16 Torsion of circular shafts and thin-walled tubes

16.1 Introduction

In Chapter 3 we introduced the concepts of shearing stress and shearing strain; these have an important application in torsion problems. Such problems arise in shafts transmitting heavy torques, in eccentrically loaded beams, in aircraft wings and fuselages, and many other instances. These problems are very complex in general, and at this elementary stage we can go no further than studying uniform torsion of circular shafts, thin-walled tubes, and thin-walled open sections.

16.2 Torsion of a thin circular tube

The simplest torsion problem is that of the twisting of a uniform thin circular tube; the tube shown in Figure 16.1 is of thickness t, and the mean radius of the wall is r, L is the length of the tube. Shearing stresses τ are applied around the circumference of the tube at each end, and in opposite directions.
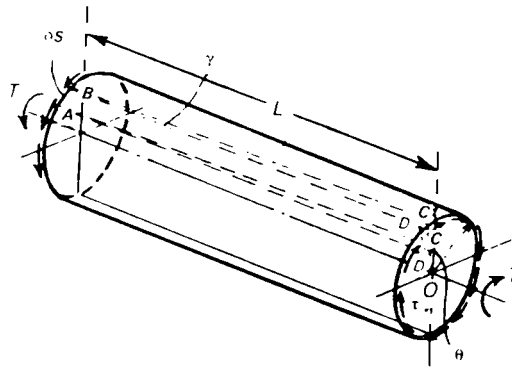

Figure 16.1 Torsion of a thin-walled circular tube.

If the stresses τ are uniform around the boundary, the total torque T at each end of the tube is

$$T = (2\pi rt)\, \tau r = 2\pi r^2 t\, \tau \qquad (16.1)$$

Thus the shearing stress around the circumference due to an applied torque T is

$$\tau = \frac{T}{2\pi r^2 t} \qquad (16.2)$$

We consider next the strains caused by these shearing stresses. We note firstly that complementary shearing stresses are set up in the wall parallel to the longitudinal axis of the tube. If δs is a small length of the circumference then an element of the wall $ABCD$, Figure 16.1, is in a state of pure shearing stress. If the remote end of the tube is assumed not to twist, then the longitudinal element $ABCD$ is distorted into the parallelogram $ABC'D'$, Figure 16.1, the angle of shearing strain being

$$\gamma = \frac{\tau}{G} \qquad (16.3)$$

if the material is elastic, and has a shearing (or rigidity) modulus G. But if θ is the angle of twist of the near end of the tube we have

$$\gamma L = r\theta \qquad (16.4)$$

Hence

$$\theta = \frac{\gamma L}{r} = \frac{\tau L}{Gr} \qquad (16.5)$$

It is sometimes more convenient to define the twist of the tube as the rate of change of twist per unit length; this is given by (θ/L), and from equation (16.5) this is equal to

$$\frac{\theta}{L} = \frac{\tau}{Gr} \qquad (16.6)$$

16.3   Torsion of solid circular shafts

The torsion of a thin circular tube is a relatively simple problem as the shearing stress may be assumed constant throughout the wall thickness. The case of a solid circular shaft is more complex because the shearing stresses are variable over the cross-section of the shaft. The solid circular shaft of Figure 16.2 has a length L and radius a in the cross-section.
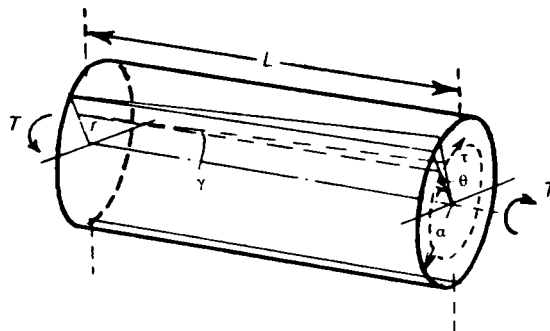

Figure 16.2 Torsion of a solid circular shaft.

When equal and opposite torques T are applied at each end about a longitudinal axis we assume that

(i) the twisting is uniform along the shaft, that is, all normal cross-sections the same distance apart suffer equal relative rotation;

(ii) cross-sections remain plane during twisting; and

(iii) radii remain straight during twisting.

