CHAPTER 11
Vector-Valued Functions

C H A P T E R 11
Vector-Valued Functions

Section 11.1 Vector-Valued Functions

Solutions to Odd-Numbered Exercises

1. $\mathbf{r}(t) = 5t\mathbf{i} - 4t\mathbf{j} - \frac{1}{t}\mathbf{k}$

Component functions: $f(t) = 5t$

$g(t) = -4t$

$h(t) = -\frac{1}{t}$

Domain: $(-\infty, 0) \cup (0, \infty)$

3. $\mathbf{r}(t) = \ln t\mathbf{i} - e^t\mathbf{j} - t\mathbf{k}$

Component functions: $f(t) = \ln t$

$g(t) = -e^t$

$h(t) = -t$

Domain: $(0, \infty)$

5. $\mathbf{r}(t) = \mathbf{F}(t) + \mathbf{G}(t) = \left(\cos t\mathbf{i} - \sin t\mathbf{j} + \sqrt{t}\mathbf{k}\right) + (\cos t\mathbf{i} + \sin t\mathbf{j}) = 2\cos t\mathbf{i} + \sqrt{t}\mathbf{k}$

Domain: $[0, \infty)$

7. $\mathbf{r}(t) = \mathbf{F}(t) \times \mathbf{G}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sin t & \cos t & 0 \\ 0 & \sin t & \cos t \end{vmatrix} = \cos^2 t\mathbf{i} - \sin t\cos t\mathbf{j} + \sin^2 t\mathbf{k}$

Domain: $(-\infty, \infty)$

9. $\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} - (t-1)\mathbf{j}$

(a) $\mathbf{r}(1) = \frac{1}{2}\mathbf{i}$

(b) $\mathbf{r}(0) = \mathbf{j}$

(c) $\mathbf{r}(s+1) = \frac{1}{2}(s+1)^2\mathbf{i} - (s+1-1)\mathbf{j} = \frac{1}{2}(s+1)^2\mathbf{i} - s\mathbf{j}$

(d)
$$\begin{aligned}\mathbf{r}(2+\Delta t) - \mathbf{r}(2) &= \tfrac{1}{2}(2+\Delta t)^2\mathbf{i} - (2+\Delta t - 1)\mathbf{j} - (2\mathbf{i} - \mathbf{j}) \\ &= \left(2 + 2\Delta t + \tfrac{1}{2}(\Delta t)^2\right)\mathbf{i} - (1+\Delta t)\mathbf{j} - 2\mathbf{i} + \mathbf{j} \\ &= \left(2\Delta t + \tfrac{1}{2}(\Delta t)^2\right)\mathbf{i} - (\Delta t)\mathbf{j}\end{aligned}$$

11. $\mathbf{r}(t) = \ln t\mathbf{i} + \frac{1}{t}\mathbf{j} + 3t\mathbf{k}$

(a) $\mathbf{r}(2) = \ln 2\mathbf{i} + \frac{1}{2}\mathbf{j} + 6\mathbf{k}$

(b) $\mathbf{r}(-3)$ is not defined. ($\ln(-3)$ does not exist.)

(c) $\mathbf{r}(t-4) = \ln(t-4)\mathbf{i} + \frac{1}{t-4}\mathbf{j} + 3(t-4)\mathbf{k}$

(d)
$$\begin{aligned}\mathbf{r}(1+\Delta t) - \mathbf{r}(1) &= \ln(1+\Delta t)\mathbf{i} + \frac{1}{1+\Delta t}\mathbf{j} + 3(1+\Delta t)\mathbf{k} - (0\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \\ &= \ln(1+\Delta t)\mathbf{i} + \left(\frac{1}{1+\Delta t} - 1\right)\mathbf{j} + (3\Delta t)\mathbf{k}\end{aligned}$$

13. $\mathbf{r}(t) = \sin 3t\mathbf{i} + \cos 3t\mathbf{j} + t\mathbf{k}$

$\|\mathbf{r}(t)\| = \sqrt{(\sin 3t)^2 + (\cos 3t)^2 + t^2} = \sqrt{1 + t^2}$

15. $\mathbf{r}(t) \cdot \mathbf{u}(t) = (3t - 1)(t^2) + \left(\frac{1}{4}t^3\right)(-8) + 4(t^3)$

$= 3t^3 - t^2 - 2t^3 + 4t^3 = 5t^3 - t^2$, a scalar.

The dot product is a scalar-valued function.

17. $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k}, -2 \le t \le 2$

$x = t, y = 2t, z = t^2$

Thus, $z = x^2$. Matches (b)

19. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + e^{0.75t}\mathbf{k}, -2 \le t \le 2$

$x = t, y = t^2, z = e^{0.75t}$

Thus, $y = x^2$. Matches (d)

21. (a) View from the negative x-axis: $(-20, 0, 0)$

(b) View from above the first octant: $(10, 20, 10)$

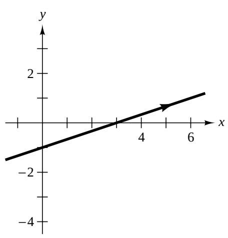
(c) View from the z-axis: $(0, 0, 20)$

(d) View from the positive x-axis: $(20, 0, 0)$

23. $x = 3t$

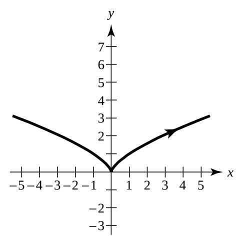
$y = t - 1$

$y = \frac{x}{3} - 1$



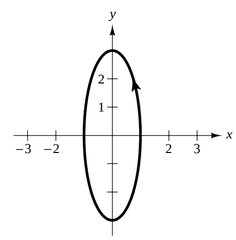
25. $x = t^3, y = t^2$

$y = x^{2/3}$



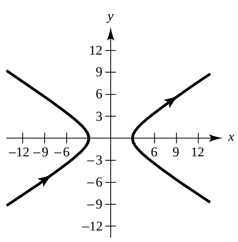
27. $x = \cos\theta, y = 3\sin\theta$

$x^2 + \frac{y^2}{9} = 1$ Ellipse



29. $x = 3\sec\theta, y = 2\tan\theta$

$\frac{x^2}{9} = \frac{y^2}{4} + 1$ Hyperbola



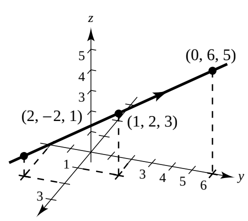
31. $x = -t + 1$

$y = 4t + 2$

$z = 2t + 3$

Line passing through the points:

$(0, 6, 5)$, $(1, 2, 3)$

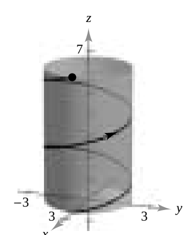


33. $x = 2\cos t,\ y = 2\sin t,\ z = t$

$\frac{x^2}{4} + \frac{y^2}{4} = 1$

$z = t$

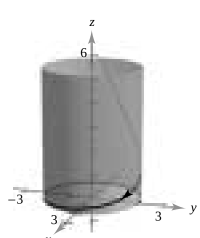
Circular helix



35. $x = 2\sin t,\ y = 2\cos t,\ z = e^{-t}$

$x^2 + y^2 = 4$

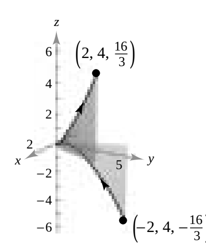
$z = e^{-t}$



37. $x = t,\ y = t^2,\ z = \frac{2}{3}t^3$

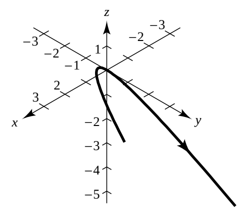
$y = x^2,\ z = \frac{2}{3}x^3$

t	-2	-1	0	1	2
x	-2	-1	0	1	2
y	4	1	0	1	4
z	$-\frac{16}{3}$	$-\frac{2}{3}$	0	$\frac{2}{3}$	$\frac{16}{3}$



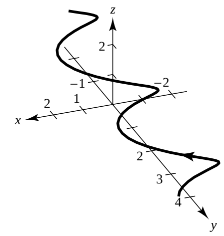
39. $\mathbf{r}(t) = -\frac{1}{2}t^2\mathbf{i} + t\mathbf{j} - \frac{\sqrt{3}}{2}t^2\mathbf{k}$

Parabola

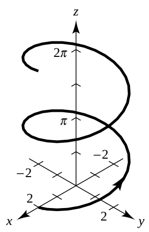


41. $\mathbf{r}(t) = \sin t\mathbf{i} + \left(\frac{\sqrt{3}}{2}\cos t - \frac{1}{2}t\right)\mathbf{j} + \left(\frac{1}{2}\cos t + \frac{\sqrt{3}}{2}\right)\mathbf{k}$

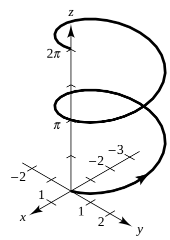
Helix



43.

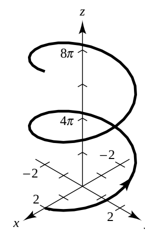


(a)



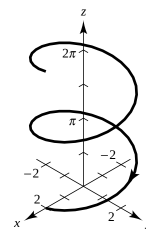
The helix is translated 2 units back on the x-axis.

(b)



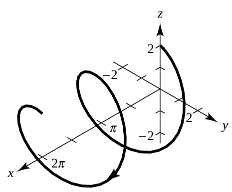
The height of the helix increases at a faster rate.

(c)



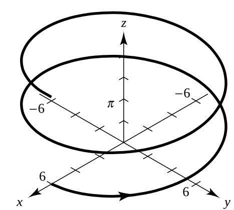
The orientation of the helix is reversed.

(d)



The axis of the helix is the x-axis.

(e)



The radius of the helix is increased from 2 to 6.

45. $y = 4 - x$

Let $x = t$, then $y = 4 - t$.

$\mathbf{r}(t) = t\mathbf{i} + (4 - t)\mathbf{j}$

47. $y = (x - 2)^2$

Let $x = t$, then $y = (t - 2)^2$.

$\mathbf{r}(t) = t\mathbf{i} + (t - 2)^2\,\mathbf{j}$

49. $x^2 + y^2 = 25$

Let $x = 5\cos t$, then $y = 5\sin t$.

$\mathbf{r}(t) = 5\cos t\mathbf{i} + 5\sin t\mathbf{j}$

51. $\frac{x^2}{16} - \frac{y^2}{4} = 1$

Let $x = 4\sec t$, $y = 2\tan t$.

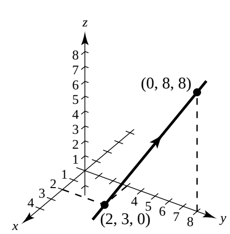
$\mathbf{r}(t) = 4\sec t\mathbf{i} + 2\tan t\mathbf{j}$

53. The parametric equations for the line are

$x = 2 - 2t,\; y = 3 + 5t,\; z = 8t.$

One possible answer is

$\mathbf{r}(t) = (2 - 2t)\mathbf{i} + (3 + 5t)\mathbf{j} + 8t\mathbf{k}.$



55. $\mathbf{r}_1(t) = t\mathbf{i}$, $\quad 0 \le t \le 4 \quad (\mathbf{r}_1(0) = \mathbf{0}, \mathbf{r}_1(4) = 4\mathbf{i})$

$\mathbf{r}_2(t) = (4 - 4t)\mathbf{i} + 6t\mathbf{j}$, $\quad 0 \le t \le 1 \quad (\mathbf{r}_2(0) = 4\mathbf{i}, \mathbf{r}_2(1) = 6\mathbf{j})$

$\mathbf{r}_3(t) = (6 - t)\mathbf{j}$, $\quad 0 \le t \le 6 \quad (\mathbf{r}_3(0) = 6\mathbf{j}, \mathbf{r}_3(6) = \mathbf{0})$

(Other answers possible)

57. $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j}, \quad 0 \le t \le 2 \ (y = x^2)$

$\mathbf{r}_2(t) = (2 - t)\mathbf{i}, \quad 0 \le t \le 2$

$\mathbf{r}_3(t) = (4 - t)\mathbf{j}, \quad 0 \le t \le 4$

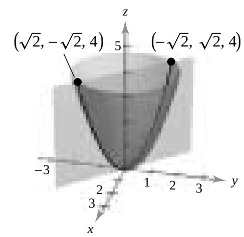
(Other answers possible)

59. $z = x^2 + y^2, \ x + y = 0$

Let $x = t$, then $y = -x = -t$ and $z = x^2 + y^2 = 2t^2$. Therefore,

$$x = t, \ y = -t, \ z = 2t^2.$$

$\mathbf{r}(t) = t\mathbf{i} - t\mathbf{j} + 2t^2\mathbf{k}$



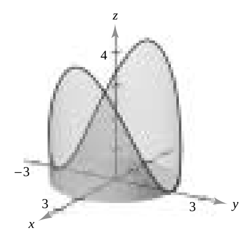
61. $x^2 + y^2 = 4, \ z = x^2$

$x = 2\sin t, \ y = 2\cos t$

$z = x^2 = 4\sin^2 t$

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
x	0	1	$\sqrt{2}$	2	$\sqrt{2}$	0
y	2	$\sqrt{3}$	$\sqrt{2}$	0	$-\sqrt{2}$	-2
z	0	1	2	4	2	0

$\mathbf{r}(t) = 2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + 4\sin^2 t\,\mathbf{k}$



63. $x^2 + y^2 + z^2 = 4, \ x + z = 2$

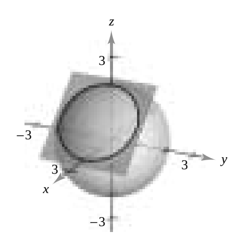
Let $x = 1 + \sin t$, then $z = 2 - x = 1 - \sin t$ and $x^2 + y^2 + z^2 = 4$.

$$(1 + \sin t)^2 + y^2 + (1 - \sin t)^2 = 2 + 2\sin^2 t + y^2 = 4$$

$y^2 = 2\cos^2 t, \quad y = \pm\sqrt{2}\cos t$

$x = 1 + \sin t, \ y = \pm\sqrt{2}\cos t$

$z = 1 - \sin t$



t	$-\frac{\pi}{2}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$
x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
y	0	$\pm\frac{\sqrt{6}}{2}$	$\pm\sqrt{2}$	$\pm\frac{\sqrt{6}}{2}$	0
z	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0

$\mathbf{r}(t) = (1 + \sin t)\mathbf{i} + \sqrt{2}\cos t\mathbf{j} + (1 + \sin t)\mathbf{k}$ and

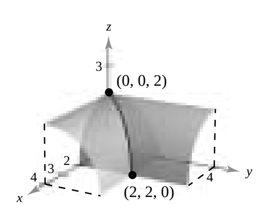
$\mathbf{r}(t) = (1 + \sin t)\mathbf{i} - \sqrt{2}\cos t\mathbf{j} + (1 - \sin t)\mathbf{k}$

65. $x^2 + z^2 = 4, \ y^2 + z^2 = 4$

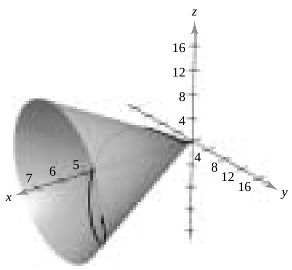
Subtracting, we have $x^2 - y^2 = 0$ or $y = \pm x$.

Therefore, in the first octant, if we let $x = t$, then $x = t, \ y = t, \ z = \sqrt{4 - t^2}$.

$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{4 - t^2}\mathbf{k}$



67. $y^2 + z^2 = (2t\cos t)^2 + (2t\sin t)^2 = 4t^2 = 4x^2$



69. $\lim_{t\to 2}\left[t\mathbf{i} + \frac{t^2-4}{t^2-2t}\mathbf{j} + \frac{1}{t}\mathbf{k}\right] = 2\mathbf{i} + 2\mathbf{j} + \frac{1}{2}\mathbf{k}$

since

$$\lim_{t\to 2}\frac{t^2-4}{t^2-2t} = \lim_{t\to 2}\frac{2t}{2t-2} = 2. \quad \text{(L'Hôpital's Rule)}$$

71. $\lim_{t\to 0}\left[t^2\mathbf{i} + 3t\mathbf{j} + \frac{1-\cos t}{t}\mathbf{k}\right] = \mathbf{0}$

since

$$\lim_{t\to 0}\frac{1-\cos t}{t} = \lim_{t\to 0}\frac{\sin t}{1} = 0. \quad \text{(L'Hôpital's Rule)}$$

73. $\lim_{t\to 0}\left[\frac{1}{t}\mathbf{i} + \cos t\mathbf{j} + \sin t\mathbf{k}\right]$

does not exist since $\lim_{t\to 0}\frac{1}{t}$ does not exist.

75. $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j}$

Continuous on $(-\infty, 0),\ (0, \infty)$

77. $\mathbf{r}(t) = t\mathbf{i} + \arcsin t\mathbf{j} + (t-1)\mathbf{k}$

Continuous on $[-1, 1]$

79. $\mathbf{r}(t) = \langle e^{-t}, t^2, \tan t\rangle$

Discontinuous at $t = \frac{\pi}{2} + n\pi$

Continuous on $\left(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right)$

81. See the definition on page 786.

83. $\mathbf{r}(t) = t^2\mathbf{i} + (t-3)\mathbf{j} + t\mathbf{k}$

(a) $\mathbf{s}(t) = \mathbf{r}(t) + 2\mathbf{k} = t^2\mathbf{i} + (t-3)\mathbf{j} + (t+3)\mathbf{k}$

(b) $\mathbf{s}(t) = \mathbf{r}(t) - 2\mathbf{i} = (t^2-2)\mathbf{i} + (t-3)\mathbf{j} + t\mathbf{k}$

(c) $\mathbf{s}(t) = \mathbf{r}(t) + 5\mathbf{j} = t^2\mathbf{i} + (t+2)\mathbf{j} + t\mathbf{k}$

85. Let $\mathbf{r}(t) = x_1(t) + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$ and $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$. Then:

$$\begin{aligned}
\lim_{t\to c}[\mathbf{r}(t)\times\mathbf{u}(t)] &= \lim_{t\to c}\{[y_1(t)z_2(t) - y_2(t)z_1(t)]\mathbf{i} - [x_1(t)z_2(t) - x_2(t)z_1(t)]\mathbf{j} + [x_1(t)y_2(t) - x_2(t)y_1(t)]\mathbf{k}\}\\
&= \left[\lim_{t\to c}y_1(t)\lim_{t\to c}z_2(t) - \lim_{t\to c}y_2(t)\lim_{t\to c}z_1(t)\right]\mathbf{i} - \left[\lim_{t\to c}x_1(t)\lim_{t\to c}z_2(t) - \lim_{t\to c}x_2(t)\lim_{t\to c}z_1(t)\right]\mathbf{j}\\
&\qquad + \left[\lim_{t\to c}x_1(t)\lim_{t\to c}y_2(t) - \lim_{t\to c}x_2(t)\lim_{t\to c}y_1(t)\right]\mathbf{k}\\
&= \left[\lim_{t\to c}x_1(t)\mathbf{i} + \lim_{t\to c}y_1(t)\mathbf{j} + \lim_{t\to c}z_1(t)\mathbf{k}\right]\times\left[\lim_{t\to c}x_2(t)\mathbf{i} + \lim_{t\to c}y_2(t)\mathbf{j} + \lim_{t\to c}z_2(t)\mathbf{k}\right]\\
&= \lim_{t\to c}\mathbf{r}(t)\times\lim_{t\to c}\mathbf{u}(t)
\end{aligned}$$

87. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Since $\mathbf{r}$ is continuous at $t = c$, then $\lim_{t\to c}\mathbf{r}(t) = \mathbf{r}(c)$.

$$\mathbf{r}(c) = x(c)\mathbf{i} + y(c)\mathbf{j} + z(c)\mathbf{k} \implies x(c),\ y(c),\ z(c)$$

are defined at c.

$$\|\mathbf{r}\| = \sqrt{(x(t))^2 + (y(t))^2 + (z(t))^2}$$

$$\lim_{t\to c}\|\mathbf{r}\| = \sqrt{(x(c))^2 + (y(c))^2 + (z(c))^2} = \|\mathbf{r}(c)\|$$

Therefore, $\|\mathbf{r}\|$ is continuous at c.

