86. $\mathbf{F} = m\mathbf{a} \implies m\mathbf{a} = \dfrac{-GmM}{r^3}\mathbf{r}$

$$\mathbf{a} = -\frac{GM}{r^3}\mathbf{r}$$

Since $\mathbf{r}$ is a constant multiple of $\mathbf{a}$, they are parallel. Since $\mathbf{a} = \mathbf{r}''$ is parallel to $\mathbf{r}$, $\mathbf{r} \times \mathbf{r}'' = \mathbf{0}$. Also,

$$\left(\frac{d}{dt}\right)(\mathbf{r} \times \mathbf{r}') = \mathbf{r}' \times \mathbf{r}' + \mathbf{r} \times \mathbf{r}'' = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Thus, $\mathbf{r} \times \mathbf{r}'$ is a constant vector which we will denote by $\mathbf{L}$.

88.
$$\begin{aligned}
\frac{d}{dt}\left[\frac{\mathbf{r}'}{GM} \times \mathbf{L} - \frac{\mathbf{r}}{r}\right] &= \frac{1}{GM}[\mathbf{r}' \times \mathbf{0} + \mathbf{r}'' \times \mathbf{L}] - \frac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\} \\
&= \frac{1}{GM}\left[\mathbf{0} + \left(\frac{-GM\mathbf{r}}{r^3}\right) \times [\mathbf{r} \times \mathbf{r}']\right] - \frac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\} \\
&= -\frac{\mathbf{r}}{r^3} \times [\mathbf{r} \times \mathbf{r}'] - \frac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\} \\
&= \frac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r} - [\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\} = \mathbf{0}
\end{aligned}$$

Thus, $\left(\dfrac{\mathbf{r}'}{GM}\right) \times \mathbf{L} - \left(\dfrac{\mathbf{r}}{r}\right)$ is a constant vector which we will denote by $\mathbf{e}$.

90. $\|\mathbf{L}\| = \|\mathbf{r} \times \mathbf{r}'\|$

Let: $\mathbf{r} = r(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$

$$\mathbf{r}' = r(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})\frac{d\theta}{dt} \qquad \left(\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{d\theta} \cdot \frac{d\theta}{dt}\right)$$

Then: $\mathbf{r} \times \mathbf{r}' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r\cos\theta & r\sin\theta & 0 \\ -r\sin\theta\dfrac{d\theta}{dt} & r\cos\theta\dfrac{d\theta}{dt} & 0 \end{vmatrix}$

$$= r^2\frac{d\theta}{dt}\mathbf{k} \text{ and } \|\mathbf{L}\| = \|\mathbf{r} \times \mathbf{r}'\| = r^2\frac{d\theta}{dt}.$$

92. Let P denote the period. Then

$$A = \int_0^P \frac{dA}{dt}\,dt = \frac{1}{2}\|\mathbf{L}\|P.$$

Also, the area of an ellipse is πab where $2a$ and $2b$ are the lengths of the major and minor axes.

$$\pi ab = \frac{1}{2}\|\mathbf{L}\|P$$

$$P = \frac{2\pi ab}{\|\mathbf{L}\|}$$

$$\begin{aligned}
P^2 &= \frac{4\pi^2a^2}{\|\mathbf{L}\|^2}(a^2 - c^2) = \frac{4\pi^2a^2}{\|\mathbf{L}\|^2}a^2(1 - e^2) \\
&= \frac{4\pi^2a^4}{\|\mathbf{L}\|^2}\left(\frac{ed}{a}\right) = \frac{4\pi^2 ed}{\|\mathbf{L}\|^2}a^3 \\
&= \frac{4\pi^2(\|\mathbf{L}\|^2/GM)}{\|\mathbf{L}\|^2}a^3 = \frac{4\pi^2}{GM}a^3 = Ka^3
\end{aligned}$$

Review Exercises for Chapter 11

2. $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + \dfrac{1}{t-4}\mathbf{j} + \mathbf{k}$

(a) Domain: $[0, 4)$ and $(4, \infty)$

(b) Continuous except at $t = 4$

4. $\mathbf{r}(t) = (2t + 1)\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$

(a) Domain: $(-\infty, \infty)$

(b) Continuous for all t

6. (a) $\mathbf{r}(0) = 3\mathbf{i} + \mathbf{j}$

(b) $\mathbf{r}\left(\dfrac{\pi}{2}\right) = -\dfrac{\pi}{2}\mathbf{k}$

(c) $\mathbf{r}(s - \pi) = 3\cos(s - \pi)\mathbf{i} + (1 - \sin(s - \pi))\mathbf{j} - (s - \pi)\mathbf{k}$

