# CHAPTER 13
# Multiple Integration

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## Section 13.1 Iterated Integrals and Area in the Plane

Solutions to Even-Numbered Exercises

**2.** $\int_x^{x^2} \frac{y}{x}\,dy = \left[\frac{1}{2}\frac{y^2}{x}\right]_x^{x^2} = \frac{1}{2}\left(\frac{x^4}{x} - \frac{x^2}{x}\right) = \frac{x}{2}(x^2 - 1)$

**4.** $\int_0^{\cos y} y\,dx = \Big[yx\Big]_0^{\cos y} = y\cos y$

**6.** $\int_{x^3}^{\sqrt{x}} (x^2 + 3y^2)\,dy = \Big[x^2y + y^3\Big]_{x^3}^{\sqrt{x}} = \left(x^2\sqrt{x} + \left(\sqrt{x}\right)^3\right) - (x^2x^3 + (x^3)^3) = x^{5/2} + x^{3/2} - x^5 - x^9$

**8.** $\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x^2 + y^2)\,dx = \left[\frac{1}{3}x^3 + y^2x\right]_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} = 2\left[\frac{1}{3}(1 - y^2)^{3/2} + y^2(1 - y^2)^{1/2}\right] = \frac{2\sqrt{1 - y^2}}{3}(1 + 2y^2)$

**10.** $\int_y^{\pi/2} \sin^3 x \cos y\,dx = \int_y^{\pi/2} (1 - \cos^2 x)\sin x \cos y\,dx$

$$= \left[\left(-\cos x + \frac{1}{3}\cos^3 x\right)\cos y\right]_y^{\pi/2} = \left(\cos y - \frac{1}{3}\cos^3 y\right)\cos y$$

**12.** $\int_{-1}^{1}\int_{-2}^{2} (x^2 - y^2)\,dy\,dx = \int_{-1}^{1}\left[x^2y - \frac{y^3}{3}\right]_{-2}^{2} dx = \int_{-1}^{1}\left[2x^2 - \frac{8}{3} + 2x^2 - \frac{8}{3}\right] dx$

$$= \int_{-1}^{1}\left(4x^2 - \frac{16}{3}\right) dx = \left[\frac{4x^3}{3} - \frac{16}{3}x\right]_{-1}^{1} = \left(\frac{4}{3} - \frac{16}{3}\right) - \left(-\frac{4}{3} + \frac{16}{3}\right) = -8$$

**14.** $\int_{-4}^{4}\int_{0}^{x^2} \sqrt{64 - x^3}\,dy\,dx = \int_{-4}^{4}\left[y\sqrt{64 - x^3}\right]_0^{x^2} dx$

$$= \int_{-4}^{4} \sqrt{64 - x^3}\,x^2\,dx = \left[-\frac{2}{9}(64 - x^3)^{3/2}\right]_{-4}^{4} = 0 + \frac{2}{9}(128)^{3/2} = \frac{2048}{9}\sqrt{2}$$

**16.** $\int_0^2\int_y^{2y} (10 + 2x^2 + 2y^2)\,dx\,dy = \int_0^2\left[10x + \frac{2x^3}{3} + 2y^2x\right]_y^{2y} dy = \int_0^2\left[\left(20y + \frac{16}{3}y^3 + 4y^3\right) - \left(10y + \frac{2}{3}y^3 + 2y^3\right)\right] dy$

$$= \int_0^2\left[10y + \frac{14}{3}y^3 + 2y^3\right] dy = \left[5y^2 + \frac{7y^4}{6} + \frac{y^4}{2}\right]_0^2 = 20 + \frac{56}{3} + 8 = \frac{140}{3}$$

**18.** $\int_0^2\int_{3y^2-6y}^{2y-y^2} 3y\,dx\,dy = \int_0^2\Big[3xy\Big]_{3y^2-6y}^{2y-y^2} dy = 3\int_0^2 (8y^2 - 4y^3)\,dy = \left[3\left(\frac{8}{3}y^3 - y^4\right)\right]_0^2 = 16$

**20.** $\int_0^{\pi/2}\int_0^{2\cos\theta} r\,dr\,d\theta = \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_0^{2\cos\theta} d\theta = \int_0^{\pi/2} 2\cos^2\theta\,d\theta = \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{\pi}{2}$

**22.** $\int_0^{\pi/4}\int_0^{\cos\theta} 3r^2\sin\theta\,dr\,d\theta = \int_0^{\pi/4}\Big[r^3\sin\theta\Big]_0^{\cos\theta} d\theta$

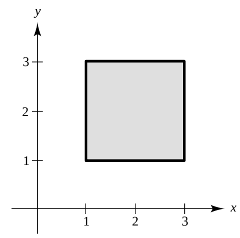
$$= \int_0^{\pi/4} \cos^3\sin\theta\,d\theta = \left[-\frac{\cos^4\theta}{4}\right]_0^{\pi/4} = -\frac{1}{4}\left[\left(\frac{1}{\sqrt{2}}\right)^4 - 1\right] = \frac{3}{16}$$

**24.** $\displaystyle\int_0^3\int_0^\infty \frac{x^2}{1+y^2}\,dy\,dx = \int_0^3 \Big[x^2 \arctan y\Big]_0^\infty dx = \int_0^3 x^2\left(\frac{\pi}{2}\right)dx = \left[\frac{\pi}{2}\cdot\frac{x^3}{3}\right]_0^3 = \frac{9\pi}{2}$

**26.** $\displaystyle\int_0^\infty\int_0^\infty xye^{-(x^2+y^2)}\,dx\,dy = \int_0^\infty\left[-\frac{1}{2}ye^{-(x^2+y^2)}\right]_0^\infty dy = \int_0^\infty \frac{1}{2}ye^{-y^2}\,dy = \left[-\frac{1}{4}e^{-y^2}\right]_0^\infty = \frac{1}{4}$

**28.** $\displaystyle A = \int_1^3\int_1^3 dy\,dx = \int_1^3 \Big[y\Big]_1^3 dx = \int_1^3 2\,dx = \Big[2x\Big]_1^3 = 4$

$\displaystyle A = \int_1^3\int_1^3 dx\,dy = \int_1^3 \Big[x\Big]_1^3 dy = \int_1^3 2\,dy = \Big[2y\Big]_1^3 = 4$



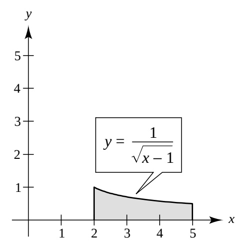

**30.** $\displaystyle A = \int_2^5\int_0^{1/\sqrt{x-1}} dy\,dx = \int_2^5 \Big[y\Big]_0^{1/\sqrt{x-1}} dx = \int_2^5 \frac{1}{\sqrt{x-1}}\,dx = \Big[2\sqrt{x-1}\Big]_2^5 = 2$

$\displaystyle A = \int_0^{1/2}\int_2^5 dx\,dy + \int_{1/2}^1\int_2^{1+(1/y^2)} dx\,dy$

$\displaystyle = \int_0^{1/2}\Big[x\Big]_2^5 dy + \int_{1/2}^1 \Big[x\Big]_2^{1+(1/y^2)} dy$

$\displaystyle = \int_0^{1/2} 3\,dy + \int_{1/2}^1\left(\frac{1}{y^2}-1\right)dy$

$\displaystyle = \Big[3y\Big]_0^{1/2} + \left[-\frac{1}{y}-y\right]_{1/2}^1 = 2$




**32.** $\displaystyle A = \int_0^2\int_0^{\sqrt{4-x^2}} dy\,dx$

$\displaystyle = \int_0^2 \sqrt{4-x^2}\,dx$

$\displaystyle = 4\int_0^{\pi/2}\cos^2\theta\,d\theta$

$\displaystyle = 2\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta$

$\displaystyle = \left[2\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_0^{\pi/2} = \pi$

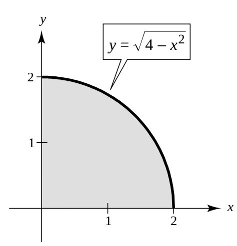
$\left(x = 2\sin\theta,\ dx = 2\cos\theta\,d\theta,\ \sqrt{4-x^2} = 2\cos\theta\right)$

$\displaystyle A = \int_0^2\int_0^{\sqrt{4-y^2}} dx\,dy = \int_0^2 \sqrt{4-y^2}\,dy$

$\displaystyle = 4\int_0^{\pi/2}\cos^2\theta\,d\theta = 2\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta$

$\displaystyle = \left[2\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_0^{\pi/2} = \pi$

$\left(y = 2\sin\theta,\ dy = 2\cos\theta\,d\theta,\ \sqrt{4-y^2} = 2\cos\theta\right)$




**34.** $\displaystyle A = \int_0^4\int_{x^{3/2}}^{2x} dy\,dx$

$\displaystyle = \int_0^4 \Big[y\Big]_{x^{3/2}}^{2x} dx$

$\displaystyle = \int_0^4 (2x - x^{3/2})\,dx$

$\displaystyle = \left[x^2 - \frac{2}{5}x^{5/2}\right]_0^4$

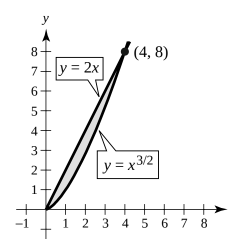
$\displaystyle = 16 - \frac{2}{5}(32) = \frac{16}{5}$

$\displaystyle A = \int_0^8\int_{y/2}^{y^{2/3}} dx\,dy$

$\displaystyle = \int_0^8 \left(y^{2/3} - \frac{y}{2}\right)dy$

$\displaystyle = \left[\frac{3}{5}y^{5/3} - \frac{y^2}{4}\right]_0^8$

$\displaystyle = \frac{3}{5}(32) - 16 = \frac{16}{5}$



**36.** $A = \int_0^3 \int_0^x dy\,dx + \int_3^9 \int_0^{9/x} dy\,dx$

$= \int_0^3 \Big[y\Big]_0^x dx + \int_3^9 \Big[y\Big]_0^{9/x} dx = \int_0^3 x\,dx + \int_3^9 \frac{9}{x}\,dx$

$= \left[\frac{1}{2}x^2\right]_0^3 + \Big[9 \ln x\Big]_3^9 = \frac{9}{2} + 9(\ln 9 - \ln 3)$

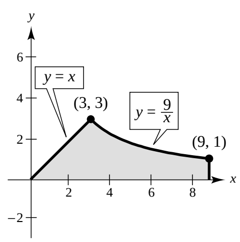
$= \frac{9}{2}(1 + \ln 9)$

$A = \int_0^1 \int_y^9 dx\,dy + \int_1^3 \int_y^{9/y} dx\,dy$

$= \int_0^1 \Big[x\Big]_y^9 dy + \int_1^3 \Big[x\Big]_y^{9/y} dy$

$= \int_0^1 (9 - y)\,dy + \int_1^3 \left(\frac{9}{y} - y\right) dy$

$= \left[9y - \frac{1}{2}y^2\right]_0^1 + \left[9 \ln y - \frac{1}{2}y^2\right]_1^3 = \frac{9}{2}(1 + \ln 9)$



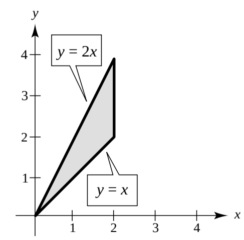

**38.** $A = \int_0^2 \int_{y/2}^{y} dx\,dy + \int_2^4 \int_{y/2}^{2} dx\,dy$

$= \int_0^2 \frac{y}{2}\,dy + \int_2^4 \left(2 - \frac{y}{2}\right) dy$

$= \left[\frac{y^2}{4}\right]_0^2 + \left[2y - \frac{y^2}{4}\right]_2^4$

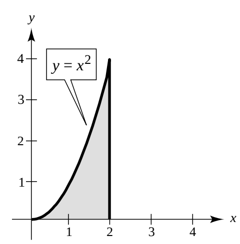
$= 1 + (4 - 3) = 2$

$A = \int_0^2 \int_x^{2x} dy\,dx = \int_0^2 (2x - x)\,dx = \left[\frac{x^2}{2}\right]_0^2 = 2$



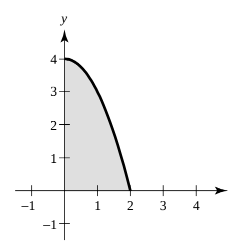

**40.** $\int_0^4 \int_{\sqrt{y}}^2 f(x, y)\,dx\,dy, \; \sqrt{y} \le x \le 2, \; 0 \le y \le 4$

$= \int_0^2 \int_0^{x^2} f(x, y)\,dy\,dx$



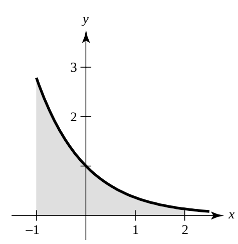

**42.** $\int_0^2 \int_0^{4-x^2} f(x, y)\,dy\,dx, \; 0 \le y \le 4 - x^2, \; 0 \le x \le 2$

$= \int_0^4 \int_0^{\sqrt{4-y}} f(x, y)\,dx\,dy$



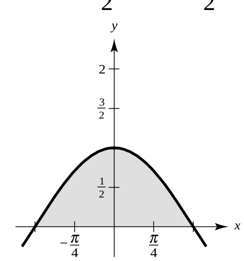

**44.** $\int_{-1}^2 \int_0^{e^{-x}} f(x, y)\,dy\,dx, \; 0 \le y \le e^{-x}, \; -1 \le x \le 2$

$= \int_0^{e^{-2}} \int_{-1}^{2} f(x, y)\,dx\,dy + \int_{e^{-2}}^{e} \int_{-1}^{-\ln y} f(x, y)\,dx\,dy$

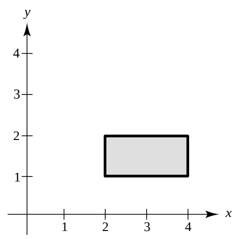



**46.** $\int_{-\pi/2}^{\pi/2} \int_0^{\cos x} f(x, y)\,dy\,dx, \; 0 \le y \le \cos x, \; -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$= \int_0^1 \int_{-\arccos y}^{\arccos y} f(x, y)\,dx\,dy$

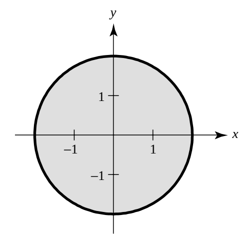


**48.** $\displaystyle\int_1^2\int_2^4 dx\,dy = \int_2^4\int_1^2 dy\,dx = 2$



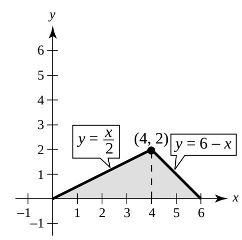
**50.** $\displaystyle\int_{-2}^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\,dx = \int_{-2}^2\left(\sqrt{4-x^2}+\sqrt{4-x^2}\right)dx = 4\pi$

$\displaystyle\int_{-2}^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dx\,dy = 4\pi$



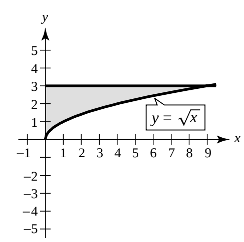
**52.** $\displaystyle\int_0^4\int_0^{x/2} dy\,dx + \int_4^6\int_0^{6-x} dy\,dx = \int_0^4 \frac{x}{2}\,dx + \int_4^6 (6-x)\,dx = 4 + 2 = 6$

$\displaystyle\int_0^2\int_{2y}^{6-y} dx\,dy = \int_0^2 (6-3y)\,dy = \left[6y - \frac{3y^2}{2}\right]_0^2 = 6$

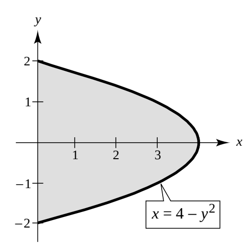



**54.** $\displaystyle\int_0^9\int_{\sqrt{x}}^3 dy\,dx = \int_0^9\left(3-\sqrt{x}\right)dx = \left[3x - \frac{2}{3}x^{3/2}\right]_0^9 = 27 - 18 = 9$

$\displaystyle\int_0^3\int_0^{y^2} dx\,dy = \int_0^3 y^2\,dy = \left[\frac{y^3}{3}\right]_0^3 = 9$




**56.** $\displaystyle\int_{-2}^2\int_0^{4-y^2} dx\,dy = \int_0^4\int_{-\sqrt{4-x}}^{\sqrt{4-x}} dy\,dx = \frac{32}{3}$



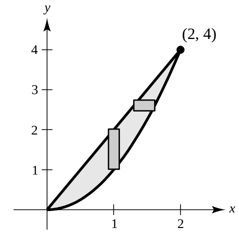

**58.** The first integral arises using vertical representative rectangles. The second integral arises using horizontal representative rectangles.

