85. Let $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $r = \|\mathbf{r}\| = \sqrt{[x(t)]^2 + [y(t)]^2 + [z(t)]^2}$ and $r' = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$. Then,

$$r\left(\frac{dr}{dt}\right) = \sqrt{[x(t)]^2 + [y(t)]^2 + [z(t)]^2}\left[\frac{1}{2}\{[x(t)]^2 + [y(t)]^2 + [z(t)]^2\}^{-1/2} \cdot (2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t))\right]$$

$$= x(t)x'(t) + y(t)y'(t) + z(t)z'(t) = \mathbf{r} \cdot \mathbf{r}'.$$

87. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ where x, y, and z are functions of t, and $r = \|\mathbf{r}\|$.

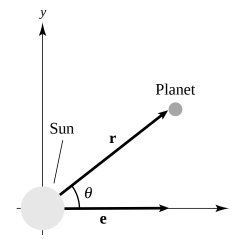
$$\frac{d}{dt}\left[\frac{\mathbf{r}}{r}\right] = \frac{r\mathbf{r}' - \mathbf{r}(dr/dt)}{r^2} = \frac{r\mathbf{r}' - \mathbf{r}[(\mathbf{r} \cdot \mathbf{r}')/r]}{r^2} = \frac{r^2\mathbf{r}' - (\mathbf{r} \cdot \mathbf{r}')\mathbf{r}}{r^3} \quad \text{(using Exercise 77)}$$

$$= \frac{(x^2 + y^2 + z^2)(x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}) - (xx' + yy' + zz')(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{r^3}$$

$$= \frac{1}{r^3}[(x'y^2 + x'z^2 - xyy' - xzz')\mathbf{i} + (x^2y' + z^2y' - xx'y - zz'y)\mathbf{j} + (x^2z' + y^2z' - xx'z - yy'z)\mathbf{k}]$$

$$= \frac{1}{r^3}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ yz' - y'z & -(xz' - x'z) & xy' - x'y \\ x & y & z \end{vmatrix} = \frac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\}$$

89. From Exercise 86, we have concluded that planetary motion is planar. Assume that the planet moves in the xy-plane with the sun at the origin. From Exercise 88, we have



$$\mathbf{r}' \times \mathbf{L} = GM\left(\frac{\mathbf{r}}{r} + \mathbf{e}\right).$$

Since $\mathbf{r}' \times \mathbf{L}$ and $\mathbf{r}$ are both perpendicular to $\mathbf{L}$, so is $\mathbf{e}$. Thus, $\mathbf{e}$ lies in the xy-plane. Situate the coordinate system so that $\mathbf{e}$ lies along the positive x-axis and θ is the angle between $\mathbf{e}$ and $\mathbf{r}$. Let $e = \|\mathbf{e}\|$. Then $\mathbf{r} \cdot \mathbf{e} = \|\mathbf{r}\|\,\|\mathbf{e}\| \cos\theta = re\cos\theta$. Also,

$$\|\mathbf{L}\|^2 = \mathbf{L} \cdot \mathbf{L} = (\mathbf{r} \times \mathbf{r}') \cdot \mathbf{L}$$

$$= \mathbf{r} \cdot (\mathbf{r}' \times \mathbf{L}) = \mathbf{r} \cdot \left[GM\left(\mathbf{e} + \frac{\mathbf{r}}{r}\right)\right] = GM\left[\mathbf{r} \cdot \mathbf{e} + \frac{\mathbf{r} \cdot \mathbf{r}}{r}\right] = GM[re\cos\theta + r]$$

Thus,

$$\frac{\|\mathbf{L}\|^2/GM}{1 + e\cos\theta} = r$$

and the planetary motion is a conic section. Since the planet returns to its initial position periodically, the conic is an ellipse.

91. $A = \frac{1}{2}\int_\alpha^\beta r^2\,d\theta$

Thus,

$$\frac{dA}{dt} = \frac{dA}{d\theta}\,\frac{d\theta}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}\|\mathbf{L}\|$$

and $\mathbf{r}$ sweeps out area at a constant rate.

