6 Thin shells under internal pressure

6.1 Thin cylindrical shell of circular cross-section

A problem in which combined stresses are present is that of a cylindrical shell under internal pressure. Suppose a long circular shell is subjected to an internal pressure p, which may be due to a fluid or gas enclosed within the cylinder, Figure 6.1. The internal pressure acting on the long sides of the cylinder gives rise to a circumferential stress in the wall of the cylinder; if the ends of the cylinder are closed, the pressure acting on these ends is transmitted to the walls of the cylinder, thus producing a longitudinal stress in the walls.

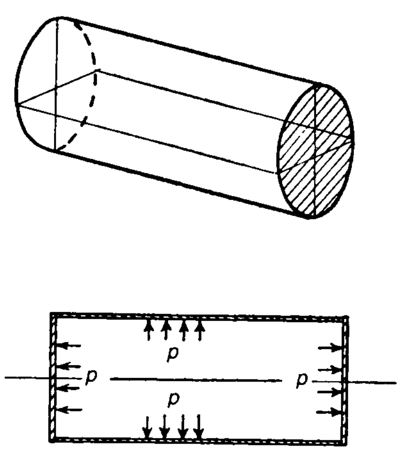


Figure 6.1 Long thin cylindrical shell with closed ends under internal pressure.

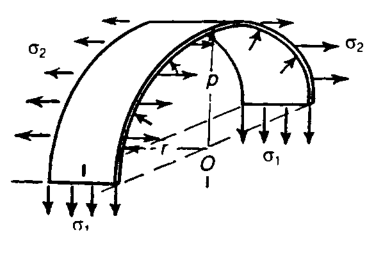


Figure 6.2 Circumferential and longitudinal stresses in a thin cylinder with closed ends under internal pressure.

Suppose r is the mean radius of the cylinder, and that its thickness t is small compared with r. Consider a unit length of the cylinder remote from the closed ends, Figure 6.2; suppose we cut this unit length with a diametral plane, as in Figure 6.2. The tensile stresses acting on the cut sections are σ_1, acting circumferentially, and σ_2, acting longitudinally. There is an internal pressure p on the inside of the half-shell. Consider equilibrium of the half-shell in a plane perpendicular to the axis of the cylinder, as in Figure 6.3; the total force due to the internal pressure p in the direction OA is

$$p \times (2r \times 1)$$

because we are dealing with a unit length of the cylinder. This force is opposed by the stresses σ_1; for equilibrium we must have

$$p \times (2r \times 1) = \sigma_1 \times 2(t \times 1)$$

Then

$$\sigma_1 = \frac{pr}{t} \tag{6.1}$$

We shall call this the *circumferential (or hoop) stress.*

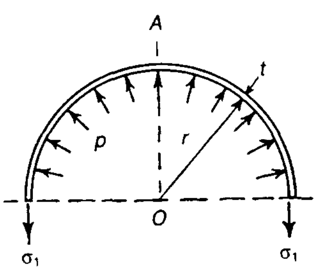


Figure 6.3 Derivation of circumferential stress.

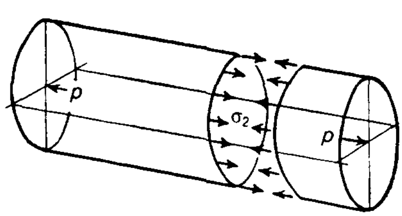


Figure 6.4 Derivation of longitudinal stress.

Now consider any transverse cross-section of the cylinder remote from the ends, Figure 6.4; the total longitudinal force on each closed end due to internal pressure is

$$p \times \pi r^2$$

At any section this is resisted by the internal stresses σ_2, Figure 6.4. For equilibrium we must have

$$p \times \pi r^2 = \sigma_2 \times 2\pi rt$$

which gives

$$\sigma_2 = \frac{pr}{2t} \tag{6.2}$$

We shall call this the *longitudinal stress*. Thus the longitudinal stress, σ_2, is only half the circumferential stress, σ_1.

The stresses acting on an element of the wall of the cylinder consist of a circumferential stress σ_1, a longitudinal stress σ_2, and a radial stress p on the internal face of the element, Figure 6.5. As (r/t) is very much greater than unity, p is small compared with σ_1 and σ_2. The state of stress in the wall of the cylinder approximates then to a simple two-dimensional system with principal stresses σ_1 and σ_2.