$\displaystyle\int_0^2\int_{x^2}^{2x} x\sin y\,dy\,dx = \int_0^2 \left(-x\cos(2x) + x\cos(x^2)\right)dx$

$\displaystyle = -\frac{1}{4}\cos(4) - \frac{1}{2}\sin(4) + \frac{1}{4}$

$\displaystyle\int_0^4\int_{y/2}^{\sqrt{y}} x\sin y\,dx\,dy = \int_0^4\left(\frac{1}{2}y\sin(y) - \frac{1}{8}y^2\sin(y)\right)dy$

$\displaystyle = -\frac{1}{4}\cos(4) - \frac{1}{2}\sin(4) + \frac{1}{4}$



**60.** $\int_0^2\int_x^2 e^{-y^2}\,dy\,dx = \int_0^2\int_0^y e^{-y^2}\,dx\,dy$

$$= \int_0^2 \left[xe^{-y^2}\right]_0^y dy = \int_0^2 ye^{-y^2}\,dy = \left[-\frac{1}{2}e^{-y^2}\right]_0^2 = -\frac{1}{2}(e^{-4}) + \frac{1}{2}e^0 = \frac{1}{2}\left(1 - \frac{1}{e^4}\right) \approx 0.4908$$

**62.** $\int_0^2\int_{y^2}^4 \sqrt{x}\sin x\,dx\,dy = \int_0^4\int_0^{\sqrt{x}} \sqrt{x}\sin x\,dy\,dx$

$$= \int_0^4 \left[y\sqrt{x}\sin x\right]_0^{\sqrt{x}} dx = \int_0^4 x\sin x\,dx = \left[\sin x - x\cos x\right]_0^4 = \sin 4 - 4\cos 4 \approx 1.858$$

**64.** $\int_0^1\int_y^{2y} \sin(x+y)\,dx\,dy = \frac{\sin 2}{2} - \frac{\sin 3}{3} \approx 0.408$

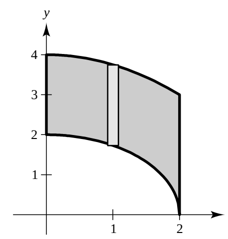
**66.** $\int_0^a\int_0^{a-x} (x^2+y^2)\,dy\,dx = \frac{a^4}{6}$

**68.** (a) $y = \sqrt{4-x^2} \iff x = \sqrt{4-y^2}$

$y = 4 - \frac{x^2}{4} \iff x = \sqrt{16-4y}$

(b) $\int_0^2\int_{\sqrt{4-y^2}}^2 \frac{xy}{x^2+y^2+1}\,dx\,dy + \int_2^3\int_0^2 \frac{xy}{x^2+y^2+1}\,dx\,dy + \int_3^4\int_0^{\sqrt{16-4y}} \frac{xy}{x^2+y^2+1}\,dx\,dy$

(c) Both orders of integration yield 1.11899.




**70.** $\int_0^2\int_x^2 \sqrt{16-x^3-y^3}\,dy\,dx \approx 6.8520$

**72.** $\int_0^{\pi/2}\int_0^{1+\sin\theta} 15\theta r\,dr\,d\theta = \frac{45\pi^2}{32} + \frac{135}{8} \approx 30.7541$

**74.** A region is vertically simple if it is bounded on the left and right by vertical lines, and bounded on the top and bottom by functions of $x$. A region is horizontally simple if it is bounded on the top and bottom by horizontal lines, and bounded on the left and right by functions of $y$.

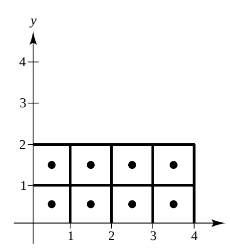
**76.** The integrations might be easier. See Exercise 59-62.

**78.** False, let $f(x, y) = x$.

## Section 13.2 Double Integrals and Volume

**For Exercises 2 and 4, $\Delta x_i = \Delta y_i = 1$ and the midpoints of the squares are**

$\left(\frac{1}{2},\frac{1}{2}\right), \left(\frac{3}{2},\frac{1}{2}\right), \left(\frac{5}{2},\frac{1}{2}\right), \left(\frac{7}{2},\frac{1}{2}\right), \left(\frac{1}{2},\frac{3}{2}\right), \left(\frac{3}{2},\frac{3}{2}\right), \left(\frac{5}{2},\frac{3}{2}\right), \left(\frac{7}{2},\frac{3}{2}\right).$




**2.** $f(x, y) = \frac{1}{2}x^2y$

$$\sum_{i=1}^{\infty} f(x_i, y_i)\Delta x_i\,\Delta y_i = \frac{1}{16} + \frac{9}{16} + \frac{25}{16} + \frac{49}{16} + \frac{3}{16} + \frac{27}{16} + \frac{75}{16} + \frac{147}{16} = 21$$

$$\int_0^4\int_0^2 \frac{1}{2}x^2y\,dy\,dx = \int_0^4 \left[\frac{x^2y^2}{4}\right]_0^2 dx = \int_0^4 x^2\,dx = \frac{x^3}{3}\Big]_0^4 = \frac{64}{3} \approx 21.3$$

**4.** $f(x, y) = \dfrac{1}{(x+1)(y+1)}$

$$\sum_{i=1}^{8} f(x_i, y_i)\,\Delta x_i\,\Delta y_i = \frac{4}{9} + \frac{4}{15} + \frac{4}{21} + \frac{4}{27} + \frac{4}{15} + \frac{4}{25} + \frac{4}{35} + \frac{4}{45} = \frac{7936}{4725} \approx 1.680$$

$$\int_0^4 \int_0^2 \frac{1}{(x+1)(y+1)}\,dy\,dx = \int_0^4 \left[\frac{1}{x+1}\ln(y+1)\right]_0^2 dx$$

$$= \int_0^4 \frac{\ln 3}{x+1}\,dx = \Big[\ln 3 \cdot \ln(x+1)\Big]_0^4 = (\ln 3)(\ln 5) \approx 1.768$$

**6.** $\displaystyle\int_0^2 \int_0^2 f(x, y)\,dy\,dx \approx 4 + 2 + 8 + 6 = 20$

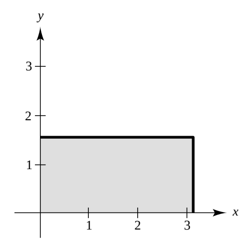
**8.** $$\int_0^{\pi} \int_0^{\pi/2} \sin^2 x \cos^2 y\,dy\,dx = \int_0^{\pi} \left[\frac{1}{2}\sin^2 x\left(y + \frac{1}{2}\sin 2y\right)\right]_0^{\pi/2} dx$$

$$= \int_0^{\pi} \frac{1}{2}\sin^2 x\left(\frac{\pi}{2}\right) dx$$

$$= \frac{\pi}{8}\int_0^{\pi} (1 - \cos 2x)\,dx$$

$$= \left[\frac{\pi}{8}\left(x - \frac{1}{2}\sin 2x\right)\right]_0^{\pi}$$

$$= \frac{\pi^2}{8}$$

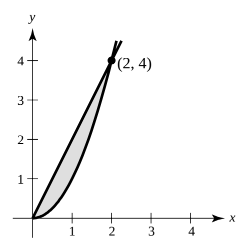



**10.** $$\int_0^4 \int_{(1/2)y}^{\sqrt{y}} x^2 y^2\,dx\,dy = \int_0^4 \left[\frac{x^3 y^2}{3}\right]_{(1/2)y}^{\sqrt{y}} dy$$

$$= \int_0^4 \left(\frac{y^{7/2}}{3} - \frac{y^5}{24}\right) dy$$

$$= \left[\frac{2y^{9/2}}{27} - \frac{y^6}{144}\right]_0^4$$

$$= \frac{1024}{27} - \frac{256}{9} = \frac{256}{27}$$

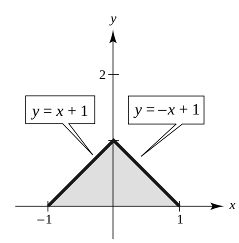



**12.** $$\int_0^1 \int_{y-1}^{0} e^{x+y}\,dx\,dy + \int_0^1 \int_0^{1-y} e^{x+y}\,dx\,dy = \int_0^1 \Big[e^{x+y}\Big]_{y-1}^{0} dy + \int_0^1 \Big[e^{x+y}\Big]_0^{1-y} dy$$

$$= \int_0^1 (e - e^{2y-1})\,dy$$

$$= \left[ey - \frac{1}{2}e^{2y-1}\right]_0^1$$

$$= \frac{1}{2}(e + e^{-1})$$

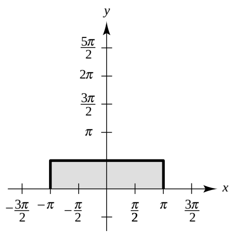


**14.** $\displaystyle\int_0^{\pi/2}\int_{-\pi}^{\pi} \sin x \sin y \, dx\, dy = \int_{-\pi}^{\pi}\int_0^{\pi/2} \sin x \sin y \, dy\, dx$

$\displaystyle = \int_{-\pi}^{\pi}\Big[-\sin x \cos y\Big]_0^{\pi/2} dx$

$\displaystyle = \int_{-\pi}^{\pi} \sin x \, dx$

$= 0$



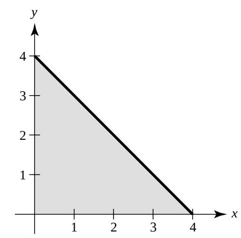
**16.** $\displaystyle\int_0^4\int_0^{4-x} xe^y \, dy\, dx = \int_0^4\int_0^{4-y} xe^y \, dx\, dy$

For the first integral, we obtain:

$\displaystyle\int_0^4\Big[xe^y\Big]_0^{4-x} - x\, dx = \int_0^{4-x}(xe^4 - x)dx$

$\displaystyle = \left[-e^{4-x}(1+x) - \frac{x^2}{2}\right]_0^4$

$= (-5 - 8) + e^4 = e^4 - 13.$

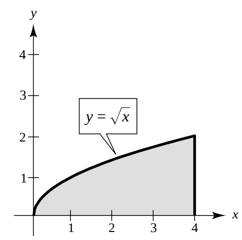


**18.** $\displaystyle\int_0^2\int_{y^2}^4 \frac{y}{1+x^2}\, dx\, dy = \int_0^4\int_0^{\sqrt{x}} \frac{y}{1+x^2}\, dy\, dx$

$\displaystyle = \frac{1}{2}\int_0^4 \left[\frac{y^2}{1+x^2}\right]_0^{\sqrt{x}} dx$

$\displaystyle = \frac{1}{2}\int_0^4 \frac{x}{1+x^2}\, dx$

$\displaystyle = \left[\frac{1}{4}\ln(1+x^2)\right]_0^4 = \frac{1}{4}\ln(17)$



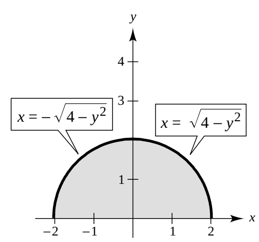

**20.** $\displaystyle\int_0^2\int_{\sqrt{4-y^2}}^{\sqrt{4-y^2}} (x^2 + y^2)\, dx\, dy$

$\displaystyle = \int_{-2}^2\int_0^{\sqrt{4-x^2}} (x^2+y^2)\, dy\, dx$

$\displaystyle = \int_{-2}^2\left[x^2y + \frac{1}{3}y^3\right]_0^{\sqrt{4-x^2}} dx$

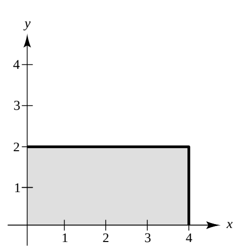
$\displaystyle = \int_{-2}^2\left[x^2\sqrt{4-x^2} + \frac{1}{3}(4-x^2)^{3/2}\right] dx$

$\displaystyle = \left[-\frac{x}{4}(4-x^2)^{3/2} + \frac{1}{2}\left(x\sqrt{4-x^2} + 4\arcsin\frac{x}{2}\right) + \frac{1}{12}\left[x(4-x^2)^{3/2} + 6x\sqrt{4-x^2} + 24\arctan\frac{x}{2}\right]\right]_{-2}^2 = 4\pi$

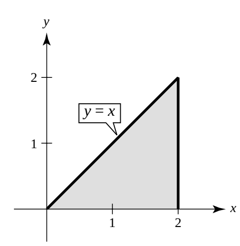



**22.** $\displaystyle\int_0^4\int_0^2 (6-2y)\, dy\, dx = \int_0^4\Big[6y - y^2\Big]_0^2 dx$

$\displaystyle = \int_0^4 8\, dx = 32$



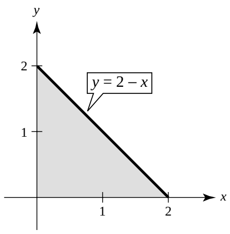
**24.** $\displaystyle\int_0^2\int_0^x 4\, dy\, dx = \int_0^2 4x\, dx = 2x^2\Big]_0^2 = 8$



**26.** $\int_0^2 \int_0^{2-x} (2 - x - y)\, dy\, dx = \int_0^2 \left[2y - xy - \frac{y^2}{2}\right]_0^{2-x} dx$

$= \int_0^2 \frac{1}{2}(2 - x)^2\, dx$

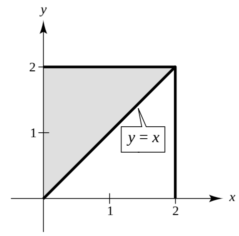
$= -\frac{1}{6}(x - 2)^3 \Big]_0^2 = \frac{4}{3}$