89. True

Section 11.2 Differentiation and Integration of Vector-Valued Functions

1. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j},\ t_0 = 2$

$x(t) = t^2,\ y(t) = t$

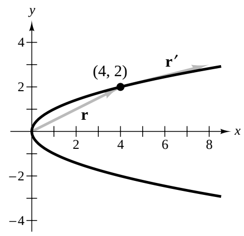
$x = y^2$

$\mathbf{r}(2) = 4\mathbf{i} + 2\mathbf{j}$

$\mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$

$\mathbf{r}'(2) = 4\mathbf{i} + \mathbf{j}$

$\mathbf{r}'(t_0)$ is tangent to the curve.



3. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j},\ t_0 = \dfrac{\pi}{2}$

$x(t) = \cos t,\ y(t) = \sin t$

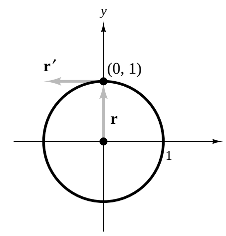
$x^2 + y^2 = 1$

$\mathbf{r}\left(\dfrac{\pi}{2}\right) = \mathbf{j}$

$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$

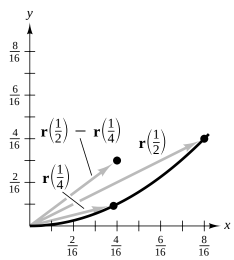
$\mathbf{r}'\left(\dfrac{\pi}{2}\right) = -\mathbf{i}$

$\mathbf{r}'(t_0)$ is tangent to the curve.



5. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$

(a)



(b) $\mathbf{r}\left(\dfrac{1}{4}\right) = \dfrac{1}{4}\mathbf{i} + \dfrac{1}{16}\mathbf{j}$

$\mathbf{r}\left(\dfrac{1}{2}\right) = \dfrac{1}{2}\mathbf{i} + \dfrac{1}{4}\mathbf{j}$

$\mathbf{r}\left(\dfrac{1}{2}\right) - \mathbf{r}\left(\dfrac{1}{4}\right) = \dfrac{1}{4}\mathbf{i} + \dfrac{3}{16}\mathbf{j}$

(c) $\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{r}'\left(\dfrac{1}{4}\right) = \mathbf{i} + \dfrac{1}{2}\mathbf{j}$

$$\frac{\mathbf{r}(1/2) - \mathbf{r}(1/4)}{(1/2) - (1/4)} = \frac{(1/4)\mathbf{i} + (3/16)\mathbf{j}}{1/4} = \mathbf{i} + \frac{3}{4}\mathbf{j}$$

This vector approximates $\mathbf{r}'\left(\frac{1}{4}\right)$.

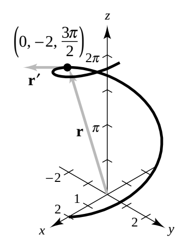
7. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k},\ t_0 = \dfrac{3\pi}{2}$

$x^2 + y^2 = 4,\ z = t$

$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \mathbf{k}$

$\mathbf{r}\left(\dfrac{3\pi}{2}\right) = -2\mathbf{j} + \dfrac{3\pi}{2}\mathbf{k}$

$\mathbf{r}'\left(\dfrac{3\pi}{2}\right) = 2\mathbf{i} + \mathbf{k}$



9. $\mathbf{r}(t) = 6t\mathbf{i} - 7t^2\mathbf{j} + t^3\mathbf{k}$

$\mathbf{r}'(t) = 6\mathbf{i} - 14t\mathbf{j} + 3t^2\mathbf{k}$

11. $\mathbf{r}(t) = a\cos^3 t\mathbf{i} + a\sin^3 t\mathbf{j} + \mathbf{k}$

$\mathbf{r}'(t) = -3a\cos^2 t\sin t\mathbf{i} + 3a\sin^2 t\cos t\mathbf{j}$

13. $\mathbf{r}(t) = e^{-t}\mathbf{i} + 4\mathbf{j}$

$\mathbf{r}'(t) = -e^{-t}\mathbf{i}$

15. $\mathbf{r}(t) = \langle t\sin t, t\cos t, t\rangle$

$\mathbf{r}'(t) = \langle \sin t + t\cos t, \cos t - t\sin t, 1\rangle$

17. $\mathbf{r}(t) = t^3\mathbf{i} + \dfrac{1}{2}t^2\mathbf{j}$

(a) $\mathbf{r}'(t) = 3t^2\mathbf{i} + t\mathbf{j}$

$\mathbf{r}''(t) = 6t\mathbf{i} + \mathbf{j}$

(b) $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = 3t^2(6t) + t = 18t^3 + t$

19. $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}$

(a) $\mathbf{r}'(t) = -4\sin t\mathbf{i} + 4\cos t\mathbf{j}$

$\mathbf{r}''(t) = -4\cos t\mathbf{i} - 4\sin t\mathbf{j}$

(b) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = (-4\sin t)(-4\cos t) + 4\cos t(-4\sin t)$

$= 0$

21. $\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} - t\mathbf{j} + \frac{1}{6}t^3\mathbf{k}$

(a) $\mathbf{r}'(t) = t\mathbf{i} - \mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

$\mathbf{r}''(t) = \mathbf{i} + t\mathbf{k}$

(b) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = t(1) - 1(0) + \frac{1}{2}t^2(t) = t + \frac{t^3}{2}$

23. $\mathbf{r}(t) = \langle \cos t + t\sin t, \sin t - t\cos t, t\rangle$

(a) $\mathbf{r}'(t) = \langle -\sin t + \sin t + t\cos t, \cos t - \cos t + t\sin t, 1\rangle$

$= \langle t\cos t, t\sin t, 1\rangle$

$\mathbf{r}''(t) = \langle \cos t - t\sin t, \sin t + t\cos t, 0\rangle$

(b) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = (t\cos t)(\cos t - t\sin t) + (t\sin t)(\sin t + t\cos t) = t$

25.

$$\mathbf{r}(t) = \cos(\pi t)\mathbf{i} + \sin(\pi t)\mathbf{j} + t^2\mathbf{k},\ t_0 = -\frac{1}{4}$$

$$\mathbf{r}'(t) = -\pi\sin(\pi t)\mathbf{i} + \pi\cos(\pi t)\mathbf{j} + 2t\mathbf{k}$$

$$\mathbf{r}'\left(-\frac{1}{4}\right) = \frac{\sqrt{2}\pi}{2}\mathbf{i} + \frac{\sqrt{2}\pi}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}$$

$$\left\|\mathbf{r}'\left(\frac{1}{4}\right)\right\| = \sqrt{\left(\frac{\sqrt{2}\pi}{2}\right)^2 + \left(\frac{\sqrt{2}\pi}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\pi^2 + \frac{1}{4}} = \frac{\sqrt{4\pi^2+1}}{2}$$

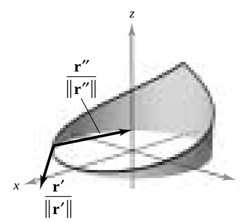
$$\frac{\mathbf{r}'(-1/4)}{\|\mathbf{r}'(-1/4)\|} = \frac{1}{\sqrt{4\pi^2+1}}\left(\sqrt{2}\pi\mathbf{i} + \sqrt{2}\pi\mathbf{j} - \mathbf{k}\right)$$

$$\mathbf{r}''(t) = -\pi^2\cos(\pi t)\mathbf{i} - \pi^2\sin(\pi t)\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{r}''\left(-\frac{1}{4}\right) = -\frac{\sqrt{2}\pi^2}{2}\mathbf{i} + \frac{\sqrt{2}\pi^2}{2}\mathbf{j} + 2\mathbf{k}$$

$$\left\|\mathbf{r}''\left(-\frac{1}{4}\right)\right\| = \sqrt{\left(-\frac{\sqrt{2}\pi^2}{2}\right)^2 + \left(\frac{\sqrt{2}\pi^2}{2}\right)^2 + (2)^2} = \sqrt{\pi^4+4}$$

$$\frac{\mathbf{r}''(-1/4)}{\|\mathbf{r}'(-1/4)\|} = \frac{1}{2\sqrt{\pi^4+4}}\left(-\sqrt{2}\pi^2\mathbf{i} + \sqrt{2}\pi^2\mathbf{j} + 4\mathbf{k}\right)$$



27. $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$

$\mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j}$

$\mathbf{r}'(0) = \mathbf{0}$

Smooth on $(-\infty, 0),\ (0, \infty)$

29. $\mathbf{r}(\theta) = 2\cos^3\theta\mathbf{i} + 3\sin^3\theta\mathbf{j}$

$\mathbf{r}'(\theta) = -6\cos^2\theta\sin\theta\mathbf{i} + 9\sin^2\theta\cos\theta\mathbf{j}$

$\mathbf{r}'\left(\frac{n\pi}{2}\right) = \mathbf{0}$

Smooth on $\left(\frac{n\pi}{2}, \frac{(n+1)\pi}{2}\right)$, n any integer.

31. $\mathbf{r}(\theta) = (\theta - 2\sin\theta)\mathbf{i} + (1 - 2\cos\theta)\mathbf{j}$

$\mathbf{r}'(\theta) = (1 - 2\cos\theta)\mathbf{i} + (1 + 2\sin\theta)\mathbf{j}$

$\mathbf{r}'(\theta) \neq \mathbf{0}$ for any value of θ

Smooth on $(-\infty, \infty)$

33. $\mathbf{r}(t) = (t-1)\mathbf{i} + \frac{1}{t}\mathbf{j} - t^2\mathbf{k}$

$\mathbf{r}'(t) = \mathbf{i} - \frac{1}{t^2}\mathbf{j} - 2t\mathbf{k} \neq \mathbf{0}$

$\mathbf{r}$ is smooth for all $t \neq 0$: $(-\infty, 0), \cup (0, \infty)$

35. $\mathbf{r}(t) = t\mathbf{i} - 3t\mathbf{j} + \tan t\mathbf{k}$

$\mathbf{r}'(t) = \mathbf{i} - 3\mathbf{j} + \sec^2 t\mathbf{k} \neq \mathbf{0}$

$\mathbf{r}$ is smooth for all $t \neq \dfrac{\pi}{2} + n\pi = \dfrac{2n+1}{2}\pi.$

Smooth on intervals of form $\left(-\dfrac{\pi}{2} + n\pi, \dfrac{\pi}{2} + n\pi\right)$

37. $\mathbf{r}(t) = t\mathbf{i} + 3t\mathbf{j} + t^2\mathbf{k},\ \mathbf{u}(t) = 4t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$

(a) $\mathbf{r}'(t) = \mathbf{i} + 3\mathbf{j} + 2t\mathbf{k}$

(b) $\mathbf{r}''(t) = 2\mathbf{k}$

(c) $\mathbf{r}(t) \cdot \mathbf{u}(t) = 4t^2 + 3t^3 + t^5$

$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = 8t + 9t^2 + 5t^4$

(d) $3\mathbf{r}(t) - \mathbf{u}(t) = -t\mathbf{i} + (9t - t^2)\mathbf{j} + (3t^2 - t^3)\mathbf{k}$

$D_t[3\mathbf{r}(t) - \mathbf{u}(t)] = -\mathbf{i} + (9 - 2t)\mathbf{j} + (6t - 3t^2)\mathbf{k}$

(e) $\mathbf{r}(t) \times \mathbf{u}(t) = 2t^4\mathbf{i} - (t^4 - 4t^3)\mathbf{j} + (t^3 - 12t^2)\mathbf{k}$

$D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = 8t^3\mathbf{i} + (12t^2 - 4t^3)\mathbf{j} + (3t^2 - 24t)\mathbf{k}$

(f) $\|\mathbf{r}(t)\| = \sqrt{10t^2 + t^4} = t\sqrt{10 + t^2}$

$D_t[\|\mathbf{r}(t)\|] = \dfrac{10 + 2t^2}{\sqrt{10 + t^2}}$

39. $\mathbf{r}(t) = 3 \sin t\mathbf{i} + 4 \cos t\mathbf{j}$

$\mathbf{r}'(t) = 3 \cos t\mathbf{i} - 4 \sin t\mathbf{j}$

$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 9 \sin t \cos t - 16 \cos t \sin t = -7 \sin t \cos t$

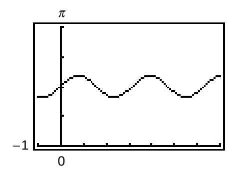
$$\cos\theta = \frac{\mathbf{r}(t) \cdot \mathbf{r}'(t)}{\|\mathbf{r}(t)\|\,\|\mathbf{r}'(t)\|} = \frac{-7 \sin t \cos t}{\sqrt{9 \sin^2 t + 16 \cos^2 t}\sqrt{9 \cos^2 t + 16 \sin^2 t}}$$

$$\theta = \arccos\left[\frac{-7 \sin t \cos t}{\sqrt{(9 \sin^2 t + 16 \cos^2 t)(9 \cos^2 t + 16 \sin^2 t)}}\right]$$

$\theta = 1.855$ maximum at $t = 3.927\left(\dfrac{5\pi}{4}\right)$ and $t = 0.785\left(\dfrac{\pi}{4}\right)$.

$\theta = 1.287$ minimum at $t = 2.356\left(\dfrac{3\pi}{4}\right)$ and $t = 5.498\left(\dfrac{7\pi}{4}\right)$.

$\theta = \dfrac{\pi}{2}(1.571)$ for $t = n\dfrac{\pi}{2}$, $n = 0, 1, 2, 3, \ldots$



41. $$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{[3(t + \Delta t) + 2]\mathbf{i} + [1 - (t + \Delta t)^2]\mathbf{j} - (3t + 2)\mathbf{i} - (1 - t^2)\mathbf{j}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{(3\Delta t)\mathbf{i} - (2t(\Delta t) + (\Delta t)^2)\mathbf{j}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} 3\mathbf{i} - (2t + \Delta t)\mathbf{j} = 3\mathbf{i} - 2t\mathbf{j}$$

43. $\displaystyle\int (2t\mathbf{i} + \mathbf{j} + \mathbf{k})\, dt = t^2\mathbf{i} + t\mathbf{j} + t\mathbf{k} + \mathbf{C}$

45. $\displaystyle\int \left(\frac{1}{t}\mathbf{i} + \mathbf{j} - t^{3/2}\mathbf{k}\right) dt = \ln t\mathbf{i} + t\mathbf{j} - \frac{2}{5}t^{5/2}\mathbf{k} + \mathbf{C}$

47. $\displaystyle\int \left[(2t - 1)\mathbf{i} + 4t^3\mathbf{j} + 3\sqrt{t}\mathbf{k}\right] dt = (t^2 - t)\mathbf{i} + t^4\mathbf{j} + 2t^{3/2}\mathbf{k} + \mathbf{C}$

49. $\displaystyle\int \left[\sec^2 t\mathbf{i} + \frac{1}{1 + t^2}\mathbf{j}\right] dt = \tan t\mathbf{i} + \arctan t\mathbf{j} + \mathbf{C}$

51. $\int_0^1 (8t\mathbf{i} + t\mathbf{j} - \mathbf{k})\,dt = \left[4t^2\mathbf{i}\right]_0^1 + \left[\frac{t^2}{2}\mathbf{j}\right]_0^1 - \left[t\mathbf{k}\right]_0^1 = 4\mathbf{i} + \frac{1}{2}\mathbf{j} - \mathbf{k}$

53. $\int_0^{\pi/2} [(a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + \mathbf{k}]\,dt = \left[a\sin t\mathbf{i}\right]_0^{\pi/2} - \left[a\cos t\mathbf{j}\right]_0^{\pi/2} + \left[t\mathbf{k}\right]_0^{\pi/2} = a\mathbf{i} + a\mathbf{j} + \frac{\pi}{2}\mathbf{k}$

55. $\mathbf{r}(t) = \int (4e^{2t}\mathbf{i} + 3e^t\mathbf{j})\,dt = 2e^{2t}\mathbf{i} + 3e^t\mathbf{j} + \mathbf{C}$

$\mathbf{r}(0) = 2\mathbf{i} + 3\mathbf{j} + \mathbf{C} = 2\mathbf{i} \implies \mathbf{C} = -3\mathbf{j}$

$\mathbf{r}(t) = 2e^{2t}\mathbf{i} + 3(e^t - 1)\mathbf{j}$

57. $\mathbf{r}'(t) = \int -32\mathbf{j}\,dt = -32t\mathbf{j} + \mathbf{C}_1$

$\mathbf{r}'(0) = \mathbf{C}_1 = 600\sqrt{3}\mathbf{i} + 600\mathbf{j}$

$\mathbf{r}'(t) = 600\sqrt{3}\mathbf{i} + (600 - 32t)\mathbf{j}$

$$\mathbf{r}(t) = \int \left[600\sqrt{3}\mathbf{i} + (600 - 32t)\mathbf{j}\right]dt$$

$$= 600\sqrt{3}t\mathbf{i} + (600t - 16t^2)\mathbf{j} + \mathbf{C}$$

$\mathbf{r}(0) = \mathbf{C} = \mathbf{0}$

$\mathbf{r}(t) = 600\sqrt{3}t\mathbf{i} + (600t - 16t^2)\mathbf{j}$

59. $\mathbf{r}(t) = \int (te^{-t^2}\mathbf{i} - e^{-t}\mathbf{j} + \mathbf{k})\,dt = -\frac{1}{2}e^{-t^2}\mathbf{i} + e^{-t}\mathbf{j} + t\mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = -\frac{1}{2}\mathbf{i} + \mathbf{j} + \mathbf{C} = \frac{1}{2}\mathbf{i} - \mathbf{j} + \mathbf{k} \implies \mathbf{C} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$