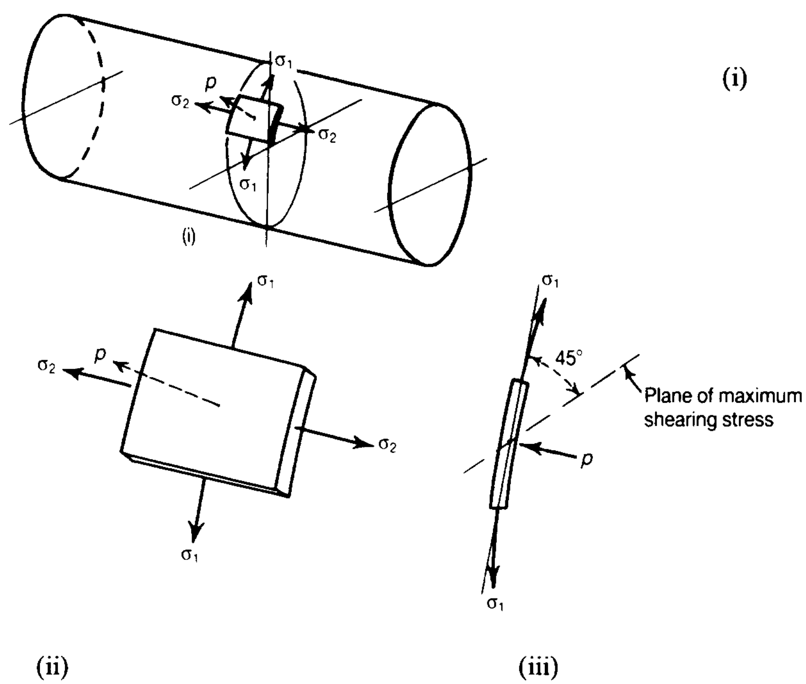


Figure 6.5 Stresses acting on an element of the wall of a circular cylindrical shell with closed ends under internal pressure.

The maximum shearing stress in the plane of σ_1 and σ_2 is therefore

$$\tau_{max} = \frac{1}{2}\left(\sigma_1 - \sigma_2\right) = \frac{pr}{4t}$$

This is not, however, the maximum shearing stress in the wall of the cylinder, for, in the plane of σ_1 and p, the maximum shearing stress is

$$\tau_{max} = \frac{1}{2}(\sigma_1) = \frac{pr}{2t} \tag{6.3}$$

since p is negligible compared with σ_1; again, in the plane of σ_2 and p, the maximum shearing stress is

$$\tau_{max} = \frac{1}{2}(\sigma_2) = \frac{pr}{4t}$$

The greatest of these maximum shearing stresses is given by equation (6.3); it occurs on a plane at 45° to the tangent and parallel to the longitudinal axis of the cylinder, Figure 6.5(iii).

The circumferential and longitudinal stresses are accompanied by direct strains. If the material of the cylinder is elastic, the corresponding strains are given by

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - v\sigma_2) = \frac{pr}{Et}\left(1 - \frac{1}{2}v\right)$$

$$\varepsilon_1 = \frac{1}{E}(\sigma_2 - v\sigma_1) = \frac{pr}{Et}\left(\frac{1}{2} - v\right) \tag{6.4}$$

The circumference of the cylinder increases therefore by a small amount $2\pi r\varepsilon_1$; the increase in mean radius is therefore $r\varepsilon_1$ The increase in length of a unit length of the cylinder is ε_2, so the change in internal volume of a unit length of the cylinder is

$$\delta V = \pi (r + r\varepsilon_1)^2 (1 + \varepsilon_2) - \pi r^2$$

The volumetric strain is therefore

$$\frac{\delta V}{\pi r^2} = (1 + \varepsilon_1)^2 (1 + \varepsilon_2) - 1$$

But ε_1 and ε_2 are small quantities, so the volumetric strain is

$$(1 + \varepsilon_1)^2 (1 + \varepsilon_2) - 1 \doteq (1 + 2\varepsilon_1)(1 + \varepsilon_2) - 1$$

$$\doteq 2\varepsilon_1 + \varepsilon_2$$

In terms of σ_1 and σ_2 this becomes

$$2\varepsilon_1 + \varepsilon_2 = \frac{pr}{Et}\left[2\left(1 - \frac{1}{2}v\right) + \left(\frac{1}{2} - v\right)\right] = \frac{pr}{Et}\left(\frac{5}{2} - 2v\right) \tag{6.5}$$

