20 Torsion of non-circular sections

20.1 Introduction

The torsional theory of circular sections (Chapter 16) cannot be applied to the torsion of non-circular sections, as the shear stresses for non-circular sections are no longer circumferential. Furthermore, plane cross-sections do not remain plane and undistorted on the application of torque, and in fact, warping of the cross-section takes place.

As a result of this behaviour, the polar second moment of area of the section is no longer applicable for static stress analysis, and it has to be replaced by a torsional constant, whose magnitude is very often a small fraction of the magnitude of the polar second moment of area.

20.2 To determine the torsional equation

Consider a prismatic bar of uniform non-circular section, subjected to twisting action, as shown in Figure 20.1.
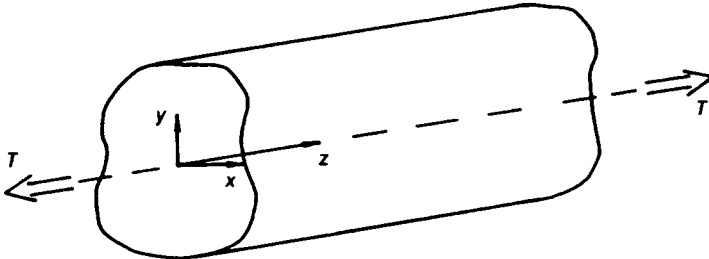

Figure 20.1 Non-circular section under twist.

Let,

T = torque

u = displacement in the x direction

v = displacement in the y direction

w = displacement in the z direction

= the warping function

θ = rotation / unit length
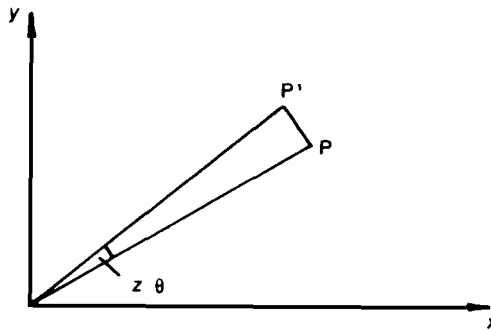
x, y, z = Cartesian co-ordinates

Figure 20.2 Displacement of P.

Consider any point P in the section, which, owing to the application of T, will rotate and warp, as shown in Figure 20.2:

$$u = -yz\theta$$
$$v = xz\theta$$

(20.1)

due to rotation, and

$$w = \theta \times \psi(x, y))$$
$$= \theta \times \psi$$

(20.2)

due to warping. The theory assumes that,

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = \gamma_{xy} = 0$$

(20.3)

and therefore the only shearing strains that exist are γ_{xz} and γ_{yz}, which are defined as follows:

$$\gamma_{xz} = \text{shear strain in the } x\text{-}z \text{ plane}$$

$$= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \theta\left(\frac{\partial \psi}{\partial x} - y\right)$$

(20.4)

$$\gamma_{yz} = \text{shear strain in the } y\text{-}z \text{ plane}$$

$$= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \theta \left(\frac{\partial \psi}{\partial y} + x \right)$$

(20.5)

The equations of equilibrium of an infinitesimal element of dimensions $dx \times dy \times dz$ can be obtained with the aid of Figure 20.3, where,

$$\tau_{xz} = \tau_{zx}$$

and

$$\tau_{yz} = \tau_{zy}$$

Resolving in the z-direction

$$\frac{\partial \tau_{yz}}{\partial y} \times dy \times dx \times dz + \frac{\partial \tau_{xz}}{\partial x} \times dx \times dy \times dz = 0$$

or

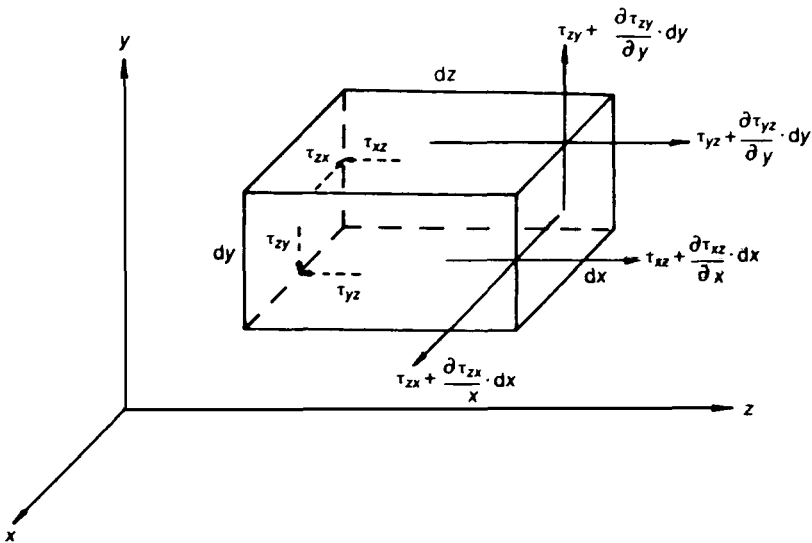$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0$$

(20.6)

Figure 20.3 Shearing stresses acting on an element.