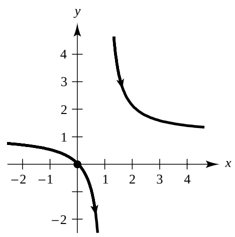
(d) $\mathbf{r}(\pi + \Delta t) - \mathbf{r}(\pi) = (3\cos(\pi + \Delta t)\mathbf{i} + (1 - \sin(\pi + \Delta t))\mathbf{j} - (\pi + \Delta t)\mathbf{k}) - (-3\mathbf{i} + \mathbf{j} - \pi\mathbf{k})$

$= (-3\cos\Delta t + 3)\mathbf{i} + \sin\Delta t - \Delta t\mathbf{k}$

8. $\mathbf{r}(t) = t\mathbf{i} + \dfrac{t}{t-1}\mathbf{j}$

$x(t) = t,\ y(t) = \dfrac{t}{t-1}$

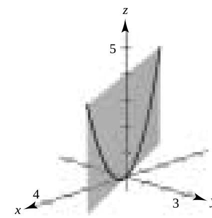
$y = \dfrac{x}{x-1}$



10. $\mathbf{r}(t) = 2t\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$

$x = 2t,\ y = t,\ z = t^2,$
$y = \frac{1}{2}x,\ z = y^2$

t	0	1	-1	2
x	0	2	-2	4
y	0	1	-1	2
z	0	1	1	4

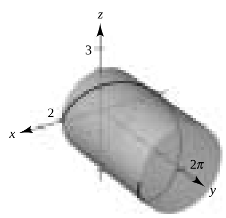


12. $\mathbf{r}(t) = 2\cos t\mathbf{i} + t\mathbf{j} + 2\sin t\mathbf{k}$

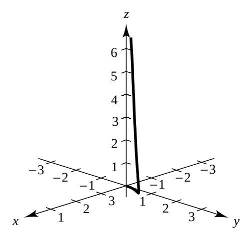
$x = 2\cos t,\ y = t,\ z = 2\sin t$

$x^2 + z^2 = 4$

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
x	2	0	-2	0
y	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
z	0	2	0	-2



14. $\mathbf{r}(t) = \frac{1}{2}t\mathbf{i} + \sqrt{t}\mathbf{j} + \frac{1}{4}t^3\mathbf{k}$



16. One possible answer is:

$\mathbf{r}_1(t) = 4t\mathbf{i}, \quad 0 \le t \le 1$

$\mathbf{r}_2(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}, \quad 0 \le t \le \dfrac{\pi}{2}$

$\mathbf{r}_3(t) = (4 - t)\mathbf{j}, \quad 0 \le t \le 4$

18. The x- and y-components are $2\cos t$ and $2\sin t$. At

$t = \dfrac{3\pi}{2},$

the staircase has made $\frac{3}{4}$ of a revolution and is 2 meters high. Thus, one answer is

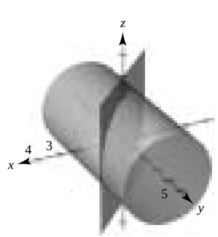
$\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + \dfrac{4}{3\pi}t\mathbf{k}.$

20. $x^2 + z^2 = 4,\ x - y = 0,\ t = x$

$x = t,\ y = t,\ z = \pm\sqrt{4 - t^2}$

$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{4 - t^2}\mathbf{k}$

$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} - \sqrt{4 - t^2}\mathbf{k}$



22. $\lim\limits_{t\to 0}\left(\dfrac{\sin 2t}{t}\mathbf{i} + e^{-t}\mathbf{j} + e^t\mathbf{k}\right) = \left(\lim\limits_{t\to 0}\dfrac{2\cos 2t}{1}\right)\mathbf{i} + \mathbf{j} + \mathbf{k}$

$= 2\mathbf{i} + \mathbf{j} + \mathbf{k}$

24. $\mathbf{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + t\mathbf{k},\ \mathbf{u}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + \frac{1}{t}\mathbf{k}$

(a) $\mathbf{r}'(t) = \cos t\mathbf{i} - \sin t\mathbf{j} + \mathbf{k}$

(b) $\mathbf{r}''(t) = -\sin t\mathbf{i} - \cos t\mathbf{j}$

(c) $\mathbf{r}(t) \cdot \mathbf{u}(t) = 2$

$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = 0$

(d) $\mathbf{u}(t) - 2\mathbf{r}(t) = -\sin t\mathbf{i} - \cos t\mathbf{j} + \left(\frac{1}{t} - 2t\right)\mathbf{k}$