**28.** $\int_0^2 \int_0^y (4 - y^2)\, dx\, dy = \int_0^2 (4y - y^3)\, dy$

$= \left[2y^2 - \frac{y^4}{4}\right]_0^2$

$= 4$



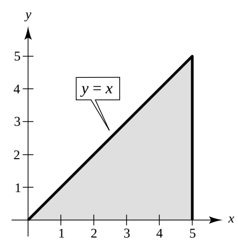

**30.** $\int_0^\infty \int_0^\infty e^{-(x+y)/2}\, dy\, dx = \int_0^\infty \left[-2e^{-(x+y)/2}\right]_0^\infty dx = \int_0^\infty 2e^{-x/2}\, dx = \left[-4e^{-x/2}\right]_0^\infty = 4$

**32.** $\int_0^1 \int_0^x \sqrt{1 - x^2}\, dy\, dx = \frac{1}{3}$

**34.** $V = \int_0^5 \int_0^x x\, dy\, dx$

$= \int_0^5 \left[xy\right]_0^x dx = \int_0^5 x^2\, dx$

$= \left[\frac{1}{3}x^3\right]_0^5 = \frac{125}{3}$



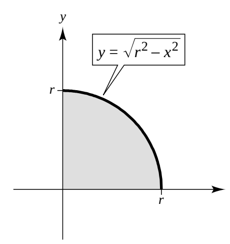

**36.** $V = 8\int_0^r \int_0^{\sqrt{r^2 - x^2}} \sqrt{r^2 - x^2 - y^2}\, dy\, dx$

$= 4\int_0^r \left[\left[y\sqrt{r^2 - x^2 - y^2} + (r^2 - x^2) \arcsin \frac{y}{\sqrt{r^2 - x^2}}\right]\right]_0^{\sqrt{r^2 - x^2}} dx$

$= 4\left(\frac{\pi}{2}\right)\int_0^r (r^2 - x^2)\, dx$

$= \left[2\pi\left(r^2x - \frac{1}{3}x^3\right)\right]_0^r$

$= \frac{4\pi r^3}{3}$



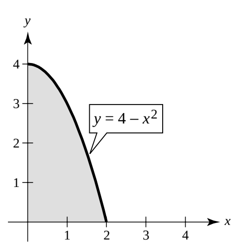

**38.** $V = \int_0^2 \int_0^{4-x^2} (4 - x^2)\, dy\, dx$

$= \int_0^2 (4 - x^2)(4 - x^2)\, dx$

$= \int_0^2 (16 - 8x^2 + x^4)\, dx$

$= \left[16x - 8\frac{x^3}{3} + \frac{x^5}{5}\right]_0^2$

$= 32 - \frac{64}{3} + \frac{32}{5} = \frac{256}{15}$

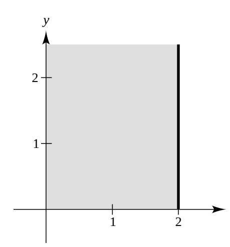



**40.** $V = \int_0^2 \int_0^\infty \frac{1}{1 + y^2}\, dy\, dx$

$= \int_0^2 \left[\arctan y\right]_0^\infty dx$

$= \int_0^2 \frac{\pi}{2}\, dx$

$= \left[\frac{\pi x}{2}\right]_0^2 = \pi$

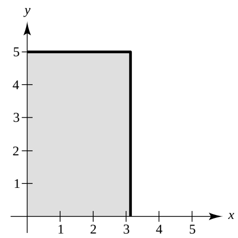


**42.** $V = \int_0^5 \int_0^{\pi} \sin^2 x\, dx\, dy$

$= \int_0^5 \frac{\pi}{2}\, dy$

$= \left[\frac{\pi}{2} y\right]_0^5$

$= \frac{5\pi}{2}$



**44.** $V = \int_0^9 \int_0^{\sqrt{9-y}} \sqrt{9-y}\, dx\, dy = \frac{81}{2}$

**46.** $V = \int_0^{16} \int_0^{4-\sqrt{y}} \ln(1 + x + y)\, dx\, dy \approx 38.25$

**48.** $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$

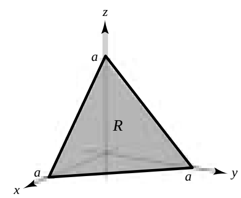
$V = \iint_R f(x, y)\, dA = \int_0^a \int_0^{b[1-(x/a)]} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dy\, dx$

$= c\int_0^a \left[y - \frac{xy}{a} - \frac{y^2}{2b}\right]_0^{b[1-(x/a)]} dx$

$= c\int_0^a \left[b\left(1 - \frac{x}{a}\right) - \frac{xb}{a}\left(1 - \frac{x}{a}\right) - \frac{b^2}{2b}\left(1 - \frac{x}{a}\right)^2\right] dx$

$= c\left[-\frac{ab}{2}\left(1 - \frac{x}{a}\right)^2 - \frac{x^2b}{2a} + \frac{x^3b}{3a^2} + \frac{ab}{6}\left(1 - \frac{x}{a}\right)^3\right]_0^a$

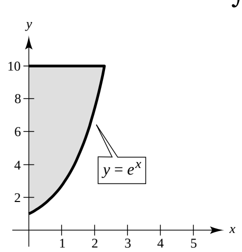
$= c\left[\left(-\frac{ab}{2} + \frac{ab}{3}\right) - \left(-\frac{ab}{2} + \frac{ab}{6}\right)\right] = \frac{abc}{6}$



**50.** $\int_0^{\ln 10} \int_{e^x}^{10} \frac{1}{\ln y}\, dy\, dx = \int_1^{10} \int_0^{\ln y} \frac{1}{\ln y}\, dx\, dy$

$= \int_1^{10} \left[\frac{x}{\ln y}\right]_0^{\ln y} dy$

$= \int_1^{10} dy = \Big[y\Big]_1^{10} = 9$



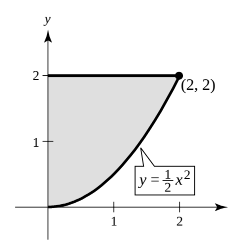

**52.** $\int_0^2 \int_{(1/2)x^2}^2 \sqrt{y} \cos y\, dy\, dx = \int_0^2 \int_0^{\sqrt{2y}} \sqrt{y} \cos y\, dx\, dy$

$= \int_0^2 \sqrt{y} \cos y \sqrt{2y}\, dy$

$= \sqrt{2} \int_0^2 y \cos y\, dy$

$= \sqrt{2}\Big[\cos y + y \sin y\Big]_0^2$

$= \sqrt{2}[\cos 2 + 2 \sin 2 - 1]$




**54.** Average $= \frac{1}{8} \int_0^4 \int_0^2 xy\, dy\, dx = \frac{1}{8} \int_0^4 2x\, dx = \left[\frac{x^2}{8}\right]_0^4 = 2$

**56.** Average $= \frac{1}{1/2} \int_0^1 \int_x^1 e^{x+y}\, dy\, dx = 2\int_0^1 e^{x+1} - e^{2x}\, dx$

$= 2\left[e^{x+1} - \frac{1}{2}e^{2x}\right]_0^1 = 2\left[e^2 - \frac{1}{2}e^2 - e + \frac{1}{2}\right]$

$= e^2 - 2e + 1$

$= (e - 1)^2$

**58.** The second is integrable. The first contains $\int \sin y^2\, dy$ which does not have an elementary antiderivation.

**60.** (a) The total snowfall in the county $R$.

(b) The average snowfall in $R$.

**62.** Average $= \dfrac{1}{150}\displaystyle\int_{45}^{60}\int_{40}^{50} \left[192x + 576y - x^2 - 5y^2 - 2xy - 5000\right] dx\, dy \approx 13{,}246.67$

**64.** $f(x, y) \geq 0$ for all $(x, y)$ and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dA = \int_0^2\int_0^2 \frac{1}{4}xy\, dy\, dx = \int_0^2 \frac{x}{2}\, dx = 1$$

$$P(0 \leq x \leq 1, 1 \leq y \leq 2) = \int_0^1\int_1^2 \frac{1}{4}xy\, dy\, dx = \int_0^1 \frac{3x}{8}\, dx = \frac{3}{16}.$$

**66.** $f(x, y) \geq 0$ for all $(x, y)$ and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dA = \int_0^{\infty}\int_0^{\infty} e^{-x-y}\, dy\, dx$$

$$= \int_0^{\infty} \lim_{b\to\infty}\left[-e^{-x-y}\right]_0^b dx = \int_0^{\infty} e^{-x}\, dx = \lim_{b\to\infty}\left[-e^{-x}\right]_0^b = 1$$

$$P(0 \leq x \leq 1, x \leq y \leq 1) = \int_0^1\int_x^1 e^{-x-y}\, dy\, dx = \int_0^1 \left[e^{-x-y}\right]_x^1 dx = \int_0^1 (e^{-2x} - e^{-x-1})\, dx$$

$$= \left[-\frac{1}{2}e^{-2x} + e^{-x-1}\right]_0^1 = \frac{1}{2}e^{-2} - e^{-1} + \frac{1}{2} = \frac{1}{2}(e^{-1} - 1)^2 \approx 0.1998.$$

**68.** Sample Program for TI-82:

Program: DOUBLE

```
: Input A
: Input B
: Input M
: Input C
: Input D
: Input N
: 0 → V
: (B − A)/M → G
: (D − C)/N → H
: For (I, 1, M, 1)
: For (J, 1, N, 1)
: A + 0.5G(2I − 1) → x
: C + 0.5H(2J − 1) → y
: V + sin(√(x + y)) × G × H → V
: End
: End
: Disp V
```

**70.** $\displaystyle\int_0^2\int_0^4 20e^{-x^3/8}\, dy\, dx \qquad m = 10, n = 20$

(a) 129.2018

(b) 129.2756

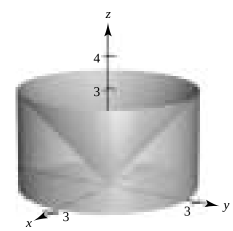
**72.** $\int_1^4 \int_1^2 \sqrt{x^3 + y^3}\, dx\, dy \qquad m = 6, n = 4$

(a) 13.956

(b) 13.9022

**74.** $V \approx 50$

Matches a.




**76.** True

**78.** $\int_1^2 e^{-xy}\, dy = \left[-\frac{1}{x}e^{-xy}\right]_1^2 = \frac{e^{-x} - e^{-2x}}{x}$

Thus,

$$\int_0^\infty \frac{e^{-x} - e^{-2x}}{x}\, dx = \int_0^\infty \int_1^2 e^{-xy}\, dx\, dy$$
$$= \int_1^2 \int_0^\infty e^{-xy}\, dx\, dy$$
$$= \int_1^2 \left[-\frac{e^{-xy}}{y}\right]_0^\infty dy$$
$$= \int_1^2 \frac{1}{y}\, dy = \left[\ln y\right]_1^2 = \ln 2.$$

## Section 13.3 Change of Variables: Polar Coordinates

**2.** Polar coordinates

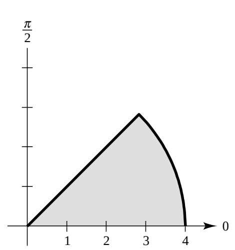
**4.** Rectangular coordinates

**6.** $R = \{(r, \theta)\colon\ 0 \le r \le 4 \sin \theta, 0 \le \theta \le \pi\}$

**8.** $R = \{(r, \theta)\colon\ 0 \le r \le r \cos 3\theta, 0 \le \theta \le \pi\}$

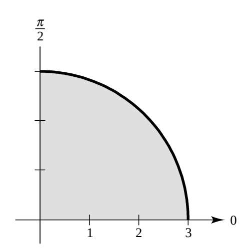
**10.** $\int_0^{\pi/4} \int_0^4 r^2 \sin \theta \cos\, dr\, d\theta = \int_0^{\pi/4} \left[\frac{r^3}{3} \sin \theta \cos \theta\right]_0^4 d\theta$

$$= \left[\left(\frac{64}{3}\right) \frac{\sin^2 \theta}{2}\right]_0^{\pi/4}$$
$$= \frac{16}{3}$$




**12.** $\int_0^{\pi/2} \int_0^3 re^{-r^2}\, dr\, d\theta = \int_0^{\pi/2} \left[-\frac{1}{2}e^{-r^2}\right]_0^3 d\theta$

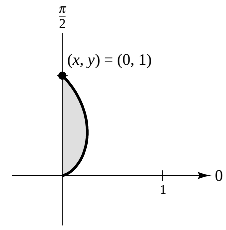
$$= \left[-\frac{1}{2}(e^{-9} - 1)\theta\right]_0^{\pi/2}$$
$$= \frac{\pi}{4}\left(1 - \frac{1}{e^9}\right)$$



**14.** $\int_0^{\pi/2}\int_0^{1-\cos\theta} (\sin\theta) r\,dr\,d\theta = \int_0^{\pi/2}\left[(\sin\theta)\frac{r^2}{2}\right]_0^{1-\cos\theta} d\theta$

$$= \int_0^{\pi/2} \frac{\sin\theta}{2}(1-\cos\theta)^2\,d\theta$$

$$= \left[\frac{1}{6}(1-\cos(\theta))^3\right]_0^{\pi/2} = \frac{1}{6}$$




**16.** $\int_0^a\int_0^{\sqrt{a^2-x^2}} x\,dy\,dx = \int_0^{\pi/2}\int_0^a r^2\cos\theta\,dr\,d\theta = \frac{a^3}{3}\int_0^{\pi/2}\cos\theta\,d\theta = \left[\frac{a^3}{3}\sin\theta\right]_0^{\pi/2} = \frac{a^3}{3}$

**18.** $\int_0^2\int_y^{\sqrt{8-y^2}} \sqrt{x^2+y^2}\,dx\,dy = \int_0^{\pi/4}\int_0^{2\sqrt{2}} r^2\,dr\,d\theta$

$$= \int_0^{\pi/4}\frac{(2\sqrt{2})^3}{3}\,d\theta = \left[\frac{(2\sqrt{2})^3}{3}\theta\right]_0^{\pi/4} = \frac{(2\sqrt{2})^3}{3}\cdot\frac{\pi}{4} = \frac{4\sqrt{2}\pi}{3}$$

**20.** $\int_0^4\int_0^{\sqrt{4y-y^2}} x^2\,dx\,dy = \int_0^{\pi/2}\int_0^{4\sin\theta} r^3\cos^2\theta\,dr\,d\theta = \int_0^{\pi/2} 64\sin^4\theta\cos^2\theta\,d\theta$

$$= 64\int_0^{\pi/2}(\sin^4\theta - \sin^6\theta)\,d\theta = \frac{64}{6}\left[\sin^5\theta\cos\theta - \frac{\sin^3\theta\cos\theta}{4} + \frac{3}{8}(\theta - \sin\theta\cos\theta)\right]_0^{\pi/2} = 2\pi$$