If θ is the relative angle of twist of the two ends of the shaft, then the shearing strain γ of an elemental tube of thickness δr and at radius r is

$$\gamma = \frac{r\theta}{L} \tag{16.7}$$

If the material is elastic, and has a shearing (or rigidity) modulus G, Section 3.4, then the circumferential shearing stress on this elemental tube is

$$\tau = G\gamma = \frac{Gr\theta}{L} \tag{16.8}$$

The thickness of the elemental tube is δr, so the total torque on this tube is

$$(2\pi r \delta r)\tau r = 2\pi r^2 \tau \delta r$$

The total torque on the shaft is then

$$T = \int_0^a 2\pi r^2 \tau \, dr$$

On substituting for τ from equation (16.8), we have

$$T = 2\pi \left(\frac{G\theta}{L} \right) \int_0^a r^3 \, dr \tag{16.9}$$

Now

$$2\pi \int_0^a r^3 \, dr = \frac{\pi a^4}{2} \tag{16.10}$$

This is the polar second moment of area of the cross-section about an axis through the centre, and is usually denoted by J. Then equation (16.9) may be written

$$T = \frac{GJ\theta}{L} \tag{16.11}$$

We may combine equations (16.8) and (16.11) in the form

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L} \tag{16.12}$$

We see from equation (16.8) that τ increases linearly with r, from zero at the centre of the shaft to $Ga\theta/L$ at the circumference. Along any radius of the cross-section, the shearing stresses are normal to the radius and in the plane of the cross-section, Figure 16.3.
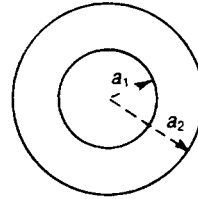
16.4   Torsion of a hollow circular shaft

It frequently arises that a torque is transmitted by a hollow circular shaft. Suppose a_1 and a_2 are the internal and external radii, respectively, of such a shaft, Figure 16.4. We make the same general assumptions as in the torsion of a solid circular shaft. If τ is the shearing stress at radius r, the total torque on the shaft is

$$T = \int_{a_1}^{a_2} 2\pi r^2 \tau \, dr \tag{16.13}$$
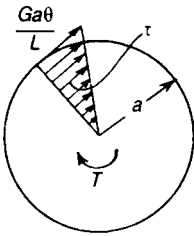
Figure 16.3 Variation of shearing stresses over the cross-section for elastic torsion of a solid circular bar.

Figure 16.4 Cross-section of a hollow circular shaft.

If we assume, as before, that radii remain straight during twisting, and that the material is elastic, we have

$$\tau = \frac{Gr\theta}{L}$$

Then equation (16.13) becomes

$$T = \int_{a_1}^{a_2} \left(\frac{G\theta}{L} \right) 2\pi r^3 \, dr = \frac{GJ\theta}{L} \tag{16.14}$$

where

$$J = \int_{a_1}^{a_2} 2\pi r^3 \, dr \tag{16.15}$$

Here, J is the polar second moment of area or, more generally, the *torsion constant* of the cross-section about an axis through the centre; J has the value

$$J = \int_{a_1}^{a_2} 2\pi r^3 \, dr = \frac{\pi}{2}\left(a_2^4 - a_1^4\right) \tag{16.16}$$

Thus, for both hollow and solid shafts, we have the relationship

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L}$$

Problem 16.1   What torque, applied to a hollow circular shaft of 25 cm outside diameter and 17.5 cm inside diameter will produce a maximum shearing stress of 75 MN/m^2 in the material *(Cambridge)*

Solution

We have

$$r_1 = 12.5 \text{ cm}, \qquad r_2 = 8.75 \text{ cm}$$

Then

$$J = \frac{\pi}{2}\left[(0.125)^4 - (0.0875)^4\right] = 0.292 \times 10^{-3} \text{ m}^4$$

If the shearing stress is limited to 75 MN/m^2, the torque is

$$T = \frac{J\tau}{r_1} = \frac{(0.292 \times 10^{-3})(75 \times 10^6)}{(0.125)} = 175.5 \text{ kNm}$$

Problem 16.2   A ship's propeller shaft has external and internal diameters of 25 cm and 15 cm. What power can be transmitted at 110 rev/minute with a maximum shearing stress of 75 MN/m^2, and what will then be the twist in degrees of a 10 m length of the shaft? $G = 80$ GN/m^2. *(Cambridge)*

Solution

In this case

$$r_1 = 0.125 \text{ m}, \qquad r_2 = 0.075 \text{ m}, \qquad l = 10 \text{ m}$$

$$J = \frac{\pi}{2} \left[(0.125)^4 - (0.075)^4 \right] = 0.335 \times 10^{-3} \text{ m}^4$$

and

$$\tau = 75 \text{ MN/m}^2$$

Then

$$T = \frac{J\tau}{r_1} = \frac{(0.335 \times 10^{-3})(75 \times 10^6)}{0.125} = 201 \text{ kNm}$$

At 110 rev/min the power generated is

$$(201 \times 10^3)\left(2\pi \times \frac{110}{60} \right) = 2.32 \times 10^6 \text{ Nm/s}$$

The angle of twist is

$$\theta = \frac{TL}{GJ} = \frac{(201 \times 10^3)(10)}{(80 \times 10^9)(0.335 \times 10^{-3})} = 0.075 \text{ radians} = 4.3°$$

Problem 16.3   A solid circular shaft of 25 cm diameter is to be replaced by a hollow shaft, the ratio of the external to internal diameters being 2 to 1. Find the size of the hollow shaft if the maximum shearing stress is to be the same as for the solid shaft. What percentage economy in mass will this change effect? *(Cambridge)*