$D_t[\mathbf{u}(t) - 2\mathbf{r}(t)] = -\cos t\mathbf{i} + \sin t\mathbf{j} + \left(-\frac{1}{t^2} - 2\right)\mathbf{k}$

(e) $\|\mathbf{r}(t)\| = \sqrt{1 + t^2}$

$D_t[\|\mathbf{r}(t)\|] = \frac{t}{\sqrt{1 + t^2}}$

(f) $\mathbf{r}(t) \times \mathbf{u}(t) = \left(\frac{1}{t}\cos t - t\cos t\right)\mathbf{i} - \left(\frac{1}{t}\sin t - t\sin t\right)\mathbf{j}$

$D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \left(-\frac{1}{t}\sin t - \frac{1}{t^2}\cos t + t\sin t - \cos t\right)\mathbf{i} - \left(\frac{1}{t}\cos t - \frac{1}{t^2}\sin t - t\cos t - \sin t\right)\mathbf{j}$

26. The graph of **u** is parallel to the *yz*-plane.

28. $\int (\ln t\mathbf{i} + t\ln t\mathbf{j} + \mathbf{k})\,dt = (t\ln t - t)\mathbf{i} + \frac{t^2}{4}(-1 + 2\ln t)\mathbf{j} + t\mathbf{k} + \mathbf{C}$

30. $\int (t\mathbf{j} + t^2\mathbf{k}) \times (\mathbf{i} + t\mathbf{j} + t\mathbf{k})\,dt = \int [(t^2 - t^3)\mathbf{i} + t^2\mathbf{j} - t\mathbf{k}]\,dt = \left(\frac{t^3}{3} - \frac{t^4}{4}\right)\mathbf{i} + \frac{t^3}{3}\mathbf{j} - \frac{t^2}{2}\mathbf{k} + \mathbf{C}$

32. $\mathbf{r}(t) = \int (\sec t\mathbf{i} + \tan t\mathbf{j} + t^2\mathbf{k})\,dt = \ln|\sec t + \tan t|\mathbf{i} - \ln|\cos t|\mathbf{j} + \frac{t^3}{3}\mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{C} = 3\mathbf{k}$

$\mathbf{r}(t) = \ln|\sec t + \tan t|\mathbf{i} - \ln|\cos t|\mathbf{j} + \left(\frac{t^3}{3} + 3\right)\mathbf{k}$

34. $\int_0^1 (\sqrt{t}\mathbf{j} + t\sin t\,\mathbf{k})\,dt = \left[\frac{2}{3}t^{3/2}\mathbf{j} + (\sin t - t\cos t)\mathbf{k}\right]_0^1 = \frac{2}{3}\mathbf{j} + (\sin 1 - \cos 1)\mathbf{k}$

36. $\int_{-1}^1 (t^3\mathbf{i} - \arcsin t\mathbf{j} - t^2\mathbf{k})\,dt = \left[\frac{t^4}{4}\mathbf{i} - \left(t\arcsin t + \sqrt{1 - t^2}\right)\mathbf{j} - \frac{t^3}{3}\mathbf{k}\right]_{-1}^1$

$= -\frac{2}{3}\mathbf{k}$

38. $\mathbf{r}(t) = \langle t, -\tan t, e^t\rangle$

$\mathbf{r}'(t) = \mathbf{v}(t) = \langle 1, -\sec^2 t, e^t\rangle$

$\|\mathbf{v}(t)\| = \sqrt{1 + \sec^4 t + e^{2t}}$

$\mathbf{r}''(t) = \mathbf{a}(t) = \langle 0, -2\sec^2 t \cdot \tan t, e^t\rangle$

40. $\mathbf{r}(t) = \langle 3\cosh t, \sinh t, -2t\rangle,\ t_0 = 0$

$\mathbf{r}'(t) = \langle 3\sinh t, \cosh t, -2\rangle$

$\mathbf{r}'(0) = \langle 0, 1, -2\rangle$ direction numbers

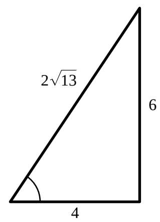
Since $\mathbf{r}(0) = \langle 3, 0, 0\rangle$, the parametric equations are $x = 3$, $y = t$, $z = -2t$.

