

www.GetPedia.com

* More than 500,000 Interesting Articles waiting for you .
* The Ebook starts from the next page : Enjoy !

* Say hello to my cat "Meme"



VECTOR CALCULUS: A summary of useful results with some examples

1. Vector Algebra in \mathbb{R}^3

Addition, multiplication by scalars, components, unit orthogonal vectors as base (right hand triad convention). Scalar and vector products:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \quad (1)$$

$$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1) \quad (2)$$

Distributive laws.

Suffix notation

a_i represents \mathbf{a} . Definitions and use of δ_{ij} and ϵ_{ijk} . The identity

$$\epsilon_{ijk} \epsilon_{pqk} = \delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp} \quad (3)$$

Scalar and Vector Triple products.

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} \quad (4)$$

Invariance of these constructions under rotation of axes.

2. Scalar and vector fields

Idea of $\phi(\mathbf{r})$ and $\mathbf{A}(\mathbf{r})$.

First derivatives

$$\text{grad} \phi = \nabla \phi \text{ and has } i\text{-component } \frac{\partial \phi}{\partial x_i} \quad (5)$$

$$\text{div} \mathbf{A} = \nabla \cdot \mathbf{A} = \delta_{ij} \frac{\partial A_i}{\partial x_j} = \frac{\partial A_i}{\partial x_i} \quad (6)$$

$$\text{curl} \mathbf{A} = \nabla \times \mathbf{A} \text{ and has } i\text{-component } \epsilon_{ijk} \frac{\partial A_k}{\partial x_j} \quad (7)$$

Second derivatives

$$\text{div grad} \phi = \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x_i \partial x_i} = \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} \quad (8)$$

$$\text{curl grad} \phi \equiv 0 \quad (9)$$

$$\operatorname{div} \operatorname{curl} \mathbf{A} \equiv 0 \quad (10)$$

$$\operatorname{curl} \operatorname{curl} \mathbf{A} \equiv \operatorname{grad} \operatorname{div} \mathbf{A} - \nabla^2 \mathbf{A} \quad (11)$$

Here $\nabla^2 \mathbf{A}$ means that the operator ∇^2 as above acts on each cartesian component of \mathbf{A} in turn. The situation is more complicated in curvilinear coordinates (see below).

Existence of scalar and vector potentials

We work with differentiable vector fields in a simply-connected volume V .

$$(i) \operatorname{curl} \mathbf{E} = 0 \text{ in } V \Leftrightarrow \exists \phi \text{ such that } \mathbf{E} = -\nabla \phi \quad (12)$$

$$(ii) \operatorname{div} \mathbf{B} = 0 \text{ in } V \Leftrightarrow \exists \mathbf{A} \text{ such that } \mathbf{B} = \operatorname{curl} \mathbf{A} \quad (13)$$

In case (i) the $-$ sign is conventional in this subject, and ϕ is fixed except for an additive constant. More will be said about case (ii) in lectures.

Derivatives of products

$$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi \quad (14)$$

$$\operatorname{div}(\phi\mathbf{A}) = \phi\operatorname{div}\mathbf{A} + \mathbf{A}\cdot\nabla\phi \quad (15)$$

$$\operatorname{curl}(\phi\mathbf{A}) = \phi\operatorname{curl}\mathbf{A} + \nabla\phi \times \mathbf{A} \quad (16)$$

$$\operatorname{div}(\mathbf{A} \times \mathbf{B}) = \mathbf{B}\cdot\operatorname{curl}\mathbf{A} - \mathbf{A}\cdot\operatorname{curl}\mathbf{B} \quad (17)$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A} \times \operatorname{curl}\mathbf{B} + \mathbf{B} \times \operatorname{curl}\mathbf{A} \quad (18)$$

$$\operatorname{curl}(\mathbf{A} \times \mathbf{B}) = \mathbf{A}\operatorname{div}\mathbf{B} - \mathbf{B}\operatorname{div}\mathbf{A} + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} \quad (19)$$

3. Integral Theorems

Green's Theorem in \mathbb{R}^2

Denote the vector field by $(u(x, y), v(x, y))$. Let $S \subset \mathbb{R}^2$ be a simply connected domain, and C its boundary curve taken anti-clockwise. Then

$$\int_S \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dx dy = \oint_C (u dy - v dx) \quad (20)$$

[Note this is the prototype of both the divergence and Stokes theorems below.]