Review Exercises for Chapter 11

1. $\mathbf{r}(t) = t\mathbf{i} + \csc t\mathbf{k}$

(a) Domain: $t \neq n\pi$, n an integer

(b) Continuous except at $t = n\pi$, n an integer

3. $\mathbf{r}(t) = \ln t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$

(a) Domain: $(0, \infty)$

(b) Continuous for all $t > 0$

5. (a) $\mathbf{r}(0) = \mathbf{i}$

(b) $\mathbf{r}(-2) = -3\mathbf{i} + 4\mathbf{j} + \frac{8}{3}\mathbf{k}$

(c) $\mathbf{r}(c-1) = (2(c-1)+1)\mathbf{i} + (c-1)^2\mathbf{j} - \frac{1}{3}(c-1)^3\mathbf{k}$

$= (2c-1)\mathbf{i} + (c-1)^2\mathbf{j} - \frac{1}{3}(c-1)^3\mathbf{k}$

(d) $\mathbf{r}(1+\Delta t) - \mathbf{r}(1) = ([2(1+\Delta t)+1]\mathbf{i} + [1+\Delta t]^2\mathbf{j} - \frac{1}{3}[1+\Delta t]^3\mathbf{k}) - (3\mathbf{i} + \mathbf{j} - \frac{1}{3}\mathbf{k})$

$= 2\Delta t\,\mathbf{i} + \Delta t(\Delta t + 2)\mathbf{j} - \frac{1}{3}(\Delta t^3 + 3\Delta t^2 + 3\Delta t)\mathbf{k}$

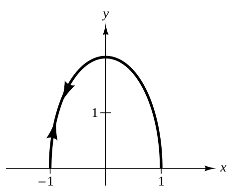
7. $\mathbf{r}(t) = \cos t\,\mathbf{i} + 2\sin^2 t\,\mathbf{j}$

$x(t) = \cos t,\; y(t) = 2\sin^2 t$

$x^2 + \frac{y}{2} = 1$

$y = 2(1 - x^2)$

$-1 \le x \le 1$

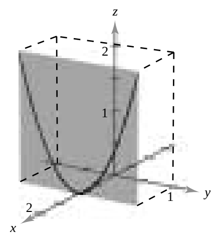


9. $\mathbf{r}(t) = \mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$

$x = 1$

$y = t$

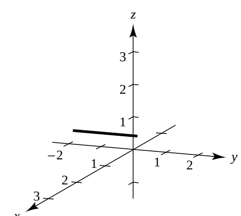
$z = t^2 \implies z = y^2$



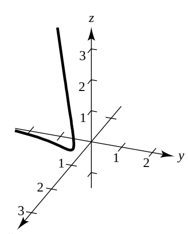
11. $\mathbf{r}(t) = \mathbf{i} + \sin t\,\mathbf{j} + \mathbf{k}$

$x = 1,\; y = \sin t,\; z = 1$

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
x	1	1	1	1
y	0	1	0	-1
z	1	1	1	1



13. $\mathbf{r}(t) = t\mathbf{i} + \ln t\,\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$



15. One possible answer is:

$\mathbf{r}_1(t) = 4t\mathbf{i} + 3t\mathbf{j}, \quad 0 \le t \le 1$

$\mathbf{r}_2(t) = 4\mathbf{i} + (3-t)\mathbf{j}, \quad 0 \le t \le 3$

$\mathbf{r}_3(t) = (4-t)\mathbf{i}, \quad 0 \le t \le 4$

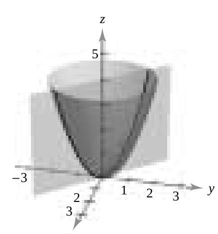
17. The vector joining the points is $\langle 7, 4, -10\rangle$. One path is

$\mathbf{r}(t) = \langle -2 + 7t, -3 + 4t, 8 - 10t\rangle$.