$\mathbf{r}(t_0 + 0.1) = \mathbf{r}(0.1) \approx \langle 3, 0.1, -0.2\rangle$

42. Range $= 4 = \dfrac{v_0^2}{16}\sin\theta\cos\theta$

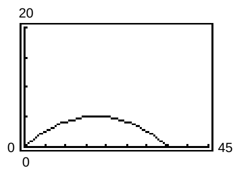
$$= \frac{v_0^2}{16}\cdot\frac{6}{2\sqrt{13}}\cdot\frac{4}{2\sqrt{13}} = \frac{3v_0^2}{104}$$

$$\frac{416}{3} = v_0^2 \implies v_0 \approx 11.776 \text{ ft/sec}$$



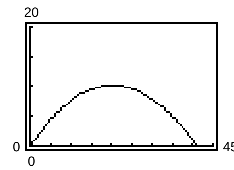
44. $\mathbf{r}(t) = [(v_0\cos\theta)t]\mathbf{i} + \left[(v_0\sin\theta)t - \frac{1}{2}(9.8)t^2\right]\mathbf{j}$

(a) $\mathbf{r}(t) = [(20\cos 30°)t]\mathbf{i} + [(20\sin 30°)t - 4.9t^2]\mathbf{j}$



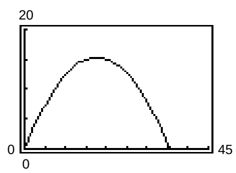
Maximum height ≈ 5.1 m; Range ≈ 35.3 m

(b) $\mathbf{r}(t) = [(20\cos 45°)t]\mathbf{i} + [(20\sin 45°)t - 4.9t^2]\mathbf{j}$



Maximum height ≈ 10.2 m; Range ≈ 40.8 m

(c) $\mathbf{r}(t) = [(20\cos 60°)t]\mathbf{i} + [(20\sin 60°)t - 4.9t^2]\mathbf{j}$



Maximum height ≈ 15.3 m; Range ≈ 35.3 m

(Note that 45° gives the longest range)

46. $\mathbf{r}(t) = (1 + 4t)\mathbf{i} + (2 - 3t)\mathbf{j}$

$\mathbf{v}(t) = 4\mathbf{i} - 3\mathbf{j}$

$\|\mathbf{v}\| = 5$

$\mathbf{a}(t) = \mathbf{0}$

$\mathbf{T}(t) = \dfrac{1}{5}(4\mathbf{i} - 3\mathbf{j})$

$\mathbf{N}(t)$ does not exist

$\mathbf{a}\cdot\mathbf{T} = 0$

$\mathbf{a}\cdot\mathbf{N}$ does not exist

48. $\mathbf{r}(t) = 2(t+1)\mathbf{i} + \dfrac{2}{t+1}\mathbf{j}$

$$\mathbf{v}(t) = 2\mathbf{i} - \frac{2}{(t+1)^2}\mathbf{j}$$

$$\|\mathbf{v}(t)\| = \frac{2\sqrt{(t+1)^4+1}}{(t+1)^2}$$

$$\mathbf{a}(t) = \frac{4}{(t+1)^3}\mathbf{j}$$

$$\mathbf{T}(t) = \frac{(t+1)^2\mathbf{i} - \mathbf{j}}{\sqrt{(t+1)^4+1}}$$

$$\mathbf{N}(t) = \frac{\mathbf{i} + (t+1)^2\mathbf{j}}{\sqrt{(t+1)^4+1}}$$

$$\mathbf{a}\cdot\mathbf{T} = \frac{-4}{(t+1)^3\sqrt{(t+1)^4+1}}$$

$$\mathbf{a}\cdot\mathbf{N} = \frac{4(t+1)^2}{(t+1)^3\sqrt{(t+1)^4+1}}$$

$$= \frac{4}{(t+1)\sqrt{(t+1)^4+1}}$$

50. $\mathbf{r}(t) = t\cos t\mathbf{i} + t\sin t\mathbf{j}$

$\mathbf{v}(t) = \mathbf{r}'(t) = (-t\sin t + \cos t)\mathbf{i} + (t\cos t + \sin t)\mathbf{j}$

$\|\mathbf{v}(t)\| = \text{speed} = \sqrt{(-t\sin t + \cos t)^2 + (t\cos t + \sin t)^2} = \sqrt{t^2+1}$

$\mathbf{a}(t) = \mathbf{r}''(t) = (-t\cos t - 2\sin t)\mathbf{i} + (-t\sin t + 2\cos t)\mathbf{j}$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{(-t\sin t + \cos t)\mathbf{i} + (t\cos t + \sin t)\mathbf{j}}{\sqrt{t^2+1}}$$

$$\mathbf{N}(t) = \frac{-(t\cos t + \sin t)\mathbf{i} + (-t\sin t + \cos t)\mathbf{j}}{\sqrt{t^2+1}}$$