However, from equations (20.4) and (20.5):

$$\tau_{xz} \;=\; G\gamma_{xz} \;=\; G\theta \left(\frac{\partial \psi}{\partial x} - y \right) \tag{20.7}$$

and

$$\tau_{yz} \;=\; G\gamma_{yz} \;=\; G\theta \left(\frac{\partial \psi}{\partial y} + x \right) \tag{20.8}$$

Let,

$$\frac{\partial \chi}{\partial y} \;=\; \frac{\partial \psi}{\partial x} - y \tag{20.9}$$

and,

$$-\frac{\partial \chi}{\partial x} \;=\; \frac{\partial \psi}{\partial y} + x \tag{20.10}$$

where χ is a shear stress function.

By differentiating equations (20.9) and (20.10) with respect to y and x, respectively, the following is obtained:

$$\frac{\partial_1^2 \chi}{\partial x^2} + \frac{\partial_1^2 \chi}{\partial y^2} \;=\; \frac{\partial^2 \Psi}{\partial x \,.\, \partial y} - 1 - \frac{\partial^2 \Psi}{\partial x \,.\, \partial y} - 1$$

or,

$$\frac{\partial_1^2 \chi}{\partial x^2} + \frac{\partial_1^2 \chi}{\partial y^2} \;=\; -2 \tag{20.11}$$

Equation (20.11) can be described as the *torsion equation for non-circular sections.*
From equations (20.7) and (20.8):

$$\tau_{xz} \;=\; G\theta \frac{\partial \chi}{\partial y} \tag{20.12}$$

and

$$\tau_{yz} \;=\; -G\theta \frac{\partial \chi}{\partial x} \tag{20.13}$$

Equation (20.11), which is known as Poisson's equation, can be put into the alternative form of equation (20.14), which is known as Laplace's equation.

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \qquad (20.14)$$

20.3   To determine expressions for the shear stress τ and the torque *T*

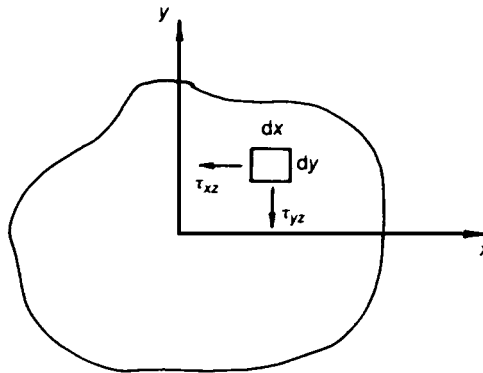Consider the non-circular cross-section of Figure 20.4.

Figure 20.4   Shearing stresses acting on an element.

From Pythagoras' theorem

τ = shearing stress at any point (x, y) on the cross-section

$$= \sqrt{\left(\tau_{xz}^2 + \tau_{yz}^2\right)} \qquad (20.15)$$

From Figure 20.4, the torque is

$$T = \int\int \left(\tau_{xz} \times y - \tau_{yz} \times x\right) dx \cdot dy \qquad (20.16)$$

To determine the *boundary value for* χ, consider an element on the boundary of the section, as shown in Figure 20.5, where the shear stress acts tangentially.   Now, as the shear stress perpendicular to the boundary is zero,

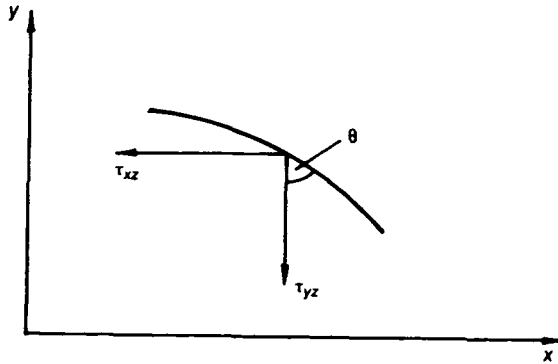$$\tau_{yz} \sin\varphi + \tau_{xz} \cos\varphi = 0$$

Figure 20.5 Shearing stresses on boundary.

or

$$-G\theta \times \frac{\partial \chi}{\partial x}\left(-\frac{dx}{ds}\right) + G\theta \times \frac{\partial \chi}{\partial y}\left(\frac{dy}{ds}\right) = 0$$

or

$$G\theta\frac{d\chi}{ds} = 0$$

where s is any distance along the boundary, i.e. χ is a constant along the boundary.

