CHAPTER 12
Functions of Several Variables

CHAPTER 12
Functions of Several Variables

Section 12.1 Introduction to Functions of Several Variables

Solutions to Even-Numbered Exercises

2. $xz^2 + 2xy - y^2 = 4$

No, z is not a function of x and y. For example, $(x, y) = (1, 0)$ corresponds to both $z = \pm 2$

4. $z + x \ln y - 8 = 0$

$z = 8 - x \ln y$

Yes, z is a function of x and y.

6. $f(x, y) = 4 - x^2 - 4y^2$

(a) $f(0, 0) = 4$

(b) $f(0, 1) = 4 - 0 - 4 = 0$

(c) $f(2, 3) = 4 - 4 - 36 = -36$

(d) $f(1, y) = 4 - 1 - 4y^2 = 3 - 4y^2$

(e) $f(x, 0) = 4 - x^2 - 0 = 4 - x^2$

(f) $f(t, 1) = 4 - t^2 - 4 = -t^2$

8. $g(x, y) = \ln|x + y|$

(a) $g(2, 3) = \ln|2 + 3| = \ln 5$

(b) $g(5, 6) = \ln|5 + 6| = \ln 11$

(c) $g(e, 0) = \ln|e + 0| = 1$

(d) $g(0, 1) = \ln|0 + 1| = 0$

(e) $g(2, -3) = \ln|2 - 3| = \ln 1 = 0$

(f) $g(e, e) = \ln|e + e| = \ln 2e$

$= \ln 2 + \ln e = (\ln 2) + 1$

10. $f(x, y, z) = \sqrt{x + y + z}$

(a) $f(0, 5, 4) = \sqrt{0 + 5 + 4} = 3$

(b) $f(6, 8, -3) = \sqrt{6 + 8 - 3} = \sqrt{11}$

12. $V(r, h) = \pi r^2 h$

(a) $V(3, 10) = \pi(3)^2(10) = 90\pi$

(b) $V(5, 2) = \pi(5)^2(2) = 50\pi$

14. $g(x, y) = \displaystyle\int_x^y \frac{1}{t}\,dt$

(a) $g(4, 1) = \displaystyle\int_4^1 \frac{1}{t}\,dt = \Big[\ln|t|\Big]_4^1 = -\ln 4$

(b) $g(6, 3) = \displaystyle\int_6^3 \frac{1}{t}\,dt = \Big[\ln|t|\Big]_6^3 = \ln 3 - \ln 6 = \ln\left(\frac{1}{2}\right)$

16. $f(x, y) = 3xy + y^2$

(a) $$\frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \frac{[3(x + \Delta x)y + y^2] - (3xy + y^2)}{\Delta x}$$

$$= \frac{3xy + 3(\Delta x)y + y^2 - 3xy - y^2}{\Delta x} = \frac{3(\Delta x)y}{\Delta x} = 3y,\ \Delta x \neq 0$$

(b) $$\frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \frac{[3x(y + \Delta y) + (y + \Delta y)^2] - (3xy + y^2)}{\Delta y}$$

$$= \frac{3xy + 3x(\Delta y) + y^2 + 2y(\Delta y) + (\Delta y)^2 - 3xy - y^2}{\Delta y}$$

$$= \frac{\Delta y(3x + 2y + \Delta y)}{\Delta y} = 3x + 2y + \Delta y,\ \Delta y \neq 0$$

18. $f(x, y) = \sqrt{4 - x^2 - 4y^2}$

Domain: $4 - x^2 - 4y^2 \geq 0$

$$x^2 + 4y^2 \leq 4$$

$$\frac{x^2}{4} + \frac{y^2}{1} \leq 1$$

$$\left\{(x, y): \frac{x^2}{4} + \frac{y^2}{1} \leq 1\right\}$$

Range: $0 \leq z \leq 2$

20. $f(x, y) = \arccos \dfrac{y}{x}$

Domain: $\left\{(x, y): -1 \leq \dfrac{y}{x} \leq 1\right\}$

Range: $0 \leq z \leq \pi$

22. $f(x, y) = \ln(xy - 6)$

Domain: $xy - 6 > 0$

$$xy > 6$$

$$\{(x, y): xy > 6\}$$

Range: all real numbers

24. $z = \dfrac{xy}{x - y}$

Domain: $\{(x, y): x \neq y\}$

Range: all real numbers

26. $f(x, y) = x^2 + y^2$

Domain: $\{(x, y)$: x is any real number, y is any real number$\}$

Range: $z \geq 0$

28. $g(x, y) = x\sqrt{y}$

Domain: $\{(x, y): y \geq 0\}$

Range: all real numbers

30. (a) Domain: $\{(x, y)$: x is any real number, y is any real number$\}$

Range: $-2 \leq z \leq 2$

(b) $z = 0$ when $x = 0$ which represents points on the y-axis.

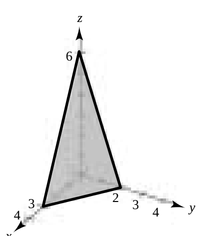
(c) No. When x is positive, z is negative. When x is negative, z is positive. The surface does not pass through the first octant, the octant where y is negative and x and z are positive, the octant where y is positive and x and z are negative, and the octant where x, y and z are all negative.

32. $f(x, y) = 6 - 2x - 3y$

Plane

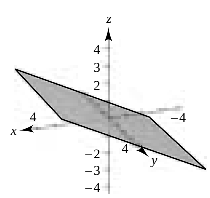
Domain: entire xy-plane

Range: $-\infty < z < \infty$



34. $g(x, y) = \frac{1}{2}x$

Plane: $z = \frac{1}{2}x$

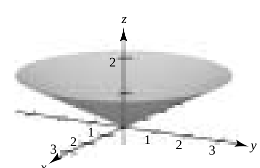


36. $z = \dfrac{1}{2}\sqrt{x^2 + y^2}$

Cone

Domain of f: entire xy-plane

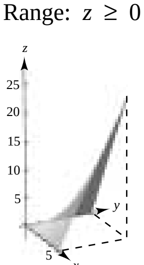
Range: $z \geq 0$



38. $f(x, y) = \begin{cases} xy, & x \geq 0,\ y \geq 0 \\ 0, & \text{elsewhere} \end{cases}$

Domain of f: entire xy-plane

Range: $z \geq 0$

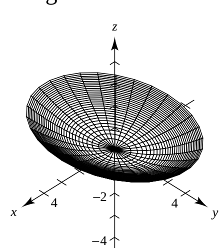


40. $f(x, y) = \frac{1}{12}\sqrt{144 - 16x^2 - 9y^2}$

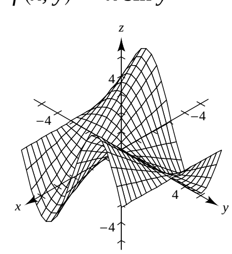
Semi-ellipsoid

Domain: set of all points lying on or inside the ellipse $(x^2/9) + (y^2/16) = 1$

Range: $0 \leq z \leq 1$

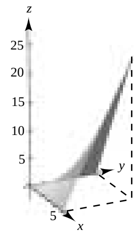


42. $f(x, y) = x \sin y$



44. $f(x, y) = xy, x \geq 0, y \geq 0$

(a)

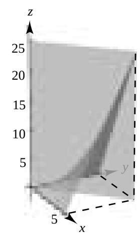


(b) g is a vertical translation of f 3 units downward

(c) g is a reflection of f in the xy-plane

(d) The graph of g is lower than the graph of f. If $z = f(x, y)$ is on the graph of f, then $\frac{1}{2}z$ is on the graph of g.

(e)



46. $z = e^{1-x^2+y^2}$

Level curves:

$$c = e^{1-x^2+y^2}$$

$$\ln c = 1 - x^2 + y^2$$

$$x^2 - y^2 = 1 - \ln c$$

Hyperbolas centered at $(0, 0)$

Matches (d)

48. $z = \cos\left(\dfrac{x + 2y^2}{4}\right)$

Level curves:

$$c = \cos\left(\frac{x^2 + 2y^2}{4}\right)$$

$$\cos^{-1} c = \frac{x^2 + 2y^2}{4}$$

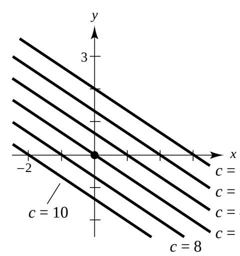
$$x^2 + 2y^2 = 4 \cos^{-1} c$$

Ellipses

Matches (a)

50. $f(x, y) = 6 - 2x - 3y$

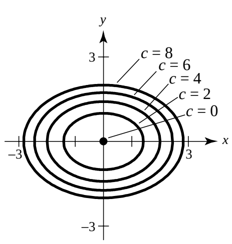
The level curves are of the form $6 - 2x - 3y = c$ or $2x + 3y = 6 - c$. Thus, the level curves are straight lines with a slope of $-\frac{2}{3}$.



52. $f(x, y) = x^2 + 2y^2$

The level curves are ellipses of the form

$x^2 + 2y^2 = c$ (except $x^2 + 2y^2 = 0$ is the point $(0, 0)$).

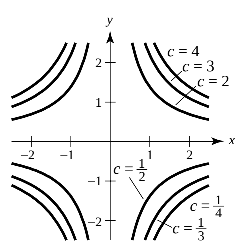


54. $f(x, y) = e^{xy/2}$

The level curves are of the form

$$e^{xy/2} = c, \text{ or } \ln c = \frac{xy}{2}.$$

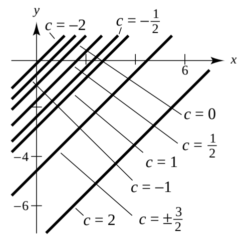
Thus, the level curves are hyperbolas.



56. $f(x, y) = \ln(x - y)$

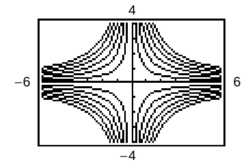
The level curves are of the form

$$c = \ln(x - y)$$
$$e^c = x - y$$
$$y = x - e^c$$

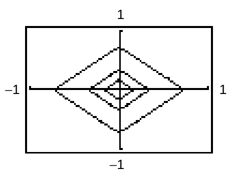


Thus, the level curves are parallel lines of slope 1 passing through the fourth quadrant.

58. $f(x, y) = |xy|$



60. $h(x, y) = 3\sin(|x| + |y|)$



62. The graph of a function of two variables is the set of all points (x, y, z) for which $z = f(x, y)$ and (x, y) is in the domain of f. The graph can be interpreted as a surface in space. Level curves are the scalar fields $f(x, y) = c$, for c, a constant.

64. $f(x, y) = \dfrac{x}{y}$

The level curves are the lines

$$c = \frac{x}{y} \text{ or } y = \frac{1}{c}x$$

These lines all pass through the origin.

66. The surface could be an ellipsoid centered at $(0, 1, 0)$. One possible function is

$$f(x, y) = x^2 + \frac{(y - 1)^2}{4} = 1.$$

68. $A(r, t) = 1000e^{rt}$

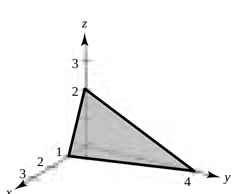
	Number of years			
Rate	5	10	15	20
0.08	\$1491.82	\$2225.54	\$3320.12	\$4953.03
0.10	\$1648.72	\$2718.28	\$4481.69	\$7389.06
0.12	\$1822.12	\$3320.12	\$6049.65	\$11,023.18
0.14	\$2013.75	\$4055.20	\$8166.17	\$16,444.65

70. $f(x, y, z) = 4x + y + 2z$

$c = 4$

$4 = 4x + y + 2z$

Plane



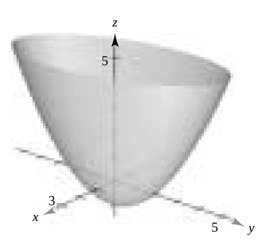
72. $f(x, y, z) = x^2 + \frac{1}{4}y^2 - z$

$c = 1$

$1 = x^2 + \frac{1}{4}y^2 - z$

Elliptic paraboloid

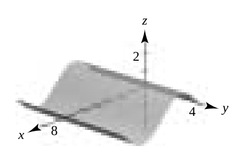
Vertex: $(0, 0, -1)$



74. $f(x, y, z) = \sin x - z$

$c = 0$

$0 = \sin x - z$ or $z = \sin x$



76. $W(x, y) = \dfrac{1}{x - y},\ y < x$

(a) $W(15, 10) = \dfrac{1}{15 - 10} = \dfrac{1}{5}\text{ hr} = 12\text{ min}$

(b) $W(12, 9) = \dfrac{1}{12 - 9} = \dfrac{1}{3}\text{ hr} = 20\text{ min}$

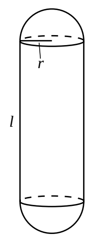
(c) $W(12, 6) = \dfrac{1}{12 - 6} = \dfrac{1}{6}\text{ hr} = 10\text{ min}$

(d) $W(4, 2) = \dfrac{1}{4 - 2} = \dfrac{1}{2}\text{ hr} = 30\text{ min}$

78. $f(x, y) = 100x^{0.6}y^{0.4}$

$$f(2x, 2y) = 100(2x)^{0.6}(2y)^{0.4}$$
$$= 100(2)^{0.6}x^{0.6}(2)^{0.4}y^{0.4} = 100(2)^{0.6}(2)^{0.4}x^{0.6}y^{0.4} = 2[100x^{0.6}y^{0.4}] = 2f(x, y)$$

80. $V = \pi r^2 l + \dfrac{4}{3}\pi r^3 = \dfrac{\pi r^2}{3}(3l + 4r)$



82. (a)

Year	1995	1996	1997	1998	1999	2000
z	12.7	14.8	17.1	18.5	21.1	25.8
Model	13.09	14.79	16.45	18.47	21.38	25.78

(b) x has the greater influence because its coefficient (0.143) is larger than that of y(0.024).

(c) $f(x, 25) = 0.143x + 0.024(25) + 0.502$

$= 0.143x + 1.102$

This function gives the shareholder's equity z in terms of net sales x and assumes constant assets of $y = 25$.

84. Southwest

86. Latitude and land versus ocean location have the greatest effect on temperature.

88. True

90. True

Section 12.2 Limits and Continuity

2. Let $\varepsilon > 0$ be given. We need to find $\delta > 0$ such that $|f(x, y) - L| = |x - 4| < \varepsilon$ whenever $0 < \sqrt{(x - a)^2 + (y - b)^2} = \sqrt{(x - 4)^2 + (y + 1)^2} < \delta$. Take $\delta = \varepsilon$. Then if $0 < \sqrt{(x - 4)^2 + (y + 1)^2} < \delta = \varepsilon$, we have

$$\sqrt{(x - 4)^2} < \varepsilon$$
$$|x - 4| < \varepsilon.$$

4. $$\lim_{(x, y)\to(a, b)} \left[\frac{4f(x, y)}{g(x, y)}\right] = \frac{4\left[\displaystyle\lim_{(x, y)\to(a, b)} f(x, y)\right]}{\displaystyle\lim_{(x, y)\to(a, b)} g(x, y)} = \frac{4(5)}{3} = \frac{20}{3}$$

6. $\displaystyle\lim_{(x, y)\to(a, b)} \left[\frac{f(x, y) - g(x, y)}{f(x, y)}\right] = \frac{\displaystyle\lim_{(x, y)\to(a, b)} f(x, y) - \lim_{(x, y)\to(a, b)} g(x, y)}{\displaystyle\lim_{(x, y)\to(a, b)} f(x, y)} = \frac{5 - 3}{5} = \frac{2}{5}$

8. $\displaystyle\lim_{(x, y)\to(0, 0)} (5x + y + 1) = 0 + 0 + 1 = 1$

Continuous everywhere

10. $\displaystyle\lim_{(x, y)\to(1, 1)} \frac{x}{\sqrt{x + y}} = \frac{1}{\sqrt{1 + 1}} = \frac{\sqrt{2}}{2}$

Continuous for $x + y > 0$

12. $\displaystyle\lim_{(x, y)\to(\pi/4, 2)} y \cos(xy) = 2 \cos \frac{\pi}{2} = 0$

Continuous everywhere

14. $\displaystyle\lim_{(x, y)\to(1, 1)} \frac{xy}{x^2 + y^2} = \frac{1}{2}$

Continuous except at $(0, 0)$

16. $\displaystyle\lim_{(x, y, z)\to(2, 0, 1)} xe^{yz} = 2e^0 = 2$

Continuous everywhere

18. $f(x, y) = \dfrac{x^2}{(x^2 + 1)(y^2 + 1)}$

$\displaystyle\lim_{(x, y)\to(0, 0)} \frac{x^2}{(x^2 + 1)(y^2 + 1)} = \frac{0}{(0 + 1)(0 + 1)} = 0$

Continuous everywhere

20. $\displaystyle\lim_{(x, y)\to(0, 0)} \left[1 - \frac{\cos(x^2 + y^2)}{x^2 + y^2}\right] = -\infty$

The limit does not exist.

Continuous except at $(0, 0)$

22. $f(x, y) = \dfrac{y}{x^2 + y^2}$

Continuous except at $(0, 0)$

Path: $y = 0$

(x, y)	$(1, 1)$	$(0.5, 0.5)$	$(0.1, 0.1)$	$(0.01, 0.01)$	$(0.001, 0.001)$
$f(x, y)$	$\frac{1}{2}$	1	5	50	500

Path: $y = x$

(x, y)	$(1, 0)$	$(0.5, 0)$	$(0.1, 0)$	$(0.01, 0)$	$(0.001, 0)$
$f(x, y)$	0	0	0	0	0

The limit does not exist because along the path $y = 0$ the function equals 0, whereas along the path $y = x$ the function tends to infinity.

24. $f(x, y) = \dfrac{2x - y^2}{2x^2 + y}$

Continuous except at $(0, 0)$

Path: $y = 0$

(x, y)	$(1, 0)$	$(0.25, 0)$	$(0.01, 0)$	$(0.001, 0)$	$(0.000001, 0)$
$f(x, y)$	1	4	100	1000	1,000,000

Path: $y = x$

(x, y)	$(1, 1)$	$(0.25, 0.25)$	$(0.01, 0.01)$	$(0.001, 0.001)$	$(0.0001, 0.0001)$
$f(x, y)$	$\frac{1}{3}$	1.17	1.95	1.995	2.0

The limit does not exist because along the line $y = 0$ the function tends to infinity, whereas along the line $y = x$ the function tends to 2.

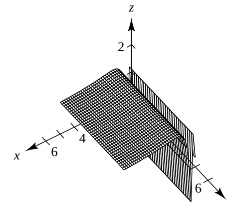
26. $\lim_{(x, y)\to(0, 0)} \frac{4x^2y^2}{(x^2 + y^2)} = 0$

Hence, $\lim_{(x, y)\to(0, 0)} f(x, y) = \lim_{(x, y)\to(0, 0)} g(x, y) = 0.$

f is continuous at (0, 0), whereas g is not continuous at (0, 0).

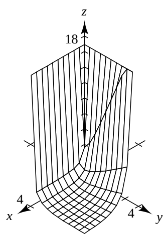
28. $\lim_{(x, y)\to(0, 0)} \left(\sin \frac{1}{x} + \cos \frac{1}{x}\right)$

Does not exist



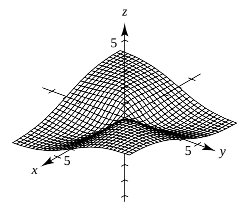
30. $\lim_{(x, y)\to(0, 0)} \frac{x^2 + y^2}{x^2y}$

Does not exist



32. $f(x, y) = \frac{2xy}{x^2 + y^2 + 1}$

The limit equals 0.



34. $\lim_{(x, y)\to(0, 0)} \frac{xy^2}{x^2 + y^2} = \lim_{r\to 0} \frac{(r \cos \theta)(r^2 \sin^2 \theta)}{r^2} = \lim_{r\to 0} (r \cos \theta \sin^2 \theta) = 0$

36. $\lim_{(x, y)\to(0, 0)} \frac{x^2y^2}{x^2 + y^2} = \lim_{r\to 0} \frac{r^4 \cos^2 \theta \sin^2 \theta}{r^2} = \lim_{r\to 0} r^2 \cos^2 \theta \sin^2 \theta = 0$

38. $f(x, y, z) = \frac{z}{x^2 + y^2 - 9}$

Continuous for $x^2 + y^2 \neq 9$

40. $f(x, y, z) = xy \sin z$

Continuous everywhere

42.
$$f(t) = \frac{1}{t}$$
$$g(x, y) = x^2 + y^2$$
$$f(g(x, y)) = f(x^2 + y^2) = \frac{1}{x^2 + y^2}$$

Continuous except at (0, 0)

44.
$$f(t) = \frac{1}{4 - t}$$
$$g(x, y) = x^2 + y^2$$
$$f(g(x, y)) = f(x^2 + y^2) = \frac{1}{4 - x^2 - y^2}$$

Continuous for $x^2 + y^2 \neq 4$

46. $f(x, y) = x^2 + y^2$

(a) $$\lim_{\Delta x\to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x\to 0} \frac{[(x + \Delta x)^2 + y^2] - (x^2 + y^2)}{\Delta x}$$
$$= \lim_{\Delta x\to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x\to 0} (2x + \Delta x) = 2x$$

(b) $$\lim_{\Delta y\to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y\to 0} \frac{[x^2 + (y + \Delta y)^2] - (x^2 + y^2)}{\Delta y}$$
$$= \lim_{\Delta y\to 0} \frac{2y\Delta y + (\Delta y)^2}{\Delta y} = \lim_{\Delta y\to 0} (2y + \Delta y) = 2y$$

48. $f(x, y) = \sqrt{y}(y + 1)$

(a) $\lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{y}(y + 1) - \sqrt{y}(y + 1)}{\Delta x} = 0$

(b)
$$\lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{(y + \Delta y)^{3/2} + (y + \Delta y)^{1/2} - (y^{3/2} + y^{1/2})}{\Delta y}$$
$$= \lim_{\Delta y \to 0} \frac{(y + \Delta y)^{3/2} - y^{3/2}}{\Delta y} + \lim_{\Delta y \to 0} \frac{(y + \Delta y)^{1/2 -} y^{1/2}}{\Delta y}$$
$$= \frac{3}{2}y^{1/2} + \frac{1}{2}y^{-1/2} \quad \text{(L'Hôpital's Rule)}$$
$$= \frac{3y + 1}{2\sqrt{y}}$$

50. See the definition on page 854.

52. $\lim_{(x, y) \to (0, 0)} \frac{x^2 + y^2}{xy}$

(a) Along $y = ax$: $\lim_{(x, ax) \to (0, 0)} \frac{x^2 + (ax)^2}{x(ax)}$

$$= \lim_{x \to 0} \frac{x^2(1 + a^2)}{ax^2} = \frac{1 + a^2}{a}, \; a \neq 0$$

If $a = 0$, then $y = 0$ and the limit does not exist.

(b) Along $y = x^2$: $\lim_{(x, x^2) \to (0, 0)} \frac{x^2 + (x^2)^2}{x(x^2)} = \lim_{x \to 0} \frac{1 + x^2}{x}$

limit does not exist

(c) No, the limit does not exist. Different paths result in different limits.

54. Given that $f(x, y)$ is continuous, then $\lim_{(x, y) \to (a, b)} f(x, y) = f(a, b) < 0$, which means that for each $\varepsilon > 0$, there corresponds a $\delta > 0$ such that $|f(x, y) - f(a, b)| < \varepsilon$ whenever

$$0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta.$$

Let $\varepsilon = |f(a, b)|/2$, then $f(x, y) < 0$ for every point in the corresponding δ neighborhood since

$$|f(x, y) - f(a, b)| < \frac{|f(a, b)|}{2} \Rightarrow -\frac{|f(a, b)|}{2} < f(x, y) - f(a, b) < \frac{|f(a, b)|}{2}$$
$$\Rightarrow \frac{3}{2}f(a, b) < f(x, y) < \frac{1}{2}f(a, b) < 0.$$

56. False. Let $f(x, y) = \frac{xy}{x^2 + y^2}$.

See Exercise 21.

