CHAPTER 12
Functions of Several Variables

CHAPTER 12
Functions of Several Variables

Section 12.1 Introduction to Functions of Several Variables

Solutions to Odd-Numbered Exercises

1. $x^2z + yz - xy = 10$

$z(x^2 + y) = 10 + xy$

$$z = \frac{10 + xy}{x^2 + y}$$

Yes, z is a function of x and y.

3. $\dfrac{x^2}{4} + \dfrac{y^2}{9} + z^2 = 1$

No, z is not a function of x and y. For example, $(x, y) = (0, 0)$ corresponds to both $z = \pm 1$.

5. $f(x, y) = \dfrac{x}{y}$

(a) $f(3, 2) = \dfrac{3}{2}$

(b) $f(-1, 4) = -\dfrac{1}{4}$

(c) $f(30, 5) = \dfrac{30}{5} = 6$

(d) $f(5, y) = \dfrac{5}{y}$

(e) $f(x, 2) = \dfrac{x}{2}$

(f) $f(5, t) = \dfrac{5}{t}$

7. $f(x, y) = xe^y$

(a) $f(5, 0) = 5e^0 = 5$

(b) $f(3, 2) = 3e^2$

(c) $f(2, -1) = 2e^{-1} = \dfrac{2}{e}$

(d) $f(5, y) = 5e^y$

(e) $f(x, 2) = xe^2$

(f) $f(t, t) = te^t$

9. $h(x, y, z) = \dfrac{xy}{z}$

(a) $h(2, 3, 9) = \dfrac{(2)(3)}{9} = \dfrac{2}{3}$

(b) $h(1, 0, 1) = \dfrac{(1)(0)}{1} = 0$

11. $f(x, y) = x \sin y$

(a) $f\left(2, \dfrac{\pi}{4}\right) = 2 \sin \dfrac{\pi}{4} = \sqrt{2}$

(b) $f(3, 1) = 3 \sin 1$

13. $g(x, y) = \displaystyle\int_x^y (2t - 3)\, dt$

(a) $g(0, 4) = \displaystyle\int_0^4 (2t - 3)\, dt = \Big[t^2 - 3t\Big]_0^4 = 4$

(b) $g(1, 4) = \displaystyle\int_1^4 (2t - 3)\, dt = \Big[t^2 - 3t\Big]_1^4 = 6$

15. $f(x, y) = x^2 - 2y$

(a) $$\frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \frac{[(x + \Delta x)^2 - 2y] - (x^2 - 2y)}{\Delta x}$$

$$= \frac{x^2 + 2x(\Delta x) + (\Delta x)^2 - 2y - x^2 + 2y}{\Delta x} = \frac{\Delta x(2x + \Delta x)}{\Delta x} = 2x + \Delta x, \Delta x \neq 0$$

(b) $$\frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \frac{[x^2 - 2(y + \Delta y)] - (x^2 - 2y)}{\Delta y} = \frac{x^2 - 2y - 2\Delta y - x^2 + 2y}{\Delta y} = \frac{-2\Delta y}{\Delta y} = -2, \Delta y \neq 0$$

17. $f(x, y) = \sqrt{4 - x^2 - y^2}$

Domain: $4 - x^2 - y^2 \geq 0$

$x^2 + y^2 \leq 4$

$\{(x, y): x^2 + y^2 \leq 4\}$

Range: $0 \leq z \leq 2$

19. $f(x, y) = \arcsin(x + y)$

Domain:
$\{(x, y): -1 \leq x + y \leq 1\}$

Range: $-\dfrac{\pi}{2} \leq z \leq \dfrac{\pi}{2}$

21. $f(x, y) = \ln(4 - x - y)$

Domain: $4 - x - y > 0$

$x + y < 4$

$\{(x, y): y < -x + 4\}$

Range: all real numbers

23. $z = \dfrac{x + y}{xy}$

Domain: $\{(x, y): x \neq 0 \text{ and } y \neq 0\}$

Range: all real numbers

25. $f(x, y) = e^{x/y}$

Domain: $\{(x, y): y \neq 0\}$

Range: $z > 0$

27. $g(x, y) = \dfrac{1}{xy}$

Domain: $\{(x, y): x \neq 0 \text{ and } y \neq 0\}$

Range: all real numbers except zero

29. $f(x, y) = \dfrac{-4x}{x^2 + y^2 + 1}$

(a) View from the positive x-axis: $(20, 0, 0)$

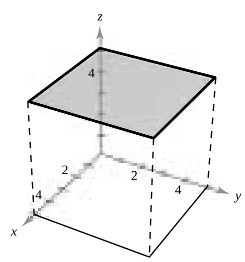
(b) View where x is negative, y and z are positive: $(-15, 10, 20)$

(c) View from the first octant: $(20, 15, 25)$

(d) View from the line $y = x$ in the xy-plane: $(20, 20, 0)$

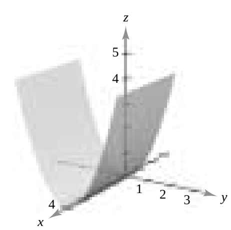
31. $f(x, y) = 5$

Plane: $z = 5$



33. $f(x, y) = y^2$

Since the variable x is missing, the surface is a cylinder with rulings parallel to the x-axis. The generating curve is $z = y^2$. The domain is the entire xy-plane and the range is $z \geq 0$.

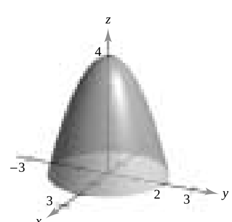


35. $z = 4 - x^2 - y^2$

Paraboloid

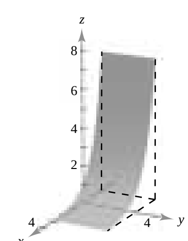
Domain: entire xy-plane

Range: $z \leq 4$



37. $f(x, y) = e^{-x}$

Since the variable y is missing, the surface is a cylinder with rulings parallel to the y-axis. The generating curve is $z = e^{-x}$. The domain is the entire xy-plane and the range is $z > 0$.

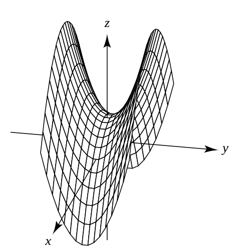


39. $z = y^2 - x^2 + 1$

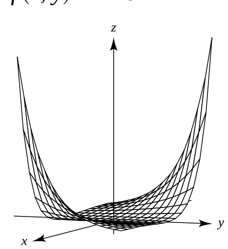
Hyperbolic paraboloid

Domain: entire xy-plane

Range: $-\infty < z < \infty$

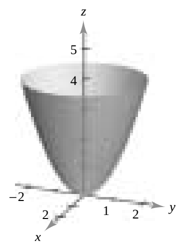


41. $f(x, y) = x^2e^{(-xy/2)}$



43. $f(x, y) = x^2 + y^2$

(a)

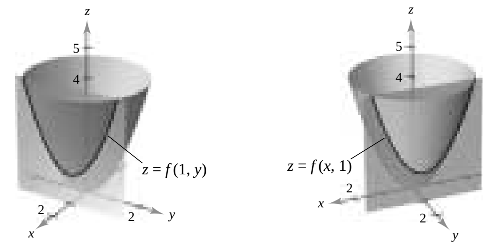


(b) g is a vertical translation of f two units upward

(c) g is a horizontal translation of f two units to the right. The vertex moves from $(0, 0, 0)$ to $(0, 2, 0)$.

(d) g is a reflection of f in the xy-plane followed by a vertical translation 4 units upward.

(e)



45. $z = e^{1-x^2-y^2}$

Level curves:

$$c = e^{1-x^2-y^2}$$
$$\ln c = 1 - x^2 - y^2$$
$$x^2 + y^2 = 1 - \ln c$$

Circles centered at $(0, 0)$

Matches (c)

47. $z = \ln|y - x^2|$

Level curves:

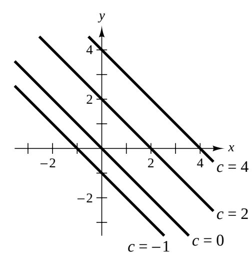
$$c = \ln|y - x^2|$$
$$\pm e^c = y - x^2$$
$$y = x^2 \pm e^c$$

Parabolas

Matches (b)

49. $z = x + y$

Level curves are parallel lines of the form $x + y = c$.

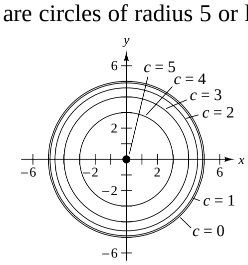


51. $f(x, y) = \sqrt{25 - x^2 - y^2}$

The level curves are of the form

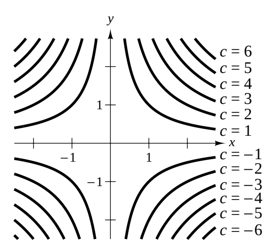
$$c = \sqrt{25 - x^2 - y^2},$$
$$x^2 + y^2 = 25 - c^2.$$

Thus, the level curves are circles of radius 5 or less, centered at the origin.



53. $f(x, y) = xy$

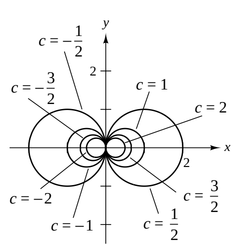
The level curves are hyperbolas of the form $xy = c$.



55. $f(x, y) = \dfrac{x}{x^2 + y^2}$

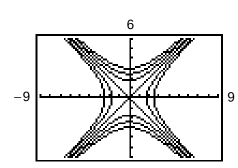
The level curves are of the form

$$c = \frac{x}{x^2 + y^2}$$
$$x^2 - \frac{x}{c} + y^2 = 0$$
$$\left(x - \frac{1}{2c}\right)^2 + y^2 = \left(\frac{1}{2c}\right)^2$$

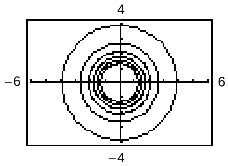


Thus, the level curves are circles passing through the origin and centered at $(1/2c, 0)$.

57. $f(x, y) = x^2 - y^2 + 2$



59. $g(x, y) = \dfrac{8}{1 + x^2 + y^2}$



61. See Definition, page 838.

63. No, The following graphs are not hemispheres.

$z = e^{-(x^2+y^2)}$

$z = x^2 + y^2$

65. The surface is sloped like a saddle. The graph is not unique. Any vertical translation would have the same level curves.
One possible function is

$f(x, y) = x^2 - y^2.$

67. $V(I, R) = 1000\left[\dfrac{1 + 0.10(1 - R)}{1 + I}\right]^{10}$

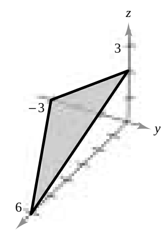
	Inflation Rate		
Tax Rate	0	0.03	0.05
0	2593.74	1929.99	1592.33
0.28	2004.23	1491.34	1230.42
0.35	1877.14	1396.77	1152.40

69. $f(x, y, z) = x - 2y + 3z$

$c = 6$

$6 = x - 2y + 3z$

Plane

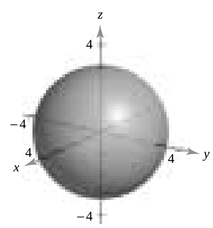


71. $f(x, y, z) = x^2 + y^2 + z^2$

$c = 9$

$9 = x^2 + y^2 + z^2$

Sphere

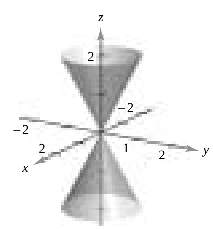


73. $f(x, y, z) = 4x^2 + 4y^2 - z^2$

$c = 0$

$0 = 4x^2 + 4y^2 - z^2$

Elliptic cone



75. $N(d, L) = \left(\dfrac{d - 4}{4}\right)^2 L$

(a) $N(22, 12) = \left(\dfrac{22 - 4}{4}\right)^2(12) = 243$ board-feet

(b) $N(30, 12) = \left(\dfrac{30 - 4}{4}\right)^2(12) = 507$ board-feet

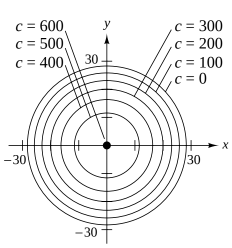
77. $T = 600 - 0.75x^2 - 0.75y^2$

The level curves are of the form

$c = 600 - 0.75x^2 - 0.75y^2$

$x^2 + y^2 = \dfrac{600 - c}{0.75}.$

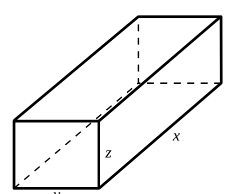
The level curves are circles centered at the origin.



79. $C = 0.75xy + 2(0.40)xz + 2(0.40)yz$

base + front & back + two ends

$= 0.75xy + 0.80(xz + yz)$



81. $PV = kT,\ 20(2600) = k(300)$

(a) $k = \dfrac{20(2600)}{300} = \dfrac{520}{3}$

(b) $P = \dfrac{kT}{V} = \dfrac{520}{3}\left(\dfrac{T}{V}\right)$

The level curves are of the form: $c = \left(\dfrac{520}{3}\right)\left(\dfrac{T}{V}\right)$

$$V = \frac{520}{3c}T$$

Thus, the level curves are lines through the origin with slope $\dfrac{520}{3c}$.

83. (a) Highest pressure at C

(b) Lowest pressure at A

(c) Highest wind velocity at B

85. (a) The boundaries between colors represent level curves

(b) No, the colors represent intervals of different lengths, as indicated in the box

(c) You could use more colors, which means using smaller intervals

87. False. Let

$f(x, y) = 2xy$

$f(1, 2) = f(2, 1)$, but $1 \neq 2$

89. False. Let

$f(x, y) = 5.$

Then, $f(2x, 2y) = 5 \neq 2^2 f(x, y)$.

Section 12.2 Limits and Continuity

1. Let $\varepsilon > 0$ be given. We need to find $\delta > 0$ such that $|f(x, y) - L| = |y - b| < \varepsilon$ whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$. Take $\delta = \varepsilon$.

Then if $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta = \varepsilon$, we have

$$\sqrt{(y-b)^2} < \varepsilon$$

$$|y - b| < \varepsilon.$$

3. $\lim\limits_{(x, y)\to(a, b)} [f(x, y) - g(x, y)] = \lim\limits_{(x, y)\to(a, b)} f(x, y) - \lim\limits_{(x, y)\to(a, b)} g(x, y) = 5 - 3 = 2$

5. $\lim\limits_{(x, y)\to(a, b)} [f(x, y)g(x, y)] = \left[\lim\limits_{(x, y)\to(a, b)} f(x, y)\right]\left[\lim\limits_{(x, y)\to(a, b)} g(x, y)\right] = 5(3) = 15$

7. $\lim\limits_{(x, y)\to(2, 1)} (x + 3y^2) = 2 + 3(1)^2 = 5$

Continuous everywhere

9. $\lim\limits_{(x, y)\to(2, 4)} \dfrac{x + y}{x - y} = \dfrac{2 + 4}{2 - 4} = -3$

Continuous for $x \neq y$

11. $\lim\limits_{(x, y)\to(0, 1)} \dfrac{\arcsin(x/y)}{1 + xy} = \arcsin 0 = 0$

Continuous for $xy \neq -1, y \neq 0, |x/y| \leq 1$

13. $\lim\limits_{(x, y)\to(-1, 2)} e^{xy} = e^{-2} = \dfrac{1}{e^2}$

Continuous everywhere

15. $\lim\limits_{(x, y, z)\to(1, 2, 5)} \sqrt{x + y + z} = \sqrt{8} = 2\sqrt{2}$

Continuous for $x + y + z \geq 0$

17. $\lim\limits_{(x, y)\to(0, 0)} e^{xy} = 1$

Continuous everywhere

19. $\lim_{(x, y)\to(0, 0)} \ln(x^2 + y^2) = \ln(0) = -\infty$

The limit does not exist.

Continuous except at $(0, 0)$

21. $f(x, y) = \dfrac{xy}{x^2 + y^2}$

Continuous except at $(0, 0)$

Path: $y = 0$

(x, y)	$(1, 0)$	$(0.5, 0)$	$(0.1, 0)$	$(0.01, 0)$	$(0.001, 0)$
$f(x, y)$	0	0	0	0	0

Path: $y = x$

(x, y)	$(1, 1)$	$(0.5, 0.5)$	$(0.1, 0.1)$	$(0.01, 0.01)$	$(0.001, 0.001)$
$f(x, y)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

The limit does not exist because along the path $y = 0$ the function equals 0, whereas along the path $y = x$ the function equals $\frac{1}{2}$.

23. $f(x, y) = -\dfrac{xy^2}{x^2 + y^4}$

Continuous except at $(0, 0)$

Path: $x = y^2$

(x, y)	$(1, 1)$	$(0.25, 0.5)$	$(0.01, 0.1)$	$(0.0001, 0.01)$	$(0.000001, 0.001)$
$f(x, y)$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

Path: $x = -y^2$

(x, y)	$(-1, 1)$	$(-0.25, 0.5)$	$(-0.01, 0.1)$	$(-0.0001, 0.01)$	$(-0.000001, 0.001)$
$f(x, y)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

The limit does not exist because along the path $x = y^2$ the function equals $-\frac{1}{2}$, whereas along the path $x = -y^2$ the function equals $\frac{1}{2}$.

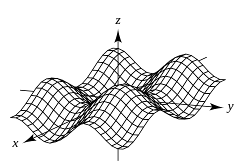
25. $\lim_{(x, y)\to(0, 0)} f(x, y) = \lim_{(x, y)\to(0, 0)} \left(\dfrac{x^2 + 2xy^2 + y^2}{x^2 + y^2}\right)$

$$= \lim_{(x, y)\to(0, 0)} \left(1 + \frac{2xy^2}{x^2 + y^2}\right) = 1$$

(same limit for g)

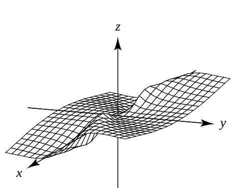
Thus, f is not continuous at $(0, 0)$, whereas g is continuous at $(0, 0)$.

27. $\lim_{(x, y)\to(0, 0)} (\sin x + \sin y) = 0$



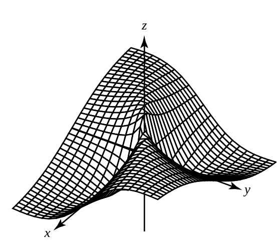
29. $\lim_{(x, y)\to(0, 0)} \dfrac{x^2y}{x^4 + 4y^2}$

Does not exist



31. $f(x, y) = \dfrac{10xy}{2x^2 + 3y^2}$

The limit does not exist. Use the paths $x = 0$ and $x = y$.



33. $\displaystyle\lim_{(x, y)\to(0, 0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = \lim_{r\to 0} \frac{\sin r^2}{r^2} = \lim_{r\to 0} \frac{2r\cos r^2}{2r} = \lim_{r\to 0} \cos r^2 = 1$

35. $\displaystyle\lim_{(x, y)\to(0, 0)} \frac{x^3 + y^3}{x^2 + y^2} = \lim_{r\to 0} \frac{r^3(\cos^3\theta + \sin^3\theta)}{r^2} = \lim_{r\to 0} r(\cos^3\theta + \sin^3\theta) = 0$

37. $f(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$

Continuous except at $(0, 0, 0)$

39. $f(x, y, z) = \dfrac{\sin z}{e^x + e^y}$

Continuous everywhere

41.
$$\begin{aligned} f(t) &= t^2 \\ g(x, y) &= 3x - 2y \\ f(g(x, y)) &= f(3x - 2y) \\ &= (3x - 2y)^2 \\ &= 9x^2 - 12xy + 4y^2 \end{aligned}$$

Continuous everywhere

43.
$$\begin{aligned} f(t) &= \frac{1}{t} \\ g(x, y) &= 3x - 2y \\ f(g(x, y)) &= f(3x - 2y) = \frac{1}{3x - 2y} \end{aligned}$$

Continuous for $y \neq \dfrac{3x}{2}$

45. $f(x, y) = x^2 - 4y$

(a) $\displaystyle\lim_{\Delta x\to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x\to 0} \frac{[(x + \Delta x)^2 - 4y] - (x^2 - 4y)}{\Delta x}$

$\displaystyle= \lim_{\Delta x\to 0} \frac{2x\Delta x - (\Delta x)^2}{\Delta x} = \lim_{\Delta x\to 0} (2x - \Delta x) = 2x$

(b) $\displaystyle\lim_{\Delta y\to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y\to 0} \frac{[x^2 - 4(y + \Delta y)] - (x^2 - 4y)}{\Delta y}$

$\displaystyle= \lim_{\Delta y\to 0} \frac{-4\Delta y}{\Delta y} = \lim_{\Delta y\to 0} (-4) = -4$

47. $f(x, y) = 2x + xy - 3y$

(a) $\displaystyle\lim_{\Delta x\to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x\to 0} \frac{[2(x + \Delta x) + (x + \Delta x)y - 3y] - (2x + xy - 3y)}{\Delta x}$

$\displaystyle= \lim_{\Delta x\to 0} \frac{2\Delta x + \Delta xy}{\Delta x} = \lim_{\Delta x\to 0} (2 + y) = 2 + y$

(b) $\displaystyle\lim_{\Delta y\to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y\to 0} \frac{[2x + x(y + \Delta y) - 3(y + \Delta y)] - (2x + xy - 3y)}{\Delta y}$

$\displaystyle= \lim_{\Delta y\to 0} \frac{x\Delta y - 3\Delta y}{\Delta y} = \lim_{\Delta y\to 0} (x - 3) = x - 3$

49. See the definition on page 851.

Show that the value of $\displaystyle\lim_{(x, y)\to(x_0, y_0)} f(x, y)$ is not the same for two different paths to (x_0, y_0).

51. No.

The existence of $f(2, 3)$ has no bearing on the existence of the limit as $(x, y)\to(2, 3)$.

53. Since $\lim_{(x, y)\to(a, b)} f(x, y) = L_1$, then for $\varepsilon/2 > 0$, there corresponds $\delta_1 > 0$ such that $|f(x, y) - L_1| < \varepsilon/2$ whenever

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta_1.$$

Since $\lim_{(x, y)\to(a, b)} g(x, y) = L_2$, then for $\varepsilon/2 > 0$, there corresponds $\delta_2 > 0$ such that $|g(x, y) - L_2| < \varepsilon/2$ whenever

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta_2.$$

Let δ be the smaller of δ_1 and δ_2. By the triangle inequality, whenever $\sqrt{(x-a)^2 + (y-b)^2} < \delta$, we have

$$|f(x, y) + g(x, y) - (L_1 + L_2)| = |(f(x, y) - L_1) + (g(x, y) - L_2)| \le |f(x, y) - L_1| + |g(x, y) - L_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, $\lim_{(x, y)\to(a, b)} [f(x, y) + g(x, y)] = L_1 + L_2$.

55. True

57. False. Let

$$f(x, y) = \begin{cases} \ln(x^2 + y^2), & (x, y) \neq (0, 0) \\ 0, & x = 0, y = 0 \end{cases}$$

See Exercise 19.