$\mathbf{r}(t) = \left(1 - \frac{1}{2}e^{-t^2}\right)\mathbf{i} + (e^{-t} - 2)\mathbf{j} + (t + 1)\mathbf{k} = \left(\frac{2 - e^{-t^2}}{2}\right)\mathbf{i} + (e^{-t} - 2)\mathbf{j} + (t + 1)\mathbf{k}$

61. See "Definition of the Derivative of a Vector-Valued Function" and Figure 11.8 on page 794.

63. At $t = t_0$, the graph of $\mathbf{u}(t)$ is increasing in the x, y, and z directions simultaneously.

65. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $c\mathbf{r}(t) = cx(t)\mathbf{i} + cy(t)\mathbf{j} + cz(t)\mathbf{k}$ and

$$D_t[c\mathbf{r}(t)] = cx'(t)\mathbf{i} + cy'(t)\mathbf{j} + cz'(t)\mathbf{k}$$
$$= c[x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}] = c\mathbf{r}'(t).$$

67. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, then $f(t)\mathbf{r}(t) = f(t)x(t)\mathbf{i} + f(t)y(t)\mathbf{j} + f(t)z(t)\mathbf{k}$.

$$D_t[f(t)\mathbf{r}(t)] = [f(t)x'(t) + f'(t)x(t)]\mathbf{i} + [f(t)y'(t) + f'(t)y(t)]\mathbf{j} + [f(t)z'(t) + f'(t)z(t)]\mathbf{k}$$
$$= f(t)[x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}] + f'(t)[x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}]$$
$$= f(t)\mathbf{r}'(t) + f'(t)\mathbf{r}(t)$$

69. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $\mathbf{r}(f(t)) = x(f(t))\mathbf{i} + y(f(t))\mathbf{j} + z(f(t))\mathbf{k}$ and

$$D_t[\mathbf{r}(f(t))] = x'(f(t))f'(t)\mathbf{i} + y'(f(t))f'(t)\mathbf{j} + z'(f(t))f'(t)\mathbf{k} \quad \text{(Chain Rule)}$$
$$= f'(t)[x'(f(t))i + y'(f(t))\mathbf{j} + z'(f(t))\mathbf{k}] = f'(t)\mathbf{r}'(f(t)).$$

71. Let $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$, $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$, and $\mathbf{v}(t) = x_3(t)\mathbf{i} + y_3(t)\mathbf{j} + z_3(t)\mathbf{k}$. Then:

$$\mathbf{r}(t) \cdot [\mathbf{u}(t) \times \mathbf{v}(t)] = x_1(t)[y_2(t)z_3(t) - z_2(t)y_3(t)] - y_1(t)[x_2(t)z_3(t) - z_2(t)x_3(t)] + z_1(t)[x_2(t)y_3(t) - y_2(t)x_3(t)]$$

$$\begin{aligned} D_t[\mathbf{r}(t) \cdot (\mathbf{u}(t) \times \mathbf{v}(t))] &= x_1(t)y_2(t)z_3'(t) + x_1(t)y_2'(t)z_3(t) + x_1'(t)y_2(t)z_3(t) - x_1(t)y_3(t)z_2'(t) - \\ &\quad x_1(t)y_3'(t)z_2(t) - x_1'(t)y_3(t)z_2(t) - y_1(t)x_2(t)z_3'(t) - y_1(t)x_2'(t)z_3(t) - y_1''(t)x_2(t)z_3(t) + \\ &\quad y_1(t)z_2(t)x_3'(t) + y_1(t)z_2'(t)x_3(t) + y_1'(t)z_2(t)x_3(t) + z_1(t)x_2(t)y_3'(t) + z_1(t)x_2'(t)y_3(t) + \\ &\quad z_1'(t)x_2(t)y_3(t) - z_1(t)y_2(t)x_3'(t) - z_1(t)y_2'(t)x_3(t) - z_1'(t)y_2(t)x_3(t) \\ &= \{x_1'(t)[y_2(t)z_3(t) - y_3(t)z_2(t)] + y_1'(t)[-x_2(t)z_3(t) + z_2(t)x_3(t)] + z_1'(t)[x_2(t)y_3(t) - y_2(t)x_3(t)]\} + \\ &\quad \{x_1(t)[y_2'(t)z_3(t) - y_3(t)z_2'(t)] + y_1(t)[-x_2'(t)z_3(t) + z_2'(t)x_3(t)] + z_1(t)[x_2'(t)y_3(t) - y_2'(t)x_3(t)]\} + \\ &\quad \{x_1(t)[y_2(t)z_3'(t) - y_3'(t)z_2(t)] + y_1(t)[-x_2(t)z_3'(t) + z_2(t)x_3'(t)] + z_1(t)[x_2(t)y_3'(t) - y_2(t)x_3'(t)]\} \\ &= \mathbf{r}'(t) \cdot [\mathbf{u}(t) \times \mathbf{v}(t)] + \mathbf{r}(t) \cdot [\mathbf{u}'(t) \times \mathbf{v}(t)] + \mathbf{r}(t) \cdot [\mathbf{u}(t) \times \mathbf{v}'(t)] \end{aligned}$$

73. False. Let $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \mathbf{k}$.

$$\|\mathbf{r}(t)\| = \sqrt{2}$$

$$\frac{d}{dt}[\|\mathbf{r}(t)\|] = 0$$

$$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = 1$$

Section 11.3 Velocity and Acceleration

1. $\mathbf{r}(t) = 3t\mathbf{i} + (t - 1)\mathbf{j}$

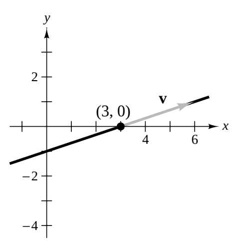
$\mathbf{v}(t) = \mathbf{r}'(t) = 3\mathbf{i} + \mathbf{j}$

$\mathbf{a}(t) = \mathbf{r}''(t) = \mathbf{0}$

$x = 3t,\ y = t - 1,\ y = \dfrac{x}{3} - 1$

At $(3, 0)$, $t = 1$.

$\mathbf{v}(1) = 3\mathbf{i} + \mathbf{j},\ \mathbf{a}(1) = \mathbf{0}$



3. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}$

$\mathbf{v}(t) = \mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$

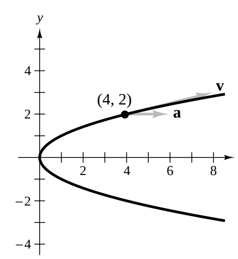
$\mathbf{a}(t) = \mathbf{r}''(t) = 2\mathbf{i}$

$x = t^2,\ y = t,\ x = y^2$

At $(4, 2)$, $t = 2$.

$\mathbf{v}(2) = 4\mathbf{i} + \mathbf{j}$

$\mathbf{a}(2) = 2\mathbf{i}$



5. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}$

$\mathbf{v}(t) = \mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j}$

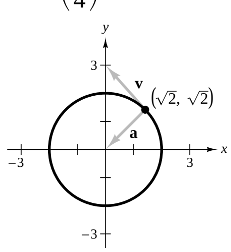
$\mathbf{a}(t) = \mathbf{r}''(t) = -2\cos t\mathbf{i} - 2\sin t\mathbf{j}$

$x = 2\cos t,\ y = 2\sin t,\ x^2 + y^2 = 4$

At $\left(\sqrt{2}, \sqrt{2}\right)$, $t = \dfrac{\pi}{4}$.

$$\mathbf{v}\left(\frac{\pi}{4}\right) = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$$

$$\mathbf{a}\left(\frac{\pi}{4}\right) = -\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j}$$



7. $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t\rangle$

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 1 - \cos t, \sin t\rangle$

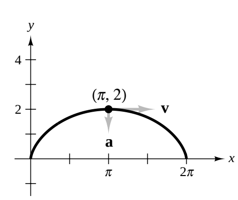
$\mathbf{a}(t) = \mathbf{r}''(t) = \langle \sin t, \cos t\rangle$

$x = t - \sin t,\ y = 1 - \cos t$ (cycloid)

At $(\pi, 2)$, $t = \pi$.

$\mathbf{v}(\pi) = \langle 2, 0\rangle = 2\mathbf{i}$

$\mathbf{a}(\pi) = \langle 0, -1\rangle = -\mathbf{j}$



9. $\mathbf{r}(t) = t\mathbf{i} + (2t - 5)\mathbf{j} + 3t\mathbf{k}$

$\mathbf{v}(t) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

$s(t) = \|\mathbf{v}(t)\| = \sqrt{1 + 4 + 9} = \sqrt{14}$

$\mathbf{a}(t) = \mathbf{0}$

11. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \dfrac{t^2}{2}\mathbf{k}$

$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$

$s(t) = \sqrt{1 + 4t^2 + t^2} = \sqrt{1 + 5t^2}$

$\mathbf{a}(t) = 2\mathbf{j} + \mathbf{k}$

13. $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{9 - t^2}\mathbf{k}$

$\mathbf{v}(t) = \mathbf{i} + \mathbf{j} - \dfrac{t}{\sqrt{9 - t^2}}\mathbf{k}$

$s(t) = \sqrt{1 + 1 + \dfrac{t^2}{9 - t^2}} = \sqrt{\dfrac{18 - t^2}{9 - t^2}}$

$\mathbf{a}(t) = -\dfrac{9}{(9 - t^2)^{3/2}}\mathbf{k}$

15. $\mathbf{r}(t) = \langle 4t, 3\cos t, 3\sin t\rangle$

$\mathbf{v}(t) = \langle 4, -3\sin t, 3\cos t\rangle = 4\mathbf{i} - 3\sin t\mathbf{j} + 3\cos t\mathbf{k}$

$s(t) = \sqrt{16 + 9\sin^2 t + 9\cos^2 t} = 5$

$\mathbf{a}(t) = \langle 0, -3\cos t, -3\sin t\rangle = -3\cos t\mathbf{j} - 3\sin t\mathbf{k}$

17. (a) $\mathbf{r}(t) = \left\langle t, -t^2, \dfrac{t^3}{4}\right\rangle, t_0 = 1$

$\mathbf{r}'(t) = \left\langle 1, -2t, \dfrac{3t^2}{4}\right\rangle$

$\mathbf{r}'(1) = \left\langle 1, -2, \dfrac{3}{4}\right\rangle$

$x = 1 + t,\ y = -1 - 2t,\ z = \dfrac{1}{4} + \dfrac{3}{4}t$

(b) $\mathbf{r}(1 + 0.1) \approx \left\langle 1 + 0.1, -1 - 2(0.1), \dfrac{1}{4} + \dfrac{3}{4}(0.1)\right\rangle$

$= \langle 1.100, -1.200, 0.325\rangle$

19. $\mathbf{a}(t) = \mathbf{i} + \mathbf{j} + \mathbf{k}, \mathbf{v}(0) = \mathbf{0}, \mathbf{r}(0) = \mathbf{0}$

$\mathbf{v}(t) = \displaystyle\int (\mathbf{i} + \mathbf{j} + \mathbf{k})\,dt = t\mathbf{i} + t\mathbf{j} + t\mathbf{k} + \mathbf{C}$

$\mathbf{v}(0) = \mathbf{C} = \mathbf{0}, \mathbf{v}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, \mathbf{v}(t) = t(\mathbf{i} + \mathbf{j} + \mathbf{k})$

$\mathbf{r}(t) = \displaystyle\int (t\mathbf{i} + t\mathbf{j} + t\mathbf{k})\,dt = \frac{t^2}{2}(\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{C} = \mathbf{0}, \mathbf{r}(t) = \dfrac{t^2}{2}(\mathbf{i} + \mathbf{j} + \mathbf{k}),$

$\mathbf{r}(2) = 2(\mathbf{i} + \mathbf{j} + \mathbf{k}) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

21. $\mathbf{a}(t) = t\mathbf{j} + t\mathbf{k}, \mathbf{v}(1) = 5\mathbf{j}, \mathbf{r}(1) = \mathbf{0}$

$\mathbf{v}(t) = \displaystyle\int (t\mathbf{j} + t\mathbf{k})\,dt = \frac{t^2}{2}\mathbf{j} + \frac{t^2}{2}\mathbf{k} + \mathbf{C}$

$\mathbf{v}(1) = \dfrac{1}{2}\mathbf{j} + \dfrac{1}{2}\mathbf{k} + \mathbf{C} = 5\mathbf{j} \implies \mathbf{C} = \dfrac{9}{2}\mathbf{j} - \dfrac{1}{2}\mathbf{k}$

$\mathbf{v}(t) = \left(\dfrac{t^2}{2} + \dfrac{9}{2}\right)\mathbf{j} + \left(\dfrac{t^2}{2} - \dfrac{1}{2}\right)\mathbf{k}$

$\mathbf{r}(t) = \displaystyle\int \left[\left(\frac{t^2}{2} + \frac{9}{2}\right)\mathbf{j} + \left(\frac{t^2}{2} - \frac{1}{2}\right)\mathbf{k}\right] dt$

$= \left(\dfrac{t^3}{6} + \dfrac{9}{2}t\right)\mathbf{j} + \left(\dfrac{t^3}{6} - \dfrac{1}{2}t\right)\mathbf{k} + \mathbf{C}$

$\mathbf{r}(1) = \dfrac{14}{3}\mathbf{j} - \dfrac{1}{3}\mathbf{k} + \mathbf{C} = \mathbf{0} \implies \mathbf{C} = -\dfrac{14}{3}\mathbf{j} + \dfrac{1}{3}\mathbf{k}$

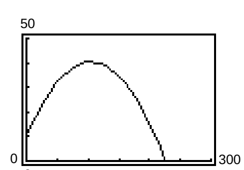
$\mathbf{r}(t) = \left(\dfrac{t^3}{6} + \dfrac{9}{2}t - \dfrac{14}{3}\right)\mathbf{j} + \left(\dfrac{t^3}{6} - \dfrac{1}{2}t + \dfrac{1}{3}\right)\mathbf{k}$

$\mathbf{r}(2) = \dfrac{17}{3}\mathbf{j} + \dfrac{2}{3}\mathbf{k}$

23. The velocity of an object involves both magnitude and direction of motion, whereas speed involves only magnitude.

25. $\mathbf{r}(t) = (88\cos 30°)t\mathbf{i} + [10 + (88\sin 30°)t - 16t^2]\mathbf{j}$

$= 44\sqrt{3}t\mathbf{i} + (10 + 44t - 16t^2)\mathbf{j}$



27. $\mathbf{r}(t) = (v_0 \cos \theta)t\mathbf{i} + \left[h + (v_0 \sin \theta)t - \frac{1}{2}gt^2\right]\mathbf{j} = \frac{v_0}{\sqrt{2}}t\mathbf{i} + \left(3 + \frac{v_0}{\sqrt{2}}t - 16t^2\right)\mathbf{j}$

$\frac{v_0}{\sqrt{2}}t = 300$ when $3 + \frac{v_0}{\sqrt{2}}t - 16t^2 = 3.$

$$t = \frac{300\sqrt{2}}{v_0},\ \frac{v_0}{\sqrt{2}}\left(\frac{300\sqrt{2}}{v_0}\right) - 16\left(\frac{300\sqrt{2}}{v_0}\right)^2 = 0,\ 300 - \frac{300^2(32)}{{v_0}^2} = 0$$

$${v_0}^2 = 300(32),\ v_0 = \sqrt{9600} = 40\sqrt{6},\ v_0 = 40\sqrt{6} \approx 97.98 \text{ ft/sec}$$

The maximum height is reached when the derivative of the vertical component is zero.

$$y(t) = 3 + \frac{tv_0}{\sqrt{2}} - 16t^2 = 3 + \frac{40\sqrt{6}}{\sqrt{2}}t - 16t^2 = 3 + 40\sqrt{3}t - 16t^2$$

$$y'(t) = 40\sqrt{3} - 32t = 0$$

$$t = \frac{40\sqrt{3}}{32} = \frac{5\sqrt{3}}{4}$$

Maximum height: $y\left(\frac{5\sqrt{3}}{4}\right) = 3 + 40\sqrt{3}\left(\frac{5\sqrt{3}}{4}\right) - 16\left(\frac{5\sqrt{3}}{4}\right)^2 = 78$ feet

29. $x(t) = t(v_0 \cos \theta)$ or $t = \frac{x}{v_0 \cos \theta}$

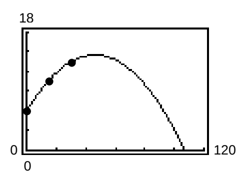
$$y(t) = t(v_0 \sin \theta) - 16t^2 + h$$

$$y = \frac{x}{v_0 \cos \theta}(v_0 \sin \theta) - 16\left(\frac{x^2}{{v_0}^2 \cos^2 \theta}\right) + h = (\tan \theta)x - \left(\frac{16}{{v_0}^2}\sec^2 \theta\right)x^2 + h$$

31. $\mathbf{r}(t) = t\mathbf{i} + (-0.004t^2 + 0.3667t + 6)\mathbf{j}$, or

(a) $y = -0.004x^2 + 0.3667x + 6$

(b)



(c) $y' = -0.008x + 0.3667 = 0 \implies x = 45.8375$ and

$y(45.8375) \approx 14.4$ feet.

(d) From Exercise 29,

$\tan \theta = 0.3667 \implies \theta \approx 20.14°$

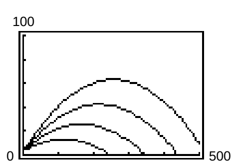
$$\frac{16 \sec^2 \theta}{{v_0}^2} = 0.004 \implies {v_0}^2 = \frac{16 \sec^2 \theta}{0.004} = \frac{4000}{\cos^2 \theta}$$

$\implies v_0 \approx 67.4$ ft/sec.

33. $100 \text{ mph} = \left(100 \frac{\text{miles}}{\text{hr}}\right)\left(5280 \frac{\text{feet}}{\text{mile}}\right)/(3600 \text{ sec/hour}) = \frac{440}{3}$ ft/sec

(a) $\mathbf{r}(t) = \left(\frac{440}{3}\cos \theta_0\right)t\mathbf{i} + \left[3 + \left(\frac{440}{3}\sin \theta_0\right)t - 16t^2\right]\mathbf{j}$

(b)



Graphing these curves together with $y = 10$ shows that $\theta_0 = 20°$.