58. True

Section 12.3 Partial Derivatives

2. $f_y(-1, -2) < 0$

4. $f_x(-1, -1) = 0$

6. $f(x, y) = x^2 - 3y^2 + 7$

$f_x(x, y) = 2x$

$f_y(x, y) = -6y$

8. $z = 2y^2\sqrt{x}$

$$\frac{\partial z}{\partial x} = \frac{y^2}{\sqrt{x}}$$

$$\frac{\partial z}{\partial y} = 4y\sqrt{x}$$

10. $z = y^3 - 4xy^2 - 1$

$$\frac{\partial z}{\partial x} = -4y^2$$

$$\frac{\partial z}{\partial y} = 3y^2 - 8xy$$

12. $z = xe^{x/y}$

$$\frac{\partial z}{\partial x} = \frac{x}{y}e^{x/y} + e^{x/y} = e^{x/y}\left(\frac{x}{y} + 1\right)$$

$$\frac{\partial z}{\partial y} = xe^{x/y}\left(-\frac{x}{y^2}\right) = -\frac{x^2}{y^2}e^{x/y}$$

14. $z = \ln\sqrt{xy} = \frac{1}{2}\ln(xy)$

$$\frac{\partial z}{\partial x} = \frac{1}{2}\frac{y}{xy} = \frac{1}{2x}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}\frac{x}{xy} = \frac{1}{2y}$$

16. $z = \ln(x^2 - y^2)$

$$\frac{\partial z}{\partial x} = \frac{1}{x^2 - y^2}(2x) = \frac{2x}{x^2 - y^2}$$

$$\frac{\partial z}{\partial y} = \frac{-2y}{x^2 - y^2}$$

18. $f(x, y) = \dfrac{xy}{x^2 + y^2}$

$$f_x(x, y) = \frac{(x^2 + y^2)(y) - (xy)(2x)}{(x^2 + y^2)^2} = \frac{y^3 - x^2y}{(x^2 + y^2)^2}$$

$$f_y(x, y) = \frac{(x^2 + y^2)(x) - (xy)(2y)}{(x^2 + y^2)^2} = \frac{x^3 - xy^2}{(x^2 + y^2)^2}$$

20. $g(x, y) = \ln\sqrt{x^2 + y^2} = \frac{1}{2}\ln(x^2 + y^2)$

$$g_x(x, y) = \frac{1}{2}\frac{2x}{x^2 + y^2} = \frac{x}{x^2 + y^2}$$

$$g_y(x, y) = \frac{1}{2}\frac{2y}{x^2 + y^2} = \frac{y}{x^2 + y^2}$$

22. $f(x, y) = \sqrt{2x + y^3}$

$$\frac{\partial f}{\partial x} = \frac{1}{2}(2x + y^3)^{-1/2}(2) = \frac{1}{\sqrt{2x + y^3}}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}(2x + y^3)^{-1/2}(3y^2) = \frac{3y^2}{2\sqrt{2x + y^3}}$$

24. $z = \sin 3x \cos 3y$

$$\frac{\partial z}{\partial x} = 3\cos 3x \cos 3y$$

$$\frac{\partial z}{\partial y} = -3\sin 3x \sin 3y$$

26. $z = \cos(x^2 + y^2)$

$$\frac{\partial z}{\partial x} = -2x\sin(x^2 + y^2)$$

$$\frac{\partial z}{\partial y} = -2y\sin(x^2 + y^2)$$

28. $f(x, y) = \int_x^y (2t + 1)\,dt + \int_y^x (2t - 1)\,dt$

$$= \int_x^y (2t + 1)\,dt - \int_x^y (2t - 1)\,dt$$

$$= \int_x^y 2\,dt = \Big[2t\Big]_x^y = 2y - 2x$$

$f_x(x, y) = -2$

$f_y(x, y) = 2$

30. $f(x, y) = x^2 - 2xy + y^2 = (x - y)^2$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - 2(x + \Delta x)y + y^2 - x^2 + 2xy - y^2}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x - 2y) = 2(x - y)$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{x^2 - 2x(y + \Delta y) + (y + \Delta y)^2 - x^2 + 2xy - y^2}{\Delta y} = \lim_{\Delta y \to 0} (-2x + 2y + \Delta y) = 2(y - x)$$

32. $f(x, y) = \dfrac{1}{x + y}$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{1}{x + \Delta x + y} - \dfrac{1}{x + y}}{\Delta x} = \lim_{\Delta x \to 0} \frac{-1}{(x + \Delta x + y)(x + y)} = \frac{-1}{(x + y)^2}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\dfrac{1}{x + y + \Delta y} - \dfrac{1}{x + y}}{\Delta y} = \lim_{\Delta y \to 0} \frac{-1}{(x + y + \Delta y)(x + y)} = \frac{-1}{(x + y)^2}$$

34. $h(x, y) = x^2 - y^2$

$h_x(x, y) = 2x$

At $(-2, 1)$: $h_x(-2, 1) = -4$

$h_y(x, y) = -2y$

At $(-2, 1)$: $h_y(-2, 1) = -2$

36. $z = \cos(2x - y)$

$\dfrac{\partial z}{\partial x} = -2\sin(2x - y)$

At $\left(\dfrac{\pi}{4}, \dfrac{\pi}{3}\right)$, $\dfrac{\partial z}{\partial x} = -2\sin\left(\dfrac{\pi}{6}\right) = -1$

$\dfrac{\partial z}{\partial y} = -\sin(2x - y)(-1) = \sin(2x - y)$

At $\left(\dfrac{\pi}{4}, \dfrac{\pi}{3}\right)$, $\dfrac{\partial z}{\partial y} = \sin\left(\dfrac{\pi}{6}\right) = \dfrac{1}{2}$

38. $f(x, y) = \arccos(xy)$

$f_x(x, y) = \dfrac{-y}{\sqrt{1 - x^2y^2}}$

At $(1, 1)$, f_x is undefined.

$f_y(x, y) = \dfrac{-x}{\sqrt{1 - x^2y^2}}$

At $(1, 1)$, f_y is undefined.

40. $f(x, y) = \dfrac{6xy}{\sqrt{4x^2 + 5y^2}}$

$f_x(x, y) = \dfrac{30y^3}{(4x^2 + 5y^2)^{3/2}}$

At $(1, 1)$, $f_x(1, 1) = \dfrac{30}{27} = \dfrac{10}{9}$

$f_y(x, y) = \dfrac{24x^3}{(4x^2 + 5y^2)^{3/2}}$

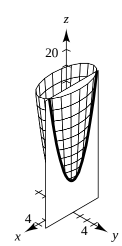
At $(1, 1)$, $f_y(1, 1) = \dfrac{8}{9}$

42. $z = x^2 + 4y^2,\ y = 1,\ (2, 1, 8)$

Intersecting curve: $z = x^2 + 4$

$\dfrac{\partial z}{\partial x} = 2x$

At $(2, 1, 8)$: $\dfrac{\partial z}{\partial x} = 2(2) = 4$

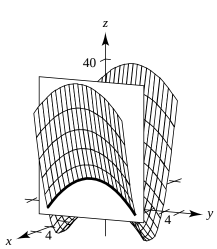


44. $z = 9x^2 - y^2,\ x = 1,\ (1, 3, 0)$

Intersecting curve: $z = 9 - y^2$

$\dfrac{\partial z}{\partial y} = -2y$

At $(1, 3, 0)$: $\dfrac{\partial z}{\partial y} = -2(3) = -6$



46. $f_x(x, y) = 9x^2 - 12y,\ f_y(x, y) = -12x + 3y^2$

$f_x = f_y = 0$: $9x^2 - 12y = 0 \implies 3x^2 = 4y$

$3y^2 - 12x = 0 \implies y^2 = 4x$

Solving for x in the second equation, $x = y^2/4$, you obtain $3(y^2/4)^2 = 4y$.

$3y^4 = 64y \implies y = 0$ or $y = \dfrac{4}{3^{1/3}}$

$\implies x = 0$ or $x = \dfrac{1}{4}\left(\dfrac{16}{3^{2/3}}\right)$

Points: $(0, 0)$, $\left(\dfrac{4}{3^{2/3}}, \dfrac{4}{3^{1/3}}\right)$

48. $f_x(x, y) = \dfrac{2x}{x^2 + y^2 + 1} = 0 \Rightarrow x = 0$

$f_y(x, y) = \dfrac{2y}{x^2 + y^2 + 1} = 0 \Rightarrow y = 0$

Points: $(0, 0)$

50. (a) The graph is that of f_x.

(b) The graph is that of f_y.

52. $w = \dfrac{3xz}{x + y}$

$\dfrac{\partial w}{\partial x} = \dfrac{(x + y)(3z) - 3xz}{(x + y)^2} = \dfrac{3yz}{(x + y)^2}$

$\dfrac{\partial w}{\partial y} = \dfrac{-3xz}{(x + y)^2}$

$\dfrac{\partial w}{\partial z} = \dfrac{3x}{x + y}$

54. $G(x, y, z) = \dfrac{1}{\sqrt{1 - x^2 - y^2 - z^2}}$

$G_x(x, y, z) = \dfrac{x}{(1 - x^2 - y^2 - z^2)^{3/2}}$

$G_y(x, y, z) = \dfrac{y}{(1 - x^2 - y^2 - z^2)^{3/2}}$

$G_z(x, y, z) = \dfrac{z}{(1 - x^2 - y^2 - z^2)^{3/2}}$

56. $f(x, y, z) = 3x^2y - 5xyz + 10yz^2$

$f_x(x, y, z) = 6xy - 5yz$

$f_y(x, y, z) = 3x^2 - 5xz + 10z^2$

$f_z(x, y, z) = -5xy + 20yz$

58. $z = x^4 - 3x^2y^2 + y^4$

$\dfrac{\partial z}{\partial x} = 4x^3 - 6xy^2$

$\dfrac{\partial^2 z}{\partial x^2} = 12x^2 - 6y^2$

$\dfrac{\partial^2 z}{\partial y \partial x} = -12xy$

$\dfrac{\partial z}{\partial y} = -6x^2y + 4y^3$

$\dfrac{\partial^2 z}{\partial y^2} = -6x^2 + 12y^2$

$\dfrac{\partial^2 z}{\partial x \partial y} = -12xy$

60. $z = \ln(x - y)$

$\dfrac{\partial z}{\partial x} = \dfrac{1}{x - y}$

$\dfrac{\partial^2 z}{\partial x^2} = -\dfrac{1}{(x - y)^2}$

$\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{1}{(x - y)^2}$

$\dfrac{\partial z}{\partial y} = \dfrac{-1}{x - y} = \dfrac{1}{y - x}$

$\dfrac{\partial^2 z}{\partial y^2} = -\dfrac{1}{(x - y)^2}$

$\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{1}{(x - y)^2}$

Therefore, $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y}$.

62. $z = 2xe^y - 3ye^{-x}$

$\dfrac{\partial z}{\partial x} = 2e^y + 3ye^{-x}$

$\dfrac{\partial^2 z}{\partial x^2} = -3ye^{-x}$

$\dfrac{\partial^2 z}{\partial y \partial x} = 2e^y + 3ye^{-x}$

$\dfrac{\partial z}{\partial y} = 2xe^y - 3e^{-x}$

$\dfrac{\partial^2 z}{\partial y^2} = 2xe^y$

$\dfrac{\partial^2 z}{\partial x \partial y} = 2e^y + 3e^{-x}$

64. $z = \sin(x - 2y)$

$\dfrac{\partial z}{\partial x} = \cos(x - 2y)$

$\dfrac{\partial^2 z}{\partial x^2} = -\sin(x - 2y)$

$\dfrac{\partial^2 z}{\partial y \partial x} = 2\sin(x - 2y)$

$\dfrac{\partial z}{\partial y} = -2\cos(x - 2y)$

$\dfrac{\partial^2 z}{\partial y^2} = -4\sin(x - 2y)$

$\dfrac{\partial^2 z}{\partial x \partial y} = 2\sin(x - 2y)$

66. $z = \sqrt{9 - x^2 - y^2}$

$\dfrac{\partial z}{\partial x} = \dfrac{-x}{\sqrt{9 - x^2 - y^2}}$

$\dfrac{\partial^2 z}{\partial x^2} = \dfrac{y^2 - 9}{(9 - x^2 - y^2)^{3/2}}$

$\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{-xy}{(9 - x^2 - y^2)^{3/2}}$

$\dfrac{\partial z}{\partial y} = \dfrac{-y}{\sqrt{9 - x^2 - y^2}}$

$\dfrac{\partial^2 z}{\partial y^2} = \dfrac{x^2 - 9}{(9 - x^2 - y^2)^{3/2}}$

$\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{-xy}{(9 - x^2 - y^2)^{3/2}}$

Therefore, $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y}$.

$\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial y} = 0$ if $x = y = 0$

68. $z = \dfrac{xy}{x - y}$

$$\frac{\partial z}{\partial x} = \frac{y(x - y) - xy}{(x - y)^2} = \frac{-y^2}{(x - y)^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{2y^2}{(x - y)^3}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{(x - y)^2(-2y) + y^2(2)(x - y)(-1)}{(x - y)^4} = \frac{-2xy}{(x - y)^3}$$

$$\frac{\partial z}{\partial y} = -\frac{x(x - y) + xy}{(x - y)^2} = \frac{x^2}{(x - y)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{2x^2}{(x - y)^3}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{(x - y)^2(2x) - x^2(2)(x - y)}{(x - y)^4} = \frac{-2xy}{(x - y)^3}$$

There are no points for which $z_x = z_y = 0$.

70. $f(x, y, z) = x^2 - 3xy + 4yz + z^3$

$f_x(x, y, z) = 2x - 3y$

$f_y(x, y, z) = -3x + 4z$

$f_{yy}(x, y, z) = 0$

$f_{xy}(x, y, z) = -3$

$f_{yx}(x, y, z) = -3$

$f_{yyx}(x, y, z) = 0$

$f_{xyy}(x, y, z) = 0$

$f_{yxy}(x, y, z) = 0$

Therefore, $f_{xyy} = f_{yxy} = f_{yyx} = 0$.

72. $f(x, y, z) = \dfrac{2z}{x + y}$

$$f_x(x, y, z) = \frac{-2z}{(x + y)^2}$$

$$f_y(x, y, z) = \frac{-2z}{(x + y)^2}$$

$$f_{yy}(x, y, z) = \frac{4z}{(x + y)^3}$$

$$f_{xy}(x, y, z) = \frac{4z}{(x + y)^3}$$

$$f_{yx}(x, y, z) = \frac{4z}{(x + y)^3}$$

$$f_{yyx}(x, y, z) = \frac{-12z}{(x + y)^4}$$

$$f_{xyy}(x, y, z) = \frac{-12z}{(x + y)^4}$$

$$f_{yxy}(x, y, z) = \frac{-12z}{(x + y)^4}$$

74. $z = \sin x\left(\dfrac{e^y - e^{-y}}{2}\right)$

$$\frac{\partial z}{\partial x} = \cos x\left(\frac{e^y - e^{-y}}{2}\right)$$

$$\frac{\partial^2 z}{\partial x^2} = -\sin x\left(\frac{e^y - e^{-y}}{2}\right)$$

$$\frac{\partial z}{\partial y} = \sin x\left(\frac{e^y + e^{-y}}{2}\right)$$

$$\frac{\partial^2 z}{\partial y^2} = \sin x\left(\frac{e^y - e^{-y}}{2}\right)$$

Therefore,

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\sin x\left(\frac{e^y - e^{-y}}{2}\right) + \sin x\left(\frac{e^y - e^{-y}}{2}\right) = 0.$$

76. $z = \arctan \dfrac{y}{x}$

From Exercise 53, we have

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{2xy}{(x^2 + y^2)^2} + \frac{-2xy}{(x^2 + y^2)^2} = 0.$$

78. $z = \sin(wct)\sin(wx)$

$$\frac{\partial z}{\partial t} = wc\cos(wct)\sin(wx)$$

$$\frac{\partial^2 z}{\partial t^2} = -w^2c^2\sin(wct)\sin(wx)$$

$$\frac{\partial z}{\partial x} = w\sin(wct)\cos(wx)$$

$$\frac{\partial^2 z}{\partial x^2} = -w^2\sin(wct)\sin(wx)$$

Therefore, $\dfrac{\partial^2 z}{\partial t^2} = c^2\dfrac{\partial^2 z}{\partial x^2}$.

80. $z = e^{-t}\sin\frac{x}{c}$

$$\frac{\partial z}{\partial t} = -e^{-t}\sin\frac{x}{c}$$

$$\frac{\partial z}{\partial x} = \frac{1}{c}e^{-t}\cos\frac{x}{c}$$

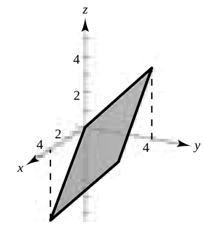
$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{c^2}e^{-t}\sin\frac{x}{c}$$

Therefore, $\frac{\partial z}{\partial t} = c^2\frac{\partial^2 z}{\partial x^2}$.

82. If $z = f(x, y)$, then to find f_x you consider y constant and differentiate with respect to x. Similarly, to find f_y, you consider x constant and differentiate with respect to y.

84. The plane $z = -x + y = f(x, y)$ satisfies

$$\frac{\partial f}{\partial x} < 0 \text{ and } \frac{\partial f}{\partial y} > 0.$$



86. In this case, the mixed partials are equal, $f_{xy} = f_{yx}$.

See Theorem 12.3.

88. $f(x, y) = 200x^{0.7}y^{0.3}$

(a) $\frac{\partial f}{\partial x} = 140x^{-0.3}y^{0.3} = 140\left(\frac{y}{x}\right)^{0.3}$

At $(x, y) = (1000, 500)$, $\frac{\partial f}{\partial x} = 140\left(\frac{500}{1000}\right)^{0.3} = 140\left(\frac{1}{2}\right)^{0.3} \approx 113.72$

(b) $\frac{\partial f}{\partial x} = 60x^{0.7}y^{-0.7} = 60\left(\frac{x}{y}\right)^{0.7}$

At $(x, y) = (1000, 500)$, $\frac{\partial f}{\partial x} = 60\left(\frac{1000}{500}\right)^{0.7} = 60(2)^{0.7} \approx 97.47$

90. $$V(I, R) = 1000\left[\frac{1 + 0.10(1 - R)}{1 + I}\right]^{10}$$

$$V_I(I, R) = 10{,}000\left[\frac{1 + 0.10(1 - R)}{1 + I}\right]^9\left[-\frac{1 + 0.10(1 - R)}{(1 + I)^2}\right] = -10{,}000\frac{[1 + 0.10(1 - R)]^{10}}{(1 + I)^{11}}$$

$V_I(0.03, 0.28) \approx -14{,}478.99$

$$V_R(I, R) = 10{,}000\left[\frac{1 + 0.10(1 - R)}{1 + I}\right]^9\left[\frac{-0.10}{1 + I}\right] = -1000\frac{[1 + 0.10(1 - R)]^9}{(1 + I)^{10}}$$

$V_R(0.03, 0.28) \approx -1391.17$

The rate of inflation has the greater negative influence on the growth of the investment. (See Exercise 61 in Section 12.1.)

92. $A = 0.885t - 22.4h + 1.20th - 0.544$

(a) $\frac{\partial A}{\partial t} = 0.885 + 1.20h$

$$\frac{\partial A}{\partial t}(30°, 0.80) = 0.885 + 1.20(0.80) = 1.845$$

$$\frac{\partial A}{\partial h} = -22.4 + 1.20t$$

$$\frac{\partial A}{\partial h}(30°, 0.80) = -22.4 + 1.20(30°) = 13.6$$

(b) The humidity has a greater effect on A since its coefficient -22.4 is larger than that of t.

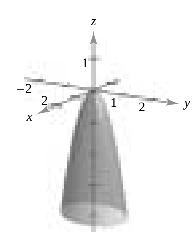
94. $U = -5x^2 + xy - 3y^2$

(a) $U_x = -10x + y$

(b) $U_y = x - 6y$

(c) $U_x(2, 3) = -17$ and $U_y(2, 3) = -16$. The person should consume one more unit of y because the rate of decrease of satisfaction is less for y.

(d)



96. (a) $\dfrac{\partial z}{\partial x} = -1.55x + 22.15$

$\dfrac{\partial^2 z}{\partial x^2} = -1.55$

$\dfrac{\partial z}{\partial y} = 0.014y - 0.54$

$\dfrac{\partial^2 z}{\partial y^2} = 0.014$

(b) Concave downward $\left(\dfrac{\partial^2 z}{\partial x^2} < 0\right)$

The rate of increase of Medicare expenses (z) is declining with respect to worker's compensation expenses (x).

(c) Concave upward $\left(\dfrac{\partial^2 z}{\partial y^2} > 0\right)$

The rate of increase of Medicare expenses (z) is increasing with respect to public assistance expenses (y).

98. False

Let $z = x + y + 1$.

100. True

102. $f(x, y) = \displaystyle\int_x^y \sqrt{1 + t^3}\, dt$

By the Second Fundamental Theorem of Calculus,

$$\frac{\partial f}{\partial x} = \frac{d}{dx}\int_x^y \sqrt{1 + t^3}\, dt = -\frac{d}{dx}\int_y^x \sqrt{1 + t^3}\, dt = -\sqrt{1 + x^3}$$

$$\frac{\partial f}{\partial y} = \frac{d}{dy}\int_x^y \sqrt{1 + t^3}\, dt = \sqrt{1 + y^3}.$$

Section 12.4 Differentials

2. $z = \dfrac{x^2}{y}$

$$dz = \frac{2x}{y}\, dx - \frac{x^2}{y^2}\, dy$$

4. $w = \dfrac{x + y}{z - 2y}$

$$dw = \frac{1}{z - 2y}\, dx + \frac{z + 2x}{(z - 2y)^2}\, dy - \frac{x + y}{(z - 2y)^2}\, dz$$

6. $z = \left(\dfrac{1}{2}\right)\left(e^{x^2+y^2} - e^{-x^2-y^2}\right)$

$$dz = 2x\left(\frac{e^{x^2+y^2} + e^{-x^2-y^2}}{2}\right) dx + 2y\left(\frac{e^{x^2+y^2} + e^{-x^2-y^2}}{2}\right) dy = \left(e^{x^2+y^2} + e^{-x^2-y^2}\right)(x\, dx + y\, dy)$$

8. $w = e^y \cos x + z^2$

$$dw = -e^y \sin x\, dx + e^y \cos x\, dy + 2z\, dz$$

10. $w = x^2yz^2 + \sin yz$

$$dw = 2xyz^2\, dx + (x^2z^2 + z\cos yz)\, dy + (2x^2yz + y\cos yz)\, dz$$

12. (a) $f(1, 2) = \sqrt{5} \approx 2.2361$

$f(1.05, 2.1) = \sqrt{5.5125} \approx 2.3479$

$\Delta z = 0.11180$

(b) $dz = \dfrac{x}{\sqrt{x^2 + y^2}}\, dx + \dfrac{y}{\sqrt{x^2 + y^2}}\, dy$

$= \dfrac{x\, dx + y\, dy}{\sqrt{x^2 + y^2}} = \dfrac{0.05 + 2(0.1)}{\sqrt{5}} \approx 0.11180$

14. (a) $f(1, 2) = e^2 \approx 7.3891$

$f(1.05, 2.1) = 1.05e^{2.1} \approx 8.5745$

$\Delta z = 1.1854$

(b) $dz = e^y\, dx + xe^y\, dy$

$= e^2(0.05) + e^2(0.1) \approx 1.1084$

16. (a) $f(1, 2) = \frac{1}{2} = 0.5$

$f(1.05, 2.1) = \frac{1.05}{2.1} = 0.5$

$\Delta z = 0$

(b) $dz = \frac{1}{y}\,dx - \frac{x}{y^2}\,dy$

$= \frac{1}{2}(0.05) - \frac{1}{4}(0.1) = 0$

18. Let $z = x^2(1 + y)^3$, $x = 2$, $y = 9$, $dx = 0.03$, $dy = -0.1$. Then: $dz = 2x(1 + y)^3\,dx + 3x^2(1 + y)^2\,dy$

$(2.03)^2(1 + 8.9)^3 - 2^2(1 + 9)^3 \approx 2(2)(1 + 9)^3(0.03) + 3(2)^2(1 + 9)^2(-0.1) = 0$

20. Let $z = \sin(x^2 + y^2)$, $x = y = 1$, $dx = 0.05$, $dy = -0.05$. Then: $dz = 2x\cos(x^2 + y^2)\,dx + 2y\cos(x^2 + y^2)\,dy$

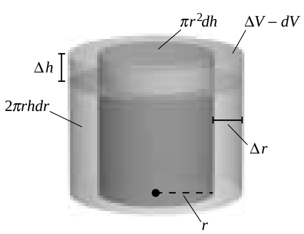
$\sin[(1.05)^2 + (0.95)^2] - \sin 2 \approx 2(1)\cos(1^2 + 1^2)(0.05) + 2(1)\cos(1^2 + 1^2)(-0.05) = 0$

22. In general, the accuracy worsens as Δx and Δy increase.

24. If $z = f(x, y)$, then $\Delta z \approx dz$ is the propagated error, and $\frac{\Delta z}{z} \approx \frac{dz}{z}$ is the relative error.