Section 12.3 Partial Derivatives

1. $f_x(4, 1) < 0$

3. $f_y(4, 1) > 0$

5. $f(x, y) = 2x - 3y + 5$

$f_x(x, y) = 2$

$f_y(x, y) = -3$

7. $z = x\sqrt{y}$

$$\frac{\partial z}{\partial x} = \sqrt{y}$$

$$\frac{\partial z}{\partial y} = \frac{x}{2\sqrt{y}}$$

9. $z = x^2 - 5xy + 3y^2$

$$\frac{\partial z}{\partial x} = 2x - 5y$$

$$\frac{\partial z}{\partial y} = -5x + 6y$$

11. $z = x^2e^{2y}$

$$\frac{\partial z}{\partial x} = 2xe^{2y}$$

$$\frac{\partial z}{\partial y} = 2x^2e^{2y}$$

13. $z = \ln(x^2 + y^2)$

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2}$$

$$\frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2}$$

15. $z = \ln\left(\frac{x+y}{x-y}\right) = \ln(x + y) - \ln(x - y)$

$$\frac{\partial z}{\partial x} = \frac{1}{x+y} - \frac{1}{x-y} = -\frac{2y}{x^2 - y^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{x+y} + \frac{1}{x-y} = \frac{2x}{x^2 - y^2}$$

17. $z = \frac{x^2}{2y} + \frac{4y^2}{x}$

$$\frac{\partial z}{\partial x} = \frac{2x}{2y} - \frac{4y^2}{x^2} = \frac{x^3 - 4y^3}{x^2y}$$

$$\frac{\partial z}{\partial y} = -\frac{x^2}{2y^2} + \frac{8y}{x} = \frac{-x^3 + 16y^3}{2xy^2}$$

19. $h(x, y) = e^{-(x^2+y^2)}$

$h_x(x, y) = -2xe^{-(x^2+y^2)}$

$h_y(x, y) = -2ye^{-(x^2+y^2)}$

21. $f(x, y) = \sqrt{x^2 + y^2}$

$$f_x(x, y) = \frac{1}{2}(x^2 + y^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + y^2}}$$

$$f_y(x, y) = \frac{1}{2}(x^2 + y^2)^{-1/2}(2y) = \frac{y}{\sqrt{x^2 + y^2}}$$

23. $z = \tan(2x - y)$

$$\frac{\partial z}{\partial x} = 2\sec^2(2x - y)$$

$$\frac{\partial z}{\partial y} = -\sec^2(2x - y)$$

25. $z = e^y \sin xy$

$$\frac{\partial z}{\partial x} = ye^y \cos xy$$

$$\frac{\partial z}{\partial y} = e^y \sin xy + xe^y \cos xy$$

$$= e^y(x \cos xy + \sin xy)$$

27. $f(x, y) = \int_x^y (t^2 - 1)\, dt$

$$= \left[\frac{t^3}{3} - t\right]_x^y = \left(\frac{y^3}{3} - y\right) - \left(\frac{x^3}{3} - x\right)$$

$$f_x(x, y) = -x^2 + 1 = 1 - x^2$$

$$f_y(x, y) = y^2 - 1$$

[You could also use the Second Fundamental Theorem of Calculus.]

29. $f(x, y) = 2x + 3y$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2(x + \Delta x) + 3y - 2x - 3y}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\Delta x}{\Delta x} = 2$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{2x + 3(y + \Delta y) - 2x - 3y}{\Delta y} = \lim_{\Delta y \to 0} \frac{3\Delta y}{\Delta y} = 3$$

31. $f(x, y) = \sqrt{x + y}$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x + y} - \sqrt{x + y}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left(\sqrt{x + \Delta x + y} - \sqrt{x + y}\right)\left(\sqrt{x + \Delta x + y} + \sqrt{x + y}\right)}{\Delta x\left(\sqrt{x + \Delta x + y} + \sqrt{x + y}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x + y} + \sqrt{x + y}} = \frac{1}{2\sqrt{x + y}}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\sqrt{x + y + \Delta y} - \sqrt{x + y}}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\left(\sqrt{x + y + \Delta y} - \sqrt{x + y}\right)\left(\sqrt{x + y + \Delta y} + \sqrt{x + y}\right)}{\Delta y\left(\sqrt{x + y + \Delta y} + \sqrt{x + y}\right)}$$

$$= \lim_{\Delta y \to 0} \frac{1}{\sqrt{x + y + \Delta y} + \sqrt{x + y}} = \frac{1}{2\sqrt{x + y}}$$

33. $g(x, y) = 4 - x^2 - y^2$

$g_x(x, y) = -2x$

At $(1, 1)$: $g_x(1, 1) = -2$

$g_y(x, y) = -2y$

At $(1, 1)$: $g_y(1, 1) = -2$

35. $z = e^{-x} \cos y$

$$\frac{\partial z}{\partial x} = -e^{-x} \cos y$$

At $(0, 0)$: $\frac{\partial z}{\partial x} = -1$

$$\frac{\partial z}{\partial y} = -e^{-x} \sin y$$

At $(0, 0)$: $\frac{\partial z}{\partial y} = 0$

37. $f(x, y) = \arctan \frac{y}{x}$

$$f_x(x, y) = \frac{1}{1 + (y^2/x^2)}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2 + y^2}$$

At $(2, -2)$: $f_x(2, -2) = \frac{1}{4}$

$$f_y(x, y) = \frac{1}{1 + (y^2/x^2)}\left(\frac{1}{x}\right) = \frac{x}{x^2 + y^2}$$

At $(2, -2)$: $f_y(2, -2) = \frac{1}{4}$

39. $f(x, y) = \frac{xy}{x - y}$

$$f_x(x, y) = \frac{y(x - y) - xy}{(x - y)^2} = \frac{-y^2}{(x - y)^2}$$

At $(2, -2)$: $f_x(2, -2) = -\frac{1}{4}$

$$f_y(x, y) = \frac{x(x - y) + xy}{(x - y)^2} = \frac{x^2}{(x - y)^2}$$

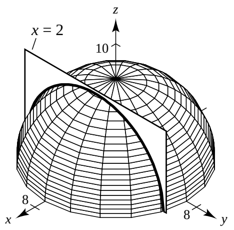
At $(2, -2)$: $f_y(2, -2) = \frac{1}{4}$

41. $z = \sqrt{49 - x^2 - y^2},\ x = 2,$
$(2, 3, 6)$

Intersecting curve: $z = \sqrt{45 - y^2}$

$$\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{45 - y^2}}$$

At $(2, 3, 6)$: $\dfrac{\partial z}{\partial y} = \dfrac{-3}{\sqrt{45 - 9}} = -\dfrac{1}{2}$

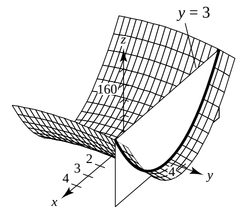


43. $z = 9x^2 - y^2,\ y = 3,\ (1, 3, 0)$

Intersecting curve: $z = 9x^2 - 9$

$$\frac{\partial z}{\partial x} = 18x$$

At $(1, 3, 0)$: $\dfrac{\partial z}{\partial x} = 18(1) = 18$



45. $f_x(x, y) = 2x + 4y - 4,\ f_y(x, y) = 4x + 2y + 16$

$f_x = f_y = 0$: $2x + 4y = 4$

$4x + 2y = -16$

Solving for x and y,

$x = -6$ and $y = 4$.

47. $f_x(x, y) = -\dfrac{1}{x^2} + y,\ f_y(x, y) = -\dfrac{1}{y^2} + x$

$f_x = f_y = 0$: $-\dfrac{1}{x^2} + y = 0$ and $-\dfrac{1}{y^2} + x = 0$

$y = \dfrac{1}{x^2}$ and $x = \dfrac{1}{y^2}$

$y = y^4 \implies y = 1 = x$

Points: $(1, 1)$

49. (a) The graph is that of f_y.

(b) The graph is that of f_x.

51. $w = \sqrt{x^2 + y^2 + z^2}$

$$\frac{\partial w}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial w}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial w}{\partial z} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

53. $F(x, y, z) = \ln \sqrt{x^2 + y^2 + z^2}$

$= \dfrac{1}{2}\ln(x^2 + y^2 + z^2)$

$$F_x(x, y, z) = \frac{x}{x^2 + y^2 + z^2}$$

$$F_y(x, y, z) = \frac{y}{x^2 + y^2 + z^2}$$

$$F_z(x, y, z) = \frac{z}{x^2 + y^2 + z^2}$$

55. $H(x, y, z) = \sin(x + 2y + 3z)$

$H_x(x, y, z) = \cos(x + 2y + 3z)$

$H_y(x, y, z) = 2\cos(x + 2y + 3z)$

$H_z(x, y, z) = 3\cos(x + 2y + 3z)$

57. $z = x^2 - 2xy + 3y^2$

$$\frac{\partial z}{\partial x} = 2x - 2y$$

$$\frac{\partial^2 z}{\partial x^2} = 2$$

$$\frac{\partial^2 z}{\partial y \partial x} = -2$$

$$\frac{\partial z}{\partial y} = -2x + 6y$$

$$\frac{\partial^2 z}{\partial y^2} = 6$$

$$\frac{\partial^2 z}{\partial x \partial y} = -2$$

59. $z = \sqrt{x^2 + y^2}$

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{y^2}{(x^2 + y^2)^{3/2}}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{-xy}{(x^2 + y^2)^{3/2}}$$

$$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{x^2}{(x^2 + y^2)^{3/2}}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-xy}{(x^2 + y^2)^{3/2}}$$

61. $z = e^x \tan y$

$\frac{\partial z}{\partial x} = e^x \tan y$

$\frac{\partial^2 z}{\partial x^2} = e^x \tan y$

$\frac{\partial^2 z}{\partial y \partial x} = e^x \sec^2 y$

$\frac{\partial z}{\partial y} = e^x \sec^2 y$

$\frac{\partial^2 z}{\partial y^2} = 2e^x \sec^2 y \tan y$

$\frac{\partial^2 z}{\partial x \partial y} = e^x \sec^2 y$

63. $z = \arctan \frac{y}{x}$

$\frac{\partial z}{\partial x} = \frac{1}{1 + (y^2/x^2)}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2 + y^2}$

$\frac{\partial^2 z}{\partial x^2} = \frac{2xy}{(x^2 + y^2)^2}$

$\frac{\partial^2 z}{\partial y \partial x} = \frac{-(x^2 + y^2) + y(2y)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$

$\frac{\partial z}{\partial y} = \frac{1}{1 + (y^2/x^2)}\left(\frac{1}{x}\right) = \frac{x}{x^2 + y^2}$

$\frac{\partial^2 z}{\partial y^2} = \frac{-2xy}{(x^2 + y^2)^2}$

$\frac{\partial^2 z}{\partial x \partial y} = \frac{(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$

65. $z = x \sec y$

$\frac{\partial z}{\partial x} = \sec y$

$\frac{\partial^2 z}{\partial x^2} = 0$

$\frac{\partial^2 z}{\partial y \partial x} = \sec y \tan y$

$\frac{\partial z}{\partial y} = x \sec y \tan y$

$\frac{\partial^2 z}{\partial y^2} = x \sec y(\sec^2 y + \tan^2 y)$

$\frac{\partial^2 z}{\partial x \partial y} = \sec y \tan y$

Therefore, $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$.

There are no points for which $z_x = 0 = z_y$, because

$\frac{\partial z}{\partial x} = \sec y \neq 0.$

67. $z = \ln\left(\frac{x}{x^2 + y^2}\right) = \ln x - \ln(x^2 + y^2)$

$\frac{\partial z}{\partial x} = \frac{1}{x} - \frac{2x}{x^2 + y^2} = \frac{y^2 - x^2}{x(x^2 + y^2)}$

$\frac{\partial^2 z}{\partial x^2} = \frac{x^4 - 4x^2y^2 - y^4}{x^2(x^2 + y^2)^2}$

$\frac{\partial^2 z}{\partial y \partial x} = \frac{4xy}{(x^2 + y^2)^2}$

$\frac{\partial z}{\partial y} = -\frac{2y}{x^2 + y^2}$

$\frac{\partial^2 z}{\partial y^2} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2}$

$\frac{\partial^2 z}{\partial x \partial y} = \frac{4xy}{(x^2 + y^2)^2}$

There are no points for which $z_x = z_y = 0$.

69. $f(x, y, z) = xyz$

$f_x(x, y, z) = yz$

$f_y(x, y, z) = xz$

$f_{yy}(x, y, z) = 0$

$f_{xy}(x, y, z) = z$

$f_{yx}(x, y, z) = z$

$f_{yyx}(x, y, z) = 0$

$f_{xyy}(x, y, z) = 0$

$f_{yxy}(x, y, z) = 0$

Therefore, $f_{xyy} = f_{yxy} = f_{yyx} = 0$.

71. $f(x, y, z) = e^{-x} \sin yz$

$f_x(x, y, z) = -e^{-x} \sin yz$

$f_y(x, y, z) = ze^{-x} \cos yz$

$f_{yy}(x, y, z) = -z^2e^{-x} \sin yz$

$f_{xy}(x, y, z) = -ze^{-x} \cos yz$

$f_{yx}(x, y, z) = -ze^{-x} \cos yz$

$f_{yyx}(x, y, z) = z^2e^{-x} \sin yz$

$f_{xyy}(x, y, z) = z^2e^{-x} \sin yz$

$f_{yxy}(x, y, z) = z^2e^{-x} \sin yz$

Therefore, $f_{xyy} = f_{yxy} = f_{yyx}$.

73. $z = 5xy$

$\frac{\partial z}{\partial x} = 5y$

$\frac{\partial^2 z}{\partial x^2} = 0$

$\frac{\partial z}{\partial y} = 5x$

$\frac{\partial^2 z}{\partial y^2} = 0$

Therefore, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 + 0 = 0.$

75. $z = e^x \sin y$

$$\frac{\partial z}{\partial x} = e^x \sin y$$

$$\frac{\partial^2 z}{\partial x^2} = e^x \sin y$$

$$\frac{\partial z}{\partial y} = e^x \cos y$$

$$\frac{\partial^2 z}{\partial y^2} = -e^x \sin y$$

Therefore, $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = e^x \sin y - e^x \sin y = 0.$

77. $z = \sin(x - ct)$

$$\frac{\partial z}{\partial t} = -c\cos(x - ct)$$

$$\frac{\partial^2 z}{\partial t^2} = -c^2 \sin(x - ct)$$

$$\frac{\partial z}{\partial x} = \cos(x - ct)$$

$$\frac{\partial^2 z}{\partial x^2} = -\sin(x - ct)$$

Therefore, $\dfrac{\partial^2 z}{\partial t^2} = c^2 \dfrac{\partial^2 z}{\partial x^2}.$

79. $z = e^{-t} \cos \dfrac{x}{c}$

$$\frac{\partial z}{\partial t} = -e^{-t} \cos \frac{x}{c}$$

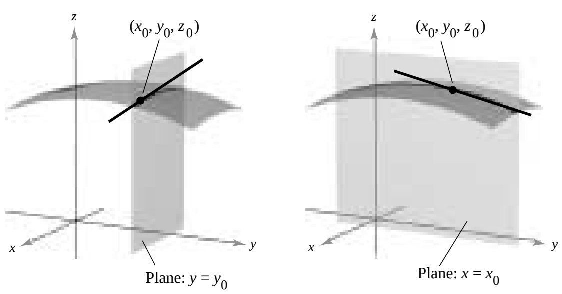
$$\frac{\partial z}{\partial x} = -\frac{1}{c} e^{-t} \sin \frac{x}{c}$$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{c^2} e^{-t} \cos \frac{x}{c}$$

Therefore, $\dfrac{\partial z}{\partial t} = c^2 \dfrac{\partial^2 z}{\partial x^2}.$

81. See the definition on page 859.

83.

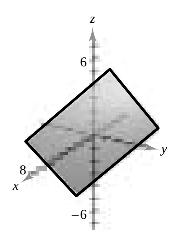


$\dfrac{\partial f}{\partial x}$ denotes the slope of the surface in the x-direction.

$\dfrac{\partial f}{\partial y}$ denotes the slope of the surface in the y-direction.

85. The plane $z = x + y = f(x, y)$ satisfies

$$\frac{\partial f}{\partial x} > 0 \text{ and } \frac{\partial f}{\partial y} > 0.$$



87. (a) $C = 32\sqrt{xy} + 175x + 205y + 1050$

$$\frac{\partial C}{\partial x} = 16\sqrt{\frac{y}{x}} + 175$$

$$\left.\frac{\partial C}{\partial x}\right]_{(80,\,20)} = 16\sqrt{\frac{1}{4}} + 175 = 183$$

$$\frac{\partial C}{\partial y} = 16\sqrt{\frac{x}{y}} + 205$$

$$\left.\frac{\partial C}{\partial y}\right]_{(80,\,20)} = 16\sqrt{4} + 205 = 237$$

(b) The fireplace-insert stove results in the cost increasing at a faster rate because

$$\frac{\partial C}{\partial y} > \frac{\partial C}{\partial x}.$$

89. An increase in either price will cause a decrease in demand.

91. $T = 500 - 0.6x^2 - 1.5y^2$

$$\frac{\partial T}{\partial x} = -1.2x, \frac{\partial T}{\partial x}(2, 3) = -2.4°/\text{m}$$

$$\frac{\partial T}{\partial y} = -3y = \frac{\partial T}{\partial y}(2, 3) = -9°/\text{m}$$

93.

$$PV = mRT$$

$$T = \frac{PV}{mR} \Rightarrow \frac{\partial T}{\partial P} = \frac{V}{mR}$$

$$P = \frac{mRT}{V} \Rightarrow \frac{\partial P}{\partial V} = -\frac{mRT}{V^2}$$

$$V = \frac{mRT}{P} \Rightarrow \frac{\partial V}{\partial T} = \frac{mR}{P}$$

$$\frac{\partial T}{\partial P} \cdot \frac{\partial P}{\partial V} \cdot \frac{\partial V}{\partial T} = \left(\frac{V}{mR}\right)\left(-\frac{mRT}{V^2}\right)\left(\frac{mR}{P}\right)$$

$$= -\frac{mRT}{VP} = -\frac{mRT}{mRT} = -1$$

95. (a) $\frac{\partial z}{\partial x} = -1.83$

$$\frac{\partial z}{\partial x} = -1.09$$

(b) As the consumption of skim milk (x) increases, the consumption of whole milk (z) decreases.

Similarly, as the consumption of reduced-fat milk (y) increases, the consumption of whole milk (z) decreases.

97. $f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

(a) $f_x(x, y) = \dfrac{(x^2 + y^2)(3x^2y - y^3) - (x^3y - xy^3)(2x)}{(x^2 + y^2)^2} = \dfrac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$

$f_y(x, y) = \dfrac{(x^2 + y^2)(x^3 - 3xy^2) - (x^3y - xy^3)(2y)}{(x^2 + y^2)^2} = \dfrac{x(x^4 - 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$

(b) $f_x(0, 0) = \lim_{\Delta x \to 0} \dfrac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{0/[(\Delta x)^2] - 0}{\Delta x} = 0$

$f_y(0, 0) = \lim_{\Delta y \to 0} \dfrac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \dfrac{0/[(\Delta y)^2] - 0}{\Delta y} = 0$

(c) $f_{xy}(0, 0) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)\bigg|_{(0, 0)} = \lim_{\Delta y \to 0} \dfrac{f_x(0, \Delta y) - f_x(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \dfrac{\Delta y(-(\Delta y)^4)}{((\Delta y)^2)^2(\Delta y)} = \lim_{\Delta y \to 0} (-1) = -1$

$f_{yx}(0, 0) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)\bigg|_{(0, 0)} = \lim_{\Delta x \to 0} \dfrac{f_y(\Delta x, 0) - f_y(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{\Delta x((\Delta x)^4)}{((\Delta x)^2)^2(\Delta x)} = \lim_{\Delta x \to 0} 1 = 1$

(d) f_{yx} or f_{xy} or both are not continuous at (0, 0).

99. True

101. True

Section 12.4 Differentials

1. $z = 3x^2y^3$

$$dz = 6xy^3\,dx + 9x^2y^2\,dy$$

3. $z = \dfrac{-1}{x^2 + y^2}$

$$dz = \frac{2x}{(x^2 + y^2)^2}\,dx + \frac{2y}{(x^2 + y^2)^2}\,dy$$

$$= \frac{2}{(x^2 + y^2)^2}(x\,dx + y\,dy)$$

5. $z = x \cos y - y \cos x$

$dz = (\cos y + y \sin x)\, dx + (-x \sin y - \cos x)\, dy = (\cos y + y \sin x)\, dx - (x \sin y + \cos x)\, dy$

7. $z = e^x \sin y$

$dz = (e^x \sin y)\, dx + (e^x \cos y)\, dy$

9. $w = 2z^3 y \sin x$

$dw = 2z^3 y \cos x\, dx + 2z^3 \sin x\, dy + 6z^2 y \sin x\, dz$

11. (a) $f(1, 2) = 4$

$f(1.05, 2.1) = 3.4875$

$\Delta z = f(1.05, 2.1) - f(1, 2) = -0.5125$

(b) $dz = -2x\, dx - 2y\, dy$

$= -2(0.05) - 4(0.1) = -0.5$

13. (a) $f(1, 2) = \sin 2$

$f(1.05, 2.1) = 1.05 \sin 2.1$

$\Delta z = f(1.05, 2.1) - f(1, 2) \approx -0.00293$

(b) $dz = \sin y\, dx + x \cos y\, dy$

$= (\sin 2)(0.05) + (\cos 2)(0.1) \approx 0.00385$

15. (a) $f(1, 2) = -5$

$f(1.05, 2.1) = -5.25$

$\Delta z = -0.25$

(b) $dz = 3\, dx - 4\, dy$

$= 3(0.05) - 4(0.1) \approx -0.25$

17. Let $z = \sqrt{x^2 + y^2}$, $x = 5$, $y = 3$, $dx = 0.05$, $dy = 0.1$. Then: $dz = \dfrac{x}{\sqrt{x^2 + y^2}}\, dx + \dfrac{y}{\sqrt{x^2 + y^2}}\, dy$

$$\sqrt{(5.05)^2 + (3.1)^2} - \sqrt{5^2 + 3^2} \approx \frac{5}{\sqrt{5^2 + 3^2}}(0.05) + \frac{3}{\sqrt{5^2 + 3^2}}(0.1) = \frac{0.55}{\sqrt{34}} \approx 0.094$$

19. Let $z = (1 - x^2)/y^2$, $x = 3$, $y = 6$, $dx = 0.05$, $dy = -0.05$. Then: $dz = -\dfrac{2x}{y^2}\, dx + \dfrac{-2(1 - x^2)}{y^3}\, dy$

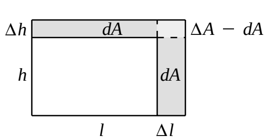
$$\frac{1 - (3.05)^2}{(5.95)^2} - \frac{1 - 3^2}{6^2} \approx -\frac{2(3)}{6^2}(0.05) - \frac{2(1 - 3^2)}{6^3}(-0.05) \approx -0.012$$

21. See the definition on page 869.

23. The tangent plane to the surface $z = f(x, y)$ at the point P is a linear approximation of z.

25. $A = lh$

$dA = l\, dh + h\, dl$



27. $V = \dfrac{\pi r^2 h}{3}$

$r = 3$

$h = 6$

$dV = \dfrac{2\pi rh}{3}\, dr + \dfrac{\pi r^2}{3}\, dh = \dfrac{\pi r}{3}(2h\, dr + r\, dh)$

Δr	Δh	dV	ΔV	$\Delta V - dV$
0.1	0.1	4.7124	4.8391	0.1267
0.1	−0.1	2.8274	2.8264	−0.0010
0.001	0.002	0.0565	0.0566	0.0001
−0.0001	0.0002	−0.0019	−0.0019	0.0000

29. (a) $dz = -1.83\,dx - 1.09\,dy$

(b) $dz = \dfrac{\partial z}{\partial x}\,dx + \dfrac{\partial z}{\partial y}\,dy$

$= -1.83(\pm 0.25) + (-1.09)(\pm 0.25)$

$= \pm 0.73$

Maximum propagated error: ± 0.73

Relative error: $\dfrac{dz}{z} = \dfrac{\pm 0.73}{(-1.83)(7.2) - 1.09(8.5) + 28.7} = \dfrac{\pm 0.73}{6.259} \approx \pm 0.1166 = 11.67\%$

31. $V = \pi r^2 h = dV = (2\pi rh)\,dr + (\pi r^2)\,dh$

$\dfrac{dV}{V} = 2\dfrac{dr}{r} + \dfrac{dh}{h}$

$= 2(0.04) + (0.02) = 0.10 = 10\%$

33. $A = \frac{1}{2}ab \sin C$

$dA = \frac{1}{2}[(b \sin C)\,da + (a \sin C)\,db + (ab \cos C)\,dC]$

$= \frac{1}{2}[4(\sin 45°)\left(\pm\frac{1}{16}\right) + 3(\sin 45°)\left(\pm\frac{1}{16}\right) + 12(\cos 45°)(\pm 0.02)] \approx \pm 0.24 \text{ in.}^2$

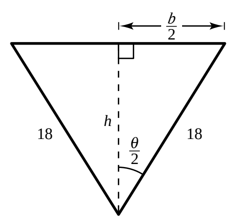
35. (a) $V = \dfrac{1}{2}bhl$

$= \left(18 \sin \dfrac{\theta}{2}\right)\left(18 \cos \dfrac{\theta}{2}\right)(16)(12)$

$= 31{,}104 \sin \theta \text{ in.}^3$

$= 18 \sin \theta \text{ ft}^3$

V is maximum when $\sin \theta = 1$ or $\theta = \pi/2$.



