45. (a) Maximize $g(\alpha, \beta, \gamma) = \cos\alpha\cos\beta\cos\gamma$ subject to the constraint $\alpha + \beta + \gamma = \pi$.

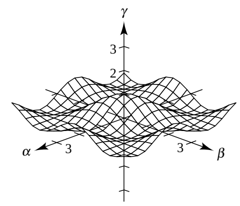
$$\left.\begin{aligned} -\sin\alpha\cos\beta\cos\gamma &= \lambda \\ -\cos\alpha\sin\beta\cos\gamma &= \lambda \\ -\cos\alpha\cos\beta\sin\gamma &= \lambda \end{aligned}\right\} \tan\alpha = \tan\beta = \tan\gamma \implies \alpha = \beta = \gamma$$

$$\alpha + \beta + \gamma = \pi \implies \alpha = \beta = \gamma = \frac{\pi}{3}$$

$$g\left(\frac{\pi}{3}, \frac{\pi}{3}. \frac{\pi}{3}\right) = \frac{1}{8}$$

(b) $\alpha + \beta + \gamma = \pi \implies \gamma = \pi - (\alpha + \beta)$

$$\begin{aligned} g(\alpha + \beta) &= \cos\alpha\cos\beta\cos(\pi - (\alpha + \beta)) \\ &= \cos\alpha\cos\beta[\cos\pi\cos(\alpha + \beta) + \sin\pi\sin(\alpha + \beta)] \\ &= -\cos\alpha\cos\beta\cos(\alpha + \beta) \end{aligned}$$



Review Exercises for Chapter 12

1. No, it is not the graph of a function.

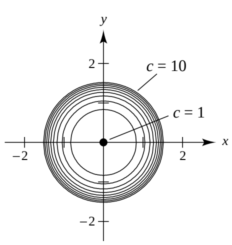
3. $f(x, y) = e^{x^2+y^2}$

The level curves are of the form

$$c = e^{x^2+y^2}$$

$$\ln c = x^2 + y^2.$$

The level curves are circles centered at the origin.



Generated by Mathematica

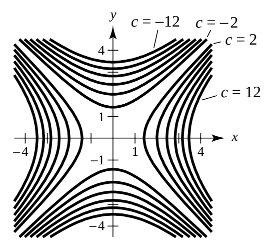
5. $f(x, y) = x^2 - y^2$

The level curves are of the form

$$c = x^2 - y^2$$

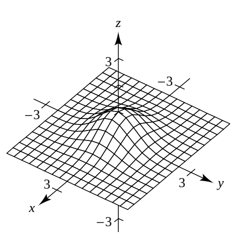
$$1 = \frac{x^2}{c} - \frac{y^2}{c}.$$

The level curves are hyperbolas.



Generated by Mathematica

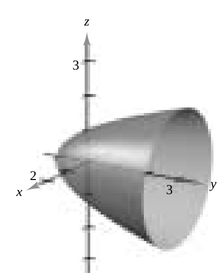
7. $f(x, y) = e^{-(x^2+y^2)}$



9. $f(x, y, z) = x^2 - y + z^2 = 1$

$$y = x^2 + z^2 - 1$$

Elliptic paraboloid



11. $\displaystyle\lim_{(x, y)\to(1, 1)} \frac{xy}{x^2 + y^2} = \frac{1}{2}$

Continuous except at $(0, 0)$.

13. $\displaystyle\lim_{(x, y)\to(0, 0)} \frac{-4x^2y}{x^4 + y^2}$

For $y = x^2$, $\dfrac{-4x^2y}{x^4 + y^2} = \dfrac{-4x^4}{x^4 + x^4} = -2$, for $x \neq 0$

For $y = 0$, $\dfrac{-4x^2y}{x^4 + y^2} = 0$, for $x \neq 0$

Thus, the limit does not exist. Continuous except at $(0, 0)$.