26. $V = \pi r^2 h$

$dV = 2\pi rh\,dr + \pi r^2\,dh$



28. $S = \pi r\sqrt{r^2 + h^2}$

$r = 8, h = 20$

$\frac{dS}{dr} = \pi(r^2 + h^2)^{1/2} + \pi r^2(r^2 + h^2)^{-1/2}$

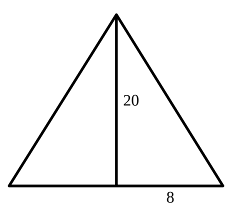
$= \frac{\pi(r^2 + h^2) + \pi r^2}{(r^2 + h^2)^{1/2}} = \pi\frac{2r^2 + h^2}{\sqrt{r^2 + h^2}}$

$\frac{dS}{dh} = \pi r(r^2 + h^2)^{-1/2}h = \pi\frac{rh}{\sqrt{r^2 + h^2}}$

$dS = \pi\frac{2r^2 + h^2}{\sqrt{r^2 + h^2}}\,dr + \pi\frac{rh}{\sqrt{r^2 + h^2}}\,dh$

$= \frac{\pi}{\sqrt{r^2 + h^2}}[(2r^2 + h^2)\,dr + (rh)\,dh]$

$S(8, 20) = 541.3758$



Δr	Δh	dS	ΔS	$\Delta S - dS$
0.1	0.1	10.0341	10.0768	0.0427
0.1	−0.1	5.3671	5.3596	−0.0075
0.001	0.002	0.12368	0.12368	0.683×10^{-5}
−0.0001	0.0002	−0.00303	−0.00303	-0.286×10^{-7}

30. $$\frac{\partial C}{\partial v} = 0.0817\left[(3.71)\frac{1}{2}v^{-1/2} - 0.25\right](T - 91.4)$$
$$= \left[\frac{0.1516}{v^{1/2}} - 0.0204\right](T - 91.4)$$
$$\frac{\partial C}{\partial T} = 0.0817\left(3.71\sqrt{v} + 5.81 - 0.25v\right)$$
$$dC = C_v\,dv + C_T\,dT$$
$$= \left(\frac{0.1516}{23^{1/2}} - 0.0204\right)(8 - 91.4)(\pm 3) + 0.0817\left(3.71\sqrt{23} + 5.81 - 0.25(23)\right)(\pm 1)$$
$$= \pm 2.79 \pm 1.46 = \pm 4.25 \text{ Maximum propagated error}$$
$$\frac{dC}{C} = \frac{\pm 4.25}{-30.24} \approx \pm 0.14$$

32. $(x, y) = (8.5, 3.2)$, $|dx| \le 0.05$, $|dy| \le 0.05$

$$r = \sqrt{x^2 + y^2} \Longrightarrow dr = \frac{x}{\sqrt{x^2 + y^2}}\,dx + \frac{y}{\sqrt{x^2 + y^2}}\,dy$$
$$= \frac{8.5}{\sqrt{8.5^2 + 3.2^2}}\,dx + \frac{3.2}{\sqrt{8.5^2 + 3.2^2}}\,dy \approx 0.9359\,dx + 0.3523\,dy$$
$$|dr| \le (1.288)(0.05) \approx 0.064$$
$$\theta = \arctan\left(\frac{y}{x}\right) \Longrightarrow d\theta = \frac{-\frac{y}{x^2}}{1 + \left(\frac{y}{x}\right)^2}\,dx + \frac{\frac{1}{x}}{1 + \left(\frac{y}{x}\right)^2}\,dy$$
$$= \frac{-y}{x^2 + y^2}\,dx + \frac{x}{x^2 + y^2}\,dy = \frac{-3.2}{8.5^2 + 3.2^2}\,dx + \frac{8.5}{8.5^2 + 3.2^2}\,dy$$

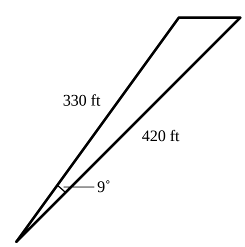
Using the worst case scenario, $dx = -0.05$ and $dy = 0.05$, you see that

$$|d\theta| \le 0.00194 + 0.00515 = 0.0071.$$

34. $$a = \frac{v^2}{r}$$
$$da = \frac{2v}{r}\,dv - \frac{v^2}{r^2}\,dr$$
$$\frac{da}{a} = 2\frac{dv}{v} - \frac{dr}{r} = 2(0.03) - (-0.02) = 0.08 = 8\%$$

Note: The maximum error will occur when dv and dr differ in signs.

36. (a) Using the Law of Cosines:



$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$= 330^2 + 420^2 - 2(330)(420)\cos 9°$$
$$a \approx 107.3 \text{ ft.}$$

(b) $a = \sqrt{b^2 + 420^2 - 2b(420)\cos\theta}$

$$da = \frac{1}{2}\left[b^2 + 420^2 - 840b\cos\theta\right]^{-1/2}\left[(2b - 840\cos\theta)\,db + 840b\sin\theta\,d\theta\right]$$
$$= \frac{1}{2}\left[330^2 + 420^2 - 840(330)\left(\cos\frac{\pi}{20}\right)\right]^{-1/2}\left[\left(2(330) - 840\cos\frac{\pi}{20}\right)(6) + 840(330)\left(\sin\frac{\pi}{20}\right)\left(\frac{\pi}{180}\right)\right]$$
$$\approx \frac{1}{2}[11512.79]^{-1/2}[\pm 1774.79] \approx \pm 8.27 \text{ ft}$$

38. $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$

$$R = \frac{R_1R_2}{R_1 + R_2}$$

$$dR_1 = \Delta R_1 = 0.5$$

$$dR_2 = \Delta R_2 = -2$$

$$\Delta R \approx dR = \frac{\partial R}{\partial R_1}\,dR + \frac{\partial R}{\partial R_2}\,dR_2 = \frac{R_2{}^2}{(R_1 + R_2)^2}\,\Delta R_1 + \frac{R_1{}^2}{(R_1 + R_2)^2}\,\Delta R_2$$

When $R_1 = 10$ and $R_2 = 15$, we have $\Delta R \approx \dfrac{15^2}{(10 + 15)^2}(0.5) + \dfrac{10^2}{(10 + 15)^2}(-2) = -0.14$ ohm.

40. $T = 2\pi\sqrt{\dfrac{L}{g}}$

$$dg = \Delta g = 32.24 - 32.09 = 0.15$$

$$dL = \Delta L = 2.48 - 2.5 = -0.02$$

$$\Delta T \approx dT = \frac{\partial T}{\partial g}\,dg + \frac{\partial T}{\partial L}\,dL = -\frac{\pi}{g}\sqrt{\frac{L}{g}}\,\Delta g + \frac{\pi}{\sqrt{Lg}}\,\Delta L$$

When $g = 32.09$ and $L = 2.5$, we have $\Delta T \approx -\dfrac{\pi}{32.09}\sqrt{\dfrac{2.5}{32.09}}(0.15) + \dfrac{\pi}{\sqrt{(2.5)(32.09)}}(-0.02) \approx -0.0111$ sec.

42. $z = f(x, y) = x^2 + y^2$

$$\begin{aligned}\Delta z &= f(x + \Delta x,\ y + \Delta y) - f(x, y)\\ &= x^2 + 2x(\Delta x) + (\Delta x)^2 + y^2 + 2y(\Delta y) + (\Delta y)^2 - (x^2 + y^2)\\ &= 2x(\Delta x) + 2y(\Delta y) + \Delta x(\Delta x) + \Delta y(\Delta y)\\ &= f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y \text{ where } \epsilon_1 = \Delta x \text{ and } \epsilon_2 = \Delta y.\end{aligned}$$

As $(\Delta x, \Delta y) \to (0, 0)$, $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$.

44. $z = f(x, y) = 5x - 10y + y^3$

$$\begin{aligned}\Delta z &= f(x + \Delta x,\ y + \Delta y) - f(x, y)\\ &= 5x + 5\Delta x - 10y - 10\Delta y + y^3 + 3y^2(\Delta y) + 3y(\Delta y)^2 + (\Delta y)^3 - (5x - 10y + y^3)\\ &= 5(\Delta x) + (3y^2 - 10)(\Delta y) + 0(\Delta x) + (3y(\Delta y) + (\Delta y)^2)\,\Delta y\\ &= f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y \text{ where } \epsilon_1 = 0 \text{ and } \epsilon_2 = 3y(\Delta y) + (\Delta y)^2.\end{aligned}$$

As $(\Delta x, \Delta y) \to (0, 0)$, $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$.

46. $f(x, y) = \begin{cases} \dfrac{5x^2y}{x^3 + y^3}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

(a) $f_x(0, 0) = \displaystyle\lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} = 0$

$f_y(0, 0) = \displaystyle\lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y} = 0$

Thus, the partial derivatives exist at (0, 0).

(b) Along the line $y = x$: $\displaystyle\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{x \to 0} \frac{5x^3}{2x^3} = \frac{5}{2}$.

Along the line $x = 0$, $\displaystyle\lim_{(x, y) \to (0, 0)} f(x, y) = 0$.

Thus, f is not continuous at (0, 0). Therefore f is not differentiable at (0, 0).

(See Theorem 12.5)

Section 12.5 Chain Rules for Functions of Several Variables

2. $w = \sqrt{x^2 + y^2}$

$x = \cos t, y = e^t$

$$\frac{dw}{dt} = \frac{x}{\sqrt{x^2 + y^2}}(-\sin t) + \frac{y}{\sqrt{x^2 + y^2}}e^t$$

$$= \frac{-x \sin t + ye^t}{\sqrt{x^2 + y^2}} = \frac{-\cos t \sin t + e^{2t}}{\sqrt{\cos^2 t + e^{2t}}}$$

4. $w = \ln \frac{y}{x}$

$x = \cos t$

$y = \sin t$

$$\frac{dw}{dt} = \left(\frac{-1}{x}\right)(-\sin t) + \left(\frac{1}{y}\right)(\cos t)$$

$$= \tan t + \cot t = \frac{1}{\sin t \cos t}$$

6. $w = \cos(x - y),\ x = t^2,\ y = 1$

(a) $\frac{dw}{dt} = -\sin(x - y)(2t) + \sin(x - y)(0)$

$= -2t \sin(x - y) = -2t \sin(t^2 - 1)$

(b) $w = \cos(t^2 - 1),\ \frac{dw}{dt} = -2t \sin(t^2 - 1)$

8. $w = xy \cos z$

$x = t$

$y = t^2$

$z = \arccos t$

(a) $\frac{dw}{dt} = (y \cos z)(1) + (x \cos z)(2t) + (-xy \sin z)\left(-\frac{1}{\sqrt{1 - t^2}}\right)$

$= t^2(t) + t(t)(2t) - t(t^2)\sqrt{1 - t^2}\left(\frac{-1}{\sqrt{1 - t^2}}\right) = t^3 + 2t^3 + t^3 = 4t^3$

(b) $w = t^4,\ \frac{dw}{dt} = 4t^3$

10. $w = xyz,\ x = t^2,\ y = 2t,\ z = e^{-t}$

(a) $\frac{dw}{dt} = yz(2t) + xz(2) + (xy)(-e^{-t})$

$= (2t)(e^{-t})(2t) + (t^2)(e^{-t})(2) + (t^2)(2t)(-e^{-t})$

$= 2t^2e^{-t}(2 + 1 - t) = 2t^2e^{-t}(3 - t)$

(b) $w = (t^2)(2t)(e^{-t}) = 2t^3e^{-t}$

$\frac{dw}{dt} = (2t^3)(-e^{-t}) + (e^{-t})(6t^2) = 2t^2e^{-t}(-t + 3)$

12. Distance $= f(t) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{\left[48 + \left(\sqrt{3} - \sqrt{2}\right)\right]^2 + \left[48t\left(1 - \sqrt{2}\right)\right]^2}$

$= 48t\sqrt{8 - 2\sqrt{2} - 2\sqrt{6}}$

$f'(t) = 48\sqrt{8 - 2\sqrt{2} - 2\sqrt{6}} = f'(1)$

14. $w = \dfrac{x^2}{y},$

$x = t^2,$

$y = t + 1,$

$t = 1$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= \frac{2x}{y}(2t) + \frac{-x^2}{y^2}(1)$$

$$= \frac{2t^2(2t)}{t+1} - \frac{t^4}{(t+1)^2}$$

$$= \frac{(t+1)(4t^3) - t^4}{(t+1)^2}$$

$$= \frac{3t^4 + 4t^3}{(t+1)^2}$$

$$\frac{d^2w}{dt^2} = \frac{(t+1)^2(12t^3 + 12t^2) - (3t^4 + 4t^3)2(t+1)}{(t+1)^4}$$

At $t = 1$: $\dfrac{d^2w}{dt^2} = \dfrac{4(24) - (7)(4)}{16} = \dfrac{68}{16} = 4.25$

16. $w = y^3 - 3x^2y$

$x = e^s$

$y = e^t$

$$\frac{\partial w}{\partial s} = -6xy(e^s) + (3y^2 - 3x^2)(0) = -6e^{2s+t}$$

$$\frac{\partial w}{\partial t} = -6xy(0) + (3y^2 - 3x^2)(e^t)$$

$$= 3e^t(e^{2t} - e^{2s})$$

When $s = 0$ and $t = 1$, $\dfrac{\partial w}{\partial s} = -6e$ and $\dfrac{\partial w}{\partial t} = 3e(e^2 - 1)$.

18. $w = \sin(2x + 3y)$

$x = s + t$

$y = s - t$

$$\frac{\partial w}{\partial s} = 2\cos(2x + 3y) + 3\cos(2x + 3y)$$

$$= 5\cos(2x + 3y) = 5\cos(5s - t)$$

$$\frac{\partial w}{\partial t} = 2\cos(2x + 3y) - 3\cos(2x + 3y)$$

$$= -\cos(2x + 3y) = -\cos(5s - t)$$

When $s = 0$ and $t = \dfrac{\pi}{2}$, $\dfrac{\partial w}{\partial s} = 0$ and $\dfrac{\partial w}{\partial t} = 0$.

20. $w = \sqrt{25 - 5x^2 - 5y^2}$, $x = r\cos\theta$, $y = r\sin\theta$

(a) $$\frac{\partial w}{\partial r} = \frac{-5x}{\sqrt{25 - 5x^2 - 5y^2}}\cos\theta + \frac{-5y}{\sqrt{25 - 5x^2 - 5y^2}}\sin\theta$$

$$= \frac{-5r\cos^2\theta - 5r\sin^2\theta}{\sqrt{25 - 5x^2 - 5y^2}} = \frac{-5r}{\sqrt{25 - 5r^2}}$$

$$\frac{\partial w}{\partial \theta} = \frac{-5x}{\sqrt{25 - 5x^2 - 5y^2}}(-r\sin\theta) + \frac{-5y}{\sqrt{25 - 5x^2 - 5y^2}}(r\cos\theta)$$

$$= \frac{-5r^2\sin^2\theta\cos\theta - 5r^2\sin\theta\cos\theta}{\sqrt{25 - 5x^2 - 5y^2}} = 0$$

(b) $w = \sqrt{25 - 5r^2}$

$$\frac{\partial w}{\partial r} = \frac{-5r}{\sqrt{25 - 5r^2}};\ \frac{\partial w}{\partial \theta} = 0$$

22. $w = \dfrac{yz}{x}, x = \theta^2, y = r + \theta, z = r - \theta$

(a) $\dfrac{\partial w}{\partial r} = \dfrac{-yz}{x^2}(0) + \dfrac{z}{x}(1) + \dfrac{y}{x}(1) = \dfrac{z+y}{x} = \dfrac{2r}{\theta^2}$

$$\frac{\partial w}{\partial \theta} = \frac{-yz}{x^2}(2\theta) + \frac{z}{x}(1) + \frac{y}{x}(-1)$$

$$= \frac{-(r+\theta)(r-\theta)}{\theta^4}(2\theta) + \frac{(r-\theta)-(r+\theta)}{\theta^2}$$

$$= \frac{2(\theta^2 - r^2)}{\theta^3} - \frac{2}{\theta} = \frac{-2r^2}{\theta^3}$$

(b) $w = \dfrac{yz}{x} = \dfrac{(r+\theta)(r-\theta)}{\theta^2} = \dfrac{r^2}{\theta^2} - 1$

$$\frac{\partial w}{\partial r} = \frac{2r}{\theta^2}$$

$$\frac{\partial w}{\partial \theta} = \frac{-2r^2}{\theta^3}$$

24. $w = x\cos yz,\ x = s^2, y = t^2,\ z = s - 2t$

$$\frac{\partial w}{\partial s} = \cos(yz)(2s) - xz\sin(yz)(0) - xy\sin(yz)(1)$$

$$= \cos(st^2 - 2t^3)2s - s^2t^2\sin(st^2 - 2t^3)$$

$$\frac{\partial w}{\partial t} = \cos(yz)(0) - xz\sin(yz)(2t) - xy\sin(yz)(-2)$$

$$= -2s^2t(s-2t)\sin(st^2 - 2t^3) + 2s^2t^2\sin(st^2 - 2t^3)$$

$$= (6s^2t^2 - 2s^3t)\sin(st^2 - 2t^3)$$

26. $w = x^2 + y^2 + z^2, x = t\sin s, y = t\cos s, z = st^2$

$$\frac{\partial w}{\partial s} = 2x + \cos s + 2y(-t\sin s) + 2z(t^2)$$

$$= 2t^2\sin s\cos s - 2t^2\sin s\cos s + 2st^4 = 2st^4$$

$$\frac{\partial w}{\partial t} = 2x\sin s + 2y\cos s + 2z(2st)$$

$$= 2t\sin^2 s + 2t\cos^2 s + 4s^2t^3 = 2t + 4s^2t^3$$

28. $\cos x + \tan xy + 5 = 0$

$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)} = -\frac{-\sin x + y\sec^2 xy}{x\sec^2 xy}$$

30. $\dfrac{x}{x^2+y^2} - y^2 - 6 = 0$

$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)}$$

$$= -\frac{(y^2 - x^2)/(x^2+y^2)^2}{(-2xy)/(x^2+y^2)^2 - 2y}$$

$$= \frac{y^2 - x^2}{2xy + 2y(x^2+y^2)^2}$$

$$= \frac{y^2 - x^2}{2xy + 2yx^4 + 4x^2y^3 + 2y^5}$$

32. $F(x, y, z) = xz + yz + xy$

$$F_x = z + y$$

$$F_y = z + x$$

$$F_z = x + y$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{y+z}{x+y}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x+z}{x+y}$$

34. $F(x, y, z) = e^x\sin(y+z) - z$

$$F_x = e^x\sin(y+z)$$

$$F_y = e^x\cos(y+z)$$

$$F_z = e^x\cos(y+z) - 1$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{e^x\sin(y+z)}{1 - e^x\cos(y+x)}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{e^x\cos(y+z)}{1 - e^x\cos(y+z)}$$

36. $x + \sin(y+z) = 0$

(i) $1 + \dfrac{\partial z}{\partial x}\cos(y+z) = 0$ implies

$$\frac{\partial z}{\partial x} = -\frac{1}{\cos(y+z)} = -\sec(y+z).$$

(ii) $\left(1 + \dfrac{\partial z}{\partial y}\right)\cos(y+z) = 0$ implies $\dfrac{\partial z}{\partial y} = -1.$

38. $x \ln y + y^2 z + z^2 - 8 = 0$

(i) $$\frac{\partial z}{\partial x} = \frac{-F_x(x, y, z)}{F_z(x, y, z)} = \frac{-\ln y}{y^2 + 2z}$$

(ii) $$\frac{\partial z}{\partial y} = \frac{-F_y(x, y, z)}{F_z(x, y, z)} = -\frac{\frac{x}{y} + 2yz}{y^2 + 2z} = -\frac{x + 2y^2 z}{y^3 + 2yz}$$

40. $x^2 + y^2 - z^2 - 5yw + 10w^2 - 2 = F(x, y, z, w)$

$F_x = 2x, F_y = 2y - 5w, F_z = 2z, F_w = -5y + 20w$

$$\frac{\partial w}{\partial x} = -\frac{F_x}{F_w} = \frac{-2x}{-5y + 20w} = \frac{2x}{5y - 20w}$$

$$\frac{\partial w}{\partial y} = -\frac{F_y}{F_w} = \frac{5w - 2y}{20w - 5y}$$

$$\frac{\partial w}{\partial z} = -\frac{F_z}{F_w} = \frac{2z}{5y - 20w}$$

42. $F(x, y, z, w) = w - \sqrt{x - y} - \sqrt{y - z} = 0$

$$\frac{\partial w}{\partial x} = \frac{-F_x}{F_w} = \frac{1}{2}\frac{(x - y)^{-1/2}}{1} = \frac{1}{2\sqrt{x - y}}$$

$$\frac{\partial w}{\partial y} = \frac{-F_y}{F_w} = \frac{-1}{2}(x - y)^{-1/2} + \frac{1}{2}(y - z)^{-1/2}$$

$$= \frac{-1}{2\sqrt{x - y}} + \frac{1}{2\sqrt{y - z}}$$

$$\frac{\partial w}{\partial z} = \frac{-F_z}{F_w} = \frac{-1}{2\sqrt{y - z}}$$

44. $f(x, y) = x^3 - 3xy^2 + y^3$

$$f(tx, ty) = (tx)^3 - 3(tx)(ty)^2 + (ty)^3$$
$$= t^3(x^3 - 3xy^2 + y^3) = t^3 f(x, y)$$

Degree: 3

$$x f_x(x, y) + y f_y(x, y) = x(3x^2 - 3y^2) + y(-6xy + 3y^2)$$
$$= 3x^3 - 9xy^2 + 3y^3 = 3f(x, y)$$

46. $f(x, y) = \dfrac{x^2}{\sqrt{x^2 + y^2}}$

$$f(tx, ty) = \frac{(tx)^2}{\sqrt{(tx)^2 + (ty)^2}} = t\left(\frac{x^2}{\sqrt{x^2 + y^2}}\right) = t f(x, y)$$

Degree: 1

$$x f_x(x, y) + y f_y(x, y) = x\left[\frac{x^3 + 2xy^2}{(x^2 + y^2)^{3/2}}\right] + y\left[\frac{-x^2 y}{(x^2 + y^2)^{3/2}}\right]$$

$$= \frac{x^4 + x^2 y^2}{(x^2 + y^2)^{3/2}} = \frac{x^2(x^2 + y^2)}{(x^2 + y^2)^{3/2}}$$

$$= \frac{x^2}{\sqrt{x^2 + y^2}} = f(x, y)$$

48. $$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$$ (Page 878)

50. $$\frac{dy}{dx} = -\frac{f_x(x, y)}{f_y(x, y)}$$

$$\frac{\partial z}{\partial x} = -\frac{f_x(x, y, z)}{f_z(x, y, z)}$$

$$\frac{\partial z}{\partial y} = -\frac{f_y(x, y, z)}{f_z(x, y, z)}$$

52. (a) $V = \pi r^2 h$

$$\frac{dV}{dt} = \pi\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right) = \pi r\left(2h\frac{dr}{dt} + r\frac{dh}{dt}\right) = \pi(12)[2(36)(6) + 12(-4)] = 4608\pi \text{ in.}^3/\text{min}$$

(b) $S = 2\pi r(r + h)$

$$\frac{dS}{dt} = 2\pi\left[(2r + h)\frac{dr}{dt} + r\frac{dh}{dt}\right] = 2\pi[(24 + 36)(6) + 12(-4)] = 624\pi \text{ in.}^2/\text{min}$$

54. (a) $V = \frac{\pi}{3}(r^2 + rR + R^2)h$

$$\frac{dV}{dt} = \frac{\pi}{3}\left[(2r + R)h\frac{dr}{dt} + (r + 2R)h\frac{dR}{dt} + (r^2 + rR + R^2)\frac{dh}{dt}\right]$$

$$= \frac{\pi}{3}\left[[2(15) + 25](10)(4) + [15 + 2(25)](10)(4) + [(15)^2 + (15)(25) + (25)^2](12)\right]$$

$$= \frac{\pi}{3}(19{,}500) = 6{,}500\pi \text{ cm}^3/\text{min}$$

(b) $S = \pi(R + r)\sqrt{(R - r)^2 + h^2}$

$$\frac{dS}{dt} = \pi\left\{\left[\sqrt{(R - r)^2 + h^2} - (R + r)\frac{(R - r)}{\sqrt{(R - r)^2 + h^2}}\right]\frac{dr}{dt} + \left[\sqrt{(R - r)^2 + h^2} + (R + r)\frac{(R - r)}{\sqrt{(R - r)^2 + h^2}}\right]\frac{dR}{dt} + (R + r)\frac{h}{\sqrt{(R - r)^2 + h^2}}\frac{dh}{dt}\right\}$$

$$= \pi\left\{\left[\sqrt{(25 - 15)^2 + 10^2} - (25 + 15)\frac{25 - 15}{\sqrt{(25 - 15)^2 + 10^2}}\right](4) + \left[\sqrt{(25 - 15)^2 + 10^2} + (25 + 15)\frac{25 - 15}{\sqrt{(25 - 15)^2 + 10^2}}\right](4) + (25 + 15)\frac{10}{\sqrt{(25 - 15)^2 + 10^2}}(12)\right\}$$

$$= 320\sqrt{2}\pi \text{ cm}^2/\text{min}$$

56. $pV = mRT$

$$T = \frac{1}{mR}(pV)$$

$$\frac{dT}{dt} = \frac{1}{mR}\left[V\frac{dp}{dt} + p\frac{dV}{dt}\right]$$

58. $g(t) = f(xt, yt) = t^n f(x, y)$

Let $u = xt,\ v = yt,$ then

$$g'(t) = \frac{\partial f}{\partial u} \cdot \frac{du}{dt} + \frac{\partial f}{\partial v} \cdot \frac{dv}{dt} = \frac{\partial f}{\partial u}x + \frac{\partial f}{\partial v}y$$

and $g'(t) = nt^{n-1}f(x, y)$.