(b) $V = \dfrac{s^2}{2}(\sin \theta)l$

$dV = s(\sin \theta)l\,ds + \dfrac{s^2}{2}l(\cos \theta)\,d\theta + \dfrac{s^2}{2}(\sin \theta)\,dl$

$= 18\left(\sin \dfrac{\pi}{2}\right)(16)(12)\left(\dfrac{1}{2}\right) + \dfrac{18^2}{2}(16)(12)\left(\cos \dfrac{\pi}{2}\right)\left(\dfrac{\pi}{90}\right) + \dfrac{18^2}{2}\left(\sin \dfrac{\pi}{2}\right)\left(\dfrac{1}{2}\right)$

$= 1809 \text{ in}^3 \approx 1.047 \text{ ft}^3$

37. $P = \dfrac{E^2}{R}$

$dP = \dfrac{2E}{R}\,dE - \dfrac{E^2}{R^2}\,dR$

$\dfrac{dP}{P} = 2\dfrac{dE}{E} - \dfrac{dR}{R} = 2(0.02) - (-0.03) = 0.07 = 7\%$

39. $L = 0.00021\left(\ln \dfrac{2h}{r} - 0.75\right)$

$$dL = 0.00021\left[\frac{dh}{h} - \frac{dr}{r}\right] = 0.00021\left[\frac{(\pm 1/100)}{100} - \frac{(\pm 1/16)}{2}\right] \approx (\pm 6.6) \times 10^{-6}$$

$$L = 0.00021(\ln 100 - 0.75) \approx 8.096 \times 10^{-4} \pm dL = 8.096 \times 10^{-4} \pm 6.6 \times 10^{-6} \text{ micro–henrys}$$

41. $z = f(x, y) = x^2 - 2x + y$

$$\begin{aligned}
\Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y)\\
&= (x^2 + 2x(\Delta x) + (\Delta x)^2 - 2x - 2(\Delta x) + y + (\Delta y)) - (x^2 - 2x + y)\\
&= 2x(\Delta x) + (\Delta x)^2 - 2(\Delta x) + (\Delta y)\\
&= (2x - 2)\,\Delta x + \Delta y + \Delta x(\Delta x) + 0(\Delta y)\\
&= f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y \text{ where } \epsilon_1 = \Delta x \text{ and } \epsilon_2 = 0.
\end{aligned}$$

As $(\Delta x, \Delta y) \rightarrow (0, 0)$, $\epsilon_1 \rightarrow 0$ and $\epsilon_2 \rightarrow 0$.

43. $z = f(x, y) = x^2 y$

$$\begin{aligned}
\Delta z &= f(x + \Delta x,\ y + \Delta y) - f(x, y)\\
&= (x^2 + 2x(\Delta x) + (\Delta x)^2)(y + \Delta y) - x^2 y\\
&= 2xy(\Delta x) + y(\Delta x)^2 + x^2\Delta y + 2x(\Delta x)(\Delta y) + (\Delta x)^2\,\Delta y\\
&= 2xy(\Delta x) + x^2\Delta y + (y\Delta x)\,\Delta x + [2x\Delta x + (\Delta x)^2]\,\Delta y\\
&= f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y \text{ where } \epsilon_1 = y(\Delta x) \text{ and } \epsilon_2 = 2x\Delta x + (\Delta x)^2.
\end{aligned}$$

As $(\Delta x, \Delta y) \rightarrow (0, 0)$, $\epsilon_1 \rightarrow 0$ and $\epsilon_2 \rightarrow 0$.

45. $f(x, y) = \begin{cases} \dfrac{3x^2 y}{x^4 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

(a) $$f_x(0, 0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{0}{(\Delta x)^4} - 0}{\Delta x} = 0$$

$$f_y(0, 0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\dfrac{0}{(\Delta y)^2} - 0}{\Delta y} = 0$$

Thus, the partial derivatives exist at $(0, 0)$.

(b) Along the line $y = x$: $\displaystyle\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{x \to 0} \frac{3x^3}{x^4 + x^2} = \lim_{x \to 0} \frac{3x}{x^2 + 1} = 0$

Along the curve $y = x^2$: $\displaystyle\lim_{(x, y) \to (0, 0)} f(x, y) = \frac{3x^4}{2x^4} = \frac{3}{2}$

f is not continuous at $(0, 0)$. Therefore, f is not differentiable at $(0, 0)$. (See Theroem 12.5)

47. Essay. For example, we can use the equation $F = ma$:

$$dF = \frac{\partial F}{\partial m}\,dm + \frac{\partial F}{\partial a}\,da = a\,dm + m\,da.$$

Section 12.5 Chain Rules for Functions of Several Variables

1. $w = x^2 + y^2$

$x = e^t$

$y = e^{-t}$

$$\frac{dw}{dt} = 2xe^t + 2y(-e^{-t}) = 2(e^{2t} - e^{-2t})$$

3. $w = x \sec y$

$x = e^t$

$y = \pi - t$

$$\frac{dw}{dt} = (\sec y)(e^t) + (x \sec y \tan y)(-1)$$
$$= e^t \sec(\pi - t)[1 - \tan(\pi - t)]$$
$$= -e^t(\sec t + \sec t \tan t)$$

5. $w = xy, \; x = 2 \sin t, \; y = \cos t$

(a) $\dfrac{dw}{dt} = 2y \cos t + x(-\sin t) = 2y \cos t - x \sin t$

$= 2(\cos^2 t - \sin^2 t) = 2 \cos 2t$

(b) $w = 2 \sin t \cos t = \sin 2t, \; \dfrac{dw}{dt} = 2 \cos 2t$

7. $w = x^2 + y^2 + z^2$

$x = e^t \cos t$

$y = e^t \sin t$

$z = e^t$

(a) $\dfrac{dw}{dt} = 2x(-e^t \sin t + e^t \cos t) + 2y(e^t \cos t + e^t \sin t) + 2ze^t = 4e^{2t}$

(b) $w = 2e^{2t}, \; \dfrac{dw}{dt} = 4e^{2t}$

9. $w = xy + xz + yz, \; x = t - 1, \; y = t^2 - 1, \; z = t$

(a) $\dfrac{dw}{dt} = (y + z) = (x + z)(2t) + (x + y)$

$= (t^2 - 1 + t) + (t - 1 + 1)(2t) + (t - 1 + t^2 - 1) = 3(2t^2 - 1)$

(b) $w = (t - 1)(t^2 - 1) + (t - 1)t + (t^2 - 1)t$

$\dfrac{dw}{dt} = 2t(t - 1) + (t^2 - 1) + 2t - 1 + 3t^2 - 1 = 3(2t^2 - 1)$

11. Distance $= f(t) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(10 \cos 2t - 7 \cos t)^2 + (6 \sin 2t - 4 \sin t)^2}$

$$f'(t) = \frac{1}{2}[(10 \cos 2t - 7 \cos t)^2 + (6 \sin 2t - 4 \sin t)^2]^{-1/2}$$
$$[[2(10 \cos 2t - 7 \cos t)(-20 \sin 2t + 7 \sin t)] + [2(6 \sin 2t - 4 \sin t)(12 \cos 2t - 4 \cos t)]]$$
$$f'\left(\frac{\pi}{2}\right) = \frac{1}{2}[(-10)^2 + 4^2]^{-1/2}[[2(-10)(7)] + (2(-4)(-12)]$$
$$= \frac{1}{2}(116)^{-1/2}(-44) = \frac{22}{2\sqrt{29}} = \frac{-11\sqrt{29}}{20} \approx -2.04$$

13. $w = \arctan(2xy),\ x = \cos t,\ y = \sin t,\ t = 0$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= \frac{2y}{1 + (4x^2y^2)}(-\sin t) + \frac{2x}{1 + (4x^2y^2)}(\cos t)$$

$$= \frac{2\sin t}{1 + 4\cos^2 t\sin^2 t}(-\sin t) + \frac{2\cos t}{1 + 4\cos^2 t\sin^2 t}(\cos t)$$

$$= \frac{2\cos^2 t - 2\sin^2 t}{1 + 4\cos^2 t\sin^2 t}$$

$$\frac{d^2w}{dt^2} = \frac{(1 + 4\cos^2 t\sin^2 t)(-8\cos t\sin t) - (2\cos^2 t - 2\sin^2 t)(8\cos^3 t\sin t - 8\sin^3 t\cos t)}{(1 + 4\cos^2 t\sin^2 t)^2}$$

$$= \frac{-8\cos t\sin t(1 + 2\sin^4 t + 2\cos^4 t)}{(1 + 4\cos^2 t\sin^2 t)^2}$$

At $t = 0$, $\dfrac{d^2w}{dt^2} = 0$.

15. $w = x^2 + y^2$

$x = s + t$

$y = s - t$

$$\frac{\partial w}{\partial s} = 2x + 2y = 2(x + y) = 4s$$

$$\frac{\partial w}{\partial t} = 2x + 2y(-1) = 2(x - y) = 4t$$

When $s = 2$ and $t = -1$,

$$\frac{\partial w}{\partial s} = 8 \text{ and } \frac{\partial w}{\partial t} = -4.$$

17. $w = x^2 - y^2$

$x = s\cos t$

$y = s\sin t$

$$\frac{\partial w}{\partial s} = 2x\cos t - 2y\sin t$$

$$= 2s\cos^2 t - 2s\sin^2 t = 2s\cos 2t$$

$$\frac{\partial w}{\partial t} = 2x(-s\sin t) - 2y(s\cos t) = -2s^2\sin 2t$$

When $s = 3$ and $t = \dfrac{\pi}{4}$, $\dfrac{\partial w}{\partial s} = 0$ and $\dfrac{\partial w}{\partial t} = -18$.

19. $w = x^2 - 2xy + y^2,\ x = r + \theta,\ y = r - \theta$

(a) $$\frac{\partial w}{\partial r} = (2x - 2y)(1) + (-2x + 2y)(1) = 0$$

$$\frac{\partial w}{\partial \theta} = (2x - 2y)(1) + (-2x + 2y)(-1)$$

$$= 4x - 4y = 4(x - y)$$

$$= 4[(r + \theta) - (r - \theta)] = 8\theta$$

(b) $$w = (r + \theta)^2 - 2(r + \theta)(r - \theta) + (r - \theta)^2$$

$$= (r^2 + 2r\theta + \theta^2) - 2(r^2 - \theta^2) + (r^2 - 2r\theta + \theta^2)$$

$$= 4\theta^2$$

$$\frac{\partial w}{\partial r} = 0$$

$$\frac{\partial w}{\partial \theta} = 8\theta$$

21. $w = \arctan\dfrac{y}{x},\ x = r\cos\theta,\ y = r\sin\theta$

(a) $\dfrac{\partial w}{\partial r} = \dfrac{-y}{x^2+y^2}\cos\theta + \dfrac{x}{x^2+y^2}\sin\theta = \dfrac{-r\sin\theta\cos\theta}{r^2} + \dfrac{r\cos\theta\sin\theta}{r^2} = 0$

$\dfrac{\partial w}{\partial \theta} = \dfrac{-y}{x^2+y^2}(-r\sin\theta) + \dfrac{x}{x^2+y^2}(r\cos\theta) = \dfrac{-(r\sin\theta)(-r\sin\theta)}{r^2} + \dfrac{(r\cos\theta)(r\cos\theta)}{r^2} = 1$

(b) $w = \arctan\dfrac{r\sin\theta}{r\cos\theta} = \arctan(\tan\theta) = \theta$

$\dfrac{\partial w}{\partial r} = 0$

$\dfrac{\partial w}{\partial \theta} = 1$

23. $w = xyz,\ x = s+t, y = s-t,\ z = st^2$

$$\begin{aligned}\frac{\partial w}{\partial s} &= yz(1) + xz(1) + xy(t^2)\\ &= (s-t)st^2 + (s+t)st^2 + (s+t)(s-t)t^2\\ &= 2s^2t^2 + s^2t^2 - t^4 = 3s^2t^2 - t^4 = t^2(3s^2 - t^2)\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial t} &= yz(1) + xz(-1) + xy(2st)\\ &= (s-t)st^2 - (s+t)st^2 + (s+t)(s-t)(2st)\\ &= -2st^3 + 2s^3t - 2st^3 = 2s^3t - 4st^3 = 2st(s^2 - 2t^2)\end{aligned}$$

25. $w = ze^{x/y}, x = s-t, y = s+t, z = st$

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{z}{y}e^{x/y}(1) + -\frac{zx}{y^2}e^{x/y}(1) + e^{x/y}(t)\\ &= e^{(s-t/s+t)}\left[\frac{st}{s+t} - \frac{(s-t)st}{(s+t)^2} + t\right]\\ &= e^{(s-t/s+t)}\left[\frac{st(s+t) - s^2t + st^2 + t(s+t)^2}{(s+t)^2}\right]\\ &= e^{(s-t/s+t)}\frac{t(s^2+4st+t^2)}{(s+t)^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{z}{y}e^{x/y}(-1) + -\frac{zx}{y^2}e^{x/y}(1) + e^{x/y}(s)\\ &= e^{(s-t/s+t)}\left[-\frac{st}{s+t} - \frac{st(s-t)}{(s+t)^2} + s\right]\\ &= e^{(s-t/s+t)}\left[\frac{-st(s+t) - st(s-t) + s(s+t)^2}{(s+t)^2}\right]\\ &= e^{(s-t/s+t)}\frac{s(s^2+t^2)}{(s+t)^2}\end{aligned}$$

27. $x^2 - 3xy + y^2 - 2x + y - 5 = 0$

$$\begin{aligned}\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)} &= -\frac{2x-3y-2}{-3x+2y+1}\\ &= \frac{3y-2x+2}{2y-3x+1}\end{aligned}$$

29. $\ln\sqrt{x^2+y^2} + xy = 4$

$\dfrac{1}{2}\ln(x^2+y^2) + xy - 4 = 0$

$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)} = -\frac{\dfrac{x}{x^2+y^2} + y}{\dfrac{y}{x^2+y^2} + x} = -\frac{x + x^2y + y^3}{y + xy^2 + x^3}$$

31. $F(x,y,z) = x^2 + y^2 + z^2 - 25$

$F_x = 2x$

$F_y = 2y$

$F_z = 2z$

$\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{x}{z}$

$\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{y}{z}$

33. $F(x,y,z) = \tan(x+y) + \tan(y+z) - 1$

$F_x = \sec^2(x+y)$

$F_y = \sec^2(x+y) + \sec^2(y+z)$

$F_z = \sec^2(y+z)$

$\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{\sec^2(x+y)}{\sec^2(y+z)}$

$$\begin{aligned}\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} &= -\frac{\sec^2(x+y) + \sec^2(y+z)}{\sec^2(y+z)}\\ &= -\left(\frac{\sec^2(x+y)}{\sec^2(y+z)} + 1\right)\end{aligned}$$

35. $x^2 + 2yz + z^2 - 1 = 0$

(i) $2x + 2y\dfrac{\partial z}{\partial x} + 2z\dfrac{\partial z}{\partial x} = 0$ implies $\dfrac{\partial z}{\partial x} = -\dfrac{x}{y + z}$.

(ii) $2y\dfrac{\partial z}{\partial y} + 2z + 2z\dfrac{\partial z}{\partial y} = 0$ implies $\dfrac{\partial z}{\partial y} = -\dfrac{z}{y + z}$.

37. $e^{xz} + xy = 0$

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} = -\frac{ze^{xz} + y}{xe^{xz}}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)} = \frac{-x}{xe^{xz}} = \frac{-1}{e^{xz}} = -e^{-xz}$$

39. $F(x, y, z, w) = xyz + xzw - yzw + w^2 - 5$

$F_x = yz + zw$

$F_y = xz - zw$

$F_z = xy + xw - yw$

$F_w = xz - yz + 2w$

$$\frac{\partial w}{\partial x} = -\frac{F_x}{F_w} = -\frac{z(y + w)}{xz - yz + 2w}$$

$$\frac{\partial w}{\partial y} = -\frac{F_y}{F_w} = -\frac{z(x - w)}{xz - yz + 2w}$$

$$\frac{\partial w}{\partial z} = -\frac{F_z}{F_w} = -\frac{xy + xw - yw}{xz - yz + 2w}$$

41. $F(x, y, z, w) = \cos xy + \sin yz + wz - 20$

$$\frac{\partial w}{\partial x} = \frac{-F_x}{F_w} = \frac{y \sin xy}{z}$$

$$\frac{\partial w}{\partial y} = \frac{-F_y}{F_w} = \frac{x \sin xy - z \cos yz}{z}$$

$$\frac{\partial w}{\partial z} = \frac{-F_z}{F_w} = -\frac{y \cos zy + w}{z}$$

43. $f(x, y) = \dfrac{xy}{\sqrt{x^2 + y^2}}$

$$f(tx, ty) = \frac{(tx)(ty)}{\sqrt{(tx)^2 + (ty)^2}} = t\left(\frac{xy}{\sqrt{x^2 + y^2}}\right) = tf(x, y)$$

Degree: 1

$$xf_x(x, y) + yf_y(x, y) = x\left(\frac{y^3}{(x^2 + y^2)^{3/2}}\right) + y\left(\frac{x^3}{(x^2 + y^2)^{3/2}}\right)$$

$$= \frac{xy}{\sqrt{x^2 + y^2}} = 1f(x, y)$$

45. $f(x, y) = e^{x/y}$

$f(tx, ty) = e^{tx/ty} = e^{x/y} = f(x, y)$

Degree: 0

$$xf_x(x, y) + yf_y(x, y) = x\left(\frac{1}{y}e^{x/y}\right) + y\left(-\frac{x}{y^2}e^{x/y}\right) = 0$$

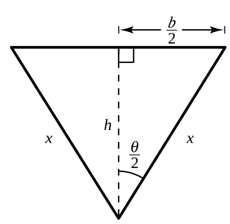
47. $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt}$ (Page 876)

49. $w = f(x, y)$ is the explicit form of a function of two variables, as in $z = x^2 + y^2$.
The implicit form is of the form $F(x, y, z) = 0$, as in $z - x^2 - y^2 = 0$.

51. $A = \dfrac{1}{2}bh = \left(x \sin\dfrac{\theta}{2}\right)\left(x \cos\dfrac{\theta}{2}\right) = \dfrac{x^2}{2}\sin\theta$

$$\frac{dA}{dt} = x \sin\theta\frac{dx}{dt} + \frac{x^2}{2}\cos\theta\frac{d\theta}{dt}$$

$$= 6\left(\sin\frac{\pi}{4}\right)\left(\frac{1}{2}\right) + \frac{6^2}{2}\left(\cos\frac{\pi}{4}\right)\left(\frac{\pi}{90}\right) = \frac{3\sqrt{2}}{2} + \frac{\pi\sqrt{2}}{10} \text{ m}^2/\text{hr}$$



53. (a) $V = \frac{1}{3}\pi r^2 h$

$$\frac{dV}{dt} = \frac{1}{3}\pi\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right) = \frac{1}{3}\pi[2(12)(36)(6) + (12)^2(-4)] = 1536\pi \text{ in.}^3/\text{min}$$

(b) $S = \pi r\sqrt{r^2 + h^2} + \pi r^2$ (Surface area includes base.)

$$\frac{dS}{dt} = \pi\left[\left(\sqrt{r^2 + h^2} + \frac{r^2}{\sqrt{r^2 + h^2}} + 2r\right)\frac{dr}{dt} + \frac{rh}{\sqrt{r^2 + h^2}}\frac{dh}{dt}\right]$$

$$= \pi\left[\left(\sqrt{12^2 + 36^2} + \frac{144}{\sqrt{12^2 + 36^2}} + 2(12)\right)(6) + \frac{36(12)}{\sqrt{12^2 + 36^2}}(-4)\right]$$

$$= \pi\left[\left(12\sqrt{10} + \frac{12}{\sqrt{10}}\right)(6) + 144 + \frac{36}{\sqrt{10}}(-4)\right]$$

$$= \frac{648\pi}{\sqrt{10}} + 144\pi \text{ in.}^2/\text{min} = \frac{36\pi}{5}\left(20 + 9\sqrt{10}\right) \text{ in.}^2/\text{min}$$

55. $I = \frac{1}{2}m(r_1{}^2 + r_2{}^2)$

$$\frac{dI}{dt} = \frac{1}{2}m\left[2r_1\frac{dr_1}{dt} + 2r_2\frac{dr_2}{dt}\right] = m[(6)(2) + (8)(2)] = 28m \text{ cm}^2/\text{sec}$$

57. (a)

$$\tan\phi = \frac{2}{x}$$

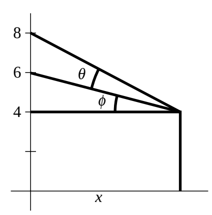
$$\tan(\theta + \phi) = \frac{4}{x}$$

$$\frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} = \frac{4}{x}$$

$$\frac{\tan\theta + (2/x)}{1 - (2/x)\tan\theta} = \frac{4}{x}$$

$$x\tan\theta + 2 = 4 - \frac{8}{x}\tan\theta$$

$$x^2\tan\theta - 2x + 8\tan\theta = 0$$



(b) $F(x, \theta) = (x^2 + 8)\tan\theta - 2x = 0$

$$\frac{d\theta}{dx} = -\frac{F_x}{F_\theta} = -\frac{2x\tan\theta - 2}{\sec^2\theta(x^2 + 8)} = \frac{2\cos^2\theta - 2x\sin\theta\cos\theta}{x^2 + 8}$$

(c) $\frac{d\theta}{dx} = 0 \implies 2\cos^2\theta = 2x\sin\theta\cos\theta \implies \cos\theta = x\sin\theta \implies \tan\theta = \frac{1}{x}$

Thus, $x^2\left(\frac{1}{x}\right) - 2x + 8\left(\frac{1}{x}\right) = 0 \implies \frac{8}{x} = x \implies x = 2\sqrt{2}$ ft.

