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NEWTON'S

INTERPOLATION FORMULAS.

BY

DUNCAN C. FRASER, M.A., F.I.A.

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I.

FOLLOWING upon a suggestion which was made in a previous number of the *Journal* (vol. l, p. 20), an endeavour has been made in the following pages to bring together the whole of Newton's work on the subject of interpolation by means of formulas of finite differences. His contributions to this subject are comprised in three items (1) the "Methodus Differentialis"; (2) a letter written in 1675, giving detailed instructions for the calculation of certain tables; and (3) the celebrated Lemma No. 5 in Book III of the "Principia."

There are also various references to the subject in the "Commercium Epistolicum", a collection of letters relating to the controversy between Newton and Leibnitz as to the origin of the differential calculus.

Methodus Differentialis.—This is a short treatise, complete in itself, on central formulas of interpolation and their applications. It was first published in the year 1711 by William Jones in a volume in which he collected a number of Newton's shorter works under the title "Analysis Per "Quantitatum Series, Fluxiones ac Differentias: cum "Enumeratio Linearum Tertii Ordinis."

A photographic reproduction of the original Latin text,*

* The scale of the original has been reduced by one-fifth.

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taken from a copy of the first edition in the Institute Library, and a translation are given. It is believed that no previous translation of this little work has been published.

Although the "Methodus Differentialis" was not printed until the year 1711, it was composed many years earlier. In the Latin preface (from which I translate), William Jones says:

> "The book is brought to a graceful close by the "addition of a little tract entitled 'Methodus Differen- "tialis', which I have transcribed by permission of the "distinguished author from his own autograph. . . . This "'Methodus Differentialis' depends upon the problem of "drawing a Parabolic Curve through a given number of "points, reference to which had been made by the "distinguished author in his letter to Oldenburg, sent in "1676, and a solution of which he gave in Lemma 5, "Book III of his 'Principia' by means of a construction "which is not at all the same as that which we now "present."

The letter to which William Jones refers is a letter dated 24 October 1676, which is included in the "Commercium Epistolicum." In the course of the letter Newton describes a method by which the function $\sqrt{a^2-ax+\frac{x^2}{2}}$ might be expanded in a series of powers of x; and then goes on to say (I translate freely from the original Latin), "But I attach "little importance to this method because when simple series "are not obtainable with sufficient ease, I have another "method not yet published by which the problem is easily "dealt with. It is based upon a convenient, ready and general "solution of this problem, *To describe a geometrical curve which "shall pass through any given points.*"

He then refers to cases in which such a problem can be solved by geometrical constructions without calculation; and adds: "but the above problem is of another kind; and "although it may seem to be intractable at first sight, it is "nevertheless quite the contrary; perhaps indeed it is one of "the prettiest problems that I can ever hope to solve." This fixes the date of composition of the "Methodus" as prior to October 1676; and there is some reason to think that its date may be several years earlier.

A short account of the "Methodus Differentialis" has previously been given in the *Journal* (vol. xv, pp. 145 and 177), by Professor Ludwig Oppermann, of Copenhagen. He there gives the date of publication in error as 1715. He gives his opinion that this little treatise was written many years before the Lemma, basing his view apparently on internal evidence only. The preface, which is conclusive as to the priority of the " Methodus ", is not printed in Horsley's complete edition of Newton's works and may not have been seen by Professor Oppermann.

NOTES.

In *Proposition I* it is shown that if the ordinate corresponding to the abscissa $A+x$ is $a+bx+cx^2+dx^3+ex^4+\ldots$ then expressions for all the divided differences can be exactly obtained. This is proved by actual division for the case when the highest power of x involved in the expression for the ordinate is the fourth.

Proposition II.—In the same case as in Proposition I, the values of five ordinates being known, full directions are given for the solution of the five simultaneous equations, from which can be obtained the values of the coefficients in terms of an ordinate and of divided differences of the ordinates.

In these two Propositions there is some confusion in the original text as to the first term in the expression for the ordinate. In the enunciation of Proposition I it is omitted, and I have supplied it. In the Table it is given as A, the abscissa being $A+x$. In the demonstrations it is not mentioned until the end of Proposition II, and an error occurs there, it being stated that the final operation in the solution of the simultaneous equation gives the first term of the abscissa A. It is quite clear that what is obtained is the first term of the ordinate, and in the translation this has been called a in correspondence with the remaining coefficients b, c, d, &c.

Proposition III. Case I.—The ordinates being equidistant and the number being odd a central difference formula is given in terms of the central ordinate and the central differences which are in line with it. The coefficients of the formula are

$$1,\ x,\ \frac{1}{2}x^2,\ \frac{x(x^2-1)}{6},\ \frac{x(x^2-1)}{24},\ \text{\&c.},$$

and it will be recognized that this is the formula which is commonly called Stirling's. The differences used in both cases of this proposition are not divided differences but simple differences without division. It will be found on examination that Newton takes his differences and measures the values of x in a sense opposite to that which is now customary. To bring the details of the work into conformity with our present practice the signs of the odd powers of x and the signs of the odd differences would have to be altered; but as it happens that the odd powers of x always occur in combination with odd differences, this makes no difference in the formula.

Case II.—The ordinates being equidistant and their number being even, a formula is given in terms of the mean of the two central ordinates and of the central differences opposite to that mean. The coefficients of the formula are

$$1, \; x, \; \frac{4x^2-1}{8}, \; \frac{x(4x^2-1)}{24}, \; \frac{(4x^2-1)(4x^2-9)}{384}, \; \&c.$$

This formula is now commonly known by the name of Bessel's formula. There is a misprint in the original, e_2+e_3 being printed for $\frac{e_2+e_3}{2}$.

Proposition IV.—A misprint occurs in the original, where the expression for the difference $b6$ ought to be $\frac{\text{A6B6}+\text{A7B7}}{\text{A6A7}}$ and not $\frac{\text{A6B6}-\text{A7B7}}{\text{A6A7}}$. Newton uses the expressions A6, B6, &c., to represent the arithmetical values of the lengths of the ordinates without reference to sign, and an example of the same practice will be found in the Lemma.

Case I.—An odd number of ordinates being given at points on the abscissa $\alpha, \beta, \gamma, \delta$, &c., which are separated by unequal intervals, a central difference formula is given in terms of the ordinate at the central point δ and of the central divided differences which are in line with it, the coefficients being

$$1, \; x-\delta, \; (x-\delta)\times\frac{1}{2}\left\{\begin{matrix}(x-\gamma)\\+(x-\epsilon)\end{matrix}\right\}, \; (x-\delta)(x-\gamma)(x-\epsilon),$$

$$(x-\delta)(x-\gamma)(x-\epsilon)\times\frac{1}{2}\left\{\begin{matrix}(x-\beta)\\+(x-\zeta)\end{matrix}\right\}, \; \&c.$$

Case II.—The number of ordinates being even and the two central ordinates being at the points δ and ϵ, a formula is given in terms of the mean of the two central ordinates and of the divided differences in line with that mean. The coefficients are

$$1,\ \frac{1}{2}\left\{\begin{matrix}(x-\delta)\\+(x-\epsilon)\end{matrix}\right\},\ (x-\delta)(x-\epsilon),\ (x-\delta)(x-\epsilon)\times\frac{1}{2}\left\{\begin{matrix}(x-\gamma)\\+(x-\zeta)\end{matrix}\right\},$$

$$(x-\delta)(x-\epsilon)(x-\gamma)(x-\zeta),\ \&c.$$

The analogy of these two formulas for divided differences with Stirling's formula and Bessel's formula will be easily seen.

In *Proposition V* Newton points out the application of the above four formulas when it is required to find any intermediate term of a series, of which certain terms are given.

In *Proposition VI* he points out that approximate expressions for the area of a curve, of which certain ordinates are known, can be derived from the preceding formulas.

In the *Scholium* Newton gives well-known formulas for the bisection of an interval, and for finding the area, when four ordinates are known. He then goes on to describe a process by which the problem of finding the approximate area when $2n+1$ ordinates are known can be reduced to the case of finding the area in terms of $n+1$ ordinates. It will be found on examination that Newton's process amounts to exactly the same thing as applying the formula for $n+1$ ordinates separately to the two halves of the curve of which $2n+1$ ordinates are given.

The meaning of his next paragraph is not entirely clear, but Newton's idea may have been to simplify the process of finding the approximate area by taking the sums of the ordinates in two's or three's, &c., using these sums as new ordinates and passing through their extremities a new curve, the area of which, taken between suitable limits, would approximate to the area required.

Letter on the Construction of Tables.—The two letters here printed, the first of which is simply a letter from William Jones to Professor Cotes, enclosing a letter from Newton, dated 8 May 1675, to Mr. John Smith, are taken from a volume published by J. Edleston, M.A., in 1850, and entitled "Correspondence of Sir Isaac Newton and Professor Cotes, "including letters of other eminent men, now first published "from the originals in the Library of Trinity College, "Cambridge", &c. The directions for the construction of tables given by Newton are of a very practical character, and will be easily followed by anyone who wishes to examine his method in detail.

The formulas employed can readily be obtained by applying the binomial theorem and by using Stirling's interpolation formula.

The two principal formulas, namely:

$$s=\omega+\frac{1}{2}st+\frac{1}{6}m$$

$$\zeta=\frac{\omega}{10}+\frac{\frac{1}{2}st}{100}+\frac{m}{6000}$$

expressed in modern notation are as follows:

$$\mathrm{F}(x+1)-\mathrm{F}(x)=\mathrm{F}'(x)+\frac{1}{2}\mathrm{F}''(x)+\frac{1}{6}\mathrm{F}'''(x)$$

$$\mathrm{F}\left(x+\frac{1}{10}\right)-\mathrm{F}(x)=\frac{1}{10}\mathrm{F}'(x)+\frac{1}{2}\cdot\frac{1}{100}\mathrm{F}''(x)+\frac{1}{6}\cdot\frac{1}{1000}\mathrm{F}'''(x)$$

The last term in this second formula, added by Mr. Edleston for the sake of completeness, appears to be superfluous.

It will be noticed that they are in fact formulas of the differential calculus and can be written down at once from Taylor's theorem. But Newton's method of obtaining them was more probably that suggested above. Brook Taylor, the discoverer of Taylor's theorem, was not born until 1685, and the theorem, which he obtained as a simple collorary to Newton's descending difference formula by making the differences indefinitely small, was first published in 1715.

Lemma No. V, Book III of the "Principia."—This has been previously translated in an English version of the "Principia" published by Motte in 1729. The version here given is new.

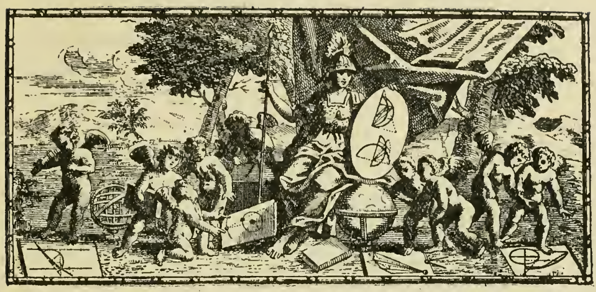
The Lemma gives the well-known propositions of interpolation by means of descending differences, for equal and for unequal intervals, which have always been regarded as laying the foundation of the calculus of finite differences. It appears from Newton's own statements that the whole of the "Principia" was written between December 1684 and May 1686, with the exception of 14 specified propositions among which the Lemma is not included. The date of the composition of the Lemma was therefore shortly before May 1686. It is difficult to suppose that he was not previously aware of the propositions stated in the Lemma; and it is remarkable that he had not included them in the "Methodus" which was composed many years before. The explanation may be that at the time he wrote the "Methodus" his mind was much engrossed with schemes for the calculation of extensive tables, for which the formulas of central differences were of greater practical use than formulas proceeding by descending differences. At the time he composed the Lemma, the particular point he had in view, as will be found by reference to the immediately succeeding proposition in the "Principia", was its application to an isolated case of interpolation.

In Newton's letter to Mr. John Smith, the notation has been reproduced without change, and it may be necessary to warn the reader that in such symbols as 2F and F2 the 2 is a suffix merely. In the versions given of the "Methodus" and of the Lemma, suffixes have been printed in the way now usual.

I have to express my acknowledgments to Mr. Walter Stott and Mr. R. O'Donovan, both of the Royal Insurance Company, Ltd., for their valuable assistance in the translation of the "Methodus." For the final form of that translation and for any defects which may be found in it, and for the translation of the Lemma, &c., I must take the entire responsibility.

In a future number of the *Journal* I hope to discuss the references to the subject of these notes which are to be found in the letters of Newton included in the "Commercium Epistolicum."

Reference may appropriately be made here to the valuable and interesting historical notes included in a contribution to vol. xviii of the Transactions of the Actuarial Society of America, by Mr. S. A Joffe, under the title "Interpolation Formulæ and Central-Difference Notation", in which he traces the history of the subject from the time of Newton, and draws particular attention to the connection between Newton's general formula for unequal differences and the general interpolation formulæ of Euler and Lagrange.



METHODUS DIFFERENTIALIS.

PROP. I.

S*I figuræ curvilineæ Abscissa componatur ex quantitate quavis data* A, *& quantitate indeterminata* x, *& Ordinata constet ex datis quotcunque quantitatibus* b, c, d, e, *&c. in totidem terminos hujus progressionis Geometricæ* x, x^2, x^3, x^4, *&c. respective ductis, & ad Abscissæ puncta totidem data erigantur Ordinatim applicatæ : dico quod Ordinatarum differentiæ primæ dividi possint per earum intervalla, & differentiarum sic divi-*

ſarum differentiæ dividi poſſint per Ordinatarum binarum intervalla, & harum differentiarum ſic diviſarum differentiæ dividi poſſint per Ordinatarum ternarum intervalla, & ſic deinceps in infinitum.

Etenim ſi pro Abſciſſæ parte indeterminata x ponantur quantitates quævis datæ p, q, r, s, t, &c. ſucceſſive, & ad Abſciſſarum ſic datarum terminos erigantur Ordinatæ α, β, γ, δ, ϵ, &c Hæ Abſciſſæ & Ordinatæ & Ordinatarum differentiæ diviſæ per Abſciſſarum differentias (quæ utique ſunt Ordinatarum intervalla) & quotorum differentiæ diviſæ per Ordinatarum alternarum differentias, & ſic deinceps, exhibentur per Tabulam ſequentem.