Now, let $t = 1$ and we have $u = x,\ v = y$. Thus,

$$\frac{\partial f}{\partial x}x + \frac{\partial f}{\partial y}y = nf(x, y).$$

60. $w = (x - y)\sin(y - x)$

$$\frac{\partial w}{\partial x} = -(x - y)\cos(y - x) + \sin(y - x)$$

$$\frac{\partial w}{\partial y} = (x - y)\cos(y - x) - \sin(y - x)$$

$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} = 0$$

62. $w = \arctan\frac{y}{x},\ x = r\cos\theta,\ y = r\sin\theta$

$$= \arctan\left(\frac{r\sin\theta}{r\cos\theta}\right) = \arctan(\tan\theta) = \theta \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\frac{\partial w}{\partial x} = \frac{-y}{x^2 + y^2},\ \frac{\partial w}{\partial y} = \frac{x}{x^2 + y^2},\ \frac{\partial w}{\partial r} = 0,\ \frac{\partial w}{\partial \theta} = 1$$

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \frac{y^2}{(x^2 + y^2)^2} + \frac{x^2}{(x^2 + y^2)^2} = \frac{1}{x^2 + y^2} = \frac{1}{r^2}$$

$$\left(\frac{\partial w}{\partial r}\right)^2 + \left(\frac{1}{r^2}\right)\left(\frac{\partial w}{\partial \theta}\right)^2 = 0 + \frac{1}{r^2}(1) = \frac{1}{r^2}$$

Therefore, $\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2$.

64. Note first that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{y}{x^2 + y^2}.$$

$$\frac{\partial u}{\partial r} = \frac{x}{x^2 + y^2}\cos\theta + \frac{y}{x^2 + y^2}\sin\theta = \frac{r\cos^2\theta + r\sin^2\theta}{r^2} = \frac{1}{r}$$

$$\frac{\partial v}{\partial \theta} = \frac{-y}{x^2 + y^2}(-r\sin\theta) + \frac{x}{x^2 + y^2}(r\cos\theta) = \frac{r^2\sin^2\theta + r^2\cos^2\theta}{r^2} = 1$$

Thus, $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$.

$$\frac{\partial v}{\partial r} = \frac{-y}{x^2 + y^2}\cos\theta + \frac{x}{x^2 + y^2}\sin\theta = \frac{-r\sin\theta\cos\theta + r\sin\theta\cos\theta}{r^2} = 0$$

$$\frac{\partial u}{\partial \theta} = \frac{x}{x^2 + y^2}(-r\sin\theta) + \frac{y}{x^2 + y^2}(r\cos\theta) = \frac{-r^2\sin\theta\cos\theta + r^2\sin\theta\cos\theta}{r^2} = 0$$

Thus, $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

Section 12.6 Directional Derivatives and Gradients

2. $f(x, y) = x^3 - y^3,\ \mathbf{v} = \frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j})$

$$\nabla f(x, y) = 3x^2\mathbf{i} - 3y^2\mathbf{j}$$

$$\nabla f(4, 3) = 48\mathbf{i} - 27\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

$$D_{\mathbf{u}} f(4, 3) = \nabla f(4, 3) \cdot \mathbf{u} = 24\sqrt{2} - \frac{27}{2}\sqrt{2} = \frac{21}{2}\sqrt{2}$$

4. $f(x, y) = \frac{x}{y}$

$$\mathbf{v} = -\mathbf{j}$$

$$\nabla f(x, y) = \frac{1}{y}\mathbf{i} - \frac{x}{y^2}\mathbf{j}$$

$$\nabla f(1, 1) = \mathbf{i} - \mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = -\mathbf{j}$$

$$D_{\mathbf{u}} f(1, 1) = \nabla f(1, 1) \cdot \mathbf{u} = 1$$

6. $g(x, y) = \arccos xy,\ \mathbf{v} = \mathbf{i} + 5\mathbf{j}$

$$\nabla g(x, y) = \frac{-y}{\sqrt{1 - (xy)^2}}\mathbf{i} + \frac{-x}{\sqrt{1 - (xy)^2}}\mathbf{j}$$

$$\nabla g(1, 0) = -\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{26}}\mathbf{i} + \frac{5}{\sqrt{26}}\mathbf{j}$$

$$D_{\mathbf{u}} g(1, 0) = \nabla g(1, 0) \cdot \mathbf{u} = \frac{-5}{\sqrt{26}} = \frac{-5\sqrt{26}}{26}$$

8. $h(x, y) = e^{-(x^2 + y^2)}$

$$\mathbf{v} = \mathbf{i} + \mathbf{j}$$

$$\nabla h = -2xe^{-(x^2+y^2)}\mathbf{i} - 2ye^{-(x^2+y^2)}\mathbf{j}$$

$$\nabla h(0, 0) = \mathbf{0}$$

$$D_{\mathbf{u}} h(0, 0) = \nabla h(0, 0) \cdot \mathbf{u} = 0$$

10. $f(x, y, z) = x^2 + y^2 + z^2$

$$\mathbf{v} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$$

$$\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$$

$$\nabla f(1, 2, -1) = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{14}}\mathbf{i} - \frac{2}{\sqrt{14}}\mathbf{j} + \frac{3}{\sqrt{14}}\mathbf{k}$$

$$D_{\mathbf{u}} f(1, 2, -1) = \nabla f(1, 2, -1) \cdot \mathbf{u} = -\frac{6}{7}\sqrt{14}$$

12. $h(x, y, z) = xyz$

$$\mathbf{v} = \langle 2, 1, 2 \rangle$$

$$\nabla h = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$$

$$\nabla h(2, 1, 1) = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$$

$$D_{\mathbf{u}} h(2, 1, 1) = \nabla h(2, 1, 1) \cdot \mathbf{u} = \frac{8}{3}$$

14. $f(x, y) = \dfrac{y}{x + y}$

$$\mathbf{u} = \frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$$

$$\nabla f = -\frac{y}{(x + y)^2}\mathbf{i} + \frac{x}{(x + y)^2}\mathbf{j}$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = -\frac{\sqrt{3}y}{2(x + y)^2} - \frac{x}{2(x + y)^2}$$

$$= -\frac{1}{2(x + y)^2}\left(\sqrt{3}y + x\right)$$

16. $g(x, y) = xe^y$

$$\mathbf{u} = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$$

$$\nabla g = e^y\,\mathbf{i} + xe^y\,\mathbf{j}$$

$$D_{\mathbf{u}} g = -\frac{1}{2}e^y + \frac{\sqrt{3}}{2}xe^y = \frac{e^y}{2}\left(\sqrt{3}x - 1\right)$$

18. $f(x, y) = \cos(x + y)$

$$\mathbf{v} = \frac{\pi}{2}\mathbf{i} - \pi\mathbf{j}$$

$$\nabla f = -\sin(x + y)\mathbf{i} - \sin(x + y)\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j}$$

$$D_{\mathbf{u}} f = -\frac{1}{\sqrt{5}}\sin(x + y) + \frac{2}{\sqrt{5}}\sin(x + y)$$

$$= \frac{1}{\sqrt{5}}\sin(x + y) = \frac{\sqrt{5}}{5}\sin(x + y)$$

At $(0, \pi)$, $D_{\mathbf{u}} f = 0$.

20. $g(x, y, z) = xye^z$

$$\mathbf{v} = -2\mathbf{i} - 4\mathbf{j}$$

$$\nabla g = ye^z\,\mathbf{i} + xe^z\,\mathbf{j} + xye^z\,\mathbf{k}$$

At $(2, 4, 0)$, $\nabla g = 4\mathbf{i} + 2\mathbf{j} + 8\mathbf{k}$.

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = -\frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j}$$

$$D_{\mathbf{u}} g = \nabla g \cdot \mathbf{u} = -\frac{4}{\sqrt{5}} - \frac{4}{\sqrt{5}} = -\frac{8}{\sqrt{5}}$$

22. $g(x, y) = 2xe^{y/x}$

$$\nabla g(x, y) = \left(-\frac{2y}{x}e^{y/x} + 2e^{y/x}\right)\mathbf{i} + 2e^{y/x}\mathbf{j}$$

$$\nabla g(2, 0) = 2\mathbf{i} + 2\mathbf{j}$$

24. $z = \ln(x^2 - y)$

$$\nabla z(x, y) = \frac{2x}{x^2 - y}\mathbf{i} - \frac{1}{x^2 - y}\mathbf{j}$$

$$\nabla z(2, 3) = 4\mathbf{i} - \mathbf{j}$$

26. $w = x\tan(y + z)$

$$\nabla w(x, y, z) = \tan(y + z)\mathbf{i} + x\sec^2(y + z)\mathbf{j} + x\sec^2(y + z)\mathbf{k}$$

$$\nabla w(4, 3, -1) = \tan 2\mathbf{i} + 4\sec^2 2\mathbf{j} + 4\sec^2 2\mathbf{k}$$

28. $\overrightarrow{PQ} = -2\mathbf{i} + 7\mathbf{j}$, $\mathbf{u} = -\dfrac{2}{\sqrt{53}}\mathbf{i} + \dfrac{7}{\sqrt{53}}\mathbf{j}$

$$\nabla f(x, y) = 6x\mathbf{i} - 2y\mathbf{j},\ \nabla f(3, 1) = 18\mathbf{i} - 2\mathbf{j}$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = -\frac{36}{\sqrt{53}} - \frac{14}{\sqrt{53}} = -\frac{50}{\sqrt{53}} = -\frac{50\sqrt{53}}{53}$$

30. $\overrightarrow{PQ} = \dfrac{\pi}{2}\mathbf{i} + \pi\mathbf{j}$, $\mathbf{u} = \dfrac{1}{\sqrt{5}}\mathbf{i} + \dfrac{2}{\sqrt{5}}\mathbf{j}$

$$\nabla f(x, y) = 2\cos 2x\cos y\mathbf{i} - \sin 2x\sin y\mathbf{j}$$

$$\nabla f(0, 0) = 2\mathbf{i}$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

32. $h(x, y) = y\cos(x - y)$

$$\nabla h(x, y) = -y\sin(x - y)\mathbf{i} + [\cos(x - y) + y\sin(x - y)]\mathbf{j}$$

$$\nabla h\left(0, \frac{\pi}{3}\right) = \frac{\sqrt{3}\pi}{6}\mathbf{i} + \left(\frac{3 - \sqrt{3}\pi}{6}\right)\mathbf{j}$$

$$\left\|\nabla h\left(0, \frac{\pi}{3}\right)\right\| = \sqrt{\frac{3\pi^2}{36} + \frac{9 - 6\sqrt{3}\pi + 3\pi^2}{36}} = \frac{\sqrt{3\left(2\pi^2 - 2\sqrt{3}\pi + 3\right)}}{6}$$

34. $g(x, y) = ye^{-x^2}$

$$\nabla g(x, y) = -2xye^{-x^2}\mathbf{i} + e^{-x^2}\mathbf{j}$$

$$\nabla g(0, 5) = \mathbf{j}$$

$$\|\nabla g(0, 5)\| = 1$$

36. $w = \dfrac{1}{\sqrt{1 - x^2 - y^2 - z^2}}$

$$\nabla w = \frac{1}{\left(\sqrt{1 - x^2 - y^2 - z^2}\right)^3}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$\nabla w(0, 0, 0) = \mathbf{0}$$

$$\|\nabla w(0, 0, 0)\| = 0$$

38. $w = xy^2z^2$

$$\nabla w = y^2z^2\mathbf{i} + 2xyz^2\mathbf{j} + 2xy^2z\mathbf{k}$$

$$\nabla w(2, 1, 1) = \mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$$

$$\|\nabla w(2, 1, 1)\| = \sqrt{33}$$

For Exercises 40–46, $f(x, y) = 3 - \frac{x}{3} - \frac{y}{2}$ and $D_\theta f(x, y) = -\left(\frac{1}{3}\right)\cos\theta - \left(\frac{1}{2}\right)\sin\theta$.

40. (a) $D_{\pi/4} f(3, 2) = -\left(\frac{1}{3}\right)\frac{\sqrt{2}}{2} - \left(\frac{1}{2}\right)\frac{\sqrt{2}}{2} = -\frac{5\sqrt{2}}{12}$

(b) $D_{2\pi/3} f(3, 2) = -\left(\frac{1}{3}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\frac{\sqrt{3}}{2} = \frac{2 - 3\sqrt{3}}{12}$

42. (a) $\mathbf{u} = \left(\frac{1}{\sqrt{2}}\right)(\mathbf{i} + \mathbf{j})$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$$

$$= -\left(\frac{1}{3}\right)\frac{1}{\sqrt{2}} - \left(\frac{1}{2}\right)\frac{1}{\sqrt{2}} = -\frac{5\sqrt{2}}{12}$$

(b) $\mathbf{v} = -3\mathbf{i} - 4\mathbf{j}$

$$\|\mathbf{v}\| = \sqrt{9 + 16} = 5$$

$$\mathbf{u} = -\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \frac{1}{5} + \frac{2}{5} = \frac{3}{5}$$

44. $\nabla f = -\left(\frac{1}{3}\right)\mathbf{i} - \left(\frac{1}{2}\right)\mathbf{j}$

46. $\nabla f = -\frac{1}{3}\mathbf{i} - \frac{1}{2}\mathbf{j}$

$$\frac{\nabla f}{\|\nabla f\|} = \frac{1}{\sqrt{13}}(-2\mathbf{i} - 3\mathbf{j})$$

Therefore, $\mathbf{u} = \left(1/\sqrt{13}\right)(3\mathbf{i} - 2\mathbf{j})$ and $D_{\mathbf{u}} f(3, 2) = \nabla f \cdot \mathbf{u} = 0$. ∇f is the direction of greatest rate of change of f. Hence, in a direction orthogonal to ∇f, the rate of change of f is 0.

For Exercises 48 and 50, $f(x, y) = 9 - x^2 - y^2$ and $D_\theta f(x, y) = -2x \cos \theta - 2y \sin \theta = -2(x \cos \theta + y \sin \theta)$.

48. (a) $D_{-\pi/4} f(1, 2) = -2\left(\frac{\sqrt{2}}{2} - \sqrt{2}\right) = \sqrt{2}$

(b) $D_{\pi/3} f(1, 2) = -2\left(\frac{1}{2} + \sqrt{3}\right) = -\left(1 + 2\sqrt{3}\right)$

50. $\nabla f(1, 2) = -2\mathbf{i} - 4\mathbf{j}$

$$\frac{\nabla f(1, 2)}{\|\nabla f(1, 2)\|} = \frac{1}{\sqrt{5}}(-\mathbf{i} - 2\mathbf{j})$$

Therefore,

$\mathbf{u} = \left(1/\sqrt{5}\right)(-2\mathbf{i} + \mathbf{j})$ and

$D_{\mathbf{u}} f(1, 2) = \nabla f(1, 2) \cdot \mathbf{u} = 0.$

52. (a) In the direction of the vector $\mathbf{i} + \mathbf{j}$.

(b) $\nabla f = \frac{1}{2}y\frac{1}{2\sqrt{x}}\mathbf{i} + \frac{1}{2}\sqrt{x}\,\mathbf{j} = \frac{y}{4\sqrt{x}}\mathbf{i} + \frac{1}{2}\sqrt{x}\,\mathbf{j}$

$\nabla f(1, 2) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$

(Same direction as in part (a).)

(c) $-\nabla f = -\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$, the direction opposite that of the gradient.

54. (a) $f(x, y) = \frac{8y}{1 + x^2 + y^2} = 2$

$\implies 4y = 1 + x^2 + y^2$

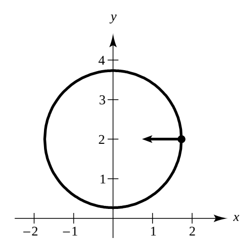
$4 = y^2 - 4y + 4 + x^2 + 1$

$(y - 2)^2 + x^2 = 3$

Circle: center: $(0, 2)$, radius: $\sqrt{3}$

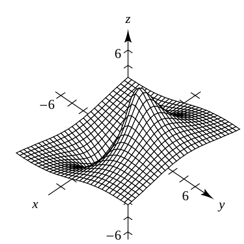
(b) $\nabla f = \frac{-16xy}{(1 + x^2 + y^2)^2}\mathbf{i} + \frac{8 + 8x^2 - 8y^2}{(1 + x^2 + y^2)^2}\mathbf{j}$

$\nabla f\left(\sqrt{3}, 2\right) = \frac{-\sqrt{3}}{2}\mathbf{i}$



(c) The directional derivative of f is 0 in the directions $\pm\mathbf{j}$.

(d)



56. $f(x, y) = 6 - 2x - 3y$

$c = 6, \; P = (0, 0)$

$\nabla f(x, y) = -2\mathbf{i} - 3\mathbf{j}$

$6 - 2x - 3y = 6$

$0 = 2x + 3y$

$\nabla f(0, 0) = -2\mathbf{i} - 3\mathbf{j}$

58. $f(x, y) = xy$

$c = -3, \; P = (-1, 3)$

$\nabla f(x, y) = y\mathbf{i} + x\mathbf{j}$

$xy = -3$

$\nabla f(-1, 3) = 3\mathbf{i} - \mathbf{j}$

60. $3x^2 - 2y^2 = 1$

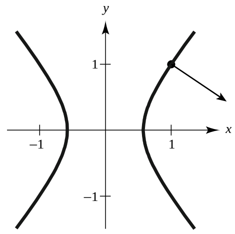
$f(x, y) = 3x^2 - 2y^2$

$\nabla f(x, y) = 6x\mathbf{i} - 4y\mathbf{j}$

$\nabla f(1, 1) = 6\mathbf{i} - 4\mathbf{j}$

$$\frac{\nabla f(1, 1)}{\|\nabla f(1, 1)\|} = \frac{1}{\sqrt{13}}(3\mathbf{i} - 2\mathbf{j})$$

$$= \frac{\sqrt{13}}{13}(3\mathbf{i} - 2\mathbf{j})$$



62. $xe^y - y = 5$

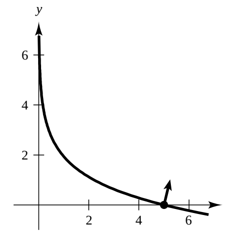
$f(x, y) = xe^y - y$

$\nabla f(x, y) = e^y\mathbf{i} + (xe^y - 1)\mathbf{j}$

$\nabla f(5, 0) = \mathbf{i} + 4\mathbf{j}$

$$\frac{\nabla f(5, 0)}{\|\nabla f(5, 0)\|} = \frac{1}{\sqrt{17}}(\mathbf{i} + 4\mathbf{j})$$

$$= \frac{\sqrt{17}}{17}(\mathbf{i} + 4\mathbf{j})$$



64. $h(x, y) = 5000 - 0.001x^2 - 0.004y^2$

$\nabla h = -0.002x\mathbf{i} - 0.008y\mathbf{j}$

$\nabla h(500, 300) = -\mathbf{i} - 2.4\mathbf{j}$ or

$5\nabla h = -(5\mathbf{i} + 12\mathbf{j})$

66. The directional derivative gives the slope of a surface at a point in an arbitrary direction $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$.

68. See the definition, page 887.

70. The gradient vector is normal to the level curves.

See Theorem 12.12.

72. The wind speed is greatest at A.

74. $T(x, y) = 100 - x^2 - 2y^2$, $P = (4, 3)$

$\frac{dx}{dt} = -2x$ $\qquad$ $\frac{dy}{dt} = -4y$

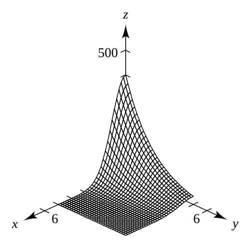
$x(t) = C_1e^{-2t}$ $\qquad$ $y(t) = C_2e^{-4t}$

$4 = x(0) = C_1$ $\qquad$ $3 = y(0) = C_2$

$x(t) = 4e^{-2t}$ $\qquad$ $y(t) = 3e^{-4t}$

$$\frac{3x^2}{16} = e^{-4t} = y \implies u = \frac{3}{16}x^2$$

76. (a)



(b) $\nabla T(x, y) = 400e^{-(x^2+y)/2}\left[(-x)\mathbf{i} - \frac{1}{2}\mathbf{j}\right]$

$\nabla T(3, 5) = 400e^{-7}\left[-3\mathbf{i} - \frac{1}{2}\mathbf{j}\right]$

There will be no change in directions perpendicular to the gradient: $\pm(\mathbf{i} - 6\mathbf{j})$

(c) The greatest increase is in the direction of the gradient: $-3\mathbf{i} - \frac{1}{2}\mathbf{j}$

78. False

$D_{\mathbf{u}} f(x, y) = \sqrt{2} > 1$ when

$$\mathbf{u} = \left(\cos\frac{\pi}{4}\right)\mathbf{i} + \left(\sin\frac{\pi}{4}\right)\mathbf{j}.$$

80. True

Section 12.7 Tangent Planes and Normal Lines

2. $F(x, y, z) = x^2 + y^2 + z^2 - 25 = 0$

$$x^2 + y^2 + z^2 = 25$$

Sphere, radius 5, centered at origin.