15. $f(x, y) = e^x \cos y$

$f_x = e^x \cos y$

$f_y = -e^x \sin y$

17. $z = xe^y + ye^x$

$\dfrac{\partial z}{\partial x} = e^y + ye^x$

$\dfrac{\partial z}{\partial y} = xe^y + e^x$

19. $g(x, y) = \dfrac{xy}{x^2 + y^2}$

$g_x = \dfrac{y(x^2 + y^2) - xy(2x)}{(x^2 + y^2)^2} = \dfrac{y(y^2 - x^2)}{(x^2 + y^2)^2}$

$g_y = \dfrac{x(x^2 - y^2)}{(x^2 + y^2)^2}$

21. $f(x, y, z) = z \arctan \dfrac{y}{x}$

$f_x = \dfrac{z}{1 + (y^2/x^2)}\left(-\dfrac{y}{x^2}\right) = \dfrac{-yz}{x^2 + y^2}$

$f_y = \dfrac{z}{1 + (y^2/x^2)}\left(\dfrac{1}{x}\right) = \dfrac{xz}{x^2 + y^2}$

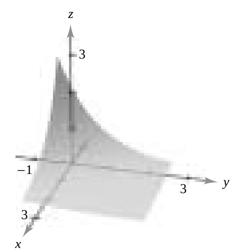
$f_z = \arctan \dfrac{y}{x}$

23. $u(x, t) = ce^{-n^2t} \sin(nx)$

$\dfrac{\partial u}{\partial x} = cne^{-n^2t} \cos(nx)$

$\dfrac{\partial u}{\partial t} = -cn^2e^{-n^2t} \sin(nx)$

25.



27. $f(x, y) = 3x^2 - xy + 2y^3$

$f_x = 6x - y$

$f_y = -x + 6y^2$

$f_{xx} = 6$

$f_{yy} = 12y$

$f_{xy} = -1$

$f_{yx} = -1$

29. $h(x, y) = x \sin y + y \cos x$

$h_x = \sin y - y \sin x$

$h_y = x \cos y + \cos x$

$h_{xx} = -y \cos x$

$h_{yy} = -x \sin y$

$h_{xy} = \cos y - \sin x$

$h_{yx} = \cos y - \sin x$

31. $z = x^2 - y^2$

$\dfrac{\partial z}{\partial x} = 2x$

$\dfrac{\partial^2 z}{\partial x^2} = 2$

$\dfrac{\partial z}{\partial y} = -2y$

$\dfrac{\partial^2 z}{\partial y^2} = -2$

Therefore, $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$.

33. $z = \dfrac{y}{x^2 + y^2}$

$\dfrac{\partial z}{\partial x} = \dfrac{-2xy}{(x^2 + y^2)^2}$

$\dfrac{\partial^2 z}{\partial x^2} = -2y\left[\dfrac{-4x^2}{(x^2 + y^2)^3} + \dfrac{1}{(x^2 + y^2)^2}\right] = 2y\dfrac{3x^2 - y^2}{(x^2 + y^2)^3}$

$\dfrac{\partial z}{\partial y} = \dfrac{(x^2 + y^2) - 2y}{(x^2 + y^2)^2} = \dfrac{x^2 - y^2}{(x^2 + y^2)^2}$

$\dfrac{\partial^2 z}{\partial y^2} = \dfrac{(x^2 + y^2)^2(-2y) - 2(x^2 - y^2)(x^2 + y^2)(2y)}{(x^2 + y^2)^4}$

$= -2y\dfrac{3x^2 - y^2}{(x^2 + y^2)^3}$

Therefore, $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$.

35. $z = x \sin \dfrac{y}{x}$

$dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy = \left(\sin\dfrac{y}{x} - \dfrac{y}{x}\cos\dfrac{y}{x}\right)dx + \left(\cos\dfrac{y}{x}\right)dy$

37. $z^2 = x^2 + y^2$

$2z\,dx = 2x\,dx + 2y\,dy$

$$dz = \frac{x}{z}\,dx + \frac{y}{z}\,dy = \frac{5}{13}\left(\frac{1}{2}\right) + \frac{12}{13}\left(\frac{1}{2}\right) = \frac{17}{26} \approx 0.654 \text{ cm}$$