Problem 20.1 Determine the shear stress function χ for an elliptical section, and hence, or otherwise, determine expressions for the torque T, the warping function w and the torsional constant J.
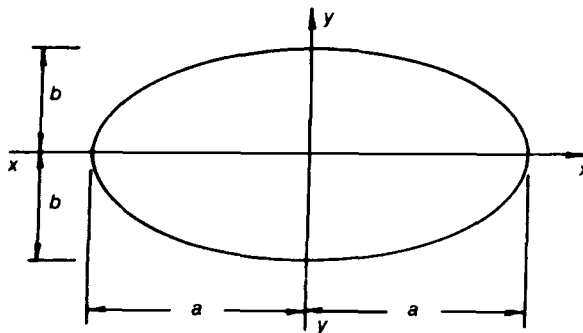

Figure 20.6 Elliptical section.

Solution

The equation for the ellipse of Figure 20.6 is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{20.17}$$

and this equation can be used for determining the shear stress function χ as follows:

$$\chi = C\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) \tag{20.18}$$

where C is a constant, to be determined.

Equation (20.18) ensures that χ is constant along the boundary, as required. The constant C can be determined by substituting equation (20.18) into (20.11), i.e.

$$C\left(\frac{2}{a^2} + \frac{2}{b^2}\right) = -2$$

therefore

$$C = \frac{-a^2 b^2}{a^2 + b^2}$$

and

$$\chi = \frac{a^2 b^2}{(a^2 + b^2)}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) \tag{20.19}$$

where χ is the required stress function for the elliptical section.

Now,

$$\tau_{xz} = G\theta \frac{\partial \chi}{\partial y} = -G\theta \frac{2ya^2}{a^2 + b^2}$$

$$\tau_{yz} = -G\theta \frac{\partial \chi}{\partial x} = \frac{G\theta \; 2xb^2}{a^2 + b^2}$$

and

$$T = \int (\tau_{xz} y - \tau_{yz} x) dA$$

$$= -G\theta \int \left(\frac{2x^2 b^2}{a^2 + b^2} + \frac{2y^2 a^2}{a^2 + b^2} \right) dA$$

$$= -2G\theta \frac{a^2 b^2}{a^2 + b^2} \left[\int \frac{x^2}{a^2} dA + \int \frac{y^2}{b^2} dA \right]$$

but

$$\int y^2 dA = I_{xx} = \frac{\pi a b^3}{4} = \text{second moment of area about } x\text{–}x$$

and,

$$\int x^2 dA = I_{yy} = \frac{\pi a^3 b}{4} = \text{second moment of area about } y\text{–}y$$

therefore

$$T = -2G\theta \frac{a^2 b^2}{a^2 + b^2} \left(\frac{\pi a b}{4} + \frac{\pi a b}{4} \right)$$

(20.20)

$$T = \frac{-G\theta \pi a^3 b^3}{a^2 + b^2}$$

therefore

$$\tau_{xz} = \frac{-2a^2 y}{(a^2 + b^2)} \cdot \frac{-(a^2 + b^2)T}{\pi a^3 b^3}$$

$$\tau_{xz} = \frac{2Ty}{\pi a b^3}$$

(20.21)

$$\tau_{yz} = \frac{-2Tx}{\pi a^3 b}$$

(20.22)

By inspection, it can be seen that $\hat{\tau}$ is obtained by substituting $y = b$ into (20.21), provided $a > b$.

$$\hat{\tau} = \text{maximum shear stress}$$

$$= \frac{2T}{\pi a b^2} \qquad (20.23)$$

and occurs at the extremities of the minor axis.

The warping function can be obtained from equation (20.2). Now,

$$\frac{\partial \chi}{\partial y} = \frac{\partial \psi}{\partial x} - y$$

or

$$\frac{2ya^2b^2}{(a^2 + b^2)b^2} = \frac{\partial \psi}{\partial x} - y$$

i.e.

