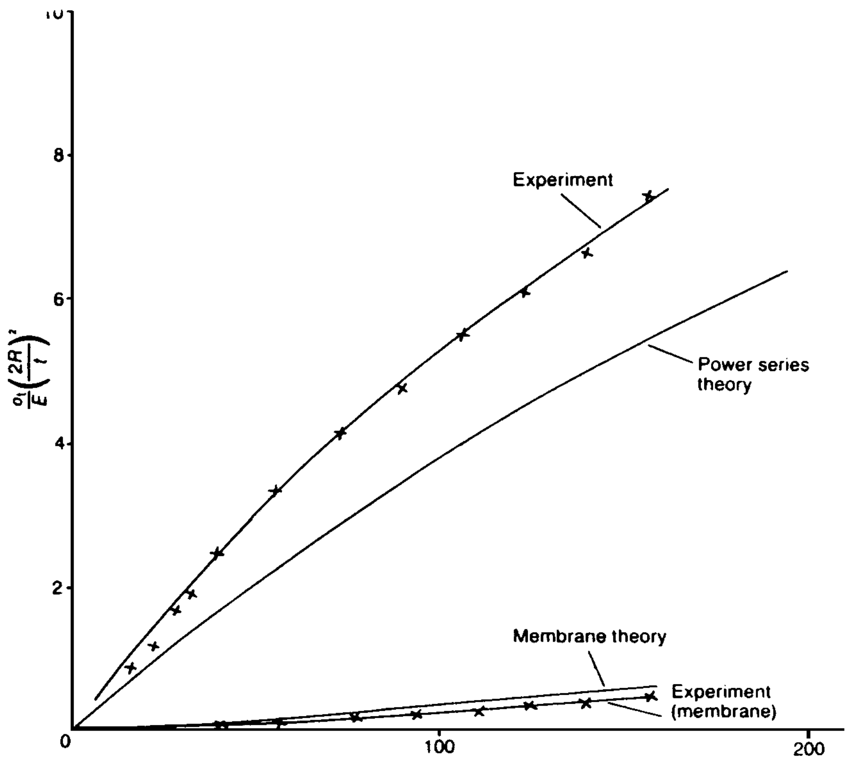

Figure 19.12 Circumferential stresses versus pressure near edge for an encastré plate.

Further problems *(answers on page 694)*

19.6 Determine an expression for the deflection of a circular plate of radius R, simply-supported around its edges, and subjected to a centrally placed concentrated load W.

19.7 Determine expressions for the deflection and circumferential bending moments for a circular plate of radius R, simply-supported around its edges and subjected to a uniform pressure p.

19.8 Determine an expression for the maximum deflection of a simply-supported circular plate, subjected to the loading shown in Figure 19.13.

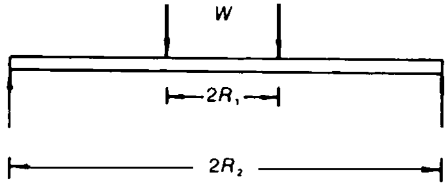

Figure 19.13 Simply-supported plate.

19.9 Determine expressions for the maximum deflection and bending moments for the concentrically loaded circular plates of Figure 19.14(a) and (b).

(a) Simply supported. (b) Clamped.

Figure 19.14 Problem 19.9

19.10 A flat circular plate of radius R is firmly clamped around its boundary. The plate has stepped variation in its thickness, where the thickness inside a radius of $(R/5)$ is so large that its flexural stiffness may be considered to approach infinity. When the plate is subjected to a pressure p over its entire surface, determine the maximum central deflection and the maximum surface stress at any radius r. $\nu = 0.3$.

19.11 If the loading of Example 19.9 were replaced by a centrally applied concentrated load W, determine expressions for the central deflection and the maximum surface stress at any radius r.