Percentage error: $\dfrac{dz}{z} = \dfrac{17/26}{13} \approx 0.0503 \approx 5\%$

39. $V = \frac{1}{3}\pi r^2 h$

$$dV = \tfrac{2}{3}\pi rh\,dr + \tfrac{1}{3}\pi r^2\,dh = \tfrac{2}{3}\pi(2)(5)\left(\pm\tfrac{1}{8}\right) + \tfrac{1}{3}\pi(2)^2\left(\pm\tfrac{1}{8}\right)$$
$$= \pm\tfrac{5}{6}\pi \pm \tfrac{1}{6}\pi = \pm\pi \text{ in.}^3$$

41. $w = \ln(x^2 + y^2)$, $x = 2t + 3$, $y = 4 - t$

Chain Rule: $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt}$

$$= \frac{2x}{x^2 + y^2}(2) + \frac{2y}{x^2 + y^2}(-1)$$

$$= \frac{2(2t + 3)2}{(2t + 3)^2 + (4 - t)^2} - \frac{2(4 - t)}{(2t + 3)^2 + (4 - t)^2}$$

$$= \frac{10t + 4}{5t^2 + 4t + 25}$$

Substitution: $w = \ln(x^2 + y^2) = \ln\left[(2t + 3)^2 + (4 - t)^2\right]$

$$\frac{dw}{dt} = \frac{2(2t + 3)(2) - 2(4 - t)}{(2t + 3)^2 + (4 - t)^2} = \frac{10t + 4}{5t^2 + 4t + 25}$$

43. $u = x^2 + y^2 + z^2$, $x = r\cos t$, $y = r\sin t$, $z = t$

Chain Rule: $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial r} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial r}$

$$= 2x\cos t + 2y\sin t + 2z(0)$$

$$= 2(r\cos^2 t + r\sin^2 t) = 2r$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}$$

$$= 2x(-r\sin t) + 2y(r\cos t) + 2z$$

$$= 2(-r^2\sin t\cos t + r^2\sin t\cos t) + 2t$$

$$= 2t$$

Substitution: $u(r, t) = r^2\cos^2 t + r^2\sin^2 t + t^2 = r^2 + t^2$

$$\frac{\partial u}{\partial r} = 2r$$

$$\frac{\partial u}{\partial t} = 2t$$

45. $x^2y - 2yz - xz - z^2 = 0$

$$2xy - 2y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial x} - z - 2z\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{-2xy + z}{-2y - x - 2z} = \frac{2xy - z}{x + 2y + 2z}$$

$$x^2 - 2y\frac{\partial z}{\partial y} - 2z - x\frac{\partial z}{\partial y} - 2z\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = \frac{-x^2 + 2z}{-2y - x - 2z} = \frac{x^2 - 2z}{x + 2y + 2z}$$

47. $f(x, y) = x^2y$

$\nabla f = 2xy\mathbf{i} + x^2\mathbf{j}$

$\nabla f(2, 1) = 4\mathbf{i} + 4\mathbf{j}$

$\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{v} = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$

$D_{\mathbf{u}} f(2, 1) = \nabla f(2, 1) \cdot \mathbf{u} = 2\sqrt{2} - 2\sqrt{2} = 0$

49. $w = y^2 + xz$

$\nabla w = z\mathbf{i} + 2y\mathbf{j} + x\mathbf{k}$

$\nabla w(1, 2, 2) = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$

$\mathbf{u} = \frac{1}{3}\mathbf{v} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$

$D_{\mathbf{u}}w(1, 2, 2) = \nabla w(1, 2, 2) \cdot \mathbf{u} = \frac{4}{3} - \frac{4}{3} + \frac{2}{3} = \frac{2}{3}$

51. $z = \frac{y}{x^2 + y^2}$

$\nabla z = -\frac{2xy}{(x^2 + y^2)^2}\mathbf{i} + \frac{x^2 - y^2}{(x^2 + y^2)^2}\mathbf{j}$