59.

$$w = f(x, y)$$

$$x = u - v$$

$$y = v - u$$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{dx}{du} + \frac{\partial w}{\partial y}\frac{dy}{du} = \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{dx}{dv} + \frac{\partial w}{\partial y}\frac{dy}{dv} = -\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}$$

$$\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} = 0$$

61. $w = f(x, y),\ x = r\cos\theta,\ y = r\sin\theta$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cos\theta + \frac{\partial w}{\partial y}\sin\theta$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}(-r\sin\theta) + \frac{\partial w}{\partial y}(r\cos\theta)$$

(a)

$$r\cos\theta\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}r\cos^2\theta + \frac{\partial w}{\partial y}r\sin\theta\cos\theta$$

$$-\sin\theta\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}(r\sin^2\theta) - \frac{\partial w}{\partial x}r\sin\theta\cos\theta$$

$$r\cos\theta\frac{\partial w}{\partial r} - \sin\theta\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}(r\cos^2\theta + r\sin^2\theta)$$

$$r\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}(r\cos\theta) - \frac{\partial w}{\partial \theta}\sin\theta$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\cos\theta - \frac{\partial w}{\partial \theta}\frac{\sin\theta}{r}$$

$$r\sin\theta\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}r\sin\theta\cos\theta + \frac{\partial w}{\partial y}r\sin^2\theta$$

$$\cos\theta\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}(-r\sin\theta\cos\theta) + \frac{\partial w}{\partial y}(r\cos^2\theta)$$

$$r\sin\theta\frac{\partial w}{\partial r} + \cos\theta\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial y}(r\sin^2\theta + r\cos^2\theta)$$

$$r\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}r\sin\theta + \frac{\partial w}{\partial \theta}\cos\theta$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\sin\theta + \frac{\partial w}{\partial \theta}\frac{\cos\theta}{r}$$

(b) $$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial w}{\partial x}\right)^2\cos^2\theta + 2\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\sin\theta\cos\theta + \left(\frac{\partial w}{\partial y}\right)^2\sin^2\theta + \left(\frac{\partial w}{\partial x}\right)^2\sin^2\theta -$$

$$2\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\sin\theta\cos\theta + \left(\frac{\partial w}{\partial y}\right)^2\cos^2\theta = \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2$$

63. Given $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, $x = r\cos\theta$ and $y = r\sin\theta$.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta = \frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\sin\theta$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta) = r\left[\frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\sin\theta\right]$$

Therefore, $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$.

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta = -\frac{\partial u}{\partial y}\cos\theta + \frac{\partial u}{\partial x}\sin\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta) = -r\left[-\frac{\partial u}{\partial y}\cos\theta + \frac{\partial u}{\partial x}\sin\theta\right]$$

Therefore, $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

Section 12.6 Directional Derivatives and Gradients

1. $f(x, y) = 3x - 4xy + 5y$

$$\mathbf{v} = \frac{1}{2}(\mathbf{i} + \sqrt{3}\,\mathbf{j})$$

$$\nabla f(x, y) = (3 - 4y)\mathbf{i} + (-4x + 5)\mathbf{j}$$

$$\nabla f(1, 2) = -5\mathbf{i} + \mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$$

$$D_{\mathbf{u}} f(1, 2) = \nabla f(1, 2) \cdot \mathbf{u} = \frac{1}{2}(-5 + \sqrt{3})$$

3. $f(x, y) = xy$

$$\mathbf{v} = \mathbf{i} + \mathbf{j}$$

$$\nabla f(x, y) = y\mathbf{i} + x\mathbf{j}$$

$$\nabla f(2, 3) = 3\mathbf{i} + 2\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

$$D_{\mathbf{u}} f(2, 3) = \nabla f(2, 3) \cdot \mathbf{u} = \frac{5\sqrt{2}}{2}$$

5. $g(x, y) = \sqrt{x^2 + y^2}$

$$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$$

$$\nabla g = \frac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}}\mathbf{j}$$

$$\nabla g(3, 4) = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$$

$$D_{\mathbf{u}} g(3, 4) = \nabla g(3, 4) \cdot \mathbf{u} = -\frac{7}{25}$$

7. $h(x, y) = e^x \sin y$

$$\mathbf{v} = -\mathbf{i}$$

$$\nabla h = e^x \sin y\,\mathbf{i} + e^x \cos y\,\mathbf{j}$$

$$h\left(1, \frac{\pi}{2}\right) = e\mathbf{i}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = -\mathbf{i}$$

$$D_{\mathbf{u}} h\left(1, \frac{\pi}{2}\right) = \nabla h\left(1, \frac{\pi}{2}\right) \cdot \mathbf{u} = -e$$

9. $f(x, y, z) = xy + yz + xz$

$$\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$$

$$\nabla f(x, y, z) = (y + z)\mathbf{i} + (x + z)\mathbf{j} + (x + y)\mathbf{k}$$

$$\nabla f(1, 1, 1) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\sqrt{6}}{3}\mathbf{i} + \frac{\sqrt{6}}{6}\mathbf{j} - \frac{\sqrt{6}}{6}\mathbf{k}$$

$$D_{\mathbf{u}} f(1, 1, 1) = \nabla f(1, 1, 1) \cdot \mathbf{u} = \frac{2\sqrt{6}}{3}$$

11. $h(x, y, z) = x \arctan yz$

$$\mathbf{v} = \langle 1, 2, -1 \rangle$$

$$\nabla h(x, y, z) = \arctan yz\,\mathbf{i} + \frac{xz}{1 + (yz)^2}\mathbf{j} + \frac{xy}{1 + (yz)^2}\mathbf{k}$$

$$\nabla h(4, 1, 1) = \frac{\pi}{4}\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right\rangle$$

$$D_{\mathbf{u}} h(4, 1, 1) = \nabla h(4, 1, 1) \cdot \mathbf{u} = \frac{\pi + 8}{4\sqrt{6}} = \frac{(\pi + 8)\sqrt{6}}{24}$$

13. $f(x, y) = x^2 + y^2$

$$\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$$\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \frac{2}{\sqrt{2}}x + \frac{2}{\sqrt{2}}y = \sqrt{2}(x + y)$$

15. $f(x, y) = \sin(2x - y)$

$$\mathbf{u} = \frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j}$$

$$\nabla f = 2\cos(2x - y)\mathbf{i} - \cos(2x - y)\mathbf{j}$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \cos(2x - y) + \frac{\sqrt{3}}{2}\cos(2x - y)$$

$$= \left(\frac{2 + \sqrt{3}}{2}\right)\cos(2x - y)$$

17. $f(x, y) = x^2 + 4y^2$

$$\mathbf{v} = -2\mathbf{i} - 2\mathbf{j}$$

$$\nabla f = 2x\,\mathbf{i} + 8y\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$$

$$D_{\mathbf{u}} f = -\frac{2}{\sqrt{2}}x - \frac{8}{\sqrt{2}}y = -\sqrt{2}(x + 4y)$$

At $P = (3, 1)$, $D_{\mathbf{u}} f = -7\sqrt{2}$.

19. $h(x, y, z) = \ln(x + y + z)$

$$\mathbf{v} = 3\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

$$\nabla h = \frac{1}{x + y + z}(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

At $(1, 0, 0)$, $\nabla h = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{19}}(3\mathbf{i} + 3\mathbf{j} + \mathbf{k})$$

$$D_{\mathbf{u}} h = \nabla h \cdot \mathbf{u} = \frac{7}{\sqrt{19}} = \frac{7\sqrt{19}}{19}$$

21. $f(x, y) = 3x - 5y^2 + 10$

$$\nabla f(x, y) = 3\mathbf{i} - 10y\mathbf{j}$$

$$\nabla f(2, 1) = 3\mathbf{i} - 10\mathbf{j}$$

23. $z = \cos(x^2 + y^2)$

$$\nabla z(x, y) = -2x\sin(x^2 + y^2)\mathbf{i} - 2y\sin(x^2 + y^2)\mathbf{j}$$

$$\nabla z(3, -4) = -6\sin 25\mathbf{i} + 8\sin 25\mathbf{j} \approx 0.7941\mathbf{i} - 1.0588\mathbf{j}$$

25. $w = 3x^2y - 5yz + z^2$

$$\nabla w(x, y, z) = 6xy\mathbf{i} + (3x^2 - 5z)\mathbf{j} + (2z - 5y)\mathbf{k}$$

$$\nabla w(1, 1, -2) = 6\mathbf{i} + 13\mathbf{j} - 9\mathbf{k}$$

27. $\overrightarrow{PQ} = 2\mathbf{i} + 4\mathbf{j},\ \mathbf{u} = \frac{1}{\sqrt{5}}\mathbf{i} = \frac{2}{\sqrt{5}}\mathbf{j}$

$$\nabla g(x, y) = 2x\mathbf{i} + 2y\mathbf{j},\ \nabla g(1, 2) = 2\mathbf{i} + 4\mathbf{j}$$

$$D_{\mathbf{u}} g = \nabla g \cdot \mathbf{u} = \frac{2}{\sqrt{5}} + \frac{8}{\sqrt{5}} = \frac{10}{\sqrt{5}} = 2\sqrt{5}$$

29. $\overrightarrow{PQ} = 2\mathbf{i} + \mathbf{j},\ \mathbf{u} = \frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}$

$$\nabla f(x, y) = -e^{-x}\cos y\mathbf{i} - e^{-x}\sin y\mathbf{j}$$

$$\nabla f(0, 0) = -\mathbf{i}$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$$

31. $h(x, y) = x\tan y$

$$\nabla h(x, y) = \tan y\mathbf{i} + x\sec^2 y\mathbf{j}$$

$$\nabla h\left(2, \frac{\pi}{4}\right) = \mathbf{i} + 4\mathbf{j}$$

$$\left\|\nabla h\left(2, \frac{\pi}{4}\right)\right\| = \sqrt{17}$$

33. $g(x, y) = \ln\sqrt[3]{x^2 + y^2} = \frac{1}{3}\ln(x^2 + y^2)$

$$\nabla g(x, y) = \frac{1}{3}\left[\frac{2x}{x^2 + y^2}\mathbf{i} + \frac{2y}{x^2 + y^2}\mathbf{j}\right]$$

$$\nabla g(1, 2) = \frac{1}{3}\left(\frac{2}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}\right) = \frac{2}{15}(\mathbf{i} + 2\mathbf{j})$$

$$\|\nabla g(1, 2)\| = \frac{2\sqrt{5}}{15}$$

35. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

$$\nabla f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$\nabla f(1, 4, 2) = \frac{1}{\sqrt{21}}(\mathbf{i} + 4\mathbf{j} + 2\mathbf{k})$$

$$\|\nabla f(1, 4, 2)\| = 1$$

37. $f(x, y, z) = xe^{yz}$

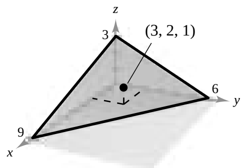
$$\nabla f(x, y, z) = e^{yz}\mathbf{i} + xze^{yz}\mathbf{j} + xye^{yz}\mathbf{k}$$

$$\nabla f(2, 0, -4) = \mathbf{i} - 8\mathbf{j}$$

$$\|\nabla f(2, 0, -4)\| = \sqrt{65}$$

For Exercises 39–45, $f(x, y) = 3 - \frac{x}{3} - \frac{y}{2}$ and $D_\theta f(x, y) = -\left(\frac{1}{3}\right)\cos\theta - \left(\frac{1}{2}\right)\sin\theta$.

39. $f(x, y) = 3 - \frac{x}{3} - \frac{y}{2}$



41. (a) $D_{4\pi/3} f(3, 2) = -\left(\frac{1}{3}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2}\right)$

$= \frac{2 + 3\sqrt{3}}{12}$

(b) $D_{-\pi/6} f(3, 2) = -\left(\frac{1}{3}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)$

$= \frac{3 - 2\sqrt{3}}{12}$

43. (a) $\mathbf{v} = -3\mathbf{i} + 4\mathbf{j}$

$\|\mathbf{v}\| = \sqrt{9 + 16} = 5$

$\mathbf{u} = -\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$

$D_\mathbf{u} f = \nabla f \cdot \mathbf{u} = \frac{1}{5} - \frac{2}{5} = -\frac{1}{5}$

(b) $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$

$\|\mathbf{v}\| = \sqrt{10}$

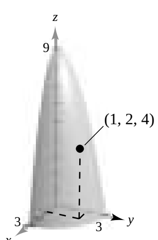
$\mathbf{u} = \frac{1}{\sqrt{10}}\mathbf{i} + \frac{3}{\sqrt{10}}\mathbf{j}$

$D_\mathbf{u} f = \nabla f \cdot \mathbf{u} = \frac{-11}{6\sqrt{10}} = -\frac{11\sqrt{10}}{60}$

45. $\|\nabla f\| = \sqrt{\frac{1}{9} + \frac{1}{4}} = \frac{1}{6}\sqrt{13}$

For Exercises 47 and 49, $f(x, y) = 9 - x^2 - y^2$ and $D_\theta f(x, y) = -2x\cos\theta - 2y\sin\theta = -2(x\cos\theta + y\sin\theta)$.

47. $f(x, y) = 9 - x^2 - y^2$



49. $\nabla f(1, 2) = -2\mathbf{i} - 4\mathbf{j}$

$\|\nabla f(1, 2)\| = \sqrt{4 + 16} = \sqrt{20} = 2\sqrt{5}$

51. (a) In the direction of the vector $-4\mathbf{i} + \mathbf{j}$.

(b) $\nabla f = \frac{1}{10}(2x - 3y)\mathbf{i} + \frac{1}{10}(-3x + 2y)\mathbf{j}$

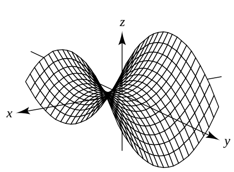
$\nabla f(1, 2) = \frac{1}{10}(-4)\mathbf{i} + \frac{1}{10}(1)\mathbf{j} = -\frac{2}{5}\mathbf{i} + \frac{1}{10}\mathbf{j}$

(Same direction as in part (a).)

(c) $-\nabla f = \frac{2}{5}\mathbf{i} - \frac{1}{10}\mathbf{j}$, the direction opposite that of the gradient.

53. $f(x, y) = x^2 - y^2$, $(4, -3, 7)$

(a)

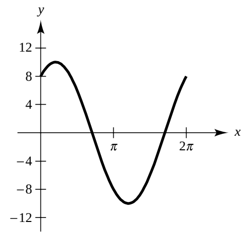


—CONTINUED–

53. —CONTINUED—

(b) $D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u} = 2x \cos \theta - 2y \sin \theta$

$D_{\mathbf{u}} f(4, -3) = 8 \cos \theta + 6 \sin \theta$



Generated by Mathematica

(c) Zeros: $\theta \approx 2.21, 5.36$

These are the angles θ for which $D_{\mathbf{u}} f(4, 3)$ equals zero.

(d) $g(\theta) = D_{\mathbf{u}} f(4, -3) = 8 \cos \theta + 6 \sin \theta$

$g'(\theta) = -8 \sin \theta + 6 \cos \theta$

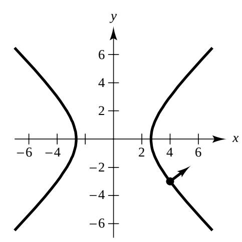
Critical numbers: $\theta \approx 0.64, 3.79$

These are the angles for which $D_{\mathbf{u}} f(4, -3)$ is a maximum (0.64) and minimum (3.79).

(e) $\|\nabla f(4, -3)\| = \|2(4)\mathbf{i} - 2(3)\mathbf{j}\| = \sqrt{64 + 36} = 10$, the maximum value of $D_{\mathbf{u}} f(4, -3)$, at $\theta = 0.64$.

(f) $f(x, y) = x^2 - y^2 = 7$

$\nabla f(4, -3) = 8\mathbf{i} + 6\mathbf{j}$ is perpendicular to the level curve at $(4, -3)$.



Generated by Mathematica

55. $f(x, y) = x^2 + y^2$

$c = 25,\ P = (3, 4)$

$\nabla f(x, y) = 2x\mathbf{i} + 2y\mathbf{j}$

$x^2 + y^2 = 25$

$\nabla f(3, 4) = 6\mathbf{i} + 8\mathbf{j}$

57. $f(x, y) = \dfrac{x}{x^2 + y^2}$

$c = \dfrac{1}{2},\ P = (1, 1)$

$\nabla f(x, y) = \dfrac{y^2 - x^2}{(x^2 + y^2)^2}\mathbf{i} - \dfrac{2xy}{(x^2 + y^2)^2}\mathbf{j}$

$\dfrac{x}{x^2 + y^2} = \dfrac{1}{2}$

$x^2 + y^2 - 2x = 0$

$\nabla f(1, 1) = -\dfrac{1}{2}\mathbf{j}$

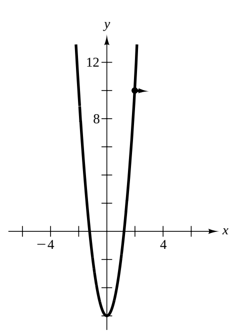
59. $4x^2 - y = 6$

$f(x, y) = 4x^2 - y$

$\nabla f(x, y) = 8x\mathbf{i} - \mathbf{j}$

$\nabla f(2, 10) = 16\mathbf{i} - \mathbf{j}$

$$\frac{\nabla f(2, 10)}{\|\nabla f(2, 10)\|} = \frac{1}{\sqrt{257}}(16\mathbf{i} - \mathbf{j}) = \frac{\sqrt{257}}{257}(16\mathbf{i} - \mathbf{j})$$



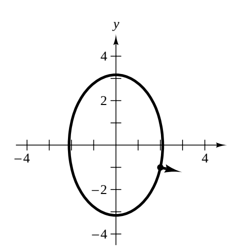
61. $9x^2 + 4y^2 = 40$

$f(x, y) = 9x^2 + 4y^2$

$\nabla f(x, y) = 18x\mathbf{i} + 8y\mathbf{j}$

$\nabla f(2, -1) = 36\mathbf{i} - 8\mathbf{j}$

$$\frac{\nabla f(2, -1)}{\|\nabla f(2, -1)\|} = \frac{1}{\sqrt{85}}(9\mathbf{i} - 2\mathbf{j}) = \frac{\sqrt{85}}{85}(9\mathbf{i} - 2\mathbf{j})$$



63. $T = \dfrac{x}{x^2 + y^2}$

$\nabla T = \dfrac{y^2 - x^2}{(x^2 + y^2)^2}\mathbf{i} - \dfrac{2xy}{(x^2 + y^2)^2}\mathbf{j}$

$\nabla T(3, 4) = \dfrac{7}{625}\mathbf{i} - \dfrac{24}{625}\mathbf{j} = \dfrac{1}{625}(7\mathbf{i} - 24\mathbf{j})$

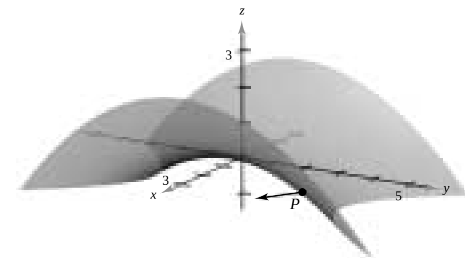
65. See the definition, page 885.

67. Let $f(x, y)$ be a function of two variables and $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ a unit vector.

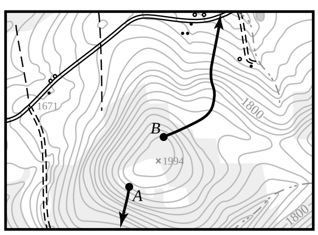
(a) If $\theta = 0°$, then $D_\mathbf{u} f = \dfrac{\partial f}{\partial x}$.

(b) If $\theta = 90°$, then $D_\mathbf{u} f = \dfrac{\partial f}{\partial y}$.

69.



71.



73. $T(x, y) = 400 - 2x^2 - y^2$, $\quad P = (10, 10)$

$\dfrac{dx}{dt} = -4x \qquad \dfrac{dy}{dt} = -2y$

$x(t) = C_1e^{-4t} \qquad y(t) = C_2e^{-2t}$

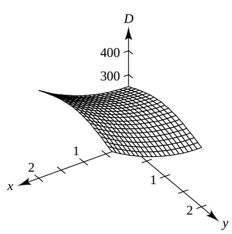
$10 = x(0) = C_1 \qquad 10 = y(0) = C_2$

$x(t) = 10e^{-4t} \qquad y(t) = 10e^{-2t}$

$x = \dfrac{y^2}{10} \qquad y^2(t) = 100e^{-4t}$

$y^2 = 10x$

75. (a)



(b) The graph of $-D = -250 - 30x^2 - 50\sin(\pi y/2)$ would model the ocean floor.

(c) $D(1, 0.5) = 250 + 30(1) + 50\sin\dfrac{\pi}{4} \approx 315.4$ ft

(d) $\dfrac{\partial D}{\partial x} = 60x$ and $\dfrac{\partial D}{\partial x}(1, 0.5) = 60$

(e) $\dfrac{\partial D}{\partial y} = 25\pi\cos\dfrac{\pi y}{2}$ and $\dfrac{\partial D}{\partial y}(1, 0.5) = 25\pi\cos\dfrac{\pi}{4} \approx 55.5$

(f) $\nabla D = 60x\mathbf{i} + 25\pi\cos\left(\dfrac{\pi y}{2}\right)\mathbf{j}$

$\nabla D(1, 0.5) = 60\mathbf{i} + 55.5\mathbf{j}$

77. True

79. True

81. Let $f(x, y, z) = e^x\cos y + \dfrac{z^2}{2} + C$. Then $\nabla f(x, y, z) = e^x\cos y\mathbf{i} - e^x\sin y\mathbf{j} + z\mathbf{k}$.

