44. Minimize $C(x, y) = 48x + 36y$ subject to the constraint $100x^{0.6}y^{0.4} = 20{,}000$.

$$48 = 60x^{-0.4}y^{0.4}\lambda \implies \left(\frac{y}{x}\right)^{0.4} = \frac{48}{60\lambda}$$

$$36 = 40x^{0.6}y^{-0.6}\lambda \implies \left(\frac{x}{y}\right)^{0.6} = \frac{36}{40\lambda}$$

$$\left(\frac{y}{x}\right)^{0.4}\left(\frac{y}{x}\right)^{0.6} = \left(\frac{48}{60\lambda}\right)\left(\frac{40\lambda}{36}\right)$$

$$\frac{y}{x} = \frac{8}{9} \implies y = \frac{8}{9}x$$

$$100x^{0.6}y^{0.4} = 20{,}000 \implies x^{0.6}\left(\frac{8}{9}x\right)^{0.4} = 200$$

$$x = \frac{200}{(8/9)^{0.4}} \approx 209.65$$

$$y = \frac{8}{9}\left[\frac{200}{(8/9)^{0.4}}\right] \approx 186.35$$

Therefore, $C(209.65, 186.35) = \$16{,}771.94$.

46. $f(x, y) = ax + by,\ x, y > 0$

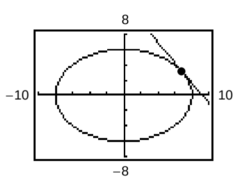
Constraint: $\dfrac{x^2}{64} + \dfrac{y^2}{36} = 1$

(a) Level curves of $f(x, y) = 4x + 3y$ are lines of form

$$y = -\frac{4}{3}x + C.$$

Using $y = -\dfrac{4}{3}x + 12.3$, you obtain

$$x \approx 7,\ y \approx 3, \quad \text{and} \quad f(7, 3) = 28 + 9 = 37.$$



Constraint is an ellipse.

(b) Level curves of $f(x, y) = 4x + 9y$ are lines of form

$$y = -\frac{4}{9}x + C.$$

Using $y = -\dfrac{4}{9}x + 7$, you obtain

$$x \approx 4,\ y \approx 5.2,\ \text{and}\ f(4, 5.2) = 62.8.$$

Review Exercises for Chapter 12

2. Yes, it is the graph of a function.

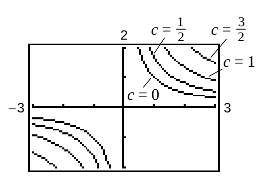
4. $f(x, y) = \ln xy$

The level curves are of the form

$$c = \ln xy$$

$$e^c = xy.$$

The level curves are hyperbolas.

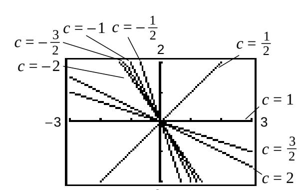


6. $f(x, y) = \dfrac{x}{x + y}$

The level curves are of the form

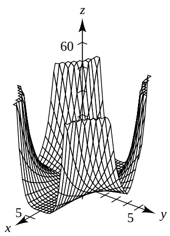
$$c = \frac{x}{x + y}$$

$$y = \left(\frac{1 - c}{c}\right)x.$$



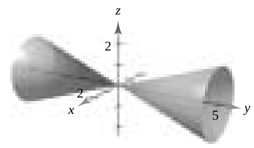
The level curves are passing through the origin with slope $\dfrac{1 - c}{c}$.

8. $g(x, y) = |y|^{1+|x|}$



10. $f(x, y, z) = 9x^2 - y^2 + 9z^2 = 0$

Elliptic cone



12. $\lim_{(x, y)\to(1, 1)} \dfrac{xy}{x^2 - y^2}$

Does not exist

Continuous except when $y = \pm x$.