$\nabla z(1, 1) = -\frac{1}{2}\mathbf{i} = \left\langle -\frac{1}{2}, 0 \right\rangle$

$\|\nabla z(1, 1)\| = \frac{1}{2}$

53. $z = e^{-x} \cos y$

$\nabla z = -e^{-x} \cos y\mathbf{i} - e^{-x} \sin y\mathbf{j}$

$\nabla z\left(0, \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j} = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$

$\left\| \nabla z\left(0, \frac{\pi}{4}\right) \right\| = 1$

55. $9x^2 - 4y^2 = 65$

$f(x, y) = 9x^2 - 4y^2$

$\nabla f(x, y) = 18x\mathbf{i} + 8y\mathbf{j}$

$\nabla f(3, 2) = 54\mathbf{i} - 16\mathbf{j}$

Unit normal: $\frac{54\mathbf{i} - 16\mathbf{j}}{\|54\mathbf{i} - 16\mathbf{j}\|} = \frac{1}{\sqrt{793}}(27\mathbf{i} - 8\mathbf{j})$

57. $F(x, y, z) = x^2y - z = 0$

$\nabla F = 2xy\mathbf{i} + x^2\mathbf{j} - \mathbf{k}$

$\nabla F(2, 1, 4) = 4\mathbf{i} + 4\mathbf{j} - \mathbf{k}$

Therefore, the equation of the tangent plane is

$4(x - 2) + 4(y - 1) - (z - 4) = 0$ or

$4x + 4y - z = 8,$

and the equation of the normal line is

$\frac{x - 2}{4} = \frac{y - 1}{4} = \frac{z - 4}{-1}.$

59. $F(x, y, z) = x^2 + y^2 - 4x + 6y + z + 9 = 0$

$\nabla F = (2x - 4)\mathbf{i} + (2y + 6)\mathbf{j} + \mathbf{k}$

$\nabla F(2, -3, 4) = \mathbf{k}$

Therefore, the equation of the tangent plane is

$z - 4 = 0$ or $z = 4,$

and the equation of the normal line is

$x = 2,\ y = -3,\ z = 4 + t.$

61. $F(x, y, z) = x^2 - y^2 - z = 0$

$G(x, y, z) = 3 - z = 0$

$\nabla F = 2x\mathbf{i} - 2y\mathbf{j} - \mathbf{k}$

$\nabla G = -\mathbf{k}$

$\nabla F(2, 1, 3) = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k}$

$$\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -2 & -1 \\ 0 & 0 & -1 \end{vmatrix} = 2(\mathbf{i} + 2\mathbf{j})$$

Therefore, the equation of the tangent line is

$\frac{x - 2}{1} = \frac{y - 1}{2},\ z = 3.$

63. $f(x, y, z) = x^2 + y^2 + z^2 - 14$

$\nabla f(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$

$\nabla f(2, 1, 3) = 4\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ Normal vector to plane.

$\cos \theta = \frac{|\mathbf{n} \cdot \mathbf{k}|}{\|\mathbf{n}\|} = \frac{6}{\sqrt{56}} = \frac{3\sqrt{14}}{14}$

$\theta = 36.7°$

65. $f(x, y) = x^3 - 3xy + y^2$

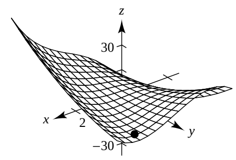
$f_x = 3x^2 - 3y = 3(x^2 - y) = 0$

$f_y = -3x + 2y = 0$

$f_{xx} = 6x$

$f_{yy} = 2$

$f_{xy} = -3$

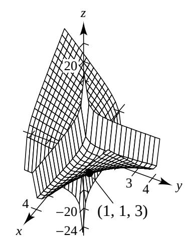


From $f_x = 0$, we have $y = x^2$. Substituting this into $f_y = 0$, we have $-3x + 2x^2 = x(2x - 3) = 0$. Thus, $x = 0$ or $\frac{3}{2}$.
At the critical point $(0, 0)$, $f_{xx}f_{yy} - (f_{xy})^2 < 0$. Therefore, $(0, 0, 0)$ is a saddle point.
At the critical point $\left(\frac{3}{2}, \frac{9}{4}\right)$, $f_{xx}f_{yy} - (f_{xy})^2 > 0$ and $f_{xx} > 0$. Therefore, $\left(\frac{3}{2}, \frac{9}{4}, -\frac{27}{16}\right)$ is a relative minimum.