Problem 6.1 A thin cylindrical shell has an internal diameter of 20 cm, and is 0.5 cm thick. It is subjected to an internal pressure of 3.5 MN/m^2. Estimate the circumferential and longitudinal stresses if the ends of the cylinders are closed.

Solution

From equations (6.1) and (6.2),

$$\sigma_1 = \frac{pr}{t} = \left(3.5 \times 10^6\right)(0.1025)/(0.005) = 71.8 \text{ MN/m}^2$$

and

$$\sigma_2 = \frac{pr}{2t} = \left(3.5 \times 10^6\right)(0.1025)/(0.010) = 35.9 \text{ MN/m}^2$$

Problem 6.2 If the ends of the cylinder in Problem 6.1 are closed by pistons sliding in the cylinder, estimate the circumferential and longitudinal stresses.

Solution

The effect of taking the end pressure on sliding pistons is to remove the force on the cylinder causing longitudinal stress. As in Problem 6.1, the circumferential stress is

$$\sigma_1 = 71.8 \text{ MN/m}^2$$

but the longitudinal stress is zero.

Problem 6.3 A pipe of internal diameter 10 cm, and 0.3 cm thick is made of mild-steel having a tensile yield stress of 375 MN/m^2. What is the maximum permissible internal pressure if the stress factor on the maximum shearing stress is to be 4?

Solution

The greatest allowable maximum shearing stress is

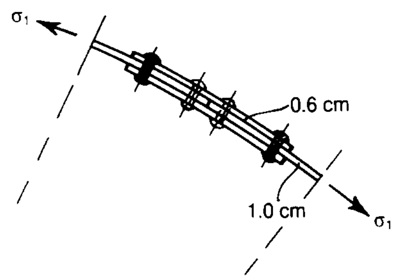
$$\tfrac{1}{4}\left(\tfrac{1}{2} \times 375 \times 10^6\right) = 46.9 \text{ MN/m}^2$$

The greatest shearing stress in the cylinder is

$$\tau_{max} = \frac{pr}{2t}$$

Then $$p = \frac{2t}{r}\left(\tau_{max}\right) = \frac{2 \times 0.003}{0.0515} \times \left(46.9 \times 10^6\right) = 5.46 \text{ MN / m}^2$$

Problem 6.4 Two boiler plates, each 1 cm thick, are connected by a double-riveted butt joint with two cover plates, each 0.6 cm thick. The rivets are 2 cm diameter and their pitch is 0.90 cm. The internal diameter of the boiler is 1.25 m, and the pressure is 0.8 MN/m^2. Estimate the shearing stress in the rivets, and the tensile stresses in the boiler plates and cover plates.



Solution

Suppose the rivets are staggered on each side of the joint. Then a single rivet takes the circumferential load associated with a ½ (0.090) = 0.045 m length of boiler. The load on a rivet is

$$\left[\frac{1}{2}(1.25)\right](0.045)\left(0.8 \times 10^6\right) = 22.5 \text{ kN}$$

Area of a rivet is

$$\frac{\pi}{4}(0.02)^2 = 0.314 \times 10^{-3} \text{ m}^2$$

The load of 22.5 kN is taken in double shear, and the shearing stress in the rivet is then

$$\frac{1}{2}(22.5 \times 10^3) / (0.314 \times 10^{-3}) = 35.8 \text{ MN/m}^2$$

The rivet holes in the plates give rise to a loss in plate width of 2 cm in each 9 cm of rivet line. The effective area of boiler plate in a 9 cm length is then

$$(0.010)(0.090 - 0.020) = (0.010)(0.070) = 0.7 \times 10^{-3} \text{ m}^2$$

The tensile load taken by this area is

$$\frac{1}{2}(1.25)(0.090)(0.8 \times 10^6) = 45.0 \text{ kN}$$

The average circumferential stress in the boiler plates is therefore

$$\sigma_1 = \frac{45.0 \times 10^3}{0.7 \times 10^{-3}} = 64.2 \text{ MN/m}^2$$

This occurs in the region of the riveted connection. Remote from the connection, the circumferential tensile stress is

$$\sigma_1 = \frac{pr}{t} = \frac{(0.8 \times 10^6)(0.625)}{(0.010)} = 50.0 \text{ MN/m}^2$$