Solution

Let r be the inside radius of the new shaft;   then = $2r$ the outside radius of the new shaft

$$J \text{ for the new shaft} = \frac{\pi}{2} \left(16r^4 - r^4 \right) = 7.5\pi r^4$$

$$J \text{ for the old shaft} = \frac{\pi}{2} \times (0.125)^4 = 0.384 \times 10^{-3} \text{ m}^4$$

If T is the applied torque, the maximum shearing stress for the old shaft is

$$\frac{T(0.125)}{0.384 \times 10^{-3}}$$

and that for the new one is

$$\frac{T(2r)}{7.5\pi r^4}$$

If these are equal,

$$\frac{T(0.125)}{0.384 \times 10^{-3}} = \frac{T(2r)}{7.5\pi r^4}$$

Then

$$r^3 = 0.261 \times 10^{-3} \text{ m}^3$$

or $r = 0.640$ m

Hence the internal diameter will be 0.128 m and the external diameter 0.256 m.

$$\frac{\text{area of new cross-section}}{\text{area of old cross-section}} = \frac{(0.128)^2 - (0.064)^2}{(0.125)^2} = 0.785$$

Thus, the saving in mass is about 21%.

Problem 16.4   A ship's propeller shaft transmits 7.5×10^6 W at 240 rev/min. The shaft has an internal diameter of 15 cm. Calculate the minimum permissible external diameter if the shearing stress in the shaft is to be limited to 150 MN/m². *(Cambridge)*

Solution

If T is the torque on the shaft, then

$$T\left(\frac{2\pi \times 240}{60}\right) = 7.5 \times 10^6$$

Thus

$$T = 298 \text{ kNm}$$

If d_1 is the outside diameter of the shaft, then

$$J = \frac{\pi}{32}\left(d_1^4 - 0.150^4\right) \text{ m}^4$$

If the shearing stress is limited to 150 MN/m², then

$$\frac{Td_1}{2J} = 150 \times 10^6$$

Thus,

$$Td_1 = \left(300 \times 10^6\right)J$$

On substituting for J and T

$$\left(298 \times 10^3\right)d_1 = \left(300 \times 10^6\right)\left(\frac{\pi}{32}\right)\left(d_1^4 - 0.150^4\right)$$

This gives

$$\left(\frac{d_1}{0.150}\right)^4 - 3\left(\frac{d_1}{0.150}\right) - 1 = 0$$

On solving this by trial-and-error, we get

$$d_1 = 1.54(0.150) = 0.231 \text{ m}$$

or $d_1 = 23.1$ cm

16.5   Principal stresses in a twisted shaft

It is important to appreciate that uniform torsion of circular shafts, of the form discussed in Section 16.3, involves no shearing between concentric elemental tubes of the shaft. Shearing stresses τ occur in a cross-section of the shaft, and complementary shearing stresses parallel to the longitudinal axis, Figure 16.5.
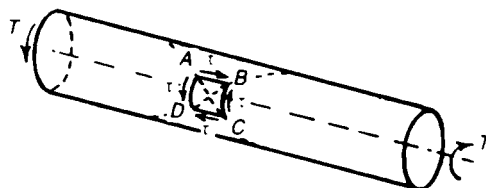

**Figure 16.5** Principal stresses in the outer surface of a twisted circular shaft.

An element $ABCD$ in the surface of the shaft is in a state of pure shear. The principal plane makes angles of $45°$ with the axis of the shaft, therefore, and the principal stresses are $\pm\tau$. If the element $ABCD$ is square, then the principal planes are AC and BD. The direct stress on AC is compressive and of magnitude τ; the direct stress on BD is tensile and of the same magnitude. Principal planes such as AC cut the surface of the shaft in a helix; for a brittle material, weak in tension, we should expect breakdown in a torsion test to occur by tensile fracture along planes such as BD. The failure of a twisted bar of a brittle material is shown in Figure 16.6.

Figure 16.6 Failure in torsion of a circular bar of brittle cast iron, showing a tendency to tensile fracture across a helix on the surface of the specimen.

The torsional failure of ductile materials occurs when the shearing stresses attain the yield stress of the material. The greatest shearing stresses in a circular shaft occur in a cross-section and along the length of the shaft. A circular bar of a ductile material usually fails by breaking off over a normal cross-section, as shown in Figure 16.7.
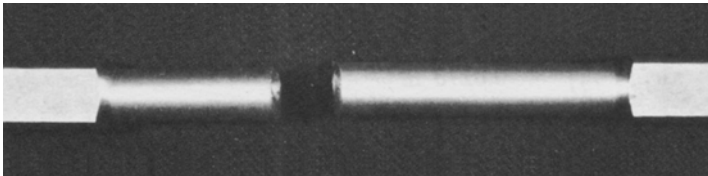
Figure 16.7 Failure of torsion of a circular bar of ductile cast iron, showing a shearing failure over a normal cross-section of the bar.