**22.** $\int_0^{(5\sqrt{2})/2}\int_0^x xy\,dy\,dx + \int_{(5\sqrt{2})/2}^5\int_0^{\sqrt{25-x^2}} xy\,dy\,dx = \int_0^{\pi/4}\int_0^5 r^3\sin\theta\cos\theta\,dr\,d\theta$

$$= \int_0^{\pi/4}\frac{625}{4}\sin\theta\cos\,d\theta$$

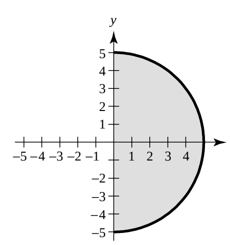
$$= \left[\frac{625}{8}\sin^2\theta\right]_0^{\pi/4}$$

$$= \frac{625}{16}$$

**24.** $\int_{-\pi/2}^{\pi/2}\int_0^5 e^{-r^2/2} r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\left[-e^{-r^2/2}\right]_0^5 d\theta$

$$= \int_{-\pi/2}^{\pi/2}(1 - e^{-25/2})\,d\theta$$

$$= \left[(1-e^{-25/2})\theta\right]_{-\pi/2}^{\pi/2} = \pi(1-e^{-25/2})$$




**26.** $\int_0^3\int_0^{\sqrt{9-x^2}} (9-x^2-y^2)\,dy\,dx = \int_0^{\pi/2}\int_0^3 (9-r^2)r\,dr\,d\theta$

$$= \int_0^{\pi/2}\int_0^3 (9r - r^3)\,dr\,d\theta = \int_0^{\pi/2}\left[\frac{9}{2}r^2 - \frac{1}{4}r^4\right]_0^3 d\theta = \frac{81}{4}\int_0^{\pi/2} d\theta = \frac{81\pi}{8}$$

**28.** $V = 4\int_0^{\pi/2}\int_0^1 (r^2+3)r\,dr\,d\theta = 4\int_0^{\pi/2}\left(\frac{r^4}{4}+\frac{3r^2}{2}\right)_0^1 d\theta$

$= 4\int_0^{\pi/4}\frac{7}{4}\,d\theta$

$= 7\left(\frac{\pi}{4}\right) = \frac{7\pi}{4}$

**30.** $\int_R\int \ln(x^2+y^2)\,dA = \int_0^{2\pi}\int_1^2 (\ln r^2)r\,dr\,d\theta$

$= 2\int_0^{2\pi}\int_1^2 r\ln r\,dr\,d\theta$

$= 2\int_0^{2\pi}\left[\frac{r^2}{4}(-1+2\ln r)\right]_1^2 d\theta$

$= 2\int_0^{2\pi}\left(\ln 4 - \frac{3}{4}\right)d\theta$

$= 4\pi\left(\ln 4 - \frac{3}{4}\right)$

**32.** $V = \int_0^{2\pi}\int_1^4 \sqrt{16-r^2}\,r\,dr\,d\theta = \int_0^{2\pi}\left[-\frac{1}{3}\left(\sqrt{16-r^2}\right)^3\right]_1^4 d\theta = \int_0^{2\pi} 5\sqrt{15}\,d\theta = 10\sqrt{15}\pi$

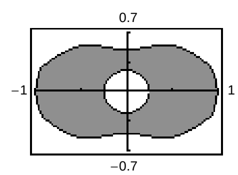
**34.** $x^2+y^2+z^2=a^2 \Longrightarrow z = \sqrt{a^2-(x^2+y^2)} = \sqrt{a^2-r^2}$

$V = 8\int_0^{\pi/2}\int_0^a \sqrt{a^2-r^2}\,r\,dr\,d\theta$ (8 times the volume in the first octant)

$= 8\int_0^{\pi/2}\left[-\frac{1}{2}\cdot\frac{2}{3}(a^2-r^2)^{3/2}\right]_0^a d\theta$

$= 8\int_0^{\pi/2}\frac{a^3}{3}\,d\theta = \left[\frac{8a^3}{3}\theta\right]_0^{\pi/2} = \frac{4\pi a^3}{3}$

**36.** $\frac{-9}{4(x^2+y^2+9)} \le z \le \frac{9}{4(x^2+y^2+9)}$; $\frac{1}{4} \le r \le \frac{1}{2}(1+\cos^2\theta)$



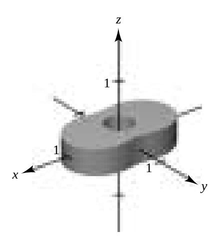

(a) $\frac{-9}{4r^2+36} \le z \le \frac{9}{4r^2+36}$

(b) Perimeter $= \int_\alpha^\beta \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta.$

$r = \frac{1}{2}(1+\cos^2\theta) = \frac{1}{2}+\frac{1}{2}\cos^2\theta$

$\frac{dr}{d\theta} = -\cos\theta\sin\theta$

Perimeter $= 2\int_0^{\pi}\sqrt{\frac{1}{4}(1+\cos^2\theta)^2+\cos^2\theta\sin^2\theta}\,d\theta \approx 5.21$




(c) $V = 2\int_0^{2\pi}\int_{1/4}^{1/2(1+\cos^2\theta)} \frac{9}{4r^2+36}\,r\,dr\,d\theta \approx 0.8000$

**38.** $A = \int_0^{2\pi}\int_2^4 r\,dr\,d\theta = \int_0^{2\pi} 6\,d\theta = 12\pi$

**40.** $\int_0^{2\pi}\int_0^{2+\sin\theta} r\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}(2+\sin\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi}(4+4\sin\theta+\sin^2\theta)\,d\theta = \frac{1}{2}\int_0^{2\pi}\left(4+4\sin\theta+\frac{1-\cos 2\theta}{2}\right)d\theta$

$= \frac{1}{2}\left[4\theta - 4\cos\theta + \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{1}{2}[8\pi - 4 + \pi + 4] = \frac{9\pi}{2}$

**42.** $8\int_0^{\pi/4}\int_0^{3\cos 2\theta} r\,dr\,d\theta = 4\int_0^{\pi/4} 9\cos^2 2\theta\,d\theta = 18\int_0^{\pi/4}(1+\cos 4\theta)\,d\theta = 18\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\pi/4} = \frac{9\pi}{2}$

**44.** See Theorem 13.3.

**46.** (a) Horizontal or polar representative elements

(b) Polar representative element

(c) Vertical or polar

**48.** (a) The volume of the subregion determined by the point $(5, \pi/16, 7)$ is base $\times$ height $= (5 \cdot 10 \cdot \pi/8)(7)$. Adding up the 20 volumes, ending with $(45 \cdot 10 \cdot \pi/8)(12)$, you obtain

$$V \approx 10 \cdot \frac{\pi}{8}[5(7+9+9+5) + 15(8+10+11+8) + 25(10+14+15+11) + 35(12+15+18+16) + 45(9+10+14+12)]$$
$$= \frac{5\pi}{4}[150 + 555 + 1250 + 2135 + 2025] \approx \frac{5\pi}{4}[6115] \approx 24013.5 \text{ ft}^3$$

(b) $(56)(24013.5) = 1{,}344{,}759$ pounds

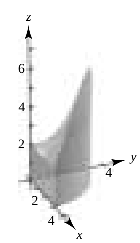
(c) $(7.48)(24103.5) \approx 179{,}621$ gallons

**50.** $\int_0^{\pi/4}\int_0^4 5e^{\sqrt{r\theta}}\, r\,dr\,d\theta \approx 87.130$

**52.** Volume $=$ base $\times$ height

$\approx \frac{9}{4}\pi \times 3 \approx 21$

Answer (a)




**54.** True

**56.** (a) Let $u = \sqrt{2}x$, then $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{\infty} e^{-u^2/2}\frac{1}{\sqrt{2}}\,du = \frac{1}{\sqrt{2}}\left(\sqrt{2\pi}\right) = \sqrt{\pi}.$

(b) Let $u = 2x$, then $\int_{-\infty}^{\infty} e^{-4x^2}\,dx = \int_{-\infty}^{\infty} e^{-u^2}\frac{1}{2}\,du = \frac{1}{2}\sqrt{\pi}.$

**58.** $\int_0^{\infty}\int_0^{\infty} ke^{-(x^2+y^2)}\,dy\,dx = \int_0^{\pi/2}\int_0^{\infty} ke^{-r^2}\,r\,dr\,d\theta = \int_0^{\pi/2}\left[-\frac{k}{2}e^{-r^2}\right]_0^{\infty} d\theta = \int_0^{\pi/2}\frac{k}{2}\,d\theta = \frac{k\pi}{4}$

For $f(x, y)$ to be a probability density function,

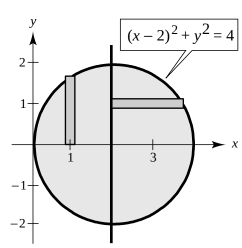
$$\frac{k\pi}{4} = 1$$

$$k = \frac{4}{\pi}.$$

**60.** (a) $4\int_0^2\int_2^{2+\sqrt{4-y^2}} f\,dx\,dy$

(b) $4\int_0^2\int_0^{\sqrt{4-(x-2)^2}} f\,dy\,dx$

(c) $2\int_0^{\pi/2}\int_0^{4\cos\theta} f r\,dr\,d\theta$



# Section 13.4 Center of Mass and Moments of Inertia

**2.** $m = \int_0^3 \int_0^{9-x^2} xy\,dy\,dx = \int_0^3 \left[\frac{xy^2}{2}\right]_0^{9-x^2} dx$

$= \int_0^3 \frac{x(9-x^2)^2}{2}\,dx$

$= \left[-\frac{1}{4}\frac{(9-x^2)^3}{3}\right]_0^3$

$= -\frac{1}{12}(0 - 9^3) = \frac{243}{4}$

**4.** $m = \int_0^3 \int_3^{3+\sqrt{9-x^2}} xy\,dy\,dx = \int_0^3 \left[x\frac{y^2}{2}\right]_3^{3+\sqrt{9-x^2}} dx$

$= \int_0^3 \frac{x}{2}\left(\left(3 + \sqrt{9-x^2}\right) - 9\right) dx$

$= \frac{1}{2}\int_0^3 \left[6x\sqrt{9-x^2} + 9x - x^3\right] dx$

$= \frac{1}{2}\left[-2(9-x^2)^{3/2} + \frac{9x^2}{2} - \frac{x^4}{4}\right]_0^3$

$= \frac{1}{2}\left[\frac{81}{2} - \frac{81}{4} + 54\right] = \frac{297}{8}$

**6.** (a) $m = \int_0^a \int_0^b kxy\,dy\,dx = \frac{ka^2b^2}{4}$

$M_x = \int_0^a \int_0^b kxy^2\,dy\,dx = \frac{ka^2b^3}{6}$

$M_y = \int_0^a \int_0^b kx^2y\,dy\,dx = \frac{ka^3b^2}{6}$

$\bar{x} = \frac{M_y}{m} = \frac{ka^3b^2/6}{ka^2b^2/4} = \frac{2}{3}a$

$\bar{y} = \frac{M_x}{m} = \frac{ka^2b^3/6}{ka^2b^2/4} = \frac{2}{3}b$

(b) $m = \int_0^a \int_0^b k(x^2 + y^2)\,dy\,dx = \frac{kab}{3}(a^2 + b^2)$

$M_x = \int_0^a \int_0^b k(x^2y + y^3)\,dy\,dx = \frac{kab^2}{12}(2a^2 + 3b^2)$

$M_y = \int_0^a \int_0^b k(x^3 + xy^2)\,dy\,dx = \frac{ka^2b}{12}(3a^2 + 2b^2)$

$\bar{x} = \frac{M_y}{m} = \frac{(ka^2b/12)(3a^2 + 2b^2)}{(kab/3)(a^2 + b^2)} = \frac{a}{4}\left(\frac{3a^2 + 2b^2}{a^2 + b^2}\right)$

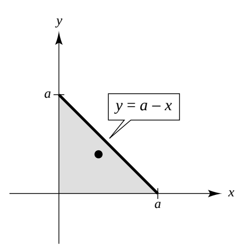
$\bar{y} = \frac{M_x}{m} = \frac{(kab^2/12)(2a^2 + 3b^2)}{(kab/3)(a^2 + b^2)} = \frac{b}{4}\left(\frac{2a^2 + 3b^2}{a^2 + b^2}\right)$

**8.** (a) $m = \frac{a^2k}{2}$

$M_x = \int_0^a \int_0^{a-x} ky\,dy\,dx = \frac{ka^3}{6}$

$M_y = M_x$ by symmetry

$\bar{x} = \bar{y} = \frac{M_x}{m} = \frac{ka^3/6}{ka^2/2} = \frac{a}{3}$




**—CONTINUED—**

**8. —CONTINUED—**

(b) $m = \int_0^a \int_0^{a-x} (x^2 + y^2)\, dy\, dx$

$= \int_0^a \left[x^2y + \frac{y^3}{3}\right]_0^{a-x} dx = \int_0^a \left[ax^2 - x^3 + \frac{1}{3}(a-x)^3\right] dx = \frac{a^4}{6}$

$M_y = \int_0^a \int_0^{a-x} (x^3 + xy^2)\, dy\, dx$

$= \int_0^a \left(ax^3 - x^4 + \frac{1}{3}a^3x - a^2x^2 + ax^3 - \frac{1}{3}x^4\right) dx = \frac{1}{3}\int_0^a (a^3x - 3a^2x^2 + 6ax^3 - 4x^4)\, dx = \frac{a^5}{15}$

$\bar{x} = \frac{M_y}{m} = \frac{a^5/15}{a^4/6} = \frac{2a}{5}$

$\bar{y} = \frac{2a}{5}$ by symmetry

**10.** The $x$-coordinate changes by $h$ units horizontally and $k$ units vertically. This is not true for variable densities.