In the cover plates, the circumferential tensile stress is

$$\frac{45.0 \times 10^3}{2(0.006)(0.070)} = 53.6 \text{ MN/m}^2$$

The longitudinal tensile stresses in the plates in the region of the connection are difficult to estimate; except very near to the rivet holes, the stress will be

$$\sigma_2 = \frac{pr}{2t} = 25.0 \text{ MN/m}^2$$

Problem 6.5 A long steel tube, 7.5 cm internal diameter and 0.15 cm thick, has closed ends, and is subjected to an internal fluid pressure of 3 MN/m^2. If $E = 200$ GN/m^2, and $\nu = 0.3$, estimate the percentage increase in internal volume of the tube.

Solution

The circumferential tensile stress is

$$\sigma_1 = \frac{pr}{t} = \frac{(3 \times 10^6)(0.0383)}{(0.0015)} = 76.6 \text{ MN/m}^2$$

The longitudinal tensile stress is

$$\sigma_2 = \frac{pr}{2t} = 38.3 \text{ MN/m}^2$$

The circumferential strain is

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2)$$

and the longitudinal strain is

$$\varepsilon_2 = \frac{1}{E}(\sigma_2 - \nu\sigma_1)$$

The volumetric strain is then

$$2\varepsilon_1 + \varepsilon_2 = \frac{1}{E}\left[2\sigma_1 - 2\nu\sigma_2 + \sigma_2 - \nu\sigma_1\right]$$

$$= \frac{1}{E}\left[\sigma_1 (2 - \nu) + \sigma_2 (1 - 2\nu)\right]$$

Thus

$$2\varepsilon_1 + \varepsilon_2 = \frac{(76.6 \times 10^6)\left[(2 - 0.3) + (1 - 0.6)\right]}{200 \times 10^9}$$

$$= \frac{(76.6 \times 10^6)(1.9)}{(200 \times 10^9)} = 0.727 \times 10^{-3}$$

The percentage increase in volume is therefore 0.0727%

Problem 6.6 An air vessel, which is made of steel, is 2 m long; it has an external diameter of 45 cm and is 1 cm thick. Find the increase of external diameter and the increase of length when charged to an internal air pressure of 1 MN/m^2.

Solution

For steel, we take

$$E = 200 \text{ GN/m}^2, \quad \nu = 0.3$$

The mean radius of the vessel is $r = 0.225$ m; the circumferential stress is then

$$\sigma_1 = \frac{pr}{t} = \frac{(1 \times 10^6)(0.225)}{0.010} = 22.5 \text{ MN/m}^2$$

The longitudinal stress is

$$\sigma_2 = \frac{pr}{2t} = 11.25 \text{ MN/m}^2$$

The circumferential strain is therefore

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2) = \frac{\sigma_1}{E}\left(1 - \frac{1}{2}\nu\right) = \frac{(22.5 \times 10^6)(0.85)}{200 \times 10^9}$$

$$= 0.957 \times 10^{-4}$$

The longitudinal strain is

$$\varepsilon_2 = \frac{1}{E}(\sigma_2 - \nu\sigma_1) = \frac{\sigma_1}{E}\left(\frac{1}{2} - \nu\right) = \frac{(22.5 \times 10^6)(0.2)}{200 \times 10^9}$$

$$= 0.225 \times 10^{-4}$$

The increase in external diameter is then

$$0.450\,(0.957 \times 10^{-4}) = 0.430 \times 10^{-4} \text{ m}$$

$$= 0.0043 \text{ cm}$$

The increase in length is

$$2\,(0.225 \times 10^{-4}) = 0.450 \times 10^{-4} \text{ m}$$

$$= 0.0045 \text{ cm}$$

Problem 6.7 A thin cylindrical shell is subjected to internal fluid pressure, the ends being closed by:

(a) two watertight pistons attached to a common piston rod;

(b) flanged ends.