$$\mathbf{a}(t)\cdot\mathbf{T}(t) = \frac{t}{\sqrt{t^2+1}}$$

$$\mathbf{a}(t)\cdot\mathbf{N}(t) = \frac{t^2+2}{\sqrt{t^2+1}}$$

52. $\mathbf{r}(t) = (t-1)\mathbf{i} + t\mathbf{j} + \frac{1}{t}\mathbf{k}$

$$\mathbf{v}(t) = \mathbf{i} + \mathbf{j} - \frac{1}{t^2}\mathbf{k}$$

$$\|\mathbf{v}(t)\| = \frac{\sqrt{2t^4+1}}{t^2}$$

$$\mathbf{a}(t) = \frac{2}{t^3}\mathbf{k}$$

$$\mathbf{T}(t) = \frac{t^2\mathbf{i} + t^2\mathbf{j} - \mathbf{k}}{\sqrt{2t^4+1}}$$

$$\mathbf{N}(t) = \frac{\mathbf{i} + \mathbf{j} + 2t^2\mathbf{k}}{\sqrt{2}\sqrt{2t^4+1}}$$

$$\mathbf{a}\cdot\mathbf{T} = \frac{-2}{t^3\sqrt{2t^4+1}}$$

$$\mathbf{a}\cdot\mathbf{N} = \frac{4}{t\sqrt{2}\sqrt{2t^4+1}}$$

54. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{2}{3}t^3\mathbf{k},\ x = t,\ y = t^2,\ z = \frac{2}{3}t^3$

When $t = 2,\ x = 2,\ y = 4,\ z = \frac{16}{3}$.

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 2t^2\mathbf{k}$

Direction numbers when $t = 2,\ a = 1,\ b = 4,\ c = 8$

$x = t + 2,\ y = 4t + 4,\ z = 8t + \frac{16}{3}$

56. Factor of 4

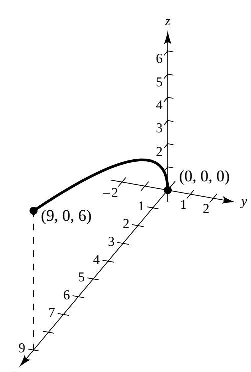
58. $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{k},\ 0 \le t \le 3$

$\mathbf{r}'(t) = 2t\mathbf{i} + 2\mathbf{k}$

$$s = \int_a^b \|\mathbf{r}'(t)\|\,dt = \int_0^3 \sqrt{4t^2+4}\,dt$$

$$= \left[\ln\left|\sqrt{t^2+1} + t\right| + t\sqrt{t^2+1}\right]_0^3$$

$$= \ln\left(\sqrt{10}+3\right) + 3\sqrt{10} \approx 11.3053$$

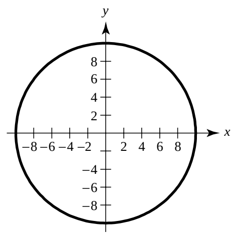


60. $\mathbf{r}(t) = 10\cos t\mathbf{i} + 10\sin t\mathbf{j}$

$\mathbf{r}'(t) = -10\sin t\mathbf{i} + 10\cos t\mathbf{j}$

$\|\mathbf{r}'(t)\| = 10$

$$s = \int_0^{2\pi} 10\,dt = 20\pi$$

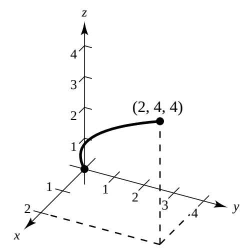


62. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 2t\mathbf{k},\ 0 \le t \le 2$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 2\mathbf{k},\ \|\mathbf{r}'(t)\| = \sqrt{5 + 4t^2}$

$$s = \int_a^b \|\mathbf{r}'(t)\|\,dt = \int_0^2 \sqrt{5 + 4t^2}\,dt$$

$$= \sqrt{21} + \frac{5}{4}\ln 5 - \frac{5}{4}\ln\left(\sqrt{105} - 4\sqrt{5}\right) \approx 6.2638$$



64. $\mathbf{r}(t) = \langle 2(\sin t - t\cos t), 2(\cos t + t\sin t), t\rangle,\ 0 \le t \le \dfrac{\pi}{2}$

$x'(t) = \langle 2t\sin t, 2t\cos t, 1\rangle,\ \|\mathbf{r}'(t)\| = \sqrt{4t^2 + 1}$

$$s = \int_a^b \|\mathbf{r}'(t)\|\,dt = \int_0^{\pi/2} \sqrt{4t^2 + 1}\,dt$$

$$= \sqrt{17} - \frac{1}{4}\ln\left(\sqrt{17} - 4\right) \approx 4.6468$$

66. $\mathbf{r}(t) = e^t\sin t\mathbf{i} + e^t\cos t\mathbf{k},\ \ 0 \le t \le \pi$