Divergence Theorem in \mathbb{R}^3

$\mathbf{F}(\mathbf{r})$ is a vector field defined in $V \subset \mathbb{R}^3$; V has surface S , and $d\mathbf{S}$ denotes a surface element parallel to the outgoing normal.

$$\int_V \operatorname{div} \mathbf{F} dV = \int_S \mathbf{F} \cdot d\mathbf{S} \quad (21)$$

Corollaries:

$$\int_V \nabla \phi dV = \int_S \phi d\mathbf{S} \quad (22)$$

(put $\mathbf{F} = \mathbf{k}\phi$, $\mathbf{k} = \text{constant}$).

$$\int_V \operatorname{curl} \mathbf{A} dV = \int_S d\mathbf{S} \times \mathbf{A} \quad (23)$$

(put $\mathbf{F} = \mathbf{A} \times \mathbf{k}$).

$$\int_V (\phi \nabla^2 \phi + (\nabla \phi)^2) dV = \int_S \phi \nabla \phi \cdot d\mathbf{S} \quad (24)$$

(put $\mathbf{F} = \phi \nabla \phi$).

Green's Reciprocal Theorem:

$$\int_V (\phi_1 \nabla^2 \phi_2 - \phi_2 \nabla^2 \phi_1) dV = \int_S (\phi_1 \nabla \phi_2 - \phi_2 \nabla \phi_1) \cdot d\mathbf{S} \quad (25)$$

Stokes's Theorem in \mathbb{R}^3

S is an open orientable surface (no Möbius bands!), bounded by a closed curve C , and lies entirely within a simply connected volume within which $\mathbf{F}(\mathbf{r})$ is defined and differentiable. The sign of the normal to S and the sign of positive progress around C are related as exemplified by Green's theorem in \mathbb{R}^2 as above. Then

$$\int_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r} \quad (26)$$

Corollaries:

$$\int_S d\mathbf{S} \times \nabla \phi = \oint_C \phi d\mathbf{r} \quad (27)$$

See also example 7 below.

4. Derivatives of r

Here $r \neq 0$ throughout.

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r} = \text{unit vector along } \mathbf{r} \text{ (as is geometrically obvious!) .} \quad (28)$$

$$\frac{\partial r^n}{\partial x_i} = nr^{n-2}x_i \quad (29)$$

In particular:

$$\frac{\partial}{\partial x_i} \left(\frac{1}{r} \right) = -\frac{x_i}{r^3} \quad (30)$$

$$\frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{1}{r} \right) = \frac{3x_i x_j - r^2 \delta_{ij}}{r^5} \quad (31)$$

$$\nabla^2 \left(\frac{1}{r} \right) = 0 \quad (32)$$

Given two points \mathbf{r}, \mathbf{r}' with $\mathbf{R} = \mathbf{r} - \mathbf{r}'$

$$\frac{\partial}{\partial x_i} \frac{1}{R} = -\frac{\partial}{\partial x'_i} \frac{1}{R} = -\frac{x_i - x'_i}{R^3} \quad (33)$$

5. Solid angle

Given a surface element $d\mathbf{S}$ not at 0, the *solid angle* that it subtends at 0 is defined by $d\Omega = \mathbf{r} \cdot d\mathbf{S} / r^3$. This is the area of the radial projection of $d\mathbf{S}$ onto the unit sphere centred at 0.