**12.** (a) $m = \int_0^a \int_0^{\sqrt{a^2-x^2}} k\, dy\, dx = \frac{k\pi a^2}{4}$

$M_y = \int_0^a \int_0^{\sqrt{a^2-x^2}} kx\, dy\, dx$

$= k\int_0^a x\sqrt{a^2 - x^2}\, dx$

$= \left[-\frac{k}{3}(a^2 - x^2)^{3/2}\right]_0^a = \frac{ka^3}{3}$

$\bar{x} = \frac{M_y}{m} = \frac{ka^3/3}{k\pi a^2/4} = \frac{4a}{3\pi}$

$\bar{y} = \frac{4a}{3\pi}$ by symmetry

(b) $m = \int_0^a \int_0^{\sqrt{a^2-x^2}} k(x^2 + y^2)\, dy\, dx$

$= \int_0^{\pi/2} \int_0^a kr^3\, dr\, d\theta = \frac{ka^4\pi}{8}$

$M_x = \int_0^a \int_0^{\sqrt{a^2-x^2}} k(x^2 + y^2)\, y\, dy\, dx$

$= \int_0^{\pi/2} \int_0^a kr^4 \sin\theta\, dr\, d\theta = \frac{ka^5}{5}$

$M_y = M_x$ by symmetry

$\bar{x} = \bar{y} = \frac{M_y}{m} = \frac{ka^5}{5} \cdot \frac{8}{ka^4\pi} = \frac{8a}{5\pi}$

**14.** $m = \int_0^2 \int_0^{x^3} kx\, dy\, dx = \int_0^2 kx^4\, dx = \frac{32k}{5}$

$M_x = \int_0^2 \int_0^{x^3} kxy\, dy\, dx = 16k$

$M_y = \int_0^2 \int_0^{x^3} kx^2\, dy\, dx = \frac{32k}{3}$

$\bar{x} = \frac{M_y}{m} = \frac{32k}{3} \cdot \frac{5}{32k} = \frac{5}{3}$

$\bar{y} = \frac{M_x}{m} = \frac{16k}{32k}(5) = \frac{5}{2}$

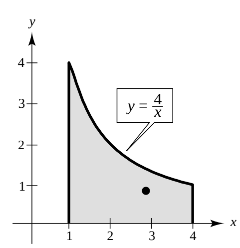
**16.** $m = \int_1^4 \int_0^{4/x} kx^2\, dy\, dx = 30k$

$M_x = \int_1^4 \int_0^{4/x} kx^2\, y\, dy\, dx = 24k$

$M_y = \int_1^4 \int_0^{4/x} kx^3\, dy\, dx = 84k$

$\bar{x} = \frac{M_y}{m} = \frac{84k}{30k} = \frac{14}{5}$

$\bar{y} = \frac{M_y}{m} = \frac{24k}{30k} = \frac{4}{5}$

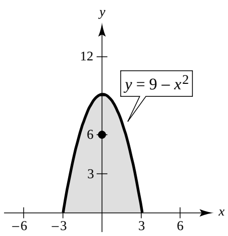


**18.** $\bar{x} = 0$ by symmetry

$$m = \int_{-3}^{3}\int_{0}^{9-x^2} ky^2\,dy\,dx = \frac{23{,}328k}{35}$$

$$M_x = \int_{-3}^{3}\int_{0}^{9-x^2} ky^3\,dy\,dx = \frac{139{,}968k}{35}$$

$$\bar{y} = \frac{M_x}{m} = \frac{139{,}968k}{35}\cdot\frac{35}{23{,}328k} = 6$$



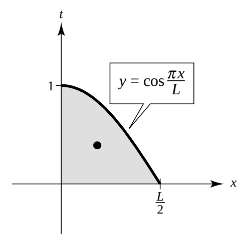

**20.** $$m = \int_{0}^{L/2}\int_{0}^{\cos \pi x/L} k\,dy\,dx = \frac{kL}{\pi}$$

$$M_x = \int_{0}^{L/2}\int_{0}^{\cos \pi x/L} ky\,dy\,dx = \frac{kL}{8}$$

$$M_y = \int_{0}^{L/2}\int_{0}^{\cos \pi x/L} kx\,dy\,dx = \frac{L^2(\pi - 2)k}{2\pi^2}$$

$$\bar{x} = \frac{M_y}{m} = \frac{L^2(\pi - 2)k}{2\pi^2}\cdot\frac{\pi}{kL} = \frac{L(\pi - 2)}{2\pi}$$

$$\bar{y} = \frac{M_x}{m} = \frac{kL}{8}\cdot\frac{\pi}{kL} = \frac{\pi}{8}$$



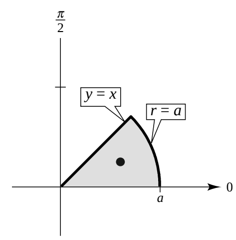

**22.** $$m = \iint_R k\sqrt{x^2 + y^2}\,dA = \int_{0}^{\pi/4}\int_{0}^{a} kr^2\,dr\,d\theta = \frac{ka^3\pi}{12}$$

$$M_x = \iint_R k\sqrt{x^2 + y^2}\,y\,dA = \int_{0}^{\pi/4}\int_{0}^{a} kr^3 \sin\theta\,d\theta = \frac{ka^4\left(2 - \sqrt{2}\right)}{8}$$

$$M_y = \iint_R k\sqrt{x^2 + y^2}\,dA = \int_{0}^{\pi/4}\int_{0}^{a} kr^3 \cos\theta\,d\theta = \frac{ka^4\sqrt{2}}{8}$$

$$\bar{x} = \frac{M_y}{m} = \frac{ka^4\sqrt{2}}{8}\cdot\frac{12}{ka^3\pi} = \frac{3\sqrt{2}a}{2\pi}$$

$$\bar{y} = \frac{M_x}{m} = \frac{ka^4\left(2 - \sqrt{2}\right)}{8}\cdot\frac{12}{ka^3\pi} = \frac{3\left(2 - \sqrt{2}\right)a}{2\pi}$$



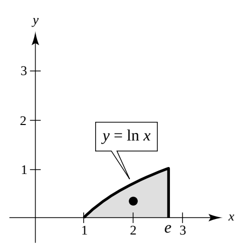

**24.** $$m = \int_{1}^{e}\int_{0}^{\ln x} \frac{k}{x}\,dy\,dx = \frac{k}{2}$$

$$M_x = \int_{1}^{e}\int_{0}^{\ln x} \frac{k}{x}y\,dy\,dx = \frac{k}{6}$$

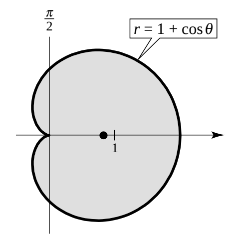
$$M_y = \int_{1}^{e}\int_{0}^{\ln x} \frac{k}{x}x\,dy\,dx = k$$

$$\bar{x} = \frac{M_y}{m} = \frac{k}{1}\cdot\frac{2}{k} = 2$$

$$\bar{y} = \frac{M_x}{m} = \frac{k}{6}\cdot\frac{2}{k} = \frac{1}{3}$$



**26.** $\bar{y} = 0$ by symmetry




$$m = \iint_R k\,dA = \int_0^{2\pi}\int_0^{1+\cos\theta} kr\,dr\,d\theta = \frac{3\pi k}{2}$$

$$M_y = \iint_R kx\,dA = \int_0^{2\pi}\int_0^{1+\cos\theta} kr^2\cos\theta\,dr\,d\theta$$

$$= \frac{k}{3}\int_0^{2\pi} \cos\theta(1 + 3\cos\theta + 3\cos^2\theta + \cos^3\theta)\,d\theta$$

$$= \frac{k}{3}\int_0^{2\pi}\left[\cos\theta + \frac{3}{2}(1 + \cos^2\theta) + 3\cos\theta(1 - \sin^2\theta) + \frac{1}{4}(1 + \cos 2\theta)^2\,d\theta\right.$$

$$= \frac{5k\pi}{4}$$

$$\bar{x} = \frac{M_y}{m} = \frac{5k\pi}{4}\cdot\frac{2}{3k\pi} = \frac{5}{6}$$

**28.** $m = \displaystyle\int_0^b\int_0^{h-(hx/b)} dy\,dx = \frac{bh}{2}$

$$I_x = \int_0^b\int_0^{h-(hx/b)} y^2\,dy\,dx = \frac{bh^3}{12}$$

$$I_y = \int_0^b\int_0^{h-(hx/b)} x^2\,dy\,dx = \frac{b^3h}{12}$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{b^3h/12}{bh/2}} = \frac{b}{\sqrt{6}} = \frac{\sqrt{6}}{6}b$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{bh^3/12}{bh/2}} = \frac{h}{\sqrt{6}} = \frac{\sqrt{6}}{6}h$$

**30.** $m = \dfrac{\pi a^2}{2}$

$$I_x = \iint_R y^2\,dA = \int_0^{\pi}\int_0^a r^3\sin^2\theta\,dr\,d\theta = \frac{a^4\pi}{8}$$

$$I_y = \iint_R x^2\,dA = \int_0^{\pi}\int_0^a r^3\cos^2\theta\,dr\,d\theta = \frac{a^4\pi}{8}$$

$$I_0 = I_x + I_x = \frac{a^4\pi}{8} + \frac{a^4\pi}{8} = \frac{a^4\pi}{4}$$

$$\bar{\bar{x}} = \bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{a^4\pi}{8}\cdot\frac{2}{\pi a^2}} = \frac{a}{2}$$

**32.** $m = \pi ab$

$$I_x = 4\int_0^a\int_0^{(b/a)\sqrt{a^2-x^2}} y^2\,dy\,dx$$

$$= 4\int_0^a \frac{b^3}{3a^3}(a^2 - x^2)^{3/2}\,dx = \frac{4b^3}{3a^3}\int_0^a\left[a^2\sqrt{a^2 - x^2} - x^2\sqrt{a^2 - x^2}\right]dx$$

$$= \frac{4b^3}{3a^3}\left[\frac{a^2}{2}\left(x\sqrt{a^2 - x^2} + a^2\arcsin\frac{x}{a}\right) - \frac{1}{8}\left[x(2x^2 - a^2)\sqrt{a^2 - x^2} + a^4\arcsin\frac{x}{a}\right]\right]_0^a = \frac{ab^3\pi}{4}$$

$$I_y = 4\int_0^b\int_0^{(a/b)\sqrt{b^2-y^2}} x^2\,dx\,dy = \frac{a^3b\pi}{4}$$

$$I_0 = I_y + I_x = \frac{a^3b\pi}{4} + \frac{ab^3\pi}{4} = \frac{ab\pi}{4}(a^2 + b^2)$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{a^3b\pi}{4}\cdot\frac{1}{\pi ab}} = \frac{a}{2}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{ab^3\pi}{4}\cdot\frac{1}{\pi ab}} = \frac{b}{2}$$

**34.** $\rho = ky$

$$m = 2k\int_0^a\int_0^{\sqrt{a^2-x^2}} y\,dy\,dx$$

$$= k\int_0^a (a^2 - x^2)\,dx = \frac{2ka^3}{3}$$

$$I_x = k\int_{-a}^a\int_0^{\sqrt{a^2-x^2}} y^3\,dy\,dx = \frac{4ka^5}{15}$$

$$I_y = k\int_{-a}^a\int_0^{\sqrt{a^2-x^2}} x^2y\,dy\,dx = \frac{2ka^5}{15}$$

$$I_0 = I_x + I_y = \frac{2ka^5}{5}$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{2ka^5/15}{2ka^3/3}} = \sqrt{\frac{a^2}{5}} = \frac{a}{\sqrt{5}}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{4ka^5/15}{2ka^3/3}} = \sqrt{\frac{2a^2}{5}} = \frac{2a}{\sqrt{10}}$$

**36.** $\rho = kxy$

$$m = k\int_0^1\int_{x^2}^x xy\,dy\,dx = \frac{k}{2}\int_0^1 (x^3 - x^5)\,dx = \frac{k}{24}$$

$$I_x = k\int_0^1\int_{x^2}^x xy^3\,dy\,dx = \frac{k}{4}\int_0^1 (x^5 - x^9)\,dx = \frac{k}{60}$$

$$I_y = k\int_0^1\int_{x^2}^x x^3y\,dy\,dx = \frac{k}{2}\int_0^1 (x^5 - x^7)\,dx = \frac{k}{48}$$

$$I_0 = I_x + I_y = \frac{9k}{240} = \frac{3k}{80}$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{k/48}{k/24}} = \frac{1}{\sqrt{2}}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}}\sqrt{\frac{k/60}{k/24}} = \sqrt{\frac{2}{5}}$$

**38.** $\rho = x^2 + y^2$

$$m = \int_0^1\int_{x^2}^{\sqrt{x}} (x^2 + y^2)\,dy\,dx = \frac{6}{35}$$

$$I_x = \int_0^1\int_{x^2}^{\sqrt{x}} (x^2 + y^2)\,y^2\,dy\,dx = \frac{158}{2079}$$

$$I_y = \int_0^1\int_{x^2}^{\sqrt{x}} (x^2 + y^2)x^2\,dy\,dx = \frac{158}{2079}$$

$$I_0 = I_x + I_y = \frac{316}{2079}$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{158}{2079}\cdot\frac{35}{6}} = \sqrt{\frac{395}{891}}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \bar{\bar{x}} = \sqrt{\frac{395}{891}}$$

**40.** $\rho = ky$

$$m = 2\int_0^2\int_{x^3}^{4x} ky\,dy\,dx = \frac{512k}{21}$$

$$I_x = 2\int_0^2\int_{x^3}^{4x} ky^3\,dy\,dx = \frac{32{,}768k}{65}$$

$$I_y = 2\int_0^2\int_{x^3}^{4x} kx^2\,y\,dy\,dx = \frac{2048k}{45}$$

$$I_0 = I_x + I_y = \frac{321{,}536k}{585}$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{2048k}{45}\cdot\frac{21}{512k}} = \sqrt{\frac{28}{15}} = \frac{2\sqrt{105}}{15}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{32{,}768k}{65}\cdot\frac{21}{512k}} = \frac{8\sqrt{1365}}{65}$$

**42.** $$I = \int_0^4\int_0^2 k(x-6)^2\,dy\,dx = \int_0^4 2k(x-6)^2\,dx = \left[\frac{2k}{3}(x-6)^3\right]_0^4 = \frac{416k}{3}$$

**44.** $$I = \int_{-a}^a\int_0^{\sqrt{a^2-x^2}} ky(y-a)^2\,dy\,dx$$

$$= \int_{-a}^a k\left[\frac{y^4}{4} - \frac{2ay^3}{3} + \frac{a^2y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_{-a}^a k\left[\frac{1}{4}(a^4 - 2a^2x^2 + x^4) - \frac{2a}{3}\left(a^2\sqrt{a^2-x^2} - x^2\sqrt{a^2-x^2}\right) + \frac{a^2}{2}(a^2 - x^2)\right] dx$$

$$= k\left[\frac{1}{4}\left(a^4x - \frac{2a^2x^3}{3} + \frac{x^5}{5}\right) - \frac{2a}{3}\left[\frac{a^2}{2}\left(x\sqrt{a^2-x^2} + a^2\arcsin\frac{x}{a}\right)\right.\right.$$

$$\left.\left. - \frac{1}{8}\left(x(2x^2 - a^2)\sqrt{a^2-x^2} + a^4\arcsin\frac{x}{a}\right)\right] + \frac{a^2}{2}\left(a^2x - \frac{x^3}{3}\right)\right]_{-a}^a$$

$$= 2k\left[\frac{1}{4}\left(a^5 - \frac{2}{3}a^5 + \frac{1}{5}a^5\right) - \frac{2a}{3}\left(\frac{a^4\pi}{4} - \frac{a^4\pi}{16}\right) + \frac{a^2}{2}\left(a^3 - \frac{a^3}{3}\right)\right] = 2k\left(\frac{7a^5}{15} - \frac{a^5\pi}{8}\right) = ka^5\left(\frac{56 - 15\pi}{60}\right)$$

**46.** $I = \int_{-2}^{2}\int_{0}^{4-x^2} k(y-2)^2\,dy\,dx = \int_{-2}^{2}\left[\frac{k}{3}(y-1)^3\right]_0^{4-x^2} dx = \int_{-2}^{2}\frac{k}{3}\left[(2-x^2)+8\right]dx$

$$= \frac{k}{3}\int_{-2}^{2}(16-12x^2+6x^4-x^6)\,dx = \left[\frac{k}{3}\left(16x-4x^3+\frac{6}{5}x^5-\frac{1}{7}x^7\right)\right]_{-2}^{2}$$

$$= \frac{2k}{3}\left(32-32+\frac{192}{5}-\frac{128}{7}\right) = \frac{1408k}{105}$$

**48.** $\rho(x, y) = k|2-x|$.

$(\bar{x}, \bar{y})$ will be the same.

