CHAPTER 11
Vector-Valued Functions

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CHAPTER 11
Vector-Valued Functions

Section 11.1 Vector-Valued Functions

Solutions to Even-Numbered Exercises

2. $\mathbf{r}(t) = \sqrt{4-t^2}\,\mathbf{i} + t^2\mathbf{j} - 6t\,\mathbf{k}$

Component functions: $f(t) = \sqrt{4-t^2}$

$g(t) = t^2$

$h(t) = -6t$

Domain: $[-2, 2]$

4. $\mathbf{r}(t) = \sin t\,\mathbf{i} + 4\cos t\,\mathbf{j} + t\,\mathbf{k}$

Component functions: $f(t) = \sin t$

$g(t) = 4\cos t$

$h(t) = t$

Domain: $(-\infty, \infty)$

6.
$$\begin{aligned}\mathbf{r}(t) = \mathbf{F}(t) - \mathbf{G}(t) &= (\ln t\,\mathbf{i} + 5t\,\mathbf{j} - 3t^2\mathbf{k}) - (\mathbf{i} + 4t\,\mathbf{j} - 3t^2\mathbf{k})\\ &= (\ln t - 1)\mathbf{i} + (5t - 4t)\mathbf{j} + (-3t^2 + 3t^2)\mathbf{k}\\ &= (\ln t - 1)\mathbf{i} + t\,\mathbf{j}\end{aligned}$$

Domain: $(0, \infty)$

8. $\mathbf{r}(t) = \mathbf{F}(t) \times \mathbf{G}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t^3 & -t & t \\ \sqrt[3]{t} & \dfrac{1}{t+1} & t+2 \end{vmatrix} = \left(-t(t+2) - \dfrac{t}{t+1}\right)\mathbf{i} - \left(t^3(t+2) - t\sqrt[3]{t}\right)\mathbf{j} + \left(\dfrac{t^3}{t+1} + t\sqrt[3]{t}\right)\mathbf{k}$

Domain: $(-\infty, -1), (-1, \infty)$

10. $\mathbf{r}(t) = \cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$

(a) $\mathbf{r}(0) = \mathbf{i}$

(b) $\mathbf{r}\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}\mathbf{i} + \sqrt{2}\,\mathbf{j}$

(c) $\mathbf{r}(\theta - \pi) = \cos(\theta - \pi)\mathbf{i} + 2\sin(\theta - \pi)\mathbf{j} = -\cos\theta\,\mathbf{i} - 2\sin\theta\,\mathbf{j}$

(d) $\mathbf{r}\left(\dfrac{\pi}{6} + \Delta t\right) - \mathbf{r}\left(\dfrac{\pi}{6}\right) = \cos\left(\dfrac{\pi}{6} + \Delta t\right)\mathbf{i} + 2\sin\left(\dfrac{\pi}{6} + \Delta t\right)\mathbf{j} - \left(\cos\left(\dfrac{\pi}{6}\right)\mathbf{i} + 2\sin\dfrac{\pi}{6}\mathbf{j}\right)$

12. $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + t^{3/2}\mathbf{j} + e^{-t/4}\mathbf{k}$

(a) $\mathbf{r}(0) = \mathbf{k}$

(b) $\mathbf{r}(4) = 2\mathbf{i} + 8\mathbf{j} + e^{-1}\mathbf{k}$

(c) $\mathbf{r}(c+2) = \sqrt{c+2}\,\mathbf{i} + (c+2)^{3/2}\mathbf{j} + e^{-[(c+2)/4]}\mathbf{k}$

(d)
$$\begin{aligned}\mathbf{r}(9+\Delta t) - \mathbf{r}(9) &= \left(\sqrt{9+\Delta t}\right)\mathbf{i} + (9+\Delta t)^{3/2}\mathbf{j} + e^{-[(9+\Delta t)/4]}\mathbf{k} - (3\mathbf{i} + 27\mathbf{j} + e^{-9/4}\mathbf{k})\\ &= \left(\sqrt{9+\Delta t} - 3\right)\mathbf{i} + ((9+\Delta t)^{3/2} - 27)\mathbf{j} + (e^{-[(9+\Delta t)/4]} - e^{-9/4})\mathbf{k}\end{aligned}$$

14. $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + 3t\,\mathbf{j} - 4t\,\mathbf{k}$

$$\begin{aligned}\|\mathbf{r}(t)\| &= \sqrt{\left(\sqrt{t}\right)^2 + (3t)^2 + (-4t)^2}\\ &= \sqrt{t + 9t^2 + 16t^2} = \sqrt{t(1+25t)}\end{aligned}$$

16. $\mathbf{r}(t) \cdot \mathbf{u}(t) = (3\cos t)(4\sin t) + (2\sin t)(-6\cos t) + (t-2)(t^2) = t^3 - 2t^2$, a scalar.

The dot product is a scalar-valued function.

18. $\mathbf{r}(t) = \cos(\pi t)\mathbf{i} + \sin(\pi t)\mathbf{j} + t^2\mathbf{k},\ -1 \le t \le 1$

$x = \cos(\pi t),\ y = \sin(\pi t),\ z = t^2$

Thus, $x^2 + y^2 = 1$. Matches (c)

20. $\mathbf{r}(t) = t\mathbf{i} + \ln t\mathbf{j} + \frac{2t}{3}\mathbf{k},\ 0.1 \le t \le 5$

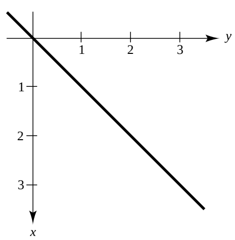
$x = t,\ y = \ln t,\ z = \frac{2t}{3}$

Thus, $z = \frac{2}{3}x$ and $y = \ln x$. Matches (a)

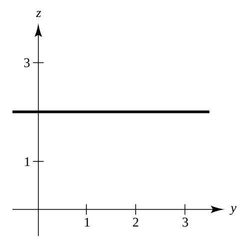
22. $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + 2\mathbf{k}$

$x = t, y = t, z = 2 \implies x = y$

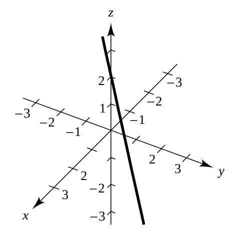
(a) $(0, 0, 20)$



(b) $(10, 0, 0)$



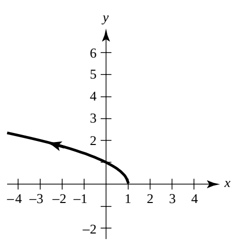
(c) $(5, 5, 5)$



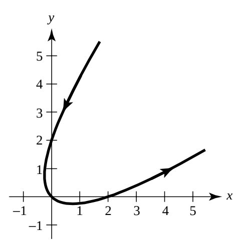
24. $x = 1 - t, y = \sqrt{t}$

$y = \sqrt{1-x}$

Domain: $t \ge 0$



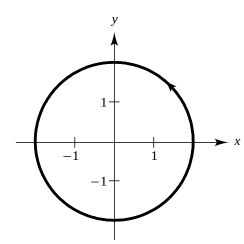
26. $x = t^2 + t, y = t^2 - t$



28. $x = 2\cos t$

$y = 2\sin t$

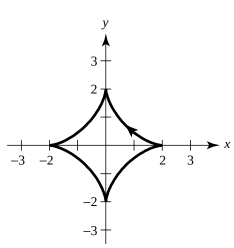
$x^2 + y^2 = 4$



30. $x = 2\cos^3 t, y = 2\sin^3 t$

$$\left(\frac{x}{2}\right)^{2/3} + \left(\frac{y}{2}\right)^{2/3} = \cos^2 t + \sin^2 t = 1$$

$$x^{2/3} + y^{2/3} = 2^{2/3}$$



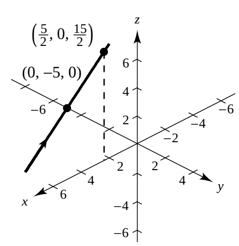
32. $x = t$

$y = 2t - 5$

$y = 3t$

Line passing through the points:

$$(0, -5, 0), \left(\frac{5}{2}, 0, \frac{15}{2}\right)$$

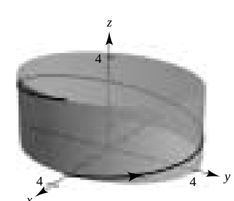


34. $x = 3\cos t,\ y = 4\sin t,\ z = \frac{t}{2}$

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

$$z = \frac{t}{2}$$

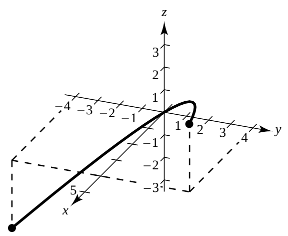
Elliptic helix



36. $x = t^2, y = 2t, z = \frac{3}{2}t$

$x = \frac{y^2}{4}, z = \frac{3}{4}y$

t	-2	-1	0	1	2
x	4	1	0	1	4
y	-4	-2	0	2	4
z	-3	$-\frac{3}{2}$	0	$\frac{3}{2}$	3



38. $x = \cos t + t \sin t$

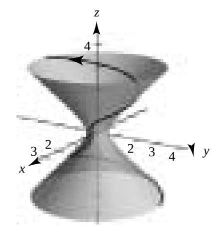
$y = \sin t - t \cos t$

$z = t$

$x^2 + y^2 = 1 + t^2 = 1 + z^2$ or $x^2 + y^2 - z^2 = 1$

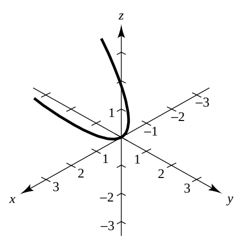
$z = t$

Helix along a hyperboloid of one sheet



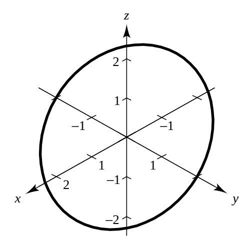
40. $\mathbf{r}(t) = t\mathbf{i} - \frac{\sqrt{3}}{2}t^2\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

Parabola

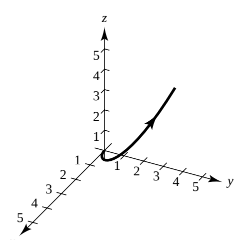


42. $\mathbf{r}(t) = -\sqrt{2}\sin t\mathbf{i} + 2\cos t\mathbf{j} + \sqrt{2}\sin t\mathbf{k}$

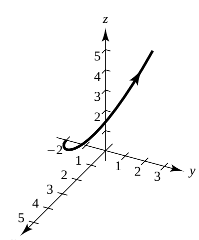
Ellipse



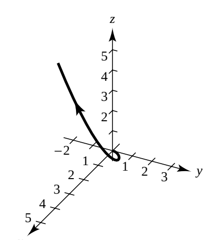
44. $r(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{2}t^3\mathbf{h}$



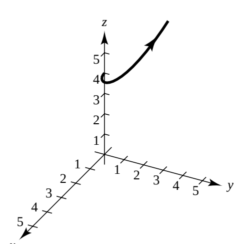
(a) $\mathbf{u}(t) = \mathbf{r}(t) - 2\mathbf{j}$ is a translation 2 units to the left along the y-axis.



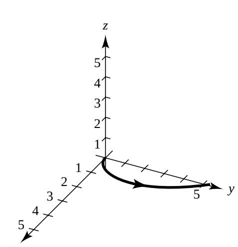
(b) $\mathbf{u}(t) = t^2\mathbf{i} + t\mathbf{j} + \frac{1}{2}t^3\mathbf{k}$ has the roles of x and y interchanged. The graph is a reflection in the plane $x = y$.



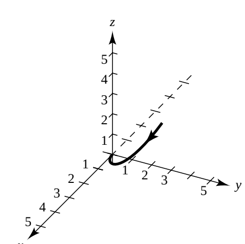
(c) $\mathbf{u}(t) = \mathbf{r}(t) + 4\mathbf{k}$ is an upward shift 4 units.



(d) $\mathbf{u}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{8}t^3\mathbf{k}$ shrinks the z-value by a factor of 4. The curve rises more slowly.



(e) $\mathbf{u}(t) = \mathbf{r}(-t)$ reverses the orientation.



46. $2x - 3y + 5 = 0$

Let $x = t$, then $y = \frac{1}{3}(2t + 5)$.

$$\mathbf{r}(t) = t\mathbf{i} + \frac{1}{3}(2\mathbf{t} + 5)\mathbf{j}$$

48. $y = 4 - x^2$

Let $x = t$, then $y = 4 - t^2$.

$$\mathbf{r}(t) = t\mathbf{i} + (4 - t^2)\mathbf{j}$$

50. $(x - 2)^2 + y^2 = 4$

Let $x - 2 = 2\cos t, y = 2\sin t$.

$$\mathbf{r}(t) = (2 + 2\cos t)\mathbf{i} + 2\sin t\,\mathbf{j}$$

52. $\frac{x^2}{16} + \frac{y^2}{9} = 1$

Let $x = 4\cos t, y = 3\sin t$.

$$\mathbf{r}(t) = 4\cos t\mathbf{i} + 3\sin t\mathbf{j}$$

54. One possible answer is

$$\mathbf{r}(t) = 1.5\cos t\mathbf{i} + 1.5\sin t\mathbf{j} + \frac{1}{\pi}t\mathbf{k},\ 0 \le t \le 2\pi$$

Note that $\mathbf{r}(2\pi) = 1.5\mathbf{i} + 2\mathbf{k}$.

56. $\mathbf{r}_1(t) = t\mathbf{i},\quad 0 \le t \le 10 \quad (\mathbf{r}_1(0) = \mathbf{0}, \mathbf{r}_1(10) = 10\mathbf{i})$

$\mathbf{r}_2(t) = 10(\cos t\mathbf{i} + \sin t\mathbf{j}),\quad 0 \le t \le \frac{\pi}{4} \quad \left(\mathbf{r}_2(0) = 10\mathbf{i}, \mathbf{r}_2\left(\frac{\pi}{4}\right) = 5\sqrt{2}\mathbf{i} + 5\sqrt{2}\mathbf{j}\right)$

$\mathbf{r}_3(t) = 5\sqrt{2}(1 - t)\mathbf{i} + 5\sqrt{2}(1 - t)\mathbf{j},\quad 0 \le t \le 1 \quad (\mathbf{r}_3(0) = 5\sqrt{2}\mathbf{i} + 5\sqrt{2}\mathbf{j}, \mathbf{r}_3(1) = \mathbf{0}$

(Other answers possible)

58. $\mathbf{r}_1(t) = t\mathbf{i} + \sqrt{t}\mathbf{j},\quad 0 \le t \le 1 \ \left(y = \sqrt{x}\right)$

$\mathbf{r}_2(t) = (1 - t)\mathbf{i} + (1 - t)\mathbf{j},\quad 0 \le t \le 1 \ (y = x)$

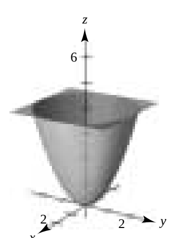
(Other answers possible)

60. $z = x^2 + y^2,\ z = 4$

Therefore, $x^2 + y^2 = 4$ or

$$x = 2\cos t, y = 2\sin t, z = 4.$$

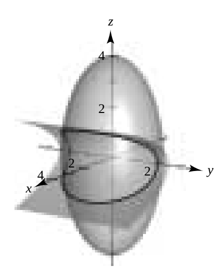
$\mathbf{r}(t) = 2\cos t\mathbf{i} + \mathbf{2}\sin t\mathbf{j} + 4\mathbf{k}$



62. $4x^2 + 4y^2 + z^2 = 16, x = z^2$

If $z = t$, then $x = t^2$ and $y = \frac{1}{2}\sqrt{16 - 4t^4 - t^2}$.

t	-1.3	-1.2	-1	0	1	1.2
x	1.69	1.44	1	0	1	1.44
y	0.85	1.25	1.66	2	1.66	1.25
z	-1.3	-1.2	-1	0	1	1.2

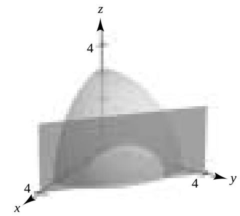


$$\mathbf{r}(t) = t^2\mathbf{i} + \frac{1}{2}\sqrt{16 - 4t^4 - t^2}\mathbf{j} + t\mathbf{k}$$

64. $x^2 + y^2 + z^2 = 10, \; x + y = 4$

Let $x = 2 + \sin t$, then $y = 2 - \sin t$ and $z = \sqrt{2(1 - \sin^2 t)} = \sqrt{2}\cos t$.

t	$-\frac{\pi}{2}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	π
x	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	2
y	3	$\frac{5}{2}$	2	$\frac{3}{2}$	1	2
z	0	$\frac{\sqrt{6}}{2}$	$\sqrt{2}$	$\frac{\sqrt{6}}{2}$	0	$-\sqrt{2}$



$\mathbf{r}(t) = (2 + \sin t)\mathbf{i} + (2 - \sin t)\mathbf{j} + \sqrt{2}\cos t\mathbf{k}$

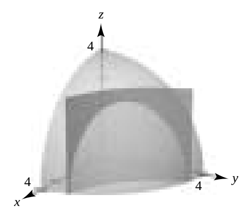
66. $x^2 + y^2 + z^2 = 16, \; xy = 4$ (first octant)

Let $x = t$, then

$$y = \frac{4}{t} \quad \text{and} \quad x^2 + y^2 + z^2 = t^2 + \frac{16}{t^2} + z^2 = 16.$$

$$z = \frac{1}{t}\sqrt{-t^4 + 16t^2 - 16}$$

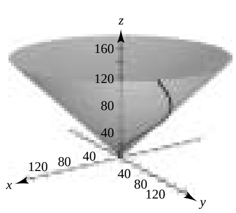
$$\left(\sqrt{8 - 4\sqrt{3}} \le t \le \sqrt{8 + 4\sqrt{3}}\right)$$



t	$\sqrt{8 + 4\sqrt{3}}$	1.5	2	2.5	3.0	3.5	$\sqrt{8 + 4\sqrt{3}}$
x	1.0	1.5	2	2.5	3.0	3.5	3.9
y	3.9	2.7	2	1.6	1.3	1.1	1.0
z	0	2.6	2.8	2.7	2.3	1.6	0

$$\mathbf{r}(t) = t\mathbf{i} + \frac{4}{t}\mathbf{j} + \frac{1}{t}\sqrt{-t^4 + 16t^2 - 16}\mathbf{k}$$

68. $x^2 + y^2 = (e^{-t}\cos t)^2 + (e^{-t}\sin t)^2 = e^{-2t} = z^2$



70. $\lim_{t\to 0}\left[e^t\mathbf{i} + \frac{\sin t}{t}\mathbf{j} + e^{-t}\mathbf{k}\right] = \mathbf{i} + \mathbf{j} + \mathbf{k}$

since

$$\lim_{t\to 0}\frac{\sin t}{t} = \lim_{t\to 0}\frac{\cos t}{1} = 1 \quad \text{(L'Hôpital's Rule)}$$

72. $\lim_{t\to 1}\left[\sqrt{t}\mathbf{i} + \frac{\ln t}{t^2 - 1}\mathbf{j} + 2t^2\mathbf{k}\right] = \mathbf{i} + \frac{1}{2}\mathbf{j} + 2\mathbf{k}$

since

$$\lim_{t\to 1}\frac{\ln t}{t^2 - 1} = \lim_{t\to 1}\frac{1/t}{2t} = \frac{1}{2}. \quad \text{(L'Hôpital's Rule)}$$

74. $\lim_{t\to\infty}\left[e^{-t}\mathbf{i} + \frac{1}{t}\mathbf{j} + \frac{t}{t^2 + 1}\mathbf{k}\right] = \mathbf{0}$

since

$$\lim_{t\to\infty} e^{-t} = 0, \; \lim_{t\to\infty}\frac{1}{t} = 0, \text{ and } \lim_{t\to\infty}\frac{t}{t^2 + 1} = 0.$$

76. $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + \sqrt{t-1}\,\mathbf{j}$

Continuous on $[1, \infty)$

78. $\mathbf{r}(t) = \langle 2e^{-t}, e^{-t}, \ln(t-1)\rangle$

Continuous on $t - 1 > 0$ or $t > 1$: $(1, \infty)$.