14. $\lim_{(x, y)\to(0, 0)} \dfrac{y + xe^{-y^2}}{1 + x^2} = \dfrac{0 + 0}{1 + 0} = 0$

Continuous everywhere

16. $f(x, y) = \dfrac{xy}{x + y}$

$$f_x = \frac{y(x + y) - xy}{(x + y)^2} = \frac{y^2}{(x + y)^2}$$

$$f_y = \frac{x^2}{(x + y)^2}$$

18. $z = \ln(x^2 + y^2 + 1)$

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2 + 1}$$

$$\frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2 + 1}$$

20. $w = \sqrt{x^2 + y^2 + z^2}$

$$\frac{\partial w}{\partial x} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial w}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial w}{\partial z} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

22. $f(x, y, z) = \dfrac{1}{\sqrt{1 - x^2 - y^2 - z^2}}$

$$f_x = -\frac{1}{2}(1 - x^2 - y^2 - z^2)^{-3/2}(-2x)$$

$$= \frac{x}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

$$f_y = \frac{y}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

$$f_z = \frac{z}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

24. $u(x, t) = c(\sin akx) \cos kt$

$$\frac{\partial u}{\partial x} = akc(\cos akx) \cos kt$$

$$\frac{\partial u}{\partial t} = -kc(\sin akx) \sin kt$$

26. $z = x^2 \ln(y + 1)$

$\dfrac{\partial z}{\partial x} = 2x \ln(y + 1)$. At $(2, 0, 0)$, $\dfrac{\partial z}{\partial x} = 0$.

Slope in x-direction.

$\dfrac{\partial z}{\partial y} = \dfrac{x^2}{1 + y}$. At $(2, 0, 0)$, $\dfrac{\partial z}{\partial y} = 4$.

Slope in y-direction.

28. $h(x, y) = \dfrac{x}{x + y}$

$$h_x = \frac{y}{(x + y)^2}$$

$$h_y = \frac{-x}{(x + y)^2}$$

$$h_{xx} = \frac{-2y}{(x + y)^3}$$

$$h_{yy} = \frac{2x}{(x + y)^3}$$

$$h_{xy} = \frac{(x + y)^2 - 2y(x + y)}{(x + y)^4} = \frac{x - y}{(x + y)^3}$$

$$h_{yx} = \frac{-(x + y)^2 + 2y(x + y)}{(x + y)^4} = \frac{x - y}{(x + y)^3}$$

30. $g(x, y) = \cos(x - 2y)$

$$g_x = -\sin(x - 2y)$$

$$g_y = 2 \sin(x - 2y)$$

$$g_{xx} = -\cos(x - 2y)$$

$$g_{yy} = -4 \cos(x - 2y)$$

$$g_{xy} = 2 \cos(x - 2y)$$

$$g_{yx} = 2 \cos(x - 2y)$$

32. $z = x^3 - 3xy^2$

$$\frac{\partial z}{\partial x} = 3x^2 - 3y^2$$

$$\frac{\partial^2 z}{\partial x^2} = 6x$$

$$\frac{\partial z}{\partial y} = -6xy$$

$$\frac{\partial^2 z}{\partial y^2} = -6x$$

Therefore, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$

34. $z = e^x \sin y$

$$\frac{\partial z}{\partial x} = e^x \sin y$$

$$\frac{\partial^2 z}{\partial x^2} = e^x \sin y$$

$$\frac{\partial z}{\partial y} = e^x \cos y$$

$$\frac{\partial^2 z}{\partial y^2} = -e^x \sin y$$

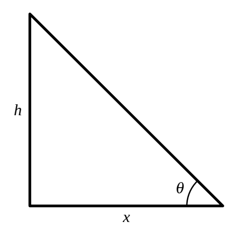
Therefore, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$

36. $z = \frac{xy}{\sqrt{x^2 + y^2}}$

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

$$= \left[\frac{\sqrt{x^2 + y^2}\,y - xy\left(x/\sqrt{x^2 + y^2}\right)}{x^2 + y^2}\right]dx + \left[\frac{\sqrt{x^2 + y^2}\,x - xy\left(y/\sqrt{x^2 + y^2}\right)}{x^2 + y^2}\right]dy = \frac{y^3}{(x^2 + y^2)^{3/2}}\,dx + \frac{x^3}{(x^2 + y^2)^{3/2}}\,dy$$

38. From the accompanying figure we observe



$$\tan\theta = \frac{h}{x} \quad \text{or} \quad h = x\tan\theta$$

$$dh = \frac{\partial h}{\partial x}\,dx + \frac{\partial h}{\partial \theta}\,d\theta = \tan\theta\,dx + x\sec^2\theta\,d\theta.$$

Letting $x = 100$, $dx = \pm\frac{1}{2}$, $\theta = \frac{11\pi}{60}$, and $d\theta = \pm\frac{\pi}{180}$.