$$\frac{\partial \psi}{\partial x} = \frac{(-2a^2 + a^2 + b^2)}{(a^2 + b^2)}y$$

therefore

$$\psi = \left(\frac{b^2 - a^2}{a^2 + b^2} \right) xy \qquad (20.24)$$

Similarly, from the expression

$$-\frac{\partial \chi}{\partial x} = \frac{\partial \psi}{\partial y} + x$$

the same equation for ψ, namely equation (20.24), can be obtained. Now,

$$w = \text{warping function}$$

$$= \theta \times \psi$$

therefore

$$w = \frac{(b^2 - a^2)}{(a^2 + b^2)} \theta xy \tag{20.25}$$

From simple torsion theory,

$$\frac{T}{J} = G\theta \tag{20.26}$$

or

$$T = G\theta J \tag{20.27}$$

Equating (20.20) and (20.27), and ignoring the negative sign in (20.20),

$$G\theta J = \frac{G\theta \pi a^3 b^3}{(a^2 + b^2)}$$

therefore

J = torsional constant for an elliptical section

$$J = \frac{\pi a^3 b^3}{(a^2 + b^2)} \tag{20.28}$$

Problem 20.2   Determine the shear stress function χ and the value of the maximum shear stress $\hat{\tau}$ for the equilateral triangle of Figure 20.7.
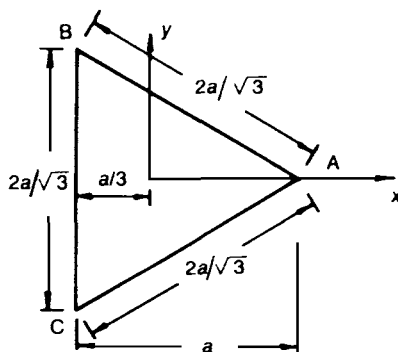

Figure 20.7 Equilateral triangle.

Solution

The equations of the three straight lines representing the boundary can be used for determining χ, as it is necessary for χ to be a constant along the boundary.

Side BC
This side can be represented by the expression

$$x = -\frac{a}{3} \quad \text{or} \quad x + \frac{a}{3} = 0 \tag{20.29}$$

Side AC
This side can be represented by the expression

$$x - \sqrt{3}\, y - \frac{2a}{3} = 0 \tag{20.30}$$

Side AB
This side can be represented by the expression

$$x + \sqrt{3}\, y - \frac{2a}{3} = 0 \tag{20.31}$$

The stress function χ can be obtained by multiplying together equations (20.29) to (20.31):

$$\chi = C(x + a/3) \times \left(x - \sqrt{3}\, y - 2a/3\right) \times \left(x + \sqrt{3}\, y - 2a/3\right) \tag{20.32}$$

$$= C\left\{\left(x^3 - 3xy^2\right) - a\left(x^2 + y^2\right) + 4a^3/27\right\}$$

From equation (20.32), it can be seen that $\chi = 0$ (i.e. constant) along the external boundary, so that the boundary condition is satisfied.
   Substituting χ into equation (20.11),

$$C(6x - 2a) + C(-6x - 2a) = -2$$

$$- 4aC = -2$$

$$C = 1/(2a)$$

therefore

$$\chi \quad \frac{1}{2a}\left(x^3 - 3xy^2\right) - \frac{1}{2}\left(x^2 + y^2\right) + \frac{2a^2}{27} \tag{20.33}$$

Now

$$\tau_{xz} = G\theta \frac{\partial \chi}{\partial y}$$

$$= G\theta \left\{ \frac{1}{2a} (-6xy) - \frac{1}{2} \times 2y \right\}$$

$$\tau_{xz} = -G\theta \left(\frac{3xy}{2a} + y \right) \tag{20.34}$$

Along

$$y = 0, \ \tau_{xz} = 0.$$

Now

$$\tau_{yz} = -G\theta \frac{\partial \chi}{\partial x} = -G\theta \left\{ \frac{1}{2a} (3x^2 - 3y^2) - \frac{1}{2} \times 2x \right\}$$

therefore

$$\tau_{yz} = -\frac{3G\theta}{2a} \left\{ (x^2 - y^2) - \frac{2ax}{3} \right\} \tag{20.35}$$

As the triangle is equilateral, the maximum shear stress $\hat{\tau}$ can be obtained by considering the variation of τ_{yz} along any edge. Consider the edge BC (i.e. $x = -a/3$):

$$\tau_{yz} \ (\text{edge} \ BC) = -\frac{3G\theta}{2a} \left(\frac{a^2}{9} - y^2 + \frac{2a^2}{9} \right)$$