80. $\mathbf{r}(t) = \left\langle 8, \sqrt{t}, \sqrt[3]{t}\right\rangle$

Continuous on $[0, \infty)$

82. No. The graph is the same because $\mathbf{r}(t) = \mathbf{u}(t + 2)$. For example, if $\mathbf{r}(0)$ is on the graph of $\mathbf{r}$, then $\mathbf{u}(2)$ is the same point.

84. A vector-valued function $\mathbf{r}$ is continuous at $t = a$ if the limit of $\mathbf{r}(t)$ exists as $t \to a$ and

$$\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a).$$

The function $\mathrm{r}(t) = \begin{cases} \mathbf{i} + \mathbf{j} & t \geq 0 \\ -\mathbf{i} + \mathbf{j} & t < 0 \end{cases}$ is not continuous at $t = 0$.

86. Let $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$ and $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$. Then:

$$\begin{aligned}\lim_{t\to c}[\mathbf{r}(t)\cdot\mathbf{u}(t)] &= \lim_{t\to c}[x_1(t)x_2(t) + y_1(t)y_2(t) + z_1(t)z_2(t)] \\ &= \lim_{t\to c} x_1(t) \lim_{t\to c} x_2(t) + \lim_{t\to c} y_1(t) \lim_{t\to c} y_2(t) + \lim_{t\to c} z_1(t) \lim_{t\to c} z_2(t) \\ &= \left[\lim_{t\to c} x_1(t)\mathbf{i} + \lim_{t\to c} y_1(t)\mathbf{j} + \lim_{t\to c} z_1(t)\mathbf{k}\right] \cdot \left[\lim_{t\to c} x_2(t)\mathbf{i} + \lim_{t\to c} y_2(t)\mathbf{j} + \lim_{t\to c} z_2(t)\mathbf{k}\right] \\ &= \lim_{t\to c}\mathbf{r}(t)\cdot\lim_{t\to c}\mathbf{u}(t)\end{aligned}$$

88. Let

$$f(t) = \begin{cases} 1, & \text{if } t \geq 0 \\ -1, & \text{if } t < 0 \end{cases}$$

and $\mathbf{r}(t) = f(t)\mathbf{i}$. Then $\mathbf{r}$ is not continuous at $c = 0$, whereas, $\|\mathbf{r}\| = 1$ is continuous for all t.

90. False. The graph of $x = y = z = t^3$ represents a line.

Section 11.2 Differentiation and Integration of Vector-Valued Functions

2. $\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j},\ t_0 = 1$

$x(t) = t,\ y(t) = t^3$

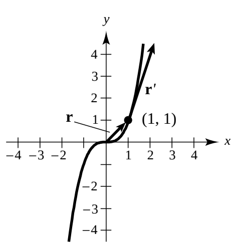
$y = x^3$

$\mathbf{r}(1) = \mathbf{i} + \mathbf{j}$

$\mathbf{r}'(t) = \mathbf{i} + 3t^2\mathbf{j}$

$\mathbf{r}'(1) = \mathbf{i} + 3\mathbf{j}$

$\mathbf{r}'(t_0)$ is tangent to the curve.



4. $\mathbf{r}(t) = t^2\mathbf{i} + \dfrac{1}{t}\mathbf{j},\ t_0 = 2$

$x(t) = t^2,\ y(t) = \dfrac{1}{t}$

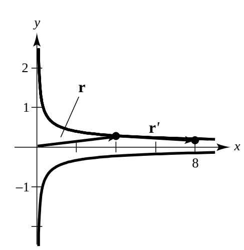
$x = \dfrac{1}{y^2}$

$\mathbf{r}(2) = 4\mathbf{i} + \dfrac{1}{2}\mathbf{j}$

$\mathbf{r}'(t) = 2t\mathbf{i} - \dfrac{1}{t^2}\mathbf{j}$

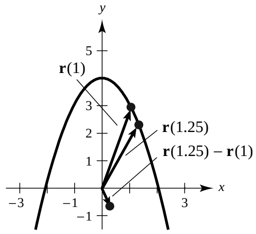
$\mathbf{r}'(2) = 4\mathbf{i} - \dfrac{1}{4}\mathbf{j}$

$\mathbf{r}'(t_0)$ is tangent to the curve.



6. $\mathbf{r}(t) = t\mathbf{i} + (4 - t^2)\mathbf{j}$

(a)



(b) $\mathbf{r}(1) = \mathbf{i} + 3\mathbf{j}$

$\mathbf{r}(1.25) = 1.25\mathbf{i} + 2.4375\mathbf{j}$

$\mathbf{r}(1.25) - \mathbf{r}(1) = 0.25\mathbf{i} - 0.5625\mathbf{j}$

(c) $\mathbf{r}'(t) = \mathbf{i} - 2t\mathbf{j}$

$\mathbf{r}'(1) = \mathbf{i} - 2\mathbf{j}$

$$\frac{\mathbf{r}(1.25) - \mathbf{r}(1)}{1.25 - 1} = \frac{0.25\mathbf{i} - 0.5625\mathbf{j}}{0.25} = \mathbf{i} - 2.25\mathbf{j}$$

This vector approximates $\mathbf{r}'(1)$.

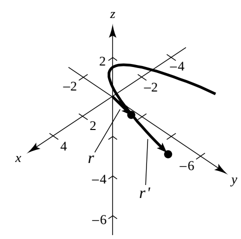
8. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{3}{2}\mathbf{k},\ t_0 = 2$

$y = x^2,\ z = \frac{3}{2}$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{r}(2) = 2\mathbf{i} + 4\mathbf{j} + \frac{3}{2}\mathbf{k}$

$\mathbf{r}'(2) = \mathbf{i} + 4\mathbf{j}$



10. $\mathbf{r}(t) = \frac{1}{t}\mathbf{i} + 16t\mathbf{j} + \frac{t^2}{2}\mathbf{k}$

$\mathbf{r}'(t) = -\frac{1}{t^2}\mathbf{i} + 16\mathbf{j} + t\mathbf{k}$

12. $\mathbf{r}(t) = 4\sqrt{t}\mathbf{i} + t^2\sqrt{t}\mathbf{j} + \ln t^2\mathbf{k}$

$\mathbf{r}'(t) = \frac{2}{\sqrt{t}}\mathbf{i} + \left(2t\sqrt{t} + \frac{t^2}{2\sqrt{t}}\right)\mathbf{j} + \frac{2}{t}\mathbf{k}$

14. $\mathbf{r}(t) = \langle \sin t - t\cos t, \cos t + t\sin t, t^2\rangle$

$\mathbf{r}'(t) = \langle t\sin t, t\cos t, 2t\rangle$

16. $\mathbf{r}(t) = \langle \arcsin t, \arccos t, 0\rangle$

$\mathbf{r}'(t) = \left\langle \frac{1}{\sqrt{1-t^2}}, -\frac{1}{\sqrt{1-t^2}}, 0\right\rangle$

18. $\mathbf{r}(t) = (t^2 + t)\mathbf{i} + (t^2 - t)\mathbf{j}$

(a) $\mathbf{r}'(t) = (2t + 1)\mathbf{i} + (2t - 1)\mathbf{j}$

$\mathbf{r}''(t) = 2\mathbf{i} + 2\mathbf{j}$

(b) $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = (2t + 1)(2) + (2t - 1)(2) = 8t$

20. $\mathbf{r}(t) = 8\cos t\mathbf{i} + 3\sin t\mathbf{j}$

(a) $\mathbf{r}'(t) = -8\sin t\mathbf{i} + 3\cos t\mathbf{j}$

$\mathbf{r}''(t) = -8\cos t\mathbf{i} - 3\sin t\mathbf{j}$

(b) $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = (-8\sin t)(-8\cos t) + 3\cos t(-3\sin t)$

$= 55\sin t\cos \text{t}$

22. $\mathbf{r}(t) = t\mathbf{i} + (2t + 3)\mathbf{j} + (3\text{t} - 5)\mathbf{k}$

(a) $\mathbf{r}'(t) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

$\mathbf{r}''(t) = 0$

(b) $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = 0$

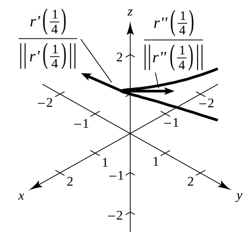
24. $\mathbf{r}(t) = \langle e^{-t}, t^2, \tan(t)\rangle$

(a) $\mathbf{r}'(t) = \langle -e^{-t}, 2t, \sec^2 t\rangle$

$\mathbf{r}''(t) = \langle e^{-t}, 2, 2\sec^2 t\tan t\rangle$

(b) $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = -e^{-2t} + 4t + 2\sec^4 t\tan t$

26. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + e^{0.75t}\mathbf{k},\ t_0 = \dfrac{1}{4}$



$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 0.75e^{0.75t}\mathbf{k}$$

$$\mathbf{r}'\left(\frac{1}{4}\right) = \mathbf{i} + \frac{1}{2}\mathbf{j} + 0.75e^{0.1875}\mathbf{k} = \mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{3}{4}e^{3/16}\mathbf{k}$$

$$\left\|\mathbf{r}'\left(\frac{1}{4}\right)\right\| = \sqrt{1^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}e^{3/16}\right)^2} = \sqrt{\frac{5}{4} + \frac{9}{16}e^{3/8}} = \frac{\sqrt{20 + 9e^{3/8}}}{4}$$

$$\frac{\mathbf{r}'(1/4)}{\|\mathbf{r}'(1/4)\|} = \frac{1}{\sqrt{20 + 9e^{3/8}}}(4\mathbf{i} + 2\mathbf{j} + 3e^{3/16}\mathbf{k})$$

$$\mathbf{r}''(t) = 2\mathbf{i} + \frac{9}{16}e^{0.75t}\mathbf{k}$$

$$\mathbf{r}''\left(\frac{1}{4}\right) = 2\mathbf{i} + \frac{9}{16}e^{3/16\mathbf{k}}$$

$$\left\|\mathbf{r}''\left(\frac{1}{4}\right)\right\| = \sqrt{2^2 + \left(\frac{9}{16}e^{3/16}\right)^2} = \sqrt{4 + \frac{81}{256}e^{3/8}} = \frac{\sqrt{1024 + 81e^{3/8}}}{16}$$

$$\frac{\mathbf{r}''(1/4)}{\|\mathbf{r}'(1/4)\|} = \frac{1}{\sqrt{1024 + 81e^{3/8}}}(32\mathbf{j} + 9e^{3/16}\mathbf{k})$$

28. $\mathbf{r}(t) = \dfrac{1}{t-1}\mathbf{i} + 3t\mathbf{j}$

$$\mathbf{r}'(t) = -\frac{1}{(t-1)^2}\mathbf{i} + 3\mathbf{j}$$

Not continuous when $t = 1$

Smooth on $(-\infty, 1),\ (1, \infty)$

30. $\mathbf{r}(\theta) = (\theta + \sin\theta)\mathbf{i} + (1 - \cos\theta)\mathbf{j}$

$$\mathbf{r}'(\theta) = (1 + \cos\theta)\mathbf{i} + \sin\theta\mathbf{j}$$

$\mathbf{r}'((2n-1)\pi) = \mathbf{0}$, n any integer

Smooth on $((2n-1)\pi, (2n+1)\pi)$

32. $\mathbf{r}(t) = \dfrac{2t}{8+t^3}\mathbf{i} + \dfrac{2t^2}{8+t^3}\mathbf{j}$

$$\mathbf{r}'(t) = \frac{16 - 4t^3}{(t^3+8)^2}\mathbf{i} + \frac{32t - 2t^4}{(t^3+8)^2}\mathbf{j}$$

$\mathbf{r}'(t) \neq \mathbf{0}$ for any value of t.

$\mathbf{r}$ is not continuous when $t = -2$.

Smooth on $(-\infty, -2), (-2, \infty)$.

34. $\mathbf{r}(t) = e^t\mathbf{i} - e^{-t}\mathbf{j} + 3t\mathbf{k}$

$$\mathbf{r}'(t) = e^t\mathbf{i} + e^{-t}\mathbf{j} + 3\mathbf{k} \neq \mathbf{0}$$

$\mathbf{r}$ is smooth for all t: $(-\infty, \infty)$

36. $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + (t^2 - 1)\mathbf{j} + \dfrac{1}{4}t\mathbf{k}$

$$\mathbf{r}'(t) = \frac{1}{2\sqrt{t}}\mathbf{i} + 2t\mathbf{j} + \frac{1}{4}\mathbf{k} \neq \mathbf{0}$$

$\mathbf{r}$ is smooth for all $t > 0$: $(0, \infty)$

38. $\mathbf{r}(t) = t\mathbf{i} + 2\sin t\mathbf{j} + 2\cos t\mathbf{k}$

$$\mathbf{u}(t) = \frac{1}{t}\mathbf{i} + 2\sin t\mathbf{j} + 2\cos t\mathbf{k}$$

(a) $\mathbf{r}'(t) = \mathbf{i} + 2\cos t\mathbf{j} - 2\sin t\mathbf{k}$

(b) $\mathbf{r}''(t) = -2\sin t\mathbf{j} - 2\cos t\mathbf{k}$

(c) $\mathbf{r}(t)\cdot\mathbf{u}(t) = 1 + 4\sin^2 t + 4\cos^2 t = 5$

$D_t[\mathbf{r}(t)\cdot\mathbf{u}(t)] = 0,\ t \neq 0$

(d) $3\mathbf{r}(t) - \mathbf{u}(t) = \left(3t - \dfrac{1}{t}\right)\mathbf{i} + 4\sin t\mathbf{j} + 4\cos t\mathbf{k}$

$$D_t[3\mathbf{r}(t) - \mathbf{u}(t)] = \left(3 - \frac{1}{t^2}\right)\mathbf{i} + 4\cos t\mathbf{j} - 4\sin t\mathbf{k}$$

—CONTINUED—

38. —CONTINUED—

(e) $$\mathbf{r}(t)\times\mathbf{u}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 2\sin t & 2\cos t \\ 1/t & 2\sin t & 2\cos t \end{vmatrix}$$

$$= 2\cos t\left(\frac{1}{t} - t\right)\mathbf{j} + 2\sin t\left(t - \frac{1}{t}\right)\mathbf{k}$$

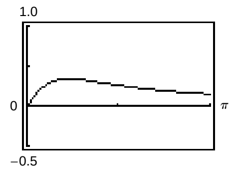
$$D_t[\mathbf{r}(t) - \mathbf{u}(t)] = \left[-2\sin t\left(\frac{1}{t} - t\right) + 2\cos t\left(-\frac{1}{t^2} - 1\right)\right]\mathbf{j}$$

$$+ \left[2\cos t\left(t - \frac{1}{t}\right) + 2\sin t\left(1 + \frac{1}{t^2}\right)\right]\mathbf{k}$$

(f) $\|\mathbf{r}(t)\| = \sqrt{t^2 + 4}$

$$D_t(\|\mathbf{r}(t)\|) = \frac{1}{2}(t^2 + 4)^{-1/2}(2t) = \frac{t}{\sqrt{t^2 + 4}}$$

40.
$$\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}$$
$$\mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$$
$$\mathbf{r}(t)\cdot\mathbf{r}'(t) = 2t^3 + t$$
$$\|\mathbf{r}(t)\| = \sqrt{t^4 + t^2},\ \|\mathbf{r}'(t)\| = \sqrt{4t^2 + 1}$$
$$\cos\theta = \frac{2t^3 + t}{\sqrt{t^4 + t^2}\sqrt{4t^2 + 1}}$$
$$\theta = \arccos\frac{2t^3 + t}{\sqrt{t^4 + t^2}\sqrt{4t^2 + 1}}$$



$\theta = 0.340$ ($\approx 19.47°$)maximum at $t = 0.707\left(\frac{\sqrt{2}}{2}\right)$.

$\theta \neq \frac{\pi}{2}$ for any t.