67. $f(x, y) = xy + \dfrac{1}{x} + \dfrac{1}{y}$

$f_x = y - \dfrac{1}{x^2} = 0, \; x^2y = 1$

$f_y = x - \dfrac{1}{y^2} = 0, \; xy^2 = 1$



Thus, $x^2y = xy^2$ or $x = y$ and substitution yields the critical point $(1, 1)$.

$f_{xx} = \dfrac{2}{x^3}$

$f_{xy} = 1$

$f_{yy} = \dfrac{2}{y^3}$

At the critical point $(1, 1)$, $f_{xx} = 2 > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 = 3 > 0$. Thus, $(1, 1, 3)$ is a relative minimum.

69. The level curves are hyperbolas. There is a critical point at $(0, 0)$, but there are no relative extrema. The gradient is normal to the level curve at any given point at (x_0, y_0).

71. $P(x_1, x_2) = R - C_1 - C_2$

$= [225 - 0.4(x_1 + x_2)](x_1 + x_2) - (0.05x_1^2 + 15x_1 + 5400) - (0.03x_2^2 + 15x_2 + 6100)$

$= -0.45x_1^2 - 0.43x_2^2 - 0.8x_1x_2 + 210x_1 + 210x_2 - 11{,}500$

$P_{x_1} = -0.9x_1 - 0.8x_2 + 210 = 0$

$0.9x_1 + 0.8x_2 = 210$

$P_{x_2} = -0.86x_2 - 0.8x_1 + 210 = 0$

$0.8x_1 + 0.86x_2 = 210$

Solving this system yields $x_1 \approx 94$ and $x_2 \approx 157$.

$P_{x_1x_1} = -0.9$

$P_{x_1x_2} = -0.8$

$P_{x_2x_2} = -0.86$

$P_{x_1x_1} < 0$

$P_{x_1x_1}P_{x_2x_2} - (P_{x_1x_2})^2 > 0$

Therefore, profit is maximum when $x_1 \approx 94$ and $x_2 \approx 157$.

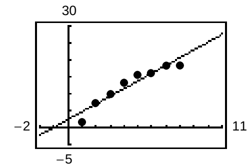
73. Maximize $f(x, y) = 4x + xy + 2y$ subject to the constraint $20x + 4y = 2000$.

$$\left.\begin{aligned} 4 + y &= 20\lambda \\ x + 2 &= 4\lambda \end{aligned}\right\} 5x - y = -6$$

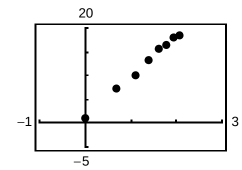
$$20x + 4y = 2000 \implies \begin{aligned} 5x + y &= 500 \\ 5x - y &= -6 \\ \hline 10x &= 494 \\ x &= 49.4 \\ y &= 253 \end{aligned}$$

$f(49.4, 253) = 13{,}201.8$

75. (a) $y = 2.29t + 2.34$



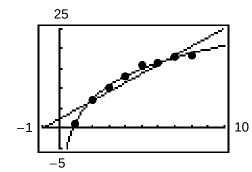
(b)



Yes, the data appears more linear.

(c) $y = 8.37 \ln t + 1.54$

(d)



The logarithmic model is a better fit.