$$\tag{20.36}$$

$$= -\frac{3G\theta}{2a} \left(\frac{a^2}{3} - y^2 \right)$$

where it can be seen from (20.36) that $\hat{\tau}$ occurs at $y = 0$. Therefore

$$\hat{\tau} = -G\theta a/2 \tag{20.37}$$

20.4   Numerical solution of the torsional equation

Equation (20.11) lends itself to satisfactory solution by either the finite element method or the finite difference method and Figure 20.8 shows the variation of χ for a rectangular section, as obtained by the computer program LAPLACE. (The solution was carried out on an Apple II + microcomputer, and the screen was then photographed.) As the rectangular section had two axes of symmetry, it was only necessary to consider the top right-hand quadrant of the rectangle.
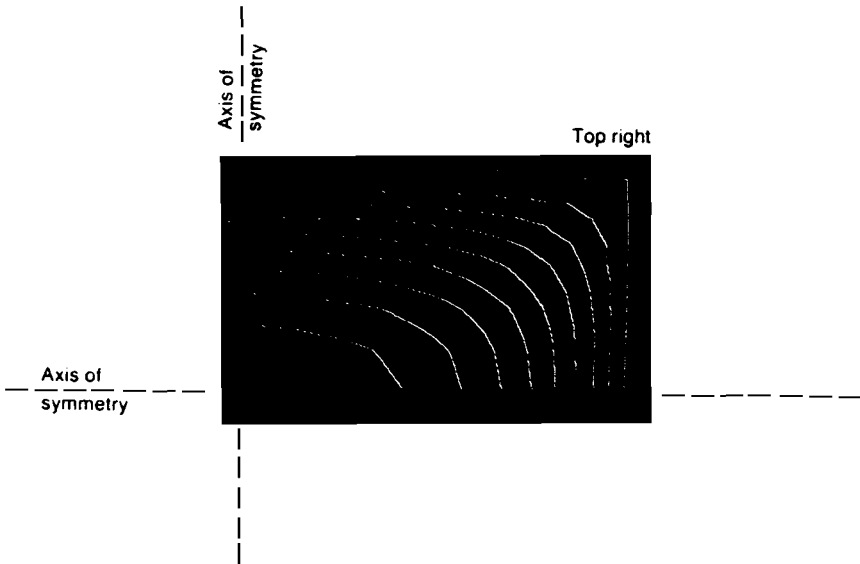

Figure 20.8   Shear stress contours.

20.5   Prandtl's membrane analogy

Prandtl noticed that the equations describing the deformation of a thin weightless membrane were similar to the torsion equation. Furthermore, he realised that as the behaviour of a thin weightless membrane under lateral pressure was more readily understood than that of the torsion of a non-circular section, the application of a membrane analogy to the torsion of non-circular sections considerably simplified the stress analysis of the latter.

Prior to using the membrane analogy, it will be necessary to develop the differential equation of a thin weightless membrane under lateral pressure. This can be done by considering the equilibrium of the element AA $'$ BB $'$ in Figure 20.9.
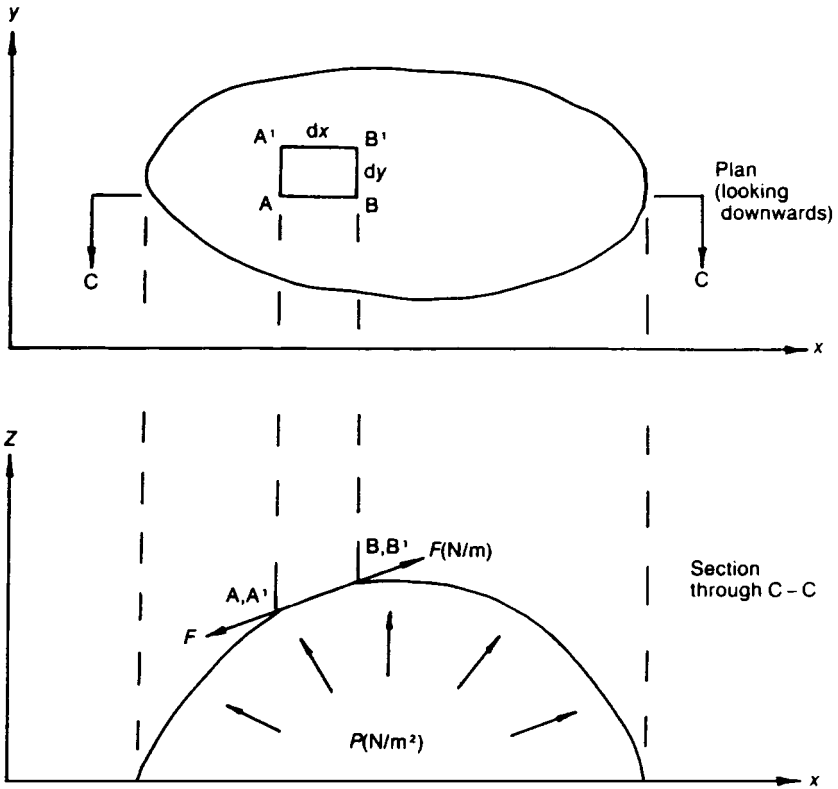
Figure 20.9 Membrane deformation.