$\mathbf{r}'(t) = (e^t\cos t + e^t\sin t)\mathbf{i} + (-e^t\sin t + e^t\cos t)\mathbf{k}$

$\|\mathbf{r}'(t)\| = \sqrt{(e^t\cos t + e^t\sin t)^2 + (-e^t\sin t + e^t\cos t)^2}$

$= \sqrt{2}e^t$

$$s = \int_0^{\pi} \|\mathbf{r}'(t)\|\,dt$$

$$= \sqrt{2}\int_0^{\pi} e^t\,dt = \Big[\sqrt{2}e^t\Big]_0^{\pi} = \sqrt{2}(e^{\pi} - 1)$$

68. $\mathbf{r}(t) = 2\sqrt{t}\mathbf{i} + 3t\mathbf{j}$

$\mathbf{r}'(t) = \dfrac{1}{\sqrt{t}}\mathbf{i} + 3\mathbf{j},\ \|\mathbf{r}'(t)\| = \sqrt{\dfrac{1}{t} + 9} = \sqrt{\dfrac{1 + 9t}{t}}$

$\mathbf{r}''(t) = -\dfrac{1}{2}t^{-3/2}\mathbf{i}$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{t}} & 3 & 0 \\ -\frac{1}{2}t^{-3/2} & 0 & 0 \end{vmatrix} = \frac{3}{2}t^{-3/2}\mathbf{k};\ \|\mathbf{r}' \times \mathbf{r}''\| = \frac{3}{2t^{3/2}}$$

$$K = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{3/2t^{3/2}}{(1 + 9t)^{3/2}/t^{3/2}} = \frac{3}{2(1 + 9t)^{3/2}}$$

70. $\mathbf{r}(t) = 2t\mathbf{i} + 5\cos t\mathbf{j} + 5\sin t\mathbf{k}$

$\mathbf{r}'(t) = 2\mathbf{i} - 5\sin t\mathbf{j} + 5\cos t\mathbf{k},\ \|\mathbf{r}'(t)\| = \sqrt{29}$

$\mathbf{r}''(t) = 5\cos t\mathbf{j} - 5\sin t\mathbf{k}$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -5\sin t & 5\cos t \\ 0 & -5\cos t & -5\sin t \end{vmatrix} = 25\mathbf{i} + 10\sin t\mathbf{j} - 10\cos t\mathbf{k}$$

$\|\mathbf{r}' \times \mathbf{r}''\| = \sqrt{725}$

$$K = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{\sqrt{725}}{(29)^{3/2}} = \frac{\sqrt{25 \cdot 29}}{29\sqrt{29}} = \frac{5}{29}$$

72. $y = e^{-x/2}$

$y' = -\frac{1}{2}e^{-x/2}, y'' = \frac{1}{4}e^{-x/2}$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{\frac{1}{4}e^{-x/2}}{\left[1 + \frac{1}{4}e^{-x}\right]^{3/2}}$$

At $x = 0$, $K = \frac{1/4}{(5/4)^{3/2}} = \frac{2}{5^{3/2}} = \frac{2}{5\sqrt{5}} = \frac{2\sqrt{5}}{25}, r = \frac{5\sqrt{5}}{2}$.

74. $y = \tan x$

$y' = \sec^2 x$

$y'' = 2\sec^2 x \tan x$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{|2\sec^2 x \tan x|}{[1 + \sec^4 x]^{3/2}}$$

At $x = \frac{\pi}{4}$, $K = \frac{4}{5^{3/2}} = \frac{4}{5\sqrt{5}} = \frac{4\sqrt{5}}{25}$ and $r = \frac{5\sqrt{5}}{4}$.

Problem Solving for Chapter 11

2. $x^{2/3} + y^{2/3} = a^{2/3}$

$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$

$y' = \frac{-y^{1/3}}{x^{1/3}}$ Slope at $P(x, y)$.

$\mathbf{r}(t) = \cos^3 t\mathbf{i} + \sin^3 t\mathbf{j}$

$\mathbf{r}'(t) = -3\cos^2 t \sin t\mathbf{i} + 3\sin^2 t \cos t\mathbf{j}$

$\|\mathbf{r}'(t)\text{i} = |3\cos t \sin t|$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = -\cos t\mathbf{i} + \sin t\mathbf{j}$$

$\mathbf{T}'(t) = \sin t\mathbf{i} + \cos t\mathbf{j}$

$Q(0, 0, 0)$ origin

$P = (\cos^3 t, \sin^3 t, 0)$ on curve.

$$\overrightarrow{PQ} \times \mathbf{T} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos^3 t & \sin^3 t & 0 \\ -\cos t & \sin t & 0 \end{vmatrix}$$

$$= (\cos^3 t \sin t - \sin^3 t \cos t)\mathbf{k}$$

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{T}\|}{\|\mathbf{T}\|} = |3\cos t \sin t|$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{1}{|3\cos t \sin t|}$$

Thus, the radius of curvature, $\frac{1}{K}$, is three times the distance from the origin to the tangent line.