77. Optimize $f(x, y, z) = xy + yz + xz$ subject to the constraint $x + y + z = 1$.

$$\left.\begin{aligned} y + z &= \lambda \\ x + z &= \lambda \\ x + y &= \lambda \end{aligned}\right\} x = y = z$$

$$x + y + z = 1 \implies x = y = z = \tfrac{1}{3}$$

Maximum: $f\left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right) = \tfrac{1}{3}$

79. $PQ = \sqrt{x^2 + 4}$, $QR = \sqrt{y^2 + 1}$, $RS = z$; $x + y + z = 10$

$C = 3\sqrt{x^2 + 4} + 2\sqrt{y^2 + 1} + 2$

Constraint: $x + y + z = 10$

$\nabla C = \lambda \nabla g$

$$\frac{3x}{\sqrt{x^2 + 4}}\mathbf{i} + \frac{2y}{\sqrt{y^2 + 1}}\mathbf{j} + \mathbf{k} = \lambda[\mathbf{i} + \mathbf{j} + \mathbf{k}]$$

$3x = \lambda\sqrt{x^2 + 4}$

$2y = \lambda\sqrt{y^2 + 1}$

$1 = \lambda$

$$9x^2 = x^2 + 4 \implies x^2 = \frac{1}{2}$$

$$4y^2 = y^2 + 1 \implies y^2 = \frac{1}{3}$$

Hence, $x = \dfrac{\sqrt{2}}{2}, y = \dfrac{\sqrt{3}}{3}, z = 10 - \dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{3}}{3} \approx 8.716$ m.

Problem Solving for Chapter 12

1. (a) The three sides have lengths 5, 6, and 5.

Thus, $s = \frac{16}{2} = 8$ and $A = \sqrt{8(3)(2)(3)} = 12$

(b) Let $f(a, b, c) = (\text{area})^2 = s(s - a)(s - b)(s - c)$, subject to the constraint $a + b + c =$ constant (perimeter).

Using Lagrange multipliers,

$-s(s - b)(s - c) = \lambda$

$-s(s - a)(s - c) = \lambda$

$-s(s - b)(s - b) = \lambda$

From the first 2 equations $s - b = s - a \implies a = b$.

Similarly, $b = c$ and hence $a = b = c$ which is an equilateral triangle.

(c) Let $f(a, b, c) = a + b + c$, subject to $(\text{Area})^2 = s(s - a)(s - b)(s - c)$ constant.

Using Lagrange multipliers,

$1 = -\lambda s(s - b)(s - c)$

$1 = -\lambda s(s - a)(s - c)$

$1 = -\lambda s(s - a)(s - b)$

Hence, $s - a = s - b \implies a = b$ and $a = b = c$.

3. (a) $F(x, y, z) = xyz - 1 = 0$

$F_x = yz, F_y = xz, F_z = xy$

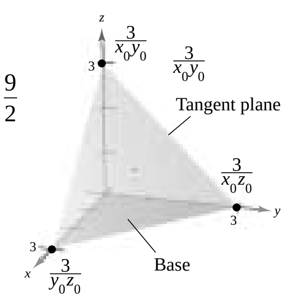
Tangent plane:

$y_0z_0(x - x_0) + x_0z_0(y - y_0) + x_0y_0(z - z_0) = 0$

$y_0z_0x + x_0z_0y + x_0y_0z = 3x_0y_0z_0 = 3$

(b) $V = \frac{1}{3}(\text{base})(\text{height})$

$$= \frac{1}{3}\left(\frac{1}{2}\,\frac{3}{y_0z_0}\,\frac{3}{x_0z_0}\right)\left(\frac{3}{x_0y_0}\right) = \frac{9}{2}$$



5. We cannot use Theorem 12.9 since f is not a differentiable function of x and y. Hence, we use the definition of directional derivatives.

$$D_{\mathbf{u}} f(x, y) = \lim_{t \to 0} \frac{f(x + t\cos\theta, y + t\sin\theta) - f(x, y)}{t}$$

$$D_{\mathbf{u}} f(0, 0) = \lim_{t \to 0} \frac{f\left[0 + \left(\frac{t}{\sqrt{2}}\right), 0 + \left(\frac{t}{\sqrt{2}}\right)\right] - f(0, 0)}{t}$$

$$= \lim_{t \to 0} \frac{1}{t}\left[\frac{4\left(\frac{t}{\sqrt{2}}\right)\left(\frac{t}{\sqrt{2}}\right)}{\left(\frac{t^2}{2}\right) + \left(\frac{t^2}{2}\right)}\right] = \lim_{t \to 0} \frac{1}{t}\left[\frac{2t^2}{t^2}\right] = \lim_{t \to 0} \frac{2}{t} \text{ which does not exist.}$$