Let,

F = membrane tension per unit length (N/m)

Z = deflection of membrane (m)

P = pressure (N/m^2)

Component of force on AA ′ in the z-direction is $F \times \dfrac{\partial Z}{\partial x} \times dy$ ↓

Component of force on BB ′in the z-direction is $F\left(\dfrac{\partial Z}{\partial x} + \dfrac{\partial^2 Z}{\partial x^2} \times dx \right) dy$ ↑

Component of force on AB in the z-direction is $F \times \dfrac{\partial Z}{\partial y} \times dx$ ↓

Component of force on A′B′in the z-direction is $F \times \left(\dfrac{\partial Z}{\partial y} + \dfrac{\partial^2 Z}{\partial y^2} \times dy \right) dx$ ↑

Resolving vertically

$$F \left(\frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2} \right) dz \times dy = -P \times dx \times dy$$

therefore

$$\frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2} = -\frac{P}{F} \tag{20.38}$$

If $Z = \chi$ in equation (20.38), and the pressure is so adjusted that $P/F = 2$, then it can be seen that equation (20.38) can be used as an analogy to equation (20.11).
From equations (20.12) and (20.13), it can be seen that

$$\tau_{xz} = G\theta \times \text{slope of the membrane in the } y \text{ direction}$$

$$\tag{20.39}$$

$$\tau_{yz} = G\theta \times \text{slope of the membrane in the } x \text{ direction}$$

Now, the torque is

$$T = \iint \left(\tau_{xz} \times y - \tau_{yz} \times x \right) dx \, dy$$

$$= G\theta \iint \left(\frac{\partial Z}{\partial y} \times y + \frac{\partial Z}{\partial x} \times x \right) dx dy \tag{20.40}$$

Consider the integral

$$\iint \frac{\partial Z}{\partial y} \times y \times dx dy = \iint \partial Z \times y \times dx$$

Now y and dx are as shown in Figure 20.10, where it can be seen that $\iint y \times dx$ is the area of section. Therefore the

$$\iint \frac{\partial Z}{\partial y} \times y \times dx \times dy = \text{volume under membrane} \tag{20.41}$$
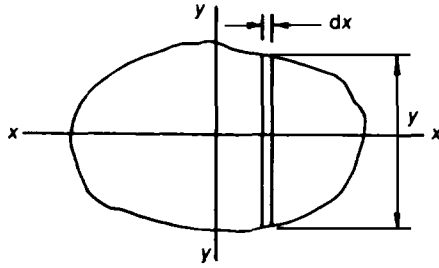
Figure 20.10

Similarly, it can be shown that the volume under membrane is

$$\iint \frac{\partial Z}{\partial x} \times x \times dx \times dy \qquad (20.42)$$

Substituting equations (20.41) and (20.42) into equation (20.40):

$$T = 2G\theta \times \text{volume under membrane} \qquad (20.43)$$

Now

$$\frac{T}{J} = G\theta$$

which, on comparison with equation (20.43), gives

$$J = \text{torsional constant}$$

$$= 2 \times \text{volume under membrane} \qquad (20.44)$$

20.6   Varying circular cross-section

Consider the varying circular section shaft of Figure 20.11, and assume that,

$$u = w = 0$$

where,

$$u = \text{radial deflection}$$
$$v = \text{circumferential deflection}$$
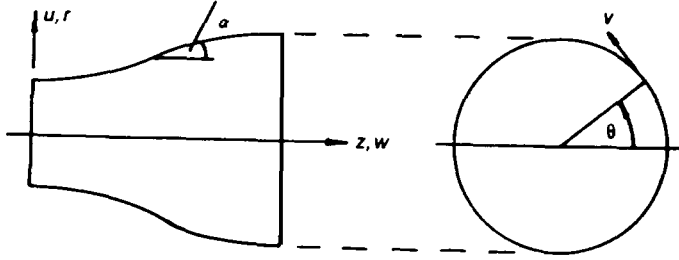$$w = \text{axial deflection}$$

Figure 20.11   Varying section shaft.