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33. —CONTINUED—

(c) We want

$$x(t) = \left(\frac{440}{3}\cos\theta\right)t \ge 400 \quad \text{and} \quad y(t) = 3 + \left(\frac{440}{3}\sin\theta\right)t - 16t^2 \ge 10.$$

From $x(t)$, the minimum angle occurs when $t = 30/(11\cos\theta)$. Substituting this for t in $y(t)$ yields:

$$3 + \left(\frac{440}{3}\sin\theta\right)\left(\frac{30}{11\cos\theta}\right) - 16\left(\frac{30}{11\cos\theta}\right)^2 = 10$$

$$400\tan\theta - \frac{14{,}400}{121}\sec^2\theta = 7$$

$$\frac{14{,}400}{121}(1 + \tan^2\theta) - 400\tan\theta + 7 = 0$$

$$14{,}400\tan^2\theta - 48{,}400\tan\theta + 15{,}247 = 0$$

$$\tan\theta = \frac{48{,}400 \pm \sqrt{48{,}400^2 - 4(14{,}400)(15{,}247)}}{2(14{,}400)}$$

$$\theta = \tan^{-1}\left(\frac{48{,}400 - \sqrt{1{,}464{,}332{,}800}}{28{,}800}\right) \approx 19.38°$$

35. $\mathbf{r}(t) = (v\cos\theta)t\,\mathbf{i} + [(v\sin\theta)t - 16t^2]\mathbf{j}$

(a) We want to find the minimum initial speed v as a function of the angle θ. Since the bale must be thrown to the position (16, 8), we have

$$16 = (v\cos\theta)t$$

$$8 = (v\sin\theta)t - 16t^2.$$

$t = 16/(v\cos\theta)$ from the first equation. Substituting into the second equation and solving for v, we obtain:

$$8 = (v\sin\theta)\left(\frac{16}{v\cos\theta}\right) - 16\left(\frac{16}{v\cos\theta}\right)^2$$

$$1 = 2\frac{\sin\theta}{\cos\theta} - 512\left(\frac{1}{v^2\cos^2\theta}\right)$$

$$512\frac{1}{v^2\cos^2\theta} = 2\frac{\sin\theta}{\cos\theta} - 1$$

$$\frac{1}{v^2} = \left(2\frac{\sin\theta}{\cos\theta} - 1\right)\frac{\cos^2\theta}{512} = \frac{2\sin\theta\cos\theta - \cos^2\theta}{512}$$

$$v^2 = \frac{512}{2\sin\theta\cos\theta - \cos^2\theta}$$

We minimize $f(\theta) = \dfrac{512}{2\sin\theta\cos\theta - \cos^2\theta}$.

$$f'(\theta) = -512\frac{2\cos^2\theta - 2\sin^2\theta + 2\sin\theta\cos\theta}{(2\sin\theta\cos\theta - \cos^2\theta)^2}$$

$$f'(\theta) = 0 \implies 2\cos(2\theta) + \sin(2\theta) = 0$$

$$\tan(2\theta) = -2$$

$$\theta \approx 1.01722 \approx 58.28°$$

Substituting into the equation for v, $v \approx 28.78$ feet per second.

(b) If $\theta = 45°$,

$$16 = (v\cos\theta)t = v\frac{\sqrt{2}}{2}t$$

$$8 = (v\sin\theta)t - 16t^2 = v\frac{\sqrt{2}}{2}t - 16t^2$$

From part (a), $v^2 = \dfrac{512}{2(\sqrt{2}/2)(\sqrt{2}/2) - (\sqrt{2}/2)^2} = \dfrac{512}{1/2} = 1024 \implies v = 32$ ft/sec.

37. $\mathbf{r}(t) = (v_0 \cos \theta)t\mathbf{i} + [(v_0 \sin \theta)t - 16t^2]\mathbf{j}$

$(v_0 \sin \theta)t - 16t^2 = 0$ when $t = 0$ and $t = \dfrac{v_0 \sin \theta}{16}$.

The range is

$x = (v_0 \cos \theta)t = (v_0 \cos \theta)\dfrac{v_0 \sin \theta}{16} = \dfrac{v_0^2}{32} \sin 2\theta.$

Hence,

$$x = \frac{1200^2}{32} \sin(2\theta) = 3000 \implies \sin 2\theta = \frac{1}{15} \implies \theta \approx 1.91°.$$

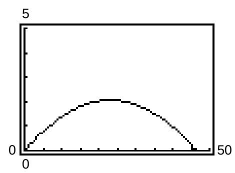
39. (a) $\theta = 10°,\ v_0 = 66$ ft/sec

$\mathbf{r}(t) = (66 \cos 10°)t\mathbf{i} + [0 + (66 \sin 10°)t - 16t^2]\mathbf{j}$

$\mathbf{r}(t) \approx (65t)\mathbf{i} + (11.46t - 16t^2)\mathbf{j}$

Maximum height: 2.052 feet

Range: 46.557 feet



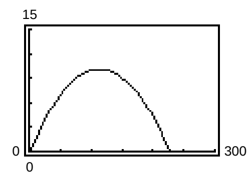
(b) $\theta = 10°,\ v_0 = 146$ ft/sec

$\mathbf{r}(t) = (146 \cos 10°)t\mathbf{i} + [0 + (146 \sin 10°)t - 16t^2]\mathbf{j}$

$\mathbf{r}(t) \approx (143.78t)\mathbf{i} + (25.35t - 16t^2)\mathbf{j}$

Maximum height: 10.043 feet

Range: 227.828 feet



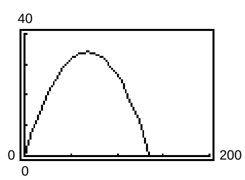
(c) $\theta = 45°,\ v_0 = 66$ ft/sec

$\mathbf{r}(t) = (66 \cos 45°)t\mathbf{i} + [0 + (66 \sin 45°)t - 16t^2]\mathbf{j}$

$\mathbf{r}(t) \approx (46.67t)\mathbf{i} + (46.67t - 16t^2)\mathbf{j}$

Maximum height: 34.031 feet

Range: 136.125 feet



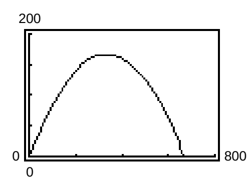
(d) $\theta = 45°,\ v_0 = 146$ ft/sec

$\mathbf{r}(t) = (146 \cos 45°)t\mathbf{i} + [0 + (146 \sin 45°)t - 16t^2]\mathbf{j}$

$\mathbf{r}(t) \approx (103.24t)\mathbf{i} + (103.24t - 16t^2)\mathbf{j}$

Maximum height: 166.531 feet

Range: 666.125 feet



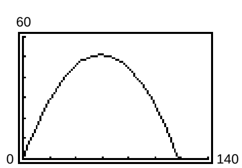
(e) $\theta = 60°,\ v_0 = 66$ ft/sec

$\mathbf{r}(t) = (66 \cos 60°)t\mathbf{i} + [0 + (66 \sin 60°)t - 16t^2]\mathbf{j}$

$\mathbf{r}(t) \approx (33t)\mathbf{i} + (57.16t - 16t^2)\mathbf{j}$

Maximum height: 51.074 feet

Range: 117.888 feet



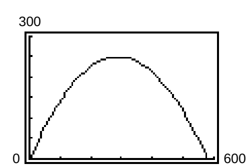
(f) $\theta = 60°,\ v_0 = 146$ ft/sec

$\mathbf{r}(t) = (146 \cos 60°)t\mathbf{i} + [0 + (146 \sin 60°)t - 16t^2]\mathbf{j}$

$\mathbf{r}(t) \approx (73t)\mathbf{i} + (126.44t - 16t^2)\mathbf{j}$

Maximum height: 249.797 feet

Range: 576.881 feet



41. $\mathbf{r}(t) = (v_0 \cos \theta)t\mathbf{i} + [h + (v_0 \sin \theta)t - 4.9t^2]\mathbf{j}$

$= (100 \cos 30°)t\mathbf{i} + [1.5 + (100 \sin 30°)t - 4.9t^2]\mathbf{j}$

The projectile hits the ground when $-4.9t^2 + 100\left(\frac{1}{2}\right)t + 1.5 = 0 \implies t \approx 10.234$ seconds.

The range is therefore $(100 \cos 30°)(10.234) \approx 886.3$ meters.

The maximum height occurs when $dy/dt = 0$.

$100 \sin 30 = 9.8t \implies t \approx 5.102$ sec

The maximum height is

$y = 1.5 + (100 \sin 30°)(5.102) - 4.9(5.102)^2 \approx 129.1$ meters.

43. $\mathbf{r}(t) = b(\omega t - \sin \omega t)\,\mathbf{i} + b(1 - \cos \omega t)\mathbf{j}$

$\mathbf{v}(t) = b(\omega - \omega \cos \omega t)\mathbf{i} + b\omega \sin \omega t\,\mathbf{j} = b\omega(1 - \cos \omega t)\mathbf{i} + b\omega \sin \omega t\mathbf{j}$

$\mathbf{a}(t) = (b\omega^2 \sin \omega t)\mathbf{i} + (b\omega^2 \cos \omega t)\mathbf{j} = b\omega^2[\sin(\omega t)\mathbf{i} + \cos(\omega t)\mathbf{j}]$

$\|\mathbf{v}(t)\| = \sqrt{2}\,b\omega\sqrt{1 - \cos(\omega t)}$

$\|\mathbf{a}(t)\| = b\omega^2$

(a) $\|\mathbf{v}(t)\| = 0$ when $\omega t = 0, 2\pi, 4\pi, \ldots$

(b) $\|\mathbf{v}(t)\|$ is maximum when $\omega t = \pi, 3\pi, \ldots,$ then $\|\mathbf{v}(t)\| = 2b\omega$.

45. $\mathbf{v}(t) = -b\omega \sin(\omega t)\mathbf{i} + b\omega \cos(\omega t)\mathbf{j}$

$\mathbf{r}(t) \cdot \mathbf{v}(t) = -b^2\omega \sin(\omega t) \cos(\omega t) + b^2\omega \sin(\omega t) \cos(\omega t) = 0$

Therefore, $\mathbf{r}(t)$ and $\mathbf{v}(t)$ are orthogonal.

47. $\mathbf{a}(t) = -b\omega^2 \cos(\omega t)\mathbf{i} - b\omega^2 \sin(\omega t)\mathbf{j} = -b\omega^2[\cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}] = -\omega^2\mathbf{r}(t)$

$\mathbf{a}(t)$ is a negative multiple of a unit vector from $(0, 0)$ to $(\cos \omega t, \sin \omega t)$ and thus $\mathbf{a}(t)$ is directed toward the origin.

49. $\|\mathbf{a}(t)\| = \omega^2 b$

$1 = m(32)$

$F = m(\omega^2 b) = \frac{1}{32}(2\omega^2) = 10$

$\omega = 4\sqrt{10}$ rad/sec

$\|\mathbf{v}(t)\| = b\omega = 8\sqrt{10}$ ft/sec

51. To find the range, set $y(t) = h + (v_0 \sin \theta)t - \frac{1}{2}gt^2 = 0$ then $0 = \left(\frac{1}{2}g\right)t^2 - (v_0 \sin \theta)t - h$.
By the Quadratic Formula, (discount the negative value)

$$t = \frac{v_0 \sin \theta + \sqrt{(-v_0 \sin \theta)^2 - 4[(1/2)g](-h)}}{2[(1/2)g]} = \frac{v_0 \sin \theta + \sqrt{v_0^2 \sin^2 \theta + 2gh}}{g}.$$

At this time,

$$x(t) = v_0 \cos \theta\left(\frac{v_0 \sin \theta + \sqrt{v_0^2 \sin^2 \theta + 2gh}}{g}\right) = \frac{v_0 \cos \theta}{g}\left(v_0 \sin \theta + \sqrt{v_0^2\left(\sin^2 \theta + \frac{2gh}{v_0^2}\right)}\right)$$

$$= \frac{v_0^2 \cos \theta}{g}\left(\sin \theta + \sqrt{\sin^2 \theta + \frac{2gh}{v_0^2}}\right).$$

53. $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ Position vector

$\mathbf{v}(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$ Velocity vector

$\mathbf{a}(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}$ Acceleration vector

$$\text{Speed} = \|\mathbf{v}(t)\| = \sqrt{(x'(t)^2 + y'(t)^2 + z'(t)^2}$$
$$= C,\ C \text{ is a constant.}$$
$$\frac{d}{dt}[x'(t)^2 + y'(t)^2 + z'(t)^2] = 0$$
$$2x'(t)x''(t) + 2y'(t)y''(t) + 2z'(t)z''(t) = 0$$
$$2[x'(t)x''(t) + y'(t)y''(t) + z'(t)z''(t)] = 0$$
$$\mathbf{v}(t) \cdot \mathbf{a}(t) = 0$$

Orthogonal

55. $\mathbf{r}(t) = 6\cos t\mathbf{i} + 3\sin t\mathbf{j}$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = -6\sin t\mathbf{i} + 3\cos t\mathbf{j}$

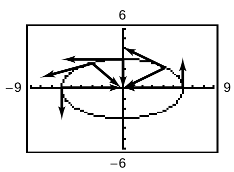
$$\|\mathbf{v}(t)\| = \sqrt{36\sin^2 t + 9\cos^2 t}$$
$$= 3\sqrt{4\sin^2 t + \cos^2 t}$$
$$= 3\sqrt{3\sin^2 t + 1}$$

$\mathbf{a}(t) = \mathbf{v}'(t) = -6\cos t\mathbf{i} - 3\sin t\mathbf{j}$

(b)

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
Speed	3	$\frac{3}{2}\sqrt{10}$	6	$\frac{3}{2}\sqrt{13}$	3

(c)



(d) The speed is increasing when the angle between $\mathbf{v}$ and $\mathbf{a}$ is in the interval

$$\left[0, \frac{\pi}{2}\right).$$

The speed is decreasing when the angle is in the interval

$$\left(\frac{\pi}{2}, \pi\right].$$

Section 11.4 Tangent Vectors and Normal Vectors

1. $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j}$

$\mathbf{r}'(t) = 2t\mathbf{i} + 2\mathbf{j},\ \|\mathbf{r}'(t)\| = \sqrt{4t^2 + 4} = 2\sqrt{t^2 + 1}$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{2t\mathbf{i} + 2\mathbf{j}}{2\sqrt{t^2 + 1}} = \frac{1}{\sqrt{t^2 + 1}}(t\mathbf{i} + \mathbf{j})$$

$$\mathbf{T}(1) = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

3. $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}$

$\mathbf{r}'(t) = -4\sin t\mathbf{i} + 4\cos t\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{16\sin^2 t + 16\cos^2 t} = 4$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

$$\mathbf{T}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

5. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + \mathbf{k}$

When $t = 0$, $\mathbf{r}'(0) = \mathbf{i} + \mathbf{k}$, $[t = 0 \text{ at } (0, 0, 0)]$.

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{k})$$

Direction numbers: $a = 1,\ b = 0,\ c = 1$

Parametric equations: $x = t,\ y = 0,\ z = t$

7. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}$

$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \mathbf{k}$

When $t = 0$, $\mathbf{r}'(0) = 2\mathbf{j} + \mathbf{k}$, $[t = 0 \text{ at } (2, 0, 0)]$.

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \frac{\sqrt{5}}{5}(2\mathbf{j} + \mathbf{k})$$

Direction numbers: $a = 0,\ b = 2,\ c = 1$

Parametric equations: $x = 2,\ y = 2t,\ z = t$

9. $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 4\rangle$

$\mathbf{r}'(t) = \langle -2\sin t, 2\cos t, 0\rangle$

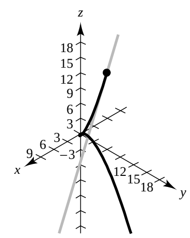
When $t = \frac{\pi}{4}$, $\mathbf{r}'\left(\frac{\pi}{4}\right) = \left\langle -\sqrt{2}, \sqrt{2}, 0\right\rangle$, $\left[t = \frac{\pi}{4} \text{ at } \left(\sqrt{2}, \sqrt{2}, 4\right)\right]$.

$$\mathbf{T}\left(\frac{\pi}{4}\right) = \frac{\mathbf{r}'(\pi/4)}{\|\mathbf{r}'(\pi/4)\|} = \frac{1}{2}\left\langle -\sqrt{2}, \sqrt{2}, 0\right\rangle$$

Direction numbers: $a = -\sqrt{2},\ b = \sqrt{2},\ c = 0$

Parametric equations: $x = -\sqrt{2}t + \sqrt{2},\ y = \sqrt{2}t + \sqrt{2},\ z = 4$

11. $\mathbf{r}(t) = \left\langle t, t^2, \frac{2}{3}t^3\right\rangle$



$\mathbf{r}'(t) = \langle 1, 2t, 2t^2\rangle$

When $t = 3$, $\mathbf{r}'(3) = \langle 1, 6, 18\rangle$, $[t = 3 \text{ at } (3, 9, 18)]$.

$$\mathbf{T}(3) = \frac{\mathbf{r}'(3)}{\|\mathbf{r}'(3)\|} = \frac{1}{19}\langle 1, 6, 18\rangle$$

Direction numbers: $a = 1,\ b = 6,\ c = 18$

Parametric equations: $x = t + 3,\ y = 6t + 9,\ z = 18t + 18$

13. $\mathbf{r}(t) = t\mathbf{i} + \ln t\mathbf{j} + \sqrt{t}\mathbf{k}, \quad t_0 = 1$

$$\mathbf{r}'(t) = \mathbf{i} + \frac{1}{t}\mathbf{j} + \frac{1}{2\sqrt{t}}\mathbf{k} \cdot \mathbf{r}'(1) = \mathbf{i} + \mathbf{j} + \frac{1}{2}\mathbf{k}$$

$$\mathbf{T}(1) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{i} + \mathbf{j} + (1/2)\mathbf{k}}{\sqrt{1 + 1 + (1/4)}} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$$

Tangent line: $x = 1 + t,\ y = t,\ z = 1 + \frac{1}{2}t$

$$\mathbf{r}(t_0 + 0.1) = \mathbf{r}(1.1) \approx 1.1\mathbf{i} + 0.1\mathbf{j} + 1.05\mathbf{k}$$

$$= \langle 1.1, 0.1, 1.05\rangle$$

15. $\mathbf{r}(4) = \langle 2, 16, 2\rangle$

$\mathbf{u}(8) = \langle 2, 16, 2\rangle$

Hence the curves intersect.

$$\mathbf{r}'(t) = \left\langle 1, 2t, \frac{1}{2}\right\rangle,\ \mathbf{r}'(4) = \left\langle 1, 8, \frac{1}{2}\right\rangle$$

$$\mathbf{u}'(s) = \left\langle \frac{1}{4}, 2, \frac{1}{3}s^{-2/3}\right\rangle,\ \mathbf{u}'(8) = \left\langle \frac{1}{4}, 2, \frac{1}{12}\right\rangle$$

$$\cos\theta = \frac{\mathbf{r}'(4)\cdot\mathbf{u}'(8)}{\|\mathbf{r}'(4)\|\,\|\mathbf{u}'(8)\|} \approx \frac{16.29167}{16.29513} \Longrightarrow \theta \approx 1.2°$$

17. $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{2}t^2\mathbf{j}, t = 2$

$\mathbf{r}'(t) = \mathbf{i} + t\mathbf{j}$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{i} + t\mathbf{j}}{\sqrt{1 + t^2}}$$

$$\mathbf{T}'(t) = \frac{-t}{(t^2 + 1)^{3/2}}\mathbf{i} + \frac{1}{(t^2 + 1)^{3/2}}\mathbf{j}$$

$$\mathbf{T}'(2) = \frac{-2}{5^{3/2}}\mathbf{i} + \frac{1}{5^{3/2}}\mathbf{j}$$

$$\mathbf{N}(2) = \frac{\mathbf{T}'(2)}{\|\mathbf{T}'(2)\|} + \frac{1}{\sqrt{5}}(-2\mathbf{i} + \mathbf{j}) = \frac{-2\sqrt{5}}{5}\mathbf{i} + \frac{\sqrt{5}}{5}\mathbf{j}$$

19. $\mathbf{r}(t) = 6\cos t\mathbf{i} + 6\sin t\mathbf{j} + \mathbf{k}, t = \frac{3\pi}{4}$

$\mathbf{r}'(t) = -6\sin t\mathbf{i} + 6\cos t\mathbf{j}$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

$\mathbf{T}'(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}, \|\mathbf{T}(t)\| = 1$

$$\mathbf{N}\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$$

21. $\mathbf{r}(t) = 4t\mathbf{i}$

$\mathbf{v}(t) = 4\mathbf{i}$

$\mathbf{a}(t) = \mathbf{O}$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{4\mathbf{i}}{4} = \mathbf{i}$$

$\mathbf{T}'(t) = \mathbf{O}$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$ is undefined.