**50.** $\rho(x, y) = k(4-x)(4-y)$. Both $\bar{x}$ and $\bar{y}$ will decrease

**52.** $I_x = \int\int_R y^2\,\rho(x, y)\,dA$ Moment of inertia about $x$-axis.

$I_y = \int\int_R x^2\,\rho(x, y)\,dA$ Moment of inertia about $y$-axis.

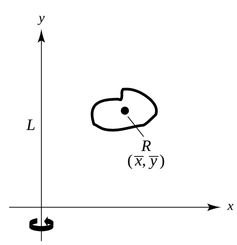
**54.** Orient the $xy$-coordinate system so that $L$ is along the $y$-axis and $R$ is in the first quadrant. Then the volume of the solid is

$$V = \int\int_R 2\pi x\,dA$$

$$= 2\pi\int\int_R x\,dA$$

$$= 2\pi\left(\frac{\int\int_R x\,dA}{\int\int_R dA}\right)\int\int_R dA$$

$$= 2\pi\bar{x}A.$$




By our positioning, $\bar{x} = r$. Therefore, $V = 2\pi rA$.

**56.** $\bar{y} = \frac{a}{2}, A = ab, h = L - \frac{a}{2}$

$$I_{\bar{y}} = \int_0^b\int_0^a\left(y-\frac{a}{2}\right)^2 dy\,dx = \frac{a^3b}{12}$$

$$y_a = \frac{a}{2} - \frac{a^3b/12}{[L-(a/2)]ab} = \frac{a(3L-2a)}{3(2L-a)}$$

**58.** $\bar{y} = 0, A = \pi a^2, h = L$

$$I_{\bar{y}} = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y^2\,dy\,dx$$

$$= \int_0^{2\pi}\int_0^a r^3\sin^2\theta\,dr\,d\theta$$

$$= \int_0^{2\pi}\frac{a^4}{4}\sin^2\theta\,d\theta$$

$$= \frac{a^4\pi}{4}$$

$$y_a = -\frac{(a^4\pi/4)}{L\pi a^2} = -\frac{a^2}{4L}$$

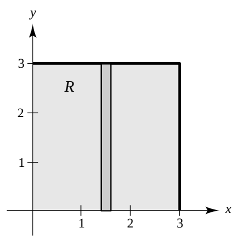
# Section 13.5 Surface Area

**2.** $f(x, y) = 15 + 2x - 3y$

$f_x = 2, f_y = -3$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{14}$

$$S = \int_0^3 \int_0^3 \sqrt{14}\, dy\, dx = \int_0^3 3\sqrt{14}\, dx = 9\sqrt{14}$$




**4.** $f(x, y) = 10 + 2x - 3y$

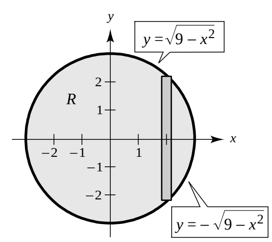
$R = \{(x, y): x^2 + y^2 \le 9\}$

$f_x = 2,\ f_y = -3$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{14}$

$$S = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \sqrt{14}\, dy\, dx$$

$$= \int_0^{2\pi} \int_0^3 \sqrt{14}\, r\, dr\, d\theta = 9\sqrt{14}\pi$$




**6.** $f(x, y) = y^2$

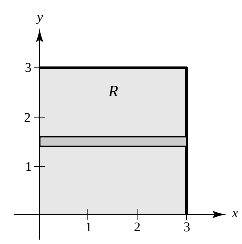
$R =$ square with vertices (0, 0), (3, 0), (0, 3), (3, 3)

$f_x = 0,\ f_y = 2y$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4y^2}$

$$S = \int_0^3 \int_0^3 \sqrt{1 + 4y^2}\, dx\, dy = \int_0^3 3\sqrt{1 + 4y^2}\, dy$$

$$= \left[\frac{3}{4}\left(2y\sqrt{1 + 4y^2} + \ln\left|2y + \sqrt{1 + 4y^2}\right|\right)\right]_0^3 = \frac{3}{4}\left(6\sqrt{37} + \ln\left|6 + \sqrt{37}\right|\right)$$




**8.** $f(x, y) = 2 + \frac{2}{3}y^{3/2}$

$f_x = 0,\ f_y = y^{1/2}$

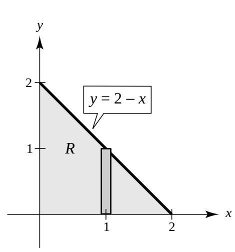
$\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + y}$

$$S = \int_0^2 \int_0^{2-y} \sqrt{1 + y}\, dx\, dy = \int_0^2 \sqrt{1 + y}(2 - y)\, dy$$

$$= \left[2(1 + y)^{3/2} - \frac{2}{5}(1 + y)^{5/2}\right]_0^2$$

$$= 2 \cdot 3^{3/2} - \frac{2}{5} \cdot 3^{5/2} - 2 + \frac{2}{5}$$

$$= \frac{12}{5}\sqrt{3} - \frac{8}{5}$$




**10.** $f(x, y) = 9 + x^2 - y^2$

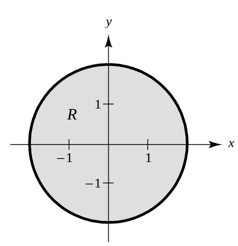
$f_x = 2x,\ f_y = -2y$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}$

$$S = \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2}\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{12}(1 + 4r^2)^{3/2}\right]_0^2 d\theta$$

$$= \int_0^{2\pi} \frac{1}{12}(17^{3/2} - 1)\, d\theta = \frac{\pi}{6}\left(17\sqrt{17} - 1\right)$$



**12.** $f(x, y) = xy$

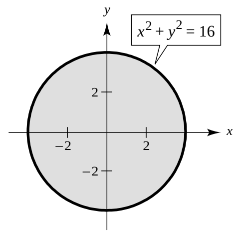
$R = \{(x, y): x^2 + y^2 \le 16\}$

$f_x = y,\ f_y = x$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + y^2 + x^2}$

$$S = \int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \sqrt{1 + y^2 + x^2}\, dy\, dx$$

$$= \int_0^{2\pi}\int_0^4 \sqrt{1 + r^2}\, r\, dr\, d\theta = \frac{2\pi}{3}\left(17\sqrt{17} - 1\right)$$




**14.** See Exercise 13.

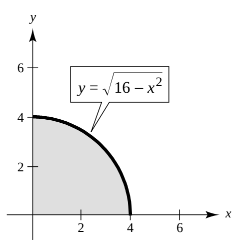
$$S = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dy\, dx = \int_0^{2\pi}\int_0^a \frac{a}{\sqrt{a^2 - r^2}}\, r\, dr\, d\theta = 2\pi a^2$$

**16.** $z = 16 - x^2 - y^2$

$\sqrt{1 + f_y^{\,2} + f_y^{\,2}} = \sqrt{1 + 4x^2 + 4y^2}$

$$S = \int_0^4\int_0^{\sqrt{16-x^2}} \sqrt{1 + 4(x^2 + y^2)}\, dy\, dx$$

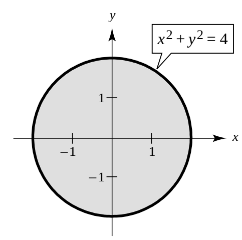
$$= \int_0^{\pi/2}\int_0^4 \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \frac{\pi}{24}\left(65\sqrt{65} - 1\right)$$




**18.** $z = 2\sqrt{x^2 + y^2}$

$$\sqrt{1 + f_x^{\,2} + f_y^{\,2}} = \sqrt{1 + \frac{4x^2}{x^2 + y^2} + \frac{4y^2}{x^2 + y^2}} = \sqrt{5}$$

$$S = \int_0^{2\pi}\int_0^2 \sqrt{5} r\, dr\, d\theta = 4\pi\sqrt{5}$$

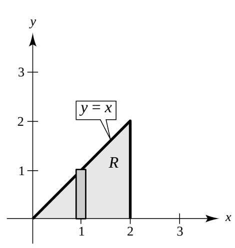



**20.** $f(x, y) = 2x + y^2$

$R$ = triangle with vertices (0, 0), (2, 0), (2, 2)

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{5 + 4y^2}$

$$S = \int_0^2\int_0^x \sqrt{5 + 4y^2}\, dy\, dx = \frac{1}{12}\left(21\sqrt{21} - 5\sqrt{5}\right)$$




**22.** $f(x, y) = x^2 + y^2$

$R = \{(x, y):\ 0 \le f(x, y) \le 16\}$

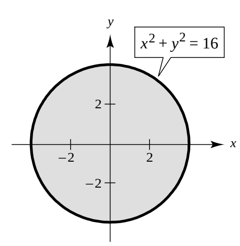
$0 \le x^2 + y^2 \le 16$

$f_x = 2x,\ f_y = 2y$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}$

$$S = \int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \sqrt{1 + 4x^2 + 4y^2}\, dy\, dx$$

$$= \int_0^{2\pi}\int_0^4 \sqrt{1 + 4r^2}\, dr\, d\theta = \frac{\left(65\sqrt{65} - 1\right)\pi}{6}$$



**24.** $f(x, y) = \frac{2}{3}x^{3/2} + \cos x$

$R = \{(x, y):\ 0 \le x \le 1, 0 \le y \le 1\}$

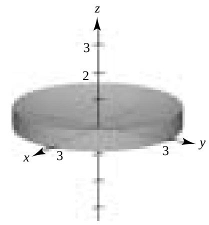
$f_x = x^{1/2} - \sin x,\ f_y = 0$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \left(\sqrt{x} - \sin x\right)^2}$

$$S = \int_0^1 \int_0^1 \sqrt{1 + \left(\sqrt{x} - \sin x\right)^2}\, dy\, dx \approx 1.02185$$

**26.** Surface area $\approx (9\pi)$

Matches (c)




**28.** $f(x, y) = \frac{2}{5}y^{5/2}$

$R = \{(x, y): 0 \le x \le 1, 0 \le y \le 1\}$

$f_x = 0,\ f_y = y^{3/2}$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + y^3}$

$$S = \int_0^1 \int_0^1 \sqrt{1 + y^3}\, dx\, dy$$

$$= \int_0^1 \sqrt{1 + y^3}\, dy \approx 1.1114$$

**30.** $f(x, y) = x^2 - 3xy - y^2$

$R = \{(x, y):\ 0 \le x \le 4,\ 0 \le y \le x\}$

$f_x = 2x - 3y,\ f_y = -3x - 2y = -(3x + 2y)$

$\sqrt{1 + (f_x) + (f_y)^2} = \sqrt{1 + (2x - 3y)^2 + (3x + 2y)^2}$

$= \sqrt{1 + 13(x^2 + y^2)}$

$$S = \int_0^4 \int_0^x \sqrt{1 + 13(x^2 + y^2)}\, dy\, dx$$

**32.** $f(x, y) = \cos(x^2 + y^2)$

$R = \left\{(x, y):\ x^2 + y^2 \le \frac{\pi}{2}\right\}$

$f_x = -2x \sin(x^2 + y^2),\ f_y = -2y \sin(x^2 + y^2)$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 \sin^2(x^2 + y^2) + 4y^2 \sin^2(x^2 + y^2)} = \sqrt{1 + 4[\sin^2(x^2 + y^2)](x^2 + y^2)}$

$$S = \int_{-\sqrt{\pi/2}}^{\sqrt{\pi/2}} \int_{-\sqrt{(\pi/2) - x^2}}^{\sqrt{(\pi/2) - x^2}} \sqrt{1 + 4(x^2 + y^2) \sin^2(x^2 + y^2)}\, dy\, dx$$

**34.** $f(x, y) = e^{-x} \sin y$

$R = \{(x, y): 0 \le x \le 4, 0 \le y \le x\}$

$f_x = -e^{-x} \sin y,\ f_y = e^{-x} \cos y$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + e^{-2x} \sin^2 y + e^{-2x} \cos^2 y}$

$= \sqrt{1 + e^{-2x}}$

$$S = \int_0^4 \int_0^x \sqrt{1 + e^{-2x}}\, dy\, dx$$

**36.** (a) Yes. For example, let $R$ be the square given by

$0 \le x \le 1, 0 \le y \le 1,$

and $S$ the square parallel to $R$ given by

$0 \le x \le 1, 0 \le y \le 1, z = 1.$

(b) Yes. Let $R$ be the region in part (a) and $S$ the surface given by $f(x, y) = xy$.

(c) No.

**38.** $f(x, y) = k\sqrt{x^2 + y^2}$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \frac{k^2x^2}{x^2 + y^2} + \frac{k^2y^2}{x^2 + y^2}} = \sqrt{k^2 + 1}$$

$$S = \iint_R \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA = \iint_R \sqrt{k^2 + 1}\, dA = \sqrt{k^2 + 1} \iint_R dA = A\sqrt{k^2 + 1} = \pi r^2\sqrt{k^2 + 1}$$

**40.** (a) $z = \frac{-1}{75}y^3 + \frac{4}{25}y^2 - \frac{16}{15}y + 25$

(b) $V \approx 2(50)\int_0^{15}\left(-\frac{1}{75}y^3 + \frac{4}{25}y^2 - \frac{16}{15}y + 25\right) dy$

$= 100(266.25) = 26{,}625$ cubic feet

(c) $f(x, y) = -\frac{1}{75}y^3 + \frac{4}{25}y^2 - \frac{16}{15}y + 25$

$f_x = 0, f_y = -\frac{1}{25}y^2 + \frac{8}{25}y - \frac{16}{15}$

$S = 2\int_0^{50}\int_0^{15} \sqrt{1 + f_y^2 + f_x^2}\, dy\, dx \approx 3087.58$ sq ft

(d) Arc length $\approx 30.8758$

Surface area of roof $\approx 2(50)(30.8758) = 3087.58$ sq ft

**42.** False. The surface area will remain the same for any vertical translation.

## Section 13.6 Triple Integrals and Applications

**2.** $\int_{-1}^1\int_{-1}^1\int_{-1}^1 x^2y^2z^2\, dx\, dy\, dz = \frac{1}{3}\int_{-1}^1\int_{-1}^1 \left[x^3y^2z^2\right]_{-1}^1 dy\, dz$

$= \frac{2}{3}\int_{-1}^1\int_{-1}^1 y^2z^2\, dy\, dz = \frac{2}{9}\int_{-1}^1 \left[y^3z^2\right]_{-1}^1 dz = \frac{4}{9}\int_{-1}^1 z^2\, dz = \left[\frac{4}{27}z^3\right]_{-1}^1 = \frac{8}{27}$

**4.** $\int_0^9\int_0^{y/3}\int_0^{\sqrt{y^2-9x^2}} z\, dz\, dx\, dy = \frac{1}{2}\int_0^9\int_0^{y/3} (y^2 - 9x^2)\, dx\, dy$

$= \frac{1}{2}\int_0^9 \left[xy^2 - 3x^3\right]_0^{y/3} dy = \frac{2}{18}\int_0^9 y^3\, dy = \left[\frac{1}{36}y^4\right]_0^9 = \frac{729}{4}$