19. $z = x^2 + y^2,\; x + y = 0,\; t = x$

$x = t,\; y = -t,\; z = 2t^2$

$\mathbf{r}(t) = t\mathbf{i} - t\mathbf{j} + 2t^2\mathbf{k}$



21. $\lim_{t \to 2^-} (t^2\mathbf{i} + \sqrt{4 - t^2}\,\mathbf{j} + \mathbf{k}) = 4\mathbf{i} + \mathbf{k}$

23. $\mathbf{r}(t) = 3t\mathbf{i} + (t-1)\mathbf{j},\ \mathbf{u}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{2}{3}t^3\mathbf{k}$

(a) $\mathbf{r}'(t) = 3\mathbf{i} + \mathbf{j}$

(b) $\mathbf{r}''(t) = \mathbf{0}$

(c) $\mathbf{r}(t)\cdot\mathbf{u}(t) = 3t^2 + t^2(t-1) = t^3 + 2t^2$

$D_t[\mathbf{r}(t)\cdot\mathbf{u}(t)] = 3t^2 + 4t$

(d) $\mathbf{u}(t) - 2\mathbf{r}(t) = -5t\mathbf{i} + (t^2 - 2t + 2)\mathbf{j} + \frac{2}{3}t^3\mathbf{k}$

$D_t[\mathbf{u}(t) - 2\mathbf{r}(t)] = -5\mathbf{i} + (2t-2)\mathbf{j} + 2t^2\mathbf{k}$

(e) $\|\mathbf{r}(t)\| = \sqrt{10t^2 - 2t + 1}$

$D_t[\|\mathbf{r}(t)\|] = \dfrac{10t - 1}{\sqrt{10t^2 - 2t + 1}}$

(f) $\mathbf{r}(t)\times\mathbf{u}(t) = \frac{2}{3}(t^4 - t^3)\mathbf{i} - 2t^4\mathbf{j} + (3t^3 - t^2 + t)\mathbf{k}$

$D_t[\mathbf{r}(t)\times\mathbf{u}(t)] = \left(\frac{8}{3}t^3 - 2t^2\right)\mathbf{i} - 8t^3\mathbf{j} + (9t^2 - 2t + 1)\mathbf{k}$

25. $x(t)$ and $y(t)$ are increasing functions at $t = t_0$, and $z(t)$ is a decreasing function at $t = t_0$.

27. $\int(\cos t\mathbf{i} + t\cos t\mathbf{j})\,dt = \sin t\mathbf{i} + (t\sin t + \cos t)\mathbf{j} + \mathbf{C}$

29. $\int\|\cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}\|\,dt = \int\sqrt{1+t^2}\,dt = \frac{1}{2}\left[t\sqrt{1+t^2} + \ln\left|t + \sqrt{1+t^2}\right|\right] + \mathbf{C}$

31. $\mathbf{r}(t) = \int(2t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k})\,dt = t^2\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{j} - \mathbf{k} + \mathbf{C} = \mathbf{i} + 3\mathbf{j} - 5\mathbf{k} \implies \mathbf{C} = \mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$

$\mathbf{r}(t) = (t^2 + 1)\mathbf{i} + (e^t + 2)\mathbf{j} - (e^{-t} + 4)\mathbf{k}$

33. $\int_{-2}^{2}(3t\mathbf{i} + 2t^2\mathbf{j} - t^3\mathbf{k})\,dt = \left[\frac{3t^2}{2}\mathbf{i} + \frac{2t^3}{3}\mathbf{j} - \frac{t^4}{4}\mathbf{k}\right]_{-2}^{2} = \frac{32}{3}\mathbf{j}$

35. $\int_0^2(e^{t/2}\mathbf{i} - 3t^2\mathbf{j} - \mathbf{k})\,dt = \left[2e^{t/2}\mathbf{i} - t^3\mathbf{j} - t\mathbf{k}\right]_0^2 = (2e - 2)\mathbf{i} - 8\mathbf{j} - 2\mathbf{k}$

37. $\mathbf{r}(t) = \langle\cos^3 t, \sin^3 t, 3t\rangle$

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -3\cos^2 t\sin t, 3\sin^2 t\cos t, 3\rangle$

$$\begin{aligned}\|\mathbf{v}(t)\| &= \sqrt{9\cos^4 t\sin^2 t + 9\sin^4 t\cos^2 t + 9}\\ &= 3\sqrt{\cos^2 t\sin^2 t(\cos^2 t + \sin^2 t) + 1}\\ &= 3\sqrt{\cos^2 t\sin^2 t + 1}\end{aligned}$$

$$\begin{aligned}\mathbf{a}(t) = \mathbf{v}'(t) &= \langle -6\cos t(-\sin^2 t) + (-3\cos^2 t)\cos t, 6\sin t\cos^2 t + 3\sin^2 t(-\sin t), 0\rangle\\ &= \langle 3\cos t(2\sin^2 t - \cos^2 t),\ 3\sin t(2\cos^2 t - \sin^2 t), 0\rangle\end{aligned}$$