The path is a line and the speed is constant.

23. $\mathbf{r}(t) = 4t^2\mathbf{i}$

$\mathbf{v}(t) = 8t\mathbf{i}$

$\mathbf{a}(t) = 8\mathbf{i}$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{8t\mathbf{i}}{8t} = \mathbf{i}$$

$\mathbf{T}'(t) = \mathbf{O}$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$ is undefined.

The path is a line and the speed is variable.

25. $\mathbf{r}(t) = t\mathbf{i} + \dfrac{1}{t}\mathbf{j},\ \mathbf{v}(t) = \mathbf{i} - \dfrac{1}{t^2}\mathbf{j},\ \mathbf{v}(1) = \mathbf{i} - \mathbf{j},$

$\mathbf{a}(t) = \dfrac{2}{t^3}\mathbf{j},\ \mathbf{a}(1) = 2\mathbf{j}$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{t^2}{\sqrt{t^4+1}}\left(\mathbf{i} - \frac{1}{t^2}\mathbf{j}\right) = \frac{1}{\sqrt{t^4+1}}(t^2\mathbf{i} - \mathbf{j})$$

$$\mathbf{T}(1) = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j}) = \frac{\sqrt{2}}{2}(\mathbf{i} - \mathbf{j})$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{\dfrac{2t}{(t^4+1)^{3/2}}\mathbf{i} + \dfrac{2t^3}{(t^4+1)^{3/2}}\mathbf{j}}{\dfrac{2t}{(t^4+1)}}$$

$$= \frac{1}{\sqrt{t^4+1}}(\mathbf{i} + t^2\mathbf{j})$$

$$\mathbf{N}(1) = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = \frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j})$$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = -\sqrt{2}$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \sqrt{2}$

27. $\mathbf{r}(t) = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j}$

$\mathbf{v}(t) = e^t(\cos t - \sin t)\mathbf{i} + e^t(\cos t + \sin t)\mathbf{j}$

$\mathbf{a}(t) = e^t(-2 \sin t)\mathbf{i} + e^t(2 \cos t)\mathbf{j}$

At $t = \dfrac{\pi}{2},\ \mathbf{T} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j}) = \dfrac{\sqrt{2}}{2}(-\mathbf{i} + \mathbf{j}).$

Motion along **r** is counterclockwise. Therefore,

$$\mathbf{N} = \frac{1}{\sqrt{2}}(-\mathbf{i} - \mathbf{j}) = -\frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j}).$$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \sqrt{2}e^{\pi/2}$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \sqrt{2}e^{\pi/2}$

29. $\mathbf{r}(t_0) = (\cos \omega t_0 + \omega t_0 \sin \omega t_0)\mathbf{i} + (\sin \omega t_0 - \omega t_0 \cos \omega t_0)\mathbf{j}$

$\mathbf{v}(t_0) = (\omega^2 t_0 \cos \omega t_0)\mathbf{i} + (\omega^2 t_0 \sin \omega t_0)\mathbf{j}$

$\mathbf{a}(t_0) = \omega^2[(\cos \omega t_0 - \omega t_0 \sin \omega t_0)\mathbf{i} + (\omega t_0 \cos \omega t_0 + \sin \omega t_0)\mathbf{j}]$

$$\mathbf{T}(t_0) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = (\cos \omega t_0)\mathbf{i} + (\sin \omega t_0)\mathbf{j}$$

Motion along **r** is counterclockwise. Therefore

$\mathbf{N}(t_0) = (-\sin \omega t_0)\mathbf{i} + (\cos \omega t_0)\mathbf{j}.$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \omega^2$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \omega^2(\omega t_0) = \omega^3 t_0$

31. $\mathbf{r}(t) = a\cos(\omega t)\mathbf{i} + a\sin(\omega t)\mathbf{j}$

$\mathbf{v}(t) = -a\omega\sin(\omega t)\mathbf{i} + a\omega\cos(\omega t)\mathbf{j}$

$\mathbf{a}(t) = -a\omega^2\cos(\omega t)\mathbf{i} - a\omega^2\sin(\omega t)\mathbf{j}$

$\mathbf{T}(t) = \dfrac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = -\sin(\omega t)\mathbf{i} + \cos(\omega t)\mathbf{j}$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = -\cos(\omega t)\mathbf{i} - \sin(\omega t)\mathbf{j}$

$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = 0$

$a_{\mathbf{N}} = \mathbf{a}\cdot\mathbf{N} = a\omega^2$

33. Speed: $\|\mathbf{v}(t)\| = a\omega$

The speed is constant since $a_{\mathbf{T}} = 0$.

35. $\mathbf{r}(t) = t\mathbf{i} + \dfrac{1}{t}\mathbf{j},\ t_0 = 2$

$x = t,\ y = \dfrac{1}{t} \Rightarrow xy = 1$

$\mathbf{r}'(t) = \mathbf{i} - \dfrac{1}{t^2}\mathbf{j}$

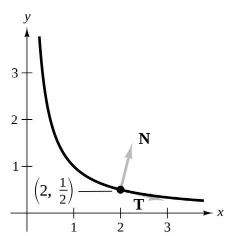
$\mathbf{T}(t) = \dfrac{t^2\mathbf{i} - \mathbf{j}}{\sqrt{t^4+1}}$

$\mathbf{N}(t) = \dfrac{\mathbf{i} + t^2\mathbf{j}}{\sqrt{t^4+1}}$

$\mathbf{r}(2) = 2\mathbf{i} + \dfrac{1}{2}\mathbf{j}$

$\mathbf{T}(2) = \dfrac{\sqrt{17}}{17}(4\mathbf{i} - \mathbf{j})$

$\mathbf{N}(2) = \dfrac{\sqrt{17}}{17}(\mathbf{i} + 4\mathbf{j})$



37. $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} - 3t\mathbf{k}$

$\mathbf{v}(t) = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$

$\mathbf{a}(t) = \mathbf{0}$

$\mathbf{T}(t) = \dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{1}{\sqrt{14}}(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) = \dfrac{\sqrt{14}}{14}(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'}{\|\mathbf{T}'\|}$ is undefined.

$a_{\mathbf{T}}, a_{\mathbf{N}}$ are not defined.

39. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \dfrac{t^2}{2}\mathbf{k}$

$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$

$\mathbf{v}(1) = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$

$\mathbf{a}(t) = 2\mathbf{j} + \mathbf{k}$

$\mathbf{T}(t) = \dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{1}{\sqrt{1+5t^2}}(\mathbf{i} + 2t\mathbf{j} + t\mathbf{k})$

$\mathbf{T}(1) = \dfrac{\sqrt{6}}{6}(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$

$$\mathbf{N}(t) = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} = \frac{\dfrac{-5t\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{(1+5t^2)^{3/2}}}{\dfrac{\sqrt{5}}{1+5t^2}} = \frac{-5t\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{5}\sqrt{1+5t^2}}$$

$\mathbf{N}(1) = \dfrac{\sqrt{30}}{30}(-5\mathbf{i} + 2\mathbf{j} + \mathbf{k})$

$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = \dfrac{5\sqrt{6}}{6}$

$a_{\mathbf{N}} = \mathbf{a}\cdot\mathbf{N} = \dfrac{\sqrt{30}}{6}$

41. $\mathbf{r}(t) = 4t\mathbf{i} + 3\cos t\mathbf{j} + 3\sin t\mathbf{k}$

$\mathbf{v}(t) = 4\mathbf{i} - 3\sin t\mathbf{j} + 3\cos t\mathbf{k}$

$\mathbf{v}\left(\dfrac{\pi}{2}\right) = 4\mathbf{i} - 3\mathbf{j}$

$\mathbf{a}(t) = -3\cos t\mathbf{j} - 3\sin t\mathbf{k}$

$\mathbf{a}\left(\dfrac{\pi}{2}\right) = -3\mathbf{k}$

$\mathbf{T}(t) = \dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{1}{5}(4\mathbf{i} - 3\sin t\mathbf{j} + 3\cos t\mathbf{k})$

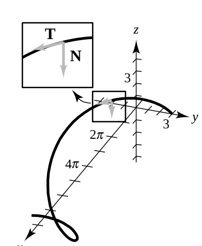
$\mathbf{T}\left(\dfrac{\pi}{2}\right) = \dfrac{1}{5}(4\mathbf{i} - 3\mathbf{j})$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'}{\|\mathbf{T}'\|} = -\cos t\mathbf{j} - \sin t\mathbf{k}$

$\mathbf{N}\left(\dfrac{\pi}{2}\right) = -\mathbf{k}$

$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = 0$

$a_{\mathbf{N}} = \mathbf{a}\cdot\mathbf{N} = 3$



43. $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$

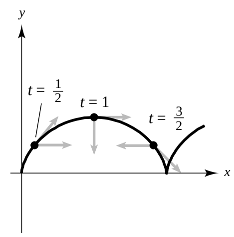
$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$

If $a(t) = a_{\mathbf{T}}\mathbf{T}(t) + a_{\mathbf{N}}\mathbf{N}(t)$, then $a_{\mathbf{T}}$ is the tangential component of acceleration and $a_{\mathbf{N}}$ is the normal component of acceleration.

45. If $a_{\mathbf{N}} = 0$, then the motion is in a straight line.

47. $\mathbf{r}(t) = \langle \pi t - \sin \pi t, 1 - \cos \pi t\rangle$

The graph is a cycloid.



(a) $\mathbf{r}(t) = \langle \pi t - \sin \pi t, 1 - \cos \pi t\rangle$

$\mathbf{v}(t) = \langle \pi - \pi \cos \pi t, \pi \sin \pi t\rangle$

$\mathbf{a}(t) = \langle \pi^2 \sin \pi t, \pi^2 \cos \pi t\rangle$

$\mathbf{T}(t) = \dfrac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \dfrac{1}{\sqrt{2(1 - \cos \pi t)}}\langle 1 - \cos \pi t, \sin \pi t\rangle$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \dfrac{1}{\sqrt{2(1 - \cos \pi t)}}\langle \sin \pi t, -1 + \cos \pi t\rangle$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \dfrac{1}{\sqrt{2(1 - \cos \pi t)}}[\pi^2 \sin \pi t(1 - \cos \pi t) + \pi^2 \cos \pi t \sin \pi t] = \dfrac{\pi^2 \sin \pi t}{\sqrt{2(1 - \cos \pi t)}}$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \dfrac{1}{\sqrt{2(1 - \cos \pi t)}}[\pi^2 \sin^2 \pi t + \pi^2 \cos \pi t(-1 + \cos \pi t)] = \dfrac{\pi^2(1 - \cos \pi t)}{\sqrt{2(1 - \cos \pi t)}} = \dfrac{\pi^2\sqrt{2(1 - \cos \pi t)}}{2}$

When $t = \dfrac{1}{2}$: $a_{\mathbf{T}} = \dfrac{\pi^2}{\sqrt{2}} = \dfrac{\sqrt{2}\pi^2}{2}$, $a_{\mathbf{N}} = \dfrac{\sqrt{2}\pi^2}{2}$

When $t = 1$: $a_{\mathbf{T}} = 0$, $a_{\mathbf{N}} = \pi^2$

When $t = \dfrac{3}{2}$: $a_{\mathbf{T}} = -\dfrac{\sqrt{2}\pi^2}{2}$, $a_{\mathbf{N}} = \dfrac{\sqrt{2}\pi^2}{2}$

(b) Speed: $s = \|\mathbf{v}(t)\| = \pi\sqrt{2(1 - \cos \pi t)}$

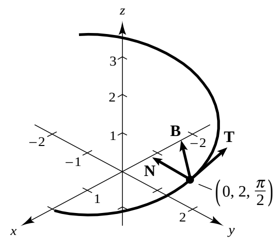
$\dfrac{ds}{dt} = \dfrac{\pi^2 \sin \pi t}{\sqrt{2(1 - \cos \pi t)}} = a_{\mathbf{T}}$

When $t = \dfrac{1}{2}$: $a_{\mathbf{T}} = \dfrac{\sqrt{2}\pi^2}{2} > 0 \implies$ the speed in increasing.

When $t = 1$: $a_{\mathbf{T}} = 0 \implies$ the height is maximum.

When $t = \dfrac{3}{2}$: $a_{\mathbf{T}} = -\dfrac{\sqrt{2}\pi^2}{2} < 0 \implies$ the speed is decreasing.

49. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + \frac{t}{2}\mathbf{k},\ t_0 = \frac{\pi}{2}$



$$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \frac{1}{2}\mathbf{k}$$

$$\mathbf{T}(t) = \frac{2\sqrt{17}}{17}\left(-2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \frac{1}{2}\mathbf{k}\right)$$

$$\mathbf{N}(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}$$

$$\mathbf{r}\left(\frac{\pi}{2}\right) = 2\mathbf{j} + \frac{\pi}{4}\mathbf{k}$$

$$\mathbf{T}\left(\frac{\pi}{2}\right) = \frac{2\sqrt{17}}{17}\left(-2\mathbf{i} + \frac{1}{2}\mathbf{k}\right) = \frac{\sqrt{17}}{17}(-4\mathbf{i} + \mathbf{k})$$

$$\mathbf{N}\left(\frac{\pi}{2}\right) = -\mathbf{j}$$

$$\mathbf{B}\left(\frac{\pi}{2}\right) = \mathbf{T}\left(\frac{\pi}{2}\right) \times \mathbf{N}\left(\frac{\pi}{2}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{4\sqrt{17}}{17} & 0 & \frac{\sqrt{17}}{17} \\ 0 & -1 & 0 \end{vmatrix} = \frac{\sqrt{17}}{17}\mathbf{i} + \frac{4\sqrt{17}}{17}\mathbf{k} = \frac{\sqrt{17}}{17}(\mathbf{i} + 4\mathbf{k})$$

51. From Theorem 11.3 we have:

$$\mathbf{r}(t) = (v_0 t\cos\theta)\mathbf{i} + (h + v_0 t\sin\theta - 16t^2)\mathbf{j}$$

$$\mathbf{v}(t) = v_0\cos\theta\mathbf{i} + (v_0\sin\theta - 32t)\mathbf{j}$$

$$\mathbf{a}(t) = -32\mathbf{j}$$

$$\mathbf{T}(t) = \frac{(v_0\cos\theta)\mathbf{i} + (v_0\sin\theta - 32t)\mathbf{j}}{\sqrt{v_0^2\cos^2\theta + (v_0\sin\theta - 32t)^2}}$$

$$\mathbf{N}(t) = \frac{(v_0\sin\theta - 32t)\mathbf{i} - v_0\cos\theta\mathbf{j}}{\sqrt{v_0^2\cos^2\theta + (v_0\sin\theta - 32t)^2}} \quad \text{(Motion is clockwise.)}$$

$$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = \frac{-32(v_0\sin\theta - 32t)}{\sqrt{v_0^2\cos^2\theta + (v_0\sin\theta - 32t)^2}}$$

$$a_{\mathbf{N}} = \mathbf{a}\cdot\mathbf{N} = \frac{32v_0\cos\theta}{\sqrt{v_0^2\cos^2\theta + (v_0\sin\theta - 32t)^2}}$$

Maximum height when $v_0\sin\theta - 32t = 0$; (vertical component of velocity)

At maximum height, $a_{\mathbf{T}} = 0$ and $a_{\mathbf{N}} = 32$.

53. $\mathbf{r}(t) = \langle 10\cos 10\pi t, 10\sin 10\pi t, 4 + 4t\rangle,\ 0 \le t \le \frac{1}{20}$

(a) $\mathbf{r}'(t) = \langle -100\pi\sin(10\pi t), 100\pi\cos(10\pi t), 4\rangle$

$$\|\mathbf{r}'(t)\| = \sqrt{(100\pi)^2\sin^2(10\pi t) + (100\pi)^2\cos^2(10\pi t) + 16}$$

$$= \sqrt{(100\pi)^2 + 16} = 4\sqrt{625\pi^2 + 1} \approx 314 \text{ mi/hr}$$

(b) $a_{\mathbf{T}} = 0$ and $a_{\mathbf{N}} = 1000\pi^2$

$a_{\mathbf{T}} = 0$ because the speed is constant.