(Note that we express the measurement of the angle in radians.) The maximum error is approximately

$$dh = \tan\left(\frac{11\pi}{60}\right)\left(\pm\frac{1}{2}\right) + 100\sec^2\left(\frac{11\pi}{60}\right)\left(\pm\frac{\pi}{180}\right) \approx \pm 0.3247 \pm 2.4814 \approx \pm 2.81 \text{ feet.}$$

40. $A = \pi r\sqrt{r^2 + h^2}$

$$dA = \left(\pi\sqrt{r^2 + h^2} + \frac{\pi r^2}{\sqrt{r^2 + h^2}}\right)dr + \frac{\pi rh}{\sqrt{r^2 + h^2}}\,dh$$

$$= \frac{\pi(2r^2 + h^2)}{\sqrt{r^2 + h^2}}\,dr + \frac{\pi rh}{\sqrt{r^2 + h^2}}\,dh = \frac{\pi(8 + 25)}{\sqrt{29}}\left(\pm\frac{1}{8}\right) + \frac{10\pi}{\sqrt{29}}\left(\pm\frac{1}{8}\right) = \pm\frac{43\pi}{8\sqrt{29}}$$

42. $u = y^2 - x$, $x = \cos t$, $y = \sin t$

Chain Rule: $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$

$$= -1(-\sin t) + 2y(\cos t)$$

$$= \sin t + 2(\sin t)\cos t$$

$$= \sin t(1 + 2\cos t)$$

Substitution: $u = \sin^2 t - \cos t$

$$\frac{du}{dt} = 2\sin t\cos t + \sin t = \sin t(1 + 2\cos t)$$

44. $w = \dfrac{xy}{z}, x = 2r + t, y = rt, z = 2r - t$

Chain Rule: $\dfrac{\partial w}{\partial r} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial r} + \dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial r}$

$$= \frac{y}{z}(2) + \frac{x}{z}(t) - \frac{xy}{z^2}(2)$$

$$= \frac{2rt}{2r - t} + \frac{(2r + t)t}{2r - t} - \frac{2(2r + t)(rt)}{(2r - t)^2}$$

$$= \frac{4r^2t - 4rt^2 - t^3}{(2r - t)^2}$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

$$= \frac{y}{z}(1) + \frac{x}{z}(r) = \frac{xy}{z^2}(-1)$$

$$= \frac{4r^2t - rt^2 + 4r^3}{(2r - t)^2}$$

Substitution: $w = \dfrac{xy}{z} = \dfrac{(2r + t)(rt)}{2r - t} = \dfrac{2r^2t + rt^2}{2r - t}$

$$\frac{\partial w}{\partial r} = \frac{4r^2t - 4rt^2 - t^3}{(2r - t)^2}$$

$$\frac{\partial w}{\partial t} = \frac{4r^2t - rt^2 - 4r^3}{(2r - t)^2}$$

46. $xz^2 - y \sin z = 0$

$$2xz\frac{\partial z}{\partial x} + z^2 - y\cos z\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{z^2}{y\cos z - 2xz}$$

$$2xz\frac{\partial z}{\partial y} - y\cos z\frac{\partial z}{\partial y} - \sin z = 0$$

$$\frac{\partial z}{\partial y} = \frac{\sin z}{2xz - y\cos z}$$

48. $f(x, y) = \dfrac{1}{4}y^2 - x^2$

$$\nabla f = -2x\mathbf{i} + \frac{1}{2}y\mathbf{j}$$

$$\nabla f(1, 4) = -2\mathbf{i} + 2\mathbf{j}$$

$$\mathbf{u} = \frac{1}{\sqrt{5}}\mathbf{v} = \frac{2\sqrt{5}}{5}\mathbf{i} + \frac{\sqrt{5}}{5}\mathbf{j}$$

$$D_{\mathbf{u}} f(1, 4) = \nabla f(1, 4) \cdot \mathbf{u} = -\frac{4\sqrt{5}}{5} + \frac{2\sqrt{5}}{5} = -\frac{2\sqrt{5}}{5}$$