Section 12.7 Tangent Planes and Normal Lines

1. $F(x, y, z) = 3x - 5y + 3z - 15 = 0$

$$3x - 5y + 3z = 15 \text{ Plane}$$

3. $F(x, y, z) = 4x^2 + 9y^2 - 4z^2 = 0$

$$4x^2 + 9y^2 = 4z^2 \text{ Elliptic cone}$$

5. $F(x, y, z) = x + y + z - 4$

$$\nabla F = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k}) = \frac{\sqrt{3}}{3}(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

7. $F(x, y, z) = \sqrt{x^2 + y^2} - z$

$$\nabla F(x, y, z) = \frac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}}\mathbf{j} - \mathbf{k}$$

$$\nabla F(3, 4, 5) = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} - \mathbf{k}$$

$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{5}{5\sqrt{2}}\left(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} - \mathbf{k}\right) = \frac{1}{5\sqrt{2}}(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}) = \frac{\sqrt{2}}{10}(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$$

9. $F(x, y, z) = x^2y^4 - z$

$$\nabla F(x, y, z) = 2xy^4\mathbf{i} + 4x^2y^3\mathbf{j} - \mathbf{k}$$

$$\nabla F(1, 2, 16) = 32\mathbf{i} + 32\mathbf{j} - \mathbf{k}$$

$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{\sqrt{2049}}(32\mathbf{i} + 32\mathbf{j} - \mathbf{k}) = \frac{\sqrt{2049}}{2049}(32\mathbf{i} + 32\mathbf{j} - \mathbf{k})$$

11. $F(x, y, z) = \ln\left(\frac{x}{y - z}\right) = \ln x - \ln(y - z)$

$$\nabla F(x, y, z) = \frac{1}{x}\mathbf{i} - \frac{1}{y - z}\mathbf{j} + \frac{1}{y - z}\mathbf{k}$$

$$\nabla F(1, 4, 3) = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{\sqrt{3}}(\mathbf{i} - \mathbf{j} + \mathbf{k}) = \frac{\sqrt{3}}{3}(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

13. $F(x, y, z) = -x \sin y + z - 4$

$$\nabla F(x, y, z) = -\sin y\mathbf{i} - x \cos y\mathbf{j} + \mathbf{k}$$

$$\nabla F\left(6, \frac{\pi}{6}, 7\right) = -\frac{1}{2}\mathbf{i} - 3\sqrt{3}\mathbf{j} + \mathbf{k}$$

$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{2}{\sqrt{113}}\left(-\frac{1}{2}\mathbf{i} - 3\sqrt{3}\mathbf{j} + \mathbf{k}\right) = \frac{1}{\sqrt{113}}\left(-\mathbf{i} - 6\sqrt{3}\mathbf{j} + 2\mathbf{k}\right) = \frac{\sqrt{113}}{113}\left(-\mathbf{i} - 6\sqrt{3}\mathbf{j} + 2\mathbf{k}\right)$$

15. $f(x, y) = 25 - x^2 - y^2,\ (3, 1, 15)$

$$F(x, y, z) = 25 - x^2 - y^2 - z$$

$F_x(x, y, z) = -2x$ $F_y(x, y, z) = -2y$ $F_z(x, y, z) = -1$

$F_x(3, 1, 15) = -6$ $F_y(3, 1, 15) = -2$ $F_z(3, 1, 15) = -1$

$$-6(x - 3) - 2(y - 1) - (z - 15) = 0$$

$$0 = 6x + 2y + z - 35$$

$$6x + 2y + z = 35$$

17. $f(x, y) = \sqrt{x^2 + y^2}, (3, 4, 5)$

$F(x, y, z) = \sqrt{x^2 + y^2} - z$

$F_x(x, y, z) = \dfrac{x}{\sqrt{x^2 + y^2}}$ $\quad F_y(x, y, z) = \dfrac{y}{\sqrt{x^2 + y^2}}$ $\quad F_z(x, y, z) = -1$

$F_x(3, 4, 5) = \dfrac{3}{5}$ $\quad F_y(3, 4, 5) = \dfrac{4}{5}$ $\quad F_z(3, 4, 5) = -1$

$\dfrac{3}{5}(x - 3) + \dfrac{4}{5}(y - 4) - (z - 5) = 0$

$3(x - 3) + 4(y - 4) - 5(z - 5) = 0$

$3x + 4y - 5z = 0$

19. $g(x, y) = x^2 - y^2, \ (5, 4, 9)$

$G(x, y, z) = x^2 - y^2 - z$

$G_x(x, y, z) = 2x$ $\quad G_y(x, y, z) = -2y$ $\quad G_z(x, y, z) = -1$

$G_x(5, 4, 9) = 10$ $\quad G_y(5, 4, 9) = -8$ $\quad G_z(5, 4, 9) = -1$

$10(x - 5) - 8(y - 4) - (z - 9) = 0$

$10x - 8y - z = 9$

21. $z = e^x(\sin y + 1), \ \left(0, \dfrac{\pi}{2}, 2\right)$

$F(x, y, z) = e^x(\sin y + 1) - z$

$F_x(x, y, z) = e^x(\sin y + 1)$ $\quad F_y(x, y, z) = e^x \cos y$ $\quad F_z(x, y, z) = -1$

$F_x\left(0, \dfrac{\pi}{2}, 2\right) = 2$ $\quad F_y\left(0, \dfrac{\pi}{2}, 2\right) = 0$ $\quad F_z\left(0, \dfrac{\pi}{2}, 2\right) = -1$

$2x - z = -2$

23. $h(x, y) = \ln \sqrt{x^2 + y^2}, \ (3, 4, \ln 5)$

$H(x, y, z) = \ln \sqrt{x^2 + y^2} - z = \dfrac{1}{2} \ln(x^2 + y^2) - z$

$H_x(x, y, z) = \dfrac{x}{x^2 + y^2}$ $\quad H_y(x, y, z) = \dfrac{y}{x^2 + y^2}$ $\quad H_z(x, y, z) = -1$

$H_x(3, 4, \ln 5) = \dfrac{3}{25}$ $\quad H_y(3, 4, \ln 5) = \dfrac{4}{25}$ $\quad H_z(3, 4, \ln 5) = -1$

$\dfrac{3}{25}(x - 3) + \dfrac{4}{25}(y - 4) - (z - \ln 5) = 0$

$3(x - 3) + 4(y - 4) - 25(z - \ln 5) = 0$

$3x + 4y - 25z = 25(1 - \ln 5)$

25. $x^2 + 4y^2 + z^2 = 36, \ (2, -2, 4)$

$F(x, y, z) = x^2 + 4y^2 + z^2 - 36$

$F_x(x, y, z) = 2x$ $\quad F_y(x, y, z) = 8y$ $\quad F_z(x, y, z) = 2z$

$F_x(2, -2, 4) = 4$ $\quad F_y(2, -2, 4) = -16$ $\quad F_z(2, -2, 4) = 8$

$4(x - 2) - 16(y + 2) + 8(z - 4) = 0$

$(x - 2) - 4(y + 2) + 2(z - 4) = 0$

$x - 4y + 2z = 18$

27. $xy^2 + 3x - z^2 = 4, \ (2, 1, -2)$

$F(x, y, z) = xy^2 + 3x - z^2 - 4$

$F_x(x, y, z) = y^2 + 3 \qquad F_y(x, y, z) = 2xy \qquad F_z(x, y, z) = -2z$

$F_x(2, 1, -2) = 4 \qquad F_y(2, 1, -2) = 4 \qquad F_Z(2, 1, -2) = 4$

$4(x - 2) + 4(y - 1) + 4(z + 2) = 0$

$x + y + z = 1$

29. $x^2 + y^2 + z = 9, \ (1, 2, 4)$

$F(x, y, z) = x^2 + y^2 + z - 9$

$F_x(x, y, z) = 2x \qquad F_y(x, y, z) = 2y \qquad F_z(x, y, z) = 1$

$F_x(1, 2, 4) = 2 \qquad F_y(1, 2, 4) = 4 \qquad F_z(1, 2, 4) = 1$

Direction numbers: 2, 4, 1

Plane: $2(x - 1) + 4(y - 2) + (z - 4) = 0, \ 2x + 4y + z = 14$

Line: $\dfrac{x - 1}{2} = \dfrac{y - 2}{4} = \dfrac{z - 4}{1}$

31. $xy - z = 0, \ (-2, -3, 6)$

$F(x, y, z) = xy - z$

$F_x(x, y, z) = y \qquad F_y(x, y, z) = x \qquad F_z(x, y, z) = -1$

$F_x(-2, -3, 6) = -3 \qquad F_y(-2, -3, 6) = -2 \qquad F_z(-2, -3, 6) = -1$

Direction numbers: 3, 2, 1

Plane: $3(x + 2) + 2(y + 3) + (z - 6) = 0, \ 3x + 2y + z = -6$

Line: $\dfrac{x + 2}{3} = \dfrac{y + 3}{2} = \dfrac{z - 6}{1}$

33. $z = \arctan\dfrac{y}{x}, \ \left(1, 1, \dfrac{\pi}{4}\right)$

$F(x, y, z) = \arctan\dfrac{y}{x} - z$

$F_x(x, y, z) = \dfrac{-y}{x^2 + y^2} \qquad F_y(x, y, z) = \dfrac{x}{x^2 + y^2} \qquad F_z(x, y, z) = -1$

$F_x\left(1, 1, \dfrac{\pi}{4}\right) = -\dfrac{1}{2} \qquad F_y\left(1, 1, \dfrac{\pi}{4}\right) = \dfrac{1}{2} \qquad F_z\left(1, 1, \dfrac{\pi}{4}\right) = -1$

Direction numbers: $1, -1, 2$

Plane: $(x - 1) - (y - 1) + 2\left(z - \dfrac{\pi}{4}\right) = 0, \ x - y + 2z = \dfrac{\pi}{2}$

Line: $\dfrac{x - 1}{1} = \dfrac{y - 1}{-1} = \dfrac{z - (\pi/4)}{2}$

35. $z = f(x, y) = \dfrac{4xy}{(x^2+1)(y^2+1)}, \; -2 \le x \le z, \; 0 \le y \le 3$

(a) Let $F(x, y, z) = \dfrac{4xy}{(x^2+1)(y^2+1)} - z$

$$\nabla F(x, y, z) = \frac{4y}{y^2+1}\left(\frac{x^2+1-2x^2}{(x^2+1)^2}\right)\mathbf{i} + \frac{4x}{x^2+1}\left(\frac{y^2+1-2y^2}{(y^2+1)^2}\right)\mathbf{j} - \mathbf{k}$$

$$= \frac{4y(1-x^2)}{(y^2+1)(x^2+1)^2}\mathbf{i} + \frac{4x(1-y^2)}{(x^2+1)(y^2+1)^2}\mathbf{j} - \mathbf{k}$$

$\nabla F(1, 1, 1) = -\mathbf{k}.$

Direction numbers: $0, 0, -1.$

Line: $x = 1, \; y = 1, \; z = 1 - t$

Tangent plane: $0(x-1) + 0(y-1) - 1(z-1) = 0 \implies z = 1$

(b) $\nabla F\left(-1, 2, -\dfrac{4}{5}\right) = 0\mathbf{i} + \dfrac{-4(-3)}{(2)(5)^2}\mathbf{j} - \mathbf{k} = \dfrac{6}{25}\mathbf{j} - \mathbf{k}$

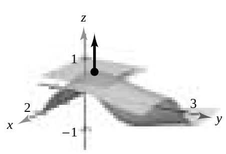
Line: $x = -1, \; y = 2 + \dfrac{6}{25}t, \; z = -\dfrac{4}{5} - t$

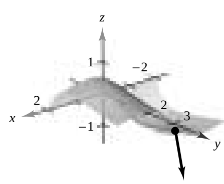
Plane: $0(x+1) + \dfrac{6}{25}(y-2) - 1\left(z + \dfrac{4}{5}\right) = 0$

$$6y - 12 - 25z - 20 = 0$$

$$6y - 25z - 32 = 0$$

(c)





(d) At $(1, 1, 1)$, the tangent plane is parallel to the xy-plane, implying that the surface is level there. At $\left(-1, 2, -\frac{4}{5}\right)$, the function does not change in the x-direction.

37. $F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_2(x_0, y_0, z_0)(z - z_0) = 0$

(Theorem 12.13)

39. $F(x, y, z) = x^2 + y^2 - 5$ $\qquad$ $G(x, y, z) = x - z$

$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j}$ $\qquad$ $\nabla G(x, y, z) = \mathbf{i} - \mathbf{k}$

$\nabla F(2, 1, 2) = 4\mathbf{i} + 2\mathbf{j}$ $\qquad$ $\nabla G(2, 1, 2) = \mathbf{i} - \mathbf{k}$

(a) $\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 2 & 0 \\ 1 & 0 & -1 \end{vmatrix} = -2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k} = -2(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

Direction numbers: $1, -2, 1, \; \dfrac{x-2}{1} = \dfrac{y-1}{-2} = \dfrac{z-2}{1}$

(b) $\cos\theta = \dfrac{|\nabla F \cdot \nabla G|}{\|\nabla F\|\,\|\nabla G\|} = \dfrac{4}{\sqrt{20}\sqrt{2}} = \dfrac{2}{\sqrt{10}} = \dfrac{\sqrt{10}}{5}$; not orthogonal

41. $F(x, y, z) = x^2 + z^2 - 25$ $\qquad$ $G(x, y, z) = y^2 + z^2 - 25$

$\nabla F = 2x\mathbf{i} + 2z\mathbf{k}$ $\qquad$ $\nabla G = 2y\mathbf{j} + 2z\mathbf{k}$

$\nabla F(3, 3, 4) = 6\mathbf{i} + 8\mathbf{k}$ $\qquad$ $\nabla G(3, 3, 4) = 6\mathbf{j} + 8\mathbf{k}$

—CONTINUED—

41. —CONTINUED—

(a) $\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 0 & 8 \\ 0 & 6 & 8 \end{vmatrix} = -48\mathbf{i} - 48\mathbf{j} + 36\mathbf{k} = -12(4\mathbf{i} + 4\mathbf{j} - 3\mathbf{k})$

Direction numbers: $4, 4, -3$, $\dfrac{x-3}{4} = \dfrac{y-3}{4} = \dfrac{z-4}{-3}$

(b) $\cos\theta = \dfrac{|\nabla F \cdot \nabla G|}{\|\nabla F\|\,\|\nabla G\|} = \dfrac{64}{(10)(10)} = \dfrac{16}{25}$; not orthogonal

43. $F(x, y, z) = x^2 + y^2 + z^2 - 6$ $\qquad G(x, y, z) = x - y - z$

$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ $\qquad \nabla G(x, y, z) = \mathbf{i} - \mathbf{j} - \mathbf{k}$

$\nabla F(2, 1, 1) = 4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ $\qquad \nabla G(2, 1, 1) = \mathbf{i} - \mathbf{j} - \mathbf{k}$

(a) $\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 2 & 2 \\ 1 & -1 & -1 \end{vmatrix} = 6\mathbf{j} - 6\mathbf{k} = 6(\mathbf{j} - \mathbf{k})$

Direction numbers: $0, 1, -1$, $x = 2$, $\dfrac{y-1}{1} = \dfrac{z-1}{-1}$

(b) $\cos\theta = \dfrac{|\nabla F \cdot \nabla G|}{\|\nabla F\|\,\|\nabla G\|} = 0$; orthogonal

45. $f(x, y) = 6 - x^2 - \dfrac{y^2}{4}$, $g(x, y) = 2x + y$

(a) $F(x, y, z) = z + x^2 + \dfrac{y^2}{4} - 6$

$G(x, y, z) = z - 2x - y$

$\nabla F(x, y, z) = 2x\mathbf{i} + \dfrac{1}{2}y\mathbf{j} + \mathbf{k}$ $\qquad \nabla G(x, y, z) = -2\mathbf{i} - \mathbf{j} + \mathbf{k}$

$\nabla F(1, 2, 4) = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ $\qquad \nabla G(1, 2, 4) = -2\mathbf{i} - \mathbf{j} + \mathbf{k}$

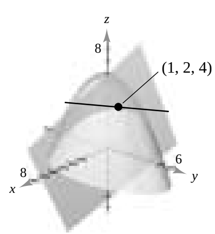
The cross product of these gradients is parallel to the curve of intersection.

$$\nabla F(1, 2, 4) \times \nabla G(1, 2, 4) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -2 & -1 & 1 \end{vmatrix} = 2\mathbf{i} - 4\mathbf{j}$$

Using direction numbers $1, -2, 0$, you get $x = 1 + t$, $y = 2 - 2t$, $z = 4$.

$$\cos\theta = \frac{\nabla F \cdot \nabla G}{\|\nabla F\|\,\|\nabla G\|} = \frac{-4 - 1 + 1}{\sqrt{6}\,\sqrt{6}} = \frac{-4}{6} \implies \theta \approx 48.2°$$

(b)



47. $F(x, y, z) = 3x^2 + 2y^2 - z - 15$, $(2, 2, 5)$

$\nabla F(x, y, z) = 6x\mathbf{i} + 4y\mathbf{j} - \mathbf{k}$

$\nabla F(2, 2, 5) = 12\mathbf{i} + 8\mathbf{j} - \mathbf{k}$

$\cos\theta = \dfrac{|\nabla F(2, 2, 5) \cdot \mathbf{k}|}{\|\nabla F(2, 2, 5)\|} = \dfrac{1}{\sqrt{209}}$

$\theta = \arccos\left(\dfrac{1}{\sqrt{209}}\right) \approx 86.03°$

49. $F(x, y, z) = x^2 - y^2 + z$, $(1, 2, 3)$

$\nabla F(x, y, z) = 2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$

$\nabla F(1, 2, 3) = 2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$

$\cos\theta = \dfrac{|\nabla F(1, 2, 3) \cdot \mathbf{k}|}{\|\nabla F(1, 2, 3)\|} = \dfrac{1}{\sqrt{21}}$

$\theta = \arccos\dfrac{1}{\sqrt{21}} \approx 77.40°$

51. $F(x, y, z) = 3 - x^2 - y^2 + 6y - z$

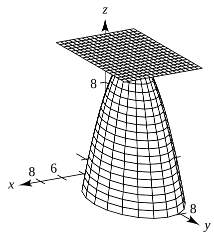
$\nabla F(x, y, z) = -2x\mathbf{i} + (-2y + 6)\mathbf{j} - \mathbf{k}$

$-2x = 0,\ x = 0$

$-2y + 6 = 0,\ y = 3$

$z = 3 - 0^2 - 3^2 + 6(3) = 12$

$(0, 3, 12)$ (vertex of paraboloid)



53. $T(x, y, z) = 400 - 2x^2 - y^2 - 4z^2,\ (4, 3, 10)$

$\frac{dx}{dt} = -4kx$ $\frac{dy}{dt} = -2ky$ $\frac{dz}{dt} = -8kz$

$x(t) = C_1e^{-4kt}$ $y(t) = C_2e^{-2kt}$ $z(t) = C_3e^{-8kt}$

$x(0) = C_1 = 4$ $y(0) = C_2 = 3$ $z(0) = C_3 = 10$

$x = 4e^{-4kt}$ $y = 3e^{-2kt}$ $z = 10e^{-8kt}$

55. $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$

$F_x(x, y, z) = \frac{2x}{a^2}$

$F_y(x, y, z) = \frac{2y}{b^2}$

$F_z(x, y, z) = \frac{2z}{c^2}$

Plane: $\frac{2x_0}{a^2}(x - x_0) + \frac{2y_0}{b^2}(y - y_0) + \frac{2z_0}{c^2}(z - z_0) = 0$

$$\frac{x_0x}{a^2} + \frac{y_0y}{b^2} + \frac{z_0z}{c^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} = 1$$

57. $F(x, y, z) = a^2x^2 + b^2y^2 - z^2$

$F_x(x, y, z) = 2a^2x$

$F_y(x, y, z) = 2b^2y$

$F_z(x, y, z) = -2z$

Plane: $2a^2x_0(x - x_0) + 2b^2y_0(y - y_0) - 2z_0(z - z_0) = 0$

$$a^2x_0x + b^2y_0y - z_0z = a^2x_0^2 + b^2y_0^2 - z_0^2 = 0$$

Hence, the plane passes through the origin.

59. $f(x, y) = e^{x-y}$

$f_x(x, y) = e^{x-y},$ $f_y(x, y) = -e^{x-y}$

$f_{xx}(x, y) = e^{x-y},$ $f_{yy}(x, y) = e^{x-y},$ $f_{xy}(x, y) = -e^{x-y}$

(a) $P_1(x, y) \approx f(0, 0) + f_x(0, 0)x + f_y(0, 0)y = 1 + x - y$

(b) $P_2(x, y) \approx f(0, 0) + f_x(0, 0)x + f_y(0,0)y + \frac{1}{2}f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \frac{1}{2}f_{yy}(0, 0)y^2$

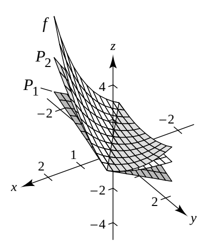
$= 1 + x - y + \frac{1}{2}x^2 - xy + \frac{1}{2}y^2$

(c) If $x = 0$, $P_2(0, y) = 1 - y + \frac{1}{2}y^2$. This is the second–degree Taylor polynomial for e^{-y}.
If $y = 0$, $P_2(x, 0) = 1 + x + \frac{1}{2}x^2$. This is the second–degree Taylor polynomial for e^x.

(d)

x	y	$f(x, y)$	$P_1(x, y)$	$P_2(x, y)$
0	0	1	1	1
0	0	0.9048	0.9000	0.9050
0.2	0.1	1.1052	1.1000	1.1050
0.2	0.5	0.7408	0.7000	0.7450
1	0.5	1.6487	1.5000	1.6250

(e)



61. Given $w = F(x, y, z)$ where F is differentiable at

(x_0, y_0, z_0) and $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$,

the level surface of F at (x_0, y_0, z_0) is of the form $F(x, y, z) = C$ for some constant C. Let

$G(x, y, z) = F(x, y, z) - C = 0.$

Then $\nabla G(x_0, y_0, z_0) = \nabla F(x_0, y_0, z_0)$ where $\nabla G(x_0, y_0, z_0)$ is normal to $F(x_0, y_0, z_0) - C = 0$.

Therefore, $\nabla F(x_0, y_0z_0)$ is normal to $F(x_0, y_0, z_0) = C$.

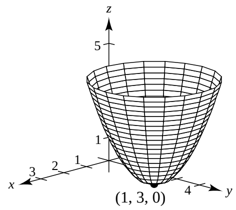
Section 12.8 Extrema of Functions of Two Variables

1. $g(x, y) = (x - 1)^2 + (y - 3)^2 \geq 0$

Relative minimum: $(1, 3, 0)$

$g_x = 2(x - 1) = 0 \implies x = 1$

$g_y = 2(y - 3) = 0 \implies y = 3$



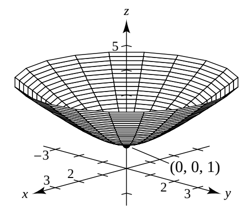
3. $f(x, y) = \sqrt{x^2 + y^2 + 1} \geq 1$

Relative minimum: $(0, 0, 1)$

Check: $f_x = \dfrac{x}{\sqrt{x^2 + y^2 + 1}} = 0 \implies x = 0$

$f_y = \dfrac{y}{\sqrt{x^2 + y^2 + 1}} = 0 \implies y = 0$

$f_{xx} = \dfrac{y^2 + 1}{(x^2 + y^2 + 1)^{3/2}},\ f_{yy} = \dfrac{x^2 + 1}{(x^2 + y^2 + 1)^{3/2}},\ f_{xy} = \dfrac{-xy}{(x^2 + y^2 + 1)^{3/2}}$



At the critical point $(0, 0)$, $f_{xx} > 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$. Therefore, $(0, 0, 1)$ is a relative minimum.

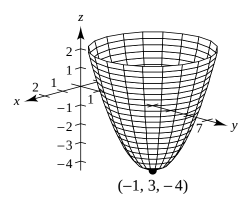
5. $f(x, y) = x^2 + y^2 + 2x - 6y + 6 = (x + 1)^2 + (y - 3)^2 - 4 \geq -4$

Relative minimum: $(-1, 3, -4)$

Check: $f_x = 2x + 2 = 0 \implies x = -1$

$f_y = 2y - 6 = 0 \implies y = 3$

$f_{xx} = 2,\ f_{yy} = 2,\ f_{xy} = 0$



At the critical point $(-1, 3)$, $f_{xx} > 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$. Therefore, $(-1, 3, -4)$ is a relative minimum.

7. $f(x, y) = 2x^2 + 2xy + y^2 + 2x - 3$

$\left.\begin{array}{l} f_x = 4x + 2y + 2 = 0 \\ f_y = 2x + 2y = 0 \end{array}\right\}$ Solving simultaneously yields $x = -1$ and $y = 1$.

$f_{xx} = 4,\ f_{yy} = 2,\ f_{xy} = 2$

At the critical point $(-1, 1)$, $f_{xx} > 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$. Therefore, $(-1, 1, -4)$ is a relative minimum.

9. $f(x, y) = -5x^2 + 4xy - y^2 + 16x + 10$

$\left.\begin{array}{l} f_x = -10x + 4y + 16 = 0 \\ f_y = 4x - 2y = 0 \end{array}\right\}$ Solving simultaneously yields $x = 8$ and $y = 16$.

$f_{xx} = -10,\ f_{yy} = -2,\ f_{xy} = 4$

At the critical point $(8, 16)$, $f_{xx} < 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$. Therefore, $(8, 16, 74)$ is a relative maximum.

11. $f(x, y) = 2x^2 + 3y^2 - 4x - 12y + 13$

$f_x = 4x - 4 = 4(x - 1) = 0$ when $x = 1$.

$f_y = 6y - 12 = 6(y - 2) = 0$ when $y = 2$.

$f_{xx} = 4,\ f_{yy} = 6,\ f_{xy} = 0$

At the critical point $(1, 2)$, $f_{xx} > 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$. Therefore, $(1, 2, -1)$ is a relative minimum.

13. $f(x, y) = 2\sqrt{x^2 + y^2} + 3$

$\left.\begin{array}{l} f_x = \dfrac{2x}{\sqrt{x^2 + y^2}} = 0 \\ f_y = \dfrac{2y}{\sqrt{x^2 + y^2}} = 0 \end{array}\right\}$ $x = 0, y = 0$

Since $f(x, y) \geq 3$ for all (x, y), $(0, 0, 3)$ is relative minimum.

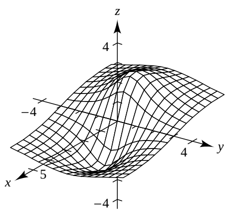
15. $g(x, y) = 4 - |x| - |y|$

$(0, 0)$ is the only critical point. Since $g(x, y) \le 4$ for all (x, y), $(0, 0, 4)$ is relative maximum.