4. Bomb: $\mathbf{r}_1(t) = \langle 5000 + 400t, 3200 - 16t^2 \rangle$

Projectile: $\mathbf{r}_2(t) = \langle (v_0 \cos\theta)t, (v_0 \sin\theta)t - 16t^2 \rangle$

At 1600 feet: Bomb:

$3200 - 16t^2 = 1600 \implies t = 10$

Projectile will travel 5 seconds:

$5(v_0 \sin\theta) - 16(25) = 1600$

$v_0 \sin\theta = 400.$

Horizontal position:

At $t = 10$, bomb is at $5000 + 400(10) = 9000$.

At $t = 5$, projectile is at $(v_0 \cos\theta)5$.

Thus,

$5v_0 \cos\theta = 9000$

$v_0 \cos\theta = 1800.$

Combining, $\frac{v_0 \sin\theta}{v_0 \cos\theta} = \frac{400}{1800} \implies \tan\theta = \frac{2}{9} \implies \theta \approx 12.5°$.

$$v_0 = \frac{1800}{\cos\theta} \approx 1843.9 \text{ ft/sec}$$

6. $r = 1 - \cos\theta$

$r' = \sin\theta$

$$s(t) = \int_\pi^t \sqrt{(1-\cos\theta)^2 + \sin^2\theta}\, d\theta = \int_\pi^t \sqrt{2 - 2\cos\theta}\, d\theta$$

$$= \int_\pi^t 2\sin\frac{\theta}{2}\, d\theta = \left[-4\cos\frac{\theta}{2}\right]_\pi^t = -4\cos\frac{t}{2}$$

$$K = \frac{\left|2(r')^2 - rr'' + r^2\right|}{\left[(r')^2 + r^2\right]^{3/2}}$$

$$= \frac{\left|2\sin^2\theta - (1-\cos\theta)(\cos\theta) + (1-\cos\theta)^2\right|}{8\sin^3\dfrac{\theta}{2}}$$

$$= \frac{\left|3 - 3\cos\theta\right|}{8\sin^3\dfrac{\theta}{2}}$$

$$= \frac{3}{4}\,\frac{\sin^2\dfrac{\theta}{2}}{\sin^3\dfrac{\theta}{2}} = \frac{3}{4\sin\dfrac{\theta}{2}}$$

$$\rho = \frac{1}{K} = \frac{4\sin\dfrac{\theta}{2}}{3}$$

$$s^2 + 9\rho^2 = 16\cos^2\frac{\theta}{2} + 16\sin^2\frac{\theta}{2} = 16$$

8. (a) $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ position vector

$$\mathbf{r} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}$$

$$\frac{d\mathbf{r}}{dt} = \left[\frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}\right]\mathbf{i} + \left[\frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}\right]\mathbf{j}$$

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \left[\frac{d^2r}{dt^2}\cos\theta - \frac{dr}{dt}\sin\theta\frac{d\theta}{dt} - \frac{dr}{dt}\sin\theta\frac{d\theta}{dt} - r\cos\theta\left(\frac{d\theta}{dt}\right)^2 - r\sin\theta\frac{d^2\theta}{dt^2}\right]\mathbf{i}$$

$$+ \left[\frac{d^2r}{dt^2}\sin\theta + \frac{dr}{dt}\cos\theta\frac{d\theta}{dt} + \frac{dr}{dt}\cos\theta\frac{d\theta}{dt} - r\sin\theta\left(\frac{d\theta}{dt}\right)^2 + r\cos\theta\frac{d^2\theta}{dt^2}\right]$$

$$a_r = \mathbf{a}\cdot\mathbf{u}_r = \mathbf{a}\cdot(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$$

$$= \left[\frac{d^2r}{dt^2}\cos^2\theta - 2\frac{dr}{dt}\sin\theta\cos\theta\frac{d\theta}{dt} - r\cos^2\theta\left(\frac{d\theta}{dt}\right)^2 - r\cos\theta\sin\theta\frac{d^2\theta}{dt^2}\right]$$

$$+ \left[\frac{d^2r}{dt^2}\sin^2\theta + 2\frac{dr}{dt}\sin\theta\cos\theta\frac{d\theta}{dt} - r\sin^2\theta\left(\frac{d\theta}{dt}\right)^2 + r\cos\theta\sin\theta\frac{d^2\theta}{dt^2}\right]$$

$$= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$

$$a_\theta = \mathbf{a}\cdot u_\theta = \mathbf{a}\cdot(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}) = 2\frac{dr}{dr}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$$

$$\mathbf{a} = (\mathbf{a}\cdot\mathbf{u}_r)\mathbf{u}_r + (\mathbf{a}\cdot\mathbf{u}_\theta)u_\theta$$

$$= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{u}_r + \left[2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right]\mathbf{u}_\theta$$