Find the increase in internal diameter in each case, given that the internal diameter is 20 cm, thickness is 0.5 cm, Poisson's ratio is 0.3, Young's modulus is 200 GN/m^2, and the internal pressure is 3.5 MN/m^2. *(RNC)*

<u>*Solution*</u>

We have

$$p = 3.5 \text{ MN/m}^2, \quad r = 0.1 \text{ m}, \quad t = 0.005 \text{ m}$$

In both cases the circumferential stress is

$$\sigma_1 = \frac{pr}{t} = \frac{\left(3.5 \times 10^6\right)(0.1)}{(0.005)} = 70 \text{ MN/m}^2$$

(a) In this case there is no longitudinal stress. The circumferential strain is then

$$\varepsilon_1 = \frac{\sigma_1}{E} = \frac{70 \times 10^6}{200 \times 10^9} = 0.35 \times 10^{-3}$$

The increase of internal diameter is

$$0.2\left(0.35 \times 10^{-3}\right) = 0.07 \times 10^{-3} \text{ m} = 0.007 \text{ cm}$$

(b) In this case the longitudinal stress is

$$\sigma_2 = \frac{pr}{2t} = 35 \text{ MN/m}^2$$

The circumferential strain is therefore

$$\varepsilon_1 = \frac{1}{E}\left(\sigma_1 - \nu\sigma_2\right) = \frac{\sigma_1}{E}\left(1 - \frac{1}{2}\nu\right) = 0.85\,\frac{\sigma_1}{E}$$

$$= 0.85\left(0.35 \times 10^{-3}\right) = 0.298 \times 10^{-3}$$

The increase of internal diameter is therefore

$$0.2\left(0.298 \times 10^{-3}\right) = 0.0596 \times 10^{-3} \text{ m} = 0.00596 \text{ cm}$$

Equations (6.1) and (6.2) are for determining stress in perfect thin-walled circular cylindrical shells. If, however, the circular cylinder is fabricated, so that its joints are weaker than the rest of the vessel, then equations (6.1) and (6.2) take on the following modified forms:

$$\sigma_1 = \text{hoop or circumferential stress} = \frac{pr}{\eta_L t} \tag{6.6}$$

$$\sigma_2 = \text{longitudinal stress} = \frac{pr}{2\eta_c t} \tag{6.7}$$

where

$$\eta_c = \text{circumferential joint efficiency} \leq 1$$

$$\eta_L = \text{longitudinal joint efficiency} \leq 1$$

NB The *circumferential stress* is associated with the *longitudinal joint* efficiency, and the *longitudinal stress* is associated with the *circumferential joint* efficiency.

6.2 Thin spherical shell

We consider next a thin spherical shell of means radius r, and thickness t, which is subjected to an internal pressure p. Consider any diameter plane through the shell, Figure 6.6; the total force normal to this plane due to p acting on a hemisphere is

$$p \times \pi r^2$$

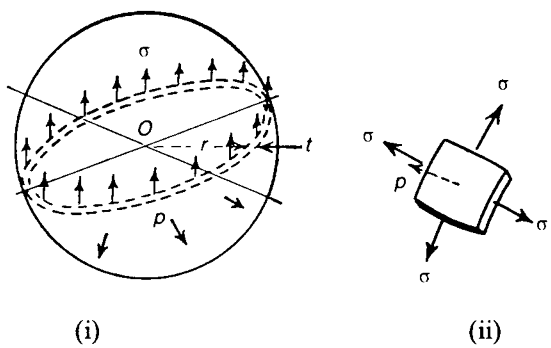


Figure 6.6 Membrane stresses in a thin spherical shell under internal pressure.

This is opposed by a tensile stress σ in the walls of the shell. By symmetry σ is the same at all points of the shell; for equilibrium of the hemisphere we must have

$$p \times \pi r^2 = \sigma \times 2\pi r t$$

This gives

$$\sigma = \frac{pr}{2t} \tag{6.8}$$

At any point of the shell the direct stress σ has the same magnitude in all directions in the plane of the surface of the shell; the state of stress is shown in Figure 6.6(ii). As p is small compared with σ, the maximum shearing stress occurs on planes at 45° to the tangent plane at any point.