17. $z = \dfrac{-4x}{x^2 + y^2 + 1}$

Relative minimum: $(1, 0, -2)$

Relative maximum: $(-1, 0, 2)$

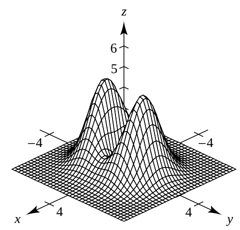


19. $z = (x^2 + 4y^2)e^{1 - x^2 - y^2}$

Relative minimum: $(0, 0, 0)$

Relative maxima: $(0, \pm 1, 4)$

Saddle points: $(\pm 1, 0, 1)$



21. $h(x, y) = x^2 - y^2 - 2x - 4y - 4$

$h_x = 2x - 2 = 2(x - 1) = 0$ when $x = 1$.

$h_y = -2y - 4 = -2(y + 2) = 0$ when $y = -2$.

$h_{xx} = 2,\ h_{yy} = -2,\ h_{xy} = 0$

At the critical point $(1, -2)$, $h_{xx} h_{yy} - (h_{xy})^2 < 0$. Therefore, $(1, -2, -1)$ is a saddle point.

23. $h(x, y) = x^2 - 3xy - y^2$

$$\left.\begin{aligned} h_x &= 2x - 3y = 0 \\ h_y &= -3x - 2y = 0 \end{aligned}\right\}$$ Solving simultaneously yields $x = 0$ and $y = 0$.

$h_{xx} = 2,\ h_{yy} = -2,\ h_{xy} = -3$

At the critical point $(0, 0)$, $h_{xx} h_{yy} - (h_{xy})^2 < 0$. Therefore, $(0, 0, 0)$ is a saddle point.

25. $f(x, y) = x^3 - 3xy + y^3$

$$\left.\begin{aligned} f_x &= 3(x^2 - y) = 0 \\ f_y &= 3(-x + y^2) = 0 \end{aligned}\right\}$$ Solving by substitution yields two critical points $(0, 0)$ and $(1, 1)$.

$f_{xx} = 6x,\ f_{yy} = 6y,\ f_{xy} = -3$

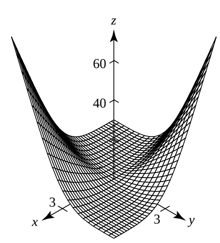
At the critical point $(0, 0)$, $f_{xx} f_{yy} - (f_{xy})^2 < 0$. Therefore, $(0, 0, 0)$ is a saddle point. At the critical point $(1, 1)$, $f_{xx} = 6 > 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$. Therefore, $(1, 1, -1)$ is a relative minimum.

27. $f(x, y) = e^{-x} \sin y$

$$\left.\begin{aligned} f_x &= -e^{-x} \sin y = 0 \\ f_y &= e^{-x} \cos y = 0 \end{aligned}\right\}$$ Since $e^{-x} > 0$ for all x and $\sin y$ and $\cos y$ are never both zero for a given value of y, there are no critical points.

29. $z = \dfrac{(x - y)^4}{x^2 + y^2} \ge 0$. $z = 0$ if $x = y \ne 0$.

Relative minimum at all points (x, x), $x \ne 0$.



31. $f_{xx} f_{yy} - (f_{xy})^2 = (9)(4) - 6^2 = 0$

Insufficient information.

33. $f_{xx} f_{yy} - (f_{xy})^2 = (-9)(6) - 10^2 < 0$

f has a saddle point at (x_0, y_0).

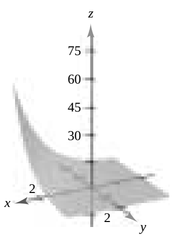
35. (a) The function f defined on a region R containing (x_0, y_0) has a relative minimum at (x_0, y_0) if $f(x, y) \geq f(x_0, y_0)$ for all (x, y) in R.

(b) The function f defined on a region R containing (x_0, y_0) has a relative maximum at (x_0, y_0) if $f(x, y) \leq f(x_0, y_0)$ for all (x, y) in R.

(c) A saddle point is a critical point which is not a relative extremum.

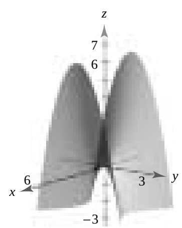
(d) See definition page 906.

37.



No extrema

39.



Saddle point

41. The point A will be a saddle point. The function could be

$$f(x, y) = x^2 - y^2.$$

43. $d = f_{xx} f_{yy} - f_{xy}{}^2 = (2)(8) - f_{xy}{}^2 = 16 - f_{xy}{}^2 > 0$

$\Rightarrow f_{xy}{}^2 < 16 \Rightarrow -4 < f_{xy} < 4$

45. $f(x, y) = x^3 + y^3$

$$\left.\begin{aligned} f_x &= 3x^2 = 0 \\ f_y &= 3y^2 = 0 \end{aligned}\right\} \text{Solving yields } x = y = 0$$

$f_{xx} = 6x,\ f_{yy} = 6y,\ f_{xy} = 0$

At $(0, 0)$, $f_{xx} f_{yy} - (f_{xy})^2 = 0$ and the test fails. $(0, 0, 0)$ is a saddle point.

47. $f(x, y) = (x - 1)^2(y + 4)^2 \geq 0$

$$\left.\begin{aligned} f_x &= 2(x - 1)(y + 4)^2 = 0 \\ f_y &= 2(x - 1)^2(y + 4) = 0 \end{aligned}\right\} \text{Solving yields the critical points } (1, a) \text{ and } (b, -4).$$

$f_{xx} = 2(y + 4)^2, f_{yy} = 2(x - 1)^2, f_{xy} = 4(x - 1)(y + 4)$

At both $(1, a)$ and $(b, -4)$, $f_{xx} f_{yy} - (f_{xy})^2 = 0$ and the test fails.

Absolute minima: $(1, a, 0)$ and $(b, -4, 0)$

49. $f(x, y) = x^{2/3} + y^{2/3} \geq 0$

$$\left.\begin{aligned} f_x &= \frac{2}{3\sqrt[3]{x}} \\ f_y &= \frac{2}{3\sqrt[3]{y}} \end{aligned}\right\} f_x \text{ and } f_y \text{ are undefined at } x = 0, y = 0. \text{ The critical point is } (0, 0).$$

$f_{xx} = -\dfrac{2}{9x\sqrt[3]{3}}, f_{yy} = -\dfrac{2}{9y\sqrt[3]{y}}, f_{xy} = 0$

At $(0, 0)$, $f_{xx} f_{yy} - (f_{xy})^2$ is undefined and the test fails.

Absolute minimum: 0 at $(0, 0)$

51. $f(x, y, z) = x^2 + (y - 3)^2 + (z + 1)^2 \geq 0$

$$\left.\begin{aligned} f_x &= 2x = 0 \\ f_y &= 2(y - 3) = 0 \\ f_z &= 2(z + 1) = 0 \end{aligned}\right\} \text{Solving yields the critical point } (0, 3, -1).$$

Absolute minimum: 0 at $(0, 3, -1)$

53. $f(x, y) = 12 - 3x - 2y$ has no critical points. On the line $y = x + 1,\ 0 \le x \le 1$,

$$f(x, y) = f(x) = 12 - 3x - 2(x + 1) = -5x + 10$$

and the maximum is 10, the minimum is 5. On the line $y = -2x + 4,\ 1 \le x \le 2$,

$$f(x, y) = f(x) = 12 - 3x - 2(-2x + 4) = x + 4$$

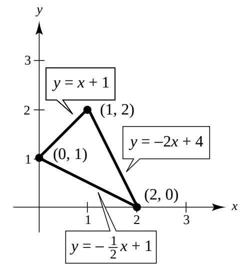
and the maximum is 6, the minimum is 5. On the line $y = -\frac{1}{2}x + 1,\ 0 \le x \le 2$,

$$f(x, y) = f(x) = 12 - 3x - 2\left(-\tfrac{1}{2}x + 1\right) = -2x + 10$$

and the maximum is 10, the minimum is 6.

Absolute maximum: 10 at $(0, 1)$

Absolute minimum: 5 at $(1, 2)$



55. $f(x, y) = 3x^2 + 2y^2 - 4y$

$$\left.\begin{aligned} f_x &= 6x = 0 &&\Rightarrow x = 0\\ f_y &= 4y - 4 = 0 &&\Rightarrow y = 1\end{aligned}\right\}\ f(0, 1) = -2$$

On the line $y = 4,\ -2 \le x \le 2$,

$$f(x, y) = f(x) = 3x^2 + 32 - 16 = 3x^2 + 16$$

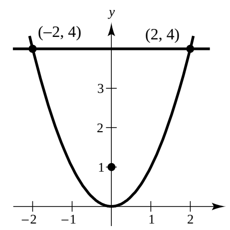
and the maximum is 28, the minimum is 16. On the curve $y = x^2,\ -2 \le x \le 2$,

$$f(x, y) = f(x) = 3x^2 + 2(x^2)^2 - 4x^2 = 2x^4 - x^2 = x^2(2x^2 - 1)$$

and the maximum is 28, the minimum is $-\frac{1}{8}$.

Absolute maximum: 28 at $(\pm 2, 4)$

Absolute minimum: -2 at $(0, 1)$



57. $f(x, y) = x^2 + xy,\ R = \{(x, y)\colon\ |x| \le 2,\ |y| \le 1\}$

$$\left.\begin{aligned} f_x &= 2x + y = 0\\ f_y &= x = 0\end{aligned}\right\}\ x = y = 0$$

$f(0, 0) = 0$

Along $y = 1,\ -2 \le x \le 2,\ f = x^2 + x,\ f' = 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$.

Thus, $f(-2, 1) = 2,\ f\left(-\frac{1}{2}, 1\right) = -\frac{1}{4}$ and $f(2, 1) = 6$.

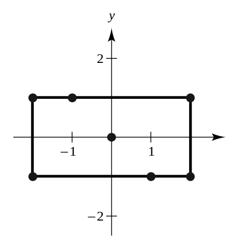
Along $y = -1,\ -2 \le x \le 2,\ f = x^2 - x,\ f' = 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$.

Thus, $f(-2, -1) = 6,\ f\left(\frac{1}{2}, -1\right) = -\frac{1}{4},\ f(2, -1) = 2$.

Along $x = 2,\ -1 \le y \le 1,\ f = 4 + 2y \Rightarrow f' = 2 \ne 0$.

Along $x = -2,\ -1 \le y \le 1,\ f = 4 - 2y \Rightarrow f' = -2 \ne 0$.

Thus, the maxima are $f(2, 1) = 6$ and $f(-2, -1) = 6$ and the minima are $f\left(-\frac{1}{2}, 1\right) = -\frac{1}{4}$ and $f\left(\frac{1}{2}, -1\right) = -\frac{1}{4}$.



59. $f(x, y) = x^2 + 2xy + y^2,\ R = \{(x, y)\colon x^2 + y^2 \le 8\}$

$$\left.\begin{aligned} f_x &= 2x + 2y = 0\\ f_y &= 2x + 2y = 0\end{aligned}\right\}\ y = -x$$

$f(x, -x) = x^2 - 2x^2 + x^2 = 0$

On the boundary $x^2 + y^2 = 8$, we have $y^2 = 8 - x^2$ and $y = \pm\sqrt{8 - x^2}$. Thus,

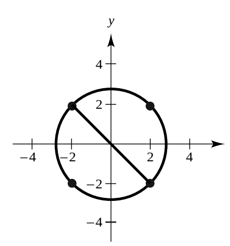
$$f = x^2 \pm 2x\sqrt{8 - x^2} + (8 - x^2) = 8 \pm 2x\sqrt{8 - x^2}$$

$$f' = \pm(8 - x^2)^{-1/2}(-2x) + 2(8 - x^2)^{1/2}) = \pm\frac{16 - 4x^2}{\sqrt{8 - x^2}}.$$

Then, $f' = 0$ implies $16 = 4x^2$ or $x = \pm 2$.

$$f(2, 2) = f(-2, -2) = 16 \quad \text{and} \quad f(2, -2) = f(-2, 2) = 0$$

Thus, the maxima are $f(2, 2) = 16$ and $f(-2, -2) = 16$, and the minima are $f(x, -x) = 0,\ |x| \le 2$.



61. $f(x, y) = \dfrac{4xy}{(x^2 + 1)(y^2 + 1)}, R = \{(x, y): 0 \leq x \leq 1, 0 \leq y \leq 1\}$

$f_x = \dfrac{4(1 - x^2)y}{(y^2 + 1)(x^2 + 1)} = 0 \implies x = 1 \text{ or } y = 0$

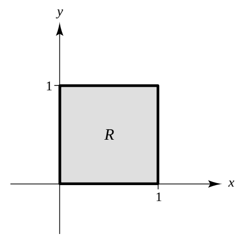
$f_y = \dfrac{4(1 - y^2)x}{(x^2 + 1)(y^2 + 1)^2} \implies x = 0 \text{ or } y = 1$

For $x = 0, y = 0$, also, and $f(0, 0) = 0$.

For $x = 1, y = 1, f(1, 1) = 1$.

The absolute maximum is $1 = f(1, 1)$.

The absolute minimum is $0 = f(0, 0)$. (In fact, $f(0, y) = f(x, 0) = 0$)



63. False

Let $f(x, y) = |1 - x - y|$.

$(0, 0, 1)$ is a relative maximum, but $f_x(0, 0)$ and $f_y(0, 0)$ do not exist.

Section 12.9 Applications of Extrema of Functions of Two Variables

1. A point on the plane is given by $(x, y, 12 - 2x - 3y)$. The square of the distance from the origin to this point is

$$S = x^2 + y^2 + (12 - 2x - 3y)^2$$

$$S_x = 2x + 2(12 - 2x - 3y)(-2)$$

$$S_y = 2y + 2(12 - 2x - 3y)(-3)$$

From the equations $S_x = 0$ and $S_y = 0$, we obtain the system

$$5x + 6y = 24$$

$$3x + 5y = 18.$$

Solving simultaneously, we have $x = \frac{12}{7}, y = \frac{18}{7}$ $z = 12 - \frac{24}{7} - \frac{54}{7} = \frac{6}{7}$. Therefore, the distance from the origin to $\left(\frac{12}{7}, \frac{18}{7}, \frac{6}{7}\right)$ is

$$\sqrt{\left(\frac{12}{7}\right)^2 + \left(\frac{18}{7}\right)^2 + \left(\frac{6}{7}\right)^2} = \frac{6\sqrt{14}}{7}.$$

3. A point on the paraboloid is given by $(x, y, x^2 + y^2)$. The square of the distance from $(5, 5, 0)$ to a point on the paraboloid is given by

$$S = (x - 5)^2 + (y - 5)^2 + (x^2 + y^2)^2$$

$$S_x = 2(x - 5) + 4x(x^2 + y^2) = 0$$

$$S_y = 2(y - 5) + 4y(x^2 + y^2) = 0.$$

From the equations $S_x = 0$ and $S_y = 0$, we obtain the system

$$2x^3 + 2xy^2 + x - 5 = 0$$

$$2y^3 + 2x^2y + y - 5 = 0$$

Multiply the first equation by y and the second equation by x, and subtract to obtain $x = y$. Then, we have $x = 1$, $y = 1$, $z = 2$ and the distance is

$$\sqrt{(1 - 5)^2 + (1 - 5)^2 + (2 - 0)^2} = 6.$$

5. Let x, y and z be the numbers. Since $x + y + z = 30$, $z = 30 - x - y$.

$$P = xyz = 30xy - x^2y - xy^2$$

$$\left.\begin{aligned} P_x &= 30y - 2xy - y^2 = y(30 - 2x - y) = 0 \\ P_y &= 30x - x^2 - 2xy = x(30 - x - 2y) = 0 \end{aligned}\right\} \begin{aligned} 2x + y &= 30 \\ x + 2y &= 30 \end{aligned}$$

Solving simultaneously yields $x = 10$, $y = 10$, and $z = 10$.

7. Let x, y, and z be the numbers and let $S = x^2 + y^2 + z^2$. Since $x + y + z = 30$, we have

$$S = x^2 + y^2 + (30 - x - y)^2$$

$$\left.\begin{aligned} S_x &= 2x + 2(30 - x - y)(-1) = 0 \\ S_y &= 2y + 2(30 - x - y)(-1) = 0 \end{aligned}\right\} \begin{aligned} 2x + y &= 30 \\ x + 2y &= 30. \end{aligned}$$

Solving simultaneously yields $x = 10$, $y = 10$, and $z = 10$.

9. Let x, y, and z be the length, width, and height, respectively. Then the sum of the length and girth is given by $x + (2y + 2z) = 108$ or $x = 108 - 2y - 2z$. The volume is given by

$$V = xyz = 108zy - 2zy^2 - 2yz^2$$

$$V_y = 108z - 4yz - 2z^2 = z(108 - 4y - 2z) = 0$$

$$V_z = 108y - 2y^2 - 4yz = y(108 - 2y - 4z) = 0.$$

Solving the system $4y + 2z = 108$ and $2y + 4z = 108$, we obtain the solution $x = 36$ inches, $y = 18$ inches, and $z = 18$ inches.

11. Let $a + b + c = k$. Then

$$V = \frac{4\pi abc}{3} = \frac{4}{3}\pi ab(k - a - b)$$

$$= \frac{4}{3}\pi(kab - a^2b - ab^2)$$

$$\left.\begin{aligned} V_a &= \frac{4\pi}{3}(kb - 2ab - b^2) = 0 \\ V_b &= \frac{4\pi}{3}(ka - a^2 - 2ab) = 0 \end{aligned}\right\} \begin{aligned} kb - 2ab - b^2 &= 0 \\ ka - a^2 - 2ab &= 0. \end{aligned}$$

Solving this system simultaneously yields $a = b$ and substitution yields $b = k/3$. Therefore, the solution is $a = b = c = k/3$.

13. Let x, y, and z be the length, width, and height, respectively and let V_0 be the given volume.

Then $V_0 = xyz$ and $z = V_0/xy$. The surface area is

$$S = 2xy + 2yz + 2xz = 2\left(xy + \frac{V_0}{x} + \frac{V_0}{y}\right)$$

$$\left.\begin{aligned} S_x &= 2\left(y - \frac{V_0}{x^2}\right) = 0 \\ S_y &= 2\left(x - \frac{V_0}{y^2}\right) = 0 \end{aligned}\right\} \begin{aligned} x^2y - V_0 &= 0 \\ xy^2 - V_0 &= 0. \end{aligned}$$

Solving simultaneously yields $x = \sqrt[3]{V_0}$, $y = \sqrt[3]{V_0}$, and $z = \sqrt[3]{V_0}$.

15. The distance from P to Q is $\sqrt{x^2 + 4}$. The distance from Q to R is $\sqrt{(y - x)^2 + 1}$. The distance from R to S is $10 - y$.

$$C = 3k\sqrt{x^2 + 4} + 2k\sqrt{(y - x)^2 + 1} + k(10 - y)$$

$$C_x = 3k\left(\frac{x}{\sqrt{x^2 + 4}}\right) + 2k\left(\frac{-(y - x)}{\sqrt{(y - x)^2 + 1}}\right) = 0$$

$$C_y = 2k\left(\frac{y - x}{\sqrt{(y - x)^2 + 1}}\right) - k = 0 \implies \frac{y - x}{\sqrt{(y - x)^2 + 1}} = \frac{1}{2}$$

$$3k\left(\frac{x}{\sqrt{x^2 + 4}}\right) + 2k\left(-\frac{1}{2}\right) = 0$$

$$\frac{x}{\sqrt{x^2 + 4}} = \frac{1}{3}$$

$$3x = \sqrt{x^2 + 4}$$

$$9x^2 = x^2 + 4$$

$$x^2 = \frac{1}{2}$$

$$x = \frac{\sqrt{2}}{2}$$

$$2(y - x) = \sqrt{(y - x)^2 + 1}$$

$$4(y - x)^2 = (y - x)^2 + 1$$

$$(y - x)^2 = \frac{1}{3}$$

$$y = \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}} = \frac{2\sqrt{3} + 3\sqrt{2}}{6}$$

Therefore, $x = \dfrac{\sqrt{2}}{2} \approx 0.707$ km and $y = \dfrac{2\sqrt{3} + 3\sqrt{2}}{6} \approx 1.284$ kms.

17. Let h be the height of the trough and r the length of the slanted sides. We observe that the area of a trapezoidal cross section is given by

$$A = h\left[\frac{(w - 2r) + [(w - 2r) + 2x]}{2}\right] = (w - 2r + x)h$$

where $x = r\cos\theta$ and $h = r\sin\theta$. Substituting these expressions for x and h, we have

$$A(r, \theta) = (w - 2r + r\cos\theta)(r\sin\theta) = wr\sin\theta - 2r^2\sin\theta + r^2\sin\theta\cos\theta$$

Now

$$A_r(r, \theta) = w\sin\theta - 4r\sin\theta + 2r\sin\theta\cos\theta = \sin\theta(w - 4r + 2r\cos\theta) = 0 \implies w = r(4 - 2\cos\theta)$$

$$A_\theta(r, \theta) = wr\cos\theta - 2r^2\cos\theta + r^2\cos 2\theta = 0.$$

Substituting the expression for w from $A_r(r, \theta) = 0$ into the equation $A_\theta(r, \theta) = 0$, we have

$$r^2(4 - 2\cos\theta)\cos\theta - 2r^2\cos\theta + r^2(2\cos^2\theta - 1) = 0$$

$$r^2(2\cos\theta - 1) = 0 \text{ or } \cos\theta = \frac{1}{2}.$$

Therefore, the first partial derivatives are zero when $\theta = \pi/3$ and $r = w/3$. (Ignore the solution $r = \theta = 0$.) Thus, the trapezoid of maximum area occurs when each edge of width $w/3$ is turned up 60° from the horizontal.

19. $R(x_1, x_2) = -5x_1^2 - 8x_2^2 - 2x_1x_2 + 42x_1 + 102x_2$

$R_{x_1} = -10x_1 - 2x_2 + 42 = 0,\ 5x_1 + x_2 = 21$

$R_{x_2} = -16x_2 - 2x_1 + 102 = 0, x_1 + 8x_2 = 51$

Solving this system yields $x_1 = 3$ and $x_2 = 6$.

$R_{x_1x_1} = -10$

$R_{x_1x_2} = -2$

$R_{x_2x_2} = -16$

$R_{x_1x_1} < 0$ and $R_{x_1x_1}R_{x_2x_2} - (R_{x_1x_2})^2 > 0$

Thus, revenue is maximized when $x_1 = 3$ and $x_2 = 6$.

21. $P(x_1, x_2) = 15(x_1 + x_2) - C_1 - C_2$

$= 15x_1 + 15x_2 - (0.02x_1^2 + 4x_1 + 500) - (0.05x_2^2 + 4x_2 + 275)$

$= -0.02x_1^2 - 0.05x_2^2 + 11x_1 + 11x_2 - 775$

$P_{x_1} = -0.04x_1 + 11 = 0, x_1 = 275$

$P_{x_2} = -0.10x_2 + 11 = 0, x_2 = 110$

$P_{x_1x_1} = -0.04$

$P_{x_1x_2} = 0$

$P_{x_2x_2} = -0.10$

$P_{x_1x_1} < 0$ and $P_{x_1x_1}P_{x_2x_2} - (P_{x_1x_2})^2 > 0$

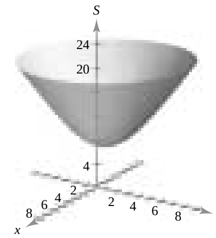
Therefore, profit is maximized when $x_1 = 275$ and $x_2 = 110$.

23. (a) $S(x, y) = d_1 + d_2 + d_3$

$$= \sqrt{(x-0)^2 + (y-0)^2} + \sqrt{(x+2)^2 + (y-2)^2} + \sqrt{(x-4)^2 + (y-2)^2}$$

$$= \sqrt{x^2 + y^2} + \sqrt{(x+2)^2 + (y-2)^2} + \sqrt{(x-4)^2 + (y-2)^2}$$

From the graph we see that the surface has a minimum.