50. $w = 6x^2 + 3xy - 4y^2z$

$$\nabla w = (12x + 3y)\mathbf{i} + (3x - 8yz)\mathbf{j} + (-4y^2)\mathbf{k}$$

$$\nabla w(1, 0, 1) = 12\mathbf{i} + 3\mathbf{j}$$

$$\mathbf{u} = \frac{1}{\sqrt{3}}\mathbf{v} = \frac{\sqrt{3}}{3}\mathbf{i} + \frac{\sqrt{3}}{3}\mathbf{j} - \frac{\sqrt{3}}{3}\mathbf{k}$$

$$D_{\mathbf{u}}w(1, 0, 1) = \nabla w(1, 0, 1) \cdot \mathbf{u}$$

$$= 4\sqrt{3} + \sqrt{3} + 0 = 5\sqrt{3}$$

52. $z = \dfrac{x^2}{x - y}$

$$\nabla z = \frac{x^2 - 2xy}{(x - y)^2}\mathbf{i} + \frac{x^2}{(x - y)^2}\mathbf{j}$$

$$\nabla z(2, 1) = 4\mathbf{j}$$

$$\|\nabla z(2, 1)\| = 4$$

54. $z = x^2y$

$$\nabla z = 2xy\mathbf{i} + x^2\mathbf{j}$$

$$\nabla z(2, 1) = 4\mathbf{i} + 4\mathbf{j}$$

$$\|\nabla z(2, 1)\| = 4\sqrt{2}$$

56. $4y \sin x - y^2 = 3$

$$f(x, y) = 4y \sin x - y^2$$

$$\nabla f(x, y) = 4y\cos x\mathbf{i} + (4\sin x - 2y)\mathbf{j}$$

$$\nabla f\left(\frac{\pi}{2}, 1\right) = 2\mathbf{j}$$

Normal vector: $\mathbf{j}$

58. $F(x, y, z) = y^2 + z^2 - 25 = 0$

$$\nabla F = 2y\mathbf{j} + 2z\mathbf{k}$$

$$\nabla F(2, 3, 4) = 6\mathbf{j} + 8\mathbf{k} = 2(3\mathbf{j} + 4\mathbf{k})$$

Therefore, the equation of the tangent plane is

$$3(y - 3) + 4(z - 4) = 0 \quad \text{or} \quad 3y + 4z = 25,$$

and the equation of the normal line is

$$x = 2,\ \frac{y - 3}{3} = \frac{z - 4}{4}.$$

60. $F(x, y, z) = x^2 + y^2 + z^2 - 9 = 0$

$$\nabla F = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$$

$$\nabla F(1, 2, 2) = 2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k} = 2(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$$

Therefore, the equation of the tangent plane is

$$(x - 1) + 2(y - 2) + 2(z - 2) = 0 \quad \text{or}$$

$$x + 2y + 2z = 9,$$

and the equation of the normal line is

$$\frac{x - 1}{1} = \frac{y - 2}{2} = \frac{z - 2}{2}.$$

62. $F(x, y, z) = y^2 + z - 25 = 0$

$$G(x, y, z) = x - y = 0$$

$$\nabla F = 2y\mathbf{i} + \mathbf{k}$$

$$\nabla G = \mathbf{i} - \mathbf{j}$$

$$\nabla F(4, 4, 9) = 8\mathbf{i} + \mathbf{k}$$

$$\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & 0 & 1 \\ 1 & -1 & 0 \end{vmatrix} = \mathbf{i} + \mathbf{j} - 8\mathbf{k}$$