If the shell remains elastic, the circumference of the sphere in any diametral plane is strained an amount

$$\varepsilon = \frac{1}{E}(\sigma - \nu\sigma) = (1 - \nu)\frac{\sigma}{E} \tag{6.9}$$

The volumetric strain of the enclosed volume of the sphere is therefore

$$3\varepsilon = 3(1 - \nu)\frac{\sigma}{E} = 3(1 - \nu)\frac{pr}{2Et} \tag{6.10}$$

Equation (6.8) is intended for determining membrane stresses in a perfect thin-walled spherical shell. If, however, the spherical shell is fabricated, so that its joint is weaker than the remainder of the shell, then equation (6.8) takes on the following modified form:

$$\sigma = \text{stress} = \frac{pr}{2\eta t} \tag{6.11}$$

where

η = joint efficiency ≤ 1

6.3 Cylindrical shell with hemispherical ends

Some pressure vessels are fabricated with hemispherical ends; this has the advantage of reducing the bending stresses in the cylinder when the ends are flat. Suppose the thicknesses t_1 and t_2 of the cylindrical section and the hemispherical end, respectively (Figure 6.7), are proportioned so that the radial expansion is the same for both cylinder and hemisphere; in this way we eliminate bending stresses at the junction of the two parts.

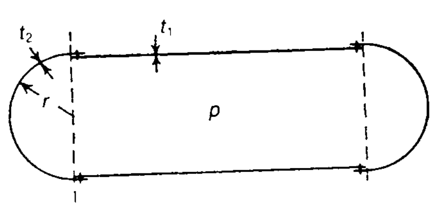


Figure 6.7 Cylindrical shell with hemispherical ends, so designed as to minimise the effects of bending stresses.

From equations (6.4), the circumferential strain in the cylinder is

$$\frac{pr}{Et_1}\left(1 - \frac{1}{2}\nu\right)$$

and from equation (6.7) the circumferential strain in the hemisphere is

$$\left(1 - \nu\right)\frac{pr}{2Et_2}$$

If these strains are equal, then

$$\frac{pr}{Et_1}\left(1 - \frac{1}{2}\nu\right) = \frac{pr}{2Et_2}\left(1 - \nu\right)$$

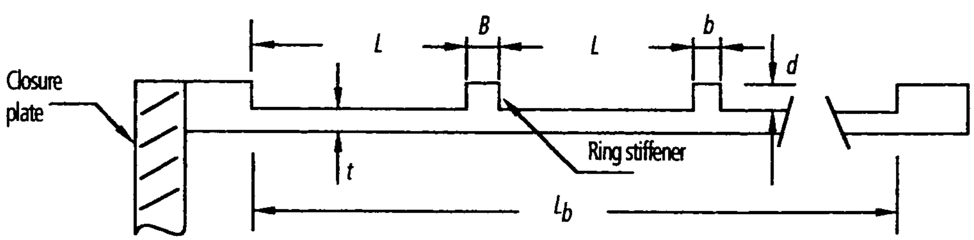
This gives

$$\frac{t_1}{t_2} = \frac{2 - \nu}{1 - \nu} \tag{6.12}$$

For most metals ν is approximately 0.3, so an average value of (t_1/t_2) is 1.7/0.7 ∔ 2.4. The hemispherical end is therefore thinner than the cylindrical section.

6.4 Bending stresses in thin-walled circular cylinders

The theory presented in Section 6.1 is based on membrane theory and neglects bending stresses due to end effects and ring stiffness. To demonstrate these effects, Figures 6.9 to 6.13 show plots of the theoretical predictions for a ring stiffened circular cylinder[3] together with experimental values, shown by crosses. This ring stiffened cylinder, which was known as Model No. 2, was firmly fixed at its ends, and subjected to an external pressure of 0.6895 MPa (100 psi), as shown by Figure 6.8.



L	L_b	B	b	d	N
95.25	616.6	10.16	8.26	15.75	5

t = 0.08　　N = number of ring stiffeners
E = Young's modulus = 71 GPa　　υ = Poisson's ratio = 0.3

Figure 6.8 Details of model No. 2 (mm).