39. $\mathbf{r}(t) = \left\langle \ln(t-3), t^2, \frac{1}{2}t\right\rangle,\ t_0 = 4$

$\mathbf{r}'(t) = \left\langle \frac{1}{t-3}, 2t, \frac{1}{2}\right\rangle$

$\mathbf{r}'(4) = \left\langle 1, 8, \frac{1}{2}\right\rangle$ direction numbers

Since $\mathbf{r}(4) = \langle 0, 16, 2\rangle$, the parametric equations are $x = t, y = 16 + 8t, z = 2 + \frac{1}{2}t$.

$\mathbf{r}(t_0 + 0.1) = \mathbf{r}(4.1) \approx \langle 0.1, 16.8, 2.05\rangle$

41. Range $= x = \dfrac{v_0^2}{32}\sin 2\theta = \dfrac{(75)^2}{32}\sin 60° \approx 152$ feet

43. Range $= x = \dfrac{v_0^2}{9.8}\sin 2\theta = 80 \implies v_0 = \sqrt{\dfrac{(80)(9.8)}{\sin 40°}} \approx 34.9$ m/sec

45. $\mathbf{r}(t) = 5t\mathbf{i}$

$\mathbf{v}(t) = 5\mathbf{i}$

$\|\mathbf{v}(t)\| = 5$

$\mathbf{a}(t) = \mathbf{0}$

$\mathbf{T}(t) = \mathbf{i}$

$\mathbf{N}(t)$ does not exist

$\mathbf{a} \cdot \mathbf{T} = 0$

$\mathbf{a} \cdot \mathbf{N}$ does not exist

(The curve is a line.)

47. $\mathbf{r}(t) = t\mathbf{i} + \sqrt{t}\mathbf{j}$

$\mathbf{v}(t) = \mathbf{i} + \dfrac{1}{2\sqrt{t}}\mathbf{j}$

$\|\mathbf{v}(t)\| = \dfrac{\sqrt{4t+1}}{2\sqrt{t}}$

$\mathbf{a}(t) = -\dfrac{1}{4t\sqrt{t}}\mathbf{j}$

$\mathbf{T}(t) = \dfrac{\mathbf{i} + \left(1/2\sqrt{t}\right)\mathbf{j}}{\left(\sqrt{4t+1}\right)/2\sqrt{t}} = \dfrac{2\sqrt{t}\mathbf{i} + \mathbf{j}}{\sqrt{4t+1}}$

$\mathbf{N}(t) = \dfrac{\mathbf{i} - 2\sqrt{t}\mathbf{j}}{\sqrt{4t+1}}$

$\mathbf{a} \cdot \mathbf{T} = \dfrac{-1}{4t\sqrt{t}\sqrt{4t+1}}$

$\mathbf{a} \cdot \mathbf{N} = \dfrac{1}{2t\sqrt{4t+1}}$

49. $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$

$\mathbf{v}(t) = e^t\mathbf{i} - e^{-t}\mathbf{j}$

$\|\mathbf{v}(t)\| = \sqrt{e^{2t} + e^{-2t}}$

$\mathbf{a}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$

$\mathbf{T}(t) = \dfrac{e^t\mathbf{i} - e^{-t}\mathbf{j}}{\sqrt{e^{2t} + e^{-2t}}}$

$\mathbf{N}(t) = \dfrac{e^{-t}\mathbf{i} + e^t\mathbf{j}}{\sqrt{e^{2t} + e^{-2t}}}$

$\mathbf{a} \cdot \mathbf{T} = \dfrac{e^{2t} - e^{-2t}}{\sqrt{e^{2t} + e^{-2t}}}$

$\mathbf{a} \cdot \mathbf{N} = \dfrac{2}{\sqrt{e^{2t} + e^{-2t}}}$

51. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \dfrac{1}{2}t^2\mathbf{k}$