For an orientable S , bounded by C , the total solid angle subtended at 0 is

$$\Omega = \int_S \frac{\mathbf{r} \cdot d\mathbf{S}}{r^3} \quad (34)$$

If S is a closed surface

$$\Omega = \begin{cases} 4\pi & \text{if 0 is inside } S, \\ 0 & \text{if 0 is outside } S \end{cases} \quad (35)$$

6. Expressions for grad, div, curl and ∇^2 in curvilinear coordinates

(Note that where items are abbreviated as “...” these should be filled in by analogy with the terms given, using obvious permutations of $\{1, 2, 3\}$.)

General

Write coordinates as $(\lambda_1, \lambda_2, \lambda_3)$ and define $\{h_i\}$ by $d\mathbf{r} = (h_1 d\lambda_1, h_2 d\lambda_2, h_3 d\lambda_3)$

$$\nabla\psi = \left(\frac{1}{h_1} \frac{\partial\psi}{\partial\lambda_1}, \dots, \dots \right) \quad (36)$$

$$\text{div}\mathbf{A} = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial\lambda_1} (h_2 h_3 A_1) + \dots + \dots \right) \quad (37)$$

$$\text{curl}\mathbf{A} = \left(\frac{1}{h_2 h_3} \left(\frac{\partial}{\partial\lambda_2} (h_3 A_3) - \frac{\partial}{\partial\lambda_3} (h_2 A_2) \right), \dots, \dots \right) \quad (38)$$

$$\nabla^2\psi = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial\lambda_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial\psi}{\partial\lambda_1} \right) + \dots + \dots \right) \quad (39)$$

Cylindrical polars

Coordinates (r, θ, z) . Note $\lambda_1 = 1, \lambda_2 = r, \lambda_3 = 1$.

$$\nabla\psi = \left(\frac{\partial\psi}{\partial r}, \frac{1}{r} \frac{\partial\psi}{\partial\theta}, \frac{\partial\psi}{\partial z} \right) \quad (40)$$

$$\operatorname{div}\mathbf{A} = \frac{1}{r} \frac{\partial}{\partial r}(rA_r) + \frac{1}{r} \frac{\partial A_\theta}{\partial\theta} + \frac{\partial A_z}{\partial z} \quad (41)$$

$$\operatorname{curl}\mathbf{A} = \left(\frac{1}{r} \frac{\partial A_z}{\partial\theta} - \frac{\partial A_\theta}{\partial z}, \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \frac{1}{r} \frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r} \frac{\partial A_r}{\partial\theta} \right) \quad (42)$$

$$\nabla^2\psi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial\psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2} \quad (43)$$

Spherical polars

Coordinates (r, θ, ϕ) . Note $\lambda_1 = 1, \lambda_2 = r, \lambda_3 = r\sin\theta$.

$$\nabla\psi = \left(\frac{\partial\psi}{\partial r}, \frac{1}{r} \frac{\partial\psi}{\partial\theta}, \frac{1}{r\sin\theta} \frac{\partial\psi}{\partial\phi} \right) \quad (44)$$

$$\operatorname{div}\mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta} \frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta} \frac{\partial A_\phi}{\partial\phi} \quad (45)$$

$$\operatorname{curl}\mathbf{A} = \left(\frac{1}{r\sin\theta} \left(\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi} \right), \frac{1}{r} \left(\frac{1}{\sin\theta} \frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi) \right), \frac{1}{r} \left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta} \right) \right) \quad (46)$$

$$\nabla^2\psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial\psi}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial\psi}{\partial\theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2\psi}{\partial\phi^2} \quad (47)$$

7. Force systems

Consider a set of forces $\{\mathbf{F}_i\}$ which act at points $\{\mathbf{r}_i\}$. Define the *total force* by

$$\mathbf{F} = \sum_i \mathbf{F}_i \quad (48)$$

The *total moment* about a point \mathbf{a} is defined by

$$\mathbf{G}(\mathbf{a}) = \sum_i (\mathbf{r}_i - \mathbf{a}) \times \mathbf{F}_i \quad (49)$$

Note

$$\mathbf{G}(\mathbf{a}) = \mathbf{G}(\mathbf{0}) - \mathbf{a} \times \mathbf{F} \quad (50)$$

A system is *null* if $\mathbf{F}=\mathbf{G}=\mathbf{0}$. Two systems are *equivalent* if they have the same values of \mathbf{F} and \mathbf{G} . If $\mathbf{F} = \mathbf{0}$, \mathbf{G} is independent of \mathbf{a} , and the system is a *couple*.