Therefore, the equation of the tangent line is

$$\frac{x - 4}{1} = \frac{y - 4}{1} = \frac{z - 9}{-8}.$$

64. (a) $f(x, y) = \cos x + \sin y, \qquad f(0, 0) = 1$

$f_x = -\sin x, \qquad f_x(0, 0) = 0$

$f_y = \cos y, \qquad f_y(0, 0) = 1$

$P_1(x, y) = 1 + y$

(b) $f_{xx} = -\cos x, \qquad f_{xx}(0, 0) = -1$

$f_{yy} = -\sin y, \qquad f_{yy}(0, 0) = 0$

$f_{xy} = 0, \qquad f_{xy}(0, 0) = 0$

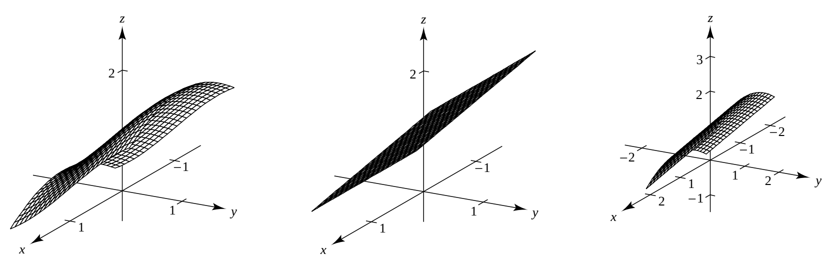
$P_2(x, y) = 1 + y - \frac{1}{2}x^2$

(c) If $y = 0$, you obtain the 2nd degree Taylor polynomial for $\cos x$.

(d)

x	y	$f(x, y)$	$P_1(x, y)$	$P_2(x, y)$
0	0	1.0	1.0	1.0
0	0.1	1.0998	1.1	1.1
0.2	0.1	1.0799	1.1	1.095
0.5	0.3	1.1731	1.3	1.175
1	0.5	1.0197	1.5	1.0

(e)



The accuracy lessens as the distance from $(0, 0)$ increases.

66. $f(x, y) = 2x^2 + 6xy + 9y^2 + 8x + 14$

$f_x = 4x + 6y + 8 = 0$

$f_y = 6x + 18y = 0, \; x = -3y$

$4(-3y) + 6y = -8 \implies y = \frac{4}{3}, \; x = -4$

$f_{xx} = 4$

$f_{yy} = 18$

$f_{xy} = 6$

$f_{xx} f_{yy} - (f_{xy})^2 = 4(18) - (6)^2 = 36 > 0$ Therefore, $\left(-4, \frac{4}{3}, -2\right)$ is a relative minimum.

68. $z = 50(x + y) - (0.1x^3 + 20x + 150) - (0.05y^3 + 20.6y + 125)$

$z_x = 50 - 0.3x^2 - 20 = 0,\ x = \pm 10$

$z_y = 50 - 0.15y^2 - 20.6 = 0,\ y = \pm 14$

Critical Points: $(10, 14),\ (10, -14),\ (-10, 14),\ (-10, -14)$

$z_{xx} = -0.6x,\ z_{yy} = -0.3y,\ z_{xy} = 0$

At $(10, 14)$, $z_{xx}z_{yy} - (z_{xy})^2 = (-6)(-4.2) - 0^2 > 0,\ z_{xx} < 0.$

$(10, 14, 199.4)$ is a relative maximum.

At $(10, -14)$, $z_{xx}z_{yy} - (z_{xy})^2 = (-6)(4.2) - 0^2 < 0.$

$(10, -14, -349.4)$ is a saddle point.

At $(-10, 14)$, $z_{xx}z_{yy} - (z_{xy})^2 = (6)(-4.2) - 0^2 < 0.$

$(-10, 14, -200.6)$ is a saddle point.

At $(-10, -14)$, $z_{xx}z_{yy} - (x_{xy})^2 = (6)(4.2) - 0^2 > 0,\ z_{xx} < 0.$

$(-10, -14, -749.4)$ is a relative minimum.

70. The level curves indicate that there is a relative extremum at A, the center of the ellipse in the second quadrant, and that there is a saddle point at B, the origin.