If $f(0, 0) = 2$, then

$$D_{\mathbf{u}} f(0, 0) = \lim_{t \to 0} \frac{f\left(0 + \frac{t}{\sqrt{2}}, 0 + \frac{t}{\sqrt{2}}\right) - 2}{t} = \lim_{t \to 0} \frac{1}{t}\left[\frac{2t^2}{t^2} - 2\right] = 0$$

which implies that the directional derivative exists.

7. $H = k(5xy + 6xz + 6yz)$

$z = \dfrac{1000}{xy} \implies H = k\left(5xy + \dfrac{6000}{y} + \dfrac{6000}{x}\right).$

$H_x = 5y - \dfrac{6000}{x^2} = 0 \implies 5yx^2 = 6000$

By symmetry, $x = y \implies x^3 = y^3 = 1200$.

Thus, $x = y = 2\sqrt[3]{150}$ and $z = \dfrac{5}{3}\sqrt[3]{150}$.

9. (a) $\dfrac{\partial f}{\partial x} = Cax^{a-1}y^{1-a}, \dfrac{\partial f}{\partial y} = C(1-a)x^a y^{-a}$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = Cax^a y^{1-a} + C(1-a)x^a y^{1-a}$$
$$= [Ca + C(1-a)]x^a y^{1-a}$$
$$= Cx^a y^{1-a} = f$$

(b) $f(tx, ty) = C(tx)^a(ty)^{1-a} = Ct^a x^a t^{1-a} y^{1-a}$

$$= Cx^a y^{1-a}(t) = tf(x, y)$$

11. (a) $x = 64(\cos 45°)t = 32\sqrt{2}t$

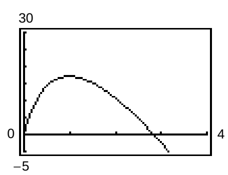
$y = 64(\sin 45°)t - 16t^2 = 32\sqrt{2}t - 16t^2$

(b) $\tan\alpha = \dfrac{y}{x + 50}$

$$\alpha = \arctan\left(\frac{y}{x+50}\right) = \arctan\left(\frac{32\sqrt{2}t - 16t^2}{32\sqrt{2}t + 50}\right)$$

(c) $$\frac{d\alpha}{dt} = \frac{1}{1 + \left(\dfrac{32\sqrt{2}t - 16t^2}{32\sqrt{2}t + 50}\right)^2} \cdot \frac{-64\left(8\sqrt{2}t^2 + 25t - 25\sqrt{2}\right)}{\left(32\sqrt{2}t + 50\right)^2} = \frac{-16\left(8\sqrt{2}t^2 + 25t - 25\sqrt{2}\right)}{64t^4 - 256\sqrt{2}t^3 + 1024t^2 + 800\sqrt{2}t + 625}$$

(d)



No. The rate of change of α is greatest when the projectile is closest to the camera.

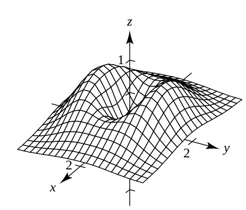
(e) $\dfrac{d\alpha}{dt} = 0$ when

$$8\sqrt{2}t^2 + 25t - 25\sqrt{2} = 0$$

$$t = \frac{-25 + \sqrt{25^2 - 4\left(8\sqrt{2}\right)\left(-25\sqrt{2}\right)}}{2\left(8\sqrt{2}\right)} \approx 0.98 \text{ second.}$$

No, the projectile is at its maximum height when $dy/dt = 32\sqrt{2} - 32t = 0$ or $t = \sqrt{2} \approx 1.41$ seconds.