$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$

$\|\mathbf{v}\| = \sqrt{1 + 5t^2}$

$\mathbf{a}(t) = 2\mathbf{j} + \mathbf{k}$

$\mathbf{T}(t) = \dfrac{\mathbf{i} + 2t\mathbf{j} + t\mathbf{k}}{\sqrt{1+5t^2}}$

$\mathbf{N}(t) = \dfrac{-5t\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{5}\sqrt{1+5t^2}}$

$\mathbf{a} \cdot \mathbf{T} = \dfrac{5t}{\sqrt{1+5t^2}}$

$\mathbf{a} \cdot \mathbf{N} = \dfrac{5}{\sqrt{5}\sqrt{1+5t^2}} = \dfrac{\sqrt{5}}{\sqrt{1+5t^2}}$

53. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}$, $x = 2\cos t$, $y = 2\sin t$, $z = t$

When $t = \dfrac{3\pi}{4}$, $x = -\sqrt{2}$, $y = \sqrt{2}$, $z = \dfrac{3\pi}{4}$.

$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \mathbf{k}$

Direction numbers when $t = \dfrac{3\pi}{4}$, $a = -\sqrt{2}$, $b = -\sqrt{2}$, $c = 1$

$x = -\sqrt{2}t - \sqrt{2}$, $y = -\sqrt{2}t + \sqrt{2}$, $z = t + \dfrac{3\pi}{4}$

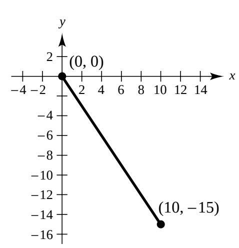
55. $v = \sqrt{\dfrac{9.56 \times 10^4}{4600}} \approx 4.56$ mi/sec

57. $\mathbf{r}(t) = 2t\mathbf{i} - 3t\mathbf{j}$, $0 \le t \le 5$

$\mathbf{r}'(t) = 2\mathbf{i} - 3\mathbf{j}$

$s = \displaystyle\int_a^b \|\mathbf{r}'(t)\|\,dt = \int_0^5 \sqrt{4+9}\,dt$

$= \sqrt{13}t\Big]_0^5 = 5\sqrt{13}$



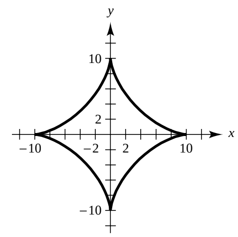
59. $\mathbf{r}(t) = 10\cos^3 t\mathbf{i} + 10\sin^3 t\mathbf{j}$

$\mathbf{r}'(t) = -30\cos^2 t\sin t\mathbf{i} + 30\sin^2 t\cos t\mathbf{j}$

$\|\mathbf{r}'(t)\| = 30\sqrt{\cos^4 t\sin^2 t + \sin^4 t\cos^2 t}$

$= 30|\cos t\sin t|$

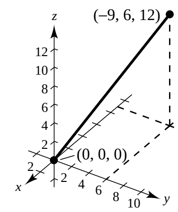
$$s = 4\int_0^{\pi/2} 30\cos t \cdot \sin t\,dt = \left[120\frac{\sin^2 t}{2}\right]_0^{\pi/2} = 60$$



61. $\mathbf{r}(t) = -3t\mathbf{i} + 2t\mathbf{j} + 4t\mathbf{k},\ 0 \le t \le 3$

$\mathbf{r}'(t) = -3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$

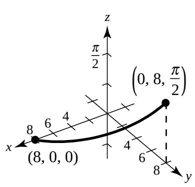
$$s = \int_a^b \|\mathbf{r}'(t)\|\,dt = \int_0^3 \sqrt{9 + 4 + 16}\,dt = \int_0^3 \sqrt{29}\,dt = 3\sqrt{29}$$



63. $\mathbf{r}(t) = \langle 8\cos t, 8\sin t, t\rangle,\ 0 \le t \le \dfrac{\pi}{2}$

$\mathbf{r}'(t) = \langle -8\sin t, 8\cos t, 1\rangle,\ \|\mathbf{r}'(t)\| = \sqrt{65}$

$$s = \int_a^b \|\mathbf{r}'(t)\|\,dt = \int_0^{\pi/2} \sqrt{65}\,dt = \frac{\pi\sqrt{65}}{2}$$