16.6 Torsion combined with thrust or tension

When a circular shaft is subjected to longitudinal thrust, or tension, as well as twisting, the direct stresses due to the longitudinal load must be combined with the shearing stresses due to torsion in order to evaluate the principal stresses in the shaft. Suppose the shaft is axially loaded in tension so that there is a longitudinal direct stress σ at all points of the shaft.
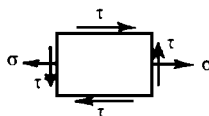
Figure 16.8 Shearing and direct stresses due to combined torsion and tension.

If τ is the shearing stress at any point, then we are interested in the principal stresses of the system shown in Figure 16.8; for this system the principal stresses, from equations (5.12), have the values

$$\frac{1}{2}\,\sigma \pm \frac{1}{2}\sqrt{\sigma^2 + 4\tau^2} \qquad\qquad (16.17)$$

and the maximum shearing stress, from equation (5.14), is

$$\tau_{max} = \frac{1}{2}\sqrt{\sigma^2 + 4\tau^2} \qquad\qquad (16.18)$$

Problem 16.5   A steel shaft, 20 cm external diameter and 7.5 cm internal, is subjected to a twisting moment of 30 kNm, and a thrust of 50 kN. Find the shearing stress due to the torque alone and the percentage increase when the thrust is taken into account. *(RNC)*

Solution

For this case, we have

$$r_1 = 0.100 \text{ m}, \qquad r_2 = 0.0375 \text{ m}$$

$$A = \pi\left(r_1^2 - r_2^2\right) = 0.0270 \text{ m}^2$$

The compressive stress is

$$\sigma = -\frac{P}{A} = -\frac{50 \times 10^3}{0.0270} = -1.85 \text{ MN/m}^2$$

Now

$$J = \frac{\pi}{2}\left(r_1^4 - r_2^4\right) = 0.00247 \text{ m}^4$$
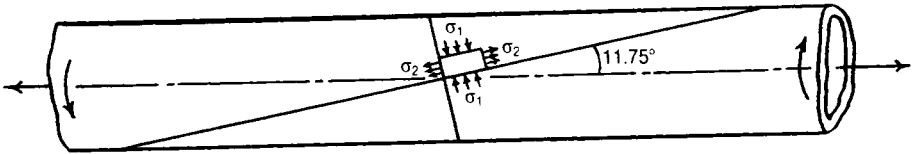
The shearing stress due to torque alone is

$$\tau = \frac{Tr_1}{J} = \frac{(30 \times 10^3)(0.100)}{0.00247} = 1.22 \text{ MN/m}^2$$

The maximum shearing stress due to the combined loading is

$$\tau_{max} = \frac{1}{2} [\sigma^2 + 4\tau^2]^{\frac{1}{2}} = 1.53 \text{ MN/m}^2$$

Problem 16.6 A thin steel tube of 2.5 cm diameter and 0.16 cm thickness has an axial pull of 10 kN, and an axial torque of 23.5 Nm applied to it. Find the magnitude and direction of the principal stresses at any point. *(Cambridge)*

Solution

It will be easier, and sufficiently accurate, to neglect the variation in the shearing stress from the inside to the outside of the tube. Let

τ = the mean shearing stress due to torsion

r = the mean radius = 0.0109 m

t = the thickness = 0.016 m

then the moment of the total resistance to shear

$$= 2 \pi r^2 \tau t = (1.19 \times 10^{-6}) \tau \text{ Nm}$$

If this is equal to 23.5 Nm, then

$$\tau = 19.75 \text{ MN/m}^2$$

The area of the cross-section is approximately

$$2\pi r t = 0.1098 \times 10^{-3} \text{ m}^2$$

Hence, the tensile stress is

$$\sigma = \frac{10 \times 10^3}{0.1098 \times 10^{-3}} = 91.1 \text{ MN/m}^2$$

The principal stresses are

$$\frac{1}{2} \left\{ \sigma \pm \sqrt{\sigma^2 + 4\tau^2} \right\} = \frac{1}{2} (91.1 \pm 99.3) \text{ MN/m}^2$$

Then

$$\sigma_1 = -4.1 \text{ MN/m}^2, \qquad \sigma_2 = +95.2 \text{ MN/m}^2$$

the positive sign denoting tension. The planes across which they act make angles θ and $(\theta + \pi/2)$ with the axis, where

$$\tan 2\theta = \frac{2\tau}{\sigma} = \frac{39.5}{91.1} = 0.434$$

giving $\theta = 11.75°$.