42. $$\mathbf{r}'(t) = \lim_{\Delta t\to 0}\frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

$$= \lim_{\Delta t\to 0}\frac{\left[\sqrt{t + \Delta t}\,\mathbf{i} + \frac{3}{t + \Delta t}\mathbf{j} - 2(t + \Delta t)\mathbf{k}\right] - \left[\sqrt{t}\,\mathbf{i} + \frac{3}{t}\mathbf{j} - 2t\mathbf{k}\right]}{\Delta t}$$

$$= \lim_{\Delta t\to 0}\left[\frac{\sqrt{t + \Delta t} - \sqrt{t}}{\Delta t}\mathbf{i} + \frac{\frac{3}{t + \Delta t} - \frac{3}{t}}{\Delta t}\mathbf{j} - 2\mathbf{k}\right]$$

$$= \lim_{\Delta t\to 0}\left[\frac{\Delta t}{\Delta t\left(\sqrt{t + \Delta t} + \sqrt{t}\right)}\mathbf{i} + \frac{-3\Delta t}{(t + \Delta t)t(\Delta t)}\mathbf{j} - 2\mathbf{k}\right]$$

$$= \lim_{\Delta t\to 0}\left[\frac{1}{\sqrt{t + \Delta t} + \sqrt{t}}\mathbf{i} - \frac{3}{(t + \Delta t)t}\mathbf{j} - 2\mathbf{k}\right]$$

$$= \frac{1}{2\sqrt{t}}\mathbf{i} - \frac{3}{t^2}\mathbf{j} - 2\mathbf{k}$$

44. $$\int\left(4t^3\mathbf{i} + 6t\mathbf{j} - 4\sqrt{t}\mathbf{k}\right)dt = t^4\mathbf{i} + 3t^2\mathbf{j} - \frac{8}{3}t^{3/2}\mathbf{k} + \mathbf{C}$$

46. $$\int\left[\ln t\,\mathbf{i} + \frac{1}{t}\mathbf{j} + \mathbf{k}\right]dt = (t\ln t - t)\mathbf{i} + \ln t\,\mathbf{j} + t\mathbf{k} + \mathbf{C}$$

(Integration by parts)

48. $$\int[e^t\mathbf{i} + \sin t\,\mathbf{j} + \cos t\,\mathbf{k}]\,dt = e^t\mathbf{i} - \cos t\,\mathbf{j} + \sin t\,\mathbf{k} + \mathbf{C}$$

50. $\int[e^{-t}\sin t\mathbf{i} + e^{-t}\cos t\mathbf{j}]\,dt = \frac{e^{-t}}{2}(-\sin t - \cos t)\mathbf{i} + \frac{e^{-t}}{2}(-\cos t + \sin t)\mathbf{j} + \mathbf{C}$

52. $\int_{-1}^{1}\left(t\mathbf{i} + t^3\mathbf{j} + \sqrt[3]{t}\mathbf{k}\right)dt = \left[\frac{t^2}{2}\mathbf{i}\right]_{-1}^{1} + \left[\frac{t^4}{4}\mathbf{j}\right]_{-1}^{1} + \left[\frac{3}{4}t^{4/3}\mathbf{k}\right]_{-1}^{1} = \mathbf{0}$

54. $\int_0^2(t\mathbf{i} + e^t\mathbf{j} - te^t\mathbf{k})\,dt = \left[\frac{t^2}{2}\mathbf{i}\right]_0^2 + \left[e^t\mathbf{j}\right]_0^2 - \left[(t-1)e^t\mathbf{k}\right]_0^2$

$= 2\mathbf{i} + (e^2 - 1)\mathbf{j} - (e^2 + 1)\mathbf{k}$

56. $\mathbf{r}(t) = \int\left(3t^2\mathbf{j} + 6\sqrt{t}\mathbf{k}\right)dt = t^3\mathbf{j} + 4t^{3/2}\mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = C = \mathbf{i} + 2\mathbf{j}$

$\mathbf{r}(t) = \mathbf{i} + (2 + t^3)\mathbf{j} + 4t^{3/2}\mathbf{k}$

58. $\mathbf{r}''(t) = -4\cos t\mathbf{i} - 3\sin t\mathbf{k}$

$\mathbf{r}'(t) = -4\sin t\mathbf{i} + 3\cos t\mathbf{k} + \mathbf{C}_1$

$\mathbf{r}'(0) = 3\mathbf{k} = 3\mathbf{k} + \mathbf{C}_1 \implies \mathbf{C}_1 = \mathbf{0}$

$\mathbf{r}(t) = 4\cos t\mathbf{i} + 3\sin t\mathbf{k} + \mathbf{C}_2$

$\mathbf{r}(0) = 4\mathbf{i} + \mathbf{C}_2 = 4\mathbf{j} \implies \mathbf{C}_2 = 4\mathbf{j} - 4\mathbf{i}$

$\mathbf{r}(t) = (4\cos t - 4)\mathbf{i} + 4\mathbf{j} + 3\sin t\mathbf{k}$

60. $\mathbf{r}(t) = \int\left[\frac{1}{1+t^2}\mathbf{i} + \frac{1}{t^2}\mathbf{j} + \frac{1}{t}\mathbf{k}\right]dt = \arctan t\mathbf{i} - \frac{1}{t}\mathbf{j} + \ln t\mathbf{k} + \mathbf{C}$

$\mathbf{r}(1) = \frac{\pi}{4}\mathbf{i} - \mathbf{j} + \mathbf{C} = 2\mathbf{i} \implies \mathbf{C} = \left(2 - \frac{\pi}{4}\right)\mathbf{i} + \mathbf{j}$

$\mathbf{r}(t) = \left[2 - \frac{\pi}{4} + \arctan t\right]\mathbf{i} + \left(1 - \frac{1}{t}\right)\mathbf{j} + \ln t\mathbf{k}$

62. To find the integral of a vector-valued function, you integrate each component function separately. The constant of integration **C** is a constant vector.

64. The graph of $\mathbf{u}(t)$ does not change position relative to the xy-plane.

66. Let $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$ and $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$.

$\mathbf{r}(t) \pm \mathbf{u}(t) = [x_1(t) \pm x_2(t)]\mathbf{i} + [y_1(t) \pm y_2(t)]\mathbf{j} + [z_1(t) \pm z_2(t)]\mathbf{k}$

$D_t[\mathbf{r}(t) \pm \mathbf{u}(t)] = [x_1'(t) \pm x_2'(t)]\mathbf{i} + [y_1'(t) \pm y_2'(t)]\mathbf{j} + [z_1'(t) \pm z_2'(t)]\mathbf{k}$

$= [x_1'(t)\mathbf{i} + y_1'(t)\mathbf{j} + z_1'(t)\mathbf{k}] \pm [x_2'(t)\mathbf{i} + y_2'(t)\mathbf{j} + z_2'(t)\mathbf{k}]$

$= \mathbf{r}'(t) \pm \mathbf{u}'(t)$

68. Let $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$ and $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$.

$\mathbf{r}(t) \times \mathbf{u}(t) = [y_1(t)z_2(t) - z_1(t)y_2(t)]\mathbf{i} - [x_1(t)z_2(t) - z_1(t)x_2(t)]\mathbf{j} + [x_1(t)y_2(t) - y_1(t)x_2(t)]\mathbf{k}$

$D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = [y_1(t)z_2'(t) + y_1'(t)z_2(t) - z_1(t)y_2'(t) - z_1'(t)y_2(t)]\mathbf{i} - [x_1(t)z_2'(t) + x_1'(t)z_2(t) - z_1(t)x_2'(t) - z_1'(t)x_2(t)]\mathbf{j} +$

$[x_1(t)y_2'(t) + x_1'(t)y_2(t) - y_1(t)x_2'(t) - y_1'(t)x_2(t)]\mathbf{k}$

$= \{[y_1(t)z_2'(t) - z_1(t)y_2'(t)]\mathbf{i} - [x_1(t)z_2'(t) - z_1(t)x_2'(t)]\mathbf{j} + [x_1(t)y_2'(t) - y_1(t)x_2'(t)]\mathbf{k}\} +$

$\{[y_1'(t)z_2(t) - z_1'(t)y_2(t)]\mathbf{i} - [x_1'(t)z_2(t) - z_1'(t)x_2(t)]\mathbf{j} + [x_1'(t)y_2(t) - y_1'(t)x_2(t)]\mathbf{k}\}$

$= \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$

70. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$.

$$\mathbf{r}(t) \times \mathbf{r}'(t) = [y(t)z'(t) - z(t)y'(t)]\mathbf{i} - [x(t)z'(t) - z(t)x'(t)]\mathbf{j} + [x(t)y'(t) - y(t)x'(t)]\mathbf{k}$$

$$D_t[\mathbf{r}(t) \times \mathbf{r}'(t)] = [y(t)z''(t) + y'(t)z'(t) - z(t)y''(t) - z'(t)y'(t)]\mathbf{i} - [x(t)z''(t) + x'(t)z'(t) - z(t)x''(t) - z'(t)x'(t)]\mathbf{j} + [x(t)y''(t) + x'(t)y'(t) - y(t)x''(t) - y'(t)x'(t)]\mathbf{k}$$

$$= [y(t)z''(t) - z(t)y''(t)]\mathbf{i} - [x(t)z''(t) - z(t)x''(t)]\mathbf{j} + [x(t)y''(t) - y(t)x''(t)]\mathbf{k} = \mathbf{r}(t) \times \mathbf{r}''(t)$$

72. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. If $\mathbf{r}(t) \cdot \mathbf{r}(t)$ is constant, then:

$$x^2(t) + y^2(t) + z^2(t) = C$$

$$D_t[x^2(t) + y^2(t) + z^2(t)] = D_t[C]$$

$$2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t) = 0$$

$$2[x(t)x'(t) + y(t)y'(t) + z(t)z'(t)] = 0$$

$$2[\mathbf{r}(t) \cdot \mathbf{r}'(t)] = 0$$

Therefore, $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.

74. False

$$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$$

(See Theorem 11.2, part 4)

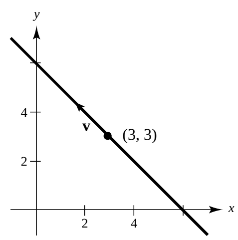
Section 11.3 Velocity and Acceleration

2. $\mathbf{r}(t) = (6 - t)\mathbf{i} + t\mathbf{j}$

$\mathbf{v}(t) = \mathbf{r}'(t) = -\mathbf{i} + \mathbf{j}$

$\mathbf{a}(t) = \mathbf{r}''(t) = \mathbf{0}$

$x = 6 - t,\ y = t,\ y = 6 - x$



4. $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$

$\mathbf{v}(t) = \mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j}$

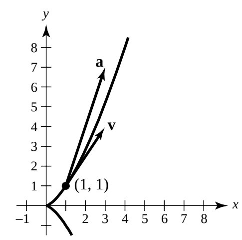
$\mathbf{a}(t) = \mathbf{r}''(t) = 2\mathbf{i} + 6t\mathbf{j}$

$x = t^2, y = t^3 \quad x = y^{2/3}$

At $(1, 1)$, $t = 1$.

$\mathbf{v}(1) = 2\mathbf{i} + 3\mathbf{j}$

$\mathbf{a}(1) = 2\mathbf{i} + 6\mathbf{j}$



6. $\mathbf{r}(t) = 3\cos t\mathbf{i} + 2\sin t\mathbf{j}$

$\mathbf{v}(t) = -3\sin t\mathbf{i} + 2\cos t\mathbf{j}$

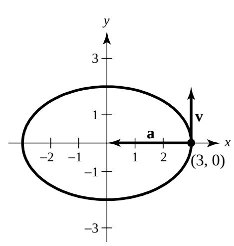
$\mathbf{a}(t) = -3\cos t\mathbf{i} - 2\sin t\mathbf{j}$

$x = 3\cos t, y = 2\sin t, \dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ Ellipse

At $(3, 0)$, $t = 0$.

$\mathbf{v}(0) = 2\mathbf{j}$

$\mathbf{a}(0) = -3\mathbf{i}$



8. $\mathbf{r}(t) = \langle e^{-t}, e^t \rangle$

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -e^{-t}, e^t \rangle$

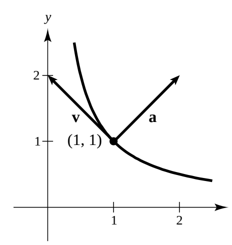
$\mathbf{a}(t) = \mathbf{r}''(t) = \langle e^{-t}, e^t \rangle$

$x = e^{-t} = \dfrac{1}{e^t},\ y = e^t,\ y = \dfrac{1}{x}$

At $(1, 1)$, $t = 0$.

$\mathbf{v}(0) = \langle -1, 1 \rangle = -\mathbf{i} + \mathbf{j}$

$\mathbf{a}(0) = \langle 1, 1 \rangle = \mathbf{i} + \mathbf{j}$



10. $\mathbf{r}(t) = 4t\mathbf{i} + 4t\mathbf{j} + 2t\mathbf{k}$

$\mathbf{v}(t) = 4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

$s(t) = \|\mathbf{v}(t)\| = \sqrt{16 + 16 + 4} = 6$

$\mathbf{a}(t) = \mathbf{0}$

12. $\mathbf{r}(t) = 3t\mathbf{i} + t\mathbf{j} + \frac{1}{4}t^2\mathbf{k}$

$\mathbf{v}(t) = 3\mathbf{i} + \mathbf{j} + \frac{1}{2}t\mathbf{k}$

$s(t) = \sqrt{9 + 1 + \frac{1}{4}t^2} = \sqrt{10 + \frac{1}{4}t^2}$

$\mathbf{a}(t) = \frac{1}{2}\mathbf{k}$

14. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + 2t^{3/2}\mathbf{k}$

$\mathbf{v}(t) = 2t\mathbf{i} + \mathbf{j} + 3\sqrt{t}\,\mathbf{k}$

$s(t) = \sqrt{4t^2 + 1 + 9t} = \sqrt{4t^2 + 9t + 1}$

$\mathbf{a}(t) = 2\mathbf{i} + \frac{3}{2\sqrt{t}}\mathbf{k}$

16. $\mathbf{r}(t) = \langle e^t \cos t, e^t \sin t, e^t \rangle$

$\mathbf{v}(t) = (e^t \cos t - e^t \sin t)\mathbf{i} + (e^t \sin t + e^t \cos t)\mathbf{j} + e^t\mathbf{k}$

$s(t) = \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\cos t + \sin t)^2 + e^{2t}}$

$= e^t\sqrt{3}$

$\mathbf{a}(t) = -2e^t \sin t\,\mathbf{i} + 2e^t \cos t\,\mathbf{j} + e^t\mathbf{k}$

18. (a) $\mathbf{r}(t) = \left\langle t, \sqrt{25 - t^2}, \sqrt{25 - t^2} \right\rangle, t_0 = 3$

$\mathbf{r}'(t) = \left\langle 1, \frac{-t}{\sqrt{25 - t^2}}, \frac{-t}{\sqrt{25 - t^2}} \right\rangle$

$\mathbf{r}'(3) = \left\langle 1, -\frac{3}{4}, -\frac{3}{4} \right\rangle$

$x = 3 + t,\ y = z = 4 - \frac{3}{4}t$

(b) $\mathbf{r}(3 + 0.1) \approx \left\langle 3 + 0.1, 4 - \frac{3}{4}(0.1), 4 - \frac{3}{4}(0.1) \right\rangle$

$= \langle 3.100, 3.925, 3.925 \rangle$

20. $\mathbf{a}(t) = 2\mathbf{i} + 3\mathbf{k}$

$\mathbf{v}(t) = \int (2\mathbf{i} + 3\mathbf{k})\,dt = 2t\mathbf{i} + 3t\mathbf{k} + \mathbf{C}$

$\mathbf{v}(0) = \mathbf{C} = 4\mathbf{j} \implies \mathbf{v}(t) = 2t\mathbf{i} + 4\mathbf{j} + 3t\mathbf{k}$

$\mathbf{r}(t) = \int (2t\mathbf{i} + 4\mathbf{j} + 3t\mathbf{k})\,dt = t^2\mathbf{i} + 4t\mathbf{j} + \frac{3}{2}t^2\mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{C} = 0 \implies \mathbf{r}(t) = t^2\mathbf{i} + 4t\mathbf{j} + \frac{3}{2}t^2\mathbf{k}$

$\mathbf{r}(2) = 4\mathbf{i} + 8\mathbf{j} + 6\mathbf{k}$

22. $\mathbf{a}(t) = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}, \mathbf{v}(0) = \mathbf{j} + \mathbf{k}, \mathbf{r}(0) = \mathbf{i}$

$\mathbf{v}(t) = \int (-\cos t\,\mathbf{i} - \sin t\,\mathbf{j})\,dt = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{C}$

$\mathbf{v}(0) = \mathbf{j} + \mathbf{C} = \mathbf{j} + \mathbf{k} \implies \mathbf{C} = \mathbf{k}$

$\mathbf{v}(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$

$\mathbf{r}(t) = \int (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k})\,dt$

$= \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{i} + \mathbf{C} = \mathbf{i} \implies \mathbf{C} = \mathbf{0}$

$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$

$\mathbf{r}(2) = (\cos 2)\mathbf{i} + (\sin 2)\mathbf{j} + 2\mathbf{k}$

24. (a) The speed is increasing.

(b) The speed is decreasing.

26. $\mathbf{r}(t) = (900 \cos 45°)t\,\mathbf{i} + [3 + (900 \sin 45°)t - 16t^2]\mathbf{j}$

$= 450\sqrt{2}t\,\mathbf{i} + \left(3 + 450\sqrt{2}t - 16t^2\right)\mathbf{j}$

The maximum height occurs when $y'(t) = 450\sqrt{2} - 32t = 0$, which implies that $t = \left(225\sqrt{2}\right)/16$. The maximum height reached by the projectile is

$$y = 3 + 450\sqrt{2}\left(\frac{225\sqrt{2}}{16}\right) - 16\left(\frac{225\sqrt{2}}{16}\right)^2 = \frac{50{,}649}{8} = 6331.125 \text{ feet.}$$

The range is determined by setting $y(t) = 3 + 450\sqrt{2}t - 16t^2 = 0$ which implies that

$$t = \frac{-450\sqrt{2} - \sqrt{405{,}192}}{-32} \approx 39.779 \text{ seconds.}$$

Range: $x = 450\sqrt{2}\left(\frac{-450\sqrt{2} - \sqrt{405{,}192}}{-32}\right) \approx 25{,}315.500$ feet

28. 50 mph $= \dfrac{220}{3}$ ft/sec

$$\mathbf{r}(t) = \left(\frac{220}{3}\cos 15°\right)t\,\mathbf{i} + \left[5 + \left(\frac{220}{3}\sin 15°\right)t - 16t^2\right]\mathbf{j}$$

The ball is 90 feet from where it is thrown when

$$x = \frac{220}{3}\cos 15° t = 90 \implies t = \frac{27}{22\cos 15°} \approx 1.2706 \text{ seconds.}$$

The height of the ball at this time is

$$y = 5 + \left(\frac{220}{3}\sin 15°\right)\left(\frac{27}{22\cos 15°}\right) - 16\left(\frac{27}{22\cos 15°}\right)^2 \approx 3.286 \text{ feet.}$$

30. $y = x - 0.005x^2$

From Exercise 34 we know that $\tan\theta$ is the coefficient of x. Therefore, $\tan\theta = 1$, $\theta = (\pi/4)$ rad $= 45°$. Also

$$\frac{16}{v_0^{\,2}}\sec^2\theta = \text{negative of coefficient of } x^2$$

$$\frac{16}{v_0^{\,2}}(2) = 0.005 \quad\text{or}\quad v_0 = 80 \text{ ft/sec}$$

$$\mathbf{r}(t) = \left(40\sqrt{2}t\right)\mathbf{i} + \left(40\sqrt{2}t - 16t^2\right)\mathbf{j}. \quad \text{Position function.}$$

When $40\sqrt{2}t = 60$,

$$t = \frac{60}{40\sqrt{2}} = \frac{3\sqrt{2}}{4}$$

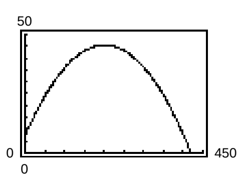
$$\mathbf{v}(t) = 40\sqrt{2}\mathbf{i} + \left(40\sqrt{2} - 32t\right)\mathbf{j}$$

$$\mathbf{v}\left(\frac{3\sqrt{2}}{4}\right) = 40\sqrt{2}\mathbf{i} + \left(40\sqrt{2} - 24\sqrt{2}\right)\mathbf{j} = 8\sqrt{2}(5\mathbf{i} + 2\mathbf{j}) \quad \text{direction}$$

$$\text{Speed} = \left\|\mathbf{v}\left(\frac{3\sqrt{2}}{4}\right)\right\| = 8\sqrt{2}\sqrt{25 + 4} = 8\sqrt{58} \text{ ft/sec}$$

32. Wind: 8 mph $= \dfrac{176}{15}$ ft/sec

$$\mathbf{r}(t) = \left(140(\cos 22°)t - \frac{176}{15}\right)\mathbf{i} + (2.5 + (140\sin 22°)t - 16t^2)\mathbf{j}$$



When $x = 375$, $t \approx 2.98$ and $y \approx 16.7$ feet.

Thus, the ball clears the 10-foot fence.