The theoretical analysis was based on *beam on elastic foundations*, and is described by Ross[3].

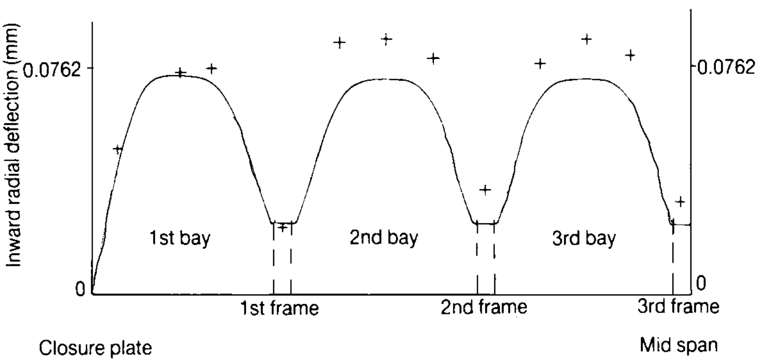


Figure 6.9 Deflection of longitudinal generator at 0.6895 MPa (100 psi), Model No. 2.

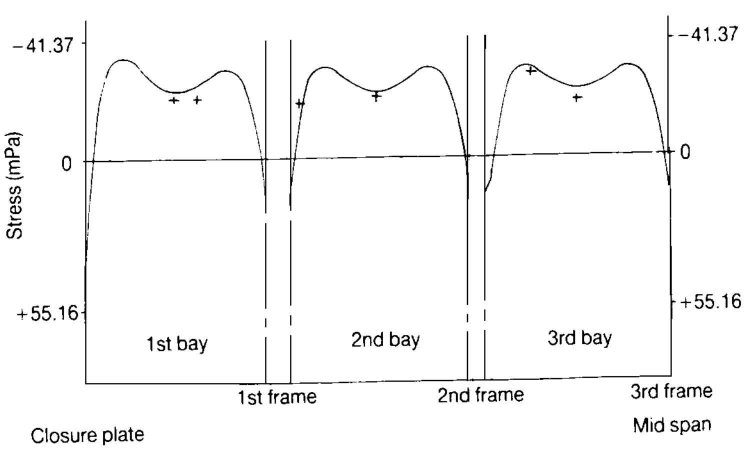


Figure 6.10 Longitudinal stress of the outermost fibre at 0.6895 MPa (100 psi), Model No. 2.

[3]Ross, C T F, ***Pressure vessels under external pressure,*** Elsevier Applied Science 1990.

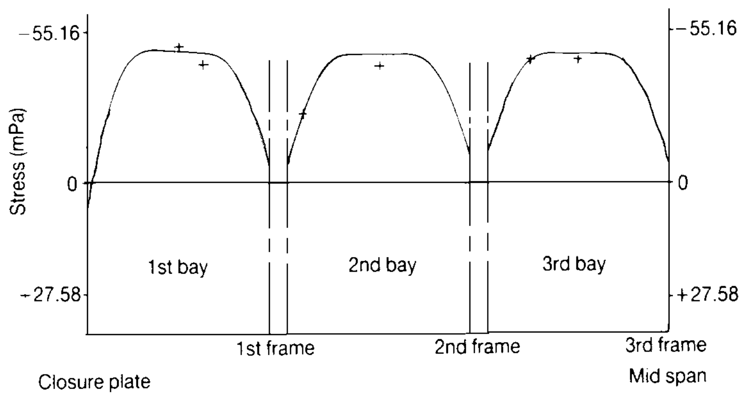


Figure 6.11 Circumferential stress of the outermost fibre at 0.6895 MPa (100 psi), Model No. 2.