72. Minimize $C(x_1, x_2) = 0.25x_1{}^2 + 10x_1 + 0.15x_2{}^2 + 12x_2$ subject to the constraint $x_1 + x_2 = 1000$.

$$\left.\begin{aligned} 0.50x_1 + 10 &= \lambda \\ 0.30x_2 + 12 &= \lambda \end{aligned}\right\} 5x_1 - 3x_2 = 20$$

$$x_1 + x_2 = 1000 \implies \begin{aligned} 3x_1 + 3x_2 &= 3000 \\ 5x_1 - 3x_2 &= 20 \\ \hline 8x_1 \quad &= 3020 \\ x_1 &= 377.5 \\ x_2 &= 622.5 \end{aligned}$$

$C(377.5, 622.5) = 104{,}997.50$

74. Minimize the square of the distance:

$$f(x, y, z) = (x - 2)^2 + (y - 2)^2 + (x^2 + y^2 - 0)^2.$$

$$\left.\begin{aligned} f_x &= 2(x - 2) + 2(x^2 + y^2)2x = 0 \\ f_y &= 2(y - 2) + 2(x^2 + y^2)2y = 0 \end{aligned}\right\} \begin{aligned} x - 2 + 2x^3 + 2xy^2 = 0 \\ y - 2 + 2y^3 + 2x^2y = 0 \end{aligned}$$

Clearly $x = y$ and hence: $4x^3 + x - 2 = 0$. Using a computer algebra system, $x \approx 0.6894$.
Thus, $(\text{distance})^2 = (0.6894 - 2)^2 + (0.6894 - 2)^2 + [2(0.6894)^2]^2 \approx 4.3389$.
distance ≈ 2.08

76. (a) $(25, 28),\ (50, 38),\ (75, 54),\ (100, 75),\ (125, 102)$

$\sum x_i = 375,\quad \sum y_i = 297,\quad \sum x_i^2 = 34{,}375,\quad \sum x_i^3 = 3{,}515{,}625$

$\sum x_i^4 = 382{,}421{,}875,\quad \sum x_i y_i = 26{,}900,\quad \sum x_i^2 y_i = 2{,}760{,}000,$

$$\begin{aligned} 382{,}421{,}875a + 3{,}515{,}625b + 34{,}375c &= 2{,}760{,}000 \\ 3{,}515{,}625a + 34{,}375b + 375c &= 26{,}900 \\ 34{,}375a + 375b + 5c &= 297 \end{aligned}$$

$a \approx 0.0045,\ b \approx 0.0717,\ c \approx 23.2914,\ y \approx 0.0045x^2 + 0.0717x + 23.2914$

(b) When $x = 80$ km/hr, $y \approx 57.8$ km.

78. Optimize $f(x, y) = x^2y$ subject to the constraint $x + 2y = 2$.

$$\left.\begin{aligned} 2xy &= \lambda \\ x^2 &= 2\lambda \end{aligned}\right\} x^2 = 4xy \implies x = 0 \text{ or } x = 4y$$

$x + 2y = 2$

If $x = 0$, $y = 1$. If $x = 4y$, then $y = \frac{1}{3}$, $x = \frac{4}{3}$.

Maximum: $f\left(\frac{4}{3}, \frac{1}{3}\right) = \frac{16}{27}$

Minimum: $f(0, 1) = 0$

Problem Solving for Chapter 12

2. $V = \frac{4}{3}\pi r^3 + \pi r^2 h$

Material $= M = 4\pi r^2 + 2\pi rh$

$V = 1000 \implies h = \dfrac{1000 - (4/3)\pi r^3}{\pi r^2}$

Hence,

$$\begin{aligned} M &= 4\pi r^2 + 2\pi r\left(\frac{1000 - (4/3)\pi r^3}{\pi r^2}\right) \\ &= 4\pi r^2 + \frac{2000}{r} - \frac{8}{3}\pi r^2 \\ \frac{dM}{dr} &= 8\pi r - \frac{2000}{r^2} - \frac{16}{3}\pi r = 0 \end{aligned}$$

$$8\pi r - \frac{16}{3}\pi r = \frac{2000}{r^2}$$

$$r^3\left(\frac{8}{3}\pi\right) = 2000$$

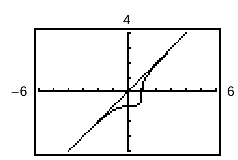
$$r^3 = \frac{750}{\pi} \implies r = 5\left(\frac{6}{\pi}\right)^{1/3}.$$

Then, $h = \dfrac{1000 - (4/3)\pi(750/\pi)}{\pi r^2} = 0.$

The tank is a sphere of radius $r = 5\left(\frac{6}{\pi}\right)^{1/3}$.