As the section is circular, it is convenient to use polar co-ordinates. Let,

ε_r =   radial strain   = 0

ε_θ =   hoop strain   = 0

ε_z =   axial strain   = 0

γ_{rz} =   shear strain in a longitudinal radial plane   = 0

r =   any radius on the cross-section

Thus, there are only two shear strains, $\gamma_{r\theta}$ and $\gamma_{\theta z}$, which are defined as follows:

$$\gamma_{r\theta} = \text{shear strain in the r--}\theta \text{ plane} = \frac{\partial v}{\partial r} - \frac{v}{r}$$

$$\gamma_{\theta z} = \text{shear strain in the }\theta\text{--}z \text{ plane} = \frac{\partial v}{\partial z}$$

But

$$\tau_{r\theta} = G\gamma_{r\theta} = G\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) \tag{20.45}$$

and

$$\tau_{\theta z} = G\gamma_{\theta z} = G\frac{\partial v}{\partial z} \tag{20.46}$$

From equilibrium considerations,

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} = 0$$

which, when rearranged, becomes

$$\frac{\partial}{\partial r}\left(r^2 \tau_{r\theta}\right) + \frac{\partial}{\partial z}\left(r^2 \tau_{\theta z}\right) = 0 \tag{20.47}$$

Let κ be the shear stress function

where

$$\frac{\partial \kappa}{\partial r} = r^2 \tau_{\theta z} \tag{20.48}$$

and

$$\frac{\partial \kappa}{\partial z} = -r^2 \tau_{r\theta} \tag{20.49}$$

which satisfies equation (20.47).
  From compatibility considerations

$$\frac{\partial \gamma_{r\theta}}{\partial z} = \frac{\partial \gamma_{\theta z}}{\partial r} - \frac{\gamma_{\theta z}}{r}$$

or

$$\frac{\partial \tau_{r\theta}}{\partial z} = \frac{\partial \tau_{\theta z}}{\partial r} - \frac{\tau_{\theta z}}{r} \tag{20.50}$$

From equation (20.49)

$$\frac{\partial \tau_{r\theta}}{\partial z} = -\frac{1}{r^2}\frac{\partial^2 \kappa}{\partial z^2} \tag{20.51}$$

From equation (20.48)

$$\frac{\partial \tau_{\theta z}}{\partial r} = \frac{1}{r^2}\frac{\partial^2 \kappa}{\partial r^2} - \frac{2}{r^3}\frac{\partial \kappa}{\partial r} \tag{20.52}$$

Substituting equations (20.59) and (20.52) into equation (20.50) gives

$$-\frac{1}{r^2}\frac{\partial^2 \kappa}{\partial z^2} = \frac{1}{r^2}\frac{\partial^2 \kappa}{\partial r^2} - \frac{2}{r^3}\frac{\partial \kappa}{\partial r} - \frac{1}{r^3}\frac{\partial \kappa}{\partial r}$$

or

$$\frac{\partial^2 \kappa}{\partial r^2} - \frac{3}{r}\frac{\partial \kappa}{\partial r} + \frac{\partial^2 \kappa}{\partial z^2} = 0 \qquad (20.53)$$

From considerations of equilibrium on the boundary,

$$\tau_{r\theta}\cos\alpha - \tau_{\theta z}\sin\alpha = 0 \qquad (20.54)$$

where

$$\cos\alpha = \frac{dz}{ds}$$

$$\qquad (20.55)$$

$$\sin\alpha = \frac{dr}{ds}$$

Substituting equations (20.48), (20.49) and (20.55) into equation (20.54),

$$-\frac{1}{r^2}\frac{\partial \kappa}{\partial z}\frac{dz}{ds} - \frac{1}{r^2}\frac{\partial \kappa}{\partial r}\frac{dr}{ds} = 0$$

or

$$\frac{2}{r^2}\frac{d\kappa}{ds} = 0$$

i.e. κ is a constant on the boundary, as required.

Equation (20.53) is the torsion equation for a tapered circular section, which is of similar form to equation (20.11).

20.7   Plastic torsion

The assumption made in this section is that the material is ideally elastic-plastic, as described in Chapter 15, so that the shear stress is everywhere equal to τ_{yp}, the yield shear stress. As the shear stress is constant, the slope of the membrane must be constant, and for this reason, the membrane analogy is now referred to as a sand-hill analogy.

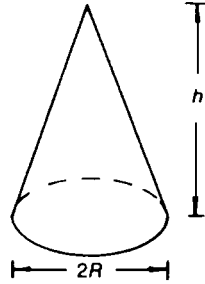Consider a *circular section*, where the sand-hill is shown in Figure 20.12.

Figure 20.12   Sand-hill for a circular section.