55. $\mathbf{r}(t) = (a\cos\omega t)\mathbf{i} + (a\sin\omega t)\mathbf{j}$

From Exercise 31, we know $\mathbf{a}\cdot\mathbf{T} = 0$ and $\mathbf{a}\cdot\mathbf{N} = a\omega^2$.

(a) Let $\omega_0 = 2\omega$. Then

$$\mathbf{a}\cdot\mathbf{N} = a\omega_0^2 = a(2\omega)^2 = 4a\omega^2$$

or the centripetal acceleration is increased by a factor of 4 when the velocity is doubled.

(b) Let $a_0 = a/2$. Then

$$\mathbf{a}\cdot\mathbf{N} = a_0\omega^2 = \left(\frac{a}{2}\right)\omega^2 = \left(\frac{1}{2}\right)a\omega^2$$

or the centripetal acceleration is halved when the radius is halved.

57. $v = \sqrt{\dfrac{9.56 \times 10^4}{4100}} \approx 4.83$ mi/sec

59. $v = \sqrt{\dfrac{9.56 \times 10^4}{4385}} \approx 4.67$ mi/sec

61. Let $\mathbf{T}(t) = \cos\phi\mathbf{i} + \sin\phi\mathbf{j}$ be the unit tangent vector. Then

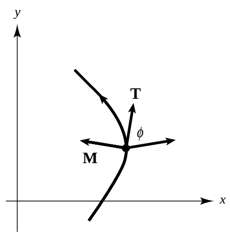
$$\mathbf{T}'(t) = \frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{d\phi}\frac{d\phi}{dt} = -(\sin\phi\mathbf{i} + \cos\phi\mathbf{j})\frac{d\phi}{dt} = \mathbf{M}\frac{d\phi}{dt}.$$

$\mathbf{M} = -\sin\phi\mathbf{i} + \cos\phi\mathbf{j} = \cos[\phi + (\pi/2)]\mathbf{i} + \sin[\phi + (\pi/2)]\mathbf{j}$ and is rotated counterclockwise through an angle of $\pi/2$ from $\mathbf{T}$.

If $d\phi/dt > 0$, then the curve bends to the left and $\mathbf{M}$ has the same direction as $\mathbf{T}'$. Thus, $\mathbf{M}$ has the same direction as

$$\mathbf{N} = \frac{\mathbf{T}'}{\|\mathbf{T}'\|},$$

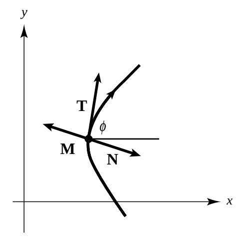
which is toward the concave side of the curve.



If $d\phi/dt < 0$, then the curve bends to the right and $\mathbf{M}$ has the opposite direction as $\mathbf{T}'$. Thus,

$$\mathbf{N} = \frac{\mathbf{T}'}{\|\mathbf{T}'\|}$$

again points to the concave side of the curve.



63. Using $\mathbf{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}$, $\mathbf{T} \times \mathbf{T} = \mathbf{O}$, and $\|\mathbf{T} \times \mathbf{N}\| = 1$, we have:

$$\begin{aligned}
\mathbf{v} \times \mathbf{a} &= \|\mathbf{v}\|\mathbf{T} \times (a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}) \\
&= \|\mathbf{v}\|a_{\mathbf{T}}(\mathbf{T} \times \mathbf{T}) + \|\mathbf{v}\|a_{\mathbf{N}}(\mathbf{T} \times \mathbf{N}) \\
&= \|\mathbf{v}\|a_{\mathbf{N}}(\mathbf{T} \times \mathbf{N}) \\
\|\mathbf{v} \times \mathbf{a}\| &= \|\mathbf{v}\|a_{\mathbf{N}}\|\mathbf{T} \times \mathbf{N}\| \\
&= \|\mathbf{v}\|a_{\mathbf{N}}
\end{aligned}$$

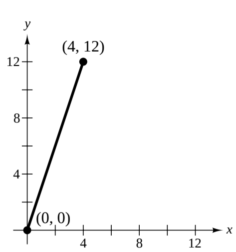
Thus, $a_{\mathbf{N}} = \dfrac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|}$.

Section 11.5 Arc Length and Curvature

1. $\mathbf{r}(t) = t\mathbf{i} + 3t\mathbf{j}$

$\dfrac{dx}{dt} = 1,\ \dfrac{dy}{dt} = 3,\ \dfrac{dz}{dt} = 0$

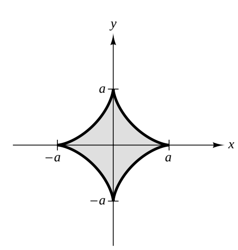
$$\begin{aligned}
s &= \int_0^4 \sqrt{1 + 9}\, dt \\
&= \sqrt{10}\int_0^4 dt \\
&= \Big[\sqrt{10}t\Big]_0^4 = 4\sqrt{10}
\end{aligned}$$



3. $\mathbf{r}(t) = a\cos^3 t\mathbf{i} + a\sin^3 t\mathbf{j}$

$\dfrac{dx}{dt} = -3a\cos^2 t\sin t,\ \dfrac{dy}{dt} = 3a\sin^2 t\cos t$

$$\begin{aligned}
s &= 4\int_0^{\pi/2} \sqrt{[3a\cos^2 t(-\sin t)]^2 + [3a\sin^2 t\cos t]^2}\, dt \\
&= 12a\int_0^{\pi/2} \sin t\cos t\, dt \\
&= 3a\int_0^{\pi/2} 2\sin 2t\, dt = \Big[-3a\cos 2t\Big]_0^{\pi/2} = 6a
\end{aligned}$$



5. (a) $\mathbf{r}(t) = (v_0 \cos \theta)t\mathbf{i} + \left[h + (v_0 \sin \theta)t - \frac{1}{2}gt^2\right]\mathbf{j}$

$$= (100 \cos 45°)t\mathbf{i} + \left[3 + (100 \sin 45°)t - \frac{1}{2}(32)t^2\right]\mathbf{j}$$

$$= 50\sqrt{2}t\mathbf{i} + \left[3 + 50\sqrt{2}t - 16t^2\right]\mathbf{j}$$

(b) $\mathbf{v}(t) = 50\sqrt{2}\mathbf{i} + \left(50\sqrt{2} - 32t\right)\mathbf{j}$

$$50\sqrt{2} - 32t = 0 \implies t = \frac{25\sqrt{2}}{16}$$

Maximum height: $3 + 50\sqrt{2}\left(\frac{25\sqrt{2}}{16}\right) - 16\left(\frac{25\sqrt{2}}{16}\right)^2 = 81.125$ ft

(c) $3 + 50\sqrt{2}t - 16t^2 = 0 \implies t \approx 4.4614$

Range: $50\sqrt{2}(4.4614) \approx 315.5$ feet

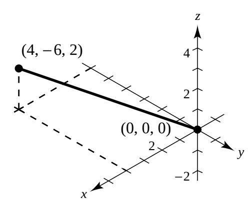
(d) $s = \int_0^{4.4614} \sqrt{\left(50\sqrt{2}\right)^2 + \left(50\sqrt{2} - 32t\right)^2}\,dt \approx 362.9$ feet

7. $\mathbf{r}(t) = 2t\mathbf{i} - 3t\mathbf{j} + t\mathbf{k}$

$$\frac{dx}{dt} = 2 \quad \frac{dy}{dt} = -3, \quad \frac{dz}{dt} = 1$$

$$s = \int_0^2 \sqrt{2^2 + (-3)^2 + 1^2}\,dt$$

$$= \int_0^2 \sqrt{14}\,dt = \left[\sqrt{14}t\right]_0^2 = 2\sqrt{14}$$

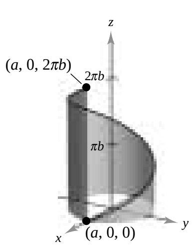


9. $\mathbf{r}(t) = a \cos t\mathbf{i} + a \sin t\mathbf{j} + bt\mathbf{k}$

$$\frac{dx}{dt} = -a \sin t, \quad \frac{dy}{dt} = a \cos t, \quad \frac{dz}{dt} = b$$

$$s = \int_0^{2\pi} \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + b^2}\,dt$$

$$= \int_0^{2\pi} \sqrt{a^2 + b^2}\,dt = \left[\sqrt{a^2 + b^2}t\right]_0^{2\pi} = 2\pi\sqrt{a^2 + b^2}$$



11. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + \ln t\mathbf{k}$

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 1, \quad \frac{dz}{dt} = \frac{1}{t}$$

$$s = \int_1^3 \sqrt{(2t)^2 + (1)^2 + \left(\frac{1}{t}\right)^2}\,dt$$

$$= \int_1^3 \sqrt{\frac{4t^4 + t^2 + 1}{t^2}}\,dt$$

$$= \int_1^3 \frac{\sqrt{4t^4 + t^2 + 1}}{t}\,dt \approx 8.37$$

13. $\mathbf{r}(t) = t\mathbf{i} + (4 - t^2)\mathbf{j} + t^3\mathbf{k}, \quad 0 \le t \le 2$

(a) $\mathbf{r}(0) = \langle 0, 4, 0\rangle, \ \mathbf{r}(2) = \langle 2, 0, 8\rangle$

distance $= \sqrt{2^2 + 4^2 + 8^2} = \sqrt{84} = 2\sqrt{21} \approx 9.165$

—CONTINUED—

13. —CONTINUED—

(b) $\mathbf{r}(0) = \langle 0, 4, 0\rangle$

$\mathbf{r}(0.5) = \langle 0.5, 3.75, .125\rangle$

$\mathbf{r}(1) = \langle 1, 3, 1\rangle$

$\mathbf{r}(1.5) = \langle 1.5, 1.75, 3.375\rangle$

$\mathbf{r}(2) = \langle 2, 0, 8\rangle$

$$\text{distance} \approx \sqrt{(0.5)^2 + (.25)^2 + (.125)^2} + \sqrt{(.5)^2 + (.75)^2 + (.875)^2} + \sqrt{(0.5)^2 + (1.25)^2 + (2.375)^2} + \sqrt{(0.5)^2 + (1.75)^2 + (4.625)^2}$$

$$\approx 0.5728 + 1.2562 + 2.7300 + 4.9702 \approx 9.529$$

(c) Increase the number of line segments.

(d) Using a graphing utility, you obtain 9.57057.

15. $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, t\rangle$

(a) $$s = \int_0^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2}\, du$$

$$= \int_0^t \sqrt{(-2\sin u)^2 + (2\cos u)^2 + (1)^2}\, du$$

$$= \int_0^t \sqrt{5}\, du = \Big[\sqrt{5}u\Big]_0^t = \sqrt{5}t$$

(b) $$\frac{s}{\sqrt{5}} = t$$

$$x = 2\cos\left(\frac{s}{\sqrt{5}}\right),\ y = 2\sin\left(\frac{s}{\sqrt{5}}\right),\ z = \frac{s}{\sqrt{5}}$$

$$\mathbf{r}(s) = 2\cos\left(\frac{s}{\sqrt{5}}\right)\mathbf{i} + 2\sin\left(\frac{s}{\sqrt{5}}\right)\mathbf{j} + \frac{s}{\sqrt{5}}\mathbf{k}$$

(c) When $s = \sqrt{5}$:

$x = 2\cos 1 \approx 1.081$

$y = 2\sin 1 \approx 1.683$

$z = 1$

$(1.081, 1.683, 1.000)$

When $s = 4$:

$x = 2\cos\dfrac{4}{\sqrt{5}} \approx -0.433$

$y = 2\sin\dfrac{4}{\sqrt{5}} \approx 1.953$

$z = \dfrac{4}{\sqrt{5}} \approx 1.789$

$(-0.433, 1.953, 1.789)$

(d) $$\|\mathbf{r}'(s)\| = \sqrt{\left(-\frac{2}{\sqrt{5}}\sin\left(\frac{s}{\sqrt{5}}\right)\right)^2 + \left(\frac{2}{\sqrt{5}}\cos\left(\frac{s}{\sqrt{5}}\right)\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2} = \sqrt{\frac{4}{5} + \frac{1}{5}} = 1$$

17. $$\mathbf{r}(s) = \left(1 + \frac{\sqrt{2}}{2}s\right)\mathbf{i} + \left(1 - \frac{\sqrt{2}}{2}s\right)\mathbf{j}$$

$$\mathbf{r}'(s) = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j} \quad \text{and} \quad \|\mathbf{r}'(s)\| = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$$

$$\mathbf{T}(s) = \frac{\mathbf{r}'(s)}{\|\mathbf{r}'(s)\|} = \mathbf{r}'(s)$$

$\mathbf{T}'(s) = \mathbf{0} \implies K = \|\mathbf{T}'(s)\| = 0$ (The curve is a line.)

19. $$\mathbf{r}(s) = 2\cos\left(\frac{s}{\sqrt{5}}\right)\mathbf{i} + 2\sin\left(\frac{s}{\sqrt{5}}\right)\mathbf{j} + \frac{s}{\sqrt{5}}\mathbf{k}$$

$$\mathbf{T}(s) = \mathbf{r}'(s) = -\frac{2}{\sqrt{5}}\sin\left(\frac{s}{\sqrt{5}}\right)\mathbf{i} + \frac{2}{\sqrt{5}}\cos\left(\frac{s}{\sqrt{5}}\right)\mathbf{j} + \frac{1}{\sqrt{5}}\mathbf{k}$$

$$\mathbf{T}'(s) = -\frac{2}{5}\cos\left(\frac{s}{\sqrt{5}}\right)\mathbf{i} - \frac{2}{5}\sin\left(\frac{s}{\sqrt{5}}\right)\mathbf{j}$$

$$K = \|\mathbf{T}'(s)\| = \frac{2}{5}$$

21. $\mathbf{r}(t) = 4t\mathbf{i} - 2t\mathbf{j}$

$\mathbf{v}(t) = 4\mathbf{i} - 2\mathbf{j}$

$\mathbf{T}(t) = \dfrac{1}{\sqrt{5}}(2\mathbf{i} - \mathbf{j})$

$\mathbf{T}'(t) = 0$

$K = \dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = 0$ (The curve is a line.)

23. $\mathbf{r}(t) = t\mathbf{i} + \dfrac{1}{t}\mathbf{j}$

$\mathbf{v}(t) = \mathbf{i} - \dfrac{1}{t^2}\mathbf{j}$

$\mathbf{v}(1) = \mathbf{i} - \mathbf{j}$

$\mathbf{a}(t) = \dfrac{2}{t^3}\mathbf{j}$

$\mathbf{a}(1) = 2\mathbf{j}$

$\mathbf{T}(t) = \dfrac{t^2\mathbf{i} - \mathbf{j}}{\sqrt{t^4 + 1}}$

$\mathbf{N}(t) = \dfrac{1}{(t^4 + 1)^{1/2}}(\mathbf{i} + t^2\mathbf{j})$

$\mathbf{N}(1) = \dfrac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$

$K = \dfrac{\mathbf{a} \cdot \mathbf{N}}{\|\mathbf{v}\|^2} = \dfrac{\sqrt{2}}{2}$

25. $\mathbf{r}(t) = 4\cos(2\pi t)\mathbf{i} + 4\sin(2\pi t)\mathbf{j}$

$\mathbf{r}'(t) = -8\pi\sin(2\pi t)\mathbf{i} + 8\pi\cos(2\pi t)\mathbf{j}$

$\mathbf{T}(t) = -\sin(2\pi t)\mathbf{i} + \cos(2\pi t)\mathbf{j}$

$\mathbf{T}'(t) = -2\pi\cos(2\pi t)\mathbf{i} - 2\pi\sin(2\pi t)\mathbf{j}$

$K = \dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \dfrac{2\pi}{8\pi} = \dfrac{1}{4}$

27. $\mathbf{r}(t) = a\cos(\omega t)\mathbf{i} + a\sin(\omega t)\mathbf{j}$

$\mathbf{r}'(t) = -a\omega\sin(\omega t)\mathbf{i} + a\omega\cos(\omega t)\mathbf{j}$

$\mathbf{T}(t) = -\sin(\omega t)\mathbf{i} + \cos(\omega t)\mathbf{j}$

$\mathbf{T}'(t) = -\omega\cos(\omega t)\mathbf{i} - \omega\sin(\omega t)\mathbf{j}$

$K = \dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \dfrac{\omega}{a\omega} = \dfrac{1}{a}$

29. $\mathbf{r}(t) = e^t\cos t\,\mathbf{i} + e^t\sin t\,\mathbf{j}$

$\mathbf{r}'(t) = (-e^t\sin t + e^t\cos t)\mathbf{i} + (e^t\cos t + e^t\sin t)\mathbf{j}$

$\mathbf{T}(t) = \dfrac{1}{\sqrt{2}}[(-\sin t + \cos t)\mathbf{i} + (\cos t + \sin t)\mathbf{j}]$

$\mathbf{T}'(t) = \dfrac{1}{\sqrt{2}}[(-\cos t - \sin t)\mathbf{i} + (-\sin t + \cos t)\mathbf{j}]$

$K = \dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \dfrac{1}{\sqrt{2}e^t} = \dfrac{\sqrt{2}}{2}e^{-t}$

31. $\mathbf{r}(t) = \langle \cos\omega t + \omega t\sin\omega t,\ \sin\omega t - \omega t\cos\omega t\rangle$

From Exercise 21, Section 11.4, we have:

$\mathbf{a} \cdot \mathbf{N} = \omega^3 t$

$K = \dfrac{\mathbf{a}(t) \cdot \mathbf{N}(t)}{\|\mathbf{v}\|^2} = \dfrac{\omega^3 t}{\omega^4 t^2} = \dfrac{1}{\omega t}$

33. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \dfrac{t^2}{2}\mathbf{k}$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$

$\mathbf{T}(t) = \dfrac{\mathbf{i} + 2t\mathbf{j} + t\mathbf{k}}{\sqrt{1 + 5t^2}}$

$\mathbf{T}'(t) = \dfrac{-5t\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{(1 + 5t^2)^{3/2}}$

$K = \dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$

$= \dfrac{\dfrac{\sqrt{5}}{(1 + 5t^2)}}{\sqrt{1 + 5t^2}} = \dfrac{\sqrt{5}}{(1 + 5t^2)^{3/2}}$

35. $\mathbf{r}(t) = 4t\mathbf{i} + 3\cos t\,\mathbf{j} + 3\sin t\,\mathbf{k}$

$\mathbf{r}'(t) = 4\mathbf{i} - 3\sin t\,\mathbf{j} + 3\cos t\,\mathbf{k}$

$\mathbf{T}(t) = \dfrac{1}{5}[4\mathbf{i} - 3\sin t\,\mathbf{j} + 3\cos t\,\mathbf{k}]$

$\mathbf{T}'(t) = \dfrac{1}{5}[-3\cos t\,\mathbf{j} - 3\sin t\,\mathbf{k}]$

$K = \dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \dfrac{3/5}{5} = \dfrac{3}{25}$

37. $y = 3x - 2$

Since $y'' = 0$, $K = 0$, and the radius of curvature is undefined.