16.7   Strain energy of elastic torsion

In Section 16.3 we found that the torque–twist relationship for a circular shaft has the form

$$T = \frac{GJ\theta}{L}$$

This shows that the angle of twist, θ, of one end relative to the other, increases linearly with T. If one end of the shaft is assumed to be fixed, then the work done in twisting the other end through an angle θ is the area under the T–θ relationship, Figure 16.9. This work is conserved in the shaft as strain energy, which has the value

$$U = \frac{1}{2} T\theta \qquad\qquad\qquad\qquad (16.19)$$
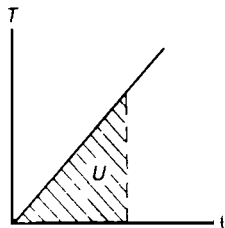
Figure 16.9   Linear torque–twist relationship and strain energy of elastic torsion.

On using equation (16.11) we may eliminate either θ or T, and we have

$$U = \left(\frac{L}{2GJ}\right) T^2 = \left(\frac{GJ}{2L}\right) \theta^2 \tag{16.20}$$

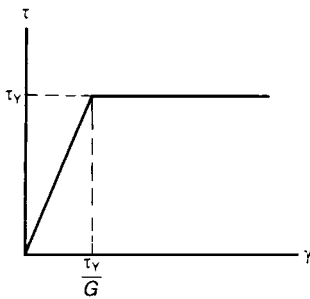## 16.8   Plastic torsion of a circular shaft

When a circular shaft is twisted the shearing stresses are greatest in the surface of the shaft. If the limit of proportionality of the material in shear is at a stress τ_Y, then this stress is first attained in the surface of the shaft at a torque

$$T = \frac{J\tau_Y}{a} \tag{16.21}$$

where J is the polar second moment of area, and a is the radius of the cross-section.

Suppose the material has the idealised shearing stress–strain curve shown in Figure 16.10; behaviour is elastic up to a shearing stress τ_Y, the shearing modulus being G. Beyond the limit of proportionality shearing proceeds at a constant stress τ_Y. This behaviour is nearly true of mild steel with a well-defined yield point.

If we are dealing with a solid circular shaft, then after the onset of plasticity in the surface fibres the shearing stresses vary radially in the form shown in Figure 16.11. The material within a radius b is still elastic; the material beyond a radius b is plastic and is everywhere stressed to the yield stress τ_Y

Figure 16.10   Idealized shearing stress–strain curve of mild steel.
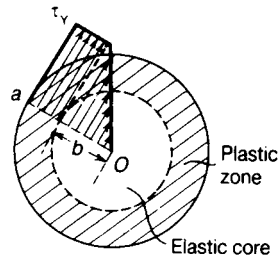
Figure 16.11   Elastic-plastic torsion of a solid circular shaft.

The torque sustained by the elastic core is

$$T_1 = \frac{J_1 \tau_Y}{b} = \frac{\pi}{2} b^3 \tau_Y \tag{16.22}$$

where subscripts 1 refer to the elastic core. The torque sustained by the outer plastic zone is

$$T_2 = \int_b^a 2\pi r^2 \tau_Y dr = \frac{2\pi}{3} \tau_Y [a^3 - b^3] \tag{16.23}$$

The total torque on the shaft is

$$T = T_1 + T_2 = \pi \tau_Y \left(\frac{2}{3} a^3 - \frac{1}{6} b^3 \right) = \frac{2\pi a^3}{3} \tau_Y \left[1 - \frac{b^3}{4a^3} \right] \tag{16.24}$$

The angle of twist of the elastic core is

$$\theta = \frac{\tau_Y L}{Gb} \tag{16.25}$$

where L is the length of the shaft. We assume that the outer plastic region suffers the same angle of twist; this is tantamount to assuming that radii remain straight during plastic torsion of the shaft. Equation (16.25) gives

$$b = \frac{\tau_Y L}{G\theta} \tag{16.26}$$

Then the torque becomes

$$T = \frac{2}{3} \pi a^3 \tau_Y \left[1 - \frac{(\tau_Y/G)^3}{4(a/L)^3} \left(\frac{1}{\theta^3} \right) \right] \tag{16.27}$$

At the onset of plasticity

$$\theta = \frac{\tau_Y L}{Ga} = \theta_Y \text{ (say)} \tag{16.28}$$

Then, for any other condition of torsion,

$$\theta = \frac{\tau_Y L}{Gb} = \theta_Y \left(\frac{a}{b} \right) \tag{16.29}$$

which gives

$$\left(\frac{b}{a} \right) = \frac{\theta_Y}{\theta} \tag{16.30}$$
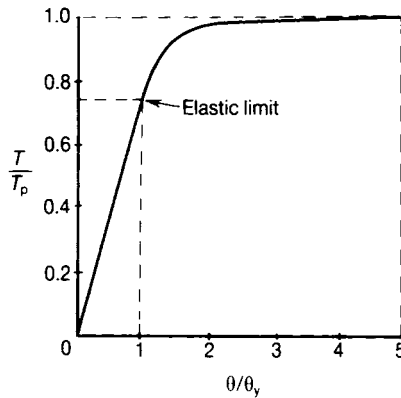
and equation (16.27) becomes

$$T = \frac{2\pi a^3}{3} \tau_Y \left[1 - \frac{1}{4} \left(\frac{\theta_Y}{\theta} \right)^3 \right]$$