65. $\mathbf{r}(t) = \dfrac{1}{2}t\mathbf{i} + \sin t\mathbf{j} + \cos t\mathbf{k},\quad 0 \le t \le \pi$

$\mathbf{r}'(t) = \dfrac{1}{2}\mathbf{i} + \cos t\mathbf{j} - \sin t\mathbf{k}$

$$s = \int_0^{\pi} \|\mathbf{r}'(t)\|\,dt$$
$$= \int_0^{\pi} \sqrt{\frac{1}{4} + \cos^2 t + \sin^2 t}\,dt$$
$$= \frac{\sqrt{5}}{2}\int_0^{\pi} dt = \left[\frac{\sqrt{5}}{2}t\right]_0^{\pi} = \frac{\sqrt{5}}{2}\pi$$

67. $\mathbf{r}(t) = 3t\mathbf{i} + 2t\mathbf{j}$

Line

$k = 0$

69. $\mathbf{r}(t) = 2t\mathbf{i} + \dfrac{1}{2}t^2\mathbf{j} + t^2\mathbf{k}$

$\mathbf{r}'(t) = 2\mathbf{i} + t\mathbf{j} + 2t\mathbf{k},\ \|\mathbf{r}'\| = \sqrt{5t^2 + 4}$

$\mathbf{r}''(t) = \mathbf{j} + 2\mathbf{k}$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & t & 2t \\ 0 & 1 & 2 \end{vmatrix} = -4\mathbf{j} + 2\mathbf{k},\ \|\mathbf{r}' \times \mathbf{r}''\| = \sqrt{20}$$

$$K = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{\sqrt{20}}{(5t^2 + 4)^{3/2}} = \frac{2\sqrt{5}}{(4 + 5t^2)^{3/2}}$$

71. $y = \dfrac{1}{2}x^2 + 2$

$y' = x$

$y'' = 1$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{1}{(1 + x^2)^{3/2}}$$

At $x = 4$, $K = \dfrac{1}{17^{3/2}}$ and $r = 17^{3/2} = 17\sqrt{17}$.

73. $y = \ln x$

$y' = \dfrac{1}{x},\ y'' = -\dfrac{1}{x^2}$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{1/x^2}{[1 + (1/x)^2]^{3/2}}$$

At $x = 1$, $K = \dfrac{1}{2^{3/2}} = \dfrac{1}{2\sqrt{2}} = \dfrac{\sqrt{2}}{4}$ and $r = 2\sqrt{2}$.

75. The curvature changes abruptly from zero to a nonzero constant at the points B and C.

Problem Solving for Chapter 11

1. $x(t) = \int_0^t \cos\left(\frac{\pi u^2}{2}\right) du, \ y(t) = \int_0^t \sin\left(\frac{\pi u^2}{2}\right) du$

$x'(t) = \cos\left(\frac{\pi t^2}{2}\right), \ y'(t) = \sin\left(\frac{\pi t^2}{2}\right)$

(a) $s = \int_0^a \sqrt{x'(t)^2 + y'(t)^2}\, dt = \int_0^a dt = a$

(b) $x''(t) = -\pi t \sin\left(\frac{\pi t^2}{2}\right), \ y''(t) = \pi t \cos\left(\frac{\pi t^2}{2}\right)$

$$K = \frac{\left|\pi t \cos^2\left(\frac{\pi t^2}{2}\right) + \pi t \sin^2\left(\frac{\pi t^2}{2}\right)\right|}{1} = \pi t$$

At $t = a, K = \pi a$.

(c) $K = \pi a = \pi(\text{length})$

3. Bomb: $\mathbf{r}_1(t) = \langle 5000 - 400t, 3200 - 16t^2\rangle$

Projectile: $\mathbf{r}_2(t) = \langle (v_0 \cos\theta)t, (v_0 \sin\theta)t - 16t^2\rangle$

At 1600 feet: Bomb:

$3200 - 16t^2 = 1600 \implies t = 10$ seconds.

Projectile will travel 5 seconds:

$$5(v_0 \sin\theta) - 16(25) = 1600$$
$$v_0 \sin\theta = 400.$$

Horizontal position:

At $t = 10$, bomb is at $5000 - 400(10) = 1000$.