34. $h = 7$ feet, $\theta = 35°$, 30 yards $= 90$ feet

$\mathbf{r}(t) = (v_0 \cos 35°)t\mathbf{i} + [7 + (v_0 \sin 35°)t - 16t^2]\mathbf{j}$

(a) $v_0 \cos 35° t = 90$ when $7 + (v_0 \sin 35°)t - 16t^2 = 4$

$$t = \frac{90}{v_0 \cos 35°}$$

$$7 + (v_0 \sin 35°)\left(\frac{90}{v_0 \cos 35°}\right) - 16\left(\frac{90}{v_0 \cos 35°}\right)^2 = 4$$

$$90 \tan 35° + 3 = \frac{129{,}600}{v_0^2 \cos^2 35°}$$

$$v_0^2 = \frac{129{,}600}{\cos^2 35°(90 \tan 35° + 3)}$$

$$v_0 \approx 54.088 \text{ feet per second}$$

(b) The maximum height occurs when

$$y'(t) = v_0 \sin 35° - 32t = 0.$$

$$t = \frac{v_0 \sin 35°}{32} \approx 0.969 \text{ second}$$

At this time, the height is $y(0.969) \approx 22.0$ feet.

(c) $x(t) = 90 \implies (v_0 \cos 35°)t = 90$

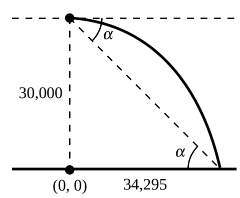
$$t = \frac{90}{54.088 \cos 35°} \approx 2.0 \text{ seconds}$$

36. Place the origin directly below the plane. Then $\theta = 0$, $v_0 = 792$ and

$$\mathbf{r}(t) = (v_0 \cos \theta)t\mathbf{i} + (30{,}000 + (v_0 \sin \theta)t - 16t^2)\mathbf{j}$$

$$= 792t\mathbf{i} + (30{,}000 - 16t^2)\mathbf{j}$$

$$\mathbf{v}(t) = 792\mathbf{i} - 32t\mathbf{j}.$$



At time of impact, $30{,}000 - 16t^2 = 0 \implies t^2 = 1875 \implies t \approx 43.3$ seconds.

$$\mathbf{r}(43.3) = 34{,}294.6\mathbf{i}$$

$$\mathbf{v}(43.3) = 792\mathbf{i} - 1385.6\mathbf{j}$$

$$\|\mathbf{v}(43.3)\| = 1596 \text{ ft/sec} = 1088 \text{ mph}$$

$$\tan \alpha = \frac{30{,}000}{34{,}294.6} \approx 0.8748 \implies \alpha \approx 0.7187(41.18°)$$

38. From Exercise 37, the range is

$$x = \frac{v_0^2}{32} \sin 2\theta.$$

Hence, $x = 150 = \dfrac{v_0^2}{32}\sin(24°) \implies v_0^2 = \dfrac{4800}{\sin 24°} \implies v_0 \approx 108.6$ ft/sec.

40. (a) $\mathbf{r}(t) = t(v_0 \cos \theta)\mathbf{i} + (tv_0 \sin \theta - 16t^2)\mathbf{j}$

$t(v_0 \sin \theta - 16t) = 0$ when $t = \dfrac{v_0 \sin \theta}{16}$.

Range: $\quad x = v_0 \cos \theta\left(\dfrac{v_0 \sin \theta}{32}\right) = \left(\dfrac{v_0^2}{32}\right)\sin 2\theta$

The range will be maximum when

$$\frac{dx}{dt} = \left(\frac{v_0^2}{32}\right)2 \cos 2\theta = 0$$

or

$$2\theta = \frac{\pi}{2}, \quad \theta = \frac{\pi}{4} \text{ rad.}$$

(b) $y(t) = tv_0 \sin \theta - 16t^2$

$\dfrac{dy}{dt} = v_0 \sin \theta - 32t = 0$ when $t = \dfrac{v_0 \sin \theta}{32}$.

Maximum height:

$$y\left(\frac{v_0 \sin \theta}{32}\right) = \frac{v_0^2 \sin^2 \theta}{32} - 16\frac{v_0^2 \sin^2 \theta}{32^2} = \frac{v_0^2 \sin^2 \theta}{64}$$

Minimum height when $\sin \theta = 1$, or $\theta = \dfrac{\pi}{2}$.

42. $\mathbf{r}(t) = (v_0 \cos\theta)t\,\mathbf{i} + [h + (v_0 \sin\theta)t - 4.9t^2]\mathbf{j}$

$= (v_0 \cos 8°)t\,\mathbf{i} + [(v_0 \sin 8°)t - 4.9t^2]\mathbf{j}$

$x = 50$ when $(v_0 \cos 8°)t = 50 \implies t = \dfrac{50}{v_0 \cos 8°}$. For this value of t, $y = 0$:

$$(v_0 \sin 8°)\left(\frac{50}{v_0 \cos 8°}\right) - 4.9\left(\frac{50}{v_0 \cos 8°}\right)^2 = 0$$

$$50 \tan 8° = \frac{(4.9)(2500)}{v_0^2 \cos^2 8°} \implies v_0^2 = \frac{(4.9)50}{\tan 8° \cos^2 8°} \approx 1777.698$$

$$\implies v_0 \approx 42.2 \text{ m/sec}$$

44. $\mathbf{r}(t) = b(\omega t - \sin\omega t)\mathbf{i} + b(1 - \cos\omega t)\mathbf{j}$

$\mathbf{v}(t) = b\omega[(1 - \cos\omega t)\mathbf{i} + (\sin\omega t)\mathbf{j}]$

Speed $= \|\mathbf{v}(t)\| = \sqrt{2}b\omega\sqrt{1 - \cos\omega t}$ and has a maximum value of $2b\omega$ when $\omega t = \pi, 3\pi, \ldots$.

55 mph $= 80.67$ ft/sec $= 80.67$ rad/sec $= \omega$ since (since $b = 1$)

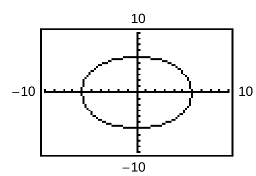
Therefore, the maximum speed of a point on the tire is twice the speed of the car:

$2(80.67)$ ft/sec $= 110$ mph

46. (a) Speed $= \|\mathbf{v}\| = \sqrt{b^2\omega^2 \sin^2(\omega t) + b^2\omega^2 \cos^2(\omega t)}$

$= \sqrt{b^2\omega^2[\sin^2(\omega t) + \cos^2(\omega t)]} = b\omega$

(b)



The graphing utility draws the circle faster for greater values of ω.

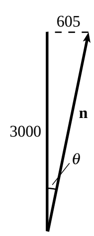
48. $\|\mathbf{a}(t)\| = b\omega^2\|\cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}\| = b\omega^2$

50. $\|\mathbf{v}(t)\| = 30$ mph $= 44$ ft/sec

$\omega = \dfrac{\|\mathbf{v}(t)\|}{b} = \dfrac{44}{300}$ rad/sec

$\|\mathbf{a}(t)\| = b\omega^2$

$F = m(b\omega^2) = \dfrac{3000}{32}(300)\left(\dfrac{44}{300}\right)^2 = 605$ lb



Let $\mathbf{n}$ be normal to the road.

$\|\mathbf{n}\| \cos\theta = 3000$

$\|\mathbf{n}\| \sin\theta = 605$

Dividing the second equation by the first:

$$\tan\theta = \frac{605}{3000}$$

$$\theta = \arctan\left(\frac{605}{3000}\right) \approx 11.4°.$$

52. $h = 6$ feet, $v_0 = 45$ feet per second, $\theta = 42.5°$. From Exercise 47,

$$t = \frac{45 \sin 42.5° + \sqrt{(45)^2 \sin^2 42.5° + 2(32)(6)}}{32} \approx 2.08 \text{ seconds.}$$

At this time, $x(t) \approx 69.02$ feet.

54. $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$

$y(t) = m(x(t)) + b$, m and b are constants.

$\mathbf{r}(t) = x(t)\mathbf{i} + [m(x(t)) + b]\mathbf{j}$

$\mathbf{v}(t) = x'(t)\mathbf{i} + mx'(t)\mathbf{j}$

$\mathbf{s}(t) = \sqrt{[x'(t)]^2 + [mx'(t)]^2} = C$, C is a constant.

Thus, $x'(t) = \dfrac{C}{\sqrt{1+m^2}}$

$x''(t) = 0$

$\mathbf{a}(t) = x''(t)\mathbf{i} + mx''(t)\mathbf{j} = \mathbf{0}$.

56. $\mathbf{r}_1(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$

$\mathbf{r}_2(t) = \mathbf{r}_1(2t)$

Velocity: $\mathbf{r}_2'(t) = 2\mathbf{r}_1'(2t)$

Acceleration: $\mathbf{r}_2''(t) = 4\mathbf{r}_1''(2t)$

In general, if $\mathbf{r}_3(t) = \mathbf{r}_1(\omega t)$, then:

Velocity: $\mathbf{r}_3'(t) = \omega\mathbf{r}_1'(\omega t)$

Acceleration: $\mathbf{r}_3''(t) = \omega^2\mathbf{r}_1''(\omega t)$

Section 11.4 Tangent Vectors and Normal Vectors

2. $\mathbf{r}(t) = t^3\mathbf{i} + 2t^2\mathbf{j}$

$\mathbf{r}'(t) = 3t^2\mathbf{i} + 4t\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{9t^4 + 16t^2}$

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \dfrac{1}{\sqrt{9t^4+16t^2}}(3t^2\mathbf{i} + 4t\mathbf{j})$

$\mathbf{T}(1) = \dfrac{1}{\sqrt{9+16}}(3\mathbf{i} + 4\mathbf{j}) = \dfrac{3}{5}\mathbf{i} + \dfrac{4}{5}\mathbf{j}$

4. $\mathbf{r}(t) = 6\cos t\mathbf{i} + 2\sin t\mathbf{j}$

$\mathbf{r}'(t) = -6\sin t\mathbf{i} + 2\cos t\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{36\sin^2 t + 4\cos^2 t}$

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \dfrac{-6\sin t\mathbf{i} + 2\cos t\mathbf{j}}{\sqrt{36\sin^2 t + 4\cos^2 t}}$

$\mathbf{T}\left(\dfrac{\pi}{3}\right) = \dfrac{-3\sqrt{3}\mathbf{i} + \mathbf{j}}{\sqrt{36(3/4) + (1/4)}} = \dfrac{1}{\sqrt{28}}\left(-3\sqrt{3}\mathbf{i} + \mathbf{j}\right)$

6. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + \dfrac{4}{3}\mathbf{k}$

$\mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$

When $t = 1$, $\mathbf{r}'(t) = \mathbf{r}'(1) = 2\mathbf{i} + \mathbf{j}$ $\left[t = 1 \text{ at } \left(1, 1, \dfrac{4}{3}\right)\right]$.

$\mathbf{T}(1) = \dfrac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \dfrac{2\mathbf{i}+\mathbf{j}}{\sqrt{5}} = \dfrac{\sqrt{5}}{5}(2\mathbf{i} + \mathbf{j})$

Direction numbers: $a = 2, b = 1, c = 0$

Parametric equations: $x = 2t + 1, y = t + 1, z = \dfrac{4}{3}$

8. $\mathbf{r}(t) = \left\langle t, t, \sqrt{4 - t^2}\right\rangle$

$\mathbf{r}'(t) = \left\langle 1, 1, -\dfrac{t}{\sqrt{4-t^2}}\right\rangle$

When $t = 1$, $\mathbf{r}'(1) = \left\langle 1, 1, -\dfrac{1}{\sqrt{3}}\right\rangle$, $\left[t = 1 \text{ at } \left(1, 1, \sqrt{3}\right)\right]$.

$\mathbf{T}(1) = \dfrac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \dfrac{\sqrt{21}}{7}\left\langle 1, 1, -\dfrac{1}{\sqrt{3}}\right\rangle$

Direction numbers: $a = 1$, $b = 1$, $c = -\dfrac{1}{\sqrt{3}}$

Parametric equations: $x = t + 1$, $y = t + 1$,

$z = -\dfrac{1}{\sqrt{3}}t + \sqrt{3}$

10. $\mathbf{r}(t) = \langle 2\sin t, 2\cos t, 4\sin^2 t\rangle$

$\mathbf{r}'(t) = \langle 2\cos t, -2\sin t, 8\sin t\cos t\rangle$

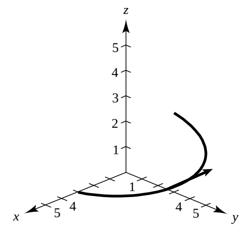
When $t = \dfrac{\pi}{6}$, $\mathbf{r}'\left(\dfrac{\pi}{6}\right) = \left\langle \sqrt{3}, -1, 2\sqrt{3}\right\rangle$, $\left[t = \dfrac{\pi}{6} \text{ at } \left(1, \sqrt{3}, 1\right)\right]$.

$\mathbf{T}\left(\dfrac{\pi}{6}\right) = \dfrac{\mathbf{r}'(\pi/6)}{\|\mathbf{r}'(\pi/6)\|} = \dfrac{1}{4}\left\langle \sqrt{3}, -1, 2\sqrt{3}\right\rangle$

Direction numbers: $a = \sqrt{3}$, $b = -1$, $c = 2\sqrt{3}$

Parametric equations: $x = \sqrt{3}t + 1$, $y = -t + \sqrt{3}$, $z = 2\sqrt{3}t + 1$

12. $\mathbf{r}(t) = 3\cos t\mathbf{i} + 4\sin t\mathbf{j} + \frac{1}{2}\mathbf{k}$



$\mathbf{r}'(t) = -3\sin t\mathbf{i} + 4\cos t\mathbf{j} + \frac{1}{2}\mathbf{k}$

When $t = \frac{\pi}{2}$, $\mathbf{r}'\left(\frac{\pi}{2}\right) = -3\mathbf{i} + \frac{1}{2}\mathbf{k}$, $\left[t = \frac{\pi}{2} \text{ at } \left(0, 4, \frac{\pi}{4}\right)\right]$.

$$\mathbf{T}\left(\frac{\pi}{2}\right) = \frac{\mathbf{r}'(\pi/2)}{\|\mathbf{r}'(\pi/2)\|} = \frac{2}{\sqrt{37}}\left(-3\mathbf{i} + \frac{1}{2}\mathbf{k}\right) = \frac{1}{\sqrt{37}}(-6\mathbf{i} + \mathbf{k})$$

Direction numbers: $a = -6$, $b = 0$, $c = 1$

Parametric equations: $x = -6t$, $y = 4$, $z = t + \frac{\pi}{4}$

14. $\mathbf{r}(t) = e^{-t}\mathbf{i} + 2\cos t\mathbf{j} + 2\sin t\mathbf{k}$, $t_0 = 0$

$\mathbf{r}'(t) = -e^{-t}\mathbf{i} - 2\sin t\mathbf{j} + 2\cos t\mathbf{k}$

$\mathbf{r}(0) = \mathbf{i} + 2\mathbf{j}$, $\mathbf{r}'(0) = -\mathbf{i} + 2\mathbf{k}$, $\|\mathbf{r}'(0)\| = \sqrt{5}$

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \frac{-\mathbf{i} + 2\mathbf{k}}{\sqrt{5}}$$

Parametric equations: $x(s) = 1 - s$, $y(s) = 2$, $z(s) = 2s$

$$\mathbf{r}(t_0 + 0.1) = \mathbf{r}(0 + 0.1) \approx \langle 1 - 0.1, 2, 2(0.1)\rangle$$
$$= \langle 0.9, 2, 0.2\rangle$$

16. $\mathbf{r}(0) = \langle 0, 1, 0\rangle$

$\mathbf{u}(0) = \langle 0, 1, 0\rangle$

Hence the curves intersect.

$\mathbf{r}'(t) = \langle 1, -\sin t, \cos t\rangle$, $\mathbf{r}'(0) = \langle 1, 0, 1\rangle$

$$\mathbf{u}'(s) = \left\langle -\sin s\cos s - \cos s, -\sin s\cos s - \cos s, \frac{1}{2}\cos 2s + \frac{1}{2}\right\rangle$$

$\mathbf{u}'(0) = \langle -1, 0, 1\rangle$

$$\cos\theta = \frac{\mathbf{r}'(0)\cdot\mathbf{u}'(0)}{\|\mathbf{r}'(0)\|\,\|\mathbf{u}'(0)\|} = 0 \implies \theta = \frac{\pi}{2}$$

18. $\mathbf{r}(t) = t\mathbf{i} + \frac{6}{t}\mathbf{j}$, $t = 3$

$\mathbf{r}'(t) = \mathbf{i} - \frac{6}{t^2}\mathbf{j}$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{1 + (36/t^4)}}\left(\mathrm{i} - \frac{6}{\mathrm{t}^2}\mathbf{j}\right)$$
$$= \frac{t^2}{\sqrt{t^4 + 36}}\left(\mathbf{i} - \frac{6}{t^2}\mathbf{j}\right)$$

$$\mathbf{T}'(t) = \frac{72t}{(t^4 + 36)^{3/2}}\mathbf{i} + \frac{12t^3}{(t^4 + 36)^{3/2}}\mathbf{j}$$

$$\mathbf{N}(2) = \frac{\mathbf{T}'(2)}{\|\mathbf{T}'(2)\|} + \frac{1}{\sqrt{13}}(3\mathbf{i} + 2\mathbf{j})$$

20. $\mathbf{r}(t) = \cos t\mathbf{i} + 2\sin t\mathbf{j} + \mathbf{k}$, $t = -\frac{\pi}{4}$

$\mathbf{r}'(t) = -\sin t\mathbf{i} + 2\cos t\mathbf{j}$

$$\mathbf{T}(t) = \frac{\mathbf{r}(t)}{\|\mathbf{r}'(t)\|} = \frac{-\sin t\mathbf{i} + 2\cos t\mathbf{j}}{\sqrt{\sin^2 t + 4\cos^2 t}}$$

The unit normal vector is perpendicular to this vector and points toward the z-axis:

$$\mathbf{N}(t) = \frac{-2\cos t\mathbf{i} - \sin t\mathbf{j}}{\sqrt{\sin^2 t + 4\cos^2 t}}.$$

22. $\mathbf{r}(t) = 4t\mathbf{i} - 2t\mathbf{j}$

$\mathbf{v}(t) = 4\mathbf{i} - 2\mathbf{j}$

$\mathbf{a}(t) = \mathbf{O}$

$\mathbf{T}(t) = \dfrac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \dfrac{1}{\sqrt{5}}(2\mathbf{i} - \mathbf{j})$

$\mathbf{T}'(t) = \mathbf{O}$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$ is undefined.

The path is a line and the speed is constant.

24. $\mathbf{r}(t) = t^2\mathbf{j} + \mathbf{k}$

$\mathbf{v}(t) = 2t\mathbf{j}$

$\mathbf{a}(t) = 2\mathbf{j}$

$\mathbf{T}(t) = \dfrac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \dfrac{2t\mathbf{j}}{2t} = \mathbf{j}$

$\mathbf{T}'(t) = \mathbf{O}$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$ is undefined.

The path is a line and the speed is variable.