39. $y = 2x^2 + 3$

$y' = 4x$

$y'' = 4$

$$K = \frac{4}{[1 + (-4)^2]^{3/2}} = \frac{4}{17^{3/2}} \approx 0.057$$

$$\frac{1}{K} = \frac{17^{3/2}}{4} \approx 17.523 \quad \text{(radius of curvature)}$$

41. $y = \sqrt{a^2 - x^2}$

$$y' = \frac{-x}{\sqrt{a^2 - x^2}}$$

$$y'' = \frac{-(2x^2 - a^2)}{(a^2 - x^2)^{3/2}}$$

At $x = 0$: $y' = 0$

$$y'' = \frac{1}{a}$$

$$K = \frac{1/a}{(1 + 0^2)^{3/2}} = \frac{1}{a}$$

$$\frac{1}{K} = a \quad \text{(radius of curvature)}$$

43. (a) Point on circle: $\left(\frac{\pi}{2}, 1\right)$

Center: $\left(\frac{\pi}{2}, 0\right)$

Equation: $\left(x - \frac{\pi}{2}\right)^2 + y^2 = 1$

(b) The circles have different radii since the curvature is different and

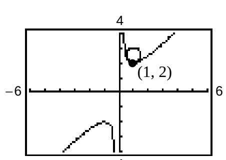
$$r = \frac{1}{K}.$$

45. $y = x + \frac{1}{x}, y' = 1 - \frac{1}{x^2}, y'' = \frac{2}{x^3}$

$$K = \frac{2}{(1 + 0^2)^{3/2}} = 2$$

Radius of curvature $= 1/2$. Since the tangent line is horizontal at $(1, 2)$, the normal line is vertical. The center of the circle is $1/2$ unit above the point $(1, 2)$ at $(1, 5/2)$.

Circle: $(x - 1)^2 + \left(y - \frac{5}{2}\right)^2 = \frac{1}{4}$



47. $y = e^x, \quad x = 0$

$y' = e^x, \quad y'' = e^x$

$y'(0) = 1, \quad y''(0) = 1$

$$K = \frac{1}{(1 + 1^2)^{3/2}} = \frac{1}{2^{3/2}} = \frac{1}{2\sqrt{2}}, \quad r = \frac{1}{K} = 2\sqrt{2}$$

The slope of the tangent line at $(0, 1)$ is $y'(0) = 1$.

The slope of the normal line is -1.

Equation of normal line: $y - 1 = -x$ or $y = -x + 1$

The center of the circle is on the normal line $2\sqrt{2}$ units away from the point $(0, 1)$.

$$\sqrt{(0 - x)^2 + (1 - y)^2} = 2\sqrt{2}$$

$$x^2 + x^2 = 8$$

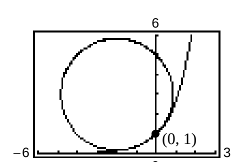
$$x^2 = 4$$

$$x = \pm 2$$

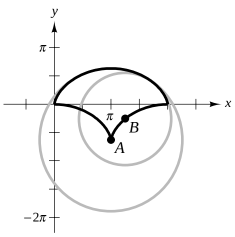
Since the circle is above the curve, $x = -2$ and $y = 3$.

Center of circle: $(-2, 3)$

Equation of circle: $(x + 2)^2 + (y - 3)^2 = 8$



49.



51. $y = (x-1)^2 + 3,\ y' = 2(x-1),\ y'' = 2$

$$K = \frac{2}{(1 + [2(x-1)]^2)^{3/2}} = \frac{2}{[1 + 4(x-1)^2]^{3/2}}$$

(a) K is maximum when $x = 1$ or at the vertex $(1, 3)$.

(b) $\lim_{x\to\infty} K = 0$

53. $y = x^{2/3},\ y' = \frac{2}{3}x^{-1/3},\ y'' = -\frac{2}{9}x^{-4/3}$

$$K = \left|\frac{(-2/9)x^{-4/3}}{[1 + (4/9)x^{-2/3}]^{3/2}}\right| = \left|\frac{6}{x^{1/3}(9x^{2/3} + 4)^{3/2}}\right|$$

(a) $K \implies \infty$ as $x \implies 0$. No maximum

(b) $\lim_{x\to\infty} K = 0$

55. $y = (x-1)^3 + 3$

$y' = 3(x-1)^2$

$y'' = 6(x-1)$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{|6(x-1)|}{[1 + 9(x-1)^4]^{3/2}} = 0 \text{ at } x = 1.$$

Curvature is 0 at $(1, 3)$.

57. $K = \dfrac{|y''|}{[1 + (y')^2]^{3/2}}$

The curvature is zero when $y'' = 0$.

59. $s = \int_a^b \|\mathbf{r}'(t)\|\, dt$

61. The curve is a line.

63. Endpoints of the major axis: $(\pm 2, 0)$

Endpoints of the minor axis: $(0, \pm 1)$

$x^2 + 4y^2 = 4$

$2x + 8yy' = 0$

$$y' = -\frac{x}{4y}$$

$$y'' = \frac{(4y)(-1) - (-x)(4y')}{16y^2} = \frac{-4y - (x^2/y)}{16y^2} = \frac{-(4y^2 + x^2)}{16y^3} = \frac{-1}{4y^3}$$

$$K = \frac{|-1/4y^3|}{[1 + (-x/4y)^2]^{3/2}} = \frac{|-16|}{(16y^2 + x^2)^{3/2}} = \frac{16}{(12y^2 + 4)^{3/2}} = \frac{16}{(16 - 3x^2)^{3/2}}$$

Therefore, since $-2 \le x \le 2$, K is largest when $x = \pm 2$ and smallest when $x = 0$.

65. $f(x) = x^4 - x^2$

(a) $K = \dfrac{2|6x^2 - 1|}{[16x^6 - 16x^4 + 4x^2 + 1]^{3/2}}$

(b) For $x = 0$, $K = 2$. $f(0) = 0$. At $(0, 0)$, the circle of curvature has radius $\frac{1}{2}$. Using the symmetry of the graph of f, you obtain

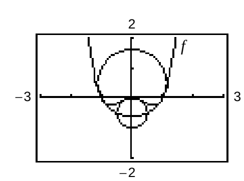
$$x^2 + \left(y + \frac{1}{2}\right)^2 = \frac{1}{4}.$$

For $x = 1$, $K = \left(2\sqrt{5}\right)/5$. $f(1) = 0$. At $(1, 0)$, the circle of curvature has radius

$$\frac{\sqrt{5}}{2} = \frac{1}{K}.$$

Using the graph of f, you see that the center of curvature is $\left(0, \frac{1}{2}\right)$. Thus,

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{5}{4}.$$



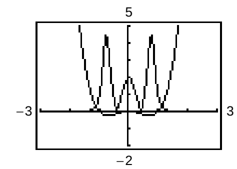
To graph these circles, use

$$y = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - x^2} \quad \text{and} \quad y = \frac{1}{2} \pm \sqrt{\frac{5}{4} - x^2}.$$

—CONTINUED—

65. —CONTINUED—

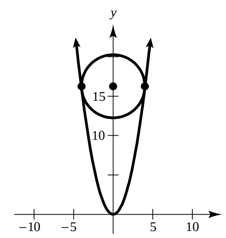
(c) The curvature tends to be greatest near the extrema of f, and K decreases as $x \to \pm\infty$. However, f and K do not have the same critical numbers.



Critical numbers of f: $x = 0, \pm\frac{\sqrt{2}}{2} \approx \pm 0.7071$

Critical numbers of K: $x = 0, \pm.7647, \pm 0.4082$

67. (a) Imagine dropping the circle $x^2 + (y - k)^2 = 16$ into the parabola $y = x^2$. The circle will drop to the point where the tangents to the circle and parabola are equal.



$$y = x^2 \quad \text{and} \quad x^2 + (y - k)^2 = 16 \implies x^2 + (x^2 - k)^2 = 16$$

Taking derivatives, $2x + 2(y - k)y' = 0$ and $y' = 2x$. Hence,

$$(y - k)y' = -x \implies y' = \frac{-x}{y - k}.$$

Thus,

$$\frac{-x}{y - k} = 2x \implies -x = 2x(y - k) \implies -1 = 2(x^2 - k) \implies x^2 - k = -\frac{1}{2}.$$

Thus,

$$x^2 + (x^2 - k)^2 = x^2 + \left(-\frac{1}{2}\right)^2 = 16 \implies x^2 = 15.75.$$

Finally, $k = x^2 + \frac{1}{2} = 16.25$, and the center of the circle is 16.25 units from the vertex of the parabola. Since the radius of the circle is 4, the circle is 12.25 units from the vertex.

(b) In 2-space, the parabola $z = y^2$ (or $z = x^2$) has a curvature of $K = 2$ at $(0, 0)$. The radius of the largest sphere that will touch the vertex has radius $= 1/K = \frac{1}{2}$.

69. Given $y = f(x)$: $K = \dfrac{|y''|}{(1 + [y']^2)^{3/2}}$

$$R = \frac{1}{K}$$

The center of the circle is on the normal line at a distance of R from (x, y).

Equation of normal line: $y - y_0 = -\dfrac{1}{y'}(x - x_0)$

$$\sqrt{(x - x_0)^2 + \left[-\frac{1}{y'}(x - x_0)\right]^2} = \frac{(1 + [y']^2)^{3/2}}{|y''|}$$

$$(x - x_0)^2\left[1 + \frac{1}{(y')^2}\right] = \frac{(1 + [y']^2)^3}{(y'')^2}$$

$$(x - x_0)^2 = \frac{(y')^2(1 + [y']^2)^2}{(y'')^2}$$

$$x - x_0 = \frac{y'(1 + [y']^2)}{y''} = y'z$$

$$x_0 = x - y'z$$

$$y - y_0 = -\frac{1}{y'}(x - (x - y'z)) = -z$$

$$y_0 = y + z$$

Thus, $(x_0, y_0) = (x - y'z, y + z)$.

For $y = e^x$, $y' = e^x$, $y'' = e^x$, $z = \dfrac{1 + e^{2x}}{e^x} = e^{-x} + e^x$.

When $x = 0$: $x_0 = x - y'z = 0 - (1)(2) = -2$

$y_0 = y + z = 1 + 2 = 3$

Center of curvature: $(-2, 3)$

(See Exercise 47)

71. $r = 1 + \sin\theta$

$r' = \cos\theta$

$r'' = -\sin\theta$

$$K = \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}}$$

$$= \frac{|2\cos^2\theta - (1 + \sin\theta)(-\sin\theta) + (1 + \sin\theta)^2|}{\sqrt{[\cos^2\theta + (1 + \sin\theta)^2]^3}}$$

$$= \frac{3(1 + \sin\theta)}{\sqrt{8(1 + \sin\theta)^3}} = \frac{3}{2\sqrt{2(1 + \sin\theta)}}$$

73. $r = a\sin\theta$

$r' = a\cos\theta$

$r'' = -a\sin\theta$

$$K = \frac{|2(r\omega)^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}}$$

$$= \frac{|2a^2\cos^2\theta + a^2\sin^2\theta + a^2\sin^2\theta|}{\sqrt{[a^2\cos^2\theta + a^2\sin^2\theta]^3}}$$

$$= \frac{2a^2}{a^3} = \frac{2}{a},\ a > 0$$

75. $r = e^{a\theta},\ a > 0$

$r' = ae^{a\theta}$

$r'' = a^2e^{a\theta}$

$$K = \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}} = \frac{|2a^2e^{2a\theta} - a^2e^{2a\theta} + e^{2a\theta}|}{[a^2e^{2a\theta} + e^{2a\theta}]^{3/2}}$$

$$= \frac{1}{e^{a\theta}\sqrt{a^2 + 1}}$$

(a) As $\theta \Rightarrow \infty,\ K \Rightarrow 0.$

(b) As $a \Rightarrow \infty,\ K \Rightarrow 0.$

77. $r = 4\sin 2\theta$

$r' = 8\cos 2\theta$

At the pole: $K = \dfrac{2}{|r'(0)|} = \dfrac{2}{8} = \dfrac{1}{4}$

79. $x = f(t)$

$y = g(t)$

$$y' = \frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \frac{g'(t)}{f'(t)}$$

$$y'' = \frac{\dfrac{d}{dt}\left[\dfrac{g'(t)}{f'(t)}\right]}{\dfrac{dx}{dt}} = \frac{\dfrac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^2}}{f'(t)}$$

$$= \frac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^3}$$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{\left|\dfrac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^3}\right|}{\left[1 + \left(\dfrac{g'(t)}{f'(t)}\right)^2\right]^{3/2}}$$

$$= \frac{\left|\dfrac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^3}\right|}{\sqrt{\left\{\dfrac{[f'(t)]^2 + [g'(t)]^2}{[f'(t)]^2}\right\}^3}}$$

$$= \frac{|f'(t)g''(t) - g'(t)f''(t)|}{([f'(t)]^2 + [g'(t)]^2)^{3/2}}$$

81. $x(\theta) = a(\theta - \sin\theta) \qquad y(\theta) = a(1 - \cos\theta)$

$x'(\theta) = a(1 - \cos\theta) \qquad y'(\theta) = a\sin\theta$

$x''(\theta) = a\sin\theta \qquad y''(\theta) = a\cos\theta$

$$K = \frac{|x'(\theta)y''(\theta) - y'(\theta)x''(\theta)|}{[x'(\theta)^2 + y'(\theta)^2]^{3/2}}$$

$$= \frac{|a^2(1 - \cos\theta)\cos\theta - a^2\sin^2\theta|}{[a^2(1 - \cos\theta)^2 + a^2\sin^2\theta]^{3/2}}$$

$$= \frac{1}{a}\,\frac{|\cos\theta - 1|}{[2 - 2\cos\theta]^{3/2}}$$

$$= \frac{1}{a}\,\frac{1 - \cos\theta}{2\sqrt{2}[1 - \cos\theta]^{3/2}} \quad (1 - \cos \geq 0)$$

$$= \frac{1}{2a\sqrt{2 - 2\cos\theta}} = \frac{1}{4a}\csc\left(\frac{\theta}{2}\right)$$

Minimum: $\dfrac{1}{4a}$ $\qquad (\theta = \pi)$

Maximum: none $\qquad (K \to \infty$ as $\theta \to 0)$

83. $$a_{\mathbf{N}} = mK\left(\frac{ds}{dt}\right)^2 = \left(\frac{5500 \text{ lb}}{32 \text{ ft/sec}^2}\right)\left(\frac{1}{100 \text{ ft}}\right)\left(\frac{30(5280) \text{ ft}}{3600 \text{ sec}}\right)^2 = 3327.5 \text{ lb}$$

85. Let $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $r = \|\mathbf{r}\| = \sqrt{[x(t)]^2 + [y(t)]^2 + [z(t)]^2}$ and $r' = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$. Then,

$$r\left(\frac{dr}{dt}\right) = \sqrt{[x(t)]^2 + [y(t)]^2 + [z(t)]^2}\left[\frac{1}{2}\{[x(t)]^2 + [y(t)]^2 + [z(t)]^2\}^{-1/2} \cdot (2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t))\right]$$

$$= x(t)x'(t) + y(t)y'(t) + z(t)z'(t) = \mathbf{r} \cdot \mathbf{r}'.$$

87. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ where x, y, and z are functions of t, and $r = \|\mathbf{r}\|$.

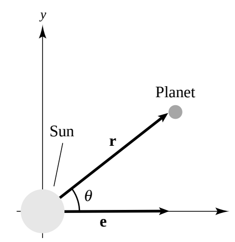
$$\frac{d}{dt}\left[\frac{\mathbf{r}}{r}\right] = \frac{r\mathbf{r}' - \mathbf{r}(dr/dt)}{r^2} = \frac{r\mathbf{r}' - \mathbf{r}[(\mathbf{r} \cdot \mathbf{r}')/r]}{r^2} = \frac{r^2\mathbf{r}' - (\mathbf{r} \cdot \mathbf{r}')\mathbf{r}}{r^3} \quad \text{(using Exercise 77)}$$

$$= \frac{(x^2 + y^2 + z^2)(x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}) - (xx' + yy' + zz')(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{r^3}$$

$$= \frac{1}{r^3}[(x'y^2 + x'z^2 - xyy' - xzz')\mathbf{i} + (x^2y' + z^2y' - xx'y - zz'y)\mathbf{j} + (x^2z' + y^2z' - xx'z - yy'z)\mathbf{k}]$$

$$= \frac{1}{r^3}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ yz' - y'z & -(xz' - x'z) & xy' - x'y \\ x & y & z \end{vmatrix} = \frac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\}$$

89. From Exercise 86, we have concluded that planetary motion is planar. Assume that the planet moves in the xy-plane with the sun at the origin. From Exercise 88, we have



$$\mathbf{r}' \times \mathbf{L} = GM\left(\frac{\mathbf{r}}{r} + \mathbf{e}\right).$$

Since $\mathbf{r}' \times \mathbf{L}$ and $\mathbf{r}$ are both perpendicular to $\mathbf{L}$, so is $\mathbf{e}$. Thus, $\mathbf{e}$ lies in the xy-plane. Situate the coordinate system so that $\mathbf{e}$ lies along the positive x-axis and θ is the angle between $\mathbf{e}$ and $\mathbf{r}$. Let $e = \|\mathbf{e}\|$. Then $\mathbf{r} \cdot \mathbf{e} = \|\mathbf{r}\|\,\|\mathbf{e}\| \cos\theta = re\cos\theta$. Also,

$$\|\mathbf{L}\|^2 = \mathbf{L} \cdot \mathbf{L} = (\mathbf{r} \times \mathbf{r}') \cdot \mathbf{L}$$

$$= \mathbf{r} \cdot (\mathbf{r}' \times \mathbf{L}) = \mathbf{r} \cdot \left[GM\left(\mathbf{e} + \frac{\mathbf{r}}{r}\right)\right] = GM\left[\mathbf{r} \cdot \mathbf{e} + \frac{\mathbf{r} \cdot \mathbf{r}}{r}\right] = GM[re\cos\theta + r]$$

Thus,

$$\frac{\|\mathbf{L}\|^2/GM}{1 + e\cos\theta} = r$$

and the planetary motion is a conic section. Since the planet returns to its initial position periodically, the conic is an ellipse.

91. $A = \frac{1}{2}\int_\alpha^\beta r^2\,d\theta$

Thus,

$$\frac{dA}{dt} = \frac{dA}{d\theta}\frac{d\theta}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}\|\mathbf{L}\|$$

and $\mathbf{r}$ sweeps out area at a constant rate.