**6.** $\int_1^4\int_1^{e^2}\int_0^{1/xz} \ln z\, dy\, dz\, dx = \int_1^4\int_1^{e^2} \left[(\ln z)y\right]_0^{1/xz} dz\, dx = \int_1^4\int_1^{e^2} \frac{\ln z}{xz}\, dz\, dx$

$= \int_1^4 \left[\frac{1}{x}\frac{(\ln z)^2}{2}\right]_1^{e^2} dx = \int_1^4 \frac{2}{x}\, dx = \left[2\ln|x|\right]_1^4 = 2\ln 4$

**8.** $\int_0^{\pi/2}\int_0^{y/2}\int_0^{1/y} \sin y\, dz\, dx\, dy = \int_0^{\pi/2}\int_0^{y/2} \frac{\sin y}{y}\, dx\, dy = \frac{1}{2}\int_0^{\pi/2} \sin y\, dy = \left[-\frac{1}{2}\cos y\right]_0^{\pi/2} = \frac{1}{2}$

**10.** $\int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}\int_{2x^2+y^2}^{4-y^2} y\, dz\, dy\, dx = \int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}} (4y - 2x^2y - 2y^3)\, dy\, dx = \frac{16\sqrt{2}}{15}$

**12.** $\int_0^3\int_0^{2-(2y/3)}\int_0^{6-2y-3z} ze^{-x^2y^2}\, dx\, dz\, dy = \int_0^6\int_0^{(6-x)/2}\int_0^{(6-x-2y)/3} ze^{-x^2y^2}\, dz\, dy\, dx$

$= \int_0^6\int_0^{3-(x/2)} \frac{1}{2}\left(\frac{6 - x - 2y}{3}\right)^2 e^{-x^2y^2}\, dy\, dx \approx 2.118$

**14.** $\int_0^3\int_0^{2x}\int_0^{9-x^2} dz\, dy\, dx$

**16.** $z = \frac{1}{2}(x^2 + y^2) \implies 2z = x^2 + y^2$

$x^2 + y^2 + z^2 = 2z + z^2 = 80 \implies z^2 + 2z - 80 = 0 \implies (z - 8)(z + 10) = 0 \implies z = 8 \implies x^2 + y^2 = 2z = 16$

$$\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{1/2(x^2+y^2)}^{\sqrt{80-x^2-y^2}} dz\,dy\,dx$$

**18.** $\int_0^1\int_0^1\int_0^{xy} dz\,dy\,dx = \int_0^1\int_0^1 xy\,dy\,dx = \int_0^1 \frac{x}{2}\,dx = \left[\frac{x^2}{4}\right]_0^1 = \frac{1}{4}$

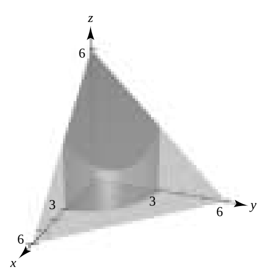
**20.**
$$4\int_0^6\int_0^{\sqrt{36-x^2}}\int_0^{36-x^2-y^2} dz\,dy\,dx = 4\int_0^6\int_0^{\sqrt{36-x^2}} (36 - x^2 - y^2)dy\,dx = 4\int_0^6\left[36y - x^2y - \frac{y^3}{3}\right]_0^{\sqrt{36-x^2}} dx$$
$$= 4\int_0^6\left[36\sqrt{36-x^2} - x^2\sqrt{36-x^2} - \frac{1}{3}(36-x^2)^{3/2}\right]dx$$
$$= 4\left[9x\sqrt{36-x^2} + 324\arcsin\left(\frac{x}{6}\right) + \frac{1}{6}x(36-x^2)^{3/2}\right]_0^6 = 4(162\pi) = 648\pi$$

**22.**
$$\int_0^2\int_0^{2-x}\int_0^{9-x^2} dz\,dy\,dx = \int_0^2\int_0^{2-x}(9 - x^2)\,dy\,dx = \int_0^2 (9 - x^2)(2 - x)\,dx$$
$$= \int_0^2 (18 - 9x - 2x^2 + x^3)\,dx = \left[18x - \frac{9}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4\right]_0^2 = \frac{50}{3}$$

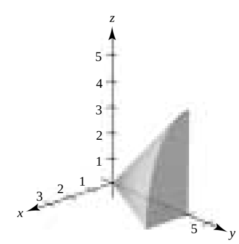
**24.** Top plane: $x + y + z = 6$

Side cylinder: $x^2 + y^2 = 9$

$$\int_0^3\int_0^{\sqrt{9-y^2}}\int_0^{6-x-y} dz\,dx\,dy$$

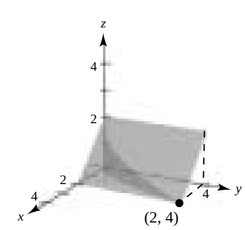



**26.** Elliptic cone: $4x^2 + z^2 = y^2$




$$\int_0^4\int_z^4\int_0^{\sqrt{y^2-z^2}/2} dx\,dy\,dz$$

**28.** $Q = \{(x, y, z): 0 \le x \le 2, x^2 \le y \le 4, 0 \le z \le 2 - x\}$




$$\iiint_Q xyz\,dV = \int_0^2\int_{x^2}^4\int_0^{2-x} xyz\,dz\,dy\,dx$$
$$= \int_0^4\int_0^{\sqrt{y}}\int_0^{2-x} xyz\,dz\,dx\,dy$$
$$= \int_0^2\int_0^{2-x}\int_{x^2}^4 xyz\,dy\,dz\,dx$$
$$= \int_0^2\int_0^{2-z}\int_{x^2}^4 xyz\,dy\,dx\,dz$$
$$= \int_0^2\int_0^{(2-z)^2}\int_0^{\sqrt{y}} xyz\,dx\,dy\,dz + \int_0^2\int_{(2-z)^2}^4\int_0^{2-z} xyz\,dx\,dy\,dz$$
$$= \int_0^4\int_0^{2-\sqrt{y}}\int_0^{\sqrt{y}} xyz\,dx\,dz\,dy + \int_0^4\int_{2-\sqrt{y}}^2\int_0^{2-z} dx\,dz\,dy\left(= \frac{104}{21}\right)$$

**30.** $Q = \{(x, y, z)\colon\ 0 \le x \le 1, y \le 1 - x^2, 0 \le z \le 6\}$

$$\iiint_Q xyz\, dV = \int_0^1\int_0^{\sqrt{1-x^2}}\int_0^6 xyz\, dz\, dy\, dx$$
$$= \int_0^1\int_0^{\sqrt{1-y^2}}\int_0^6 xyz\, dz\, dx\, dy$$
$$= \int_0^1\int_0^6\int_0^{\sqrt{1-y^2}} xyz\, dx\, dz\, dy$$
$$= \int_0^6\int_0^1\int_0^{\sqrt{1-y^2}} xyz\, dx\, dy\, dz$$
$$= \int_0^1\int_0^6\int_0^{\sqrt{1-x^2}} xyz\, dy\, dz\, dx$$
$$= \int_0^6\int_0^1\int_0^{\sqrt{1-x^2}} xyz\, dy\, dx\, dz$$


**32.** $$m = k\int_0^5\int_0^{5-x}\int_0^{1/5(15-3x-3y)} y\, dz\, dy\, dz = \frac{125}{8}k$$
$$M_{xz} = k\int_0^5\int_0^{5-x}\int_0^{1/5(15-3x-3y)} y^2\, dz\, dy\, dx = \frac{125}{4}k$$
$$\bar{y} = \frac{M_{xz}}{m} = 2$$

**34.** $$m = k\int_0^b\int_0^{a[1-(y/b)]}\int_0^{c[1-(y/b)-(x/a)]} dz\, dx\, dy = \frac{kabc}{6}$$
$$M_{xz} = k\int_0^b\int_0^{a[1-(y/b)]}\int_0^{c[1-(y/b)-(x/a)]} y\, dz\, dx\, dy = \frac{kab^2c}{24}$$
$$\bar{y} = \frac{M_{xz}}{m} = \frac{kab^2c/24}{kabc/6} = \frac{b}{4}$$

**36.** $$m = k\int_0^a\int_0^b\int_0^c z\, dz\, dy\, dx = \frac{kabc^2}{2}$$
$$M_{xy} = k\int_0^a\int_0^b\int_0^c z^2\, dz\, dy\, dx = \frac{kabc^3}{3}$$
$$M_{yz} = k\int_0^a\int_0^b\int_0^c xz\, dz\, dy\, dx = \frac{ka^2bc^2}{4}$$
$$M_{xz} = k\int_0^a\int_0^b\int_0^c yz\, dz\, dy\, dx = \frac{kab^2c^2}{4}$$
$$\bar{x} = \frac{M_{yz}}{m} = \frac{ka^2bc^2/4}{kabc^2/2} = \frac{a}{2}$$
$$\bar{y} = \frac{M_{xz}}{m} = \frac{kab^2c^2/4}{kabc^2/2} = \frac{b}{2}$$
$$\bar{z} = \frac{M_{xy}}{m} = \frac{kabc^3/3}{kabc^2/2} = \frac{2c}{3}$$

**38.** $\bar{z}$ will be greater than $8/5$, whereas $\bar{x}$ and $\bar{y}$ will be unchanged.

**40.** $\bar{x}, \bar{y}$ and $\bar{z}$ will all be greater than their original values.

**42.** $m = 2k\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^y dz\,dy\,dx$

$$= k\int_0^2 (4 - x^2)\,dx = \frac{16k}{3}$$

$$M_{yz} = k\int_{-2}^2\int_0^{\sqrt{4-x^2}}\int_0^y x\,dz\,dy\,dx = 0$$

$$M_{xz} = 2k\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^y y\,dz\,dy\,dx = 2k\pi$$

$$M_{xy} = 2k\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^y z\,dz\,dy\,dx = k\pi$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{0}{16k/3} = 0$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{2k\pi}{16k/3} = \frac{3\pi}{8}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{k\pi}{16k/3} = \frac{3\pi}{16}$$

**44.** $\bar{x} = 0$

$$m = 2k\int_0^2\int_0^1\int_0^{1/(y^2+1)} dz\,dy\,dx = 2k\int_0^2\int_0^1 \frac{1}{y^2+1}\,dy\,dx = 2k\left(\frac{\pi}{4}\right)\int_0^2 dx = k\pi$$

$$M_{xz} = 2k\int_0^2\int_0^1\int_0^{1/(y^2+1)} y\,dz\,dy\,dx = 2k\int_0^2\int_0^1 \frac{y}{y^2+1}\,dy\,dx = k\int_0^2 (\ln 2)\,dx = k\ln 4$$

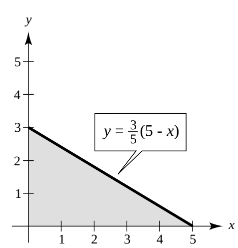
$$M_{xy} = 2k\int_0^2\int_0^1\int_0^{1/(y^2+1)} z\,dz\,dy\,dx$$

$$= k\int_0^2\int_0^1 \frac{1}{(y^2+1)^2}\,dy\,dx = k\int_0^2\left[\frac{y}{2(y^2+1)} + \frac{1}{2}\arctan y\right]_0^1 dx = k\left(\frac{1}{4} + \frac{\pi}{8}\right)\int_0^2 dx = k\left(\frac{1}{2} + \frac{\pi}{4}\right)$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{k\ln 4}{k\pi} = \frac{\ln 4}{\pi}$$

$$\bar{z} = \frac{M_{xy}}{m} = k\left(\frac{1}{2} + \frac{\pi}{4}\right)/k\pi = \frac{2+\pi}{4\pi}$$

**46.** $f(x, y) = \frac{1}{15}(60 - 12x - 20y)$




$$m = k\int_0^5\int_0^{-(3/5)x+3}\int_0^{(1/15)(60-12x-20y)} dz\,dy\,dx = 10k$$

$$M_{yz} = k\int_0^5\int_0^{-(3/5)x+3}\int_0^{(1/15)(60-12x-20y)} x\,dz\,dy\,dx = \frac{25k}{2}$$

$$M_{xz} = k\int_0^5\int_0^{-(3/5)x+3}\int_0^{-(1/15)(60-12x-20y)} y\,dz\,dy\,dx = \frac{15k}{2}$$

$$M_{xy} = k\int_0^5\int_0^{-(3/5)x+3}\int_0^{(1/15)(60-12x-20y)} z\,dz\,dy\,dx = 10k$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{25k/2}{10k} = \frac{5}{4}$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{15k/2}{10k} = \frac{3}{4}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{10k}{10k} = 1$$

**48.** (a) $I_{xy} = k\int_{-a/2}^{a/2}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} z^2\,dz\,dy\,dx = \dfrac{ka^5}{12}$

$I_{xz} = I_{yz} = \dfrac{ka^5}{12}$ by symmetry

$I_x = I_y = I_z = \dfrac{ka^5}{12} + \dfrac{ka^5}{12} = \dfrac{ka^5}{6}$

(b) $I_{xy} = k\int_{-a/2}^{a/2}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} z^2(x^2 + y^2)\,dz\,dy\,dx = \dfrac{a^3k}{12}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2}(x^2 + y^2)\,dy\,dx = \dfrac{a^7k}{72}$

$I_{xz} = k\int_{-a/2}^{a/2}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} y^2(x^2 + y^2)\,dz\,dy\,dx = ka\int_{-a/2}^{a/2}\int_{-a/2}^{a/2}(x^2y^2 + y^4)\,dy\,dx = \dfrac{7ka^7}{360}$

$I_{yz} = I_{xz}$ by symmetry

$I_x = I_{xy} + I_{xz} = \dfrac{a^7k}{30}$

$I_y = I_{xy} + I_{yz} = \dfrac{a^7k}{30}$

$I_z = I_{yz} + I_{xz} = \dfrac{7ka^7}{180}$

**50.** (a) $I_{xy} = k\int_0^4\int_0^2\int_0^{4-y^2} z^3\,dz\,dy\,dx = k\int_0^4\int_0^2 \dfrac{1}{4}(4 - y^2)^4\,dy\,dx$

$= \dfrac{k}{4}\int_0^4\int_0^2 (256 - 256y^2 + 96y^4 - 16y^6 + y^8)\,dy\,dx$

$= \dfrac{k}{4}\int_0^4 \left[256y - \dfrac{256y^3}{3} + \dfrac{96y^5}{5} - \dfrac{16y^7}{7} + \dfrac{y^9}{9}\right]_0^2 dx = k\int_0^4 \dfrac{16,384}{945}\,dx = \dfrac{65,536k}{315}$

$I_{xz} = k\int_0^4\int_0^2\int_0^{4-y^2} y^2z\,dz\,dy\,dx = k\int_0^4\int_0^2 \dfrac{1}{2}y^2(4 - y^2)^2\,dy\,dx$

$= k\int_0^4\int_0^2 \dfrac{1}{2}(16y^2 - 8y^4 + y^6)\,dy\,dx = \dfrac{k}{2}\int_0^4\left[\dfrac{16y^3}{3} - \dfrac{8y^5}{5} + \dfrac{y^7}{7}\right]_0^2 dx = \dfrac{k}{2}\int_0^4 \dfrac{1024}{105}\,dx = \dfrac{2048k}{105}$