26. $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j},\ t = 1$

$\mathbf{v}(t) = 2t\mathbf{i} + 2\mathbf{j},\ \mathbf{v}(1) = 2\mathbf{i} + 2\mathbf{j}$

$\mathbf{a}(t) = 2\mathbf{i},\ \mathbf{a}(1) = 2\mathbf{i}$

$\mathbf{T}(t) = \dfrac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \dfrac{1}{\sqrt{4t^2 + 4}}(2t\mathbf{i} + 2\mathbf{j}) = \dfrac{1}{\sqrt{t^2 + 1}}(t\mathbf{i} + \mathbf{j})$

$\mathbf{T}(1) = \dfrac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = \dfrac{\sqrt{2}}{2}\mathbf{i} + \dfrac{\sqrt{2}}{2}\mathbf{j}$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{\dfrac{1}{(t^2+1)^{3/2}}\mathbf{i} + \dfrac{-t}{(t^2+1)^{3/2}}\mathbf{j}}{\dfrac{1}{t^2+1}}$$

$$= \frac{1}{\sqrt{t^2 + 1}}(\mathbf{i} + t\mathbf{j})$$

$\mathbf{N}(1) = \dfrac{\sqrt{2}}{2}\mathbf{i} - \dfrac{\sqrt{2}}{2}\mathbf{j}$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \sqrt{2}$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \sqrt{2}$

28. $\mathbf{r}(t) = a\cos(\omega t)\mathbf{i} + b\sin(\omega t)\mathbf{j}$

$\mathbf{v}(t) = -a\omega\sin(\omega t)\mathbf{i} + b\omega\cos(\omega t)\mathbf{j}$

$\mathbf{v}(0) = b\omega\mathbf{j}$

$\mathbf{a}(t) = -a\omega^2\cos(\omega t)\mathbf{i} - b\omega^2\sin(\omega t)\mathbf{j}$

$\mathbf{a}(0) = -a\omega^2\mathbf{i}$

$\mathbf{T}(0) = \dfrac{\mathbf{v}(0)}{\|\mathbf{v}(0)\|} = \mathbf{j}$

Motion along $\mathbf{r}(t)$ is counterclockwise. Therefore,

$\mathbf{N}(0) = -\mathbf{i}.$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = 0$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = a\omega^2$

30. $\mathbf{r}(t_0) = (\omega t_0 - \sin\omega t_0)\mathbf{i} + (1 - \cos\omega t_0)\mathbf{j}$

$\mathbf{v}(t_0) = \omega[(1 - \cos\omega t_0)\mathbf{i} + (\sin\omega t_0)\mathbf{j}]$

$\mathbf{a}(t_0) = \omega^2[(\sin\omega t_0)\mathbf{i} + (\cos\omega t_0)\mathbf{j}]$

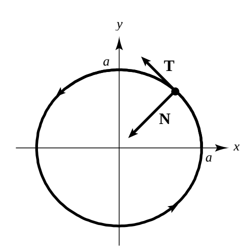
$\mathbf{T} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{(1 - \cos\omega t_0)\mathbf{i} + (\sin\omega t_0)\mathbf{j}}{\sqrt{2}\sqrt{1 - \cos\omega t_0}}$

Motion along $\mathbf{r}$ is clockwise. Therefore, $\mathbf{N} = \dfrac{(\sin\omega t_0)\mathbf{i} - (1 - \cos\omega t_0)\mathbf{j}}{\sqrt{2}\sqrt{1 - \cos\omega t_0}}$.

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \dfrac{\omega^2\sin\omega t_0}{\sqrt{2}\sqrt{1 - \cos\omega t_0}} = \dfrac{\omega^2}{\sqrt{2}}\sqrt{1 + \cos\omega t_0}$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \dfrac{\omega^2}{\sqrt{2}}\sqrt{1 - \cos\omega t_0}$

32. $\mathbf{T}(t)$ points in the direction that $\mathbf{r}$ is moving. $\mathbf{N}(t)$ points in the direction that $\mathbf{r}$ is turning, toward the concave side of the curve.



34. If the angular velocity ω is halved,

$$a_{\mathbf{N}} = a\left(\frac{\omega}{2}\right)^2 = \frac{a\omega^2}{4}.$$

$a_{\mathbf{N}}$ is changed by a factor of $\frac{1}{4}$.

36. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j},\ t_0 = \dfrac{\pi}{4}$

$$x = 2\cos t,\ y = 2\sin t \implies x^2 + y^2 = 4$$

$$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j}$$

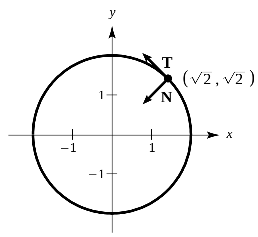
$$\mathbf{T}(t) = \frac{1}{2}(-2\sin t\mathbf{i} + 2\cos t\mathbf{j}) = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

$$\mathbf{N}(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}$$

$$\mathbf{r}\left(\frac{\pi}{4}\right) = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$$

$$\mathbf{T}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(-\mathbf{i} + \mathbf{j})$$

$$\mathbf{N}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(-\mathbf{i} - \mathbf{j})$$



38. $\mathbf{r}(t) = 4t\mathbf{i} - 4t\mathbf{j} + 2t\mathbf{k}$

$$\mathbf{v}(t) = 4\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{a}(t) = \mathbf{0}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'}{\|\mathbf{T}'\|}$ is undefined.

$a_{\mathbf{T}}$, $a_{\mathbf{N}}$ are not defined.

40. $\mathbf{r}(t) = e^t \sin t\,\mathbf{i} + e^t\cos t\mathbf{j} + e^t\mathbf{k}$

$$\mathbf{v}(t) = (e^t\cos t + e^t\sin t)\mathbf{i} + (-e^t\sin t + e^t\cos t)\mathbf{j} + e^t\mathbf{k}$$

$$\mathbf{v}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

$$\mathbf{a}(t) = 2e^t\cos t\mathbf{i} - 2e^t\sin t\mathbf{j} + e^t\mathbf{k}$$

$$\mathbf{a}(0) = 2\mathbf{i} + \mathbf{k}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{3}}[(\cos t + \sin t)\mathbf{i} + (-\sin t + \cos t)\mathbf{j} + \mathbf{k}]$$

$$\mathbf{T}(0) = \frac{1}{\sqrt{3}}[\mathbf{i} + \mathbf{j} + \mathbf{k}]$$

$$\mathbf{N}(t) = \frac{1}{\sqrt{2}}[(-\sin t + \cos t)\mathbf{i} + (-\cos t - \sin t)\mathbf{j}]$$

$$\mathbf{N}(0) = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$$

$$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = \sqrt{3}$$

$$a_{\mathbf{N}} = \mathbf{a}\cdot\mathbf{N} = \sqrt{2}$$

42. $\mathbf{r}(t) = t\mathbf{i} + 3t^2\mathbf{j} + \dfrac{t^2}{2}\mathbf{k}$

$$\mathbf{v}(t) = \mathbf{i} + 6t\mathbf{j} + t\mathbf{k}$$

$$\mathbf{v}(2) = \mathbf{i} + 12\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{a}(t) = 6\mathbf{j} + \mathbf{k}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{1 + 37t^2}}(\mathbf{i} + 6t\mathbf{j} + t\mathbf{k})$$

$$\mathbf{T}(2) = \frac{1}{\sqrt{149}}(\mathbf{i} + 12\mathbf{j} + 2\mathbf{k})$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} = \frac{\dfrac{1}{(1 + 37t^2)^{3/2}}[-37t\mathbf{i} + 6\mathbf{j} + \mathbf{k}]}{\dfrac{\sqrt{37}}{1 + 37^2}}$$

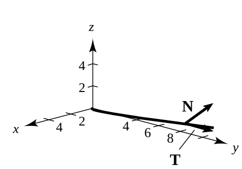
$$= \frac{1}{\sqrt{37}\sqrt{1 + 37t^2}}[-37t\mathbf{i} + 6\mathbf{j} + \mathbf{k}]$$

$$\mathbf{N}(2) = \frac{1}{\sqrt{37}\sqrt{149}}[-74\mathbf{i} + 6\mathbf{j} + \mathbf{k}]$$

$$= \frac{1}{\sqrt{5513}}(-74\mathbf{i} + 6\mathbf{j} + \mathbf{k})$$

$$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = \frac{74}{\sqrt{149}}$$

$$a_{\mathbf{N}} = \mathbf{a}\cdot\mathbf{N} = \frac{37}{\sqrt{5513}} = \frac{\sqrt{37}}{\sqrt{149}}$$



44. The unit tangent vector points in the direction of motion.

46. If $a_{\mathbf{T}} = 0$, then the speed is constant.

48. (a) $\mathbf{r}(t) = \langle \cos \pi t + \pi t \sin \pi t, \sin \pi t - \pi t \cos \pi t \rangle$

$\mathbf{v}(t) = \langle -\pi \sin \pi t + \pi \sin \pi t + \pi^2 t \cos \pi t, \pi \cos \pi t - \pi \cos \pi t + \pi^2 t \sin \pi t \rangle = \langle \pi^2 t \cos \pi t, \pi^2 t \sin \pi t \rangle$

$\mathbf{a}(t) = \langle \pi^2 \cos \pi t - \pi^3 t \sin \pi t, \pi^2 \sin \pi t + \pi^3 t \cos \pi t \rangle$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \langle \cos \pi t, \sin \pi t \rangle$$

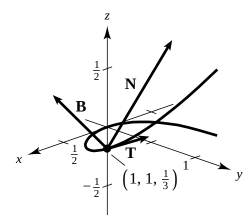
$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \cos \pi t(\pi^2 \cos \pi t - \pi^3 t \sin \pi t) + \sin \pi t(\pi^2 \sin \pi t + \pi^3 t \cos \pi t) = \pi^2$

$a_{\mathbf{N}} = \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^2} = \sqrt{\pi^4(1 + \pi^2 t^2) - \pi^4} = \pi^3 t$

When $t = 1$, $a_{\mathbf{T}} = \pi^2$, $a_{\mathbf{N}} = \pi^3$. When $t = 2$, $a_{\mathbf{T}} = \pi^2$, $a_{\mathbf{N}} = 2\pi^3$.

(b) Since $a_{\mathbf{T}} = \pi^2 > 0$ for all values of t, the speed is increasing when $t = 1$ and $t = 2$.

50. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \dfrac{t^3}{3}\mathbf{k}, \; t_0 = 1$



$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k}$

$$\mathbf{T}(t) = \frac{1}{\sqrt{1 + 4t^2 + t^4}}(\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k})$$

$$\mathbf{N}(t) = \frac{1}{\sqrt{1 + 4t^2 + t^4}\sqrt{1 + t^2 + t^4}}[(-2t - t^3)\mathbf{i} + (1 - t^4)\mathbf{j} + (t + 2t^3)\mathbf{k}]$$

$$\mathbf{r}(1) = \mathbf{i} + \mathbf{j} + \frac{1}{3}\mathbf{k}$$

$$\mathbf{T}(1) = \frac{1}{\sqrt{6}}(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

$$\mathbf{N}(1) = \frac{1}{\sqrt{6}\sqrt{3}}(-3\mathbf{i} + 3\mathbf{k}) = \frac{\sqrt{2}}{2}(-\mathbf{i} + \mathbf{k})$$

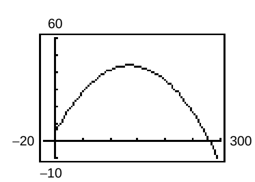
$$\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{vmatrix} = \frac{\sqrt{3}}{3}\mathbf{i} - \frac{\sqrt{3}}{3}\mathbf{j} + \frac{\sqrt{3}}{3}\mathbf{k} = \frac{\sqrt{3}}{3}(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

52. (a) $\mathbf{r}(t) = (v_0 \cos \theta)t\mathbf{i} + \left[h + (v_0 \sin \theta)t - \dfrac{1}{2}gt^2\right]\mathbf{j}$

$= (100 \cos 30°)t\mathbf{i} + [5 + (100 \sin 30°)t - 16t^2]\mathbf{j}$

$= 50\sqrt{3}t\mathbf{i} + [5 + 50t - 16t^2]\mathbf{j}$

(b)



Maximum height ≈ 44.0625

Range ≈ 279.0325

(c) $\mathbf{v}(t) = 50\sqrt{3}\mathbf{i} + (50 - 32t)\mathbf{j}$

Speed $= \|\mathbf{v}(t)\| = \sqrt{2500(3) + (50 - 32t)^2}$

$= 4\sqrt{64t^2 - 200t + 625}\mathbf{a}(t) = -32\mathbf{j}$

(d)

t	0.5	1.0	1.5	2.0	2.5	3.0
Speed	93.04	88.45	86.63	87.73	91.65	98.06

—CONTINUED—

52. —CONTINUED—

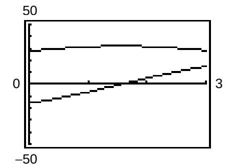
(e) $$\mathbf{T}(t) = \frac{25\sqrt{3}\mathbf{i} + (25 - 16t)\mathbf{j}}{2\sqrt{64t^2 - 200t + 625}}$$

$$\mathbf{N}(t) = \frac{(25 - 16t)\mathbf{i} - 25\sqrt{3}}{2\sqrt{64t^2 - 200t + 625}}$$

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \frac{16(16t - 25)}{\sqrt{64t^2 - 200t + 625}}$$

$$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \frac{400\sqrt{3}}{\sqrt{64t^2 - 200t + 625}}$$

$$a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N} = -32\mathbf{j}$$

(f) 

The speed is increasing when $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ have opposite signs.

54. 600 mph = 880 ft/sec

$$\mathbf{r}(t) = 880t\mathbf{i} + (-16t^2 + 36{,}000)\mathbf{j}$$

$$\mathbf{v}(t) = 880\mathbf{i} - 32t\mathbf{j}$$

$$\mathbf{a}(t) = -32\mathbf{j}$$

$$\mathbf{T}(t) = \frac{880\mathbf{i} - 32t\mathbf{j}}{16\sqrt{4t^2 + 3025}} = \frac{55\mathbf{i} - 2t\mathbf{j}}{\sqrt{4t^2 + 3025}}$$

Motion along **r** is clockwise, therefore

$$\mathbf{N}(t) = \frac{-2t\mathbf{i} - 55\mathbf{j}}{\sqrt{4t^2 + 3025}}$$

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \frac{64t}{\sqrt{4t^2 + 3025}}$$

$$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \frac{1760}{\sqrt{4t^2 + 3025}}$$

56. $$\mathbf{r}(t) = (r\cos\omega t)\mathbf{i} + (r\sin\omega t)\mathbf{j}$$

$$\mathbf{v}(t) = (-r\omega\sin\omega t)\mathbf{i} + (r\omega\cos\omega t)\mathbf{j}$$

$$\|\mathbf{v}(t)\| = r\omega\sqrt{1} = r\omega = v$$

$$\mathbf{a}(t) = (-r\omega^2\cos\omega t)\mathbf{i} - (r\omega^2\sin\omega t)\mathbf{j}$$

$$\|\mathbf{a}(t)\| = r\omega^2$$

(a) $F = m\|\mathbf{a}(t)\| = m(r\omega^2) = \frac{m}{r}(r^2\omega^2) = \frac{mv^2}{r}$

(b) By Newton's Law:

$$\frac{mv^2}{r} = \frac{GMm}{r^2},\ v^2 = \frac{GM}{r},\ v = \sqrt{\frac{GM}{r}}$$

58. $v = \sqrt{\frac{9.56 \times 10^4}{4200}} \approx 4.77$ mi/sec

60. Let x = distance from the satellite to the center of the earth ($x = r + 4000$). Then:

$$v = \frac{2\pi x}{t} = \frac{2\pi x}{24(3600)} = \sqrt{\frac{9.56 \times 10^4}{x}}$$

$$\frac{4\pi^2 x^2}{(24)^2(3600)^2} = \frac{9.56 \times 10^4}{x}$$

$$x^3 = \frac{(9.56 \times 10^4)(24)^2(3600)^2}{4\pi^2} \implies x \approx 26{,}245 \text{ mi}$$

$$v \approx \frac{2\pi(26{,}245)}{24(3600)} \approx 1.92 \text{ mi/sec} \approx 6871 \text{ mph}$$

62. $$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$$

$y(t) = m(x(t)) + b$, m and b are constants.

$$\mathbf{r}(t) = x(t)\mathbf{i} + [m(x(t)) + b]\mathbf{j}$$

$$\mathbf{v}(t) = x'(t)\mathbf{i} + mx'(t)\mathbf{j}$$

$$\|\mathbf{v}(t)\| = \sqrt{[x'(t)]^2 + [mx'(t)]^2} = |x'(t)|\sqrt{1 + m^2}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{\pm(\mathbf{i} + m\mathbf{j})}{\sqrt{1 + m^2}}, \text{ constant}$$

Hence, $\mathbf{T}'(t) = \mathbf{0}$.

64. $$\|\mathbf{a}\|^2 = \mathbf{a} \cdot \mathbf{a}$$

$$= (a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}) \cdot (a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N})$$

$$= a_{\mathbf{T}}^2\|\mathbf{T}\|^2 + 2a_{\mathbf{T}}a_{\mathbf{N}}\mathbf{T} \cdot \mathbf{N} + a_{\mathbf{N}}^2\|\mathbf{N}\|^2$$

$$= a_{\mathbf{T}}^2 + a_{\mathbf{N}}^2$$

$$a_{\mathbf{N}}^2 = \|\mathbf{a}\|^2 - a_{\mathbf{T}}^2$$

Since $a_{\mathbf{N}} > 0$, we have $a_{\mathbf{N}} = \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^2}$.

Section 11.5 Arc Length and Curvature

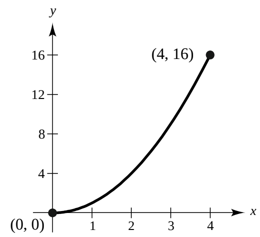
2. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{k}$

$$\frac{dx}{dt} = 1,\ \frac{dy}{dt} = 0,\ \frac{dz}{dt} = 2t$$

$$s = \int_0^4 \sqrt{1 + 4t^2}\, dt$$

$$= \frac{1}{4}\left[2t\sqrt{1 + 4t^2} + \ln\left|2t + \sqrt{1 + 4t}\right|\right]_0^4$$

$$= \frac{1}{4}\left[8\sqrt{65} + \ln\left(8 + \sqrt{65}\right)\right] \approx 16.819$$

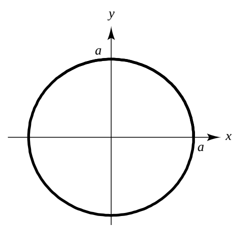


4. $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j}$

$$\frac{dx}{dt} = -a\sin t,\ \frac{dy}{dt} = a\cos t$$

$$s = \int_0^{2\pi} \sqrt{a^2\sin^2 t + a^2\cos^2 t}\, dt$$

$$= \int_0^{2\pi} a\, dt = \Big[at\Big]_0^{2\pi} = 2\pi a$$



6. (a) $\mathbf{r}(t) = (v_0\cos\theta)t\mathbf{i} + \left[(v_0\sin\theta)t - \frac{1}{2}gt^2\right]\mathbf{j}$

$$y(t) = (v_0\sin\theta)t - \frac{1}{2}gt^2$$

$$y'(t) = v_0\sin\theta - gt = 0 \text{ when } t = \frac{v_0\sin\theta}{g}.$$

Maximum height when $\sin\theta = 1$, or $\theta = \frac{\pi}{2}$.

(b) $y(t) = (v_0\sin\theta)t - \frac{1}{2}gt^2 = 0 \implies t = \frac{2v_0\sin\theta}{g}$

Range: $x(t) = (v_0\cos\theta)\left(\frac{2v_0\sin\theta}{g}\right) = \frac{{v_0}^2}{g}\sin^2\theta$

The range $x(t)$ is a maximum for $\sin 2\theta = 1$, or $\theta = \frac{\pi}{4}$.