Abſciſſæ	Ordinatæ
$A+p$	$A+bp+cp^2+dp^3+ep^4=\alpha$
$A+q$	$A+bq+cq^2+dq^3+eq^4=\beta$
$A+r$	$A+br+cr^2+dr^3+er^4=\gamma$
$A+s$	$A+bs+cs^2+ds^3+es^4=\delta$
$A+t$	$A+bt+ct^2+dt^3+et^4=\epsilon$
Diviſor. Diff. Ord.	Quoti per diviſionem prodeuntes.
$p-q)\ \alpha-\beta$	$b+c\times\overline{p+q}+d\times\overline{pp+pq+qq}+e\times\overline{p^3+p^2q+pq^2+q^3}=\zeta$
$q-r)\ \beta-\gamma$	$b+c\times\overline{q+r}+d\times\overline{qq+qr+rr}+e\times\overline{q^3+q^2r+qr^2+r^3}=\eta$
$r-s)\ \gamma-\delta$	$b+c\times\overline{r+s}+d\times\overline{rr+rs+ss}+e\times\overline{r^3+r^2s+rs^2+s^3}=\theta$
$s-t)\ \delta-\epsilon$	$b+c\times\overline{s+t}+d\times\overline{ss+st+tt}+e\times\overline{s^3+s^2t+st^2+t^3}=\kappa$
$p-r)\ \zeta-\eta$	$c+d\times\overline{p+q+r}+e\times\overline{pp+pq+qq+pr+qr+rr}=\lambda$
$q-s)\ \eta-\theta$	$c+d\times\overline{q+r+s}+e\times\overline{qq+qr+rr+qs+rs+ss}=\mu$
$r-t)\ \theta-\kappa$	$c+d\times\overline{r+s+t}+e\times\overline{rr+rs+ss+rt+st+tt}=\nu$
$p-s)\ \lambda-\mu$	$d+e\times\overline{p+q+r+s}=\xi.$
$q-t)\ \mu-\nu$	$d+e\times\overline{q+r+s+t}=\pi.$
$p-t)\ \xi-\pi$	$e=\sigma.$

PROP.

PROP. II.

Iisdem positis, & quod numerus terminorum b, c, d, e, *&c. sit finitus, dico quod Quotorum ultimus æqualis erit ultimo terminorum* b, c, d, e, *&c. et quod per Quotos reliquos dabuntur termini reliqui* b, c, d, e, *&c. et his datis dabitur Linea Curva generis Parabolici quæ per Ordinatarum omnium terminos transibit.*

Etenim in Tabula superiore Quotus ultimus σ æqualis erat termino ultimo e. Et hic terminus ductus in summam datam $\overline{p+q+r+s}$, & ablatus de Quoto ξ relinquit terminum penultimum d. Et quantitates jam datæ $d\times\overline{p+q+r}+e\times\overline{pp+pq+qq+pr+qr+rr}$, si auferantur de Quoto λ, relinquunt terminorum antepenultimum c. Et quantitates jam datæ $c\times\overline{p+q}+d\times\overline{pp+pq+qq}+e\times\overline{p^3+ppq+pqq+q^3}$, si auferantur de Quoto ζ, relinquunt terminum b. Et simili computo si plures essent termini, colligerentur omnes per Quotorum Ordines totidem. Deinde quantitates datæ $bp+cpp+dp^3+ep^4$, si subducantur de Ordinata prima α, relinquunt Abscissæ terminum primum A. Et quantitas $A+bx+cx^2+dx^3+ex^4+$ &c. est Ordinata Curvæ generis Parabolici quæ per Ordinatarum omnium datarum terminos transibit, existente Abscissa $A+x$.

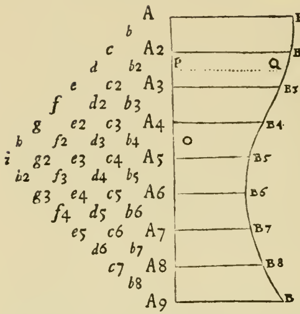
Ex his Propositionibus quæ sequuntur facile colligi possunt.

PROP. III.

Si Recta aliqua AA9 *in æquales quotcunque partes* AA2, A2A3, A3A4, A4A5, *&c. dividatur, & ad puncta divisionum erigantur parallelæ* AB, A2B2, A3B3, *&c. Invenire curvam Geometricam generis Parabolici quæ per omnium erectarum terminos* B, B2, B3, *&c. transibit.*

Erectarum AB, A2B2, A3B3, &c. quære differentias Primas, b, $b2$, $b3$, &c. Secundas c, $c2$, $c3$, &c. Tertias d, $d2$, $d3$, &c. et sic deinceps usque dum veneris ad ultimam differentiam, quæ hic sit i.

Tunc.



Tunc incipiendo ab ultima differentia excerpe medias differentias in alternis Columnis vel Ordinibus differentiarum, & Arithmetica media inter duas medias reliquarum, Ordine pergendo usque ad Seriem primorum terminorum AB, A2B2, A3B3, &c. sint hæc $k, l, m, n, o, p, q, r, s$, &c. quorum ultimus significet ultimam differentiam; penultimus medium Arithmeticum inter duas penultimas differentias; antepenultimus mediam trium antepenultimarum differentiarum, & sic deinceps usque ad primum quod erit vel medius terminorum A, A2, A3, &c. vel Arithmeticus medius inter duos medios. Prius accidit ubi numerus terminorum A, A2, A3, &c. est impar; posterius ubi par.

C A S. I.

In Casu priori, sit A5B5 iste medius terminus, hoc est, $A5B5 = k$, $\frac{b4+b5}{2} = l$, $c4 = m$, $\frac{d3+d4}{2} = n$, $e3 = o$, $\frac{f2+f3}{2} = p$, $g2 = q$, $\frac{b+b2}{2} = r$, $i = s$. Et erecta Ordinatim applicata PQ, dic $A5P = x$; & duc terminos hujus Progressionis

$$1 \times \frac{x}{1} \times \frac{x}{2} \times \frac{x^2-1}{3x} \times \frac{x}{4} \times \frac{x^2-4}{5x} \times \frac{x}{6} \times \frac{x^2-9}{7x} \times \frac{x}{8} \times \frac{x^2-16}{9x} \times \frac{x}{10} \times \frac{x^2-25}{11x} \times \frac{x}{12} \times \frac{x^2-36}{13x} \text{ \&c.}$$

in se continuo; & orientur termini

$$1.\ x.\ \frac{x^2}{2}.\ \frac{x^3-x}{6}.\ \frac{x^4-x^2}{24}.\ \frac{x^5-5x^3+4x}{120}.\ \frac{x^6-5x^4+4x^2}{720}.\ \frac{x^7-14x^5+49x^3-36x}{5040}.\ \text{\&c.}$$

per quos si termini seriei k, l, m, n, o, p, &c. respective multiplicentur, aggregatum factorum $k + xl + \frac{x^2}{2}m + \frac{x^3-x}{6}n + \frac{x^4-x^2}{24}o + \frac{x^5-5x^3+4x}{120}p +$ &c. erit longitudo Ordinatim applicatæ PQ.

C A S. II.

In Casu posteriori, sint A4B4, A5B5 duo medii termini, hoc est, sit $\frac{A4B4+A5B5}{2} = k$, $b4 = l$, $\frac{c3+c4}{2} = m$, $d3 = n$, $e2 + e3 = o$, $f2 = p$, $\frac{g+g2}{2} = q$, & $h = r$.

& $b = r$. Et erecta Ordinatim applicata PQ, biseca $A4A5$ in O, & dicto $OP = x$, duc Terminos hujus Progressionis

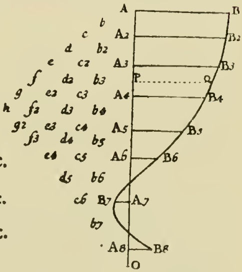
$1 \times \frac{x}{1} \times \frac{xx - \frac{1}{4}}{2x} \times \frac{x}{3} \times \frac{xx - \frac{9}{4}}{4x} \times \frac{x}{5} \times \frac{xx - \frac{25}{4}}{6x} \times \frac{x}{7} \times \frac{xx - \frac{49}{4}}{8x}$, &c. in se continuo; et orientur termini $1.\ x.\ \frac{4xx-1}{8}.\ \frac{4x^3 - x}{24}.\ \frac{16x^4 - 40x^2 + 9}{384}.$ &c. per quos si termini series k, l, m, n, o, p, q, &c. respective multiplicentur, aggregatum factorum $k + xl + \frac{4x^2-1}{8}m + \frac{4x^3-x}{24}n + \frac{16x^4 - 40x^2 + 9}{384}o +$ &c. erit Longitudo Ordinatim applicatæ PQ.

Sed hic notandum est quod intervalla $AA2$, $A2A3$, $A3A4$, &c. hic supponantur esse unitates, & quod differentiæ colligi debent auferendo inferiores quantitates de superioribus, $A2B2$ de AB, $A3B3$ de $A2B2$, $b2$ de b, &c. et faciendo ut sint $AB - A2B2 = b$, $A2B2 - A3B3 = b2$, $b - b2 = c$, &c. adeoque quando differentiæ illæ hoc modo prodeunt negativæ signa earum mutanda sunt.

PROP. IV.

Si recta aliqua in partes quotcunque inæquales AA2, A2A3, A3A4, A4A5, *&c. dividatur, & ad puncta divisionum erigantur parallelæ* AB, A2B2, A3B3, *&c. Invenire Curvam Geometricam generis Parabolici quæ per omnium erectarum terminos* B, B2, B3, *&c. transibit.*

Sunto puncta data B, B2, B3, B4, B5, B6, B7, &c. et ad Abscissam quamvis $AA7$ demitte Ordinatas perpendiculariter BA, $B2A2$, &c.



Et fac $\frac{AB - A2B2}{AA2} = b$, $\frac{A2B2 - A3B3}{A2A3} = b2$,

$\frac{A3B3 - A4B4}{A3A4} = b3$, $\frac{A4B4 - A5B5}{A4A5} = b4$,

$\frac{A5B5 - A6B6}{A5A6} = b5$, $\frac{A6B6 - A7B7}{A6A7} = b6$,

$\frac{-A7B7 - A8B8}{A7A8} = b7$.

Deinde $\frac{b-b2}{AA3} = c$, $\frac{b2-b3}{A2A4} = c2$, $\frac{b3-b4}{A3A5} = c3$, &c.

Tunc $\frac{c-c2}{AA4} = d$, $\frac{c2-c3}{A2A5} = d2$, $\frac{c3-c4}{A3A6} = d3$, &c.

Et $\frac{d-d2}{AA5} = e$, $\frac{d2-d3}{A2A6} = e2$, $\frac{d3-d4}{A3A7} = e3$, &c.

Sic pergendum est ad ultimam differentiam.

Differentiis sic collectis & divisis per intervalla Ordinatim applicatarum; in alternis earum Columnis sive Seriebus vel Ordinibus excerpe medias, incipiendo ab ultima, & in reliquis Columnis excerpe media Arithmetica inter duas medias, pergendo usque ad seriem primorum terminorum, AB, A2B2, &c. Sunto hæc k, l, m, n, o, p, q, r, &c. quorum ultimus terminus significet ultimam differentiam; penultimus medium Arithmeticum inter duas penultimas; antepenultimus mediam trium antepenultimarum, &c. Et primus k erit media Ordinatim applicata, si numerus datorum punctorum est impar; vel medium Arithmeticum inter duas medias, si numerus eorum est par.

CAS. I.

In Casu priori, sit A4B4 ista media Ordinatim applicata, hoc est, sit $A4B4 = k$, $\frac{b3+b4}{2} = l$, $c3 = m$, $\frac{d2+d3}{2} = r$, $e2 = o$, $\frac{f+f2}{2} = p$, $g = q$. Et erecta Ordinatim applicata PQ, & in Basi AA5 sumpto quovis puncto O, dic $OP = x$, & duc in se gradatim terminos hujus Progressionis

$$1 \times \overline{x - OA4} \times \overline{x - \frac{OA3+OA5}{2}} \times \frac{\overline{x-OA3} \times \overline{x-OA5}}{\overline{x - \frac{1}{2}OA3+OA5}} \times \overline{x - \frac{OA2+OA6}{2}} \times \text{\&c.}$$

et ortam Progressionem asserva; vel quod perinde est duc terminos hujus Progressionis

$$1 \times \overline{x-OA4} \times \overline{\overline{x-OA3} \times \overline{x-OA5}} \times \overline{\overline{x-OA2} \times \overline{x-OA6}} \times \overline{\overline{x-OA} \times \overline{x-OA7}} \times \text{\&c.}$$

in se gradatim, & terminos exinde ortos duc respective in terminos hujus Progressionis

$$1.\ x - \frac{+OA3+OA5}{2}.\ x - \frac{+OA2+OA6}{2}.\ x - \frac{+OA+OA7}{2},$$ &c. et orientur termini intermedii tota Progressione existente

$$1.\ x - OA4.\ x^2 - \frac{+OA3+2OA4+OA5}{2}x + \frac{OA3+OA5}{2} \times OA4, \text{ \&c.}$$

Vel dic $OA = \alpha$, $OA2 = \beta$, $OA3 = \gamma$, $OA4 = \delta$, $OA5 = \epsilon$, $OA6 = \zeta$, $OA7 = \eta$: $\frac{OA3+OA5}{2} = \theta$, $\frac{OA2+OA6}{2} = \chi$, $\frac{OA+OA7}{2} = \lambda$. Et ex Progressione $1 \times \overline{x-\delta} \times \overline{\overline{x-\gamma} \times \overline{x-\epsilon}} \times \overline{\overline{x-\beta} \times \overline{x-\zeta}} \times \overline{\overline{x-\alpha} \times \overline{x-\eta}}$ &c. collige terminos quibus multiplicatis per $1.\ x-\theta,\ x-\chi,\ x-\lambda$, &c collige alios terminos intermedios, tota serie prodeunte

$$1,\ x-\delta,\ x^2 - \overline{\delta+\theta}x + \delta\theta,\ x^3 - \overline{\delta+2\theta}x^2 + \overline{\gamma\epsilon+2\delta\theta}x - \gamma\delta\epsilon, \text{ \&c.}$$

per cujus terminos multiplica series k, l, m, n, o, &c. Et aggregatum productorum $k + \overline{x-\delta} \times l + \overline{x^2 - \overline{\delta+\theta}x + \delta\theta} \times m +$ &c. erit longitudo Ordinatim applicatæ PQ.