Review Exercises for Chapter 11

1. $\mathbf{r}(t) = t\mathbf{i} + \csc t\mathbf{k}$

(a) Domain: $t \neq n\pi$, n an integer

(b) Continuous except at $t = n\pi$, n an integer

3. $\mathbf{r}(t) = \ln t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$

(a) Domain: $(0, \infty)$

(b) Continuous for all $t > 0$

5. (a) $\mathbf{r}(0) = \mathbf{i}$

(b) $\mathbf{r}(-2) = -3\mathbf{i} + 4\mathbf{j} + \frac{8}{3}\mathbf{k}$

(c) $\mathbf{r}(c-1) = (2(c-1)+1)\mathbf{i} + (c-1)^2\mathbf{j} - \frac{1}{3}(c-1)^3\mathbf{k}$

$= (2c-1)\mathbf{i} + (c-1)^2\mathbf{j} - \frac{1}{3}(c-1)^3\mathbf{k}$

(d) $\mathbf{r}(1+\Delta t) - \mathbf{r}(1) = ([2(1+\Delta t)+1]\mathbf{i} + [1+\Delta t]^2\mathbf{j} - \frac{1}{3}[1+\Delta t]^3\mathbf{k}) - (3\mathbf{i} + \mathbf{j} - \frac{1}{3}\mathbf{k})$

$= 2\Delta t\,\mathbf{i} + \Delta t(\Delta t + 2)\mathbf{j} - \frac{1}{3}(\Delta t^3 + 3\Delta t^2 + 3\Delta t)\mathbf{k}$

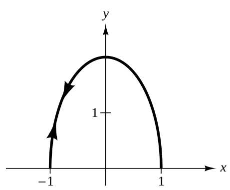
7. $\mathbf{r}(t) = \cos t\,\mathbf{i} + 2\sin^2 t\,\mathbf{j}$

$x(t) = \cos t,\ y(t) = 2\sin^2 t$

$x^2 + \frac{y}{2} = 1$

$y = 2(1 - x^2)$

$-1 \le x \le 1$

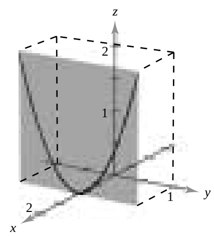


9. $\mathbf{r}(t) = \mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$

$x = 1$

$y = t$

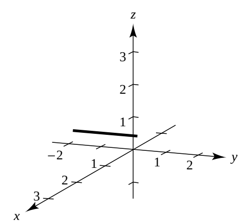
$z = t^2 \implies z = y^2$



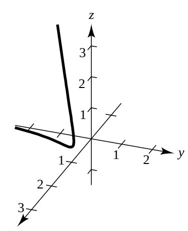
11. $\mathbf{r}(t) = \mathbf{i} + \sin t\,\mathbf{j} + \mathbf{k}$

$x = 1,\ y = \sin t,\ z = 1$

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
x	1	1	1	1
y	0	1	0	-1
z	1	1	1	1



13. $\mathbf{r}(t) = t\mathbf{i} + \ln t\,\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$



15. One possible answer is:

$\mathbf{r}_1(t) = 4t\mathbf{i} + 3t\mathbf{j}, \quad 0 \le t \le 1$

$\mathbf{r}_2(t) = 4\mathbf{i} + (3 - t)\mathbf{j}, \quad 0 \le t \le 3$

$\mathbf{r}_3(t) = (4 - t)\mathbf{i}, \quad 0 \le t \le 4$

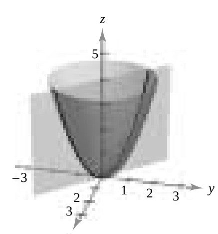
17. The vector joining the points is $\langle 7, 4, -10\rangle$. One path is

$\mathbf{r}(t) = \langle -2 + 7t, -3 + 4t, 8 - 10t\rangle$.

19. $z = x^2 + y^2,\ x + y = 0,\ t = x$

$x = t,\ y = -t,\ z = 2t^2$

$\mathbf{r}(t) = t\mathbf{i} - t\mathbf{j} + 2t^2\mathbf{k}$



21. $\lim_{t\to 2^-} (t^2\mathbf{i} + \sqrt{4 - t^2}\mathbf{j} + \mathbf{k}) = 4\mathbf{i} + \mathbf{k}$

23. $\mathbf{r}(t) = 3t\mathbf{i} + (t-1)\mathbf{j},\ \mathbf{u}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{2}{3}t^3\mathbf{k}$

(a) $\mathbf{r}'(t) = 3\mathbf{i} + \mathbf{j}$

(b) $\mathbf{r}''(t) = \mathbf{0}$

(c) $\mathbf{r}(t) \cdot \mathbf{u}(t) = 3t^2 + t^2(t-1) = t^3 + 2t^2$

$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = 3t^2 + 4t$

(d) $\mathbf{u}(t) - 2\mathbf{r}(t) = -5t\mathbf{i} + (t^2 - 2t + 2)\mathbf{j} + \frac{2}{3}t^3\mathbf{k}$

$D_t[\mathbf{u}(t) - 2\mathbf{r}(t)] = -5\mathbf{i} + (2t-2)\mathbf{j} + 2t^2\mathbf{k}$

(e) $\|\mathbf{r}(t)\| = \sqrt{10t^2 - 2t + 1}$

$D_t[\|\mathbf{r}(t)\|] = \dfrac{10t - 1}{\sqrt{10t^2 - 2t + 1}}$

(f) $\mathbf{r}(t) \times \mathbf{u}(t) = \frac{2}{3}(t^4 - t^3)\mathbf{i} - 2t^4\mathbf{j} + (3t^3 - t^2 + t)\mathbf{k}$

$D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \left(\frac{8}{3}t^3 - 2t^2\right)\mathbf{i} - 8t^3\mathbf{j} + (9t^2 - 2t + 1)\mathbf{k}$

25. $x(t)$ and $y(t)$ are increasing functions at $t = t_0$, and $z(t)$ is a decreasing function at $t = t_0$.

27. $\int (\cos t\mathbf{i} + t\cos t\mathbf{j})\,dt = \sin t\mathbf{i} + (t\sin t + \cos t)\mathbf{j} + \mathbf{C}$

29. $\int \|\cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}\|\,dt = \int \sqrt{1+t^2}\,dt = \frac{1}{2}\left[t\sqrt{1+t^2} + \ln\left|t + \sqrt{1+t^2}\right|\right] + \mathbf{C}$

31. $\mathbf{r}(t) = \int (2t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k})\,dt = t^2\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{j} - \mathbf{k} + \mathbf{C} = \mathbf{i} + 3\mathbf{j} - 5\mathbf{k} \Rightarrow \mathbf{C} = \mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$

$\mathbf{r}(t) = (t^2+1)\mathbf{i} + (e^t + 2)\mathbf{j} - (e^{-t} + 4)\mathbf{k}$

33. $\int_{-2}^{2} (3t\mathbf{i} + 2t^2\mathbf{j} - t^3\mathbf{k})\,dt = \left[\frac{3t^2}{2}\mathbf{i} + \frac{2t^3}{3}\mathbf{j} - \frac{t^4}{4}\mathbf{k}\right]_{-2}^{2} = \frac{32}{3}\mathbf{j}$

35. $\int_0^2 (e^{t/2}\mathbf{i} - 3t^2\mathbf{j} - \mathbf{k})\,dt = \left[2e^{t/2}\mathbf{i} - t^3\mathbf{j} - t\mathbf{k}\right]_0^2 = (2e - 2)\mathbf{i} - 8\mathbf{j} - 2\mathbf{k}$

37. $\mathbf{r}(t) = \langle \cos^3 t, \sin^3 t, 3t\rangle$

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -3\cos^2 t\sin t, 3\sin^2 t\cos t, 3\rangle$

$$\begin{aligned}\|\mathbf{v}(t)\| &= \sqrt{9\cos^4 t\sin^2 t + 9\sin^4 t\cos^2 t + 9}\\ &= 3\sqrt{\cos^2 t\sin^2 t(\cos^2 t + \sin^2 t) + 1}\\ &= 3\sqrt{\cos^2 t\sin^2 t + 1}\end{aligned}$$

$$\begin{aligned}\mathbf{a}(t) = \mathbf{v}'(t) &= \langle -6\cos t(-\sin^2 t) + (-3\cos^2 t)\cos t, 6\sin t\cos^2 t + 3\sin^2 t(-\sin t), 0\rangle\\ &= \langle 3\cos t(2\sin^2 t - \cos^2 t),\ 3\sin t(2\cos^2 t - \sin^2 t), 0\rangle\end{aligned}$$

39. $\mathbf{r}(t) = \left\langle \ln(t-3), t^2, \frac{1}{2}t\right\rangle,\ t_0 = 4$

$\mathbf{r}'(t) = \left\langle \frac{1}{t-3}, 2t, \frac{1}{2}\right\rangle$

$\mathbf{r}'(4) = \left\langle 1, 8, \frac{1}{2}\right\rangle$ direction numbers

Since $\mathbf{r}(4) = \langle 0, 16, 2\rangle$, the parametric equations are $x = t, y = 16 + 8t, z = 2 + \frac{1}{2}t$.

$\mathbf{r}(t_0 + 0.1) = \mathbf{r}(4.1) \approx \langle 0.1, 16.8, 2.05\rangle$

41. Range $= x = \dfrac{v_0^2}{32}\sin 2\theta = \dfrac{(75)^2}{32}\sin 60° \approx 152$ feet

43. Range $= x = \dfrac{v_0^2}{9.8}\sin 2\theta = 80 \Rightarrow v_0 = \sqrt{\dfrac{(80)(9.8)}{\sin 40°}} \approx 34.9$ m/sec

45. $\mathbf{r}(t) = 5t\mathbf{i}$

$\mathbf{v}(t) = 5\mathbf{i}$

$\|\mathbf{v}(t)\| = 5$

$\mathbf{a}(t) = \mathbf{0}$

$\mathbf{T}(t) = \mathbf{i}$

$\mathbf{N}(t)$ does not exist

$\mathbf{a} \cdot \mathbf{T} = 0$

$\mathbf{a} \cdot \mathbf{N}$ does not exist

(The curve is a line.)

47. $\mathbf{r}(t) = t\mathbf{i} + \sqrt{t}\mathbf{j}$

$$\mathbf{v}(t) = \mathbf{i} + \frac{1}{2\sqrt{t}}\mathbf{j}$$

$$\|\mathbf{v}(t)\| = \frac{\sqrt{4t+1}}{2\sqrt{t}}$$

$$\mathbf{a}(t) = -\frac{1}{4t\sqrt{t}}\mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{i} + \left(1/2\sqrt{t}\right)\mathbf{j}}{\left(\sqrt{4t+1}\right)/2\sqrt{t}} = \frac{2\sqrt{t}\mathbf{i} + \mathbf{j}}{\sqrt{4t+1}}$$

$$\mathbf{N}(t) = \frac{\mathbf{i} - 2\sqrt{t}\mathbf{j}}{\sqrt{4t+1}}$$

$$\mathbf{a} \cdot \mathbf{T} = \frac{-1}{4t\sqrt{t}\sqrt{4t+1}}$$

$$\mathbf{a} \cdot \mathbf{N} = \frac{1}{2t\sqrt{4t+1}}$$

49. $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$

$\mathbf{v}(t) = e^t\mathbf{i} - e^{-t}\mathbf{j}$

$\|\mathbf{v}(t)\| = \sqrt{e^{2t} + e^{-2t}}$

$\mathbf{a}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$

$$\mathbf{T}(t) = \frac{e^t\mathbf{i} - e^{-t}\mathbf{j}}{\sqrt{e^{2t} + e^{-2t}}}$$

$$\mathbf{N}(t) = \frac{e^{-t}\mathbf{i} + e^t\mathbf{j}}{\sqrt{e^{2t} + e^{-2t}}}$$

$$\mathbf{a} \cdot \mathbf{T} = \frac{e^{2t} - e^{-2t}}{\sqrt{e^{2t} + e^{-2t}}}$$

$$\mathbf{a} \cdot \mathbf{N} = \frac{2}{\sqrt{e^{2t} + e^{-2t}}}$$

51. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$

$\|\mathbf{v}\| = \sqrt{1 + 5t^2}$

$\mathbf{a}(t) = 2\mathbf{j} + \mathbf{k}$

$$\mathbf{T}(t) = \frac{\mathbf{i} + 2t\mathbf{j} + t\mathbf{k}}{\sqrt{1+5t^2}}$$

$$\mathbf{N}(t) = \frac{-5t\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{5}\sqrt{1+5t^2}}$$

$$\mathbf{a} \cdot \mathbf{T} = \frac{5t}{\sqrt{1+5t^2}}$$

$$\mathbf{a} \cdot \mathbf{N} = \frac{5}{\sqrt{5}\sqrt{1+5t^2}} = \frac{\sqrt{5}}{\sqrt{1+5t^2}}$$

53. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k},\ x = 2\cos t,\ y = 2\sin t,\ z = t$

When $t = \frac{3\pi}{4},\ x = -\sqrt{2},\ y = \sqrt{2},\ z = \frac{3\pi}{4}.$

$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \mathbf{k}$

Direction numbers when $t = \frac{3\pi}{4},\ a = -\sqrt{2},\ b = -\sqrt{2},\ c = 1$

$$x = -\sqrt{2}t - \sqrt{2},\ y = -\sqrt{2}t + \sqrt{2},\ z = t + \frac{3\pi}{4}$$

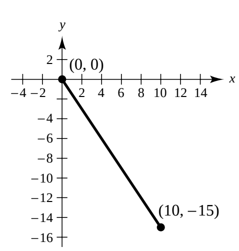
55. $v = \sqrt{\dfrac{9.56 \times 10^4}{4600}} \approx 4.56$ mi/sec

57. $\mathbf{r}(t) = 2t\mathbf{i} - 3t\mathbf{j},\ 0 \le t \le 5$

$\mathbf{r}'(t) = 2\mathbf{i} - 3\mathbf{j}$

$$s = \int_a^b \|\mathbf{r}'(t)\|\,dt = \int_0^5 \sqrt{4+9}\,dt$$

$$= \sqrt{13}t\Big]_0^5 = 5\sqrt{13}$$



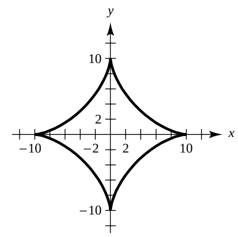
59. $\mathbf{r}(t) = 10\cos^3 t\mathbf{i} + 10\sin^3 t\mathbf{j}$

$\mathbf{r}'(t) = -30\cos^2 t\sin t\mathbf{i} + 30\sin^2 t\cos t\mathbf{j}$

$\|\mathbf{r}'(t)\| = 30\sqrt{\cos^4 t\sin^2 t + \sin^4 t\cos^2 t}$

$= 30|\cos t\sin t|$

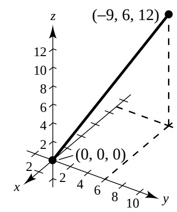
$$s = 4\int_0^{\pi/2} 30\cos t \cdot \sin t\,dt = \left[120\frac{\sin^2 t}{2}\right]_0^{\pi/2} = 60$$



61. $\mathbf{r}(t) = -3t\mathbf{i} + 2t\mathbf{j} + 4t\mathbf{k},\ 0 \le t \le 3$

$\mathbf{r}'(t) = -3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$

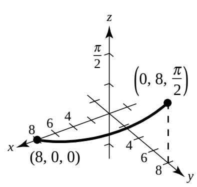
$$s = \int_a^b \|\mathbf{r}'(t)\|\,dt = \int_0^3 \sqrt{9 + 4 + 16}\,dt = \int_0^3 \sqrt{29}\,dt = 3\sqrt{29}$$



63. $\mathbf{r}(t) = \langle 8\cos t, 8\sin t, t\rangle,\ 0 \le t \le \frac{\pi}{2}$

$\mathbf{r}'(t) = \langle -8\sin t, 8\cos t, 1\rangle,\ \|\mathbf{r}'(t)\| = \sqrt{65}$

$$s = \int_a^b \|\mathbf{r}'(t)\|\,dt = \int_0^{\pi/2} \sqrt{65}\,dt = \frac{\pi\sqrt{65}}{2}$$



65. $\mathbf{r}(t) = \frac{1}{2}t\mathbf{i} + \sin t\mathbf{j} + \cos t\mathbf{k},\quad 0 \le t \le \pi$

$\mathbf{r}'(t) = \frac{1}{2}\mathbf{i} + \cos t\mathbf{j} - \sin t\mathbf{k}$

$$s = \int_0^\pi \|\mathbf{r}'(t)\|\,dt$$

$$= \int_0^\pi \sqrt{\frac{1}{4} + \cos^2 t + \sin^2 t}\,dt$$

$$= \frac{\sqrt{5}}{2}\int_0^\pi dt = \left[\frac{\sqrt{5}}{2}t\right]_0^\pi = \frac{\sqrt{5}}{2}\pi$$

67. $\mathbf{r}(t) = 3t\mathbf{i} + 2t\mathbf{j}$

Line

$k = 0$

69. $\mathbf{r}(t) = 2t\mathbf{i} + \frac{1}{2}t^2\mathbf{j} + t^2\mathbf{k}$

$\mathbf{r}'(t) = 2\mathbf{i} + t\mathbf{j} + 2t\mathbf{k},\ \|\mathbf{r}'\| = \sqrt{5t^2 + 4}$

$\mathbf{r}''(t) = \mathbf{j} + 2\mathbf{k}$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & t & 2t \\ 0 & 1 & 2 \end{vmatrix} = -4\mathbf{j} + 2\mathbf{k},\ \|\mathbf{r}' \times \mathbf{r}''\| = \sqrt{20}$$

$$K = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{\sqrt{20}}{(5t^2 + 4)^{3/2}} = \frac{2\sqrt{5}}{(4 + 5t^2)^{3/2}}$$

71. $y = \frac{1}{2}x^2 + 2$

$y' = x$

$y'' = 1$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{1}{(1 + x^2)^{3/2}}$$

At $x = 4$, $K = \frac{1}{17^{3/2}}$ and $r = 17^{3/2} = 17\sqrt{17}$.

73. $y = \ln x$

$y' = \frac{1}{x},\ y'' = -\frac{1}{x^2}$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{1/x^2}{[1 + (1/x)^2]^{3/2}}$$

At $x = 1$, $K = \frac{1}{2^{3/2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$ and $r = 2\sqrt{2}$.

75. The curvature changes abruptly from zero to a nonzero constant at the points B and C.