13. (a) There is a minimum at $(0, 0, 0)$, maxima at $(0, \pm 1, 2/e)$ and saddle point at $(\pm 1, 0, 1/e)$:



$$f_x = (x^2 + 2y^2)e^{-(x^2+y^2)}(-2x) + (2x)e^{-(x^2+y^2)}$$
$$= e^{-(x^2+y^2)}[(x^2 + 2y^2)(-2x) + 2x]$$
$$= e^{-(x^2+y^2)}[-2x^3 + 4xy^2 + 2x] = 0 \implies x^3 + 2xy^2 - x = 0$$

$$f_y = (x^2 + 2y^2)e^{-(x^2+y^2)}(-2y) + (4y)e^{-(x^2+y^2)}$$
$$= e^{-(x^2+y^2)}[(x^2 + 2y^2)(-2y) + 4y]$$
$$= e^{-(x^2+y^2)}[-4y^3 - 2x^2y + 4y] = 0 \implies 2y^3 + x^2y - 2y = 0$$

Solving the two equations $x^3 + 2xy^2 - x = 0$ and $2y^3 + x^2y - 2y = 0$, you obtain the following critical points: $(0, \pm 1)$, $(\pm 1, 0)$, $(0, 0)$. Using the second derivative test, you obtain the results above.

—CONTINUED—

13. —CONTINUED—

(b) As in part (a), you obtain

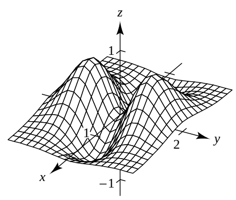
$$f_x = e^{-(x^2+y^2)}[2x(x^2 - 1 - 2y^2)]$$
$$f_y = e^{-(x^2+y^2)}[2y(2 + x^2 - 2y^2)]$$

The critical numbers are $(0, 0)$, $(0, \pm 1)$, $(\pm 1, 0)$. These yield

$(\pm 1, 0, -1/e)$ minima

$(0, \pm 1, 2/e)$ maxima

$(0, 0, 0)$ saddle



(c) In general, for $\alpha > 0$ you obtain

$(0, 0, 0)$ minimum

$(0, \pm 1, \beta/e)$ maxima

$(\pm 1, 0, \alpha/e)$ saddle

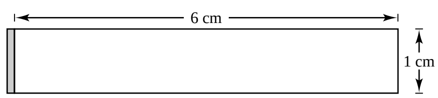
For $\alpha < 0$, you obtain

$(\pm 1, 0, \alpha/e)$ minima

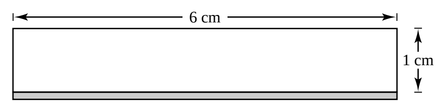
$(0, \pm 1, \beta/e)$ maxima

$(0, 0, 0)$ saddle

15. (a)



(b)



(c) The height has more effect since the shaded region in (b) is larger than the shaded region in (a).

(d) $A = hl \implies dA = l\,dh + h\,dl$

If $dl = 0.01$ and $dh = 0$, then $dA = 1(0.01) = 0.01$.

If $dh = 0.01$ and $dl = 0$, then $dA = 6(0.01) = 0.06$.

17. Essay

19. $u(x, t) = \frac{1}{2}[f(x - ct) + f(x + ct)]$

Let $r = x - ct$ and $s = x + ct$. Then $u(r, s) = \frac{1}{2}[f(r) + f(s)]$.

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial t} = \frac{1}{2}\frac{df}{dr}(-c) + \frac{1}{2}\frac{df}{ds}(c)$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{2}\frac{d^2f}{dr^2}(-c)^2 + \frac{1}{2}\frac{d^2f}{ds^2}(c)^2 = \frac{c^2}{2}\left[\frac{d^2f}{dr^2} + \frac{d^2f}{ds^2}\right]$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = \frac{1}{2}\frac{df}{dr}(1) + \frac{1}{2}\frac{df}{ds}(1)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\frac{d^2f}{dr^2}(1)^2 + \frac{1}{2}\frac{d^2f}{ds^2}(1)^2 = \frac{1}{2}\left[\frac{d^2f}{dr^2} + \frac{d^2f}{ds^2}\right]$$

Thus, $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$.