(b) $$S_x(x, y) = \frac{x}{\sqrt{x^2 + y^2}} + \frac{x+2}{\sqrt{(x+2)^2 + (y-2)^2}} + \frac{x-4}{\sqrt{(x-4)^2 + (y-2)^2}}$$

$$S_y(x, y) = \frac{y}{\sqrt{x^2 + y^2}} + \frac{y-2}{\sqrt{(x+2)^2 + (y-2)^2}} + \frac{y-2}{\sqrt{(x-4)^2 + (y-2)^2}}$$

(c) $$-\nabla S(1, 1) = -S_x(1, 1)\mathbf{i} - S_y(1, 1)\mathbf{j} = -\frac{1}{\sqrt{2}}\mathbf{i} - \left(\frac{1}{\sqrt{2}} - \frac{2}{\sqrt{10}}\right)\mathbf{j}$$

$$\tan\theta = \frac{\left(2/\sqrt{10}\right) - \left(1/\sqrt{2}\right)}{-1/\sqrt{2}} = 1 - \frac{2}{\sqrt{5}} \implies \theta \approx 186.027°$$

(d) $$(x_2, y_2) = (x_1 - S_x(x_1, y_1)t, y_1 - S_y(x_1, y_1)t) = \left(1 - \frac{1}{\sqrt{2}}t, 1 + \left(\frac{2}{\sqrt{10}} - \frac{1}{\sqrt{2}}\right)t\right)$$

$$S\left(1 - \frac{1}{\sqrt{2}}t, 1 + \left(\frac{2}{\sqrt{10}} - \frac{1}{\sqrt{2}}\right)t\right) = \sqrt{2 + \left(\frac{2\sqrt{10}}{5} - 2\sqrt{2}\right)t + \left(1 - \frac{2\sqrt{5}}{5} + \frac{2}{5}\right)t^2}$$

$$+ \sqrt{10 - \left(\frac{2\sqrt{10}}{5} + 2\sqrt{2}\right)t + \left(1 - \frac{2\sqrt{5}}{5} + \frac{2}{5}\right)t^2}$$

$$+ \sqrt{10 - \left(\frac{2\sqrt{10}}{5} - 4\sqrt{2}\right)t + \left(1 - \frac{2\sqrt{5}}{5} + \frac{2}{5}\right)t^2}$$

Using a computer algebra system, we find that the minimum occurs when $t \approx 1.344$. Thus, $(x_2, y_2) \approx (0.05, 0.90)$.

(e) $(x_3, y_3) = (x_2 - S_x(x_2, y_2)t, y_2 - S_y(x_2, y_2)t) \approx (0.05 + 0.03t, 0.90 - 0.26t)$

$$S(0.05 + 0.03t, 0.90 - 0.26t) = \sqrt{(0.05 + 0.03t)^2 + (0.90 - 0.26t)^2} + \sqrt{(2.05 + 0.03t)^2 + (-1.10 - 0.26t)^2}$$

$$+ \sqrt{(-3.95 + 0.03t)^2 + (-1.10 - 0.26t)^2}$$

Using a computer algebra system, we find that the minimum occurs when $t \approx 1.78$. Thus $(x_3, y_3) \approx (0.10, 0.44)$.

$(x_4, y_4) = (x_3 - S_x(x_3, y_3)t, y_3 - S_y(x_3, y_3)t) \approx (0.10 - 0.09t, 0.44 - 0.01t)$

$$S(0.10 - 0.09t, 0.45 - 0.01t) = \sqrt{(0.10 - 0.09t)^2 + (0.45 - 0.01t)^2} + \sqrt{(2.10 - 0.09t)^2 + (-1.55 - 0.01t)^2}$$

$$+ \sqrt{(-3.90 - 0.09t)^2 + (-1.55 - 0.01t)^2}$$

Using a computer algebra system, we find that the minimum occurs when $t \approx 0.44$. Thus, $(x_4, y_4) \approx (0.06, 0.44)$.

Note: The minimum occurs at $(x, y) = (0.0555, 0.3992)$

(f) $-\nabla S(x, y)$ points in the direction that S *decreases* most rapidly. You would use $\nabla S(x, y)$ for maximization problems.

25. Write the equation to be maximized or minimized as a function of two variables. Set the partial derivatives equal to zero (or undefined) to obtain the critical points. Use the Second Partials Test to test for relative extrema using the critical points. Check the boundary points, too.

27. (a)

x	y	xy	x^2
-2	0	0	4
0	1	0	0
2	3	6	4
$\sum x_i = 0$	$\sum y_i = 4$	$\sum x_iy_i = 6$	$\sum x_i^2 = 8$

$a = \dfrac{3(6) - 0(4)}{3(8) - 0^2} = \dfrac{3}{4},\ b = \dfrac{1}{3}\left[4 - \dfrac{3}{4}(0)\right] = \dfrac{4}{3},$

$y = \dfrac{3}{4}x + \dfrac{4}{3}$

(b) $S = \left(-\dfrac{3}{2} + \dfrac{4}{3} - 0\right)^2 + \left(\dfrac{4}{3} - 1\right)^2 + \left(\dfrac{3}{2} + \dfrac{4}{3} - 3\right)^2$

$= \dfrac{1}{6}$

29. (a)

x	y	xy	x^2
0	4	0	0
1	3	3	1
1	1	1	1
2	0	0	4
$\sum x_i = 4$	$\sum y_i = 8$	$\sum x_iy_i = 4$	$\sum x_i^2 = 6$

$a = \dfrac{4(4) - 4(8)}{4(6) - 4^2} = -2,\ b = \dfrac{1}{4}[8 + 2(4)] = 4,$

$y = -2x + 4$

(b) $S = (4 - 4)^2 + (2 - 3)^2 + (2 - 1)^2 + (0 - 0)^2 = 2$

31. (0, 0), (1, 1), (3, 4), (4, 2), (5, 5)

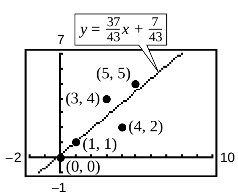
$\sum x_i = 13,\qquad \sum y_i = 12,$

$\sum x_iy_i = 46,\qquad \sum x_i^2 = 51$

$a = \dfrac{5(46) - 13(12)}{5(51) - (13)^2} = \dfrac{74}{86} = \dfrac{37}{43}$

$b = \dfrac{1}{5}\left[12 - \dfrac{37}{43}(13)\right] = \dfrac{7}{43}$

$y = \dfrac{37}{43}x + \dfrac{7}{43}$



33. (0, 6), (4, 3), (5, 0), (8, −4), (10, −5)

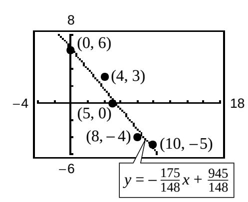
$\sum x_i = 27,\qquad \sum y_i = 0,$

$\sum x_iy_i = -70,\qquad \sum x_i^2 = 205$

$a = \dfrac{5(-70) - (27)(0)}{5(205) - (27)^2} = \dfrac{-350}{296} = -\dfrac{175}{148}$

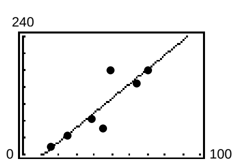
$b = \dfrac{1}{5}\left[0 - \left(-\dfrac{175}{148}\right)(27)\right] = \dfrac{945}{148}$

$y = -\dfrac{175}{148}x + \dfrac{945}{148}$



35. (a) $y = 1.7236x + 79.7334$

(b)



(c) For each one-year increase in age, the pressure changes by 1.7236 (slope of line).

37. (1.0, 32), (1.5, 41), (2.0, 48), (2.5, 53)

$\sum x_i = 7, \sum y_i = 174, \sum x_iy_i = 322, \sum x_i^2 = 13.5$

$a = 14, b = 19, y = 14x + 19$

When $x = 1.6, y = 41.4$ bushels per acre.

39. $S(a, b, c) = \sum_{i=1}^{n} (y_i - ax_i^2 - bx_i - c)^2$

$$\frac{\partial S}{\partial a} = \sum_{i=1}^{n} -2x_i^2(y_i - ax_i^2 - bx_i - c) = 0$$

$$\frac{\partial S}{\partial b} = \sum_{i=1}^{n} -2x_i(y_i - ax_i^2 - bx_i - c) = 0$$

$$\frac{\partial S}{\partial c} = -2\sum_{i=1}^{n}(y_i - ax_i^2 - bx_i - c) = 0$$

$$a\sum_{i=1}^{n}x_i^4 + b\sum_{i=1}^{n}x_i^3 + c\sum_{i=1}^{n}x_i^2 = \sum_{i=1}^{n}x_i^2y_i$$

$$a\sum_{i=1}^{n}x_i^3 + b\sum_{i=1}^{n}x_i^2 + c\sum_{i=1}^{n}x_i = \sum_{i=1}^{n}x_iy_i$$

$$a\sum_{i=1}^{n}x_i^2 + b\sum_{i=1}^{n}x_i + cn = \sum_{i=1}^{n}y_i$$

41. $(-2, 0), (-1, 0), (0, 1), (1, 2), (2, 5)$

$\sum x_i = 0$

$\sum y_i = 8$

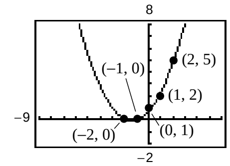
$\sum x_i^2 = 10$

$\sum x_i^3 = 0$

$\sum x_i^4 = 34$

$\sum x_iy_i = 12$

$\sum x_i^2y_i = 22$



$34a + 10c = 22,\ 10b = 12,\ 10a + 5c = 8$

$a = \frac{3}{7},\ b = \frac{6}{5},\ c = \frac{26}{35},\ y = \frac{3}{7}x^2 + \frac{6}{5}x + \frac{26}{35}$

43. $(0, 0), (2, 2), (3, 6), (4, 12)$

$\sum x_i = 9$

$\sum y_i = 20$

$\sum x_i^2 = 29$

$\sum x_i^3 = 99$

$\sum x_i^4 = 353$

$\sum x_iy_i = 70$

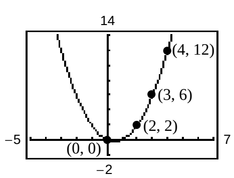
$\sum x_i^2y_i = 254$

$353a + 99b + 29c = 254$

$99a + 29b + 9c = 70$

$29a + 9b + 4c = 20$

$a = 1,\ b = -1,\ c = 0,\ y = x^2 - x$



45. $(0, 0), (2, 15), (4, 30), (6, 50), (8, 65), (10, 70)$

$\sum x_i = 30,$

$\sum y_i = 230,$

$\sum x_i^2 = 220,$

$\sum x_i^3 = 1{,}800,$

$\sum x_i^4 = 15{,}664,$

$\sum x_iy_i = 1{,}670,$

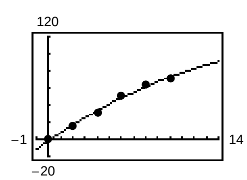
$\sum x_i^2y_i = 13{,}500$

$15{,}664a + 1{,}800b + 220c = 13{,}500$

$1{,}800a + 220b + 30c = 1{,}670$

$220a + 30b + 6c = 230$

$y = -\frac{25}{112}x^2 + \frac{541}{56}x - \frac{25}{14} \approx -0.22x^2 + 9.66x - 1.79$

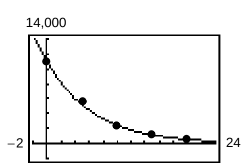


47. (a) $\ln P = -0.1499h + 9.3018$

(b) $\ln P = -0.1499h + 9.3018$

$P = e^{-0.1499h + 9.3018} = 10{,}957.7e^{-0.1499h}$

(c)



(d) Same answers.

Section 12.10 Lagrange Multipliers

1. Maximize $f(x, y) = xy$.

Constraint: $x + y = 10$

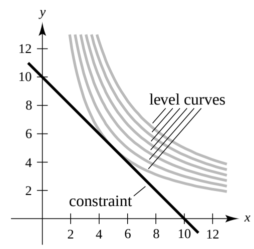
$\nabla f = \lambda \nabla g$

$y\mathbf{i} + x\mathbf{j} = \lambda(\mathbf{i} + \mathbf{j})$

$$\left.\begin{aligned} y &= \lambda \\ x &= \lambda \end{aligned}\right\} x = y$$

$x + y = 10 = \implies x = y = 5$

$f(5, 5) = 25$



3. Minimize $f(x, y) = x^2 + y^2$.

Constraint: $x + y = 4$

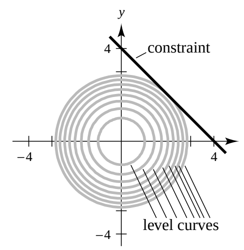
$\nabla f = \lambda \nabla g$

$2x\mathbf{i} + 2y\mathbf{j} = \lambda\mathbf{i} + \lambda\mathbf{j}$

$$\left.\begin{aligned} 2x &= \lambda \\ 2y &= \lambda \end{aligned}\right\} x = y$$

$x + y = 4 \implies x = y = 2$

$f(2, 2) = 8$



5. Minimize $f(x, y) = x^2 - y^2$.

Constraint: $x - 2y = -6$

$\nabla f = \lambda \nabla g$

$2x\mathbf{i} - 2y\mathbf{j} = \lambda\mathbf{i} - 2\lambda\mathbf{j}$

$$2x = \lambda \implies x = \frac{\lambda}{2}$$

$$-2y = -2\lambda \implies y = \lambda$$

$$x - 2y = -6 \implies -\frac{3}{2}\lambda = -6$$

$$\lambda = 4,\ x = 2,\ y = 4$$

$f(2, 4) = -12$

7. Maximize $f(x, y) = 2x + 2xy + y$.

Constraint: $2x + y = 100$

$\nabla f = \lambda \nabla g$

$(2 + 2y)\mathbf{i} + (2x + 1)\mathbf{j} = 2\lambda\mathbf{i} + \lambda\mathbf{j}$

$$\left.\begin{aligned} 2 + 2y = 2\lambda &\implies y = \lambda - 1 \\ 2x + 1 = \lambda &\implies x = \frac{\lambda - 1}{2} \end{aligned}\right\} y = 2x$$

$$2x + y = 100 \implies 4x = 100$$

$$x = 25,\ y = 50$$

$f(25, 50) = 2600$

9. **Note:** $f(x, y) = \sqrt{6 - x^2 - y^2}$ is maximum when $g(x, y)$ is maximum.

Maximize $g(x, y) = 6 - x^2 - y^2$.

Constraint: $x + y = 2$

$$\left.\begin{aligned} -2x &= \lambda \\ -2y &= \lambda \end{aligned}\right\} x = y$$

$x + y = 2 \implies x = y = 1$

$f(1, 1) = \sqrt{g(1, 1)} = 2$

11. Maximize $f(x, y) = e^{xy}$.

Constraint: $x^2 + y^2 = 8$

$$\left.\begin{aligned} ye^{xy} &= 2x\lambda \\ xe^{xy} &= 2y\lambda \end{aligned}\right\} x = y$$

$x^2 + y^2 = 8 \implies 2x^2 = 8$

$$x = y = 2$$

$f(2, 2) = e^4$

13. Maximize or minimize $f(x, y) = x^2 + 3xy + y^2$.

Constraint: $x^2 + y^2 \le 1$

Case 1: On the circle $x^2 + y^2 = 1$

$$\left.\begin{aligned} 2x + 3y &= 2x\lambda \\ 3x + 2y &= 2y\lambda \end{aligned}\right\} x^2 = y^2$$

$$x^2 + y^2 = 1 \implies x = \pm\frac{\sqrt{2}}{2}, y = \pm\frac{\sqrt{2}}{2}$$

Maxima: $f\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right) = \frac{5}{2}$

Minima: $f\left(\pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2}\right) = -\frac{1}{2}$

Case 2: Inside the circle

$$\left.\begin{aligned} f_x &= 2x + 3y = 0 \\ f_y &= 3x + 2y = 0 \end{aligned}\right\} x = y = 0$$

$f_{xx} = 2,\ f_{yy} = 2,\ f_{xy} = 3,\ f_{xx}f_{yy} - (f_{xy})^2 \le 0$

Saddle point: $f(0, 0) = 0$

By combining these two cases, we have a maximum of $\frac{5}{2}$ at

$$\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$$

and a minimum of $-\frac{1}{2}$ at

$$\left(\pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2}\right).$$

15. Minimize $f(x, y, z) = x^2 + y^2 + z^2$.

Constraint: $x + y + z = 6$

$$\left.\begin{aligned} 2x &= \lambda \\ 2y &= \lambda \\ 2z &= \lambda \end{aligned}\right\} x = y = z$$

$x + y + z = 6 \implies x = y = z = 2$

$f(2, 2, 2) = 12$

17. Minimize $f(x, y, z) = x^2 + y^2 + z^2$.

Constraint: $x + y + z = 1$

$$\left.\begin{aligned} 2x &= \lambda \\ 2y &= \lambda \\ 2z &= \lambda \end{aligned}\right\} x = y = z$$

$x + y + z = 1 \implies x = y = z = \frac{1}{3}$

$f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3}$

19. Maximize $f(x, y, z) = xyz$.

Constraints: $x + y + z = 32$

$x - y + z = 0$

$\nabla f = \lambda \nabla g + \mu \nabla h$

$yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mu(\mathbf{i} - \mathbf{j} + \mathbf{k})$

$$\left.\begin{aligned} yz &= \lambda + \mu \\ xz &= \lambda - \mu \\ xy &= \lambda + \mu \end{aligned}\right\} yz = xy \implies x = z$$

$$\left.\begin{aligned} x + y + z &= 32 \\ x - y + z &= 0 \end{aligned}\right\} 2x + 2z = 32 \implies x = z = 8$$

$y = 16$

$f(8, 16, 8) = 1024$

21. Maximize $f(x, y, z) = xy + yz$.

Constraints: $x + 2y = 6$

$x - 3z = 0$

$\nabla f = \lambda \nabla g + \mu \nabla h$

$y\mathbf{i} + (x + z)\mathbf{j} + y\mathbf{k} = \lambda(\mathbf{i} + 2\mathbf{j}) + \mu(\mathbf{i} - 3\mathbf{k})$

$$\left.\begin{aligned} y &= \lambda + \mu \\ x + z &= 2\lambda \\ y &= -3\mu \end{aligned}\right\} y = \frac{3}{4}\lambda \implies x + z = \frac{8}{3}y$$

$x + 2y = 6 \implies y = 3 - \frac{x}{2}$

$x - 3z = 0 \implies z = \frac{x}{3}$

$$x + \frac{x}{3} = \frac{8}{3}\left(3 - \frac{x}{2}\right)$$

$$x = 3, y = \frac{3}{2}, z = 1$$

$f\left(3, \frac{3}{2}, 1\right) = 6$

23. Minimize the square of the distance $f(x, y) = x^2 + y^2$ subject to the constraint $2x + 3y = -1$.

$$\left.\begin{aligned} 2x &= 2\lambda \\ 2y &= 3\lambda \end{aligned}\right\} y = \frac{3x}{2}$$

$2x + 3y = -1 \implies x = -\frac{2}{13}, y = -\frac{3}{13}$

The point on the line is $\left(-\frac{2}{13}, -\frac{3}{13}\right)$ and the desired distance is

$$d = \sqrt{\left(-\frac{2}{13}\right)^2 + \left(-\frac{3}{13}\right)^2} = \frac{\sqrt{13}}{13}.$$

25. Minimize the square of the distance

$$f(x, y, z) = (x - 2)^2 + (y - 1)^2 + (z - 1)^2$$

subject to the constraint $x + y + z = 1$.

$$\left.\begin{aligned} 2(x - 2) &= \lambda \\ 2(y - 1) &= \lambda \\ 2(z - 1) &= \lambda \end{aligned}\right\} y = z \text{ and } y = x - 1$$

$x + y + z = 1 \implies x + 2(x - 1) = 1$

$x = 1, y = z = 0$

The point on the plane is $(1, 0, 0)$ and the desired distance is

$$d = \sqrt{(1 - 2)^2 + (0 - 1)^2 + (0 - 1)^2} = \sqrt{3}.$$

27. Maximize $f(x, y, z) = z$ subject to the constraints $x^2 + y^2 + z^2 = 36$ and $2x + y - z = 2$.

$$\left.\begin{aligned} 0 &= 2x\lambda + 2\mu \\ 0 &= 2y\lambda + \mu \\ 1 &= 2z\lambda - \mu \end{aligned}\right\} x = 2y$$

$$x^2 + y^2 + z^2 = 36$$

$$2x + y - z = 2 \implies z = 2x + y - 2 = 5y - 2$$

$$(2y)^2 + y^2 + (5y - 2)^2 = 36$$

$$30y^2 - 20y - 32 = 0$$

$$15y^2 - 10y - 16 = 0$$

$$y = \frac{5 \pm \sqrt{265}}{15}$$

Choosing the positive value for y we have the point

$$\left(\frac{10 + 2\sqrt{265}}{15}, \frac{5 + \sqrt{265}}{15}, \frac{-1 + \sqrt{265}}{3}\right).$$

29. Optimization problems that have restrictions or contstraints on the values that can be used to produce the optimal solution are called contrained optimization problems.

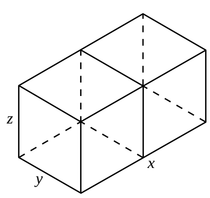
31. Maximize $V(x, y, z) = xyz$ subject to the constraint $x + 2y + 2z = 108$.

$$\left.\begin{aligned} yz &= \lambda \\ xz &= 2\lambda \\ xy &= 2\lambda \end{aligned}\right\} y = z \text{ and } x = 2y$$

$$x + 2y + 2z = 108 \implies 6y = 108, y = 18$$

$$x = 36, y = z = 18$$

Volume is maximum when the dimensions are $36 \times 18 \times 18$ inches



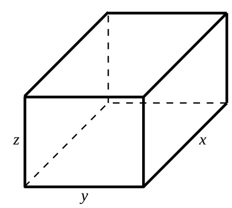
33. Minimize $C(x, y, z) = 5xy + 3(2xz + 2yz + xy)$ subject to the constraint $xyz = 480$.

$$\left.\begin{aligned} 8y + 6z &= yz\lambda \\ 8x + 6z &= xz\lambda \\ 6x + 6y &= xy\lambda \end{aligned}\right\} x = y, 4y = 3z$$

$$xyz = 480 \implies \tfrac{4}{3}y^3 = 480$$

$$x = y = \sqrt[3]{360}, z = \tfrac{4}{3}\sqrt[3]{360}$$

Dimensions: $\sqrt[3]{360} \times \sqrt[3]{360} \times \frac{4}{3}\sqrt[3]{360}$ feet



35. Maximize $V(x, y, z) = (2x)(2y)(2z) = 8xyz$ subject to the constraint $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

$$\left.\begin{aligned} 8yz &= \frac{2x}{a^2}\lambda \\ 8xz &= \frac{2y}{b^2}\lambda \\ 8xy &= \frac{2z}{c^2}\lambda \end{aligned}\right\} \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \implies \frac{3x^2}{a^2} = 1, \frac{3y^2}{b^2} = 1, \frac{3z^2}{c^2} = 1$$

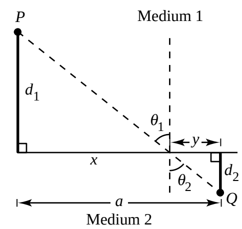
$$x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}$$

Therefore, the dimensions of the box are $\dfrac{2\sqrt{3}a}{3} \times \dfrac{2\sqrt{3}b}{3} \times \dfrac{2\sqrt{3}c}{3}$.