(16.31)

When θ becomes very large, T approaches the value

$$\frac{2\pi a^3}{3} \tau_Y = T_P \text{ (say)}$$

(16.32)

which is the torque on the shaft when it is fully plastic. For smaller values of θ, we have then

$$\frac{T}{T_P} = 1 - \frac{1}{4} \left(\frac{\theta_Y}{\theta} \right)^3$$

(16.33)

Figure 16.12 Development of full plasticity in the torsion of a solid circular shaft.

This relationship, which is plotted in Figure 16.12 for values of θ/θ_p up to 5, shows that the fully plastic torque T_Y is approached rapidly after the elastic limit is exceeded. The torque T_Y at the elastic limit is

$$T_Y = \frac{\cdot \cdot}{4} T_F$$

(16.34)

If a torsion test is carried out on a thin-walled circular tube of mean radius r and thickness t, the average shearing stress due to a torque T is

$$\tau = \frac{T}{2\pi r^2 t}$$

from equation (16.2). If θ is the angle of twist of a length L of the tube, the shearing strain is

$$\gamma = \frac{r\theta}{L}$$

from equation (16.4). Thus, from a torsion test, in which values of T and θ are measured, the shearing stress τ and strain γ can be deduced. The resulting variation of τ and γ is called the *shearing stress–strain curve* of the material; the forms of these stress–strain curves are similar to tensile and compressive stress–strain curves, as shown in Figure 16.13. In the elastic range of a material

$$\tau = G\gamma$$

where G is the *shearing modulus* of the material (Section 3.4).
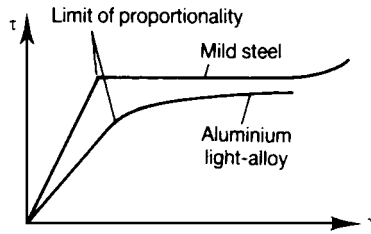
Figure 16.13   Forms of shearing stress–strain curves for mild steel
and for aluminium light alloys.

It is important to appreciate that the shearing stress–strain curve cannot be directly deduced from a torsion test of a *solid* circular bar, although the limit of proportionality can be estimated reasonably accurately.

16.9   Torsion of thin tubes of non-circular cross-section

In general the problem of the torsion of a shaft of non-circular cross-section is a complex one; in the particular case when the shaft is a hollow thin tube we can develop, however, a simple theory giving results that are sufficiently accurate for engineering purposes.

   Consider a thin-walled closed tube of uniform section throughout its length. The thickness of the wall at any point is t, Figure 16.14, although this may vary at points around the circumference of the tube. Suppose torques T are applied to each end so that the tube twists about a longitudinal axis Cz. We assume that the torque T is distributed over the end of the tube in the form of shearing stresses which are parallel to the tangent to the wall at any point, Figure 16.14, and that the ends of the tube are free from axial restraint. If the shearing stress at any point of the circumference is τ, then an equal complementary shearing stress is set up along the length of the tube. Consider the equilibrium of the section $ABCD$ of the wall: if the shearing stress τ at any point is uniform throughout the wall thickness then the shearing force transmitted over the edge BC is τt per unit length.
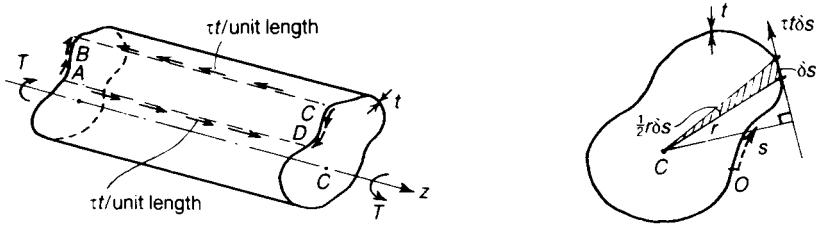
Figure 16.14   Torsion of a thin-walled tube of any cross-section.

For longitudinal equilibrium of $ABCD$ we must have that τt on BC is equal and opposite to τt on AD; but the section $ABCD$ is an arbitrary one, and we must have that τt is constant for all parts of the tube. Suppose this constant value of τt is

$$\tau t \;=\; q \tag{16.35}$$

The symbol 'q' is called the *shear flow*; it has the units of a load per unit length of the circumference of the tube.