C A S.

C A S. II.

In Casu posteriori, sint $A4B4$, $A5B5$ duæ mediæ Ordinatim applicatæ, hoc est, $\frac{A4B4+A5B5}{2} = k$, $b4 = l$, $\frac{c3+c4}{2} = m$, $d3 = n$, $\frac{e2+e3}{2} = o$, $f2 = p$, &c. Et alternorum k, m, o, q, &c. Coefficientes orientur ex multiplicatione terminorum hujus Progressionis in se

$$1 \times \overline{\overline{x - OA4} \times \overline{x - OA5}} \times \overline{\overline{x - OA3} \times \overline{x - OA6}} \times \overline{\overline{x - OA2} \times \overline{x - OA7}} \times \overline{\overline{x - OA} \times \overline{x - OA8}} \text{ \&c.}$$

Et reliquorum Coefficientes ex multiplicatione horum per terminos hujus Progressionis

$$x - \frac{+OA4+OA5}{2},\ x - \frac{+OA3+OA6}{2},\ x - \frac{+OA2+OA7}{2},\ x - \frac{+OA+OA8}{2}, \text{ \&c.}$$

Hoc est, erit $k + \overline{x - \frac{+OA4+OA5}{2}} \times l + \overline{x^2 - \overline{OA4+OA5}\,x + OA4 \times OA5} \times m$, &c. Ordinatim applicata PQ,

vel $PQ = k + \begin{array}{c} x \\ -\frac{1}{2}OA4 \\ -\frac{1}{2}OA5 \end{array} \times l + \begin{array}{c} x \\ -OA4 \end{array} \times \begin{array}{c} x \\ -OA5 \end{array} \times m + \begin{array}{c} x \\ -OA4 \end{array} \times \begin{array}{c} x \\ -OA5 \end{array} \times \begin{array}{c} x \\ -\frac{1}{2}OA3 \\ -\frac{1}{2}OA6 \end{array} \times n$ &c.

Sive dic $x - \frac{+OA4+OA5}{2} = \pi$, $\quad \overline{x - OA4} \times \overline{x - OA5} = \rho$,

$$\rho \times \overline{x - \frac{+OA3+OA6}{2}} = \sigma,\ \rho \times \overline{x - OA3} \times \overline{x - OA6} = \tau,$$

$$\tau \times \overline{x - \frac{+OA2+OA7}{2}} = \upsilon,\ \tau \times \overline{x - OA2} \times \overline{x - OA7} = \varphi,$$

$$\varphi \times \overline{x - \frac{+OA+OA8}{2}} = \chi,\ \varphi \times \overline{x - OA} \times \overline{x - OA8} = \psi,$$

Et erit $k + \pi l + \rho m + \sigma n + \tau o + \upsilon p + \varphi q + \chi r + \psi s = PQ$.

P R O P. V.

Datis aliquot terminis seriei cujuscunque ad data intervalla dispositis, invenire terminum quemvis intermedium quamproxime.

Ad rectam positione datam erigantur termini dati in dato angulo, interpositis datis intervallis, & per eorum puncta extima, per Propositiones præcedentes, ducatur linea Curva generis Parabolici. Hæc enim continget terminos omnes intermedios per seriem totam.

PROP.

PROP. VI.

Figuram quamcunque Curvilineam quadrare quamproxime, cujus Ordinatæ aliquot inveniri possunt.

Per terminos Ordinatarum ducatur linea Curva generis Parabolici ope Propositionum præcedentium. Hæc enim figuram terminabit quæ semper quadrari potest, et cujus Area æquabitur Areæ figuræ propositæ quamproxime.

SCHOLIUM.

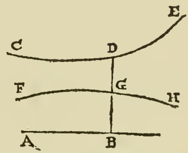
Utiles sunt hæ Propositiones ad Tabulas construendas per interpolationem Serierum, ut & ad solutiones Problematum quæ a quadraturis Curvarum dependent, præsertim si Ordinatarum intervalla & parva sint & æqualia inter se, & Regulæ computentur, & in usum reserventur pro dato quocunque numero Ordinatarum. Ut si quatuor sint Ordinatæ ad æqualia intervalla sitæ, sit A summa primæ & quartæ, B summa secundæ & tertiæ, & R intervallum inter primam & quartam, & Ordinata nova in medio omnium erit $\frac{9B-A}{16}$, & Area tota inter primam & quartam erit $\frac{A+3B}{8}R$.

Et nota quod ubi Ordinatæ stant ad æquales ab invicem distantias, sumendo summas Ordinatarum quæ ab Ordinata media hinc inde æqualiter distant, & duplum Ordinatæ mediæ, componitur Curva nova cujus Area per pauciores Ordinatas determinatur, & æqualis est Areæ Curvæ prioris quam invenire oportuit. Quinetiam si pro Ordinatis novis sumantur summa Ordinatæ primæ & secundæ, et summa tertiæ & quartæ, et summa quintæ & sextæ, & sic deinceps; vel si sumantur summa trium primarum Ordinatarum, & summa trium proximarum, & summa trium quæ sunt deinceps; vel si sumantur summæ quaternarum Ordinatarum, vel summæ quinarum: Area Curvæ novæ æqualis erit Areæ Curvæ primo propositæ. Et sic habitis Curvæ quadrandæ Ordinatis quotcunque quadratura ejus ad quadraturam Curvæ alterius per pauciores Ordinatas reducetur.

Per

Per data vero puncta quotcunque non ſolum Curvæ lineæ generis *Parabolici*, ſed etiam Curvæ aliæ innumeræ diverſorum generum duci poſſunt.

Sunto CDE, FGH Curvæ duæ Abſciſſam habentes communem AB, et Ordinatas in eadem recta jacentes BD, BG; & relatio inter has Ordinatas definiatur per æquationem quamcunque. Dentur puncta quotcunque per quæ Curva CDE tranſire debet, & per æquationem illam dabuntur puncta totidem nova per quæ Curva FGH tranſibit. Per Propoſitiones ſuperiores deſcribatur Curva FGH generis *Parabolici* quæ per puncta illa omnia nova tranſeat, & per æquationem eandem dabitur Curva CDE quæ per puncta omnia primo data tranſibit.



F I N I S.

I.

METHODUS DIFFERENTIALIS.

(Translation.)

Prop. I.

If the abscissa of a curve consist of a given quantity A *and an indeterminate quantity x, and if the ordinate consist of any number of quantities b, c, d, e, &c., multiplied respectively into a corresponding number of terms of the G. P. x, x^2, x^3, x^4, &c., and if ordinates be erected at as many points of the abscissa; then the first differences of the ordinates are divisible by their intervals; and the differences of the differences so divided are divisible by the intervals between alternate ordinates; and the differences of these differences so divided are divisible by the intervals between every third ordinate, and so on indefinitely.*

Thus if given quantities p, q, r, s, t, &c., be substituted in succession for the indeterminate portion of the abscissa, and ordinates α, β, γ, δ, ϵ, &c., be erected at the extremities of the abscissæ so determined; the abscissæ and the ordinates, and the differences of ordinates divided by the differences of abscissæ (which are in fact the intervals of the ordinates), and the differences of the quotients divided by the differences of alternate ordinates, and so on, are shewn in the following table. (See p. 10.)

Prop. II.

Making the same suppositions and assuming the number of terms b, c, d, e, &c., to be finite, the last quotient will be equal to the last of the terms b, c, d, e, &c., and the remaining terms will be found by means of the remaining quotients; and, when these terms are known, a parabolic curve is determined which passes through the extremities of all the ordinates.

Thus, in the preceding table, the last quotient σ is equal to the last term e; and the product of this term by the known sum $p+q+r+s$ when subtracted from the quotient ξ leaves as remainder d, the last term but one. The quantities $d(p+q+r) + e(p^2+pq+q^2+pr+qr+r^2)$ which are then known, being deducted from the quotient λ, give the term c. The quantities $c(p+q) + d(p^2+pq+q^2) + e(p^3+p^2q+pq^2+q^3)$, which are then known, being deducted from the quotient ζ, leave the term b. By a similar calculation, other terms, if any, would be obtainable by means of a corresponding series of quotients. Finally, the ascertained quantities $bp+cp^2+dp^3+ep^4$, when deducted from the first ordinate α, leave the first term a of the expression for the ordinate. And the quantity $a+bx+cx^2+dx^3+ex^4$, &c., is the ordinate of a parabolic curve which passes through the extremities of all the given ordinates, the abscissa being $\mathrm{A}+x$.

From these propositions those which follow are easily inferred.

Prop. III.

If a straight line A_1A_9 *be divided into any number of equal parts* A_1A_2, A_2A_3, A_3A_4, A_4A_5, *&c., and if parallel straight lines* A_1B_1, A_2B_2, A_3B_3, *&c., be erected at the points of division, it is required to find a parabolic curve which shall pass through the extremities of all these lines.*

(See figure on p. 12).

Find the first differences b_1, b_2, b_3, &c., of the ordinates A_1B_1, A_2B_2, A_3B_3, &c. ; the second differences c_1, c_2, c_3, &c. ; the third differences d_1, d_2, d_3, &c. ; and so on, up to the last difference and let that difference be called i.

Then beginning at the last difference, take the central differences in the alternate columns or orders of differences, and the arithmetic means between the two central differences in the remaining columns, proceeding in order up to the series of primary terms A_1B_1, A_2B_2, A_3B_3, A_4B_4, &c. Call the terms extracted $k, l, m, n, o, p, q, r, s$, &c. ; the last of these symbols representing the last difference ; the last but one, the mean between the two differences in the last column but one, the last but two, the central difference of the three differences in the last column but two, and so on to the first of the symbols, which will represent either the central ordinate of the series A_1B_1, A_2B_2, A_3B_3, &c., or the mean between the two central ordinates, the former happening when the number of the ordinates is odd, and the latter when the number is even.

CASE I.

In the former case let A_5B_5 be the central ordinate, that is put

$$A_5B_5 = k,\ \frac{b_4+b_5}{2} = l,\ c_4 = m,\ \frac{d_3+d_4}{2} = n,$$

$$e_3 = o,\ \frac{f_2+f_3}{2} = p,\ g_2 = q,\ \frac{b_1+b_2}{2} = r,\ i = s.$$

Erect the ordinate PQ and let $AP = x$. Now multiply continuously into one another the terms of the progression,

$$1,\ x,\ \frac{x}{2},\ \frac{x^2-1}{3x},\ \frac{x}{4},\ \frac{x^2-4}{5x},\ \frac{x}{6},\ \frac{x^2-9}{7x},\ \frac{x}{8},\ \frac{x^2-16}{9x},\ \frac{x}{10},\ \frac{x^2-25}{11x},\ \frac{x}{12},\ \frac{x^2-36}{13x},$$

&c., the resulting terms being

$$1,\ x,\ \frac{x^2}{2},\ \frac{x^3-x}{6},\ \frac{x^4-x^2}{24},\ \frac{x^5-5x^3+4x}{120},\ \frac{x^6-5x^4+4x^2}{720},$$

$$\frac{x^7-14x^5+49x^3-36x}{5040},\ \&c.$$

Then if these terms be respectively multiplied into the terms of the series k, l, m, n, o, p, &c., the sum of the products, namely

$$k + x \,.\, l + \frac{x^2}{2} \cdot m + \frac{x^3 - x}{6} \cdot n + \frac{x^4 - x^2}{24} \cdot o + \frac{x^5 - 5x^3 + 4x}{120} \cdot p + \&\text{c.}$$

will be the length of the ordinate PQ.

CASE II.

In the latter case let A_4B_4 and A_5B_5 be the two central ordinates, that is, put

$$\frac{A_4B_4 + A_5B_5}{2} = k,\ b_4 = l,\ \frac{c_3 + c_4}{2} = m,\ \ d_3 = n,$$

$$\frac{e_2 + e_3}{2} = o,\ f_2 = p,\ \frac{g_1 + g_2}{2} = q,\ \&\ h = r.$$

Erecting the ordinate PQ take the middle point O of A_4A_5, and call $OP = x$.
Now multiply continuously into one another the terms of the progression

$$1,\ x,\ \frac{x^2 - \frac{1}{4}}{2x},\ \frac{x}{3},\ \frac{x^2 - \frac{9}{4}}{4x},\ \frac{x}{5},\ \frac{x^2 - \frac{25}{4}}{6x},\ \frac{x}{7},\ \frac{x^2 - \frac{49}{4}}{8x},\ \&\text{c.}$$

the resulting terms being

$$1,\ x,\ \frac{4x^2 - 1}{8},\ \frac{4x^3 - x}{24},\ \frac{16x^4 - 40x^2 + 9}{384},\ \&\text{c.}$$

Then if these terms be respectively multiplied into the terms of the series k, l, m, n, o, p, q, &c., the sum of the products, namely,

$$k + x \,.\, l + \frac{4x^2 - 1}{8} \cdot m + \frac{4x^3 - x}{24} \cdot n + \frac{16x^4 - 40x^2 + 9}{384} \cdot o + \&\text{c.}$$

will be the length of the ordinate PQ.

It is to be noted that each of the intervals A_1A_2, A_2A_3, A_3A_4, &c., is here assumed to be unity; also that the differences are to be obtained by deducting the lower quantities from the upper, A_2B_2 from A_1B_1, A_3B_3 from A_2B_2, b_2 from b_1, &c., so that $A_1B_1 - A_2B_2 = b_1$, $A_2B_2 - A_3B_3 = b_2$, $b_1 - b_2 = c_1$, &c., and further that when any of the differences taken in this way turn out to be negative, effect must be given to the negative signs.

Prop. IV.

If a straight line be divided into any number of unequal parts A_1A_2, A_2A_3, A_3A_4, A_4A_5, *&c., and if parallel straight lines* A_1B_1, A_2B_2, A_3B_3, *&c., be erected at the points of division; it is required to find a parabolic curve which shall pass through the extremities of all the lines so erected.*

(See figure on p. 13.)