37. Using the formula Time $= \dfrac{\text{Distance}}{\text{Rate}}$, minimize $T(x, y) = \dfrac{\sqrt{d_1^2 + x^2}}{v_1} + \dfrac{\sqrt{d_2^2 + y^2}}{v_2}$ subject to the constraint $x + y = a$.

$$\left.\begin{aligned} \frac{x}{v_1\sqrt{d_2^2 + x^2}} &= \lambda \\ \frac{y}{v_2\sqrt{d_2^2 + y^2}} &= \lambda \end{aligned}\right\} \frac{x}{v_1\sqrt{d_1^2 + x^2}} = \frac{y}{v_2\sqrt{d_2^2 + y^2}}$$

$$x + y = a$$



Since $\sin\theta_1 = \dfrac{x}{\sqrt{d_1^2 + x^2}}$ and $\sin\theta_2 = \dfrac{y}{\sqrt{d_2^2 + y^2}}$, we have

$$\frac{x/\sqrt{d_1^2 + x^2}}{v_1} = \frac{y/\sqrt{d_2^2 + y^2}}{v_2} \quad \text{or} \quad \frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}.$$

39. Maximize $P(p, q, r) = 2pq + 2pr + 2qr$.

Constraint: $p + q + r = 1$

$\nabla P = \lambda \nabla g$

$$\left.\begin{aligned} 2q + 2r &= \lambda \\ 2p + 2r &= \lambda \\ 2p + 2q &= \lambda \end{aligned}\right\} \begin{aligned} &\Rightarrow 3\lambda = 4(p + q + r) = 4(1) \\ &\Rightarrow \lambda = \tfrac{4}{3} \end{aligned}$$

$p + q + r = 1$

$$\left.\begin{aligned} q + r &= \tfrac{2}{3} \\ p + q + r &= 1 \end{aligned}\right\} \Rightarrow p = \tfrac{1}{3}, q = \tfrac{1}{3}, r = \tfrac{1}{3}$$

$P\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = 2\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{2}{3}$.

41. Maximize $P(x, y) = 100x^{0.25}y^{0.75}$

subject to the constraint $48x + 36y = 100{,}000$.

$$25x^{-0.75}y^{0.75} = 48\lambda \Rightarrow \left(\frac{y}{x}\right)^{0.75} = \frac{48\lambda}{25}$$

$$75x^{0.25}y^{-0.25} = 36\lambda \Rightarrow \left(\frac{x}{y}\right)^{0.25} = \frac{36\lambda}{75}$$

$$\left(\frac{y}{x}\right)^{0.75}\left(\frac{y}{x}\right)^{0.25} = \left(\frac{48\lambda}{25}\right)\left(\frac{75}{36\lambda}\right)$$

$$\frac{y}{x} = 4$$

$$y = 4x$$

$$48x + 36y = 100{,}000 \Rightarrow 192x = 100{,}000$$

$$x = \frac{3125}{6}, y = \frac{6250}{3}$$

Therefore, $P\left(\frac{3125}{6}, \frac{6250}{3}\right) \approx 147{,}314$.

43. Minimize $C(x, y) = 48x + 36y$ subject to the constraint $100x^{0.25}y^{0.75} = 20{,}000$.

$$48 = 25x^{-0.75}y^{0.75}\lambda \Rightarrow \left(\frac{y}{x}\right)^{0.75} = \frac{48}{25\lambda}$$

$$36 = 75x^{0.25}y^{-0.25}\lambda \Rightarrow \left(\frac{x}{y}\right)^{0.25} = \frac{36}{75\lambda}$$

$$\left(\frac{y}{x}\right)^{0.75}\left(\frac{y}{x}\right)^{0.25} = \left(\frac{48}{25\lambda}\right)\left(\frac{75\lambda}{36}\right)$$

$$\frac{y}{x} = 4 \Rightarrow y = 4x$$

$$100x^{0.25}y^{0.75} = 20{,}000 \Rightarrow x^{0.25}(4x)^{0.75} = 200$$

$$x = \frac{200}{4^{0.75}} = \frac{200}{2\sqrt{2}} = 50\sqrt{2}$$

$$y = 4x = 200\sqrt{2}$$

Therefore, $C\left(50\sqrt{2}, 200\sqrt{2}\right) \approx \$13{,}576.45$.

45. (a) Maximize $g(\alpha, \beta, \gamma) = \cos\alpha\cos\beta\cos\gamma$ subject to the constraint $\alpha + \beta + \gamma = \pi$.

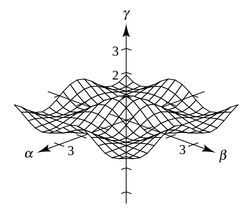
$$\left.\begin{aligned} -\sin\alpha\cos\beta\cos\gamma &= \lambda \\ -\cos\alpha\sin\beta\cos\gamma &= \lambda \\ -\cos\alpha\cos\beta\sin\gamma &= \lambda \end{aligned}\right\} \tan\alpha = \tan\beta = \tan\gamma \implies \alpha = \beta = \gamma$$

$$\alpha + \beta + \gamma = \pi \implies \alpha = \beta = \gamma = \frac{\pi}{3}$$

$$g\left(\frac{\pi}{3}, \frac{\pi}{3}. \frac{\pi}{3}\right) = \frac{1}{8}$$

(b) $\alpha + \beta + \gamma = \pi \implies \gamma = \pi - (\alpha + \beta)$

$$\begin{aligned} g(\alpha + \beta) &= \cos\alpha\cos\beta\cos(\pi - (\alpha + \beta)) \\ &= \cos\alpha\cos\beta[\cos\pi\cos(\alpha + \beta) + \sin\pi\sin(\alpha + \beta)] \\ &= -\cos\alpha\cos\beta\cos(\alpha + \beta) \end{aligned}$$



Review Exercises for Chapter 12

1. No, it is not the graph of a function.

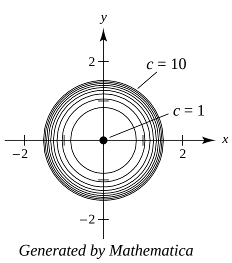
3. $f(x, y) = e^{x^2+y^2}$

The level curves are of the form

$$c = e^{x^2+y^2}$$

$$\ln c = x^2 + y^2.$$

The level curves are circles centered at the origin.



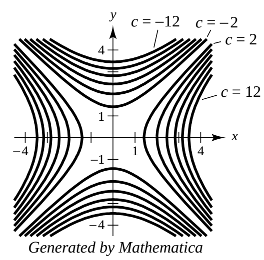
5. $f(x, y) = x^2 - y^2$

The level curves are of the form

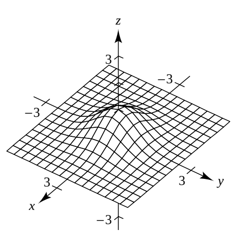
$$c = x^2 - y^2$$

$$1 = \frac{x^2}{c} - \frac{y^2}{c}.$$

The level curves are hyperbolas.



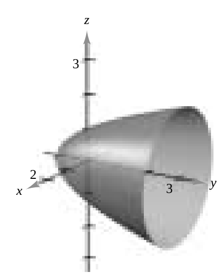
7. $f(x, y) = e^{-(x^2+y^2)}$



9. $f(x, y, z) = x^2 - y + z^2 = 1$

$$y = x^2 + z^2 - 1$$

Elliptic paraboloid



11. $\displaystyle\lim_{(x, y)\to(1, 1)} \frac{xy}{x^2 + y^2} = \frac{1}{2}$

Continuous except at (0, 0).

13. $\displaystyle\lim_{(x, y)\to(0, 0)} \frac{-4x^2y}{x^4 + y^2}$

For $y = x^2$, $\dfrac{-4x^2y}{x^4 + y^2} = \dfrac{-4x^4}{x^4 + x^4} = -2$, for $x \neq 0$

For $y = 0$, $\dfrac{-4x^2y}{x^4 + y^2} = 0$, for $x \neq 0$

Thus, the limit does not exist. Continuous except at (0, 0).

15. $f(x, y) = e^x \cos y$

$$f_x = e^x \cos y$$

$$f_y = -e^x \sin y$$

17. $z = xe^y + ye^x$

$$\frac{\partial z}{\partial x} = e^y + ye^x$$

$$\frac{\partial z}{\partial y} = xe^y + e^x$$

19. $g(x, y) = \dfrac{xy}{x^2 + y^2}$

$$g_x = \frac{y(x^2 + y^2) - xy(2x)}{(x^2 + y^2)^2} = \frac{y(y^2 - x^2)}{(x^2 + y^2)^2}$$

$$g_y = \frac{x(x^2 - y^2)}{(x^2 + y^2)^2}$$

21. $f(x, y, z) = z \arctan \dfrac{y}{x}$

$$f_x = \frac{z}{1 + (y^2/x^2)}\left(-\frac{y}{x^2}\right) = \frac{-yz}{x^2 + y^2}$$

$$f_y = \frac{z}{1 + (y^2/x^2)}\left(\frac{1}{x}\right) = \frac{xz}{x^2 + y^2}$$

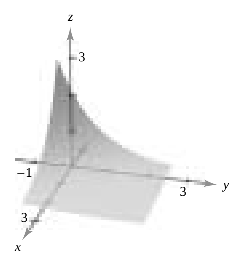
$$f_z = \arctan \frac{y}{x}$$

23. $u(x, t) = ce^{-n^2t} \sin(nx)$

$$\frac{\partial u}{\partial x} = cne^{-n^2t} \cos(nx)$$

$$\frac{\partial u}{\partial t} = -cn^2e^{-n^2t} \sin(nx)$$

25.



27. $f(x, y) = 3x^2 - xy + 2y^3$

$$f_x = 6x - y$$

$$f_y = -x + 6y^2$$

$$f_{xx} = 6$$

$$f_{yy} = 12y$$

$$f_{xy} = -1$$

$$f_{yx} = -1$$

29. $h(x, y) = x \sin y + y \cos x$

$$h_x = \sin y - y \sin x$$

$$h_y = x \cos y + \cos x$$

$$h_{xx} = -y \cos x$$

$$h_{yy} = -x \sin y$$

$$h_{xy} = \cos y - \sin x$$

$$h_{yx} = \cos y - \sin x$$

31. $z = x^2 - y^2$

$$\frac{\partial z}{\partial x} = 2x$$

$$\frac{\partial^2 z}{\partial x^2} = 2$$

$$\frac{\partial z}{\partial y} = -2y$$

$$\frac{\partial^2 z}{\partial y^2} = -2$$

Therefore, $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0.$

33. $z = \dfrac{y}{x^2 + y^2}$

$$\frac{\partial z}{\partial x} = \frac{-2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial x^2} = -2y\left[\frac{-4x^2}{(x^2 + y^2)^3} + \frac{1}{(x^2 + y^2)^2}\right] = 2y\,\frac{3x^2 - y^2}{(x^2 + y^2)^3}$$

$$\frac{\partial z}{\partial y} = \frac{(x^2 + y^2) - 2y}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{(x^2 + y^2)^2(-2y) - 2(x^2 - y^2)(x^2 + y^2)(2y)}{(x^2 + y^2)^4}$$

$$= -2y\,\frac{3x^2 - y^2}{(x^2 + y^2)^3}$$

Therefore, $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0.$

35. $z = x \sin \dfrac{y}{x}$

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = \left(\sin \frac{y}{x} - \frac{y}{x}\cos \frac{y}{x}\right)dx + \left(\cos \frac{y}{x}\right)dy$$

37. $z^2 = x^2 + y^2$

$2z\,dx = 2x\,dx + 2y\,dy$

$$dz = \frac{x}{z}\,dx + \frac{y}{z}\,dy = \frac{5}{13}\left(\frac{1}{2}\right) + \frac{12}{13}\left(\frac{1}{2}\right) = \frac{17}{26} \approx 0.654 \text{ cm}$$

Percentage error: $\dfrac{dz}{z} = \dfrac{17/26}{13} \approx 0.0503 \approx 5\%$

39. $V = \frac{1}{3}\pi r^2 h$

$$dV = \tfrac{2}{3}\pi rh\,dr + \tfrac{1}{3}\pi r^2\,dh = \tfrac{2}{3}\pi(2)(5)\left(\pm\tfrac{1}{8}\right) + \tfrac{1}{3}\pi(2)^2\left(\pm\tfrac{1}{8}\right)$$

$$= \pm\tfrac{5}{6}\pi \pm \tfrac{1}{6}\pi = \pm\pi \text{ in.}^3$$

41. $w = \ln(x^2 + y^2),\ x = 2t + 3,\ y = 4 - t$

Chain Rule: $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt}$

$$= \frac{2x}{x^2 + y^2}(2) + \frac{2y}{x^2 + y^2}(-1)$$

$$= \frac{2(2t + 3)2}{(2t + 3)^2 + (4 - t)^2} - \frac{2(4 - t)}{(2t + 3)^2 + (4 - t)^2}$$

$$= \frac{10t + 4}{5t^2 + 4t + 25}$$

Substitution: $w = \ln(x^2 + y^2) = \ln\left[(2t + 3)^2 + (4 - t)^2\right]$

$$\frac{dw}{dt} = \frac{2(2t + 3)(2) - 2(4 - t)}{(2t + 3)^2 + (4 - t)^2} = \frac{10t + 4}{5t^2 + 4t + 25}$$

43. $u = x^2 + y^2 + z^2,\ x = r\cos t,\ y = r\sin t,\ z = t$

Chain Rule: $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial r} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial r}$

$$= 2x\cos t + 2y\sin t + 2z(0)$$

$$= 2(r\cos^2 t + r\sin^2 t) = 2r$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}$$

$$= 2x(-r\sin t) + 2y(r\cos t) + 2z$$

$$= 2(-r^2\sin t\cos t + r^2\sin t\cos t) + 2t$$

$$= 2t$$

Substitution: $u(r, t) = r^2\cos^2 t + r^2\sin^2 t + t^2 = r^2 + t^2$

$$\frac{\partial u}{\partial r} = 2r$$

$$\frac{\partial u}{\partial t} = 2t$$

45. $x^2y - 2yz - xz - z^2 = 0$

$$2xy - 2y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial x} - z - 2z\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{-2xy + z}{-2y - x - 2z} = \frac{2xy - z}{x + 2y + 2z}$$

$$x^2 - 2y\frac{\partial z}{\partial y} - 2z - x\frac{\partial z}{\partial y} - 2z\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = \frac{-x^2 + 2z}{-2y - x - 2z} = \frac{x^2 - 2z}{x + 2y + 2z}$$

47. $f(x, y) = x^2y$

$$\nabla f = 2xy\mathbf{i} + x^2\mathbf{j}$$

$$\nabla f(2, 1) = 4\mathbf{i} + 4\mathbf{j}$$

$$\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{v} = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$$

$$D_{\mathbf{u}} f(2, 1) = \nabla f(2, 1) \cdot \mathbf{u} = 2\sqrt{2} - 2\sqrt{2} = 0$$

49. $w = y^2 + xz$

$$\nabla w = z\mathbf{i} + 2y\mathbf{j} + x\mathbf{k}$$

$$\nabla w(1, 2, 2) = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$$

$$\mathbf{u} = \frac{1}{3}\mathbf{v} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$$

$$D_{\mathbf{u}}w(1, 2, 2) = \nabla w(1, 2, 2) \cdot \mathbf{u} = \frac{4}{3} - \frac{4}{3} + \frac{2}{3} = \frac{2}{3}$$

51. $z = \dfrac{y}{x^2 + y^2}$

$$\nabla z = -\frac{2xy}{(x^2 + y^2)^2}\mathbf{i} + \frac{x^2 - y^2}{(x^2 + y^2)^2}\mathbf{j}$$

$$\nabla z(1, 1) = -\frac{1}{2}\mathbf{i} = \left\langle -\frac{1}{2}, 0 \right\rangle$$

$$\|\nabla z(1, 1)\| = \frac{1}{2}$$

53. $z = e^{-x}\cos y$

$$\nabla z = -e^{-x}\cos y\mathbf{i} - e^{-x}\sin y\mathbf{j}$$

$$\nabla z\left(0, \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j} = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$$

$$\left\| \nabla z\left(0, \frac{\pi}{4}\right) \right\| = 1$$

55. $9x^2 - 4y^2 = 65$

$$f(x, y) = 9x^2 - 4y^2$$

$$\nabla f(x, y) = 18x\mathbf{i} + 8y\mathbf{j}$$

$$\nabla f(3, 2) = 54\mathbf{i} - 16\mathbf{j}$$

Unit normal: $\dfrac{54\mathbf{i} - 16\mathbf{j}}{\|54\mathbf{i} - 16\mathbf{j}\|} = \dfrac{1}{\sqrt{793}}(27\mathbf{i} - 8\mathbf{j})$

57. $F(x, y, z) = x^2y - z = 0$

$$\nabla F = 2xy\mathbf{i} + x^2\mathbf{j} - \mathbf{k}$$

$$\nabla F(2, 1, 4) = 4\mathbf{i} + 4\mathbf{j} - \mathbf{k}$$

Therefore, the equation of the tangent plane is

$4(x - 2) + 4(y - 1) - (z - 4) = 0$ or

$4x + 4y - z = 8,$

and the equation of the normal line is

$$\frac{x - 2}{4} = \frac{y - 1}{4} = \frac{z - 4}{-1}.$$

59. $F(x, y, z) = x^2 + y^2 - 4x + 6y + z + 9 = 0$

$$\nabla F = (2x - 4)\mathbf{i} + (2y + 6)\mathbf{j} + \mathbf{k}$$

$$\nabla F(2, -3, 4) = \mathbf{k}$$

Therefore, the equation of the tangent plane is

$z - 4 = 0$ or $z = 4,$

and the equation of the normal line is

$x = 2,\ y = -3,\ z = 4 + t.$

61. $F(x, y, z) = x^2 - y^2 - z = 0$

$$G(x, y, z) = 3 - z = 0$$

$$\nabla F = 2x\mathbf{i} - 2y\mathbf{j} - \mathbf{k}$$

$$\nabla G = -\mathbf{k}$$

$$\nabla F(2, 1, 3) = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k}$$

$$\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -2 & -1 \\ 0 & 0 & -1 \end{vmatrix} = 2(\mathbf{i} + 2\mathbf{j})$$

Therefore, the equation of the tangent line is

$$\frac{x - 2}{1} = \frac{y - 1}{2},\ z = 3.$$

63. $f(x, y, z) = x^2 + y^2 + z^2 - 14$

$$\nabla f(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$$

$\nabla f(2, 1, 3) = 4\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ Normal vector to plane.

$$\cos\theta = \frac{|\mathbf{n} \cdot \mathbf{k}|}{\|\mathbf{n}\|} = \frac{6}{\sqrt{56}} = \frac{3\sqrt{14}}{14}$$

$$\theta = 36.7°$$

65. $f(x, y) = x^3 - 3xy + y^2$

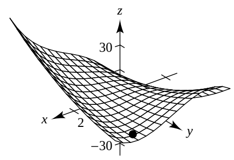
$f_x = 3x^2 - 3y = 3(x^2 - y) = 0$

$f_y = -3x + 2y = 0$

$f_{xx} = 6x$

$f_{yy} = 2$

$f_{xy} = -3$

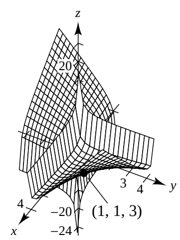


From $f_x = 0$, we have $y = x^2$. Substituting this into $f_y = 0$, we have $-3x + 2x^2 = x(2x - 3) = 0$. Thus, $x = 0$ or $\frac{3}{2}$.
At the critical point $(0, 0)$, $f_{xx}f_{yy} - (f_{xy})^2 < 0$. Therefore, $(0, 0, 0)$ is a saddle point.
At the critical point $\left(\frac{3}{2}, \frac{9}{4}\right)$, $f_{xx}f_{yy} - (f_{xy})^2 > 0$ and $f_{xx} > 0$. Therefore, $\left(\frac{3}{2}, \frac{9}{4}, -\frac{27}{16}\right)$ is a relative minimum.

67. $f(x, y) = xy + \dfrac{1}{x} + \dfrac{1}{y}$

$f_x = y - \dfrac{1}{x^2} = 0, \ x^2y = 1$

$f_y = x - \dfrac{1}{y^2} = 0, \ xy^2 = 1$



Thus, $x^2y = xy^2$ or $x = y$ and substitution yields the critical point $(1, 1)$.

$f_{xx} = \dfrac{2}{x^3}$

$f_{xy} = 1$

$f_{yy} = \dfrac{2}{y^3}$

At the critical point $(1, 1)$, $f_{xx} = 2 > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 = 3 > 0$. Thus, $(1, 1, 3)$ is a relative minimum.

69. The level curves are hyperbolas. There is a critical point at $(0, 0)$, but there are no relative extrema. The gradient is normal to the level curve at any given point at (x_0, y_0).

71. $P(x_1, x_2) = R - C_1 - C_2$

$= [225 - 0.4(x_1 + x_2)](x_1 + x_2) - (0.05x_1^2 + 15x_1 + 5400) - (0.03x_2^2 + 15x_2 + 6100)$

$= -0.45x_1^2 - 0.43x_2^2 - 0.8x_1x_2 + 210x_1 + 210x_2 - 11{,}500$

$P_{x_1} = -0.9x_1 - 0.8x_2 + 210 = 0$

$0.9x_1 + 0.8x_2 = 210$

$P_{x_2} = -0.86x_2 - 0.8x_1 + 210 = 0$

$0.8x_1 + 0.86x_2 = 210$

Solving this system yields $x_1 \approx 94$ and $x_2 \approx 157$.

$P_{x_1x_1} = -0.9$

$P_{x_1x_2} = -0.8$

$P_{x_2x_2} = -0.86$

$P_{x_1x_1} < 0$

$P_{x_1x_1}P_{x_2x_2} - (P_{x_1x_2})^2 > 0$

Therefore, profit is maximum when $x_1 \approx 94$ and $x_2 \approx 157$.

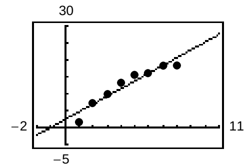
73. Maximize $f(x, y) = 4x + xy + 2y$ subject to the constraint $20x + 4y = 2000$.

$$\left.\begin{aligned} 4 + y &= 20\lambda \\ x + 2 &= 4\lambda \end{aligned}\right\} 5x - y = -6$$

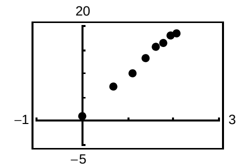
$$20x + 4y = 2000 \implies \begin{aligned} 5x + y &= 500 \\ 5x - y &= -6 \\ \hline 10x \quad &= 494 \\ x &= 49.4 \\ y &= 253 \end{aligned}$$

$f(49.4, 253) = 13{,}201.8$

75. (a) $y = 2.29t + 2.34$



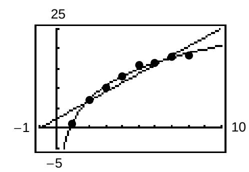
(b)



Yes, the data appears more linear.

(c) $y = 8.37 \ln t + 1.54$

(d)



The logarithmic model is a better fit.

77. Optimize $f(x, y, z) = xy + yz + xz$ subject to the constraint $x + y + z = 1$.

$$\left.\begin{aligned} y + z &= \lambda \\ x + z &= \lambda \\ x + y &= \lambda \end{aligned}\right\} x = y = z$$

$$x + y + z = 1 \implies x = y = z = \tfrac{1}{3}$$

Maximum: $f\left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right) = \tfrac{1}{3}$

79. $PQ = \sqrt{x^2 + 4}$, $QR = \sqrt{y^2 + 1}$, $RS = z$; $x + y + z = 10$

$C = 3\sqrt{x^2 + 4} + 2\sqrt{y^2 + 1} + 2$

Constraint: $x + y + z = 10$

$\nabla C = \lambda \nabla g$

$$\frac{3x}{\sqrt{x^2 + 4}}\mathbf{i} + \frac{2y}{\sqrt{y^2 + 1}}\mathbf{j} + \mathbf{k} = \lambda[\mathbf{i} + \mathbf{j} + \mathbf{k}]$$

$3x = \lambda\sqrt{x^2 + 4}$

$2y = \lambda\sqrt{y^2 + 1}$

$1 = \lambda$

$$9x^2 = x^2 + 4 \implies x^2 = \frac{1}{2}$$

$$4y^2 = y^2 + 1 \implies y^2 = \frac{1}{3}$$

Hence, $x = \dfrac{\sqrt{2}}{2}, y = \dfrac{\sqrt{3}}{3}, z = 10 - \dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{3}}{3} \approx 8.716$ m.