From Figure 20.12, it can be seen that the volume (*Vol*) of the sand-hill is

$$Vol = \frac{1}{3}\pi R^2 h$$

but

$$\tau_{yp} = G\theta \times \text{slope of the sand-hill}$$

where

$$\theta = \text{twist/unit length} \rightarrow \infty$$

$$G = \text{modulus of rigidity} \rightarrow 0$$

$$\therefore \tau_{yp} = G\theta\,\frac{h}{R}$$

or

$$h = R\tau_{yp}/G\theta$$

and

$$Vol = \frac{\pi R^3 \tau_{yp}}{3G\theta}$$

Now

$$J = 2 \times Vol = 2\pi R^3 \tau_{yp}/(3G\theta)$$

and

$$T_p = G\theta J = G\theta \times 2\pi R^3 \tau_{yp}/(3G\theta)$$

therefore

$$T_p = 2\pi R^3 \tau_{yp}/3$$

where T_p is the fully plastic torsional moment of resistance of the section, which agrees with the value obtained in Chapter 4.

Consider a *rectangular section,* where the sand-hill is shown in Figure 20.13.
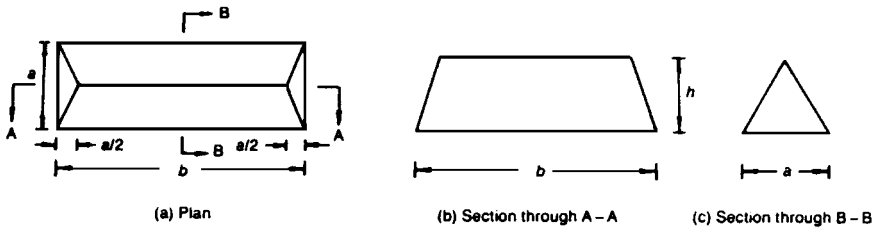

(a) Plan   (b) Section through A – A   (c) Section through B – B

Figure 20.13 Sand-hill for rectangular section.

The volume under sand-hill is

$$Vol = \frac{1}{2}abh - \frac{1}{3}\left(\frac{1}{2}a \times \frac{a}{2}\right) \times h \times 2$$

$$= \frac{1}{2}abh - \frac{a^2 h}{6}$$

$$= \frac{ah}{6}(3b - a)$$

and $\quad \tau_{yp} = G\theta \times \text{slope of sand-hill} = G\theta \times 2h/a$

or

$$h = \frac{a\tau_{yp}}{2G\theta}$$

therefore

$$Vol = \frac{a\,(3b - a)a\tau_{yp}}{12\,G\theta}$$

Now

$$J = 2 \times \text{vol} = a^2(3b - a)\tau_{yp}/(6G\theta)$$

and

$$T_p = G\theta J$$

therefore

$$T_p = a^2(3b - a)\tau_{yp}/6$$

where T_p is the fully plastic moment of resistance of the rectangular section.

Consider an equilateral triangular section, where the sand-hill is shown in Figure 20.14.
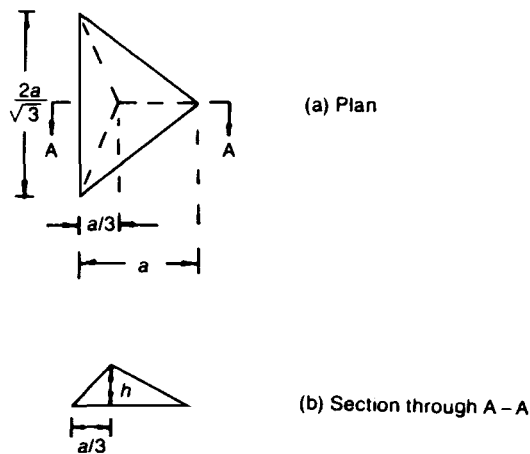

(a) Plan

(b) Section through A – A

Figure 20.14 Sand-hill for triangular section.

Now

$$\tau_{yp} = G\theta \times \text{slope of sand-hill}$$

or

$$\tau_{yp} = G\theta \times \frac{3h}{a}$$

and

$$h = \frac{a\tau_{yp}}{3G\theta}$$

therefore, the volume of the sand-hill is

$$Vol = \frac{1}{3}\left(\frac{1}{2} \times \frac{2a}{\sqrt{3}} \times a\right) \times h$$

$$= \frac{a^2}{3\sqrt{3}} \times \frac{a\tau_{yp}}{3G\theta}$$

$$= \frac{a^3\tau_{yp}}{9\sqrt{3}G\theta}$$

and

$$T_p = 2G\theta \times \frac{a^3\tau_{yp}}{9\sqrt{3}G\theta}$$

$$T_p = \frac{2a^3\tau_{yp}}{9\sqrt{3}}$$

where T_p is the fully plastic torsional resistance of the triangular section.