Let the given points be B_1, B_2, B_3, B_4, B_5, B_6, B_7, &c., and let fall ordinates B_1A_1, B_2A_2, &c., perpendicularly on the abscissa A_1A_7. Put

$$\frac{A_1B_1 - A_2B_2}{A_1A_2} = b_1, \quad \frac{A_2B_2 - A_3B_3}{A_2A_3} = b_2, \quad \frac{A_3B_3 - A_4B_4}{A_3A_4} = b_3,$$

$$\frac{A_4B_4 - A_5B_5}{A_4A_5} = b_4, \quad \frac{A_5B_5 - A_6B_6}{A_5A_6} = b_5, \quad \frac{A_6B_6 + A_7B_7}{A_6A_7} = b_6, \quad \frac{-A_7B_7 - A_8B_8}{A_7A_8} = b_7.$$

Thence derive

$$\frac{b_1 - b_2}{A_1A_3} = c_1, \qquad \frac{b_2 - b_3}{A_2A_4} = c_2, \qquad \frac{b_3 - b_4}{A_3A_5} = c_3, \text{ \&c.};$$

and then

$$\frac{c_1 - c_2}{A_1A_4} = d_1, \qquad \frac{c_2 - c_3}{A_2A_5} = d_2, \qquad \frac{c_3 - c_4}{A_3A_6} = d_3, \text{ \&c.};$$

and

$$\frac{d_1 - d_2}{A_1A_5} = e_1, \qquad \frac{d_2 - d_3}{A_2A_6} = e_2, \qquad \frac{d_3 - d_4}{A_3A_7} = e_3, \text{ \&c.};$$

the process being continued in the same way until the last difference is reached.

After the differences have been collected and divided by the intervals between the ordinates, the next step is to pick out the central terms in the alternate columns (or series, or lines), reckoning from the last difference, and the arithmetic means between the two central terms in the remaining columns, right up to the series of primary terms A_1B_1, A_2B_2, &c. Let the terms extracted be k, l, m, n, o, p, q, r, &c., of which the last symbol denotes the last difference; the last but one, the mean between the two differences in the last column but one; the last but two, the central difference of the three differences in the last column but two, &c. Then the first symbol k will represent the central ordinate if the number of ordinates is odd, or the arithmetic mean between the two central ordinates if their number is even.

CASE I.

In the former case let A_4B_4 be the central ordinate, that is, put $A_4B_4 = k$, $\frac{b_3+b_4}{2} = l$, $c_3 = m$, $\frac{d_2+d_3}{2} = n$, $e_2 = o$, $\frac{f_1+f_2}{2} = p$, $g = q$. Having erected the ordinate PQ, take a fixed point O in the base A_1A_5 and call $OP = x$. Then multiply into one another in succession the terms of the progression,

$$1,\quad x - OA_4,\quad x - \frac{OA_3 + OA_5}{2},\quad \frac{(x - OA_3)(x - OA_5)}{x - \frac{1}{2}(OA_3 + OA_5)},$$

$$x - \frac{OA_2 + OA_6}{2},\ \text{\&c.},$$

and take the resulting progression.

Or, what comes to the same thing, multiply into one another in succession the terms of the progression,

$$1,\ x - OA_4,\quad (x - OA_3)(x - OA_5),\quad (x - OA_2)(x - OA_6),$$

$$(x - OA_1)(x - OA_7),\ \text{\&c.}$$

Then multiply the resulting terms respectively into the terms of the progression,

$$1,\ x - \frac{OA_3 + OA_5}{2},\ x - \frac{OA_2 + OA_6}{2},\ x - \frac{OA_1 + OA_7}{2},\ \text{\&c.},$$

and intermediate terms will be obtained, the complete progression being

$$1,\ x - OA_4,\ x^2 - \frac{OA_3 + 2OA_4 + OA_5}{2} \cdot x + \frac{OA_3 + OA_5}{2} \times OA_4,\ \text{\&c.}$$

Otherwise, let $OA_1 = \alpha$, $OA_2 = \beta$, $OA_3 = \gamma$, $OA_4 = \delta$, $OA_5 = \epsilon$,

$$OA_6 = \zeta,\ OA_7 = \eta,\ \frac{OA_3 + OA_5}{2} = \theta,\ \frac{OA_2 + OA_6}{2} = \chi,\ \frac{OA_1 + OA_7}{2} = \lambda.$$

Obtain terms by continuous multiplication from the progression

$$1,\ x - \delta,\ x - \gamma,\ x - \epsilon,\ x - \beta,\ x - \zeta,\ x - \alpha,\ x - \eta,\ \text{\&c.},$$

and obtain intermediate terms by multiplying the results respectively by

$$1,\ x - \theta,\ x - \chi,\ x - \lambda,\ \text{\&c.},$$

the whole series being

$$1,\ x - \delta,\ x^2 - (\delta + \theta)x + \delta\theta,\ x^3 - (\delta + 2\theta)x^2 + (\gamma\epsilon + 2\delta\theta)x - \gamma\delta\epsilon,\ \text{\&c.},$$

the terms of which are to be multiplied respectively by k, l, m, n, o, p, &c. Then the sum of the products, namely

$$k+(x-\delta).l+\{x^2-(\delta+\theta)x+\delta\theta\}.m+\&c.$$

will be the length of the ordinate PQ.

CASE II.

In the latter case let A_4B_4, A_5B_5, be the two middle ordinates, that is, put

$$\frac{A_4B_4+A_5B_5}{2}=k,\quad b_4=l,\quad \frac{c_3+c_4}{2}=m,\quad d_3=n,\quad \frac{e_2+e_3}{2}=o,\quad f_2=p,\quad \&c.$$

The co-efficients of the alternate terms are obtained by the continuous multiplication of the terms of the progression,

$$1,\ (x-OA_4)(x-OA_5),\ (x-OA_3)(x-OA_6),\ (x-OA_2)(x-OA_7),$$
$$(x-OA_1)(x-OA_8),\ \&c.,$$

and those of the remaining terms are obtained by multiplying the above co-efficients by the terms of the progression,

$$x-\frac{OA_4+OA_5}{2},\ x-\frac{OA_3+OA_6}{2},\ x-\frac{OA_2+OA_7}{2},\ x-\frac{OA_1+OA_8}{2}.$$

Then

$$k+\left(x-\frac{OA_4+OA_5}{2}\right).l+\{x^2-(OA_4+OA_5).x+OA_4\times OA_5\}.m+\&c.,$$

will be the ordinate PQ;

$$\text{or,}\quad PQ=k+\left(x-\tfrac{1}{2}OA_4-\tfrac{1}{2}OA_5\right).l+(x-OA_4)(x-OA_5).m$$
$$+(x-OA_4)(x-OA_5)\left(x-\tfrac{1}{2}OA_5-\tfrac{1}{2}OA_6\right).n,\ \&c.$$

Thus putting

$$x-\frac{OA_4+OA_5}{2}=\pi,\qquad (x-OA_4)(x-OA_5)=\rho,$$
$$\rho.\left(x-\frac{OA_3+OA_6}{2}\right)=\sigma,\quad \rho.(x-OA_3)(x-OA_6)=\tau,$$
$$\tau.\left(x-\frac{OA_2+OA_7}{2}\right)=\upsilon,\quad \tau.(x-OA_2)(x-OA_7)=\phi,$$
$$\phi.\left(x-\frac{OA_1+OA_8}{2}\right)=\chi,\quad \phi.(x-OA_1)(x-OA_8)=\psi.$$

the equation to the curve will be

$$k+\pi.l+\rho.m+\sigma.n+\tau.o+\upsilon.p+\phi.q+\chi.r+\psi.s=PQ.$$

PROP. V.

Certain terms out of a sequence of values being given, arranged at known intervals, it is required to find any intermediate term as closely as possible.

On a fixed straight line erect at a constant angle the given terms arranged at the given intervals ; and let a parabolic curve be drawn through their extremities by means of the preceding propositions. This curve will pass through the extremities of all the intermediate terms.

PROP. VI.

To find the approximate area of any curve a number of whose ordinates can be ascertained.

Let a parabolic curve be drawn through the extremities of the ordinates by means of the preceding propositions. This will form the boundary of a figure whose area can always be ascertained, and its area will be approximately equal to the area required.

SCHOLIUM.

These propositions are useful for the construction of tables by the interpolation of series, as also for the solution of problems which depend on finding the areas of curves, especially if the intervals between the ordinates are small and equal to one another ; and rules applicable to any given number of ordinates can be derived and recorded for reference. For example: If there are four ordinates at equal intervals, let A be the sum of the first and fourth, B the sum of the second and third, and R the interval between the first and fourth ; then the central ordinate will be $\frac{9B - A}{16}$, and the area between the first and fourth ordinates will be $\frac{A + 3B}{8}$ R.

Note also that when the ordinates stand at equal distances from one another, the sums of ordinates which are equally distant from the central ordinate, along with twice the central ordinate, supply data for a new curve the area of which is determined by means of a smaller number of ordinates and is equal to the area of the original curve. Moreover, if the sum of the first and second ordinates, the sum of the third and fourth, the sum of the fifth and sixth, and so on in succession are taken as new ordinates ; or if the sum of the first three ordinates, the sum of the next three, and the sum of the succeeding three are taken ; or if the sums of the ordinates are taken four at a time, or five at a time, the area of the new curve will be equal to that of the original curve. And in this way, when a number of ordinates are given of a curve whose area is required, the calculation of the area is reduced to that of the area of another curve by means of a smaller number of ordinates.

Through a given number of points not only parabolic curves but an infinity of other curves of different kinds can be drawn.

(See figure on page 17.)

Let CDE, FGH, be two curves having a common abscissa AB, and ordinates BD, BG, lying in the same straight line; and let the relation between the ordinates be defined by any equation whatever. Let any number of points be given through which the curve CDE is required to pass, and by that equation an equal number of points will be given through which the curve FGH will pass. By means of the foregoing propositions let a parabolic curve FGH be described passing through those points, and by the same equation a curve CDE will be given which will pass through all the points first given.

II.

LETTER ON CONSTRUCTION OF TABLES.

LETTER CIX.

W. Jones to Prof. Cotes.

LONDON, *Jan. 1st*, $17\frac{11}{12}$.

Dr. Sr.

I have sent you here inclos'd, the Coppy of a Letter, that I found among Mr. Collins's papers, from Sr. Is. Newton to one Mr. Smith; the contents thereof seem to have, in some measure, relation to what you are about, as being the application of the Doctrine of Differences to the making of Tables; and for that reason I thought it might be of use to you, so far as to see what has bin done already: I shew'd this to Sr. Isaac, he remembers yt. he apply'd it to all sorts of Tables, but has nothing by him more than what is printed: I have more papers of Mr. Mercator's and others, upon this subject, tho, I think, none so material, as this. I shou'd be very glad to see what you have done of this kind all publish'd. And I must confess, that, unless you design a considerable large Volume, 'twere much better to put them into the Transactions; for that wou'd sufficiently preserve them from being lost, which is ye. common fate of small single Tracts; and at ye. same time save the trouble and expense of printing them, since the subject is too curious to expect any profit by it: and besides, now, as the R. Society having done themselves the honour of choosing you a Member, something from you cannot but be acceptable to them: Sr. Isaac himself expects those things of yours that I formerly mentioned to him as your promise.

I am, Sr. your much oblig'd

friend, & humble Servt.

W. JONES.

LETTER CIX. (*bis*).

Newton to J. Smith.

[Enclosed in Letter CIX.] [COPY.]

TRIN. COLL. CAMBRIDGE, *May 8th*, 1675.

Sr.

I have consider'd ye. buisiness of computing Tables of Square, Cube, & Sq. Sqr. Roots; and ye. best way of p'forming it, yt. I can think of is yt. which follows:

If yu. wo'd compute a Table to 8 decimal places, let ye. roots of every hundredth number be extracted to ten decimal places, and then compute every tenth numbr. and afterwards every number by the following methods:

TAB. I.

$n-50$	$*o$	α	
	m	o	
$n-40$	op	β	
	m	π	
$n-30$	pq	γ	
	m	χ	
$n-20$	qr	δ	
	m	ρ	
$n-10$	rs	ϵ	
	m	σ	
n	st	ζ	$\frac{m}{10}$
	m	τ	
$n+10$	tv	η	
	m	υ	
$n+20$	vx	θ	
	m	ϕ	
$n+30$	xy	ι	
	m	ψ	
$n+40$	yz	κ	
	m	ω	
$n+50$	$z*$	λ	

TAB. II.

$n-6$	4E	
	5ϵ	
$n-5$	5E. F5	
	$\zeta 4$	
$n-4$	F4	
	$\zeta 3$	
$n-3$	F3	
	$\zeta 2$	
$n-2$	F2	
	$\zeta 1$	
$n-1$	F1	
	ζ	$\frac{st}{100}$
n	F	
	1ζ	
$n+1$	1F	
	2ζ	
$n+2$	2F	
	3ζ	
$n+3$	3F	
	4ζ	
$n+4$	4F	
	5ζ	
$n+5$	5F. G5	
	$\eta 4$	
$n+6$	G4	
	$\eta 3$	
$n+7$	G3	
	$\eta 2$	
$n+8$	G2	
	$\eta 1$	
$n+9$	G1	
	η	$\frac{tv}{100}$
$n+10$	G	
	1η	
$n+11$	1G	
	2η	
$n+12$	2G	

In the First Table.

Let n signify every 100th numbr. & F its root, whether Square, Cube, or Sq. Square; and $n-50$, $n-40$, $n-30$, &c., every tenth numbr.; and A, B, C, D, &c., their roots; and o, p, q, r, &c., the differences of these roots; and op, pq, qr, &c., their second differences, (that is op, the diff. of o & p, pq the diff. of p & q, &c.) and m their third difference, that is ye. common difference of *o & op, op & pq, pq & qr, &c.

Further, let α, β, γ, δ, &c., signify ye. the differences of these Roots from those next less, namely, α the difference of ye. root of $n-50$ & ye. like root of $n-51$, β, the diff. of ye. roots $n-40$ & $n-41$, ζ the diff. of ye. roots of n & $n-1$, η the diff. of ye. roots of $n+10$ & $n+9$, &c. And let o, π, χ, ρ, &c., signify the diff. of α, β, γ, δ, &c. And $\frac{m}{10}$ the common diff. of o, π, χ, ρ, &c.