(c) $x'(t) = v_0\cos\theta$

$$y'(t) = v_0\sin\theta - gt$$

$$x'(t)^2 + y'(t)^2 = {v_0}^2\cos^2\theta + (v_0\sin\theta - gt)^2$$

$$= {v_0}^2\cos^2\theta + {v_0}^2\sin^2\theta - 2{v_0}^2 g\sin\theta t + g^2t^2$$

$$= {v_0}^2 - 2v_0 g\sin\theta t + g^2t^2$$

$$s(\theta) = \int_0^{2v_0\sin\theta/g}\left[{v_0}^2 - 2v_0 g\sin\theta t + g^2t^2\right]^{1/2}_{dt}$$

Since $v_0 = 96$ ft/sec, we have

$$s(\theta) = \int_0^{6\sin\theta}\left[96^2 - (6144\sin\theta)t + 1024t^2\right]^{1/2}_{dt}.$$

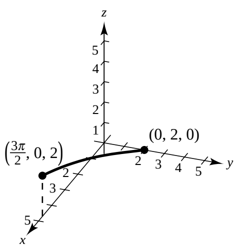
Using a computer algebra system, $s(\theta)$ is a maximum for $\theta \approx 0.9855 \approx 56.5°$.

8. $\mathbf{r}(t) = \langle 3t, 2\cos t, 2\sin t\rangle$

$$\frac{dx}{dt} = 3,\ \frac{dy}{dt} = -2\sin t,\ \frac{dz}{dt} = 2\cos t$$

$$s = \int_0^{\pi/2} \sqrt{3^2 + (-2\sin t)^2 + (2\cos t)^2}\,dt$$

$$= \int_0^{\pi/2} \sqrt{13}\,dt = \sqrt{13}t\Big]_0^{\pi/2} = \frac{\sqrt{13}\pi}{2}$$

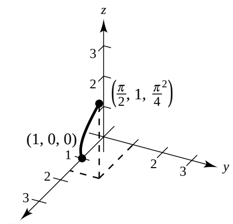


10. $\mathbf{r}(t) = \langle \cos t + t\sin t, \sin t - t\cos t, t^2\rangle$

$$\frac{dx}{dt} = t\cos t,\ \frac{dy}{dt} = t\sin t,\ \frac{dz}{dt} = 2t$$

$$s = \int_0^{\pi/2} \sqrt{(t\cos t)^2 + (t\sin t)^2 + (2t)^2}\,dt$$

$$= \int_0^{\pi/2} \sqrt{5t^2}\,dt = \sqrt{5}\,\frac{t^2}{2}\Big]_0^{\pi/2} = \frac{\sqrt{5}\pi^2}{8}$$



12. $\mathbf{r}(t) = \sin \pi t\mathbf{i} + \cos \pi t\mathbf{j} + t^3\mathbf{k}$

$$\frac{dx}{dt} = \pi\cos \pi t,\ \frac{dy}{dt} = -\pi\sin \pi t,\ \frac{dz}{dt} = 3t^2$$

$$s = \int_0^2 \sqrt{(\pi\cos \pi t)^2 + (-\pi\sin \pi t)^2 + (3t^2)^2}\,dt$$

$$= \int_0^2 \sqrt{\pi^2 + 9t^4}\,dt \approx 11.15$$

14. $\mathbf{r}(t) = 6\cos\left(\frac{\pi t}{4}\right)\mathbf{i} + 2\sin\left(\frac{\pi t}{4}\right)\mathbf{j} + t\mathbf{k},\ 0 \le t \le 2$

(a) $\mathbf{r}(0) = 6\mathbf{i} = \langle 6, 0, 0\rangle$

$\mathbf{r}(2) = 2\mathbf{j} + 2\mathbf{k} = \langle 0, 2, 2\rangle$

$\text{distance} = \sqrt{6^2 + 2^2 + 2^2} = \sqrt{44} = 2\sqrt{11} \approx 6.633$

(b) $\mathbf{r}(0) = \langle 6, 0, 0\rangle$

$\mathbf{r}(0.5) = \langle 5.543, 0.765, 0.5\rangle$

$\mathbf{r}(1.0) = \langle 4.243, 1.414, 1.0\rangle$

$\mathbf{r}(1.5) = \langle 2.296, 1.848, 1.5\rangle$

$\mathbf{r}(2.0) = \langle 0, 2, 2\rangle$

(c) Increase the number of line segments.

(d) Using a graphing utility, you obtain

$$s = \int_0^2 \|\mathbf{r}'(t)\|\,dt \approx 7.0105.$$

16. $\mathbf{r}(t) = \left\langle 4(\sin t - t\cos t), 4(\cos t + t\sin t), \frac{3}{2}t^2\right\rangle$

(a) $s = \int_0^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2}\,du$

$$= \int_0^t \sqrt{(4u\sin u)^2 + (4u\cos u)^2 + (3u)^2}\,du = \int_0^t \sqrt{16u + 9u^2}\,du = \int_0^t 5u\,du = \frac{5}{2}t^2$$

—CONTINUED—

16. —CONTINUED—

(b) $t = \sqrt{\frac{2s}{5}}$

$$x = 4\left(\sin\sqrt{\frac{2s}{5}} - \sqrt{\frac{2s}{5}}\cos\sqrt{\frac{2s}{5}}\right)$$

$$y = 4\left(\cos\sqrt{\frac{2s}{5}} + \sqrt{\frac{2s}{5}}\sin\sqrt{\frac{2s}{5}}\right)$$

$$z = \frac{3}{2}\left(\sqrt{\frac{2s}{5}}\right)^2 = \frac{3s}{5}$$

$$\mathbf{r}(s) = 4\left(\sin\sqrt{\frac{2s}{5}} - \sqrt{\frac{2s}{5}}\cos\sqrt{\frac{2s}{5}}\right)\mathbf{i} + 4\left(\cos\sqrt{\frac{2s}{5}} + \sqrt{\frac{2s}{5}}\sin\sqrt{\frac{2s}{5}}\right)\mathbf{j} + \frac{3s}{5}\mathbf{k}$$

(c) When $s = \sqrt{5}$:

$$x = 4\left(\sin\sqrt{\frac{2\sqrt{5}}{5}} - \sqrt{\frac{2\sqrt{5}}{5}}\cos\sqrt{\frac{2\sqrt{5}}{5}}\right) \approx -6.956$$

$$y = 4\left(\cos\sqrt{\frac{2\sqrt{5}}{5}} + \sqrt{\frac{2\sqrt{5}}{5}}\sin\sqrt{\frac{2\sqrt{5}}{5}}\right) \approx 14.169$$

$$z = \frac{3\sqrt{5}}{5} \approx 1.342$$

$(-6.956, 14.169, 1.342)$

When $s = 4$:

$$x = 4\left(\sin\sqrt{\frac{8}{5}} - \sqrt{\frac{8}{5}}\cos\sqrt{\frac{8}{5}}\right) \approx 2.291$$

$$y = 4\left(\cos\sqrt{\frac{8}{5}} + \sqrt{\frac{8}{5}}\sin\sqrt{\frac{8}{5}}\right) \approx 6.029$$

$$z = \frac{12}{5} = 2.4$$

$(2.291, 6.029, 2.400)$

(d) $\|\mathbf{r}'(s)\| = \sqrt{\left(\frac{4}{5}\sin\sqrt{\frac{2s}{5}}\right)^2 + \left(\frac{4}{5}\cos\sqrt{\frac{2s}{5}}\right)^2 + \left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25} + \frac{9}{25}} = 1$

18. $\mathbf{r}(s) = (3 + s)\mathbf{i} + \mathbf{j}$

$\mathbf{r}'(s) = \mathbf{i}$ and $\|\mathbf{r}'(s)\| = 1$

$\mathbf{T}(s) = \mathbf{r}'(s)$

$\mathbf{T}'(s) = \mathbf{0} \implies K = \|\mathbf{T}'(s)\| = 0$ (The curve is a line.)

20. $\mathbf{r}(s) = 4\left(\sin\sqrt{\frac{2s}{5}} - \sqrt{\frac{2s}{5}}\cos\sqrt{\frac{2s}{5}}\right)\mathbf{i} + 4\left(\cos\sqrt{\frac{2s}{5}} + \sqrt{\frac{2s}{5}}\sin\sqrt{\frac{2s}{5}}\right)\mathbf{j} + \frac{3s}{5}\mathbf{k}$

$$\mathbf{T}(s) = \mathbf{r}'(s) = \frac{4}{5}\sin\sqrt{\frac{2s}{5}}\,\mathbf{i} + \frac{4}{5}\cos\sqrt{\frac{2s}{5}}\,\mathbf{j} + \frac{3}{5}\mathbf{k}$$

$$\mathbf{T}'(s) = \frac{4}{25}\sqrt{\frac{5}{2s}}\cos\sqrt{\frac{2s}{5}}\,\mathbf{i} - \frac{4}{25}\sqrt{\frac{5}{2s}}\sin\sqrt{\frac{2s}{5}}\,\mathbf{j}$$

$$K = \|\mathbf{T}'(s)\| = \frac{4}{25}\sqrt{\frac{5}{2s}} = \frac{2\sqrt{10s}}{25s}$$

22. $\mathbf{r}(t) = t^2\mathbf{j} + \mathbf{k}$

$\mathbf{v}(t) = 2t\mathbf{j}$

$\mathbf{T}(t) = \mathbf{j}$

$\mathbf{T}'(t) = 0$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = 0$$

24. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$

$$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j}$$
$$\mathbf{v}(1) = \mathbf{i} + 2\mathbf{j}$$
$$\mathbf{a}(t) = 2\mathbf{j}$$
$$\mathbf{a}(1) = 2\mathbf{j}$$
$$\mathbf{T}(t) = \frac{\mathbf{i} + 2t\mathbf{j}}{\sqrt{1 + 4t^2}}$$
$$\mathbf{N}(t) = \frac{1}{\sqrt{1 + 4t^2}}(-2t\mathbf{i} + \mathbf{j})$$
$$\mathbf{N}(1) = \frac{1}{\sqrt{5}}(-2\mathbf{i} + \mathbf{j})$$
$$K = \frac{\mathbf{a} \cdot \mathbf{N}}{\|\mathbf{v}\|^2} = \frac{2}{5\sqrt{5}}$$

26. $\mathbf{r}(t) = 2\cos \pi t\mathbf{i} + \sin \pi t\mathbf{j}$

$$\mathbf{r}'(t) = -2\pi \sin \pi t\mathbf{i} + \pi \cos \pi t\mathbf{j}$$
$$\|\mathbf{r}'(t)\| = \pi\sqrt{4\sin^2 \pi t + \cos^2 \pi t}$$
$$\mathbf{T}(t) = \frac{-2\sin \pi t\mathbf{i} + \cos \pi t\mathbf{j}}{\sqrt{4\sin^2 \pi t + \cos^2 \pi t}}$$
$$\mathbf{T}'(t) = \frac{-2\pi \cos \pi t\mathbf{i} - 4\pi \sin \pi t\mathbf{j}}{(4\sin^2 \pi t + \cos^2 \pi t)^{3/2}}$$
$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\dfrac{2\pi}{4\sin^2 \pi t + \cos^2 \pi t}}{\pi\sqrt{4\sin^2 \pi t + \cos^2 \pi t}}$$
$$= \frac{2}{(4\sin^2 \pi t + \cos^2 \pi t)^{3/2}}$$

28. $\mathbf{r}(t) = a\cos(\omega t)\mathbf{i} + b\sin(\omega t)\mathbf{j}$

$$\mathbf{r}'(t) = -a\omega \sin(\omega t)\mathbf{i} + b\omega \cos(\omega t)\mathbf{j}$$
$$\mathbf{T}(t) = \frac{-a\sin(\omega t)\mathbf{i} + b\cos(\omega t)\mathbf{j}}{\sqrt{a^2\sin^2(\omega t) + b^2\cos^2(\omega t)}}$$
$$\mathbf{T}'(t) = \frac{-ab^2\omega\cos(\omega t)\mathbf{i} - a^2b\omega\sin(\omega t)\mathbf{j}}{[a^2\sin^2(\omega t) + b^2\cos^2(\omega t)]^{3/2}}$$
$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\dfrac{ab\omega}{a^2\sin^2(\omega t) + b^2\cos^2(\omega t)}}{\omega\sqrt{a^2\sin^2(\omega t) + b^2\cos^2(\omega t)}}$$
$$= \frac{ab}{[a^2\sin^2(\omega t) + b^2\cos^2(\omega t)]^{3/2}}$$

30. $\mathbf{r}(t) = \langle a(\omega t - \sin \omega t), a(1 - \cos \omega t)\rangle$

From Exercise 22, Section 11.4, we have:

$$\mathbf{a} \cdot \mathbf{N} = \frac{a\omega^2}{\sqrt{2}} \cdot \sqrt{1 - \cos \omega t}$$
$$K = \frac{\mathbf{a}(t) \cdot \mathbf{N}(t)}{\|\mathbf{v}(t)\|^2}$$
$$= \frac{\left(\dfrac{a\omega^2}{\sqrt{2}}\right)\sqrt{1 - \cos \omega t}}{2a^2\omega^2(1 - \cos \omega t)} = \frac{\sqrt{2}}{4a\sqrt{1 - \cos \omega t}}$$

32. $\mathbf{r}(t) = 4t\mathbf{i} - 4t\mathbf{j} + 2t\mathbf{k}$

$$\mathbf{r}'(t) = 4\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$$
$$\mathbf{T}(t) = \frac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$
$$\mathbf{T}'(t) = \mathbf{0}$$
$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = 0$$

34. $\mathbf{r}(t) = 2t^2\mathbf{i} + t\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

$$\mathbf{r}'(t) = 4t\mathbf{i} + \mathbf{j} + t\mathbf{k}$$
$$\mathbf{T}(t) = \frac{4t\mathbf{i} + \mathbf{j} + t\mathbf{k}}{\sqrt{1 + 17t^2}}$$
$$\mathbf{T}'(t) = \frac{4\mathbf{i} - 17t\mathbf{j} + \mathbf{k}}{(1 + 17t^2)^{3/2}}$$
$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\sqrt{289t^2 + 17}}{(1 + 17t^2)^{3/2}}\Big/(1 + 17t^2)^{1/2}$$
$$= \frac{\sqrt{17}}{(1 + 17t^2)^{3/2}}$$

36. $\mathbf{r}(t) = e^t\cos t\mathbf{i} + e^t\sin t\mathbf{j} + e^t\mathbf{k}$

$$\mathbf{r}'(t) = (-e^t\sin t + e^t\cos t)\mathbf{i} + (e^t\cos t + e^t\sin t)\mathbf{j} + e^t\mathbf{k}$$
$$\mathbf{T}(t) = \frac{1}{\sqrt{3}}[(-\sin t + \cos t)\mathbf{i} + (\cos t + \sin t)\mathbf{j} + \mathbf{k}]$$
$$\mathbf{T}'(t) = \frac{1}{\sqrt{3}}[(-\cos t - \sin t)\mathbf{i} + (-\sin t + \cos t)\mathbf{j}]$$
$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\left(1/\sqrt{3}\right)\sqrt{(-\cos t - \sin t)^2 + (-\sin t + \cos t)^2}}{\sqrt{3}e^t} = \frac{\sqrt{2}}{3e^t}$$

38. $y = mx + b$

Since $y'' = 0$, $K = 0$, and the radius of curvature is undefined.

40. $y = 2x + \frac{4}{x}, x = 1$

$y' = 2 - \frac{4}{x^2}, y'(1) = -2$

$y'' = \frac{8}{x^3}, y''(1) = 8$

$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{8}{(1 + 4)^{3/2}} = \frac{8}{5^{3/2}}$

$\frac{1}{K} = \frac{5^{3/2}}{8}$ (radius of curvature)

42. $y = \frac{3}{4}\sqrt{16 - x^2}$

$y' = \frac{-9x}{16y}$

$y'' = \frac{-[9 + (16y')^2]}{16y}$

At $x = 0$: $y' = 0$

$y'' = -\frac{3}{16}$

$K = \left|\frac{-3/16}{(1 + 0^2)^{3/2}}\right| = \frac{3}{16}$

$\frac{1}{K} = \frac{16}{3}$ (radius of curvature)

44. (a) $y = \frac{4x^2}{x^2 + 3}$

$y' = \frac{24x}{(x^2 + 3)^2}$

$y'' = \frac{72(1 - x^2)}{(x^2 + 3)^3}$

At $x = 0$: $y' = 0$

$y'' = \frac{72}{27} = \frac{8}{3}$

$K = \frac{8/3}{(1 + 0^2)^{3/2}} = \frac{8}{3}$

$r = \frac{1}{K} = \frac{3}{8}$

Center: $\left(0, \frac{3}{8}\right)$

Equation: $x^2 + \left(y - \frac{3}{8}\right)^2 = \frac{9}{64}$

(b) The circles have different radii since the curvature is different and

$r = \frac{1}{K}.$

46. $y = \ln x, \quad x = 1$

$y' = \frac{1}{x}, \quad y'' = -\frac{1}{x^2}$

$y'(1) = 1, \quad y''(1) = -1$

$K = \frac{|-1|}{(1 + (1)^2)^{3/2}} = \frac{1}{2^{3/2}}, \; r = \frac{1}{K} = 2^{3/2} = 2\sqrt{2}$

The slope of the tangent line at $(1, 0)$ is $y'(1) = 1$.

The slope of the normal line is -1.

Equation of normal line: $y = -(x - 1) = -x + 1$

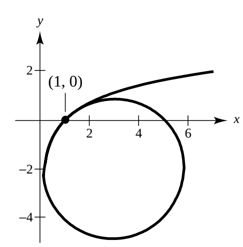
The center of the circle is on the normal line $2\sqrt{2}$ units away from the point $(1, 0)$.

$$\sqrt{(1 - x)^2 + (0 - y)^2} = 2\sqrt{2}$$
$$(1 - x)^2 + (x - 1)^2 = 8$$
$$2x^2 - 4x + 2 = 8$$
$$2(x^2 - 2x - 3) = 0$$
$$2(x - 3)(x + 1) = 0$$
$$x = 3 \text{ or } x = -1$$

Since the circle is below the curve, $x = 3$ and $y = -2$.

Center of circle: $(3, -2)$

Equation of circle: $(x - 3)^2 + (y + 2)^2 = 8$



48. $y = \frac{1}{3}x^3, \qquad x = 1$

$y' = x^2, \quad y''(1) = 2x$

$y'(1) = 1, \quad y''(1) = 2$

$$K = \frac{2}{(1+1)^{3/2}} = \frac{1}{\sqrt{2}},\ r = \frac{1}{K} = \sqrt{2}$$

The slope of the tangent line at $\left(1, \frac{1}{3}\right)$ is $y'(1) = 1$.

The slope of the normal line is -1.

Equation of normal line: $y - \frac{1}{3} = -(x-1)$ or $y = -x + \frac{4}{3}$

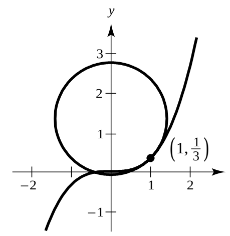
The center of the circle is on the normal line $\sqrt{2}$ units away from the point $\left(1, \frac{1}{3}\right)$.

$$\sqrt{(1-x)^2 + \left(\frac{1}{3} - y\right)^2} = \sqrt{2}$$

$$(1-x)^2 + (x-1)^2 = 2$$

$$(x-1)^2 = 1$$

$$x = 0 \text{ or } x = 2$$

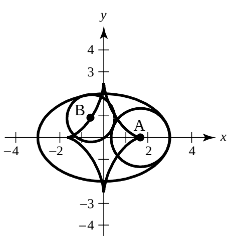


Since the circle is above the curve, $x = 0$ and $y = \frac{4}{3}$.