—CONTINUED—

8. —CONTINUED—

(b) $\mathbf{r} = 42{,}000\cos\left(\frac{\pi t}{12}\right)\mathbf{i} + 42{,}000\sin\left(\frac{\pi t}{12}\right)\mathbf{j}$

$$r = 42{,}000, \frac{dr}{dt} = 0, \frac{d^2r}{dt^2} = 0$$

$$\frac{d\theta}{dt} = \frac{\pi}{12}, \frac{d^2\theta}{dt^2} = 0$$

Therefore, $\mathbf{a} = -42000\left(\frac{\pi}{12}\right)^2\mathbf{u}_r = -\frac{875}{3}\pi^2\mathbf{u}_r$.

Radial component: $-\frac{875}{3}\pi^2$

Angular component: 0

10. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} - \mathbf{k}, t = \frac{\pi}{4}$

$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}, \|\mathbf{r}'(t)\| = 1$

$\mathbf{T} = -\sin t\mathbf{i} + \cos t\mathbf{j}$

$\mathbf{T}' = -\cos t\mathbf{i} - \sin t\mathbf{j}$

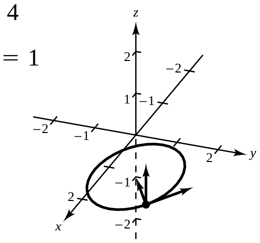
$\mathbf{N} = -\cos t\mathbf{i} - \sin t\mathbf{j}$

$\mathbf{B} = \mathbf{T}\times\mathbf{N} = \mathbf{k}$

At $t = \frac{\pi}{4}$, $\mathbf{T}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$

$\mathbf{N}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$

$\mathbf{B}\left(\frac{\pi}{4}\right) = \mathbf{k}$



12. $y = \frac{1}{32}x^{5/2}$

$y' = \frac{5}{64}x^{3/2}$

$y'' = \frac{15}{128}x^{1/2}$

$$K = \left|\frac{\frac{15}{128}x^{1/2}}{\left(1 + \frac{25}{4096}x^3\right)^{3/2}}\right|$$

At the point $(4, 1)$, $K = \frac{120}{(89)^{3/2}} \implies r = \frac{1}{K} = \frac{(89)^{3/2}}{120} \approx 7$.

14. (a) Eliminate the parameter to see that the Ferris wheel has a radius of 15 meters and is centered at $16\mathbf{j}$. At $t = 0$, the friend is located at $\mathbf{r}_1(0) = \mathbf{j}$, which is the low point on the Ferris wheel.

(b) If a revolution takes Δt seconds, then

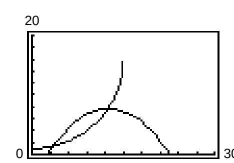
$$\frac{\pi(t + \Delta t)}{10} = \frac{\pi t}{10} + 2\pi$$

and so $\Delta t = 20$ seconds. The Ferris wheel makes three revolutions per minute.

(c) The initial velocity is $r'_2(t_0) = -8.03\mathbf{i} + 11.47\mathbf{j}$. The speed is $\sqrt{8.03^2 + 11.47^2} \approx 14$ m/sec. The angle of inclination is $\arctan(11.47/8/03) \approx 0.96$ radians or 55°.

(d) Although you may start with other values, $t_0 = 0$ is a fine choice. The graph at the right shows two points of intersection. At $t = 3.15$ sec the friend is near the vertex of the parabola, which the object reaches when

$$t - t_0 = -\frac{11.47}{2(-4.9)} \approx 1.17 \text{ sec.}$$



Thus, after the friend reaches the low point on the Ferris wheel, wait $t_0 = 2$ sec before throwing the object in order to allow it to be within reach.

(e) The approximate time is 3.15 seconds after starting to rise from the low point on the Ferris wheel. The friend has a constant speed of $\|\mathbf{r}'_1(t)\| = 15$ m/sec. The speed of the object at that time is

$$\|\mathbf{r}'_2(3.15)\| = \sqrt{8.03^2 + [11.47 - 9.8(3.15 - 2)]^2} \approx 8.03 \text{ m/sec.}$$