In the Second Table.

Let $n-6$, $n-5$, $n-4$, $n-3$ &c. signify ye. single numbers,

4E, 5E, or F5, F4, F3 &c. their roots,

5ϵ, $\zeta 4$, $\zeta 3$, $\zeta 2$ &c. the diff. of those roots;

$\frac{st}{100}$ the common diff. of those differences for ye. ten numbers between $n-5$ & $n+5$.

And so for ye. ten numbers between $n+5$ & $n+15$; let G5, G4, G3, &c. signify ye. roots; $\eta 4$, $\eta 3$, $\eta 2$ &c. their first differences, and $\frac{tv}{100}$ their second differences; and the like for every denarie between $n-50$ & $n+50$.

This explication of the Tables being p'mis'd, you may compute them thus;

$$\text{Out of } n, \text{ extract F ye.} \left\{\begin{matrix}\text{Square}\\ \text{Cube}\\ \text{Sq. Sq.}\end{matrix}\right\} \text{Root, make} \left\{\begin{matrix} \frac{10\text{F}}{2n}=\omega, & \frac{10\omega}{2n}=st, & \frac{30st}{2n}=m \\ \frac{10\text{F}}{3n}=\omega, & \frac{20\omega}{3n}=st, & \frac{50st}{3n}=m \\ \frac{10\text{F}}{4n}=\omega, & \frac{30\omega}{4n}=st, & \frac{70st}{4n}=m \end{matrix}\right.$$

$$\omega+\frac{1}{2}st+\frac{1}{6}m=s,\quad \frac{\omega}{10}+\frac{\frac{1}{2}st}{100}+\text{‘}\frac{m}{6000}\text{’}\dagger=\zeta, \text{ and } \frac{st}{10}+\frac{55m}{1000}=\sigma$$

And these quantities F, st, m, s, ζ, & σ, being thus found, ye. rest are given by Additn. & Subduct.

† *Note by Editor* (J. Edleston).—I have added the ‘$\frac{m}{6000}$’. I have also corrected some other errors of transcription.

Note by D. C. F.—The quantity $\frac{m}{6000}$ is so small that Newton properly omitted it.

For $st + m = rs$, $rs + m = qr$, &c. $st + m = tr$, $tr + m = vx$, &c.

Again $s + rs = r$, $r + rq = q$, &c. $s - st = t$, $t - tv = v$, &c.

And $F - s = E$, $E - r = D$, &c. $F + t = G$, $G + v = H$, &c.

Further

$\sigma + \frac{m}{10} = \rho$, $\rho + \frac{m}{10} = \chi$, &c. $\sigma - \frac{m}{10} = \tau$, $\tau - \frac{m}{10} = \upsilon$, &c.

Lastly $\zeta + \sigma = \epsilon$, $\epsilon + \rho = \delta$, &c. $\zeta - \tau = \eta$, $\eta - \upsilon = \theta$, &c.

These quantities being thus computed in the first Table, to every 10^{th} number, the roots may be computed in y^e. 2^d Table to every numb^r. by Addition and Subduction only ;

For $\zeta + \frac{st}{100} = \zeta 1$, $\zeta 1 + \frac{st}{100} = \zeta 2$, &c.

$\zeta - \frac{st}{100} = 1\zeta$, $1\zeta - \frac{st}{100} = 2\zeta$, &c.

Again $F - \zeta = F1$, $F1 - \zeta 1 = F2$, &c.

$F + 1\zeta = 1F$, $1F + 2\zeta = 2F$, &c.

Thus you must proceed to five Figures on either hand, and then do the like in the next ten Figures, saying

$$\eta + \frac{tv}{100} = \eta 1,\ \eta 1 + \frac{tv}{100} = \eta 2, \text{ \&c.}$$

And the like for every Denarie between $n - 50$ & $n + 50$.

In these Computations, Note, 1^st.—That they must be done everywhere to 10 or 11 decimal places, if you will have a Table of Roots exact to 8 of these places.

2^dly.—If 5F & G5, the roots of $n + 5$ found two ways agree to 8 decimal places, it argues the whole works from which they were derived to be true. And so of y^e. roots of $n + 15$, $n + 25$, $n - 5$, &c. And also of y^e. Terms A, $^*o$, & α ; L, z^*, & λ wherê two works meet. Let this therefore be y^e. Proof of y^e. work.

This S^r. is w^t. has occurred to me about your design, which I hope will do your business, the whole work being p'form'd by Addit. & Subduct : excepting y^t. in y^e. computation of every 100^th number, there is required y^e. Extraction of one root, & three divisions to find F, ω, st, & m.

S^r. I am,

Your humble Serv^t.

IS. NEWTON.

[*Note appended by Editor*, J. EDLESTON.]

The person to whom this letter is written may be conjectured to be "John Smith, Philo-Accomptant", author of Stereometrie, Lond. 1673. (He must not be confounded with Cotes's uncle). In the Macclesfield Correspondence, II, 370–374, there are two other letters on the extraction of roots from Newton to this same person (not to Collins, as there printed) dated July 24 and Aug. 27, 1675, in the former of which he refers to the method given in the foregoing letter. Mr. J. Smith seems to have had a design of constructing Tables of Square, Cube and Biquadr. Roots, and consulted Newton as to the best mode of computing them. The Tables, if ever made, do not appear to have been published. The earliest Tables of Roots are Briggs's MS. Tables of the Square Roots of Numbers up to 1000 mentioned in Mayne's Merchant's Companion (London, 1674), p. 80.

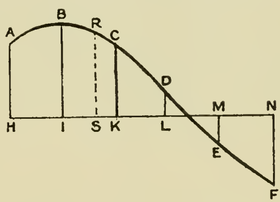
III.

PRINCIPIA. BOOK III. LEMMA V.

(*Translation.*)

To find a parabolic curve which shall pass through any given points :

$$\begin{array}{ccccccccc} b & & b_2 & & b_3 & & b_4 & & b_5 \\ & c & & c_2 & & c_3 & & c_4 & \\ & & d & & d_2 & & d_3 & & \\ & & & e & & e_2 & & & \\ & & & & f & & & & \end{array}$$



Let the points be A, B, C, D, E, F, &c , and from them let fall perpendiculars AH, BI, CK, DL, EM, FN, on any given straight line HN.

Case I.—If the intervals HI, IK, KL, &c., between the points H, I, K, L, M, N, are equal, find the first differences b, b_2, b_3, b_4, b_5, &c., of the perpendiculars ; the second differences c, c_2, c_3, c_4, &c. ; the third differences d, d_2, d_3, &c., so that $AH - BI = b$, $BI - CK = b_2$, $CK - DL = b_3$, $DL + EM = b_4$, $-EM + FM = b_5$, &c. ; $b - b_2 = c$, &c. ; and so on, to the last difference, which in this case is f. Then having erected any perpendicular RS, which shall be an ordinate to the required curve, in order to find its length put each of the intervals HI, IK, KL, LM, equal to unity and let $AH = a$,

$$-HS = p,\quad \frac{1}{2}p \times (-IS) = q,\quad \frac{1}{3}q \times SK = r,\quad \frac{1}{4}r \times SL = s,\quad \frac{1}{5}s \times SM = t,$$

proceeding in the same way to the last but one of the perpendiculars ME, and prefixing the negative sign to the terms HS, IS, &c., which lie on the same side of the point S as A, and the positive signs to the terms SK, SL, &c., which lie on the other side of the point S. Then, observing the correct signs, RS will be $= a + bp + cq + dr + es + ft$, &c.

Case II.—But if the intervals HI, IK, &c., between the points H, I, K, L are unequal, find b, b_2, b_3, b_4, b_5, being the first differences of the perpendiculars AH, BI, CK, &c., divided by the intervals between the perpendiculars; c, c_2, c_3, c_4, &c., the second differences divided by the intervals between the alternate perpendiculars; d, d_2, d_3, &c., the third differences divided by the intervals between every third perpendicular; e, e_2, &c., the fourth differences divided by the intervals between every fourth perpendicular, and so on; so that $b = \frac{\text{AH} - \text{BI}}{\text{HI}}$, $b_2 = \frac{\text{BI} - \text{CK}}{\text{IK}}$, $b_3 = \frac{\text{CK} - \text{DL}}{\text{KL}}$, &c.; $c = \frac{b - b_2}{\text{HK}}$, $c_2 = \frac{b_2 - b}{\text{IL}}$, $c_3 = \frac{b_3 - b_4}{\text{KM}}$, &c.; $d = \frac{c - c_2}{\text{HL}}$, $d_2 = \frac{c_2 - c_3}{\text{IM}}$, &c. The differences having been ascertained, put $\text{AH} = a$, $-\text{HS} = p$, $p \times (-\text{IS}) = q$, $q \times \text{SK} = r$, $r \times \text{SL} = s$, $s \times \text{SM} = t$; proceeding, it will be understood, to the last perpendicular but one, ME; then the ordinate RS will be $= a + bp + cq + dr + es + ft$, &c.

Corollary.—Hence the areas of all curves can be ascertained approximately. For if several points are found of the curve whose area is required, and a parabolic curve be supposed drawn through these points, its area will be approximately the same as that of the given curve. But the area of the parabolic curve can always be found geometrically by methods that are very well known.

II.

THE following references to the subject will be found in the "Commercium Epistolicum":

Letter from Leibnitz to Oldenburg, dated 3 *February* 1672/3. —The "Commercium Epistolicum" includes ten letters from Leibnitz to Oldenburg, and references to five others, besides four letters from Leibnitz to other mathematicians.

Only one of these, the earliest of the series, discusses questions relating to Finite Differences; and as it is entirely devoted to this subject and furnishes an interesting indication of the state of knowledge at the time, I have given a version of it in full. It will be observed that while the heading clearly states that it is addressed to Oldenburg, he is referred to in the course of the letter in the third person and by name. It seems to have the form of a memorandum rather than of a letter, and we may suppose it to have been written for private circulation, or possibly with a view to publication in the *Transactions* of the Royal Society, of which Oldenburg was Secretary.

The paragraph in which Leibnitz explains a general rule relating to the differences of any kind of powers is merely a verbal description of the formula

$$a^n - b^n = a^{n-1}(a-b) + b(a^{n-1} - b^{n-1}).$$

In his final paragraph he gives some examples of series with fractional terms, the general term of the rth series being

$$\frac{r}{(n+2)(n+3)\ \ldots\ (n+r+1)}.$$

The rest of the letter is taken up with a discussion of the fundamental formula of Finite Differences, $u_n = (1+\Delta)^n u_0$, for the case when n is an integer; and with remarks on the properties of the coefficients which are employed.

It appears that Leibnitz and writers before him found no difficulty in writing down the formula for any given integral value of n, the necessary coefficients being obtained by reference to a table. Leibnitz employed a table drawn from Pascal's Treatise on the Arithmetical Triangle (printed in 1654, but not published until 1665); an account of which is given in the "History of the Theory of Probability" by Isaac

Todhunter, who mentions the application of the Arithmetical Triangle by Pascal in solving questions of Combinations and Probabilities, and in obtaining the powers [*i.e.*, integral powers] of binomial quantities. The particular point of interest to us in the present connection is that mathematicians before Newton were in the habit of using one and the same table for obtaining the integral powers of binomial quantities, and for expressing any one of a series of values in terms of the initial value and its leading differences.

In vol. xiv of the *Journal* (pp. 1 and 73) references will be found to the early use of methods of Finite Differences by Briggs and by Mouton, both of whom are mentioned in Leibnitz' letter.

Newton's letter of 13 *June* 1676.—This letter needs little comment. In it the first enunciation appears of what we now call the Binomial Theorem, a name not used by Newton; and he gives detailed examples of its application. A series is also given for finding a number from its logarithm, *i.e.*, the exponential series. There are some remarks of a vague and general character on methods of obtaining approximate series which probably refer to methods of differences. In my version I have omitted a short section on the roots of equations.

Newton's letter of 24 *October* 1676.—This letter constitutes the principal document in the controversy between Newton and Leibnitz as to priority in the discovery of the Differential Calculus. The letter is a long one; and in my version I have omitted more than half of it, retaining only those sections (fortunately non-controversial) which contain matter of interest in relation to the subject of interpolation by methods of finite differences. It opens with a clear and detailed account of the process of discovery of the Binomial Theorem; and it is interesting to note that the discovery arose out of a problem in interpolation, the question which Newton set himself to solve being to ascertain the form of the series which would represent the function $\int (1-x^2)^m \,.\, dx$ for a fractional value of m, the forms of the series for a number of integral values of m being known. This was an example of the "interpolation of series", *i.e.*, of the insertion of a new series among a number of known series. The solution of this question suggested the solution of the simpler problem of finding the general form of the series for the expansion of $(1-x^2)^m$, for any value of m integral or fractional, which led at once to the Binomial Theorem; and the Binomial Theorem in turn suggested many of the familiar processes which are included in the general description of "Algebra up to the Binomial Theorem."

In a letter dated 21 June 1677 to Oldenburg, Leibnitz says, "His description of the way in which he was led to some of "his very elegant theorems is singularly happy; and what he "says on the interpolations of Wallis is especially pleasing, "because by this argument a proof of these interpolations is "obtained which (so far as I know) had previously been given "by induction only."

The date of discovery of the Binomial Theorem is fixed by a note left by Newton in which he says: "In the beginning "of the year 1665 I found the method of approximating series "and the rule for reducing any dignity " [*i.e.*, power] " of a "binomial into such a series."

Newton's proof depends solely on the properties of the coefficients, and is not affected in any way by the nature of the quantities, algebraical or otherwise, with which they are associated. Remembering that the mathematicians of the time were in the habit of familiarly using the same table of coefficients for obtaining the expansions of the integral powers of binomials, and for expressing any one of a series of known values in terms of the initial value and its leading differences, it will be appreciated that Newton's discovery of a general formula for the expansion of the fractional powers of binomials gave him command at the same time of the fundamental formula of interpolation $u_n=(1+\Delta)^n . u_0$, which he embodied more than twenty years later in Lemma V of Book III of the Principia.