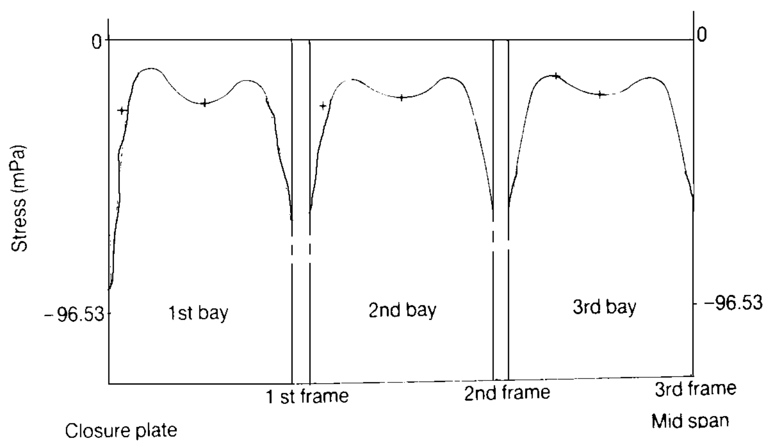


Figure 6.12 Longitudinal stress of the innermost fibre at 0.6895 MPa (100 psi), Model No. 2.

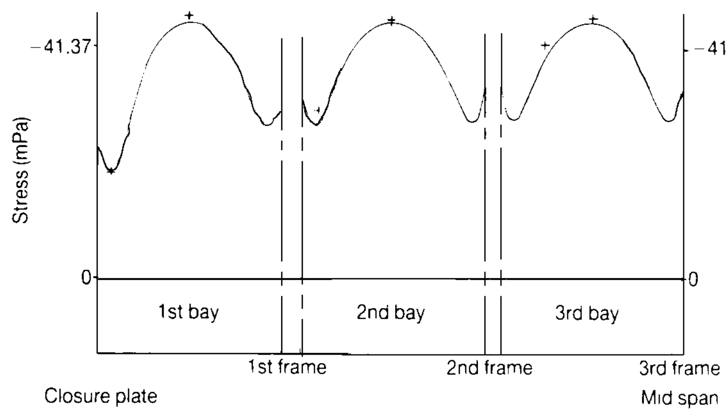


Figure 6.13 Circumferential stress of the innermost fibre at 0.6895 MPa (100 psi), Model No.2.

From Figures 6.9 to 6.13, it can be seen that bending stresses in thin-walled circular cylinders are very localised.

Further problems *(answers on page 692)*

6.8 A pipe has an internal diameter of 10 cm and is 0.5 cm thick. What is the maximum allowable internal pressure if the maximum shearing stress does not exceed 55 MN/m^2? Assume a uniform distribution of stress over the cross-section. *(Cambridge)*

6.9 A long boiler tube has to withstand an internal test pressure of 4 MN/m^2, when the mean circumferential stress must not exceed 120 MN/m^2. The internal diameter of the tube is 5 cm and the density is 7840 kg/m^3. Find the mass of the tube per metre run. *(RNEC)*

6.10 A long, steel tube, 7.5 cm internal diameter and 0.15 cm thick, is plugged at the ends and subjected to internal fluid pressure such that the maximum direct stress in the tube is 120 MN/m^2. Assuming ν = 0.3 and E = 200 GN/m^2, find the percentage increase in the capacity of the tube. *(RNC)*

6.11 A copper pipe 15 cm internal diameter and 0.3 cm thick is closely wound with a single layer of steel wire of diameter 0.18 cm, the initial tension of the wire being 10 N. If the pipe is subjected to an internal pressure of 3 MN/m^2 find the stress in the copper and in the wire (a) when the temperature is the same as when the tube was wound, (b) when the temperature throughout is raised 200°C. E for steel = 200 GN/m^2, E for copper = 100 GN/m^2, coefficient of linear expansion for steel = 11×10^{-6}, for copper 18×10^{-6} per 1°C. *(Cambridge)*

6.12 A thin spherical copper shell of internal diameter 30 cm and thickness 0.16 cm is just full of water at atmospheric pressure. Find how much the internal pressure will be increased if 25 cc of water are pumped in. Take $\nu = 0.3$ for copper and $K = 2$ GN/m^2 for water. *(Cambridge)*

6.13 A spherical shell of 60 cm diameter is made of steel 0.6 cm thick. It is closed when just full of water at 15°C, and the temperature is raised to 35°C. For this range of temperature, water at atmospheric pressure increases 0.0059 per unit volume. Find the stress induced in the steel. The bulk modulus of water is 2 GN/m^2, E for steel is 200 GN/m^2, and the coefficient of linear expansion of steel is 12×10^{-6} per 1°C, and Poisson's ratio = 0.3. *(Cambridge)*