4. (a) As $x \to \pm\infty$, $f(x) = (x^3 - 1)^{1/3} \to x$ and hence

$$\lim_{x\to\infty} [f(x) - g(x)] = \lim_{x\to-\infty} [f(x) - g(x)] = 0.$$



(b) Let $(x_0, (x_0^3 - 1)^{1/3})$ be a point on the graph of f.

The line through this point perpendicular to g is

$$y = -x + x_0 + \sqrt[3]{x_0^3 - 1}.$$

This line intersects g at the point

$$\left(\frac{1}{2}\left[x_0 + \sqrt[3]{x_0^3 - 1}\right], \frac{1}{2}\left[x_0 + \sqrt[3]{x_0^3 - 1}\right]\right).$$

The square of the distance between these two points is

$$h(x_0) = \frac{1}{2}\left(x_0 - \sqrt[3]{x_0^3 - 1}\right)^2.$$

h is a maximum for $x_0 = \dfrac{1}{\sqrt[3]{2}}$. Hence, the point on f farthest from g is $\left(\dfrac{1}{\sqrt[3]{2}}, -\dfrac{1}{\sqrt[3]{2}}\right)$.

6. Heat Loss $= H = k(5xy + xy + 3xz + 3xz + 3yz + 3yz)$

$$= k(6xy + 6xz + 6yz)$$

$V = xyz = 1000 \implies z = \dfrac{1000}{xy}$.

Then $H = 6k\left(xy + \dfrac{1000}{y} + \dfrac{1000}{x}\right)$.

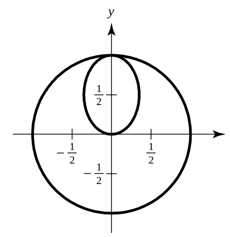
Setting $H_x = H_y = 0$, you obtain $x = y = z = 10$.

8. (a) $T(x, y) = 2x^2 + y^2 - y + 10 = 10$

$$2x^2 + y^2 - y + \frac{1}{4} = \frac{1}{4}$$

$$2x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$$

$$\frac{x^2}{1/8} + \frac{(y - (1/2))^2}{1/4} = 1 \quad \text{ellipse}$$



(b) On $x^2 + y^2 = 1$, $T(x, y) = T(y) = 2(1 - y^2) + y^2 - y + 10 = 12 - y^2 - y$

$$T'(y) = -2y - 1 = 0 \implies y = -\frac{1}{2}, x = \pm\frac{\sqrt{3}}{2}.$$

Inside: $T_x = 4x - 0$, $T_y = 2y - 1 = 0 \implies \left(0, \dfrac{1}{2}\right)$

$T\left(0, \dfrac{1}{2}\right) = \dfrac{39}{4}$ minimum

$T\left(\pm\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2}\right) = \dfrac{49}{4}$ maximum

10. $x = r\cos\theta, y = r\sin\theta, z = z$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \theta}$$

$$= \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}r\cos\theta \text{ Similarly,}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta.$$

$$\frac{\partial^2 u}{\partial \theta^2} = (-r\sin\theta)\left[\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial x \partial y}\frac{\partial y}{\partial \theta} + \frac{\partial^2 u}{\partial x \partial z}\frac{\partial z}{\partial \theta}\right] - r\frac{\partial u}{\partial x}\cos\theta$$

$$+ (r\cos\theta)\left[\frac{\partial^2 u}{\partial y \partial x}\frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial \theta} + \frac{\partial^2 u}{\partial y \partial z}\frac{\partial z}{\partial \theta}\right] - r\frac{\partial u}{\partial y}\sin\theta$$

$$= \frac{\partial^2 u}{\partial x^2}r^2\sin^2\theta + \frac{\partial^2 u}{\partial y^2}r^2\cos^2\theta - 2\frac{\partial^2 u}{\partial x \partial y}r^2\sin\theta\cos\theta - \frac{\partial u}{\partial x}r\cos\theta - \frac{\partial u}{\partial y}r\sin\theta$$

Similarly, $\dfrac{\partial^2 u}{\partial r^2} = \dfrac{\partial^2 u}{\partial x^2}\cos^2\theta + \dfrac{\partial^2 u}{\partial y^2}\sin^2\theta + 2\dfrac{\partial^2 u}{\partial x \partial y}\cos\theta\sin\theta$.