4. $F(x, y, z) = 16x^2 - 9y^2 + 144z = 0$

$$16x^2 - 9y^2 + 144z = 0 \text{ Hyperbolic paraboloid}$$

6. $F(x, y, z) = x^2 + y^2 + z^2 - 11$

$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$

$\nabla F(3, 1, 1) = 6\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{\sqrt{44}}(6\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = \frac{1}{\sqrt{11}}(3\mathbf{i} + \mathbf{j} + \mathbf{k}) = \frac{\sqrt{11}}{11}(3\mathbf{i} + \mathbf{j} + \mathbf{k})$$

8. $F(x, y, z) = x^3 - z$

$\nabla F(x, y, z) = 3x^2\mathbf{i} - \mathbf{k}$

$\nabla F(2, 1, 8) = 12\mathbf{i} - \mathbf{k}$

$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{\sqrt{145}}(12\mathbf{i} - \mathbf{k}) = \frac{\sqrt{145}}{145}(12\mathbf{i} - \mathbf{k})$$

10. $F(x, y, z) = x^2 + 3y + z^3 - 9$

$\nabla F(x, y, z) = 2x\mathbf{i} + 3\mathbf{j} + 3z^2\mathbf{k}$

$\nabla F(2, -1, 2) = 4\mathbf{i} + 3\mathbf{j} + 12\mathbf{k}$

$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{13}(4\mathbf{i} + 3\mathbf{j} + 12\mathbf{k})$$

12. $F(x, y, z) = ze^{x^2 - y^2} - 3$

$\nabla F(x, y, z) = 2xze^{x^2 - y^2}\mathbf{i} - 2yze^{x^2 - y^2}\mathbf{j} + e^{x^2 - y^2}\mathbf{k}$

$\nabla F(2, 2, 3) = 12\mathbf{i} - 12\mathbf{j} + \mathbf{k}$

$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{17}(12\mathbf{i} - 12\mathbf{j} + \mathbf{k})$$

14. $F(x, y, z) = \sin(x - y) - z - 2$

$\nabla F(x, y, z) = \cos(x - y)\mathbf{i} - \cos(x - y)\mathbf{j} - \mathbf{k}$

$$\nabla F\left(\frac{\pi}{3}, \frac{\pi}{6}, -\frac{3}{2}\right) = \frac{\sqrt{3}}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j} - \mathbf{k}$$

$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{2}{\sqrt{10}}\left(\frac{\sqrt{3}}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j} - \mathbf{k}\right) = \frac{1}{\sqrt{10}}\left(\sqrt{3}\,\mathbf{i} - \sqrt{3}\,\mathbf{j} - 2\mathbf{k}\right) = \frac{\sqrt{10}}{10}\left(\sqrt{3}\,\mathbf{i} - \sqrt{3}\,\mathbf{j} - 2\mathbf{k}\right)$$

16. $f(x, y) = \dfrac{y}{x}$, $(1, 2, 2)$

$F(x, y, z) = \dfrac{y}{x} - z$

$F_x(x, y, z) = -\dfrac{y}{x^2}$ $\quad F_y(x, y, z) = \dfrac{1}{x}$ $\quad F_z(x, y, z) = -1$

$F_x(1, 2, 2) = -2$ $\quad F_y(1, 2, 2) = 1$ $\quad F_z(1, 2, 2) = -1$

$$-2(x - 1) + (y - 2) - (z - 2) = 0$$
$$-2x + y - z + 2 = 0$$
$$2x - y + z = 2$$

18. $g(x, y) = \arctan\dfrac{y}{x}$, $(1, 0, 0)$

$G(x, y, z) = \arctan\dfrac{y}{x} - z$

$G_x(x, y, z) = \dfrac{-(y/x^2)}{1 + (y^2/x^2)} = \dfrac{-y}{x^2 + y^2}$ $\quad G_y(x, y, z) = \dfrac{1/x}{1 + (y^2/x^2)} = \dfrac{x}{x^2 + y^2}$ $\quad G_z(x, y, z) = -1$

$G_x(1, 0, 0) = 0$ $\quad G_y(1, 0, 0) = 1$ $\quad G_z(1, 0, 0) = -1$

$y - z = 0$

20. $f(x, y) = 2 - \frac{2}{3}x - y, \ (3, -1, 1)$

$F(x, y, z) = 2 - \frac{2}{3}x - y - z$

$F_x(x, y, z) = -\frac{2}{3}, \qquad F_y(x, y, z) = -1, \qquad F_z(x, y, z) = -1$

$$-\frac{2}{3}(x - 3) - (y + 1) - (z - 1) = 0$$
$$-\frac{2}{3}x - y - z + 2 = 0$$
$$2x + 3y + 3z = 6$$

22. $z = x^2 - 2xy + y^2, (1, 2, 1)$

$F(x, y, z) = x^2 - 2xy + y^2 - z$

$F_x(x, y, z) = 2x - 2y \qquad F_y(x, y, z) = -2x + 2y \qquad F_z(x, y, z) = -1$

$F_x(1, 2, 1) = -2 \qquad F_y(1, 2, 1) = 2 \qquad F_z(1, 2, 1) = -1$

$$-2(x - 1) + 2(y - 2) - (z - 1) = 0$$
$$-2x + 2y - z - 1 = 0$$
$$2x - 2y + z = -1$$

24. $h(x, y) = \cos y, \ \left(5, \frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$

$H(x, y, z) = \cos y - z$

$H_x(x, y, z) = 0 \qquad H_y(x, y, z) = -\sin y \qquad H_z(x, y, z) = -1$

$H_x\left(5, \frac{\pi}{4}, \frac{\sqrt{2}}{2}\right) = 0 \qquad H_y\left(5, \frac{\pi}{4}, \frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2} \qquad H_z\left(5, \frac{\pi}{4}, \frac{\sqrt{2}}{2}\right) = -1$

$$-\frac{\sqrt{2}}{2}\left(y - \frac{\pi}{4}\right) - \left(z - \frac{\sqrt{2}}{2}\right) = 0$$
$$-\frac{\sqrt{2}}{2}y - z + \frac{\sqrt{2}\pi}{8} + \frac{\sqrt{2}}{2} = 0$$
$$4\sqrt{2}y + 8z = \sqrt{2}(\pi + 4)$$

26. $x^2 + 2z^2 = y^2, \ (1, 3, -2)$

$F(x, y, z) = x^2 - y^2 + 2z^2$

$F_x(x, y, z) = 2x \qquad F_y(x, y, z) = -2y \qquad F_z(x, y, z) = 4z$

$F_x(1, 3, -2) = 2 \qquad F_y(1, 3, -2) = -6 \qquad F_z(1, 3, -2) = -8$

$$2(x - 1) - 6(y - 3) - 8(z + 2) = 0$$
$$(x - 1) - 3(y - 3) - 4(z + 2) = 0$$
$$x - 3y - 4z = 0$$

28. $x = y(2z - 3), (4, 4, 2)$

$F(x, y, z) = x - 2yz + 3y$

$F_x(x, y, z) = 1 \qquad F_y(x, y, z) = -2z + 3 \qquad F_z(x, y, z) = -2y$

$F_x(4, 4, 2) = 1 \qquad F_y(4, 4, 2) = -1 \qquad F_z(4, 4, 2) = -8$

$$(x - 4) - 1(y - 4) - 8(z - 2) = 0$$
$$x - y - 8z = -16$$
$$-x + y + 8z = 16$$

30. $x^2 + y^2 + z^2 = 9,\ (1, 2, 2)$

$F(x, y, z) = x^2 + y^2 + z^2 - 9$

$F_x(x, y, z) = 2x \qquad F_y(x, y, z) = 2y \qquad F_z(x, y, z) = 2z$

$F_x(1, 2, 2) = 2 \qquad F_y(1, 2, 2) = 4 \qquad F_z(1, 2, 2) = 4$

Direction numbers: 1, 2, 2

Plane: $(x - 1) + 2(y - 2) + 2(z - 2) = 0,\ x + 2y + 2z = 9$

Line: $\dfrac{x-1}{1} = \dfrac{y-2}{2} = \dfrac{z-2}{2}$

32. $x^2 - y^2 + z^2 = 0, (5, 13, -12)$

$F(x, y, z) = x^2 - y^2 + z^2$

$F_x(x, y, z) = 2x \qquad F_y(x, y, z) = -2y \qquad F_z(x, y, z) = 2z$

$F_x(5, 13, -12) = 10 \qquad F_y(5, 13, -12) = -26 \qquad F_z(x, y, z) = -24$

Direction numbers: $5, -13, -12$

Plane

$$5(x - 5) - 13(y - 13) - 12(z + 12) = 0$$
$$5x - 13y - 12z = 0$$

Line: $\dfrac{x-5}{5} = \dfrac{y-13}{-13} = \dfrac{z+12}{-12}$

34. $xyz = 10,\ (1, 2, 5)$

$F(x, y, z) = xyz - 10$

$F_x(x, y, z) = yz \qquad F_y(x, y, z) = xz \qquad F_z(x, y, z) = xy$

$F_x(1, 2, 5) = 10 \qquad F_y(1, 2, 5) = 5 \qquad F_z(1, 2, 5) = 2$

Direction numbers: 10, 5, 2

Plane: $10(x - 1) + 5(y - 2) + 2(z - 5) = 0,\ 10x + 5y + 2z = 30$

Line: $\dfrac{x-1}{10} = \dfrac{y-2}{5} = \dfrac{z-5}{2}$

36. See the definition on page 897.

38. For a sphere, the common object is the center of the sphere. For a right circular cylinder, the common object is the axis of the cylinder.

40. $F(x, y, z) = x^2 + y^2 - z \qquad G(x, y, z) = 4 - y - z$

$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \qquad \nabla G(x, y, z) = -\mathbf{j} - \mathbf{k}$

$\nabla F(2, -1, 5) = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k} \qquad \nabla G(2, -1, 5) = -\mathbf{j} - \mathbf{k}$

(a) $\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -2 & -1 \\ 0 & -1 & -1 \end{vmatrix} = \mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$

Direction numbers: $1, 4, -4,\ \dfrac{x-2}{1} = \dfrac{y+1}{4} = \dfrac{z-5}{-4}$

(b) $\cos\theta = \dfrac{|\nabla F \cdot \nabla G|}{\|\nabla F\|\,\|\nabla G\|} = \dfrac{3}{\sqrt{21}\sqrt{2}} = \dfrac{3}{\sqrt{42}} = \dfrac{\sqrt{42}}{14}$; not orthogonal

42. $F(x, y, z) = \sqrt{x^2 + y^2} - z$ $\qquad G(x, y, z) = 5x - 2y + 3z = 22$

$\nabla F(x, y, z) = \dfrac{x}{\sqrt{x^2+y^2}}\mathbf{i} + \dfrac{y}{\sqrt{x^2+y^2}}\mathbf{j} - \mathbf{k}$ $\qquad \nabla G(x, y, z) = 5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

$\nabla F(3, 4, 5) = \dfrac{3}{5}\mathbf{i} + \dfrac{4}{5}\mathbf{j} - \mathbf{k}$ $\qquad \nabla G(3, 4, 5) = 5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

$$\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3/5 & 4/5 & -1 \\ 5 & -2 & 3 \end{vmatrix} = \frac{2}{5}\mathbf{i} - \frac{34}{5}\mathbf{j} - \frac{26}{5}\mathbf{k}$$

Direction numbers: $1, -17, -13$

$$\frac{x-3}{1} = \frac{y-4}{-17} = \frac{z-5}{-13} \quad \text{Tangent line}$$

$$\cos\theta = \frac{|\nabla F \cdot \nabla G|}{\|\nabla F\|\,\|\nabla G\|} = \frac{-(8/5)}{\sqrt{2}\sqrt{38}} = \frac{-8}{5\sqrt{76}} \quad \text{Not orthogonal}$$

44. $F(x, y, z) = x^2 + y^2 - z$ $\qquad G(x, y, z) = x + y + 6z - 33$

$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}$ $\qquad \nabla G(x, y, z) = \mathbf{i} + \mathbf{j} + 6\mathbf{k}$

$\nabla F(1, 2, 5) = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ $\qquad \nabla G(1, 2, 5) = \mathbf{i} + \mathbf{j} + 6\mathbf{k}$

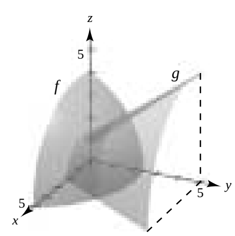
(a) $\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & -1 \\ 1 & 1 & 6 \end{vmatrix} = 25\mathbf{i} - 13\mathbf{j} - 2\mathbf{k}$

Direction numbers: $25, -13, -2$, $\dfrac{x-1}{25} = \dfrac{y-2}{-13} = \dfrac{z-5}{-2}$

(b) $\cos\theta = \dfrac{|\nabla F \cdot \nabla G|}{\|\nabla F\|\,\|\nabla G\|} = 0$; orthogonal

46. (a) $f(x, y) = \sqrt{16 - x^2 - y^2 + 2x - 4y}$

$g(x, y) = \dfrac{\sqrt{2}}{2}\sqrt{1 - 3x^2 + y^2 + 6x + 4y}$



(b)
$$f(x, y) = g(x, y)$$
$$16 - x^2 - y^2 + 2x - 4y = \frac{1}{2}(1 - 3x^2 + y^2 + 6x + 4y)$$
$$32 - 2x^2 - 2y^2 + 4x - 8y = 1 - 3x^2 + y^2 + 6x + 4y$$
$$x^2 - 2x + 31 = 3y^2 + 12y$$
$$(x^2 - 2x + 1) + 42 = 3(y^2 + 4y + 4)$$
$$(x-1)^2 + 42 = 3(y+2)^2$$

To find points of intersection, let $x = 1$. Then

$$3(y+2)^2 = 42$$
$$(y+2)^2 = 14$$
$$y = -2 \pm \sqrt{14}$$

$\nabla f\left(1, -2 + \sqrt{14}\right) = -\sqrt{2}\mathbf{j}$, $\nabla g\left(1, -2 + \sqrt{14}\right) = \left(1/\sqrt{2}\right)\mathbf{j}$. The normals to f and g at this point are $-\sqrt{2}\mathbf{j} - \mathbf{k}$ and $\left(1/\sqrt{2}\right)\mathbf{j} - \mathbf{k}$, which are orthogonal.

Similarly, $\nabla f\left(1, -2 - \sqrt{14}\right) = \sqrt{2}\mathbf{j}$ and $\nabla g\left(1, -2 - \sqrt{14}\right) = \left(-1/\sqrt{2}\right)\mathbf{j}$ and the normals are $\sqrt{2}\mathbf{j} - \mathbf{k}$ and $\left(-1/\sqrt{2}\right)\mathbf{j} - \mathbf{k}$, which are also orthogonal.

(c) No, showing that the surfaces are orthogonal at 2 points does not imply that they are orthogonal at every point of intersection.

48. $F(x, y, z) = 2xy - z^3, (2, 2, 2)$

$\nabla F = 2y\mathbf{i} + 2x\mathbf{j} - 3z^2\mathbf{k}$

$\nabla F(2, 2, 2) = 4\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}$

$$\cos\theta = \frac{|\nabla F(2, 2, 2) \cdot \mathbf{k}|}{\|\nabla F(2, 2, 2)\|} = \frac{|-12|}{\sqrt{176}} = \frac{3\sqrt{11}}{11}$$

$$\theta = \arccos\left(\frac{3\sqrt{11}}{11}\right) \approx 25.24°$$

50. $F(x, y, z) = x^2 + y^2 - 5,\ (2, 1, 3)$

$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j}$

$\nabla F(2, 1, 3) = 4\mathbf{i} + 2\mathbf{j}$

$$\cos\theta = \frac{|\nabla F(2, 1, 3) \cdot \mathbf{k}|}{\|\nabla F(2, 1, 3)\|} = 0$$

$\theta = \arccos 0 = 90°$

52. $F(x, y, z) = 3x^2 + 2y^2 - 3x + 4y - z - 5$

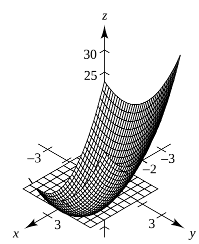
$\nabla F(x, y, z) = (6x - 3)\mathbf{i} + (4y + 4)\mathbf{j} - \mathbf{k}$

$6x - 3 = 0,\ x = \frac{1}{2}$

$4y + 4 = 0,\ y = -1$

$z = 3\left(\frac{1}{2}\right)^2 + 2(-1)^2 - 3\left(\frac{1}{2}\right) + 4(-1) - 5 = -\frac{31}{4}$

$\left(\frac{1}{2}, -1, -\frac{31}{4}\right)$



54. $T(x, y, z) = 100 - 3x - y - z^2,\ (2, 2, 5)$

$\frac{dx}{dt} = -3$	$\frac{dy}{dt} = -1$	$\frac{dz}{dt} = -2z$
$x(t) = -3t + C_1$	$y(t) = -t + C_2$	$z(t) = C_3e^{-2t}$
$x(0) = C_1 = 2$	$y(0) = C_2 = 2$	$z(0) = C_3 = 5$
$x = -3t + 2$	$y = -t + 2$	$z = 5e^{-2t}$

56. $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1$

$F_x(x, y, z) = \frac{2x}{a^2}$

$F_y(x, y, z) = \frac{2y}{b^2}$

$F_z(x, y, z) = \frac{-2z}{c^2}$

Plane: $\frac{2x_0}{a^2}(x - x_0) + \frac{2y_0}{b^2}(y - y_0) - \frac{2z_0}{c^2}(z - z_0) = 0$

$$\frac{x_0x}{a^2} + \frac{y_0y}{b^2} - \frac{z_0z}{c^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} - \frac{z_0^2}{c^2} = 1$$

58. $z = xf\left(\frac{y}{x}\right)$

$F(x, y, z) = xf\left(\frac{y}{x}\right) - z$

$F_x(x, y, z) = f\left(\frac{y}{x}\right) + xf'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) = f\left(\frac{y}{x}\right) - \frac{y}{x}f'\left(\frac{y}{x}\right)$

$F_y(x, y, z) = xf'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) = f'\left(\frac{y}{x}\right)$

$F_x(x, y, z) = -1$

Tangent plane at (x_0, y_0, z_0):

$$\left[f\left(\frac{y_0}{x_0}\right) - \frac{y_0}{x_0}f'\left(\frac{y_0}{x_0}\right)\right](x - x_0) + f'\left(\frac{y_0}{x_0}\right)(y - y_0) - (z - z_0) = 0$$

$$\left[f\left(\frac{y_0}{x_0}\right) - \frac{y_0}{x_0}f'\left(\frac{y_0}{x_0}\right)\right]x - x_0f\left(\frac{y_0}{x_0}\right) + y_0f'\left(\frac{y_0}{x_0}\right) + yf'\left(\frac{y_0}{x_0}\right) - y_0f'\left(\frac{y_0}{x_0}\right) - z + x_0f\left(\frac{y_0}{x_0}\right) = 0$$

$$\left[f\left(\frac{y_0}{x_0}\right) - \frac{y_0}{x_0}f'\left(\frac{y_0}{x_0}\right)\right]x + f'\left(\frac{y_0}{x_0}\right)y - z = 0$$

Therefore, the plane passes through the origin $(x, y, z) = (0, 0, 0)$.

60. $f(x, y) = \cos(x + y)$

$f_x(x, y) = -\sin(x + y)$ $\quad$ $f_y(x, y) = -\sin(x + y)$

$f_{xx}(x, y) = -\cos(x + y),$ $\quad$ $f_{yy}(x, y) = -\cos(x + y),$ $\quad$ $f_{xy}(x, y) = -\cos(x + y)$

(a) $P_1(x, y) \approx f(0, 0) + f_x(0, 0)x + f_y(0, 0)y = 1$

(b) $P_2(x, y) \approx f(0, 0) + f_x(0, 0)x + f_y(0,0)y + \frac{1}{2} f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \frac{1}{2} f_{yy}(0, 0)y^2$

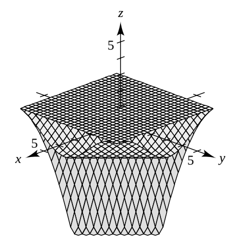
$= 1 - \frac{1}{2}x^2 - xy - \frac{1}{2}y^2$

(c) If $x = 0$, $P_2(0, y) = 1 - \frac{1}{2}y^2$. This is the second–degree Taylor polynomial for $\cos y$.
If $y = 0$, $P_2(x, 0) = 1 - \frac{1}{2}x^2$. This is the second–degree Taylor polynomial for $\cos x$.

(d)

x	y	$f(x, y)$	$P_1(x, y)$	$P_2(x, y)$
0	0	1	1	1
0	0.1	0.9950	1	0.9950
0.2	0.1	0.9553	1	0.9950
0.2	0.5	0.7648	1	0.7550
1	0.5	0.0707	1	−0.1250

(e)



62. Given $z = f(x, y)$, then:

$$F(x, y, z) = f(x, y) - z = 0$$

$$\nabla F(x_0, y_0, z_0) = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j} - \mathbf{k}$$

$$\cos\theta = \frac{|\nabla F(x_0, y_0, z_0) \cdot \mathbf{k}|}{\|\nabla F(x_0, y_0, z_0)\| \, \|\mathbf{k}\|}$$

$$= \frac{|-1|}{\sqrt{[f_x(x_0, y_0)]^2 + [f_y(x_0, y_0)]^2 + (-1)^2}}$$

$$= \frac{1}{\sqrt{[f_x(x_0, y_0)]^2 + [f_y(x_0, y_0)]^2 + 1}}$$

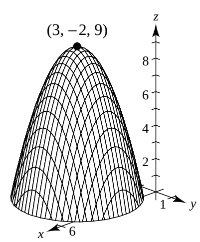
Section 12.8 Extrema of Functions of Two Variables

2. $g(x, y) = 9 - (x - 3)^2 - (y + 2)^2 \le 9$

Relative maximum: $(3, -2, 9)$

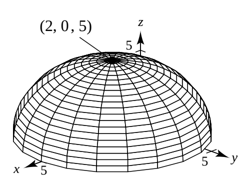
$g_x = -2(x - 3) = 0 \implies x = 3$

$g_y = -2(y + 2) = 0 \implies y = -2$



4. $f(x, y) = \sqrt{25 - (x - 2)^2 - y^2} \le 5$

Relative maximum: $(2, 0, 5)$



Check: $f_x = -\dfrac{x - 2}{\sqrt{25 - (x - 2)^2 - y^2}} = 0 \implies x = 2$

$f_y = -\dfrac{y}{\sqrt{25 - (x - 2)^2 - y^2}} = 0 \implies y = 0$

$$f_{xx} = -\frac{25 - y^2}{[25 - (x - 2)^2 - y^2]^{3/2}}, f_{yy} = -\frac{25 - (x - 2)^2}{[25 - (x - 2)^2 - y^2]^{3/2}}, f_{xy} = -\frac{y(x - 2)}{[25 - (x - 2)^2 - y^2]^{3/2}}$$

At the critical point $(2, 0)$, $f_{xx} < 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$. Therefore, $(2, 0, 5)$ is a relative maximum.

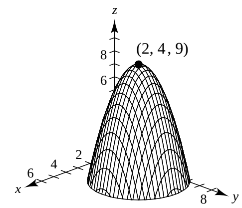
6. $f(x, y) = -x^2 - y^2 + 4x + 8y - 11 = -(x - 2)^2 - (y - 4)^2 + 9 \le 9$

Relative maximum: $(2, 4, 9)$

Check: $f_x = -2x + 4 = 0 \Rightarrow x = 2$

$f_y = -2y + 8 = 0 \Rightarrow y = 4$

$f_{xx} = -2,\ f_{yy} = -2,\ f_{xy} = 0$



At the critical point $(2, 4)$, $f_{xx} < 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$. Therefore, $(2, 4, 9)$ is a relative maximum.

8. $f(x, y) = -x^2 - 5y^2 + 10x - 30y - 62$

$$\left.\begin{aligned} f_x &= -2x + 10 = 0 \\ f_y &= -10y - 30 = 0 \end{aligned}\right\} x = 5, y = -3$$

$f_{xx} = -2, f_{yy} = -10, f_{xy} = 0$

At the critical point $(5, -3)$, $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$.

Therefore, $(5, -3, 8)$ is a relative maximum.

10. $f(x, y) = x^2 + 6xy + 10y^2 - 4y + 4$

$$\left.\begin{aligned} f_x &= 2x + 6y = 0 \\ f_y &= 6x + 20y - 4 = 0 \end{aligned}\right\} \quad \text{Solving simultaneously yields } x = -6 \text{ and } y = 2.$$

$f_{xx} = 2,\ f_{yy} = 20,\ f_{xy} = 6$

At the critical point $(-6, 2)$, $f_{xx} > 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$. Therefore, $(-6, 2, 0)$ is a relative minimum.

12. $f(x, y) = -3x^2 - 2y^2 + 3x - 4y + 5$

$f_x = -6x + 3 = 0$ when $x = \frac{1}{2}$.

$f_y = -4y - 4 = 0$ when $y = -1$.

$f_{xx} = -6,\ f_{yy} = -4,\ f_{xy} = 0$

At the critical point $\left(\frac{1}{2}, -1\right)$, $f_{xx} < 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$. Therefore, $\left(\frac{1}{2}, -1, \frac{31}{4}\right)$ is a relative maximum.

14. $h(x, y) = (x^2 + y^2)^{1/3} + 2$

$$\left.\begin{aligned} h_x &= \frac{2x}{3(x^2 + y^2)^{2/3}} = 0 \\ h_y &= \frac{2y}{3(x^2 + y^2)^{2/3}} = 0 \end{aligned}\right\} x = 0, y = 0$$

Since $h(x, y) \ge 2$ for all (x, y), $(0, 0, 2)$ is a relative minimum.

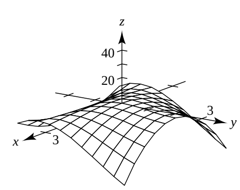
16. $f(x, y) = |x + y| - 2$

Since $f(x, y) \ge -2$ for all (x, y), the relative minima of f consist of all points (x, y) satisfying $x + y = 0$.

18. $f(x, y) = y^3 - 3yx^2 - 3y^2 - 3x^2 + 1$

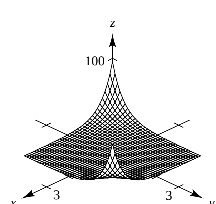
Relative maximum: $(0, 0, 1)$

Saddle points: $(0, 2, -3)$, $\left(\pm\sqrt{3}, -1, -3\right)$



20. $z = e^{xy}$

Saddle point: $(0, 0, 1)$



22. $g(x, y) = 120x + 120y - xy - x^2 - y^2$

$$\left.\begin{aligned} g_x &= 120 - y - 2x = 0 \\ g_y &= 120 - x - 2y = 0 \end{aligned}\right\} \text{ Solving simultaneously yields } x = 40 \text{ and } y = 40.$$

$g_{xx} = -2,\ g_{yy} = -2,\ g_{xy} = -1$

At the critical point $(40, 40)$, $g_{xx} < 0$ and $g_{xx}\,g_{yy} - (g_{xy})^2 > 0$. Therefore, $(40, 40, 4800)$ is a relative maximum.