Center of circle: $\left(0, \frac{4}{3}\right)$

Equation of circle: $x^2 + \left(y - \frac{4}{3}\right)^2 = 2$

50.



52. $y = x^3,\ y' = 3x^2,\ y'' = 6x$

$$K = \left|\frac{6x}{(1+9x^4)^{3/2}}\right|$$

(a) K is maximum at $\left(\frac{1}{\sqrt[4]{45}}, \frac{1}{\sqrt[4]{45^3}}\right), \left(\frac{-1}{\sqrt[4]{45}}, \frac{-1}{\sqrt[4]{45^3}}\right)$.

(b) $\lim_{x \to \infty} K = 0$

54. $y = \ln x,\ y' = \frac{1}{x},\ y'' = -\frac{1}{x^2}$

$$K = \left|\frac{-1/x^2}{[1 + (1/x)^2]^{3/2}}\right| = \frac{x}{(x^2+1)^{3/2}}$$

$$\frac{dK}{dx} = \frac{-2x^2+1}{(x^2+1)^{5/2}}$$

(a) K has a maximum when $x = \frac{1}{\sqrt{2}}$.

(b) $\lim_{x \to \infty} K = 0$

56. $y = \cos x$

$y' = -\sin x$

$y'' = -\cos x$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{|-\cos x|}{(1 + \sin^2 x)^{3/2}} = 0 \text{ for } x = \frac{\pi}{2} + K\pi.$$

Curvature is 0 at $\left(\frac{\pi}{2} + K\pi, 0\right)$.

58. $y = \cosh x = \frac{e^x + e^{-x}}{2}$

$$y' = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$y'' = \frac{e^x + e^{-x}}{2} = \cosh x$$

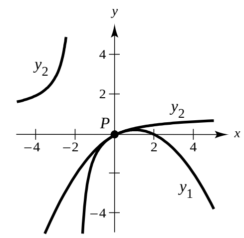
$$K = \frac{|\cosh x|}{[1 + (\sinh x)^2]^{3/2}} = \frac{\cosh x}{(\cosh^2 x)^{3/2}} = \frac{1}{\cosh^2 x} = \frac{1}{y^2}$$

60. See page 828.

62. $K = \dfrac{|y''|}{[1 + (y')^2]^{3/2}}$

At the smooth relative extremum $y' = 0$, so $K = |y''|$. Yes, for example, $y = x^4$ has a curvature of 0 at its relative minimum (0, 0). The curvature is positive for any other point of the curvature.

64. $y_1 = ax(b - x),\ y_2 = \dfrac{x}{x + 2}$



We observe that (0, 0) is a solution point to both equations. Therefore, the point P is the origin.

$$y_1 = ax(b - x), \quad y_1' = a(b - 2x), \quad y_1'' = -2a$$

$$y_2 = \frac{x}{x + 2}, \quad y_2' = \frac{2}{(x + 2)^2}, \quad y_2'' = \frac{-4}{(x + 2)^3}$$

At P,

$$y_1'(0) = ab \text{ and } y_2'(0) = \frac{2}{(0 + 2)^2} = \frac{1}{2}.$$

Since the curves have a common tangent at P, $y_1'(0) = y_2'(0)$ or $ab = \frac{1}{2}$. Therefore, $y_1'(0) = \frac{1}{2}$. Since the curves have the same curvature at P, $K_1(0) = K_2(0)$.

$$K_1(0) = \left|\frac{y_1''(0)}{[1 + (y_1(0))^2]^{3/2}}\right| = \left|\frac{-2a}{[1 + (1/2)^2]^{3/2}}\right|$$

$$K_2(0) = \left|\frac{y_2''(0)}{[1 + (y_2(0))^2]^{3/2}}\right| = \left|\frac{-1/2}{[1 + (1/2)^2]^{3/2}}\right|$$

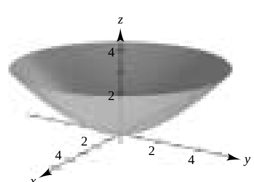
Therefore, $2a = \pm\frac{1}{2}$ or $a = \pm\frac{1}{4}$. In order that the curves intersect at only one point, the parabola must be concave downward. Thus,

$$a = \frac{1}{4} \quad \text{and} \quad b = \frac{1}{2a} = 2.$$

$$y_1 = \frac{1}{4}x(2 - x) \quad \text{and} \quad y_2 = \frac{x}{x + 2}$$

66. $y = \dfrac{1}{4}x^{8/5},\ 0 \le x \le 5$

(a)



(rotated about y-axis)

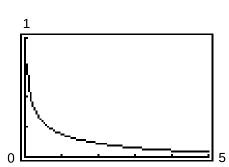
(b) $V = \displaystyle\int_0^5 2\pi x\left(\frac{1}{4}x^{8/5}\right) dx$ (shells)

$$= \frac{\pi}{2}\int_0^5 x^{13/5}\,dx = \frac{\pi}{2}\left[\frac{x^{18/5}}{18/5}\right]_0^5$$

$$= \frac{5\pi}{36}5^{18/5} \approx 143.25 \text{ cm}^3$$

(c) $y' = \dfrac{2}{5}x^{3/5},\ y'' = \dfrac{6}{25}x^{-2/5} = \dfrac{6}{25x^{2/5}}$

$$K = \frac{\dfrac{6}{25x^{2/5}}}{\left[1 + \dfrac{4}{25}x^{6/5}\right]^{3/2}} = \frac{6}{25x^{2/5}\left[1 + \dfrac{4}{25}x^{6/5}\right]^{3/2}}$$



(d) No, the curvature approaches ∞ as $x \to 0^+$. Hence, any spherical object will hit the sides of the goblet before touching the bottom (0, 0).

68. $s = \dfrac{c}{\sqrt{K}}$

$y = \dfrac{1}{3}x^3$

$y' = x^2$

$y'' = 2x$

$K = \left|\dfrac{2x}{(1 + x^4)^{3/2}}\right|$

When $x = 1$: $K = \dfrac{1}{\sqrt{2}}$

$$s = \frac{c}{\sqrt{1/\sqrt{2}}} = \sqrt[4]{2}c$$

$$30 = \sqrt[4]{2}c \implies c = \frac{30}{\sqrt[4]{2}}$$

At $x = \dfrac{3}{2}$, $K = \dfrac{3}{[1 + (81/16)]^{3/2}} \approx 0.201$

$$s\left(\frac{3}{2}\right) = \frac{c}{\sqrt{K}} = \frac{30/\sqrt[4]{2}}{\sqrt{K}} \approx 56.27 \text{ mi/hr.}$$

70. $r(\theta) = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} = f(\theta)\cos\theta\mathbf{i} + f(\theta)\sin\theta\mathbf{j}$

$x(\theta) = f(\theta)\cos\theta$

$y(\theta) = f(\theta)\sin\theta$

$x'(\theta) = -f(\theta)\sin\theta + f'(\theta)\cos\theta$

$y'(\theta) = f(\theta)\cos\theta + f'(\theta)\sin\theta$

$x''(\theta) = -f(\theta)\cos\theta - f'(\theta)\sin\theta - f'(\theta)\sin\theta + f''(\theta)\cos\theta = -f(\theta)\cos\theta - 2f'(\theta)\sin\theta + f''(\theta)\cos\theta$

$y''(\theta) = -f(\theta)\sin\theta + f'(\theta)\cos\theta + f'(\theta)\cos\theta + f''(\theta)\sin\theta = -f(\theta)\sin\theta + 2f'(\theta)\cos\theta + f''(\theta)\sin\theta$

$$K = \frac{|x'y'' - y'x''|}{[(x')^2 + (y')^2]^{3/2}} = \frac{|f^2(\theta) - f(\theta)f''(\theta) + 2(f'(\theta))^2|}{[f^2(\theta) + (f'(\theta))^2]^{3/2}} = \frac{|r^2 - rr'' + 2(r')^2|}{[r^2 + (r')^2]^{3/2}}$$

72. $r = \theta$

$r' = 1$

$r'' = 0$

$K = \dfrac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}} = \dfrac{2 + \theta^2}{(1 + \theta^2)^{3/2}}$

74. $r = e^\theta$

$r' = e^\theta$

$r'' = e^\theta$

$K = \dfrac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}} = \dfrac{2e^{2\theta}}{(2e^{2\theta})^{3/2}} = \dfrac{1}{\sqrt{2}e^\theta}$

76. At the pole, $r = 0$.

$$K = \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}}$$

$$= \frac{|2(r')^2|}{|r'|^3} = \frac{2}{|r'|}$$

78. $r = 6\cos 3\theta$

$r' = -18\sin 3\theta$

At the pole,

$$\theta = \frac{\pi}{6},\ r'\left(\frac{\pi}{6}\right) = -18,$$

and

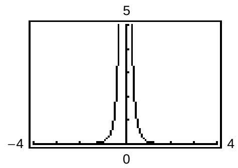
$$K = \frac{2}{|r'(\pi/6)|} = \frac{2}{|-18|} = \frac{1}{9}.$$

80. $x(t) = t^3,\ x'(t) = 3t^2,\ x''(t) = 6t$

$y(t) = \frac{1}{2}t^2,\ y'(t) = t,\ y''(t) = 1$

$$K = \frac{|(3t^2)(1) - (t)(6t)|}{[(3t^2)^2 + (t)^2]^{3/2}}$$

$$= \frac{3t^2}{|t^3|(9t^2+1)^{3/2}} = \frac{3}{|t|(9t^2+1)^{3/2}}$$



$K \to 0$ as $t \to \pm\infty$

82. (a) $\mathbf{r}(t) = 3t^2\mathbf{i} + (3t - t^3)\mathbf{j}$

$\mathbf{v}(t) = 6t\mathbf{i} + (3 - 3t^2)\mathbf{j}$

$\frac{ds}{dt} = \|\mathbf{v}(t)\| = 3(1 + t^2),\ \frac{d^2s}{dt^2} = 6t$

$$K = \frac{2}{3(1+t^2)^2}$$

$$a_{\mathbf{T}} = \frac{d^2s}{dt^2} = 6t$$

$$a_{\mathbf{N}} = K\left(\frac{ds}{dt}\right)^2 = \frac{2}{3(1+t^2)^2} \cdot 9(1+t^2)^2 = 6$$

(b) $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$

$\frac{ds}{dt} = \|\mathbf{v}(t)\| = \sqrt{5t^2+1}$

$$\frac{d^2s}{dt^2} = \frac{5t}{\sqrt{5t^2+1}}$$

$\mathbf{a}(t) = 2\mathbf{j} + \mathbf{k}$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \mathbf{v}(t) \times \mathbf{a}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & t \\ 0 & 2 & 1 \end{vmatrix} = -\mathbf{j} + 2\mathbf{k}$$

$$K = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\sqrt{5}}{(5t^2+1)^{3/2}}$$

$$a_{\mathbf{T}} = \frac{d^2s}{dt^2} = \frac{5t}{\sqrt{5t^2+1}}$$

$$a_{\mathbf{N}} = K\left(\frac{ds}{dt}\right)^2 = \frac{\sqrt{5}}{(5t^2+1)^{3/2}}(5t^2+1) = \frac{\sqrt{5}}{\sqrt{5t^2+1}}$$

84. (a) $K = \|\mathbf{T}'(s)\| = \left\|\frac{d\mathbf{T}}{ds}\right\| = \left\|\frac{d\mathbf{T}}{dt} \cdot \frac{dt}{ds}\right\|$, by the Chain Rule

$$= \left\|\frac{d\mathbf{T}/dt}{ds/dt}\right\| = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{v}(t)\|} = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$$

(b) $$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{r}'(t)}{ds/dt}$$

$$\mathbf{r}'(t) = \frac{ds}{dt}\mathbf{T}(t)$$

$$\mathbf{r}''(t) = \left(\frac{d^2s}{dt^2}\right)\mathbf{T}(t) + \frac{ds}{dt}\mathbf{T}'(t)$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \left(\frac{ds}{dt}\right)\left(\frac{d^2s}{dt^2}\right)[\mathbf{T}(t) \times \mathbf{T}(t)] + \left(\frac{ds}{dt}\right)^2[\mathbf{T}(t) \times \mathbf{T}'(t)]$$

Since $\mathbf{T}(t) \times \mathbf{T}(t) = \mathbf{0}$ and $\frac{ds}{dt} = \|\mathbf{r}'(t)\|$, we have:

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \|\mathbf{r}'(t)\|^2[\mathbf{T}(t) \times \mathbf{T}'(t)]$$

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \|\mathbf{r}'(t)\|^2\|\mathbf{T}(t) \times \mathbf{T}'(t)\| = \|\mathbf{r}'(t)\|^2\|\mathbf{T}(t)\|\,\|\mathbf{T}'(t)\| = \|\mathbf{r}'(t)\|^2(1)K\|\mathbf{r}'(t)\| \quad \text{from (a)}$$

Therefore, $\frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = K.$

(c) $$K = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)^3\|} = \frac{\dfrac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}}{\|\mathbf{r}'(t)\|^2} = \frac{\dfrac{\|\mathbf{v}(t) \times \mathbf{a}(t)\|}{\|\mathbf{v}(t)\|}}{\|\mathbf{r}'(t)\|^2} = \frac{\mathbf{a}(t) \cdot \mathbf{N}(t)}{\|\mathbf{r}'(t)\|^2}$$

86. $\mathbf{F} = m\mathbf{a} \implies m\mathbf{a} = \dfrac{-GmM}{r^3}\mathbf{r}$

$$\mathbf{a} = -\frac{GM}{r^3}\mathbf{r}$$

Since $\mathbf{r}$ is a constant multiple of $\mathbf{a}$, they are parallel. Since $\mathbf{a} = \mathbf{r}''$ is parallel to $\mathbf{r}$, $\mathbf{r} \times \mathbf{r}'' = \mathbf{0}$. Also,

$$\left(\frac{d}{dt}\right)(\mathbf{r} \times \mathbf{r}') = \mathbf{r}' \times \mathbf{r}' + \mathbf{r} \times \mathbf{r}'' = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Thus, $\mathbf{r} \times \mathbf{r}'$ is a constant vector which we will denote by $\mathbf{L}$.

88.
$$\begin{aligned}\frac{d}{dt}\left[\frac{\mathbf{r}'}{GM} \times \mathbf{L} - \frac{\mathbf{r}}{r}\right] &= \frac{1}{GM}[\mathbf{r}' \times \mathbf{0} + \mathbf{r}'' \times \mathbf{L}] - \frac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\}\\ &= \frac{1}{GM}\left[\mathbf{0} + \left(\frac{-GM\mathbf{r}}{r^3}\right) \times [\mathbf{r} \times \mathbf{r}']\right] - \frac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\}\\ &= -\frac{\mathbf{r}}{r^3} \times [\mathbf{r} \times \mathbf{r}'] - \frac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\}\\ &= \frac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r} - [\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\} = \mathbf{0}\end{aligned}$$

Thus, $\left(\dfrac{\mathbf{r}'}{GM}\right) \times \mathbf{L} - \left(\dfrac{\mathbf{r}}{r}\right)$ is a constant vector which we will denote by $\mathbf{e}$.

90. $\|\mathbf{L}\| = \|\mathbf{r} \times \mathbf{r}'\|$

Let: $\mathbf{r} = r(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$

$$\mathbf{r}' = r(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})\frac{d\theta}{dt} \qquad \left(\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{d\theta} \cdot \frac{d\theta}{dt}\right)$$

Then: $\mathbf{r} \times \mathbf{r}' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r\cos\theta & r\sin\theta & 0 \\ -r\sin\theta\dfrac{d\theta}{dt} & r\cos\theta\dfrac{d\theta}{dt} & 0 \end{vmatrix}$

$$= r^2\frac{d\theta}{dt}\mathbf{k} \text{ and } \|\mathbf{L}\| = \|\mathbf{r} \times \mathbf{r}'\| = r^2\frac{d\theta}{dt}.$$

92. Let P denote the period. Then

$$A = \int_0^P \frac{dA}{dt}\,dt = \frac{1}{2}\|\mathbf{L}\|P.$$

Also, the area of an ellipse is πab where $2a$ and $2b$ are the lengths of the major and minor axes.

$$\pi ab = \frac{1}{2}\|\mathbf{L}\|P$$

$$P = \frac{2\pi ab}{\|\mathbf{L}\|}$$

$$\begin{aligned}P^2 &= \frac{4\pi^2a^2}{\|\mathbf{L}\|^2}(a^2 - c^2) = \frac{4\pi^2a^2}{\|\mathbf{L}\|^2}a^2(1 - e^2)\\ &= \frac{4\pi^2a^4}{\|\mathbf{L}\|^2}\left(\frac{ed}{a}\right) = \frac{4\pi^2ed}{\|\mathbf{L}\|^2}a^3\\ &= \frac{4\pi^2(\|\mathbf{L}\|^2/GM)}{\|\mathbf{L}\|^2}a^3 = \frac{4\pi^2}{GM}a^3 = Ka^3\end{aligned}$$

Review Exercises for Chapter 11

2. $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + \dfrac{1}{t - 4}\mathbf{j} + \mathbf{k}$

(a) Domain: $[0, 4)$ and $(4, \infty)$

(b) Continuous except at $t = 4$

4. $\mathbf{r}(t) = (2t + 1)\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$

(a) Domain: $(-\infty, \infty)$

(b) Continuous for all t

6. (a) $\mathbf{r}(0) = 3\mathbf{i} + \mathbf{j}$

(b) $\mathbf{r}\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}\mathbf{k}$

(c) $\mathbf{r}(s - \pi) = 3\cos(s - \pi)\mathbf{i} + (1 - \sin(s - \pi))\mathbf{j} - (s - \pi)\mathbf{k}$

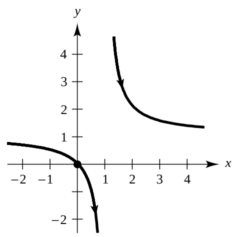
(d) $\mathbf{r}(\pi + \Delta t) - \mathbf{r}(\pi) = (3\cos(\pi + \Delta t)\mathbf{i} + (1 - \sin(\pi + \Delta t))\mathbf{j} - (\pi + \Delta t)\mathbf{k}) - (-3\mathbf{i} + \mathbf{j} - \pi\mathbf{k})$