That he at once proceeded to apply his theorem in this direction is suggested by the fact that in the summer of the same year he engaged in extensive calculations of logarithms. In two separate sections of the letter he gives elaborate details of his methods, and in a note dated 4 July 1699, quoted in Brewster's Life of Newton, he says: "In summer 1665, being "forced from Cambridge by the plague, I computed the area "of the hyperbola at Boothby, in Lincolnshire, to two and fifty "figures." The MS. is still preserved in the University Library at Cambridge, and is mentioned in the "Catalogue "of the Portsmouth collection of books and papers written "by or belonging to Sir Isaac Newton" (Cambridge, 1888), under the title "Calculation of the Area of the Hyberbola", being item No. 4 of sub-section "Early papers by Newton. (Holograph)." An examination of this MS. and of other MSS. in the same collection, for example the "Regula Differentiarum", which is No. 5 of sub-section "Miscellaneous Mathematical subjects" might bring to light valuable information.

Newton remarks in his letter that at a later date he used other methods which gave logarithms more exactly, and the editors of the "Commercium Epistolicum" refer to the "Geometria Analytica." This work was first published in 1736 in a translation by Colson and it contains formulas for the interpolation of logarithms derived from the properties of the logarithmic series.

The "Logarithmotechnia, sive methodus construendi logarithma nova, accurata et facilis" (4to. London), of Nicholas Mercator (not Mercator of the Maps), which is mentioned by Newton was published in September 1668. In this work Mercator took the equation to the hyperbola in the form $y=\frac{1}{1+x}$; and by simple division, explaining each step of the process in great detail, he obtained the series $1-x+x^2-x^3+$, &c., the integration of which term by term gave the logarithmic series. This was the first time that the operation of division with algebraical symbols had appeared in print and it excited extraordinary interest among mathematicians, though as the editors of the "Commercium Epistolicum" explain it was already known that the expression $\frac{1}{1-x}$, x being less than unity, represented the sum of the series $1+x+x^2+x^3+$, &c., taken to infinity. On his attention being drawn to this publication, Barrow, then Professor of Mathematics at Cambridge, made it known that a young friend of his, by name Newton, had previously arrived at general propositions of which Mercator's example was only a particular case.

Newton mentions that he wrote some papers on series in 1671. Wallis states that these papers, or some of them, were destroyed by fire. Some portions of them appear to have been included in the "Geometria Analytica."

In a following section of the letter will be found the references, which I have already quoted, to the "Methodus Differentialis."

In the second of the two sections on the subject of logarithms, the process of interpolation by intervals of one-tenth is repeatedly mentioned, and there can be little doubt that this is the process described in detail in the letter of 8 May 1675 to J. Smith. We may reasonably suppose that the same process is referred to in the final paragraph of my version where Newton says that he had almost decided to describe his method of inserting intermediate terms in the construction of trigonometrical and other tables.

It is a remarkable conclusion that as early as the year 1665, when he was still under the age of 23, Newton appears to have had at his command practically all the methods and facilities of computation which are now in use, with the exception of calculating machines; and when we consider that we owe the originating germ of such machines to Newton's contemporary Pascal, we may realize what a direct and vital connection there is between the ideas of the mathematicians of the 17th century and the practical work of the present-day Actuary.

Extract from Letter of February 1672/3.

Leibnitz to Oldenburg.

Recently when I happened to meet the eminent mathematician, Pell, at the house of the famous Boyle, we began to talk about numbers, and I was reminded by our conversation that I had a method of my own of constructing the terms of a series of any kind, either increasing or decreasing continuously, by a class of differences which I call generating differences. If the differences of a given series are found, and the differences of the differences, and the differences arising from the differences of the differences, &c.; and if a series be constructed consisting of the first term, and the first difference, and the first difference of the differences, and the first difference arising from the differences of the differences, &c., that will be the series of generating differences; so that if the continuously increasing or decreasing series be a, b, c, d, then putting $\backsim$ as the sign of the difference the generating differences will be:

1. a 2. $a \backsim b$ 3. $\overline{a \backsim b} \backsim \overline{b \backsim c}$ 4. $\overline{\overline{a \backsim b} \backsim \overline{b \backsim c}} \backsim \overline{\overline{b \backsim c} \backsim \overline{c \backsim d}}$

$$
\begin{array}{lcccc}
4. & \overline{\overline{a \backsim b} \backsim \overline{b \backsim c}} & \backsim & \overline{\overline{b \backsim c} \backsim \overline{c \backsim d}} & \\
3. & \overline{a \backsim b} \backsim \overline{b \backsim c} & & \overline{b \backsim c} \backsim \overline{c \backsim d} & \\
2. & a \backsim b & & b \backsim c & c \backsim d \\
1. & a & b & c & d
\end{array}
$$

Or, in numbers, if the series be the series of cubes increasing in succession from unity, the generating differences will be 0, 1, 6, 6. I call them by this name, because the terms of the series are produced from them when multiplied in a particular way. Their use appears to be greatest when the generating differences are finite in number, but the terms of the series infinite; as in the example proposed of cube numbers:

				0		0		0				
			6		6		6		6			
		6		12		18		24		30		
	1		7		19		37		61		91	
0		1		8		27		64		125		216

When the eminent Pell heard this, he replied that it had already been described by Mouton, Canon of Leyden, from the observation of the most noble Francis Reynald of Leyden, a man long celebrated in the literary world, in a book of Mouton on the apparent diameters of the Sun and Moon. From a letter of Reynald's addressed to Monconisius, and from a diary of the journeys of Monconisius, I had become acquainted with the name of Mouton, and the two works he had in view; namely, the apparent diameters of the luminaries, and a scheme for transmitting the measures of things to posterity; but I did not know that the book had actually appeared. Wherefore I hurried off at once to Oldenburg, the Secretary of the Royal Society, and borrowed the book and found that Pell had spoken truly. But nevertheless I thought that I ought to take pains that no suspicion should remain in their minds of my having wished to appropriate the credit of another man's ideas by suppressing the name of the discoverer, and I hope it will be apparent that I am not in such want of ideas as to be compelled to pilfer those of others. Moreover, I shall vindicate my honour by two arguments; first by showing my rough notes, in which not only my discovery but also the manner and the occasion of the discovery appear; and then by adding some things of the greatest importance not remarked by Reynald and Mouton, which it is not very likely should have been contrived by me in a night, and which could not reasonably be expected to be produced by a mere transcriber.

From my papers it appears that the occasion of the discovery was as follows: I was seeking a method of finding the differences of every kind of powers; just as it is known that the differences of the square numbers are the odd numbers; and I had found a general rule of this kind.

The preceding power of a given order being known, to find the power following (or the reverse) at a given distance, that is the powers of given terms; or to find the differences of the powers of a given order, whatever their distances apart. In the powers of the next lower order, let the power of the greater term be multiplied by the difference of the terms ; and let the difference of the powers (still in the next lower order) be multiplied by the smaller term. The sum of the products will be the required difference of the powers of the given terms. I had adapted the same rule in such a way that to ascertain the powers of the terms for a higher order it was sufficient to know the powers of the given terms for any lower order. And I showed that what is observed to be the case for squares, namely, that their differences are the odd numbers, is not outside the basis of the rule proposed.

My mind being fixed on these ideas, as in the case of square numbers the differences are the odd numbers, so also I enquired what might be the differences of the cubes ; and since these appeared to be irregular I sought the differences of the differences, until I found the third differences to be all sixes. This observation produced another. For I saw that the terms and the successive differences were generated from the preceding differences in the same way as all the successive terms arise from the primary differences, which I call on that account the generating differences, namely, in this case 0, 1, 6, 6. Having come to this conclusion it remained to find by what kind of addition or multiplication, or combination of these, the successive terms could be produced from the generating differences. And thus by solution and experiment, I perceived the first term, 0, to be composed of the first generating difference, 0, taken once or by itself ; the second term, 1, to be composed of the first generating difference, 0, taken once ; and the second, 1, taken once ; the third term, 8, of the first generating difference, 0, taken once ; the second, 1, taken twice ; and the third, 6, taken once ; for

$$0 \times 1 + 1 \times 2 + 6 \times 1 = 8\ ;$$

the fourth term, 27, of the first generating difference, 0, taken once ; the second, 1, taken three times ; the third, 6, taken three times ; and the fourth, 6, taken once ; for

$$0 \times 1 + 1 \times 3 + 6 \times 3 + 6 \times 1 = 27\ ;$$

and further calculation proved to me that this was general. This was the occasion of my observation, far otherwise from Mouton's way of approaching it ; who happened upon this convenient method of calculation along with Reynald, when he was at work on the construction of his tables. Nor should either he or Reynald have any less praise because Briggs also had in some degree turned his attention to certain methods of this kind in his logarithmic tables, as Pell observes. For me, this much remains ; that I may add some things not remarked by them so as to avoid the reputation of being a transcriber merely ; for in the commonwealth of knowledge it does not matter who made an observation ; the thing that matters is what was observed. First then I direct attention to a question which is not noticed in Mouton's works, and yet is the head of the whole matter, namely, what are those numbers of which he gives a table to be continued to infinity, by the multiplication of which into the generating differences and by combining the products, the terms of the series may be produced. For you may see, from the very way in which the table is set out on p. 385 of his book, that it has not been sufficiently examined by him; for otherwise it is likely that the table would have been set forth in such a way that the connection and harmony of its numbers would be apparent; unless one is to say that he has been at pains to conceal it; for a part of the table is as follows:

1	1					
2	1	1				
3	1	2	1			
4	1	3	3	1		
5	1	4	6	4	1	
6	1	5	10	10	5	1
7	1	6	15	20	15	6
8	1	7	21	35	35	21
9	1	8	28	56	70	56
10	1	9	36	84	126	126
11	1	10	45	120	210	252

It appears from this table that the relationship of correspondence of the generating numbers is only with the number of the term generated; so that when the term is the fourth it is produced from the first difference taken once, the second difference taken three times, the third taken three times, and the fourth once; and therefore in the same transverse line (4) are placed the numbers 1, 3, 3, 1. But the author has either not observed, or if he has observed it he has concealed that he knew the correspondence of the numbers if they are arranged in columns proceeding from the top downwards in the following manner:

1	1					
2	1	1				
3	1	2	1			
4	1	3	3	1		
5	1	4	6	4	1	
6	1	5	10	10	5	1
7	1	6	15	20	15	6
8	1	7	21	35	35	21
9	1	8	28	56	70	56
10	1	9	36	84	126	126
11	1	10	45	120	210	252

For in this way their real and genuine nature and origin are apparent; that they are in fact the numbers, which I am accustomed to call combinatorial numbers, of which I have written at length in my dissertation on the art of combination, and which others call the numerical orders; unities in the first column; natural numbers in the second column; triangular numbers in the third column; pyramidal numbers in the fourth; triangulo-triangular in the fifth, &c., of which a whole treatise of Pascal's deals under the title of the Arithmetical Triangle; in which nevertheless I have wondered that such a conspicuous and natural property of these numbers has not been observed.

[*Note by Editors of the "Commercium Epistolicum."*—On the contrary, it has been observed. See Pascal's Arithmetical Triangle, published in Paris, in the year 1665, p. 2, where the last definition but one is this:

"The number in each cell is equal to that of the cell which precedes it in "the perpendicular column, added to that of the cell which precedes it in its "parallel column. So the cell F, that is the number in the cell F, is equal to "the cell C plus the cell E; and similarly for the other cells."]

But there is indeed an element of fortune in discovery, which does not always offer the best things to the greatest abilities, but often gives some of them to moderate abilities.

Hence the true nature of these numbers and the construction of the table is perceived, whether concealed or not by Mouton or by Reynald; for any given term of a given column is composed of the preceding term in the same column and of that in the previous column; and it also appears, that it is not a work involving any troublesome calculation to continue the table set forth by Mouton as he demands, since these series of numbers are now everywhere described and used in calculation.

Moreover, from the observation of Mouton for interpolating proportional means between two extremes, I drew the conclusion that it could be used for continuing the extreme numbers themselves to infinity. He found a use for the rule only when the ultimate differences vanish, or almost vanish; I, however, detected innumerable cases, included in a rule which had been overlooked, where although the differences do not vanish I can from given finite numbers multiplied in a certain way, produce the numbers to infinity of very many series.

From the same foundations, I can work out many problems in progressions either in integers or fractions. For I can add and subtract progressions, and even multiply and divide them, and that very conveniently:

$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$
$\frac{1}{6}$	$\frac{1}{10}$	$\frac{1}{15}$	$\frac{1}{21}$
$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{35}$	$\frac{1}{56}$
$\frac{1}{15}$	$\frac{1}{35}$	$\frac{1}{70}$	$\frac{1}{126}$
&c.	&c.	&c.	&c.

Many other points about these numbers have been noticed by me, of which the above is one of the most important; I have a method of finding the sum of a series of fractions decreasing to infinity, of which the numerator is unity, and the denominators are the triangular or pyramidal numbers, or triangulo-triangular numbers, &c.

Letter of 13 June 1676.

Newton to Oldenburg.

(To be communicated to Leibnitz.)

Although the modesty of Leibnitz, in the extracts which you have lately sent me from his letter, attributes great credit to our countrymen for their investigations into Infinite series of which there begins to be talk; nevertheless I feel no doubt that he has not only discovered, as he claims, a method of reducing all sorts of quantities into series of this kind, but also a variety of convenient processes very like ours if not even better.

But since he wishes to know what discoveries have been made in England in the subject, and it happens that several years ago

I engaged in investigations of this kind, I have sent you some points that have occurred to me so that I might meet his wishes at least in some degree.

Fractions are reduced into infinite series by division, and radical quantities by the extraction of roots; by performing these operations on symbols in the same way as they are usually performed on decimal numbers. These are the fundamental principles of such reductions.