24. $g(x, y) = xy$

$$\left.\begin{aligned} g_x &= y \\ g_y &= x \end{aligned}\right\} \ x = 0 \text{ and } y = 0$$

$g_{xx} = 0,\ g_{yy} = 0,\ g_{xy} = 1$

At the critical point $(0, 0)$, $g_{xx}\,g_{yy} - (g_{xy})^2 < 0$. Therefore, $(0, 0, 0)$ is a saddle point.

26. $f(x, y) = 2xy - \dfrac{1}{2}(x^4 + y^2) + 1$

$$\left.\begin{aligned} f_x &= 2y - 2x^3 \\ f_y &= 2x - 2y^3 \end{aligned}\right\} \begin{aligned} &\text{Solving by substitution yields 3 critical points:} \\ &(0, 0), (1, 1), (-1, -1) \end{aligned}$$

$f_{xx} = -6x^2, f_{yy} = -6y^2, f_{xy} = 2$

At $(0, 0)$, $f_{xx} f_{yy} - (f_{xy})^2 < 0 \implies (0, 0, 1)$ saddle point.

At $(1, 1)$, $f_{xx} f_{yy} - (f_{xy})^2 > 0$ and $f_{xx} < 0 \implies (1, 1, 2)$ relative maximum.

At $(-1, -1)$, $f_{xx} f_{yy} - (f_{xy})^2 > 0$ and $f_{xx} < 0 \implies (-1, -1, 2)$ relative maximum.

28. $f(x, y) = \left(\dfrac{1}{2} - x^2 + y^2\right)e^{1-x^2-y^2}$

$$\left.\begin{aligned} f_x &= (2x^3 - 2xy^2 - 3x)e^{1-x^2-y^2} = 0 \\ f_y &= (2x^2y - 2y^3 + y)e^{1-x^2-y^2} = 0 \end{aligned}\right\} \text{ Solving yields the critical points } (0, 0), \left(0, \pm\frac{\sqrt{2}}{2}\right), \left(\pm\frac{\sqrt{6}}{2}, 0\right).$$

$f_{xx} = (-4x^4 + 4x^2y^2 + 12x^2 - 2y^2 - 3)e^{1-x^2-y^2}$

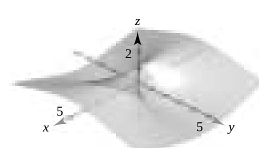
$f_{yy} = (4y^4 - 4x^2y^2 + 2x^2 - 8y^2 + 1)e^{1-x^2-y^2}$

$f_{xy} = (-4x^3y + 4xy^3 + 2xy)e^{1-x^2-y^2}$

At the critical point $(0, 0)$, $f_{xx}\,f_{yy} - (f_{xy})^2 < 0$. Therefore, $(0, 0, e/2)$ is a saddle point. At the critical points $\left(0, \pm\sqrt{2}/2\right)$, $f_{xx} < 0$ and $f_{xx}\,f_{yy} - (f_{xy})^2 > 0$. Therefore, $\left(0, \pm\sqrt{2}/2, \sqrt{e}\right)$ are relative maxima. At the critical points $\left(\pm\sqrt{6}/2, 0\right)$, $f_{xx} > 0$ and $f_{xx}\,f_{yy} - (f_{xy})^2 > 0$. Therefore, $\left(\pm\sqrt{6}/2, 0, -\sqrt{e}/e\right)$ are relative minima.

30. $z = \dfrac{(x^2 - y^2)^2}{x^2 + y^2} \ge 0.$ $z = 0$ if $x^2 = y^2 \neq 0.$

Relative minima at all points (x, x) and $(x, -x)$, $x \neq 0$.



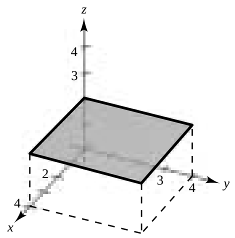
32. $f_{xx} < 0$ and $f_{xx}\,f_{yy} - (f_{xy})^2 = (-3)(-8) - 2^2 > 0$

f has a relative maximum at (x_0, y_0).

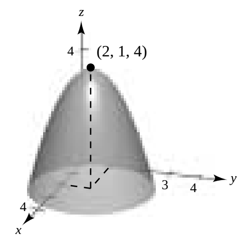
34. $f_{xx} > 0$ and $f_{xx}\,f_{yy} - (f_{xy})^2 = (25)(8) - 10^2 > 0$

f has a relative minimum at (x_0, y_0).

36. See Theorem 12.17.

38. 

Extrema at all (x, y)

40. 

Relative maximum

42. A and B are relative extrema. C and D are saddle points.

44. $d = f_{xx} f_{yy} - f_{xy}{}^2 < 0$ if f_{xx} and f_{yy} have opposite signs. Hence, $(a, b, f(a, b))$ is a saddle point. For example, consider $f(x, y) = x^2 - y^2$ and $(a, b) = (0, 0)$.

46. $f(x, y) = x^3 + y^3 - 6x^2 + 9y^2 + 12x + 27y + 19$

$$\left.\begin{aligned} f_x &= 3x^2 - 12x + 12 = 0 \\ f_y &= 3y^2 + 18y + 27 = 0 \end{aligned}\right\} \text{Solving yields } x = 2 \text{ and } y = -3.$$

$f_{xx} = 6x - 12,\ f_{yy} = 6y + 18,\ f_{xy} = 0$

At $(2, -3)$, $f_{xx} f_{yy} - (f_{xy})^2 = 0$ and the test fails. $(1, -2, 0)$ is a saddle point.

48. $f(x, y) = \sqrt{(x - 1)^2 + (y + 2)^2} \geq 0$

$$\left.\begin{aligned} f_x &= \frac{x - 1}{\sqrt{(x - 1)^2 + (y + 2)^2}} = 0 \\ f_y &= \frac{y + 2}{\sqrt{(x - 1)^2 + (y + 2)^2}} = 0 \end{aligned}\right\} \text{Solving yields } x = 1 \text{ and } y = -2.$$

$$f_{xx} = \frac{(y + 2)^2}{[(x - 1)^2 + (y + 2)^2]^{3/2}},\ f_{yy} = \frac{(x - 1)^2}{[(x - 1)^2 + (y + 2)^2]^{3/2}},\ f_{xy} = \frac{(x - 1)(y + 2)}{[(x - 1)^2 + (y + 2)^2]^{3/2}}$$

At $(1, -2)$, $f_{xx} f_{yy} - (f_{xy})^2$ is undefined and the test fails.

Absolute minimum: $(1, -2, 0)$

50. $f(x, y) = (x^2 + y^2)^{2/3} \geq 0$

$$\left.\begin{aligned} f_x &= \frac{4x}{3(x^2 + y^2)^{1/3}} \\ f_y &= \frac{4y}{3(x^2 + y^2)^{1/3}} \end{aligned}\right\} f_x \text{ and } f_y \text{ are undefined at } x = 0, y = 0. \text{ The critical point is } (0, 0).$$

$$f_{xx} = \frac{4(x^2 + 3y^2)}{9(x^2 + y^2)^{4/3}},\ f_{yy} = \frac{4(3x^2 + y^2)}{9(x^2 + y^2)^{4/3}},\ f_{xy} = \frac{-8xy}{9(x^2 + y^2)^{4/3}}$$

At $(0, 0)$, $f_{xx} f_{yy} - (f_{xy})^2$ is undefined and the test fails.

Absolute minimum: $(0, 0, 0)$

52. $f(x, y, z) = 4 - [x(y - 1)(z + 2)]^2 \leq 4$

$$\left.\begin{aligned} f_x &= -2x(y - 1)^2(z + 2)^2 = 0 \\ f_y &= -2x^2(y - 1)(z + 2)^2 = 0 \\ f_z &= -2x(y - 1)^2(z + 2) = 0 \end{aligned}\right\}$$

Solving yields the critical points $(0, a, b)$, $(c, 1, d)$, $(e, f, -2)$. These points are all absolute maxima.

54. $f(x, y) = (2x - y)^2$

$f_x = 4(2x - y) = 0 \implies 2x = y$

$f_y = -2(2x - y) = 0 \implies 2x = y$

On the line $y = x + 1,\ 0 \le x \le 1$,

$$f(x, y) = f(x) = (2x - (x + 1))^2 = (x - 1)^2$$

and the maximum is 1, the minimum is 0. On the line $y = -\frac{1}{2}x + 1,\ 0 \le x \le 2$,

$$f(x, y) = f(x) = \left(2x - \left(-\tfrac{1}{2}x + 1\right)\right)^2 = \left(\tfrac{5}{2}x - 1\right)^2$$

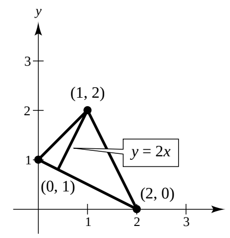
and the maximum is 16, the minimum is 0. On the line $y = -2x + 4,\ 1 \le x \le 2$,

$$f(x, y) = f(x) = (2x - (-2x + 4))^2 = (4x - 4)^2$$

and the maximum is 16, the minimum is 0.

Absolute maximum: 16 at $(2, 0)$

Absolute minimum: 0 at $(1, 2)$ and along the line $y = 2x$.



56. $f(x, y) = 2x - 2xy + y^2$

$$\left.\begin{array}{l} f_x = 2 - 2y = 0 \implies y = 1 \\ f_y = 2y - 2x = 0 \implies y = x \implies x = 1 \end{array}\right\} f(1, 1) = 1$$

On the line $y = 1,\ -1 \le x \le 1$,

$$f(x, y) = f(x) = 2x - 2x + 1 = 1.$$

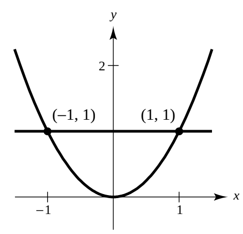
On the curve $y = x^2,\ -1 \le x \le 1$

$$f(x, y) = f(x) = 2x - 2x(x^2) + (x^2)^2 = x^4 - 2x^3 + 2x$$

and the maximum is 1, the minimum is $-\frac{11}{16}$.

Absolute maximum: 1 at $(1, 1)$ and on $y = 1$

Absolute minimum: $-\frac{11}{16} = -0.6875$ at $\left(-\frac{1}{2}, \frac{1}{4}\right)$



58. $f(x, y) = x^2 + 2xy + y^2,\ R = \{(x, y): |x| \le 2, |y| \le 1\}$

$$\left.\begin{array}{l} f_x = 2x + 2y = 0 \\ f_y = 2x + 2y = 0 \end{array}\right\} y = -x$$

$f(x, -x) = x^2 - 2x^2 + x^2 = 0$

Along $y = 1,\ -2 \le x \le 2$,

$$f = x^2 + 2x + 1,\ f' = 2x + 2 = 0 \implies x = -1,\ f(-2, 1) = 1,\ f(-1, 1) = 0,\ f(2, 1) = 9.$$

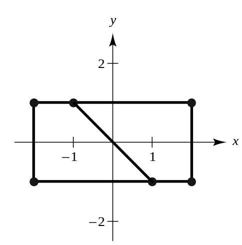
Along $y = -1,\ -2 \le x \le 2$,

$$f = x^2 - 2x + 1,\ f' = 2x - 2 = 0 \implies x = 1,\ f(-2, -1) = 9,\ f(1, -1) = 0,\ f(2, -1) = 1.$$

Along $x = 2,\ -1 \le y \le 1,\ f = 4 + 4y + y^2,\ f' = 2y + 4 \ne 0$.

Along $x = -2,\ -1 \le y \le 1,\ f = 4 - 4y + y^2,\ f' = 2y - 4 \ne 0$.

Thus, the maxima are $f(-2, -1) = 9$ and $f(2, 1) = 9$, and the minima are $f(x, -x) = 0,\ -1 \le x \le 1$.



60. $f(x, y) = x^2 - 4xy + 5,\ R = \left\{(x, y): 0 \le x \le 4, 0 \le y \le \sqrt{x}\right\}$

$$\left.\begin{array}{l} f_x = 2x - 4y = 0 \\ f_y = -4x = 0 \end{array}\right\} x = y = 0$$

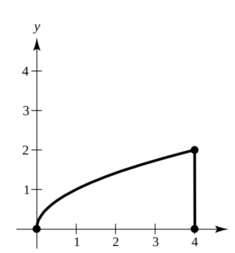
$f(0, 0) = 5$

Along $y = 0,\ 0 \le x \le 4,\ f = x^2 + 5$ and $f(4, 0) = 21$.

Along $x = 4,\ 0 \le y \le 2,\ f = 16 - 16y + 5,\ f' = -16 \ne 0$ and $f(4, 2) = -11$.

Along $y = \sqrt{x},\ 0 \le x \le 4,\ f = x^2 - 4x^{3/2} + 5,\ f' = 2x - 6x^{1/2} \ne 0$ on $[0, 4]$.

Thus, the maximum is $f(4, 0) = 21$ and the minimum is $f(4, 2) = -11$.



62. $f(x, y) = \dfrac{4xy}{(x^2 + 1)(y^2 + 1)}, R = \{(x, y): x \geq 0, y \geq 0, x^2 + y^2 \leq 1\}$

$$f_x = \frac{4(1 - x^2)y}{(y^2 + 1)(x^2 + 1)^2} = 0 \implies x = 1 \text{ or } y = 0$$

$$f_y = \frac{4(1 - y^2)x}{(x^2 + 1)(y^2 + 1)^2} = 0 \implies y = 1 \text{ or } x = 0$$

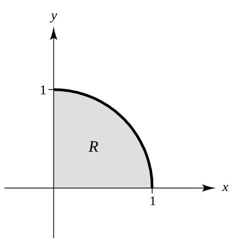
For $x = 0$, $y = 0$, also, and $f(0, 0) = 0$.

For $x = 1$ and $y = 1$, the point $(1, 1)$ is outside R.

For $x^2 + y^2 = 1$, $f(x, y) = f\left(x, \sqrt{1 - x^2}\right) = \dfrac{4x\sqrt{1 - x^2}}{2 + x^2 - x^4}$, and the maximum occurs at $x = \dfrac{\sqrt{2}}{2}, y = \dfrac{\sqrt{2}}{2}$.

Absolute maximum is $\dfrac{8}{9} = f\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$.

The absolute minimum is $0 = f(0, 0)$. (In fact, $f(0, y) = f(x, 0) = 0$)



64. False

Let $f(x, y) = x^4 - 2x^2 + y^2$.

Relative minima: $(\pm 1, 0, -1)$

Saddle point: $(0, 0, 0)$

Section 12.9 Applications of Extrema of Functions of Two Variables

2. A point on the plane is given by $(x, y, 12 - 2x - 3y)$. The square of the distance from $(1, 2, 3)$ to a point on the plane is given by

$$S = (x - 1)^2 + (y - 2)^2 + (9 - 2x - 3y)^2$$
$$S_x = 2(x - 1) + 2(9 - 2x - 3y)(-2)$$
$$S_y = 2(y - 2) + 2(9 - 2x - 3y)(-3).$$

From the equations $S_x = 0$ and $S_y = 0$, we obtain the system

$$5x + 6y = 19$$
$$6x + 10y = 29.$$

Solving simultaneously, we have $x = \frac{16}{14}, y = \frac{31}{14}, z = \frac{43}{14}$ and the distance is

$$\sqrt{\left(\frac{16}{14} - 1\right)^2 + \left(\frac{31}{14} - 2\right)^2 + \left(\frac{43}{14} - 3\right)^2} = \frac{1}{\sqrt{14}}.$$

4. A point on the paraboloid is given by $(x, y, x^2 + y^2)$. The square of the distance from $(5, 0, 0)$ to a point on the paraboloid is given by

$$S = (x - 5)^2 + y^2 + (x^2 + y^2)^2$$
$$S_x = 2(x - 5) + 4x(x^2 + y^2) = 0$$
$$S_y = 2y + 4y(x^2 + y^2) = 0.$$

From the equations $S_x = 0$ and $S_y = 0$, we obtain the system

$$2x^3 + 2xy^2 + x - 5 = 0$$
$$2y^3 + 2x^2y + y = 0.$$

Solving as in Exercise 3, we have $x \approx 1.235$, $y = 0$, $z \approx 1.525$ and the distance is

$$\sqrt{(1.235 - 5)^2 + (1.525)^2} \approx 4.06.$$

6. Since $x + y + z = 32$, $z = 32 - x - y$. Therefore,

$$P = xy^2z = 32xy^2 - x^2y^2 - xy^3$$
$$P_x = 32y^2 - 2xy^2 - y^3 = y^2(32 - 2x - y) = 0$$
$$P_y = 64xy - 2x^2y - 3xy^2 = y(64x - 2x^2 - 3xy) = 0.$$

Ignoring the solution $y = 0$ and substituting $y = 32 - 2x$ into $P_y = 0$, we have

$$64x - 2x^2 - 3x(32 - 2x) = 0$$
$$4x(x - 8) = 0.$$

Therefore, $x = 8$, $y = 16$, and $z = 8$.

8. Let x, y, and z be the numbers and let $S = x^2 + y^2 + z^2$. Since $x + y + z = 1$, we have

$$S = x^2 + y^2 + (1 - x - y)^2$$
$$\left.\begin{aligned} S_x &= 2x - 2(1 - x - y) = 0 \\ S_y &= 2y - 2(1 - x - y) = 0 \end{aligned}\right\} \begin{aligned} 2x + y &= 1 \\ x + 2y &= 1. \end{aligned}$$

Solving simultaneously yields $x = \frac{1}{3}$, $y = \frac{1}{3}$, and $z = \frac{1}{3}$.

10. Let x, y, and z be the length, width, and height, respectively. Then $C_0 = 1.5xy + 2yz + 2xz$ and $z = \dfrac{C_0 - 1.5xy}{2(x + y)}$. The volume is given by

$$V = xyz = \frac{C_0xy - 1.5x^2y^2}{2(x + y)}$$
$$V_x = \frac{y^2(2C_0 - 3x^2 - 6xy)}{4(x + y)^2}$$
$$V_y = \frac{x^2(2C_0 - 3y^2 - 6xy)}{4(x + y)^2}.$$

In solving the system $V_x = 0$ and $V_y = 0$, we note by the symmetry of the equations that $y = x$. Substituting $y = x$ into $V_x = 0$ yields

$$\frac{x^2(2C_0 - 9x^2)}{16x^2} = 0,\ 2C_0 = 9x^2,\ x = \frac{1}{3}\sqrt{2C_0},\ y = \frac{1}{3}\sqrt{2C_0},\ \text{and } z = \frac{1}{4}\sqrt{2C_0}.$$

12. Consider the sphere given by $x^2 + y^2 + z^2 = r^2$ and let a vertex of the rectangular box be $\left(x, y, \sqrt{r^2 - x^2 - y^2}\right)$. Then the volume is given by

$$V = (2x)(2y)\left(2\sqrt{r^2 - x^2 - y^2}\right) = 8xy\sqrt{r^2 - x^2 - y^2}$$
$$V_x = 8\left(xy\frac{-x}{\sqrt{r^2 - x^2 - y^2}} + y\sqrt{r^2 - x^2 - y^2}\right) = \frac{8y}{\sqrt{r^2 - x^2 - y^2}}(r^2 - 2x^2 - y^2) = 0$$
$$V_y = 8\left(xy\frac{-y}{\sqrt{r^2 - x^2 - y^2}} + x\sqrt{r^2 - x^2 - y^2}\right) = \frac{8x}{\sqrt{r^2 - x^2 - y^2}}(r^2 - x^2 - 2y^2) = 0.$$

Solving the system

$$2x^2 + y^2 = r^2$$
$$x^2 + 2y^2 = r^2$$

yields the solution $x = y = z = r/\sqrt{3}$.

14. Let x, y, and z be the length, width, and height, respectively. Then the sum of the two perimeters of the two cross sections is given by

$$(2x + 2z) + (2y + 2z) = 108 \text{ or } x = 54 - y - 2z.$$

The volume is given by

$$V = xyz = 54yz - y^2z - 2yz^2$$
$$V_y = 54z - 2yz - 2z^2 = z(54 - 2y - 2z) = 0$$
$$V_z = 54y - y^2 - 4yz = y(54 - y - 4z) = 0.$$

Solving the system $2y + 2z = 54$ and $y + 4z = 54$, we obtain the solution

$x = 18$ inches, $y = 18$ inches, and $z = 9$ inches.

16. $A = \frac{1}{2}[(30 - 2x) + (30 - 2x) + 2x\cos\theta]x\sin\theta$

$= 30x\sin\theta - 2x^2\sin\theta + x^2\sin\theta\cos\theta$

$\frac{\partial A}{\partial x} = 30\sin\theta - 4x\sin\theta + 2x\sin\theta\cos\theta = 0$

$\frac{\partial A}{\partial \theta} = 30\cos\theta - 2x^2\cos\theta + x^2(2\cos^2\theta - 1) = 0$

From $\frac{\partial A}{\partial x} = 0$ we have $15 - 2x + x\cos\theta = 0 \implies \cos\theta = \frac{2x - 15}{x}$.

From $\frac{\partial A}{\partial \theta} = 0$ we obtain

$$30x\left(\frac{2x - 15}{x}\right) - 2x^2\left(\frac{2x - 15}{x}\right) + x^2\left(2\left(\frac{2x - 15}{x}\right)^2 - 1\right) = 0$$

$$30(2x - 15) - 2x(2x - 15) + 2(2x - 15)^2 - x^2 = 0$$

$$3x^2 - 30x = 0$$

$$x = 10$$

Then $\cos\theta = \frac{1}{2} \implies \theta = 60°$.

18. $P(p, q, r) = 2pq + 2pr + 2qr$.

$p + q + r = 1$ implies that $r = 1 - p - q$.

$P(p, q) = 2pq + 2p(1 - p - q) + 2q(1 - p - q)$

$= 2pq + 2p - 2p^2 - 2pq + 2q - 2pq - 2q^2$

$= -2pq + 2p + 2q - 2p^2 - 2q^2$

$\frac{\partial P}{\partial p} = -2q + 2 - 4p$; $\frac{\partial P}{\partial q} = -2p + 2 - 4q$

Solving $\frac{\partial P}{\partial p} = \frac{\partial P}{\partial q} = 0$ gives

$q + 2p = 1$

$p + 2q = 1$

and hence $p = q = \frac{1}{3}$ and

$$P\left(\frac{1}{3}, \frac{1}{3}\right) = -2\left(\frac{1}{9}\right) + 2\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right) - 2\left(\frac{1}{9}\right) - 2\left(\frac{1}{9}\right)$$

$$= \frac{6}{9} = \frac{2}{3}.$$

20. $R = 515p_1 + 805p_2 + 1.5p_1p_2 - 1.5p_1^2 - p_2^2$

$R_{p_1} = 515 + 1.5p_2 - 3p_1 = 0$

$R_{p_2} = 805 + 1.5p_1 - p_2 = 0$

$3p_1 - 1.5p_2 = 515$

$-1.5p_1 + p_2 = 805$

Solving this system yields $p_1 = \$2296.67$, $p_2 = \$4250$.

22. $S = d_1 + d_2 + d_3 = \sqrt{(0 - 0)^2 + (y - 0)^2} + \sqrt{(0 - 2)^2 + (y - 2)^2} + \sqrt{(0 + 2)^2 + (y - 2)^2}$

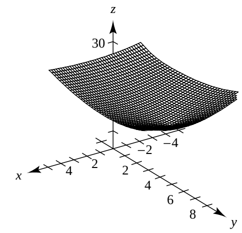
$= y + 2\sqrt{4 + (y - 2)^2}$

$\frac{dS}{dy} = 1 + \frac{2(y - 2)}{\sqrt{4 + (y - 2)^2}} = 0$ when $y = 2 - \frac{2\sqrt{3}}{3} = \frac{6 - 2\sqrt{3}}{3}$.

The sum of the distance is minimized when $y = \frac{2(3 - \sqrt{3})}{3} \approx 0.845$.

24. (a) $S = \sqrt{(x+4)^2 + y^2} + \sqrt{(x-1)^2 + (y-6)^2} + \sqrt{(x-12)^2 + (y-2)^2}$

The surface appears to have a minimum near $(x, y) = (1, 5)$.