At $t = 5$, projectile is at $5v_0 \cos\theta$.

Thus, $v_0 \cos\theta = 200$.

Combining, $\frac{v_0 \sin\theta}{v_0 \cos\theta} = \frac{400}{200} \implies \tan\theta = 2 \implies \theta \approx 63.4°$.

$v_0 = \frac{200}{\cos\theta} \approx 447.2$ ft/sec

5. $x'(\theta) = 1 - \cos\theta, \ y'(\theta) = \sin\theta, \ 0 \le \theta \le 2\pi$

$$\sqrt{x'(\theta)^2 + y'(\theta)^2} = \sqrt{(1-\cos\theta)^2 + \sin^2\theta} = \sqrt{2 - 2\cos\theta} = \sqrt{4\sin^2\frac{\theta}{2}}$$

$$s(t) = \int_\pi^t 2\sin\frac{\theta}{2}\, d\theta = \left[-4\cos\frac{\theta}{2}\right]_\pi^t = -4\cos\frac{t}{2}$$

$x''(\theta) = \sin\theta, \ y''(\theta) = \cos\theta$

$$K = \frac{1\left|(1-\cos\theta)\cos\theta - \sin\theta\sin\theta\right|}{\left(2\sin\frac{\theta}{2}\right)^3} = \frac{|1\cos\theta - 1|}{8\sin^3\frac{\theta}{2}} = \frac{1}{4\sin\frac{\theta}{2}}$$

Thus, $\rho = \frac{1}{K} = 4\sin\frac{t}{2}$ and

$$s^2 + \rho^2 = 16\cos^2\left(\frac{t}{2}\right) + 16\sin^2\left(\frac{t}{2}\right) = 16.$$

7. $\|\mathbf{r}^2(t)\| = \mathbf{r}(t)\cdot\mathbf{r}(t)$

$$\frac{d}{dt}(\|\mathbf{r}(t)\|)^2 = 2\|\mathbf{r}(t)\|\frac{d}{dt}\|\mathbf{r}(t)\|$$
$$= \mathbf{r}(t)\cdot\mathbf{r}'(t) + \mathbf{r}'(t)\cdot\mathbf{r}(t) \implies \frac{d}{dt}\|\mathbf{r}(t)\| = \frac{\mathbf{r}(t)\cdot\mathbf{r}'(t)}{\|\mathbf{r}(t)\|}$$

9. $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j} + 3t\mathbf{k},\ t = \dfrac{\pi}{2}$

$\mathbf{r}'(t) = -4\sin t\mathbf{i} + 4\cos t\mathbf{j} + 3\mathbf{k},\ \|\mathbf{r}'(t)\| = 5$

$\mathbf{r}''(t) = -4\cos t\mathbf{i} - 4\sin t\mathbf{j}$

$$\mathbf{T} = -\frac{4}{5}\sin t\mathbf{i} + \frac{4}{5}\cos t\mathbf{j} + \frac{3}{5}\mathbf{k}$$

$$\mathbf{T}' = -\frac{4}{5}\cos t\mathbf{i} - \frac{4}{5}\sin t\mathbf{j}$$

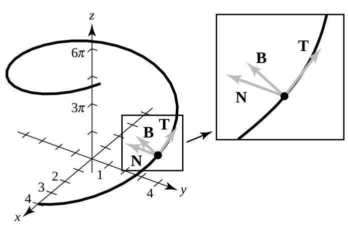
$$\mathbf{N} = -\cos t\mathbf{i} - \sin t\mathbf{j}$$

$$\mathbf{B} = \mathbf{T}\times\mathbf{N} = \frac{3}{5}\sin t\mathbf{i} - \frac{3}{5}\cos t\mathbf{j} + \frac{4}{5}\mathbf{k}$$

At $t = \dfrac{\pi}{2}$, $\mathbf{T}\left(\dfrac{\pi}{2}\right) = -\dfrac{4}{5}\mathbf{i} + \dfrac{3}{5}\mathbf{k}$

$$\mathbf{N}\left(\frac{\pi}{2}\right) = -\mathbf{j}$$

$$\mathbf{B}\left(\frac{\pi}{2}\right) = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k}$$