But the extractions of roots are greatly shortened by this theorem

$$\overline{P+P.Q.}|^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}A.Q. + \frac{m-n}{2n}B.Q. + \frac{m-2n}{3n}C.Q + \frac{m-3n}{4n}D.Q + \&c.$$

where $P+P.Q.$ signifies the quantity of which a root, or a power, or a root of a power is to be found; P is the first term; Q the remaining terms divided by the first. Also $\frac{m}{n}$ is the numerical index of the power of $P+P.Q.$ whether the power be integral or fractional, positive or negative. For as mathematicians are accustomed to write a^2, a^3, &c., for $a.a$, $a.a.a$, &c., so for $\sqrt{a}$, $\sqrt{a^3}$, $\sqrt{c.a^5}$, &c., I write $a^{\frac{1}{2}}$, $a^{\frac{3}{2}}$, $a^{\frac{5}{3}}$, &c., and for $\frac{1}{a}$, $\frac{1}{a.a}$, $\frac{1}{a.a.a}$, I write a^{-1}, a^{-2}, a^{-3}. And in the same way for $\frac{a.a}{\sqrt{c.\,a^3+b.b.x}}$ I write $a.a\times\overline{a^3+b.b.x}|^{-\frac{1}{3}}$ and for $\frac{a.a.b}{\sqrt{c:(a^3+b.b.x)\times(a^3+b.b.x)}}$ I write $a.a.b\times\overline{a^3+b.b.x}|^{-\frac{2}{3}}$. In the last case if $\overline{a^3+b.b.x}|^{-\frac{2}{3}}$ be taken to stand for $\overline{P+P.Q.}|^{\frac{m}{n}}$ in the formula, P will be $=a^3$; Q will be $=\frac{b.b.x}{a^3}$; $m=-2$, $n=3$. Finally for the terms found in the quotient in the course of the work, I employ the symbols A, B, C, D, &c.; namely, A for the first term, $P^{\frac{m}{n}}$; B for the second, $\frac{m}{n}\cdot A.Q$; and so on in succession. In other respects the use of the formula will be plain from examples.

[The examples, each of which is expanded to several terms, are

(1) $(c^2+x^2)^{\frac{1}{2}}$

(2) $(c^5+c^4x-x^5)^{\frac{1}{5}}$. Here P may be taken as c^5 or as $-x^5$. "The former is to be preferred if x is very small; the latter if x is very great."

(3) $N\times\overline{y^3-a^2y}|^{-\frac{1}{2}}$

(4) $(d+e)^{\frac{4}{3}}$

(5) "In the same way simple powers also are produced."

$$\text{Ex}: (d+e)^5$$

(6) "Moreover, Division, whether simple or repeated, is accomplished by the same rule."

$$\text{Ex}: \frac{1}{d+e} = (d+e)^{-1}$$

(7) $(d+e)^{-3}$

(8) $\mathrm{N} \times (d+e)^{-\frac{1}{3}}$

(9) $\mathrm{N} \times (d+e)^{-\frac{3}{5}}$]

By the same rule, expansions of powers, divisions by powers or by radical quantities, and extractions of the higher roots of numbers, are also conveniently performed.

*   *   *   *   *

It would take too long to describe how, from equations so reduced to infinite series, the areas and the lengths of curves, the volumes and the surfaces of solids, or of any segments of any figures, and their centres of gravity, may be determined ; and also how all mechanical curves can be reduced to equations of infinite series of this kind, and problems concerning them resolved just as if they were geometrical curves. It may suffice to review some specimens of such problems ; and in these I shall sometimes for the sake of brevity use the letters A, B, C, D, &c., to indicate the successive terms of a series taken from the beginning.

(Nine problems are discussed. The first three relate to trigonometrical functions, and the next three to the ellipse ; the eighth to the quadratrix, and the ninth to the spheroid. The seventh is as follows) :

(7) if C D be an hyperbola, whose asymptotes EB, EF make the right angle BEF ; and if on EB are erected perpendiculars AC, BD, meeting the hyperbola in C and D ; and if EA be called a ; AC, b ; and the area CADB, z ;

$$\text{AB will be} = \frac{z}{b} + \frac{z^2}{2ab^2} + \frac{z^3}{6a^2b^3} + \frac{z^4}{24a^3b^4} + \frac{z^5}{120a^4b^5} +, \&\text{c.}$$

Where the coefficients of the denominators arise from the multiplication into one another continuously of the terms of the A.P. 1, 2, 3, 4, 5, &c. And hence from a given logarithm the number corresponding to it can be found.

From these examples it may be seen how the bounds of Analysis are enlarged by means of infinite equations of this kind ; indeed, by their aid the method extends, I had almost said, to all problems except Diophantine and similar questions. The method however is not quite general except by means of certain further methods of forming series. For there are problems, in which one cannot arrive at infinite series by division, or by the extraction of roots, simple or affected. But there is no time to say what the procedure should be in such cases; nor to describe some other things which I had devised relating to the reduction of infinite into finite series, when the nature of the case permits. For I am writing somewhat briefly because these speculations have begun for some time past to be less interesting to me, so that indeed I have abstained from them now for almost five years.

Nevertheless I shall add one point; that after any problem is expressed in terms of an infinite equation, then various approximations for mechanical use can be found with hardly any labour; which when sought by other methods usually involve much labour and the expenditure of time.

[The letter concludes with an investigation of Huygen's approximate formula for the length of the arc of a circle:

If A be the chord of the arc, B the chord of half the arc, and r the radius, then if z be the length of the arc;

$$z = \frac{8B - A}{3} \text{ with an error less than } \frac{z^5}{7680r^4} \text{ in excess};$$

and some further approximations are given relating to the circle, the ellipse, and the hyperbola].

Letter of 24 October 1676.

Newton to Oldenburg.

(To be communicated to Leibnitz.)

I could hardly say with how much pleasure I have read the letters of those illustrious men, Leibnitz and Tschirnhausius.

The method by which Leibnitz obtains convergent series is very elegant indeed, and would have shown the ability of the author if he had written nothing else. But remarks, most worthy of his reputation, which occur here and there in his letter, lead us to hope for the greatest things from him. The diversity of methods by which the same results are obtained is all the more attractive because three methods had already become known to me of arriving at series of this kind; so that I hardly expected that anything new could be communicated to us.

One of my methods I described in my former letter; now I add another; that, indeed, by which I was first led to these series; for I was led to them before I discovered the division (of fractions) and the extractions of roots which I now use. And in the explanation of the Theorem which was set forth at the beginning of my former letter is to be found the foundation which Leibnitz desires from me.

At the beginning of my mathematical studies, when I had fallen in with the works of our celebrated Wallis and came to consider the series by the interpolation of which he brings out the areas of the circle and of the hyperbola, since in the series of curves whose basis or common axis is x and whose ordinates are :

$$(1-x^2)^{\frac{0}{2}},\ (1-x^2)^{\frac{1}{2}},\ (1-x^2)^{\frac{2}{2}},\ (1-x^2)^{\frac{3}{2}},\ (1-x^2)^{\frac{4}{2}},\ (1-x^2)^{\frac{5}{2}},\ \&c.,$$

if the areas of the alternate curves, which are

$$x,\ x-\frac{1}{3}x^3,\ x-\frac{2}{3}x^3+\frac{1}{5}x^5,\ x-\frac{3}{3}x^3+\frac{3}{5}x^5-\frac{1}{7}x^7,\ \&c.,$$

can be interpolated we should have the areas of the intermediate curves the first of which, $(1-x^2)^{\frac{1}{2}}$, is the circle; I noted for these interpolations that in every case the first term was x, that the second terms $\frac{0}{3}x^3$, $\frac{1}{3}x^3$, $\frac{2}{3}x^3$, $\frac{3}{3}x^3$, &c., were in A.P.; and accordingly that the two first terms of the series to be interpolated must be

$$x-\frac{1}{2}\cdot\frac{x^3}{3},\ x-\frac{3}{2}\frac{x^3}{3},\ x-\frac{5}{2}\frac{x^3}{3},\ \&c.$$

For the insertion of the remaining terms I considered that the denominators 1, 3, 5, 7, &c., were in A.P.; and so the numerical coefficients had to be investigated for the numerators only. But in the given alternate areas these were the figures which express the powers of the number 11; namely, 1; 1, 1; 1, 2, 1; 1, 3, 3, 1; 1, 4, 6, 4, 1, &c.

Then I set myself to enquire how in these groups of figures when the first two terms of a group were given the rest could be derived. And I found that assuming the second figure to be m the rest would be produced by the continuous multiplication of the terms,

$$\frac{m-0}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5}\times,\ \&c.$$

For example, let the second term m be $=4$; then the third term will be $4\times\frac{m-1}{2}$, that is 6; and the fourth $6\times\frac{m-2}{3}$, that is 4; and the fifth, $4\times\frac{m-3}{4}$, that is 1; and the sixth, $1\times\frac{m-4}{5}$, that is 0; with which the series in this case terminates.

I applied this rule therefore to obtain the intermediate series. And since for the circle the second term was $\frac{1}{2}\cdot\frac{x}{3}$, I put $m=\frac{1}{2}$ and the resulting terms were $\frac{1}{2}\times\frac{\frac{1}{2}-1}{2}$, or $-\frac{1}{8}$; $-\frac{1}{8}\times\frac{\frac{1}{2}-2}{3}$, or $\frac{1}{16}$; $\frac{1}{16}\times\frac{\frac{1}{2}-3}{4}$ or $-\frac{5}{128}$; and so on to infinity, whence I found that the area sought for the segment of a circle was

$$x - \frac{1}{2}\frac{x^3}{3} - \frac{1}{8}\frac{x^5}{5} - \frac{1}{16}\frac{x^7}{7} - \frac{5}{128}\frac{x^9}{9}, \text{ \&c.}$$

By the same method also the intermediate areas of the remaining curves are obtained; as also the area of the hyperbola and other alternate terms in this series,—

$$(1+x^2)^{\frac{0}{2}},\ (1+x^2)^{\frac{1}{2}},\ (1+x^2)^{\frac{2}{2}},\ (1+x^2)^{\frac{3}{2}}, \text{ \&c.}$$

And the method for obtaining intermediate series in other cases is the same, whether the intervals between two terms or more are wanting.

This was my first entrance into these speculations, which would have quite passed out of my memory had I not cast my eyes on certain memoranda a few weeks ago.

But when I had obtained these results, I soon began to consider that the terms $(1-x^2)^{\frac{0}{2}}$, $(1-x^2)^{\frac{2}{2}}$, $(1-x^2)^{\frac{4}{2}}$, $(1-x^2)^{\frac{6}{2}}$, could be interpolated in the same manner as the areas generated by them; and for this nothing more was necessary than the omission of the denominators 1, 3, 5, 7, &c., in the terms expressing the areas. That is to say, the coefficients of the terms of the quantity to be interpolated, $(1-x^2)^{\frac{1}{2}}$, or $(1-x^2)^{\frac{3}{2}}$, or generally $(1-x^2)^m$, arise from the continuous multiplication of the terms of this series,

$$m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$$

So, for example,

$$(1-x^2)^{\frac{1}{2}} = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6, \text{ \&c.}$$

$$(1-x^2)^{\frac{3}{2}} = 1 - \frac{3}{2}x^2 + \frac{3}{8}x^4 + \frac{1}{16}x^6, \text{ \&c.}$$

In this way therefore the general reduction of radical expressions into infinite series became known to me by the rule which I set forth at the beginning of my former letter, before I discovered the extraction of roots. But when I had learned this, the other could not long be concealed. For in order that I might prove these operations I multiplied $1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6$, &c., into itself and the result was $(1-x^2)$, the remaining terms to infinity vanishing throughout the continuation of the series. And in the same way $1 - \frac{1}{3}x^2 - \frac{1}{9}x^4 - \frac{5}{81}x^6$, &c., multiplied twice into itself produced $(1-x^2)$.

When I was sure of the demonstration of these conclusions I was led to try on the other hand whether these series which it proved to be roots of the quantity $(1-x^2)$ could not be extracted from it in the arithmetical manner ; and the attempt was quite successful. The form of the operation in the case of the square root was this :

$$1 - x^2\left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \text{, \&c.}\right.$$

$$\frac{1}{0 - x^2}$$

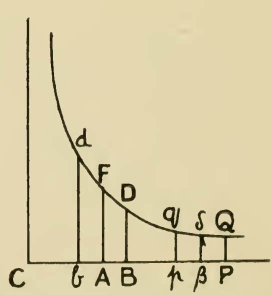
$$\frac{-x^2 + \frac{1}{4}x^4}{-\frac{1}{4}x^4}$$

$$\frac{-\frac{1}{4}x^4 + \frac{1}{8}x^6 + \frac{1}{64}x^8}{-\frac{1}{8}x^6 - \frac{1}{64}x^8}$$

After realizing these consequences I entirely neglected the interpolation of series ; and employed only these operations, as being more essentially fundamental. Nor was reduction by division concealed from me, a method very easy to use.

But I soon attacked the solution of affected equations, and obtained that also ; from which, the ordinates, the segments of the areas and any other lines became known at once when the areas or arcs of the curves were given. For the regression to these results required nothing beyond the solution of the equations by which the areas or arcs were given in terms of the given lines.

At that time the increasing plague (which fell in the years 1665, 1666) compelled me to fly from this place and to turn my thoughts to other things. Nevertheless, immediately afterwards I produced a scheme, which I here subjoin, for the calculation of logarithms from the area of the hyperbola. Let dFD be a hyperbola whose centre is C, vertex F, and intercepted square CAFE = 1.



[The completion of the square, and the letter E do not occur in Newton's diagram.]