(b) $$S_x = \frac{x+4}{\sqrt{(x+4)^2 + y^2}} + \frac{x-1}{\sqrt{(x-1)^2 + (y-6)^2}} + \frac{x-12}{\sqrt{(x-12)^2 + (y-2)^2}}$$

$$S_y = \frac{y}{\sqrt{(x+4)^2 + y^2}} + \frac{y-6}{\sqrt{(x-1)^2 + (y-6)^2}} + \frac{y-2}{\sqrt{(x-12)^2 + (y-2)^2}}$$

(c) Let $(x_1, y_1) = (1, 5)$. Then

$-\nabla S(1, 5) = 0.258\mathbf{i} + 0.03\mathbf{j}$

Direction $\approx 6.6°$

(d) $t \approx 0.94$ $x_2 \approx 1.24$ $y_2 \approx 5.03$

(e) $t \approx 3.56$, $x_3 \approx 1.24$, $y_3 \approx 5.06$,

$t \approx 1.04$, $x_4 \approx 1.23$, $y_4 \approx 5.06$

Note: Minimum occurs at $(x, y) = (1.2335, 5.0694)$

(f) $-\nabla S(x, y)$ points in the direction that S *decreases* most rapidly.

26. See the last paragraph on page 915 and Theorem 12.18.

28. (a)

x	y	xy	x^2
-3	0	0	9
-1	1	-1	1
1	1	1	1
3	2	6	9
$\sum x_i = 0$	$\sum y_i = 4$	$\sum x_i y_i = 6$	$\sum x_i^2 = 20$

$$a = \frac{4(6) - 0(4)}{4(20) - (0)^2} = \frac{3}{10},\ b = \frac{1}{4}\left[4 - \frac{3}{10}(0)\right] = 1,$$

$$y = \frac{3}{10}x + 1$$

(b) $$S = \left(\frac{1}{10} - 0\right)^2 + \left(\frac{7}{10} - 1\right)^2 + \left(\frac{13}{10} - 1\right)^2 + \left(\frac{19}{10} - 2\right)^2$$

$$= \frac{1}{5}$$

30. (a)

x	y	xy	x^2
3	0	0	9
1	0	0	1
2	0	0	4
3	1	3	9
4	1	4	16
4	2	8	16
5	2	10	25
6	2	12	36
$\sum x_i = 28$	$\sum y_i = 8$	$\sum x_i y_i = 37$	$\sum x_i^2 = 116$

$$a = \frac{8(37) - (28)(8)}{8(116) - (28)^2} = \frac{72}{144} = \frac{1}{2},\ b = \frac{1}{8}\left[8 - \frac{1}{2}(28)\right] = -\frac{3}{4}, y = \frac{1}{2}x - \frac{3}{4}$$

(b) $$S = \left(\frac{3}{4} - 0\right)^2 + \left(-\frac{1}{4} - 0\right)^2 + \left(\frac{1}{4} - 0\right)^2 + \left(\frac{3}{4} - 1\right)^2 + \left(\frac{5}{4} - 1\right)^2 + \left(\frac{5}{4} - 2\right)^2 + \left(\frac{7}{4} - 2\right)^2 + \left(\frac{9}{4} - 2\right)^2 = \frac{3}{2}$$

32. (1, 0), (3, 3), (5, 6)

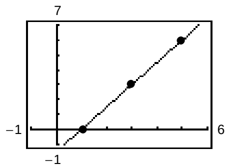
$\sum x_i = 9, \quad \sum y_i = 9,$

$\sum x_i y_i = 39, \quad \sum x_i^2 = 35$

$$a = \frac{3(39) - 9(9)}{3(35) - (9)^2} = \frac{36}{24} = \frac{3}{2}$$

$$b = \frac{1}{3}\left[9 - \frac{3}{2}(9)\right] = -\frac{9}{6} = -\frac{3}{2}$$

$$y = \frac{3}{2}x - \frac{3}{2}$$



34. (6, 4), (1, 2), (3, 3), (8, 6), (11, 8), (13, 8); $n = 6$

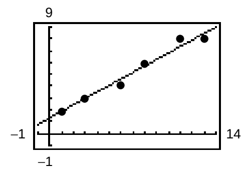
$\sum x_i = 42 \quad \sum y_i = 31$

$\sum x_i y_i = 275 \quad \sum x_i^2 = 400$

$$a = \frac{6(275) - (42)(31)}{6(400) - (42)^2} = \frac{29}{53} \approx 0.5472$$

$$b = \frac{1}{6}\left(31 - \frac{29}{53}42\right) = \frac{425}{318} \approx 1.3365$$

$$y = \frac{29}{53}x + \frac{425}{318}$$



36. (a) (1.00, 450), (1.25, 375), (1.50, 330)

$\sum x_i = 3.75, \sum y_i = 1{,}155, \sum x_i^2 = 4.8125,$

$\sum x_i y_i = 1{,}413.75$

$$a = \frac{3(1{,}413.75) - (3.75)(1{,}155)}{3(4.8125) - (3.75)^2} = -240$$

$$b = \frac{1}{3}[1{,}155 - (-240)(3.75)] = 685$$

$$y = -240x + 685$$

(b) When $x = 1.40$, $y = -240(1.40) + 685 = 349$.

38. (a) $y = 1.8311x - 47.1067$

(b) For each 1 point increase in the percent (x), y increases by about 1.83 (slope of line).

40. $S(a, b) = \sum_{i=1}^{n}(ax_i + b - y_i)^2$

$$S_a(a, b) = 2a\sum_{i=1}^{n}x_i^2 + 2b\sum_{i=1}^{n}x_i - 2\sum_{i=1}^{n}x_i y_i$$

$$S_b(a, b) = 2a\sum_{i=1}^{n}x_i + 2nb - 2\sum_{i=1}^{n}y_i$$

$$S_{aa}(a, b) = 2\sum_{i=1}^{n}x_i^2$$

$$S_{bb}(a, b) = 2n$$

$$S_{ab}(a, b) = 2\sum_{i=1}^{n}x_i$$

$S_{aa}(a, b) > 0$ as long as $x_i \neq 0$ for all i. (**Note:** If $x_i = 0$ for all i, then $x = 0$ is the least squares regression line.)

$$d = S_{aa}S_{bb} - S_{ab}^2 = 4n\sum_{i=1}^{n}x_i^2 - 4\left(\sum_{i=1}^{n}x_i\right)^2 = 4\left[n\sum_{i=1}^{n}x_i^2 - \left(\sum_{i=1}^{n}x_i\right)^2\right] \geq 0 \text{ since } n\sum_{i=1}^{n}x_i^2 \geq \left(\sum_{i=1}^{n}x_i\right)^2.$$

As long as $d \neq 0$, the given values for a and b yield a minimum.

42. $(-4, 5),\ (-2, 6),\ (2, 6),\ (4, 2)$

$\sum x_i = 0$

$\sum y_i = 19$

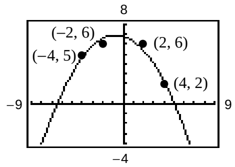
$\sum x_i^2 = 40$

$\sum x_i^3 = 0$

$\sum x_i^4 = 544$

$\sum x_i y_i = -12$

$\sum x_i^2 y_i = 160$



$544a + 40c = 160,\ 40b = -12,\ 40a + 4c = 19$

$a = -\frac{5}{24},\ b = -\frac{3}{10},\ c = \frac{41}{6},\ y = -\frac{5}{24}x^2 - \frac{3}{10}x + \frac{41}{6}$

44. $(0, 10), (1, 9), (2, 6), (3, 0)$

$\sum x_i = 6$

$\sum y_i = 25$

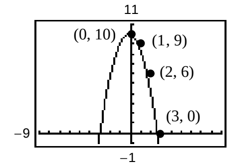
$\sum x_i^2 = 14$

$\sum x_i^3 = 36$

$\sum x_i^4 = 98$

$\sum x_i y_i = 21$

$\sum x_i^2 y_i = 33$



$98a + 36b + 14c = 33$

$36a + 14b + 6c = 21$

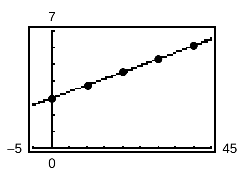
$14a + 6b + 4c = 25$

$a = -\frac{5}{4},\ b = \frac{9}{20},\ c = \frac{199}{20},\ y = -\frac{5}{4}x^2 + \frac{9}{20}x + \frac{199}{20}$

46. (a) $y = 0.078x + 2.96$

(b) $y = 0.0001429x^2 + 0.07229x + 2.9886$

(c)



(d) For the linear model, $x = 50$ gives $y \approx 6.86$ billion.

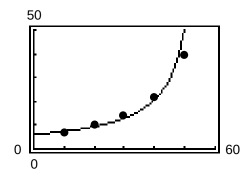
For the quadratic model, $x = 50$ gives $y \approx 6.96$ billion.

As you extrapolate into the future, the quadratic model increases more rapidly.

48. (a) $\frac{1}{y} = ax + b = -0.0029x + 0.1640$

$$y = \frac{1}{-0.0029x + 0.1640}$$

(b)



(c) No. For $x = 60$, $y \approx -100$. Note that there is a vertical asymptote at $x \approx 56.6$.

Section 12.10 Lagrange Multipliers

2. Maximize $f(x, y) = xy$.

Constraint: $2x + y = 4$

$\nabla f = \lambda \nabla g$

$y\mathbf{i} + x\mathbf{j} = 2\lambda\,\mathbf{i} + \lambda\,\mathbf{j}$

$y = 2\lambda$

$x = \lambda$

$2x + y = 4 \implies 4\lambda = 4$

$\lambda = 1,\ x = 1,\ y = 2$

$f(1, 2) = 2$

4. Minimize $f(x, y) = x^2 + y^2$.

Constraint: $2x + 4y = 5$

$\nabla f = \lambda \nabla g$

$2x\mathbf{i} + 2y\mathbf{j} = 2\lambda\,\mathbf{i} + 4\lambda\,\mathbf{j}$

$2x = 2\lambda \implies x = \lambda$

$2y = 4\lambda \implies y = 2\lambda$

$2x + 4y = 5 \implies 10\lambda = 5$

$\lambda = \frac{1}{2},\ x = \frac{1}{2},\ y = 1$

$f\left(\frac{1}{2}, 1\right) = \frac{5}{4}$

6. Maximize $f(x, y) = x^2 - y^2$.

Constraint: $2y - x^2 = 0$

$\nabla f = \lambda \nabla g$

$2x\mathbf{i} - 2y\mathbf{j} = -2x\lambda\,\mathbf{i} + 2\lambda\,\mathbf{j}$

$2x = -2x\lambda \implies x = 0$ or $\lambda = -1$

If $x = 0$, then $y = 0$ and $f(0, 0) = 0$.

If $\lambda = -1$,

$$-2y = 2\lambda = -2 \implies y = 1 \implies x^2 = 2 \implies x = \sqrt{2}.$$

$f(\sqrt{2}, 1) = 2 - 1 = 1$ Maximum.

8. Minimize $f(x, y) = 3x + y + 10$.

Constraint: $x^2y = 6$

$\nabla f = \lambda \nabla g$

$3\mathbf{i} + \mathbf{j} = 2xy\lambda\,\mathbf{i} + x^2\lambda\,\mathbf{j}$

$$\left.\begin{aligned} 3 = 2xy\lambda &\implies \lambda = \frac{3}{2xy} \\ 1 = x^2\lambda &\implies \lambda = \frac{1}{x^2} \end{aligned}\right\} 3x^2 = 2xy \implies y = \frac{3x}{2} \quad (x \neq 0)$$

$$x^2y = 6 \implies x^2\left(\frac{3x}{2}\right) = 6$$

$$x^3 = 4$$

$$x = \sqrt[3]{4},\ y = \frac{3\sqrt[3]{4}}{2}$$

$$f\left(\sqrt[3]{4}, \frac{3\sqrt[3]{4}}{2}\right) = \frac{9\sqrt[3]{4} + 20}{2}$$

10. Note: $f(x, y) = \sqrt{x^2 + y^2}$ is minimum when $g(x, y)$ is minimum.

Minimize $g(x, y) = x^2 + y^2$.

Constraint: $2x + 4y = 15$

$$\left.\begin{aligned} 2x &= 2\lambda \\ 2y &= 4\lambda \end{aligned}\right\} y = 2x$$

$2x + 4y = 15 \implies 10x = 15$

$$x = \frac{3}{2},\ y = 3$$

$$f\left(\frac{3}{2}, 3\right) = \sqrt{g\left(\frac{3}{2}, 3\right)} = \frac{3\sqrt{5}}{2}$$

12. Minimize $f(x, y) = 2x + y$.

Constraint: $xy = 32$

$$\left.\begin{aligned} 2 &= y\lambda \\ 1 &= x\lambda \end{aligned}\right\} y = 2x$$

$xy = 32 \implies 2x^2 = 32$

$$x = 4, y = 8$$

$f(4, 8) = 16$

14. Maximize or minimize $f(x, y) = e^{-xy/4}$.

Constraint: $x^2 + y^2 \leq 1$

Case 1: On the circle $x^2 + y^2 = 1$

$$\left.\begin{aligned} -(y/4)e^{-xy/4} &= 2x\lambda \\ -(x/4)e^{-xy/4} &= 2y\lambda \end{aligned}\right\} \implies x^2 = y^2$$

$$x^2 + y^2 = 1 \implies x = \pm\frac{\sqrt{2}}{2}$$

Maxima: $f\left(\pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2}\right) = e^{1/8} \approx 1.1331$

Minima: $f\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right) = e^{-1/8} \approx 0.8825$

Case 2: Inside the circle

$$\left.\begin{aligned} f_x &= -(y/4)e^{-xy/4} = 0 \\ f_y &= -(x/4)e^{-xy/4} = 0 \end{aligned}\right\} \implies x = y = 0$$

$$f_{xx} = \frac{y^2}{16}e^{-xy/4},\ f_{yy} = \frac{x^2}{16}e^{-xy/4},\ f_{xy} = e^{-xy}\left[\frac{1}{16}xy - \frac{1}{4}\right)$$

At $(0, 0)$, $f_{xx}f_{yy} - (f_{xy})^2 < 0$.

Saddle point: $f(0, 0) = 1$

Combining the two cases, we have a maximum of $e^{1/8}$ at $\left(\pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2}\right)$ and a minimum of $e^{-1/8}$ at $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$.

16. Maximize $f(x, y, z) = xyz$.

Constraint: $x + y + z = 6$

$$\left.\begin{array}{r} yz = \lambda \\ xz = \lambda \\ xy = \lambda \end{array}\right\} x = y = z$$

$x + y + z = 6 \implies x = y = z = 2$

$f(2, 2, 2) = 8$

18. Minimize $x^2 - 10x + y^2 - 14y + 70$

Constraint: $x + y = 10$

$$\left.\begin{array}{r} 2x - 10 = \lambda \\ 2y - 14 = \lambda \\ x + y = 8 \end{array}\right\} \begin{array}{l} x = (1/2)(\lambda + 10) \\ y = (1/2)(\lambda + 14) \end{array}$$

$$x + y = \frac{1}{2}(\lambda + 10) + \frac{1}{2}(\lambda + 14)$$

$$= \lambda + 12 = 8 \implies \lambda = -4$$

Then $x = 3, y = 5$.

$f(3, 5) = 9 - 30 + 25 - 70 + 70 = 4$

20. Minimize $f(x, y, z) = x^2 + y^2 + z^2$.

Constraints: $x + 2z = 6$

$x + y = 12$

$\nabla f = \lambda \nabla g + \mu \nabla h$

$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(\mathbf{i} + 2\mathbf{k}) + \mu(\mathbf{i} + \mathbf{j})$

$$\left.\begin{array}{l} 2x = \lambda + \mu \\ 2y = \mu \\ 2z = 2\lambda \end{array}\right\} 2x = 2y + z$$

$x + 2z = 6 \implies z = \dfrac{6 - x}{2} = 3 - \dfrac{x}{2}$

$x + y = 12 \implies y = 12 - x$

$2x = 2(12 - x) + \left(3 - \dfrac{x}{2}\right) \implies \dfrac{9}{2}x = 27 \implies x = 6$

$x = 6, z = 0$

$f(6, 6, 0) = 72$

22. Maximize $f(x, y, z) = xyz$.

Constraints: $x^2 + z^2 = 5$

$x - 2y = 0$

$\nabla f = \lambda \nabla g + \mu \nabla h$

$yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \lambda(2x\mathbf{i} + 2z\mathbf{k}) + \mu(\mathbf{i} - 2\mathbf{j})$

$$yz = 2x\lambda + \mu$$

$$xz = -2\mu \implies \mu = -\frac{xy}{2}$$

$$xy = 2z\lambda \implies \lambda = \frac{xy}{2z}$$

$$x^2 + z^2 = 5 \implies z = \sqrt{5 - x^2}$$

$$x - 2y = 0 \implies y = \frac{x}{2}$$

$$yz = 2x\left(\frac{xy}{2z}\right) - \frac{xz}{2}$$

$$\frac{x\sqrt{5 - x^2}}{2} = \frac{x^3}{2\sqrt{5 - x^2}} - \frac{x\sqrt{5 - x^2}}{2}$$

$$x\sqrt{5 - x^2} = \frac{x^3}{2\sqrt{5 - x^2}}$$

$$2x(5 - x^2) = x^3$$

$$0 = 3x^3 - 10x = x(3x^2 - 10)$$

$$x = 0 \text{ or } x = \sqrt{\frac{10}{3}}, y = \frac{1}{2}\sqrt{\frac{10}{3}}, z = \sqrt{\frac{5}{3}}$$

$$f\left(\sqrt{\frac{10}{3}}, \frac{1}{2}\sqrt{\frac{10}{3}}, \sqrt{\frac{5}{3}}\right) = \frac{5\sqrt{15}}{9}$$

Note: $f\left(0, 0, \sqrt{5}\right) = 0$ does not yield a maximum.

24. Minimize the square of the distance $f(x, y) = x^2 + (y - 10)^2$ subject to the constraint $(x - 4)^2 + y^2 = 4$.

$$\left.\begin{aligned} 2x &= 2(x-4)\lambda \\ 2(y-10) &= 2y\lambda \end{aligned}\right\} \frac{x}{x-4} = \frac{y-10}{y} \implies y = -\frac{5}{2}x + 10$$

$$(x-4)^2 + y^2 = 4 \implies (x^2 - 8x + 16) + \left(\frac{25}{4}x^2 - 50x + 100\right) = 4$$

$$\frac{29}{4}x^2 - 58x + 112 = 0$$

Using a graphing utility, we obtain $x \approx 3.2572$ and $x \approx 4.7428$ or, by the Quadratic Formula,

$$x = \frac{58 \pm \sqrt{58^2 - 4(29/4)(112)}}{2(29/4)} = \frac{58 \pm 2\sqrt{29}}{29/2} = 4 \pm \frac{4\sqrt{29}}{29}.$$

Using the smaller value, we have $x = 4\left(1 - \frac{\sqrt{29}}{29}\right)$ and $y = \frac{10\sqrt{29}}{29} \approx 1.8570.$

The point on the circle is $\left[4\left(1 - \frac{\sqrt{29}}{29}\right), \frac{10\sqrt{29}}{29}\right]$

and the desired distance is $d = \sqrt{16\left(1 - \frac{\sqrt{29}}{29}\right)^2 + \left(\frac{10\sqrt{29}}{29} - 10\right)^2} \approx 8.77.$

The larger x-value does not yield a minimum.

26. Minimize the square of the distance

$$f(x, y, z) = (x - 4)^2 + y^2 + z^2$$

subject to the constraint $\sqrt{x^2 + y^2} - z = 0$.

$$\left.\begin{aligned} 2(x-4) &= \frac{x}{\sqrt{x^2+y^2}}\lambda = \frac{x}{z}\lambda \\ 2y &= \frac{y}{\sqrt{x^2+y^2}}\lambda = \frac{y}{z}\lambda \\ 2z &= -\lambda \end{aligned}\right\} \begin{aligned} 2(x-4) &= -2x \\ 2y &= -2y \end{aligned}$$

$$\sqrt{x^2 + y^2} - z = 0, \; x = 2, \; y = 0, \; z = 2$$

The point on the plane is $(2, 0, 2)$ and the desired distance is

$$d = \sqrt{(2-4)^2 + 0^2 + 2^2} = 2\sqrt{2}.$$

28. Maximize $f(x, y, z) = z$ subject to the constraints $x^2 + y^2 - z^2 = 0$ and $x + 2z = 4$.

$$0 = 2x\lambda + \mu$$

$$0 = 2y\lambda \implies y = 0$$

$$1 = -2z\lambda + 2\mu$$

$$x^2 + y^2 - z^2 = 0$$

$$x + 2z = 4 \implies x = 4 - 2z$$

$$(4 - 2z)^2 + 0^2 - z^2 = 0$$

$$3z^2 - 16z + 16 = 0$$

$$(3z - 4)(z - 4) = 0$$

$$z = \tfrac{4}{3} \text{ or } z = 4$$

The maximum value of f occurs when $z = 4$ at the point of $(-4, 0, 4)$.

30. See explanation at the bottom of page 922.

32. Maximize $V(x, y, z) = xyz$ subject to the constraint $1.5xy + 2xz + 2yz = C$.

$$\left.\begin{aligned} yz &= (1.5y + 2z)\lambda \\ xz &= (1.5x + 2z)\lambda \\ xy &= (2x + 2y)\lambda \end{aligned}\right\} x = y \text{ and } z = \frac{3}{4}x$$

$$1.5xy + 2xz + 2yz = C \implies 1.5x^2 + \frac{3}{2}x^2 + \frac{3}{2}x^2 = C$$

$$x = \frac{\sqrt{2C}}{3}$$

Volume is maximum when

$$x = y = \frac{\sqrt{2C}}{3} \quad \text{and} \quad z = \frac{\sqrt{2C}}{4}.$$

34. Minimize $A(\pi, r) = 2\pi rh + 2\pi r^2$ subject to the constraint $\pi r^2 h = V_0$.

$$\left.\begin{aligned} 2\pi h + 4\pi r &= 2\pi rh\lambda \\ 2\pi r &= \pi r^2\lambda \end{aligned}\right\} h = 2r$$

$$\pi r^2 h = V_0 \implies 2\pi r^3 = V_0$$

Dimensions: $r = \sqrt[3]{\frac{V_0}{2\pi}}$ and $h = 2\sqrt[3]{\frac{V_0}{2\pi}}$

36. (a) Maximize $P(x, y, z) = xyz$ subject to the constraint

$$x + y + z = S.$$

$$\left.\begin{aligned} yz &= \lambda \\ xz &= \lambda \\ xy &= \lambda \end{aligned}\right\} x = y = z$$

$$x + y + z = S \implies x = y = z = \frac{S}{3}$$

Therefore,

$$xyz \le \left(\frac{S}{3}\right)\left(\frac{S}{3}\right)\left(\frac{S}{3}\right), \quad x, y, z > 0$$

$$xyz \le \frac{S^3}{27}$$

$$\sqrt[3]{xyz} \le \frac{S}{3}$$

$$\sqrt[3]{xyz} \le \frac{x + y + z}{3}.$$

(b) Maximize $P = x_1x_2x_3 \ldots x_n$ subject to the constraint

$$\sum_{i=1}^{n} x_i = S.$$

$$\left.\begin{aligned} x_2x_3 \ldots x_n &= \lambda \\ x_1x_3 \ldots x_n &= \lambda \\ x_1x_2 \ldots x_n &= \lambda \\ &\vdots \\ x_1x_2x_3 \ldots x_{n-1} &= \lambda \end{aligned}\right\} x_1 = x_2 = x_3 = \cdots = x_n$$

$$\sum_{i=1}^{n} x_i = S \implies x_1 = x_2 = x_3 = \cdots = x_n = \frac{S}{n}$$

Therefore,

$$x_1x_2x_3 \ldots x_n \le \left(\frac{S}{n}\right)\left(\frac{S}{n}\right)\left(\frac{S}{n}\right) \cdots \left(\frac{S}{n}\right), \quad x_i \ge 0$$

$$x_1x_2x_3 \ldots x_n \le \left(\frac{S}{n}\right)^n$$

$$\sqrt[n]{x_1x_2x_3 \ldots x_n} \le \frac{S}{n}$$

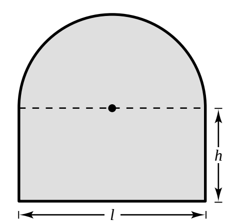
$$\sqrt[n]{x_1x_2x_3 \ldots x_n} \le \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}.$$

38. Case 1: Minimize $P(l, h) = 2h + l + \left(\frac{\pi l}{2}\right)$ subject to the constraint $lh + \left(\frac{\pi l^2}{8}\right) = A$.

$$1 + \frac{\pi}{2} = \left(h + \frac{\pi l}{4}\right)\lambda$$

$$2 = l\lambda \implies \lambda = \frac{2}{l}, 1 + \frac{\pi}{2} = \frac{2h}{l} + \frac{\pi}{2}$$

$$l = 2h$$



Case 2: Minimize $A(l, h) = lh + \left(\frac{\pi l^2}{8}\right)$ subject to the constraint $2h + l + \left(\frac{\pi l}{2}\right) = P$.

$$h + \frac{\pi l}{4} = \left(\perp + \frac{\pi}{2}\right)\lambda$$

$$l = 2\lambda \implies \lambda = \frac{l}{2}, h + \frac{\pi l}{4} = \frac{l}{2} + \frac{\pi l}{4}$$

$$h = \frac{l}{2} \text{ or } l = 2h$$

40. Maximize $T(x, y, z) = 100 + x^2 + y^2$ subject to the constraints $x^2 + y^2 + z^2 = 50$ and $x - z = 0$.

$$\left.\begin{aligned} 2x &= 2x\lambda + \mu \\ 2y &= 2y\lambda \\ 0 &= 2z\lambda - \mu \end{aligned}\right\}$$

If $y \neq 0$, then $\lambda = 1$ and $\mu = 0$, $z = 0$.