Suppose we measure a distance s round the tube from some point O on the circumference, Figure 16.14. The force acting along the tangent to an element of length δs in the cross-section is $\tau t \delta s$. Suppose r is the length of the perpendicular from the centre of twist C onto the tangent. Then the moment of the force $\tau t \delta s$ about C is

$$\tau t r \delta s$$

The total torque on the cross-section of the tube is therefore

$$T = \oint \tau t r\,ds \tag{16.36}$$

where the integration is carried out over the whole of the circumference. But τt is constant and equal to q for all values of s. Then

$$T = \tau t \oint r\,ds = q \oint r\,ds \tag{16.37}$$

Now $\oint r\,ds$ is twice the area, A, enclosed by the centre line of the wall of the tube, and so

$$T \;=\; 2Aq \tag{16.38}$$

The shearing stress at any point is then

$$\tau = \frac{q}{t} = \frac{T}{2At} \tag{16.39}$$

To find the angle of twist of the tube we consider the strain energy stored in the tube, and equate this to the work done by the torques T in twisting the tube. When a material is subjected to shearing stresses τ the strain energy stored per unit volume of material is, from equation (3.5),

$$\frac{\tau^2}{2G}$$

where G is the shearing (or rigidity) modulus of the material. In the tube the shearing stresses are varying around the circumference but not along the length of the tube. Then the strain energy stored in a longitudinal element of length L, width δs and thickness t is

$$\left(\frac{\tau^2}{2G}\right) Lt\delta s$$

The total strain energy stored in the tube is therefore

$$U = \oint \frac{\tau^2}{2G} Lt\,ds \tag{16.40}$$

where the integration is carried out over the whole circumference of the tube. But τt is constant, and equal to q, and we may write

$$U = \oint \frac{\tau^2}{2G} Lt^2 \frac{ds}{t} = \frac{q^2 L}{2G} \oint \frac{ds}{t} \tag{16.41}$$

If the ends of the tube twist relative to each other by an angle θ, then the work done by the torques T is

$$W = \frac{1}{2}T\theta \tag{16.42}$$

On equating U and W, we have

$$\theta = \frac{q^2 L}{GT} \int \frac{ds}{t} \tag{16.43}$$

But from equation (16.38) we have

$$q = \frac{T}{2A} \tag{16.44}$$

Then equation (16.43) may be written

$$\theta = \frac{TL}{2A^2G} \int \frac{ds}{t}$$ (16.45)

For a tube of uniform thickness t,

$$\theta = \frac{TL}{4A^2G} \left(\frac{S}{t} \right)$$ (16.46)

where S is the total circumference of the tube.

Equation (16.45) can be written in the form

$$\theta = \frac{TL}{GJ}$$

where

$$J = \frac{4A^2}{\int \frac{ds}{t}}$$

J is the *torsion constant* for the section; for circular cross-sections J is equal to the polar second moment of area, but this is not true in general.

16.10 Torsion of a flat rectangular strip

A long flat strip of rectangular cross-section has a breadth b, thickness t, and length L. For uniform torsion about the centroid of the cross-section, the strip may be treated as a set of concentric thin hollow tubes, all twisted by the same amount.
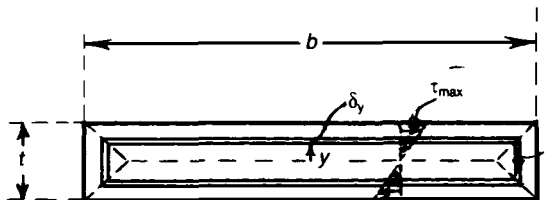

Figure 16.15 Torsion of a thin strip.

Consider such an elemental tube which is rectangular in shape the longer sides being a distance y from the central axis of the strip; the thickness of the tube is δy, Figure 16.15.

If δT is the torque carried by this elemental tube then the shearing stress in the longer sides of the tube is

$$\tau = \frac{\delta T}{4by\,\delta y} \tag{16.47}$$

where b is assumed very much greater than t. This relationship gives

$$\frac{dT}{dy} = 4by\tau \tag{16.48}$$

For the angle of twist of the elemental tube we have, from equation (16.46),

$$\theta = \frac{2bL\delta T}{16b^2 y^2 G\delta y} \tag{16.49}$$

where L is the length of the strip. This gives the further relationship

$$\frac{dT}{dy} = 8by^2 G\frac{\theta}{L} \tag{16.50}$$

On comparing equations (16.48) and (16.50), we have

$$\tau = 2yG\left(\frac{\theta}{L}\right) \tag{16.51}$$

This shows that the shearing stress τ varies linearly throughout the thickness of the strip, having a maximum value in the surface of

$$\tau_{max} = Gt\left(\frac{\theta}{L}\right) \tag{16.52}$$

An important feature is that the shearing stresses τ act parallel to the longer side b of the strip, and that their directions reverse over the thickness of the strip. This approximate solution gives an inexact picture of the shearing stresses near the corners of the cross-section.
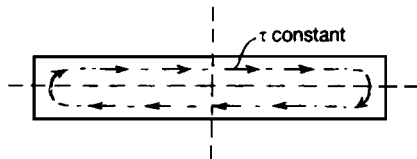

Figure 16.16 Directions of shearing stresses in the torsion of a thin strip.