Now observe that

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = \left[\frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta + 2\frac{\partial^2 u}{\partial x \partial y}\cos\theta\sin\theta\right] + \frac{1}{r}\left[\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right]$$

$$+ \left[\frac{\partial^2 u}{\partial x^2}\sin^2\theta + \frac{\partial^2 u}{\partial y^2}\cos^2\theta - 2\frac{\partial^2 u}{\partial x \partial y}\sin\theta\cos\theta - \frac{1}{r}\frac{\partial u}{\partial x}\cos\theta - \frac{1}{r}\frac{\partial u}{\partial y}\sin\theta\right] + \frac{\partial^2 u}{\partial z^2}$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

Thus, Laplaces equation in cylindrical coordinates, is $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$.

12. (a) $d = \sqrt{x^2 + y^2} = \sqrt{\left(32\sqrt{2}t\right)^2 + \left(32\sqrt{2}t - 16t^2\right)^2}$

$= \sqrt{4096t^2 - 1024\sqrt{2}t^3 + 256t^4}$

$= 16t\sqrt{t^2 - 4\sqrt{2}t + 16}$

(b) $\dfrac{dd}{dt} = \dfrac{32\left(t^2 - 3\sqrt{2}t + 8\right)}{\sqrt{t^2 - 4\sqrt{2}t + 16}}$

(c) When $t = 2$:

$$\frac{dd}{dt} = \frac{32\left(12 - 6\sqrt{2}\right)}{\sqrt{20 - 8\sqrt{2}}} \approx 38.16 \text{ ft/sec}$$

(d) $\dfrac{d^2d}{dt^2} = \dfrac{32\left(t^3 - 6\sqrt{2}t^2 + 36t - 32\sqrt{12}\right)}{\left(t^2 - 4\sqrt{2}t + 16\right)^{3/2}} = 0$

when $t \approx 1.943$ seconds. No. The projectile is at its maximum height when $t = \sqrt{2}$.

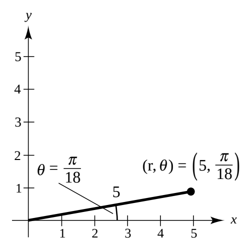
14. Given that f is a differentiable function such that $\nabla f(x_0, y_0) = \mathbf{0}$, then $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$. Therefore, the tangent plane is $-(z - z_0) = 0$ or $z = z_0 = f(x_0, y_0)$ which is horizontal.

16. $(r, \theta) = \left(5, \dfrac{\pi}{18}\right)$

$dr = \pm 0.05$, $d\theta = \pm 0.05$

$x = r\cos\theta = 5\cos\dfrac{\pi}{18} \approx 4.924$

$y = r\sin\theta = 5\sin\dfrac{\pi}{18} \approx 0.868$



(a) dx should be more effected by changes in r.

$$dx = (\cos\theta)dr + (-r\sin\theta)d\theta$$
$$\approx (0.985)dr - 0.868\,d\theta$$

dx is more effected by changes in r because $0.985 > 0.868$.

(b) dy should be more effected by changes in θ.

$$dy = \sin\theta\,dr + r\cos\theta\,d\theta$$
$$\approx 0.174\,dr + 4.924\,d\theta$$

dy is more effected by θ because $4.924 > 0.174$.

18. $\dfrac{\partial u}{\partial t} = \dfrac{1}{2}[-\cos(x - t) + \cos(x + t)]$

$\dfrac{\partial^2 u}{\partial t^2} = \dfrac{1}{2}[-\sin(x - t) - \sin(x + t)]$

$\dfrac{\partial u}{\partial x} = \dfrac{1}{2}[\cos(x - t) + \cos(x + t)]$

$\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{2}[-\sin(x - t) - \sin(x + t)]$

Then, $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$.