In AC take AB and Ab on one side and the other, $=\frac{1}{10}$ or 0·1; and having erected the perpendiculars BD, bd, with their extremities on the hyperbola, the semi-sum of the spaces, AD and Ad

$$=0{\cdot}1+\frac{0{\cdot}001}{3}+\frac{0{\cdot}00001}{5}+\frac{0{\cdot}0000001}{7}, \text{ \&c.};$$

and their semi-difference

$$=\frac{0{\cdot}01}{2}+\frac{0{\cdot}0001}{4}+\frac{0{\cdot}000001}{6}+\frac{0{\cdot}00000001}{8}, \text{ \&c.}$$

And the calculation of these terms gives the following results:

0·100, 000, 000, 000, 0	0·005, 000, 000, 000, 0
333, 333, 333, 3	25, 000, 000, 0
2, 000, 000, 0	166, 666, 6
14, 285, 7	1, 250, 0
111, 1	10, 0
9	1
0·100, 335, 347, 731, 0	0·005, 025, 167, 926, 7

The sum of these 0·105, 360, 515, 657, 7, is Ad; and their difference 0·095, 310, 179, 804, 3, is AD. And in the same way AB and Ab being taken on one side and the other $=0{\cdot}2$, Ad will be found $=0{\cdot}223$, 143, 551, 314, 2; and AD$=0{\cdot}182$, 321, 556, 793, 9. After the hyperbolic logarithms of the four decimal numbers 0·8, 0·9, 1·1, and 1·2, have been obtained in this way; since $\frac{1{\cdot}2}{0{\cdot}8}\times\frac{1{\cdot}2}{0{\cdot}9}=2$; and 0·8 and 0·9 are less than unity, by adding their logarithms to twice the logarithm of 1·2, you get 0·693, 147, 180, 559, 7 as the hyperbolic logarithm of the number 2. Since $\frac{2\times2\times2}{0{\cdot}8}=10$, by adding the logarithm of 0·8 to three times the logarithm of 2 you will get 2·302, 585, 092, 993, 3 the logarithm of the number 10; and then by addition the logarithms of the numbers 9 and 11 are at once obtained, and thus the logarithms of all these prime numbers, 2, 3, 5, 11, are ready at hand. And further, simply by the depression of the numbers used in the decimal calculation detailed above, and by addition, the logarithms of the decimal numbers 0·98, 0·99, 1·01, 1·02, are obtained; as also of the decimal numbers 0·998, 0·999, 1·001, 1·002; and thence by addition and subtraction, the logarithms of the prime numbers 7, 13, 17, 37, &c., are obtained. Which, together with those found above, when divided by the logarithm of the number 10 give the true logarithms for insertion in a table. But these I afterwards obtained more exactly.

[*Footnote by Editors of "Commercium Epistolicum." Vide* "Geometriam Analyticam", Cap. ix, pp. 36–48.]

I am ashamed to say to what a number of decimal places I carried these calculations being then at leisure. For, indeed, I took too much pleasure at that time in these investigations. But when the ingenious Logarithmotechnia of Nicholas Mercator appeared (whom I suppose to have been the first discoverer of his methods), I began to take less interest in them; imagining that he was acquainted with the extraction of roots as well as the division of fractions; or that others, once the method of division had been brought to light, would soon discover the rest before I was of mature age for writing.

[*Footnote by Editors of "Commercium Epistolicum."*—Earlier mathematicians discovered this theorem, that the sum of the terms of a Geometrical Progression proceeding to infinity, is in the same ratio to the first and greatest of the terms, as this term is to the difference between the first and second terms. This is proved arithmetically by multiplying the extremes and means of the ratio. Wallis proved it by dividing the product of the means by the last term of the ratio. See the Opus Arithmeticum of Wallis, published in 1657, Cap. xxxiii, § 36. By using Wallis' method of division, Mercator proved and extended the quadrature of the hyperbola, previously discovered by Brounker. And Gregory proved the same geometrically. But none of these discovered the general method of finding the areas of curves. Mercator nowhere claimed this. Gregory who was admittedly a man of the greatest ability, and who had his attention drawn to the subject by the letters of Collins, at length and with difficulty found a method of this kind. Newton found it by the interpolation of series, and afterwards by the divisions and extractions of roots as being more familiar.]

At the time when that book appeared, a compendium of the method of these series was communicated to Collins by my friend Barrow, the Professor of Mathematics at Cambridge. In that compendium I had shown how the areas and lengths of all curves and the surfaces and volumes of solid bodies could be found from their given ordinates, and *vice versâ*, how the ordinates could be found if the areas, &c., were given; also I had illustrated the method there given by various series.

A regular correspondence having thereupon sprung up between us, Collins, who was devoted to the promotion of mathematical knowledge, did not cease to urge that I should make these results public. And five years ago (1671) when on the advice of my friends I was planning the publication of a treatise on the refraction of light and on colours which I then had in readiness, I began again to think of these series and wrote a treatise on them with the intention of publishing both together.

But, arising out of the catadioptric telescope, after I had written you a letter in which I briefly explained my ideas on the subject of light; something unexpected brought it about that I felt it to be of importance to me to write to you hastily about the printing of that letter. And the number of questions that immediately arose, through letters of various people crammed with objections, &c., deterred me entirely from my plan; and had the result that I accused myself of imprudence, because by grasping at the shadow I had lost my peace, a thing of real substance.

About that time James Gregory, from a single series of mine which Collins had sent him, arrived at the same method, after, as he said in a letter to Collins, a great deal of consideration ; and he left a treatise on the subject which we hope will be published by his friends. For, from the ability he possessed he could not but add much new matter of his own, and it is of importance in the interests of mathematical knowledge that it should not be lost. Moreover, I had not completely finished my treatise when I desisted from my plan ; nor to this day has my mind returned again to the things that were left to be added. That part indeed was wanting in which I had proposed to explain the method of solving problems which could not be reduced to quadratures ; granted that I might have done something towards the foundation of it. Moreover, infinite series did not occupy much space in that treatise.

* * * * *

Although many things remain to be investigated about methods of approximation, and about different kinds of series which may serve for that purpose ; nevertheless I should hardly hope with Tschirnhausius that simpler or more general methods of reducing quantities to the kind of series in question can be given than the divisions and the extractions of roots, which Leibnitz and myself use ; at any rate not more general, because for quadrature and rectification of curves and similar questions, no series can be given depending on these simple algebraical terms (involving only one indefinite quantity) which it is not possible to obtain by this method.

For the number of convergent series for the determination of the same quantity cannot be greater than the number of indefinite quantities from the powers of which the series are produced, and I am acquainted with methods of obtaining a series from any indefinite quantity that may be employed ; and I believe that Leibnitz also has that in his power.

For although by my method there is a free choice, for the construction of the series, of any indefinite quantity on which the question depends, and the method which he has communicated to us seems to be adapted for the choice of such indefinite quantities as can conveniently be reduced to fractions, which, by division only, produce infinite series ; nevertheless any other indefinite quantities whatever can be employed for the construction of series by means of the method used for the solution of affected equations, provided they are solved in appropriate terms, that is by constructing the series only from terms which are involved in the question.

Moreover, I do not see why it should be said that by using these divisions and resolutions, problems are solved by accident, since these operations have the same relation to this kind of algebra as the common operations of arithmetic to ordinary algebra.

But as regards simplicity of method ; I would not have fractions and radicals resolved invariably into infinite series without previous reduction ; when complicated quantities occur, all kinds of reductions are to be tried ; whether by increasing, diminishing, multiplying or dividing the indefinite quantities ; or by Leibnitz' method of transmutation ; or by any other method which may happen to fit the case ; and then resolution into series will be suitably employed by division and extraction.

Moreover, efforts should specially be made to reduce the denominators of fractions, and quantities under the radical sign, to the fewest and least complicated terms possible ; and to such also as are most rapidly expanded into convergent series, although the roots may be neither converted into fractions nor depressed. For by the rule given at the beginning of the earlier letter, the extraction of the highest roots is as simple and easy as the extraction of the square root or division ; and series which result from division are usually the least convergent of all.

Hitherto I have spoken of series involving only one indefinite quantity. But series can also by the method investigated be constructed at pleasure from two or from more indefinite quantities. Moreover, by the aid of the same method series can be formed for all curves, of a character similar to the series given by Gregory for the circle and hyperbola ; that is series of which the final term gives the area sought. But I would not willingly undertake this calculation.

Finally, series can be derived from complex expressions by the same method. As for example, if $\sqrt{a^2 - ax + \frac{x^3}{a}}$ be the ordinate of any curve, I put $a^2 - ax = z^2$, and the extraction of the square root of the binomial, $z^2 + \frac{x^3}{a}$, will produce $z + \frac{x^3}{2az} - \frac{x^6}{8a^2z^3}$, &c.

All the terms of this series can be quadrated by the theorem already described. But I attach little importance to this method because when simple series are not obtainable with sufficient ease, I have another method not yet published by which the problem is easily dealt with. It is based upon a convenient, ready and general solution of this problem, *To describe a geometrical curve which shall pass through any given points.*

Euclid has shown how to describe a circle through three given points. Also a conic section can be described through five given points and a curve of three dimensions through seven given points ; so that I have in my power a description of all the curves of that order which are determined by seven points only. These are done at once by geometrical methods, without any calculation. But the above problem is of another kind, and although it may seem to be intractable at first sight, it is nevertheless quite the contrary ; perhaps indeed it is one of the prettiest problems that I can ever hope to solve.

* * * * *

Nor when he [*i.e.*, Leibnitz] divides this series

$$\frac{z}{b} + \frac{z^2}{2ab^2} + \frac{z^3}{6a^2b^3} + \frac{z^4}{24a^3b^4} +, \text{\&c.},$$

does he seem to have observed my general method of using letters in the place of quantities affected with their signs + and −. For since the hyperbolic area AD, here signified by z, is positive or negative according as it lies on one or the other side of the ordinate AF; if that area given in numbers is l, and l is substituted for z in the series, the result will be either $\frac{l}{b} + \frac{l^2}{2ab^2} + \frac{l^3}{6a^2b^3} + \frac{l^4}{24a^3b^4} +$, &c., or $-\frac{l}{b} + \frac{l^2}{2ab^2} - \frac{l^3}{6a^2b^3} + \frac{l^4}{24a^3b^4}$, &c., according as l is positive or negative. This being understood, if $a = 1 = b$, and l stands for the hyperbolic logarithm, the number corresponding to it will be $1 + \frac{l}{1} + \frac{l^2}{2} + \frac{l^3}{6} + \frac{l^4}{24}$, &c., if l be positive; and $1 - \frac{l}{1} + \frac{l^2}{2} - \frac{l^3}{6} + \frac{l^4}{24}$, &c., if l be negative. In this way I avoid the multiplication of theorems, which otherwise would increase to an extraordinary degree. For to take an example, that one theorem which I gave above [omitted from the present version] for the quadrature of curves would be resolved into 32, if it were multiplied in accordance with the variations of sign.

Moreover, I do not yet understand what my eminent friend says about finding a number, greater than unity, from its hyperbolic logarithm by the use of the series

$$\frac{l}{1} - \frac{l^2}{1 \times 2} + \frac{l^3}{1 \times 2 \times 3} - \frac{l^4}{1 \times 2 \times 3 \times 4} +, \text{\&c.},$$

rather than by the use of the series

$$\frac{l}{1} + \frac{l^2}{1 \times 2} + \frac{l^3}{1 \times 2 \times 3} + \frac{l^4}{1 \times 2 \times 3 \times 4} +$$

For if one term more be added to the latter series than to the former, the latter will give a better approximation. And certainly it is less laborious to calculate one or two figures of this additional term than to divide unity by the number extended to many decimal places, derived from the hyperbolic logarithm, in order that the required number greater than unity may then be obtained. Therefore let either series, if it be right to speak of two, be employed for its appropriate work. Nevertheless the series

$$\frac{l}{1} + \frac{l^3}{1 \times 2 \times 3} + \frac{l^5}{1 \times 2 \times 3 \times 4 \times 5}, \text{\&c.},$$

depending on half the terms can be best employed, since this will give the semi-difference of two numbers; from which, and from the given rectangle either number is given. So also from the series

$$1 - \frac{l^2}{1 \times 2} + \frac{l^4}{1 \times 2 \times 3 \times 4}, \text{\&c.},$$

the semi-sum of the numbers is given, and thence the numbers themselves. From which arises a relation between the two series so that when one is given the other is found.

From the following simple process which depends on such series, you will readily agree that the construction of logarithms need not be attempted in any other way.

By the method previously explained the hyperbolic logarithms of the numbers 10, ·98, ·99, 1·01, 1·02 are investigated, which would occupy an hour or so. Then, dividing the logarithms of the four last numbers by the logarithms of the number 10, and adding the index 2, the true logarithms of the numbers 98, 99, 100, 101, 102 are obtained for entry in a table. These are to be interpolated by intervals of one-tenth (*Hi per dena intervalla interpolandi sunt*) and the logarithms of all numbers between 980 and 1,020 will be found ; the numbers between 980 and 1,000 being again interpolated by intervals of one-tenth, the table will so far be constructed. Then from these are to be collected the logarithms of all the primes, less than 100, and their multiples ; for which nothing but addition and subtraction are required. Thus :

$$\sqrt[10]{\frac{9984 \times 1020}{9945}} = 2 \; ; \; \sqrt[4]{\frac{8 \times 9963}{984}} = 3 \; ; \; \frac{10}{2} = 5 \; ; \; \sqrt{\frac{98}{2}} = 7 \; ; \; \frac{99}{9} = 11 \; ;$$

$$\frac{1001}{7 \times 11} = 13 \; ; \; \frac{102}{6} = 17 \; ; \; \frac{988}{4 \times 13} = 19 \; ; \; \frac{9936}{16 \times 27} = 23 \; ; \; \frac{986}{2 \times 17} = 29 \; ;$$

$$\frac{992}{32} = 31 \; ; \; \frac{999}{27} = 37 \; ; \; \frac{984}{24} = 41 \; ; \; \frac{989}{23} = 43 \; ; \; \frac{987}{21} = 47 \; ;$$

$$\frac{9911}{11 \times 17} = 53 \; ; \; \frac{9971}{13 \times 13} = 59 \; ; \; \frac{9882}{2 \times 81} = 61 \; ; \; \frac{9949}{3 \times 49} = 67 \; ; \; \frac{994}{14} = 71 \; ;$$

$$\frac{9928}{8 \times 17} = 73 \; ; \; \frac{9954}{7 \times 18} = 79 \; ; \; \frac{996}{12} = 83 \; ; \; \frac{9968}{7 \times 16} = 89 \; ; \; \frac{9894}{6 \times 17} = 97.$$

And so, having found the logarithms of all numbers less than 100, it remains only to interpolate these also once and again by intervals of one-tenth.

* * * * *

[After a section on the construction of Trigonometrical Tables the following paragraph occurs] :

What has been said about Tables of this kind can be applied to others where Geometrical considerations have no place. Moreover, it is sufficient by means of these series to calculate 30 or 20, or even fewer terms at suitable distances apart, since the intermediate terms are easily inserted by a method which I had almost decided to describe here for the use of computers. But I pass on to other matters.

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