We ought to consider not rectangular elemental tubes but flat tubes with curved ends. The contours of constant shearing stress are then continuous curves, Figure 16.16.

The total torque on the cross-section is

$$T = \int_0^{\frac{1}{2}t} 8by^2 G\left(\frac{\theta}{L}\right) dy = \frac{1}{3} bt^3 G\frac{\theta}{L} \tag{16.53}$$

The polar second moment of area of the cross-section about its centre is

$$J = \frac{1}{12}\left(bt^3 + b^3t\right) \tag{16.54}$$

If b is very much greater than t, then, approximately,

$$J = \frac{1}{12} b^3t \tag{16.55}$$

The geometrical constant occurring in equation (16.53) is $bt^3/3$; thus, in the torsion of a thin strip we cannot use the polar second moment of area for J in the relationship

$$\frac{T}{J} = \frac{G\theta}{L} \tag{16.56}$$

Instead we must use

$$J = \frac{1}{3} bt^3 \tag{16.57}$$

16.11 Torsion of thin-walled open sections

We may extend the analysis of the preceding section to the uniform torsion of thin-walled open-sections of any cross-sectional form.
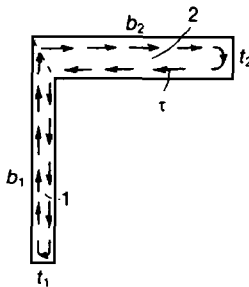

Figure 16.17 Torsion of an angle section.

In the angle section of Figure 16.17, we take elemental tubes inside the two limbs of the section. If t_1 and t_2 are small compared with b_1 and b_2, the maximum shearing stresses in limbs 1 and 2 are

$$\tau_1 \;=\; Gt_1\left(\frac{\theta}{L}\right) \qquad \tau_2 \;=\; Gt_2\left(\frac{\theta}{L}\right) \tag{16.58}$$

where the angle of twist per unit length, θ/L, is common to both limbs.

The greatest shearing stress occurs then in the surface of the thicker limb of the cross-section. The total torque is the summation of the torques carried by the two limbs, and has the value

$$T \;=\; \frac{1}{3}\left(b_1 t_1^3 + b_2 t_2^3\right) G\left(\frac{\theta}{L}\right) \tag{16.59}$$

In general, for a thin-walled open-section of any shape the shearing stress in the surface of a section of thickness t is

$$\tau \;=\; Gt\left(\frac{\theta}{L}\right) \tag{16.60}$$

The total torque on the section is

$$T \;=\; G\left(\frac{\theta}{L}\right) \sum \frac{1}{3} bt^3 \tag{16.61}$$

where the summation is carried out for all limbs of the cross-section.

Further problems (answers on page 693)

16.7   Find the maximum shearing stress in a propeller shaft 40 cm external, and 20 cm internal diameter, when subjected to a torque of 450 kNm. If $G = 80$ GN/m², what is the angle of twist in a length of 20 diameters? What diameter would be required for a solid shaft with the same maximum stress and torque? (RNC)

16.8   A propeller shaft, 45 m long, transmits 10 MW at 80 rev/min. The external diameter of the shaft is 57 cm, and the internal diameter 24 cm. Assuming that the maximum torque is 1.19 times the mean torque, find the maximum shearing stress produced. Find also the relative angular movement of the ends of the shaft when transmitting the average torque. Take $G = 80$ GN/m². (RNC)

16.9   A steel tube, 3 m long, 3.75 cm diameter, 0.06 cm thick, is twisted by a couple of 50 Nm. Find the maximum shearing stress, the maximum tensile stress, and the angle through which the tube twists. Take $G = 80$ GN/m². (Cambridge)

16.10 Compare the mass of a solid shaft with that of a hollow one to transmit a given power at a given speed with a given maximum shearing stress, the inside diameter of the hollow shaft being two-thirds of the outside diameter. *(Cambridge)*

16.11 A 2.5 cm circular steel shaft is provided with enlarged portions A and B. On to this enlarged portion a steel tube 0.125 cm thick is shrunk. While the shrinking process is going on, the 2.5 cm shaft is held twisted by a couple of magnitude 50 Nm. When the tube is firmly set on the shaft this twisting couple is removed. Calculate what twisting couple is left on the shaft, the shaft and tube being made of the same material. *(Cambridge)*
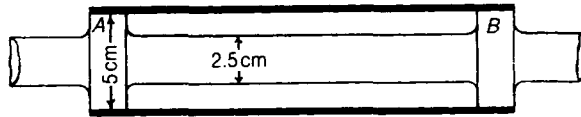
16.12 A thin tube of mean diameter 2.5 cm and thickness 0.125 cm is subjected to a pull of 7.5 kN, and an axial twisting moment of 125 Nm. Find the magnitude and direction of the principal stresses. *(Cambridge)*