REVISION EXAMPLES

These are intended to restore fluency and confidence; do some, all or none of questions 1-7, as you feel appropriate. It is not intended that this should occupy a whole week's supervision time.

1. Show that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$.

2. Force Systems. If $\mathbf{F} \neq \mathbf{0}$, $\mathbf{F} \cdot \mathbf{G}(\mathbf{a})$ is invariant to the choice of \mathbf{a} and it is possible to choose \mathbf{a} such that \mathbf{F} and \mathbf{G} are parallel. Verify this by taking

$$\mathbf{a} = \frac{\mathbf{F} \times \mathbf{G}(\mathbf{0})}{\mathbf{F}^2} + k\mathbf{F}$$

where k is arbitrary. The system is then equivalent to a single force \mathbf{F} acting at \mathbf{a} together with a couple parallel to \mathbf{F} (cf. corkscrew!).

3. If $\mathbf{B} = (0, 0, B_0)$ in cartesians, verify that the following possible vector potentials yield $\mathbf{B} = \text{curl}\mathbf{A}$:

(i) In cartesians, $\mathbf{A} = (0, xB_0, 0)$

(ii) In cylindrical polars, $\mathbf{A} = (0, \frac{1}{2}B_0r, 0)$

(iii) In spherical polars, $\mathbf{A} = (0, 0, \frac{1}{2}B_0r\sin\theta)$

4. If $\mathbf{B} = \text{curl}\mathbf{A}$ where (for $r \neq 0$), the vector potential is $\mathbf{A} = \mathbf{M} \times \mathbf{r}/r^3$, $\mathbf{M} = \text{constant}$, show that \mathbf{B} can also be written as $\mathbf{B} = -\nabla\chi$, where the scalar potential is $\chi = \mathbf{M} \cdot \mathbf{r}/r^3$.

[Hints: relate \mathbf{r}/r^3 to $\nabla(1/r)$ and work throughout in cartesian suffix notation; recall that $\nabla^2(1/r) = 0$].

5. Show that:

$$(i) \int_S r^n \mathbf{r} \cdot d\mathbf{S} = \int_V (n+3)r^n dV, \quad (ii) \int_S r^n d\mathbf{S} = \int_V nr^{n-2} \mathbf{r} dV$$

6. Show that the solid angle of a cone of semi-vertical angle α is $2\pi(1 - \cos\alpha)$. What is the solid angle subtended by one face of a cube at its centre?

7. Establish a second corollary to Stokes's theorem:

$$\int_S \left((-\text{div}\mathbf{A})d\mathbf{S} + \nabla(\mathbf{A} \cdot d\mathbf{S}) \right) = \oint_C d\mathbf{r} \times \mathbf{A}$$

[Here the “ ∇ ” acts only on \mathbf{A}].

8. *This is very hard and long, aimed at enthusiasts or budding Lucasian Professors.*

Do not plague your supervisors with it!

Prove Lundgren's identity:

$$\begin{aligned} \nabla [\mathbf{A}, \mathbf{B}, \mathbf{C}] &= \mathbf{A} \times \text{curl}(\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times \text{curl}(\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times \text{curl}(\mathbf{A} \times \mathbf{B}) \\ &\quad - (\mathbf{B} \times \mathbf{C}) \text{div}\mathbf{A} - (\mathbf{C} \times \mathbf{A}) \text{div}\mathbf{B} - (\mathbf{A} \times \mathbf{B}) \text{div}\mathbf{C} \end{aligned}$$

(Here [...] denotes the scalar triple product).

[Reference: T.S. Lundgren, *Physics of Fluids* **6**, 898 (1963)].

Comments on both the Notes and Examples welcomed! email: jpd2@damtp.cam.ac.uk