11. (a) $\|\mathbf{B}\| = \|\mathbf{T}\times\mathbf{N}\| = 1$ constant length $\Longrightarrow \dfrac{dB}{ds} \perp \mathbf{B}$

$$\frac{d\mathbf{B}}{ds} = \frac{d}{ds}(\mathbf{T}\times\mathbf{N}) = (\mathbf{T}\times\mathbf{N}') + (\mathbf{T}'\times\mathbf{N})$$

$$\mathbf{T}\cdot\frac{d\mathbf{B}}{ds} = \mathbf{T}\cdot(\mathbf{T}\times\mathbf{N}') + \mathbf{T}\cdot(\mathbf{T}'\times\mathbf{N})$$

$$= (\mathbf{T}\times\mathbf{T})\cdot\mathbf{N}' + \mathbf{T}\cdot\left(\mathbf{T}'\times\frac{\mathbf{T}'}{\|\mathbf{T}'\|}\right) = 0$$

Hence, $\dfrac{d\mathbf{B}}{ds}\perp\mathbf{B}$ and $\dfrac{d\mathbf{B}}{ds}\perp\mathbf{T} \Longrightarrow \dfrac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$
for some scalar τ.

(b) $\mathbf{B} = \mathbf{T}\times\mathbf{N}$. Using Exercise 10.3, number 64,

$$\mathbf{B}\times\mathbf{N} = (\mathbf{T}\times\mathbf{N})\times\mathbf{N} = -\mathbf{N}\times(\mathbf{T}\times\mathbf{N})$$
$$= -[(\mathbf{N}\cdot\mathbf{N})\mathbf{T} - (\mathbf{N}\cdot\mathbf{T})\mathbf{N}]$$
$$= -\mathbf{T}$$

$$\mathbf{B}\times\mathbf{T} = (\mathbf{T}\times\mathbf{N})\times\mathbf{T} = -\mathbf{T}\times(\mathbf{T}\times\mathbf{N})$$
$$= -[(\mathbf{T}\cdot\mathbf{N})\mathbf{T} - (\mathbf{T}\cdot\mathbf{T})\mathbf{N}]$$
$$= \mathbf{N}.$$

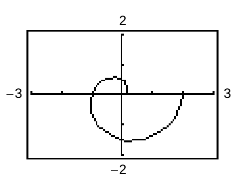
Now, $K\mathbf{N} = \left\|\dfrac{d\mathbf{T}}{ds}\right\|\dfrac{\mathbf{T}'(s)}{\|\mathbf{T}'(s)\|} = \mathbf{T}'(s) = \dfrac{d\mathbf{T}}{ds}.$

Finally,

$$\mathbf{N}'(s) = \frac{d}{ds}(\mathbf{B}\times\mathbf{T}) = (\mathbf{B}\times\mathbf{T}') + (\mathbf{B}'\times\mathbf{T})$$
$$= (\mathbf{B}\times K\mathbf{N}) + (-\tau\mathbf{N}\times\mathbf{T})$$
$$= -K\mathbf{T} + \tau\mathbf{B}.$$

13. $\mathbf{r}(t) = \langle t\cos\pi t, t\sin\pi t\rangle,\ 0 \le t \le 2$

(a)



(b) Length $= \displaystyle\int_0^2 \|\mathbf{r}'(t)\|\,dt$

$$= \int_0^2 \sqrt{\pi^2t^2+1}\,dt \approx 6.766 \quad \text{(graphing utility)}$$

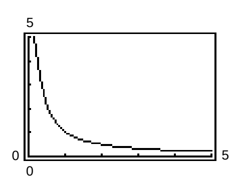
(c) $$K = \frac{\pi(\pi^2t^2+2)}{[\pi^2t^2+1]^{3/2}}$$

$$K(0) = 2\pi$$

$$K(1) = \frac{\pi(\pi^2+2)}{(\pi^2+1)^{3/2}} \approx 1.04$$

$$K(2) \approx 0.51$$

(d)



(e) $\displaystyle\lim_{t\to\infty} K = 0$

(f) As $t\to\infty$, the graph spirals outward and the curvature decreases.