$= (-3\cos\Delta t + 3)\mathbf{i} + \sin\Delta t - \Delta t\mathbf{k}$

8. $\mathbf{r}(t) = t\mathbf{i} + \dfrac{t}{t-1}\mathbf{j}$

$x(t) = t,\ y(t) = \dfrac{t}{t-1}$

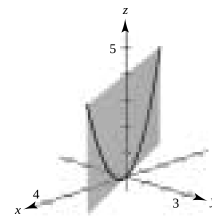
$y = \dfrac{x}{x-1}$



10. $\mathbf{r}(t) = 2t\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$

$x = 2t,\ y = t,\ z = t^2,$
$y = \frac{1}{2}x,\ z = y^2$

t	0	1	-1	2
x	0	2	-2	4
y	0	1	-1	2
z	0	1	1	4

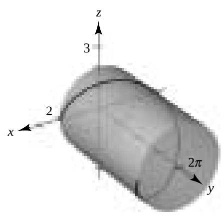


12. $\mathbf{r}(t) = 2\cos t\mathbf{i} + t\mathbf{j} + 2\sin t\mathbf{k}$

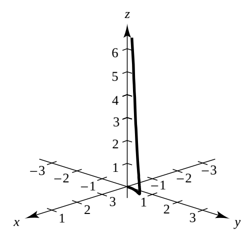
$x = 2\cos t,\ y = t,\ z = 2\sin t$

$x^2 + z^2 = 4$

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
x	2	0	-2	0
y	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
z	0	2	0	-2



14. $\mathbf{r}(t) = \frac{1}{2}t\mathbf{i} + \sqrt{t}\mathbf{j} + \frac{1}{4}t^3\mathbf{k}$



16. One possible answer is:

$\mathbf{r}_1(t) = 4t\mathbf{i},\qquad 0 \le t \le 1$

$\mathbf{r}_2(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j},\qquad 0 \le t \le \dfrac{\pi}{2}$

$\mathbf{r}_3(t) = (4 - t)\mathbf{j},\qquad 0 \le t \le 4$

18. The x- and y-components are $2\cos t$ and $2\sin t$. At

$$t = \frac{3\pi}{2},$$

the staircase has made $\frac{3}{4}$ of a revolution and is 2 meters high. Thus, one answer is

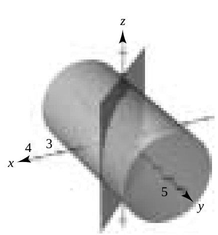
$$\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + \frac{4}{3\pi}t\mathbf{k}.$$

20. $x^2 + z^2 = 4,\ x - y = 0,\ t = x$

$x = t,\ y = t,\ z = \pm\sqrt{4 - t^2}$

$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{4 - t^2}\mathbf{k}$

$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} - \sqrt{4 - t^2}\mathbf{k}$



22. $\displaystyle\lim_{t\to 0}\left(\frac{\sin 2t}{t}\mathbf{i} + e^{-t}\mathbf{j} + e^t\mathbf{k}\right) = \left(\lim_{t\to 0}\frac{2\cos 2t}{1}\right)\mathbf{i} + \mathbf{j} + \mathbf{k}$

$= 2\mathbf{i} + \mathbf{j} + \mathbf{k}$

24. $\mathbf{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + t\mathbf{k}, \ \mathbf{u}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + \frac{1}{t}\mathbf{k}$

(a) $\mathbf{r}'(t) = \cos t\mathbf{i} - \sin t\mathbf{j} + \mathbf{k}$

(b) $\mathbf{r}''(t) = -\sin t\mathbf{i} - \cos t\mathbf{j}$

(c) $\mathbf{r}(t) \cdot \mathbf{u}(t) = 2$

$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = 0$

(d) $\mathbf{u}(t) - 2\mathbf{r}(t) = -\sin t\mathbf{i} - \cos t\mathbf{j} + \left(\frac{1}{t} - 2t\right)\mathbf{k}$

$D_t[\mathbf{u}(t) - 2\mathbf{r}(t)] = -\cos t\mathbf{i} + \sin t\mathbf{j} + \left(-\frac{1}{t^2} - 2\right)\mathbf{k}$

(e) $\|\mathbf{r}(t)\| = \sqrt{1 + t^2}$

$D_t[\|\mathbf{r}(t)\|] = \frac{t}{\sqrt{1 + t^2}}$

(f) $\mathbf{r}(t) \times \mathbf{u}(t) = \left(\frac{1}{t}\cos t - t\cos t\right)\mathbf{i} - \left(\frac{1}{t}\sin t - t\sin t\right)\mathbf{j}$

$D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \left(-\frac{1}{t}\sin t - \frac{1}{t^2}\cos t + t\sin t - \cos t\right)\mathbf{i} - \left(\frac{1}{t}\cos t - \frac{1}{t^2}\sin t - t\cos t - \sin t\right)\mathbf{j}$

26. The graph of **u** is parallel to the yz-plane.

28. $\int (\ln t\mathbf{i} + t\ln t\mathbf{j} + \mathbf{k})\, dt = (t\ln t - t)\mathbf{i} + \frac{t^2}{4}(-1 + 2\ln t)\mathbf{j} + t\mathbf{k} + \mathbf{C}$

30. $\int (t\mathbf{j} + t^2\mathbf{k}) \times (\mathbf{i} + t\mathbf{j} + t\mathbf{k})\, dt = \int [(t^2 - t^3)\mathbf{i} + t^2\mathbf{j} - t\mathbf{k}]\, dt = \left(\frac{t^3}{3} - \frac{t^4}{4}\right)\mathbf{i} + \frac{t^3}{3}\mathbf{j} - \frac{t^2}{2}\mathbf{k} + \mathbf{C}$

32. $\mathbf{r}(t) = \int (\sec t\mathbf{i} + \tan t\mathbf{j} + t^2\mathbf{k})\, dt = \ln|\sec t + \tan t|\mathbf{i} - \ln|\cos t|\mathbf{j} + \frac{t^3}{3}\mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{C} = 3\mathbf{k}$

$\mathbf{r}(t) = \ln|\sec t + \tan t|\mathbf{i} - \ln|\cos t|\mathbf{j} + \left(\frac{t^3}{3} + 3\right)\mathbf{k}$

34. $\int_0^1 (\sqrt{t}\mathbf{j} + t\sin t\,\mathbf{k})\, dt = \left[\frac{2}{3}t^{3/2}\mathbf{j} + (\sin t - t\cos t)\mathbf{k}\right]_0^1 = \frac{2}{3}\mathbf{j} + (\sin 1 - \cos 1)\mathbf{k}$

36. $\int_{-1}^1 (t^3\mathbf{i} - \arcsin t\mathbf{j} - t^2\mathbf{k})\, dt = \left[\frac{t^4}{4}\mathbf{i} - \left(t\arcsin t + \sqrt{1 - t^2}\right)\mathbf{j} - \frac{t^3}{3}\mathbf{k}\right]_{-1}^1$

$= -\frac{2}{3}\mathbf{k}$

38. $\mathbf{r}(t) = \langle t, -\tan t, e^t\rangle$

$\mathbf{r}'(t) = \mathbf{v}(t) = \langle 1, -\sec^2 t, e^t\rangle$

$\|\mathbf{v}(t)\| = \sqrt{1 + \sec^4 t + e^{2t}}$

$\mathbf{r}''(t) = \mathbf{a}(t) = \langle 0, -2\sec^2 t \cdot \tan t, e^t\rangle$

40. $\mathbf{r}(t) = \langle 3\cosh t, \sinh t, -2t\rangle, \ t_0 = 0$

$\mathbf{r}'(t) = \langle 3\sinh t, \cosh t, -2\rangle$

$\mathbf{r}'(0) = \langle 0, 1, -2\rangle$ direction numbers

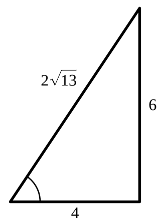
Since $\mathbf{r}(0) = \langle 3, 0, 0\rangle$, the parametric equations are $x = 3$, $y = t$, $z = -2t$.

$\mathbf{r}(t_0 + 0.1) = \mathbf{r}(0.1) \approx \langle 3, 0.1, -0.2\rangle$

42. Range $= 4 = \dfrac{v_0^2}{16}\sin\theta\cos\theta$

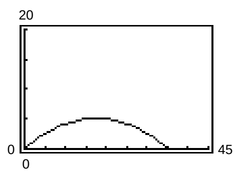
$$= \frac{v_0^2}{16}\cdot\frac{6}{2\sqrt{13}}\cdot\frac{4}{2\sqrt{13}} = \frac{3v_0^2}{104}$$

$$\frac{416}{3} = v_0^2 \implies v_0 \approx 11.776 \text{ ft/sec}$$



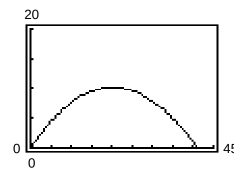
44. $\mathbf{r}(t) = [(v_0\cos\theta)t]\mathbf{i} + \left[(v_0\sin\theta)t - \frac{1}{2}(9.8)t^2\right]\mathbf{j}$

(a) $\mathbf{r}(t) = [(20\cos 30°)t]\mathbf{i} + [(20\sin 30°)t - 4.9t^2]\mathbf{j}$



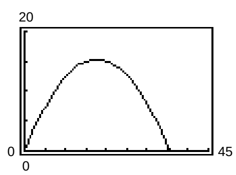
Maximum height ≈ 5.1 m; Range ≈ 35.3 m

(b) $\mathbf{r}(t) = [(20\cos 45°)t]\mathbf{i} + [(20\sin 45°)t - 4.9t^2]\mathbf{j}$



Maximum height ≈ 10.2 m; Range ≈ 40.8 m

(c) $\mathbf{r}(t) = [(20\cos 60°)t]\mathbf{i} + [(20\sin 60°)t - 4.9t^2]\mathbf{j}$



Maximum height ≈ 15.3 m; Range ≈ 35.3 m

(Note that 45° gives the longest range)

46. $\mathbf{r}(t) = (1 + 4t)\mathbf{i} + (2 - 3t)\mathbf{j}$

$\mathbf{v}(t) = 4\mathbf{i} - 3\mathbf{j}$

$\|\mathbf{v}\| = 5$

$\mathbf{a}(t) = \mathbf{0}$

$\mathbf{T}(t) = \dfrac{1}{5}(4\mathbf{i} - 3\mathbf{j})$

$\mathbf{N}(t)$ does not exist

$\mathbf{a}\cdot\mathbf{T} = 0$

$\mathbf{a}\cdot\mathbf{N}$ does not exist

48. $\mathbf{r}(t) = 2(t+1)\mathbf{i} + \dfrac{2}{t+1}\mathbf{j}$

$$\mathbf{v}(t) = 2\mathbf{i} - \frac{2}{(t+1)^2}\mathbf{j}$$

$$\|\mathbf{v}(t)\| = \frac{2\sqrt{(t+1)^4+1}}{(t+1)^2}$$

$$\mathbf{a}(t) = \frac{4}{(t+1)^3}\mathbf{j}$$

$$\mathbf{T}(t) = \frac{(t+1)^2\mathbf{i} - \mathbf{j}}{\sqrt{(t+1)^4+1}}$$

$$\mathbf{N}(t) = \frac{\mathbf{i} + (t+1)^2\mathbf{j}}{\sqrt{(t+1)^4+1}}$$

$$\mathbf{a}\cdot\mathbf{T} = \frac{-4}{(t+1)^3\sqrt{(t+1)^4+1}}$$

$$\mathbf{a}\cdot\mathbf{N} = \frac{4(t+1)^2}{(t+1)^3\sqrt{(t+1)^4+1}}$$

$$= \frac{4}{(t+1)\sqrt{(t+1)^4+1}}$$

50.
$$\mathbf{r}(t) = t\cos t\mathbf{i} + t\sin t\mathbf{j}$$
$$\mathbf{v}(t) = \mathbf{r}'(t) = (-t\sin t + \cos t)\mathbf{i} + (t\cos t + \sin t)\mathbf{j}$$
$$\|\mathbf{v}(t)\| = \text{speed} = \sqrt{(-t\sin t + \cos t)^2 + (t\cos t + \sin t)^2} = \sqrt{t^2+1}$$
$$\mathbf{a}(t) = \mathbf{r}''(t) = (-t\cos t - 2\sin t)\mathbf{i} + (-t\sin t + 2\cos t)\mathbf{j}$$
$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{(-t\sin t + \cos t)\mathbf{i} + (t\cos t + \sin t)\mathbf{j}}{\sqrt{t^2+1}}$$
$$\mathbf{N}(t) = \frac{-(t\cos t + \sin t)\mathbf{i} + (-t\sin t + \cos t)\mathbf{j}}{\sqrt{t^2+1}}$$
$$\mathbf{a}(t)\cdot\mathbf{T}(t) = \frac{t}{\sqrt{t^2+1}}$$
$$\mathbf{a}(t)\cdot\mathbf{N}(t) = \frac{t^2+2}{\sqrt{t^2+1}}$$

52.
$$\mathbf{r}(t) = (t-1)\mathbf{i} + t\mathbf{j} + \frac{1}{t}\mathbf{k}$$
$$\mathbf{v}(t) = \mathbf{i} + \mathbf{j} - \frac{1}{t^2}\mathbf{k}$$
$$\|\mathbf{v}(t)\| = \frac{\sqrt{2t^4+1}}{t^2}$$
$$\mathbf{a}(t) = \frac{2}{t^3}\mathbf{k}$$
$$\mathbf{T}(t) = \frac{t^2\mathbf{i} + t^2\mathbf{j} - \mathbf{k}}{\sqrt{2t^4+1}}$$
$$\mathbf{N}(t) = \frac{\mathbf{i} + \mathbf{j} + 2t^2\mathbf{k}}{\sqrt{2}\sqrt{2t^4+1}}$$
$$\mathbf{a}\cdot\mathbf{T} = \frac{-2}{t^3\sqrt{2t^4+1}}$$
$$\mathbf{a}\cdot\mathbf{N} = \frac{4}{t\sqrt{2}\sqrt{2t^4+1}}$$

54. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{2}{3}t^3\mathbf{k},\ x = t,\ y = t^2,\ z = \frac{2}{3}t^3$

When $t = 2,\ x = 2,\ y = 4,\ z = \frac{16}{3}$.

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 2t^2\mathbf{k}$$

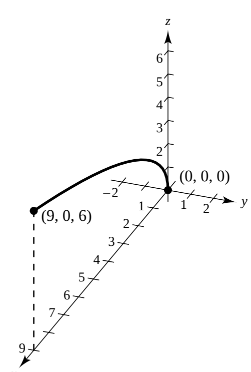
Direction numbers when $t = 2,\ a = 1,\ b = 4,\ c = 8$

$$x = t + 2,\ y = 4t + 4,\ z = 8t + \frac{16}{3}$$

56. Factor of 4

58. $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{k},\ 0 \le t \le 3$

$$\mathbf{r}'(t) = 2t\mathbf{i} + 2\mathbf{k}$$
$$s = \int_a^b \|\mathbf{r}'(t)\|\,dt = \int_0^3 \sqrt{4t^2+4}\,dt$$
$$= \left[\ln\left|\sqrt{t^2+1} + t\right| + t\sqrt{t^2+1}\right]_0^3$$
$$= \ln\left(\sqrt{10}+3\right) + 3\sqrt{10} \approx 11.3053$$

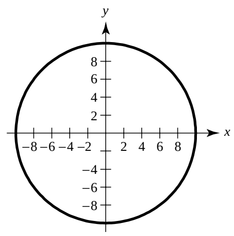


60. $\mathbf{r}(t) = 10\cos t\mathbf{i} + 10\sin t\mathbf{j}$

$\mathbf{r}'(t) = -10\sin t\mathbf{i} + 10\cos t\mathbf{j}$

$\|\mathbf{r}'(t)\| = 10$

$$s = \int_0^{2\pi} 10\,dt = 20\pi$$

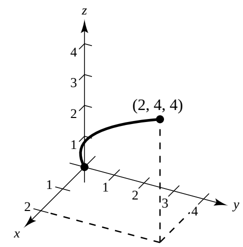


62. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 2t\mathbf{k},\ 0 \le t \le 2$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 2\mathbf{k},\ \|\mathbf{r}'(t)\| = \sqrt{5 + 4t^2}$

$$s = \int_a^b \|\mathbf{r}'(t)\|\,dt = \int_0^2 \sqrt{5 + 4t^2}\,dt$$

$$= \sqrt{21} + \frac{5}{4}\ln 5 - \frac{5}{4}\ln\left(\sqrt{105} - 4\sqrt{5}\right) \approx 6.2638$$



64. $\mathbf{r}(t) = \langle 2(\sin t - t\cos t), 2(\cos t + t\sin t), t\rangle,\ 0 \le t \le \dfrac{\pi}{2}$

$x'(t) = \langle 2t\sin t, 2t\cos t, 1\rangle,\ \|\mathbf{r}'(t)\| = \sqrt{4t^2 + 1}$

$$s = \int_a^b \|\mathbf{r}'(t)\|\,dt = \int_0^{\pi/2} \sqrt{4t^2 + 1}\,dt$$

$$= \sqrt{17} - \frac{1}{4}\ln\left(\sqrt{17} - 4\right) \approx 4.6468$$

66. $\mathbf{r}(t) = e^t\sin t\mathbf{i} + e^t\cos t\mathbf{k},\ 0 \le t \le \pi$

$\mathbf{r}'(t) = (e^t\cos t + e^t\sin t)\mathbf{i} + (-e^t\sin t + e^t\cos t)\mathbf{k}$

$$\|\mathbf{r}'(t)\| = \sqrt{(e^t\cos t + e^t\sin t)^2 + (-e^t\sin t + e^t\cos t)^2} = \sqrt{2}e^t$$

$$s = \int_0^{\pi} \|\mathbf{r}'(t)\|\,dt = \sqrt{2}\int_0^{\pi} e^t\,dt = \left[\sqrt{2}e^t\right]_0^{\pi} = \sqrt{2}(e^{\pi} - 1)$$

68. $\mathbf{r}(t) = 2\sqrt{t}\mathbf{i} + 3t\mathbf{j}$

$$\mathbf{r}'(t) = \frac{1}{\sqrt{t}}\mathbf{i} + 3\mathbf{j},\ \|\mathbf{r}'(t)\| = \sqrt{\frac{1}{t} + 9} = \sqrt{\frac{1 + 9t}{t}}$$

$$\mathbf{r}''(t) = -\frac{1}{2}t^{-3/2}\mathbf{i}$$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{t}} & 3 & 0 \\ -\frac{1}{2}t^{-3/2} & 0 & 0 \end{vmatrix} = \frac{3}{2}t^{-3/2}\mathbf{k};\ \|\mathbf{r}' \times \mathbf{r}''\| = \frac{3}{2t^{3/2}}$$

$$K = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{3/2t^{3/2}}{(1 + 9t)^{3/2}/t^{3/2}} = \frac{3}{2(1 + 9t)^{3/2}}$$

70. $\mathbf{r}(t) = 2t\mathbf{i} + 5\cos t\mathbf{j} + 5\sin t\mathbf{k}$

$\mathbf{r}'(t) = 2\mathbf{i} - 5\sin t\mathbf{j} + 5\cos t\mathbf{k},\ \|\mathbf{r}'(t)\| = \sqrt{29}$

$\mathbf{r}''(t) = 5\cos t\mathbf{j} - 5\sin t\mathbf{k}$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -5\sin t & 5\cos t \\ 0 & -5\cos t & -5\sin t \end{vmatrix} = 25\mathbf{i} + 10\sin t\mathbf{j} - 10\cos t\mathbf{k}$$

$\|\mathbf{r}' \times \mathbf{r}''\| = \sqrt{725}$

$$K = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{\sqrt{725}}{(29)^{3/2}} = \frac{\sqrt{25 \cdot 29}}{29\sqrt{29}} = \frac{5}{29}$$