Thus, $x = z = 0$ and $y = \sqrt{50}$.

$$T\left(0, \sqrt{50}, 0\right) = 100 + 50 = 150$$

If $y = 0$, then $x^2 + z^2 = 2x^2 = 50$ and $x = z = \sqrt{50}/2$.

$$T\left(\frac{\sqrt{50}}{2}, 0, \frac{\sqrt{50}}{2}\right) = 100 + \frac{50}{4} = 112.5$$

Therefore, the maximum temperature is 150.

42. Maximize $P(x, y) = 100x^{0.4}y^{0.6}$

Constraint: $48x + 36y = 100{,}000$.

$$40x^{-0.6}y^{0.6} = 48\lambda \implies \left(\frac{y}{x}\right)^{0.6} = \frac{48\lambda}{40}$$

$$60x^{0.4}y^{-0.4} = 36\lambda \implies \left(\frac{x}{y}\right)^{0.4} = \frac{36\lambda}{60}$$

$$\left(\frac{y}{x}\right)^{0.6}\left(\frac{y}{x}\right)^{0.4} = \left(\frac{48\lambda}{40}\right)\left(\frac{60}{36\lambda}\right)$$

$$\frac{y}{x} = 2 \implies y = 2x$$

$$48x + 36y(2x) = 100{,}000 \implies x = \frac{2500}{3}, y = \frac{5000}{3}$$

$$P\left(\frac{2500}{3}, \frac{5000}{3}\right) \approx \$126{,}309.71.$$

44. Minimize $C(x, y) = 48x + 36y$ subject to the constraint $100x^{0.6}y^{0.4} = 20{,}000$.

$$48 = 60x^{-0.4}y^{0.4}\lambda \implies \left(\frac{y}{x}\right)^{0.4} = \frac{48}{60\lambda}$$

$$36 = 40x^{0.6}y^{-0.6}\lambda \implies \left(\frac{x}{y}\right)^{0.6} = \frac{36}{40\lambda}$$

$$\left(\frac{y}{x}\right)^{0.4}\left(\frac{y}{x}\right)^{0.6} = \left(\frac{48}{60\lambda}\right)\left(\frac{40\lambda}{36}\right)$$

$$\frac{y}{x} = \frac{8}{9} \implies y = \frac{8}{9}x$$

$$100x^{0.6}y^{0.4} = 20{,}000 \implies x^{0.6}\left(\frac{8}{9}x\right)^{0.4} = 200$$

$$x = \frac{200}{(8/9)^{0.4}} \approx 209.65$$

$$y = \frac{8}{9}\left[\frac{200}{(8/9)^{0.4}}\right] \approx 186.35$$

Therefore, $C(209.65, 186.35) = \$16{,}771.94$.

46. $f(x, y) = ax + by,\ x, y > 0$

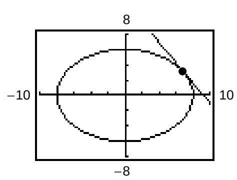
Constraint: $\dfrac{x^2}{64} + \dfrac{y^2}{36} = 1$

(a) Level curves of $f(x, y) = 4x + 3y$ are lines of form

$$y = -\frac{4}{3}x + C.$$

Using $y = -\frac{4}{3}x + 12.3$, you obtain

$$x \approx 7,\ y \approx 3, \quad \text{and} \quad f(7, 3) = 28 + 9 = 37.$$



Constraint is an ellipse.

(b) Level curves of $f(x, y) = 4x + 9y$ are lines of form

$$y = -\frac{4}{9}x + C.$$

Using $y = -\frac{4}{9}x + 7$, you obtain

$$x \approx 4,\ y \approx 5.2,\ \text{and}\ f(4, 5.2) = 62.8.$$

Review Exercises for Chapter 12

2. Yes, it is the graph of a function.

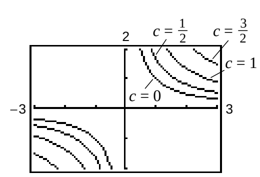
4. $f(x, y) = \ln xy$

The level curves are of the form

$$c = \ln xy$$

$$e^c = xy.$$

The level curves are hyperbolas.

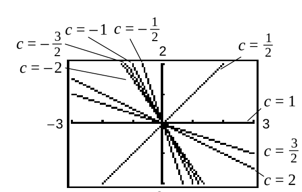


6. $f(x, y) = \dfrac{x}{x + y}$

The level curves are of the form

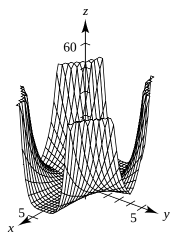
$$c = \frac{x}{x + y}$$

$$y = \left(\frac{1 - c}{c}\right)x.$$



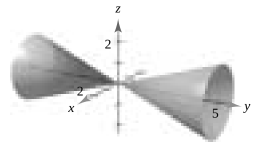
The level curves are passing through the origin with slope $\dfrac{1 - c}{c}$.

8. $g(x, y) = |y|^{1+|x|}$



10. $f(x, y, z) = 9x^2 - y^2 + 9z^2 = 0$

Elliptic cone



12. $\lim_{(x, y)\to(1, 1)} \frac{xy}{x^2 - y^2}$

Does not exist

Continuous except when $y = \pm x$.

14. $\lim_{(x, y)\to(0, 0)} \frac{y + xe^{-y^2}}{1 + x^2} = \frac{0 + 0}{1 + 0} = 0$

Continuous everywhere

16. $f(x, y) = \frac{xy}{x + y}$

$$f_x = \frac{y(x + y) - xy}{(x + y)^2} = \frac{y^2}{(x + y)^2}$$

$$f_y = \frac{x^2}{(x + y)^2}$$

18. $z = \ln(x^2 + y^2 + 1)$

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2 + 1}$$

$$\frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2 + 1}$$

20. $w = \sqrt{x^2 + y^2 + z^2}$

$$\frac{\partial w}{\partial x} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial w}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial w}{\partial z} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

22. $f(x, y, z) = \frac{1}{\sqrt{1 - x^2 - y^2 - z^2}}$

$$f_x = -\frac{1}{2}(1 - x^2 - y^2 - z^2)^{-3/2}(-2x)$$

$$= \frac{x}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

$$f_y = \frac{y}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

$$f_z = \frac{z}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

24. $u(x, t) = c(\sin akx) \cos kt$

$$\frac{\partial u}{\partial x} = akc(\cos akx) \cos kt$$

$$\frac{\partial u}{\partial t} = -kc(\sin akx) \sin kt$$

26. $z = x^2 \ln(y + 1)$

$\frac{\partial z}{\partial x} = 2x \ln(y + 1)$. At $(2, 0, 0)$, $\frac{\partial z}{\partial x} = 0$.

Slope in x-direction.

$\frac{\partial z}{\partial y} = \frac{x^2}{1 + y}$. At $(2, 0, 0)$, $\frac{\partial z}{\partial y} = 4$.

Slope in y-direction.

28. $h(x, y) = \frac{x}{x + y}$

$$h_x = \frac{y}{(x + y)^2}$$

$$h_y = \frac{-x}{(x + y)^2}$$

$$h_{xx} = \frac{-2y}{(x + y)^3}$$

$$h_{yy} = \frac{2x}{(x + y)^3}$$

$$h_{xy} = \frac{(x + y)^2 - 2y(x + y)}{(x + y)^4} = \frac{x - y}{(x + y)^3}$$

$$h_{yx} = \frac{-(x + y)^2 + 2y(x + y)}{(x + y)^4} = \frac{x - y}{(x + y)^3}$$

30. $g(x, y) = \cos(x - 2y)$

$$g_x = -\sin(x - 2y)$$

$$g_y = 2 \sin(x - 2y)$$

$$g_{xx} = -\cos(x - 2y)$$

$$g_{yy} = -4 \cos(x - 2y)$$

$$g_{xy} = 2 \cos(x - 2y)$$

$$g_{yx} = 2 \cos(x - 2y)$$

32. $z = x^3 - 3xy^2$

$$\frac{\partial z}{\partial x} = 3x^2 - 3y^2$$

$$\frac{\partial^2 z}{\partial x^2} = 6x$$

$$\frac{\partial z}{\partial y} = -6xy$$

$$\frac{\partial^2 z}{\partial y^2} = -6x$$

Therefore, $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$.

34. $z = e^x \sin y$

$$\frac{\partial z}{\partial x} = e^x \sin y$$

$$\frac{\partial^2 z}{\partial x^2} = e^x \sin y$$

$$\frac{\partial z}{\partial y} = e^x \cos y$$

$$\frac{\partial^2 z}{\partial y^2} = -e^x \sin y$$

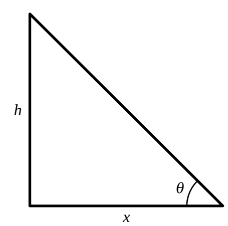
Therefore, $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$.

36. $z = \dfrac{xy}{\sqrt{x^2 + y^2}}$

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

$$= \left[\frac{\sqrt{x^2+y^2}\,y - xy\left(x/\sqrt{x^2+y^2}\right)}{x^2+y^2}\right]dx + \left[\frac{\sqrt{x^2+y^2}\,x - xy\left(y/\sqrt{x^2+y^2}\right)}{x^2+y^2}\right]dy = \frac{y^3}{(x^2+y^2)^{3/2}}\,dx + \frac{x^3}{(x^2+y^2)^{3/2}}\,dy$$

38. From the accompanying figure we observe



$$\tan\theta = \frac{h}{x} \quad \text{or} \quad h = x\tan\theta$$

$$dh = \frac{\partial h}{\partial x}\,dx + \frac{\partial h}{\partial \theta}\,d\theta = \tan\theta\,dx + x\sec^2\theta\,d\theta.$$

Letting $x = 100$, $dx = \pm\dfrac{1}{2}$, $\theta = \dfrac{11\pi}{60}$, and $d\theta = \pm\dfrac{\pi}{180}$.

(Note that we express the measurement of the angle in radians.) The maximum error is approximately

$$dh = \tan\left(\frac{11\pi}{60}\right)\left(\pm\frac{1}{2}\right) + 100\sec^2\left(\frac{11\pi}{60}\right)\left(\pm\frac{\pi}{180}\right) \approx \pm 0.3247 \pm 2.4814 \approx \pm 2.81 \text{ feet.}$$

40. $A = \pi r\sqrt{r^2 + h^2}$

$$dA = \left(\pi\sqrt{r^2+h^2} + \frac{\pi r^2}{\sqrt{r^2+h^2}}\right)dr + \frac{\pi rh}{\sqrt{r^2+h^2}}\,dh$$

$$= \frac{\pi(2r^2+h^2)}{\sqrt{r^2+h^2}}\,dr + \frac{\pi rh}{\sqrt{r^2+h^2}}\,dh = \frac{\pi(8+25)}{\sqrt{29}}\left(\pm\frac{1}{8}\right) + \frac{10\pi}{\sqrt{29}}\left(\pm\frac{1}{8}\right) = \pm\frac{43\pi}{8\sqrt{29}}$$

42. $u = y^2 - x$, $x = \cos t$, $y = \sin t$

Chain Rule: $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial t}$

$$= -1(-\sin t) + 2y(\cos t)$$

$$= \sin t + 2(\sin t)\cos t$$

$$= \sin t(1 + 2\cos t)$$

Substitution: $u = \sin^2 t - \cos t$

$$\frac{du}{dt} = 2\sin t\cos t + \sin t = \sin t(1 + 2\cos t)$$

44. $w = \dfrac{xy}{z}, x = 2r + t, y = rt, z = 2r - t$

Chain Rule: $\dfrac{\partial w}{\partial r} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial r} + \dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial r}$

$$= \frac{y}{z}(2) + \frac{x}{z}(t) - \frac{xy}{z^2}(2)$$

$$= \frac{2rt}{2r - t} + \frac{(2r + t)t}{2r - t} - \frac{2(2r + t)(rt)}{(2r - t)^2}$$

$$= \frac{4r^2t - 4rt^2 - t^3}{(2r - t)^2}$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

$$= \frac{y}{z}(1) + \frac{x}{z}(r) = \frac{xy}{z^2}(-1)$$

$$= \frac{4r^2t - rt^2 + 4r^3}{(2r - t)^2}$$

Substitution: $w = \dfrac{xy}{z} = \dfrac{(2r + t)(rt)}{2r - t} = \dfrac{2r^2t + rt^2}{2r - t}$

$$\frac{\partial w}{\partial r} = \frac{4r^2t - 4rt^2 - t^3}{(2r - t)^2}$$

$$\frac{\partial w}{\partial t} = \frac{4r^2t - rt^2 - 4r^3}{(2r - t)^2}$$

46. $xz^2 - y\sin z = 0$

$$2xz\frac{\partial z}{\partial x} + z^2 - y\cos z\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{z^2}{y\cos z - 2xz}$$

$$2xz\frac{\partial z}{\partial y} - y\cos z\frac{\partial z}{\partial y} - \sin z = 0$$

$$\frac{\partial z}{\partial y} = \frac{\sin z}{2xz - y\cos z}$$

48. $f(x, y) = \dfrac{1}{4}y^2 - x^2$

$$\nabla f = -2x\mathbf{i} + \frac{1}{2}y\mathbf{j}$$

$$\nabla f(1, 4) = -2\mathbf{i} + 2\mathbf{j}$$

$$\mathbf{u} = \frac{1}{\sqrt{5}}\mathbf{v} = \frac{2\sqrt{5}}{5}\mathbf{i} + \frac{\sqrt{5}}{5}\mathbf{j}$$

$$D_{\mathbf{u}}f(1, 4) = \nabla f(1, 4)\cdot\mathbf{u} = -\frac{4\sqrt{5}}{5} + \frac{2\sqrt{5}}{5} = -\frac{2\sqrt{5}}{5}$$

50. $w = 6x^2 + 3xy - 4y^2z$

$$\nabla w = (12x + 3y)\mathbf{i} + (3x - 8yz)\mathbf{j} + (-4y^2)\mathbf{k}$$

$$\nabla w(1, 0, 1) = 12\mathbf{i} + 3\mathbf{j}$$

$$\mathbf{u} = \frac{1}{\sqrt{3}}\mathbf{v} = \frac{\sqrt{3}}{3}\mathbf{i} + \frac{\sqrt{3}}{3}\mathbf{j} - \frac{\sqrt{3}}{3}\mathbf{k}$$

$$D_{\mathbf{u}}w(1, 0, 1) = \nabla w(1, 0, 1)\cdot\mathbf{u}$$

$$= 4\sqrt{3} + \sqrt{3} + 0 = 5\sqrt{3}$$

52. $z = \dfrac{x^2}{x - y}$

$$\nabla z = \frac{x^2 - 2xy}{(x - y)^2}\mathbf{i} + \frac{x^2}{(x - y)^2}\mathbf{j}$$

$$\nabla z(2, 1) = 4\mathbf{j}$$

$$\|\nabla z(2, 1)\| = 4$$

54. $z = x^2y$

$$\nabla z = 2xy\mathbf{i} + x^2\mathbf{j}$$

$$\nabla z(2, 1) = 4\mathbf{i} + 4\mathbf{j}$$

$$\|\nabla z(2, 1)\| = 4\sqrt{2}$$

56. $4y\sin x - y^2 = 3$

$$f(x, y) = 4y\sin x - y^2$$

$$\nabla f(x, y) = 4y\cos x\mathbf{i} + (4\sin x - 2y)\mathbf{j}$$

$$\nabla f\left(\frac{\pi}{2}, 1\right) = 2\mathbf{j}$$

Normal vector: $\mathbf{j}$

58. $F(x, y, z) = y^2 + z^2 - 25 = 0$

$$\nabla F = 2y\mathbf{j} + 2z\mathbf{k}$$

$$\nabla F(2, 3, 4) = 6\mathbf{j} + 8\mathbf{k} = 2(3\mathbf{j} + 4\mathbf{k})$$

Therefore, the equation of the tangent plane is

$$3(y - 3) + 4(z - 4) = 0 \quad \text{or} \quad 3y + 4z = 25,$$

and the equation of the normal line is

$$x = 2, \ \frac{y - 3}{3} = \frac{z - 4}{4}.$$

60. $F(x, y, z) = x^2 + y^2 + z^2 - 9 = 0$

$\nabla F = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$

$\nabla F(1, 2, 2) = 2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k} = 2(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$

Therefore, the equation of the tangent plane is

$(x - 1) + 2(y - 2) + 2(z - 2) = 0$ or

$x + 2y + 2z = 9,$

and the equation of the normal line is

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-2}{2}.$$

62. $F(x, y, z) = y^2 + z - 25 = 0$

$G(x, y, z) = x - y = 0$

$\nabla F = 2y\mathbf{i} + \mathbf{k}$

$\nabla G = \mathbf{i} - \mathbf{j}$

$\nabla F(4, 4, 9) = 8\mathbf{i} + \mathbf{k}$

$$\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & 0 & 1 \\ 1 & -1 & 0 \end{vmatrix} = \mathbf{i} + \mathbf{j} - 8\mathbf{k}$$

Therefore, the equation of the tangent line is

$$\frac{x-4}{1} = \frac{y-4}{1} = \frac{z-9}{-8}.$$

64. (a) $f(x, y) = \cos x + \sin y, \quad f(0, 0) = 1$

$f_x = -\sin x, \quad f_x(0, 0) = 0$

$f_y = \cos y, \quad f_y(0, 0) = 1$

$P_1(x, y) = 1 + y$

(b) $f_{xx} = -\cos x, \quad f_{xx}(0, 0) = -1$

$f_{yy} = -\sin y, \quad f_{yy}(0, 0) = 0$

$f_{xy} = 0, \quad f_{xy}(0, 0) = 0$

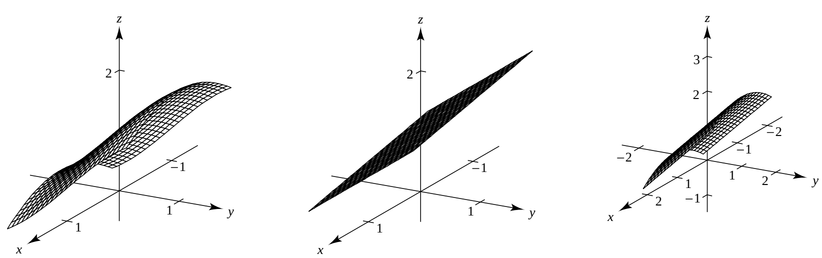
$P_2(x, y) = 1 + y - \frac{1}{2}x^2$

(c) If $y = 0$, you obtain the 2nd degree Taylor polynomial for $\cos x$.

(d)

x	y	$f(x, y)$	$P_1(x, y)$	$P_2(x, y)$
0	0	1.0	1.0	1.0
0	0.1	1.0998	1.1	1.1
0.2	0.1	1.0799	1.1	1.095
0.5	0.3	1.1731	1.3	1.175
1	0.5	1.0197	1.5	1.0

(e)



The accuracy lessens as the distance from $(0, 0)$ increases.

66. $f(x, y) = 2x^2 + 6xy + 9y^2 + 8x + 14$

$f_x = 4x + 6y + 8 = 0$

$f_y = 6x + 18y = 0, \; x = -3y$

$4(-3y) + 6y = -8 \implies y = \frac{4}{3}, \; x = -4$

$f_{xx} = 4$

$f_{yy} = 18$

$f_{xy} = 6$

$f_{xx}f_{yy} - (f_{xy})^2 = 4(18) - (6)^2 = 36 > 0$ Therefore, $\left(-4, \frac{4}{3}, -2\right)$ is a relative minimum.

68. $z = 50(x + y) - (0.1x^3 + 20x + 150) - (0.05y^3 + 20.6y + 125)$

$z_x = 50 - 0.3x^2 - 20 = 0, \; x = \pm 10$

$z_y = 50 - 0.15y^2 - 20.6 = 0, \; y = \pm 14$

Critical Points: $(10, 14), \; (10, -14), \; (-10, 14), \; (-10, -14)$

$z_{xx} = -0.6x, \; z_{yy} = -0.3y, \; z_{xy} = 0$

At $(10, 14)$, $z_{xx}z_{yy} - (z_{xy})^2 = (-6)(-4.2) - 0^2 > 0, \; z_{xx} < 0.$

$(10, 14, 199.4)$ is a relative maximum.

At $(10, -14)$, $z_{xx}z_{yy} - (z_{xy})^2 = (-6)(4.2) - 0^2 < 0.$

$(10, -14, -349.4)$ is a saddle point.

At $(-10, 14)$, $z_{xx}z_{yy} - (z_{xy})^2 = (6)(-4.2) - 0^2 < 0.$

$(-10, 14, -200.6)$ is a saddle point.

At $(-10, -14)$, $z_{xx}z_{yy} - (x_{xy})^2 = (6)(4.2) - 0^2 > 0, \; z_{xx} < 0.$

$(-10, -14, -749.4)$ is a relative minimum.

70. The level curves indicate that there is a relative extremum at *A*, the center of the ellipse in the second quadrant, and that there is a saddle point at *B*, the origin.

72. Minimize $C(x_1, x_2) = 0.25x_1{}^2 + 10x_1 + 0.15x_2{}^2 + 12x_2$ subject to the constraint $x_1 + x_2 = 1000$.

$$\left.\begin{aligned} 0.50x_1 + 10 &= \lambda \\ 0.30x_2 + 12 &= \lambda \end{aligned}\right\} 5x_1 - 3x_2 = 20$$

$$x_1 + x_2 = 1000 \implies \begin{aligned} 3x_1 + 3x_2 &= 3000 \\ 5x_1 - 3x_2 &= 20 \\ \hline 8x_1 \quad\quad &= 3020 \\ x_1 &= 377.5 \\ x_2 &= 622.5 \end{aligned}$$

$C(377.5, 622.5) = 104{,}997.50$

74. Minimize the square of the distance:

$$f(x, y, z) = (x - 2)^2 + (y - 2)^2 + (x^2 + y^2 - 0)^2.$$

$$\left.\begin{aligned} f_x &= 2(x - 2) + 2(x^2 + y^2)2x = 0 \\ f_y &= 2(y - 2) + 2(x^2 + y^2)2y = 0 \end{aligned}\right\} \begin{aligned} x - 2 + 2x^3 + 2xy^2 &= 0 \\ y - 2 + 2y^3 + 2x^2y &= 0 \end{aligned}$$

Clearly $x = y$ and hence: $4x^3 + x - 2 = 0$. Using a computer algebra system, $x \approx 0.6894$.
Thus, $(\text{distance})^2 = (0.6894 - 2)^2 + (0.6894 - 2)^2 + [2(0.6894)^2]^2 \approx 4.3389$.
distance ≈ 2.08

76. (a) $(25, 28), \; (50, 38), \; (75, 54), \; (100, 75), \; (125, 102)$

$\sum x_i = 375, \quad \sum y_i = 297, \quad \sum x_i^2 = 34{,}375, \quad \sum x_i^3 = 3{,}515{,}625$

$\sum x_i^4 = 382{,}421{,}875, \quad \sum x_i y_i = 26{,}900, \quad \sum x_i^2 y_i = 2{,}760{,}000,$

$$\begin{aligned} 382{,}421{,}875a + 3{,}515{,}625b + 34{,}375c &= 2{,}760{,}000 \\ 3{,}515{,}625a + 34{,}375b + 375c &= 26{,}900 \\ 34{,}375a + 375b + 5c &= 297 \end{aligned}$$

$a \approx 0.0045, \; b \approx 0.0717, \; c \approx 23.2914, \; y \approx 0.0045x^2 + 0.0717x + 23.2914$

(b) When $x = 80$ km/hr, $y \approx 57.8$ km.