$I_{yz} = k\int_0^4\int_0^2\int_0^{4-y^2} x^2z\,dz\,dy\,dx = k\int_0^4\int_0^2 \dfrac{1}{2}x^2(4 - y^2)^2\,dy\,dx$

$= k\int_0^4\int_0^2 \dfrac{1}{2}x^2(16 - 8y^2 + y^4)\,dy\,dx = \dfrac{k}{2}\int_0^4 \left[x^2\left(16y - \dfrac{8y^3}{3} + \dfrac{y^5}{5}\right)\right]_0^2 dx = \dfrac{k}{2}\int_0^4 \dfrac{256}{15}x^2\,dx = \dfrac{8192k}{45}$

$I_x = I_{xz} + I_{xy} = \dfrac{2048k}{9}, I_y = I_{yz} + I_{xy} = \dfrac{8192k}{21}, I_z = I_{yz} + I_{xz} = \dfrac{63,488k}{315}$

**—CONTINUED—**

**50. —CONTINUED—**

(b) $I_{xy} = \int_0^4\int_0^2\int_0^{4-y^2} z^2(4-z)\,dz\,dy\,dx$

$$= k\int_0^4\int_0^2\int_0^{4-y^2} 4z^2\,dz\,dy\,dx - k\int_0^4\int_0^2\int_0^{4-y^2} z^3\,dz\,dy\,dx = \frac{32{,}768k}{105} - \frac{65{,}536k}{315} = \frac{32{,}768k}{315}$$

$$I_{xz} = \int_0^4\int_0^2\int_0^{4-y^2} y^2(4-z)\,dz\,dy\,dx$$

$$= k\int_0^4\int_0^2\int_0^{4-y^2} 4y^2\,dz\,dy\,dx - k\int_0^4\int_0^2\int_0^{4-y^2} y^2z\,dz\,dy\,dx = \frac{1024k}{15} - \frac{2048k}{105} = \frac{1024k}{21}$$

$$I_{yz} = k\int_0^4\int_0^2\int_0^{4-y^2} x^2(4-z)\,dz\,dy\,dx$$

$$= k\int_0^4\int_0^2\int_0^{4-y^2} 4x^2\,dz\,dy\,dx - k\int_0^4\int_0^2\int_0^{4-y^2} x^2z\,dz\,dy\,dx = \frac{4096k}{9} - \frac{8192k}{45} = \frac{4096k}{15}$$

$$I_x = I_{xz} + I_{xy} = \frac{48{,}128k}{315},\ I_y = I_{yz} + I_{xy} = \frac{118{,}784k}{315},\ I_z = I_{xz} + I_{yz} = \frac{11{,}264k}{35}$$

**52.** $I_{xy} = \int_{-c/2}^{c/2}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2} z^2\,dz\,dy\,dx = \frac{b^3}{12}\int_{-c/2}^{c/2}\int_{-a/2}^{a/2} dy\,dx = \frac{1}{12}b^2(abc) = \frac{1}{12}mb^2$

$$I_{xz} = \int_{-c/2}^{c/2}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2} y^2\,dz\,dy\,dx = b\int_{-c/2}^{c/2}\int_{-a/2}^{a/2} y^2\,dy\,dx = \frac{ba^3}{12}\int_{-c/2}^{c/2} dx = \frac{ba^3c}{12} = \frac{1}{12}a^2(abc) = \frac{1}{12}ma^2$$

$$I_{yz} = \int_{-c/2}^{c/2}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2} x^2\,dz\,dy\,dx = ab\int_{-c/2}^{c/2} x^2\,dx = \frac{abc^3}{12} = \frac{1}{12}c^2(abc) = \frac{1}{12}mc^2$$

$$I_x = I_{xy} + I_{xz} = \frac{1}{12}m(a^2+b^2)$$

$$I_y = I_{xy} + I_{yz} = \frac{1}{12}m(b^2+c^2)$$

$$I_z = I_{xz} + I_{yz} = \frac{1}{12}m(a^2+c^2)$$

**54.** $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{4-x^2-y^2} kx^2(x^2+y^2)\,dz\,dy\,dx$

**56.** 6

**58.** Because the density increases as you move away from the axis of symmetry, the moment of intertia will increase.

## Section 13.7 Triple Integrals in Cylindrical and Spherical Coordinates

**2.** $\int_0^{\pi/4}\int_0^2\int_0^{2-r} rz\,dz\,dr\,d\theta = \int_0^{\pi/4}\int_0^2\left[\frac{rz^2}{2}\right]_0^{2-r} dr\,d\theta$

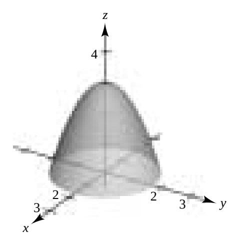
$$= \frac{1}{2}\int_0^{\pi/4}\int_0^2 (4r - 4r^2 + r^3)dr\,d\theta = \frac{1}{2}\int_0^{\pi/4}\left[2r^2 - \frac{4r^3}{3} + \frac{r^4}{4}\right]_0^2 d\theta = \frac{2}{3}\int_0^{\pi/4} d\theta = \frac{\pi}{6}$$

**4.** $\int_0^{\pi/2}\int_0^{\pi}\int_0^2 e^{-\rho^3}\rho^2\,d\rho\,d\theta\,d\phi = \int_0^{\pi/2}\int_0^{\pi}\left[-\frac{1}{3}e^{-\rho^3}\right]_0^2 d\theta\,d\phi = \int_0^{\pi/2}\int_0^{\pi}\frac{1}{3}(1-e^{-8})\,d\theta\,d\phi = \frac{\pi^2}{6}(1-e^{-8})$

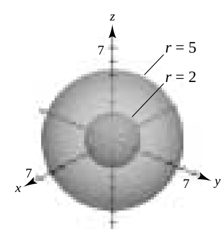
**6.** $$\int_0^{\pi/4}\int_0^{\pi/4}\int_0^{\cos\theta} \rho^2 \sin\phi \cos\phi \, d\rho \, d\theta \, d\phi = \frac{1}{3}\int_0^{\pi/4}\int_0^{\pi/4} \cos^3\theta \sin\phi\cos\phi \, d\theta \, d\phi$$
$$= \frac{1}{3}\int_0^{\pi/4}\int_0^{\pi/4} \sin\phi\cos\phi[\cos\theta(1-\sin^2\theta)]d\theta \, d\phi$$
$$= \frac{1}{3}\int_0^{\pi/4} \sin\phi\cos\phi\left[\sin\theta - \frac{\sin^3\theta}{3}\right]_0^{\pi/4} d\phi$$
$$= \frac{5\sqrt{2}}{36}\int_0^{\pi/4} \sin\phi\cos\phi \, d\phi = \left[\frac{5\sqrt{2}}{36}\frac{\sin^2\phi}{2}\right]_0^{\pi/4} = \frac{5\sqrt{2}}{144}$$

**8.** $$\int_0^{\pi/2}\int_0^{\pi}\int_0^{\sin\theta} (2\cos\phi)\rho^2 \, d\rho \, d\theta \, d\phi = \frac{8}{9}$$

**10.** $$\int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^{3-r^2} r \, dz \, dr \, d\theta = \int_0^{2\pi}\int_0^{\sqrt{3}} r(3-r^2)dr \, d\theta$$
$$= \int_0^{2\pi}\left(\frac{3r^2}{2} - \frac{r^4}{4}\right)\Bigg]_0^{\sqrt{3}} d\theta$$
$$= \int_0^{2\pi}\frac{9}{4}d\theta = \frac{9\pi}{2}$$




**12.** $$\int_0^{2\pi}\int_0^{\pi}\int_2^{5} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{117}{3}\int_0^{2\pi}\int_0^{\pi} \sin\phi \, d\phi \, d\theta$$
$$= \frac{117}{3}\int_0^{2\pi}\Big[-\cos\phi\Big]_0^{\pi} d\theta$$
$$= \frac{468\pi}{3}$$




**14.** (a) $$\int_0^{\pi/2}\int_0^{2}\int_0^{\sqrt{16-r^2}} r^2 \, dz \, dr \, d\theta = \frac{8\pi^2}{3} - 2\pi\sqrt{3}$$

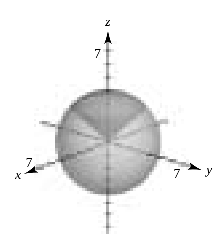
(b) $$\int_0^{\pi/2}\int_0^{\pi/6}\int_0^{4} \rho^3 \sin^2\phi \, d\rho \, d\phi \, d\theta + \int_0^{\pi/2}\int_{\pi/6}^{\pi/2}\int_4^{2\csc\phi} \rho^3 \sin^2\phi \, d\rho \, d\phi \, d\theta = \frac{8\pi^2}{3} - 2\pi\sqrt{3}$$

**16.** (a) $$\int_0^{\pi/2}\int_0^{1}\int_0^{\sqrt{1-r^2}} r\sqrt{r^2+z^2} \, dz \, dr \, d\theta = \frac{\pi}{8}$$

(b) $$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{1} \rho^3 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{\pi}{8}$$

**18.** $$V = \frac{2}{3}\pi(4)^3 + 4\left[\int_0^{\pi/2}\int_0^{2\sqrt{2}}\int_0^{r} r \, dz \, dr \, d\theta + \int_0^{\pi/2}\int_{2\sqrt{2}}^{4}\int_0^{\sqrt{16-r^2}} r \, dz \, dr \, d\theta\right]$$

(Volume of lower hemisphere) + 4(Volume in the first octant)




$$V = \frac{128\pi}{3} + 4\left[\int_0^{\pi/2}\int_0^{2\sqrt{2}} r^2 \, dr \, d\theta + \int_0^{\pi/2}\int_{2\sqrt{2}}^{4} r\sqrt{16-r^2} \, dr \, d\theta\right]$$
$$= \frac{128\pi}{3} + 4\left[\frac{8\sqrt{2}\pi}{3} + \int_0^{\pi/2}\left[-\frac{1}{3}(16-r^2)^{3/2}\right]_{2\sqrt{2}}^{4} d\theta\right]$$
$$= \frac{128\pi}{3} + 4\left[\frac{8\sqrt{2}\pi}{3} + \frac{8\sqrt{2}\pi}{3}\right]$$
$$= \frac{128\pi}{3} + \frac{64\sqrt{2}\pi}{3} = \frac{64\pi}{3}\left(2+\sqrt{2}\right)$$

**20.** $V = \int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^{\sqrt{2}} \left(r\sqrt{4-r^2} - r^2\right) dr\,d\theta$

$= \int_0^{2\pi}\left[-\frac{1}{3}(4-r^2)^{3/2} - \frac{r^3}{3}\right]_0^{\sqrt{2}} d\theta$

$= \frac{8\pi}{3}\left(2 - \sqrt{2}\right)$

**22.** $\int_0^{\pi/2}\int_0^{2}\int_0^{12e^{-r^2}} k\,r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^{2} 12ke^{-r^2}\,r\,dr\,d\theta$

$= \int_0^{\pi/2}\left[-6ke^{-r^2}\right]_0^2$

$= \int_0^{\pi/2}(-6ke^{-4} + 6k)\,d\theta$

$= 3k\pi(1 - e^{-4})$

**24.** $\bar{x} = \bar{y} = 0$ by symmetry

$m = \frac{1}{3}\pi r_0^2 hk$ from Exercise 23

$M_{xy} = 4k\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} zr\,dz\,dr\,d\theta$

$= \frac{2kh^2}{r_0^2}\int_0^{\pi/2}\int_0^{r_0}(r_0^2 r - 2r_0 r^2 + r^3)dr\,d\theta$

$= \frac{2kh^2}{r_0^2}\left(\frac{r_0^4}{12}\right)\left(\frac{\pi}{2}\right) = \frac{kr_0^2\,h^2\pi}{12}$

$\bar{z} = \frac{M_{xy}}{m} = \frac{kr_0^2h^2\pi}{12}\left(\frac{3}{\pi r_0^2 hk}\right) = \frac{h}{4}$

**26.** $\rho = kz$

$\bar{x} = \bar{y} = 0$ by symmetry

$m = 4k\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} zr\,dz\,dr\,d\theta$

$= \frac{1}{12}k\pi r_0^2 h^2$

$M_{xy} = 4k\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} z^2 r\,dz\,dr\,d\theta$

$= \frac{1}{30}k\pi r_0^2 h^3$

$\bar{z} = \frac{M_{xy}}{m} = \frac{k\pi r_0^2 h^3/30}{k\pi r_0^2 h^2/12} = \frac{2h}{5}$

**28.** $I_z = \iiint_Q (x^2 + y^2)\rho(x, y, z)\,dV$

$= 4k\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} r^4\,dz\,dr\,d\theta$

$= 4kh\int_0^{\pi/2}\int_0^{r_0}\frac{r_0 - r}{r_0}r^4\,dr\,d\theta$

$= 4kh\int_0^{\pi/2}\left[\frac{r^5}{5} - \frac{r^6}{6r_0}\right]_0^{r_0} d\theta$

$= 4kh\int_0^{\pi/2}\left[\frac{r_0^5}{5} - \frac{r_0^5}{6}\right] d\theta$

$= 4kh\int_0^{\pi/2}\frac{1}{30}r_0^5\,d\theta$

$= 4kh\frac{1}{30}r_0^5\,\frac{\pi}{2}$

$= \frac{1}{15}r_0^5\pi kh$

**30.** $m = k\pi a^2 h$

$I_z = 2k\int_0^{\pi/2}\int_0^{2a\sin\theta}\int_0^{h} r^3\,dz\,dr\,d\theta$

$= \frac{3}{2}k\pi a^4 h$

$= \frac{3}{2}ma^2$

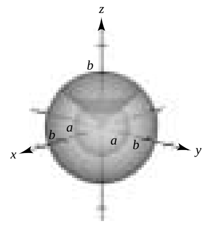
**32.** $V = 8\int_0^{\pi/4}\int_0^{\pi/2}\int_a^b \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$ (includes upper and lower cones)

$$= \frac{8}{3}(b^3 - a^3)\int_0^{\pi/4}\int_0^{\pi/2} \sin\phi \, d\theta \, d\phi$$

$$= \frac{4\pi}{3}(b^3 - a^3)\int_0^{\pi/4} \sin\phi \, d\phi$$

$$= \left[\frac{4\pi}{3}(b^3 - a^3)(-\cos\phi)\right]_0^{\pi/4}$$

$$= \left(1 - \frac{\sqrt{2}}{2}\right)\frac{4\pi}{3}(b^3 - a^3) = \frac{2\pi}{3}\left(2 - \sqrt{2}\right)(b^3 - a^3)$$




**34.** $m = 8k\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \rho^3 \sin^2\phi \, d\rho \, d\theta \, d\phi$

$$= 2ka^4\int_0^{\pi/2}\int_0^{\pi/2} \sin^2\phi \, d\theta \, d\phi$$

$$= k\pi a^4\int_0^{\pi/2} \sin^2\phi \, d\phi$$

$$= \left[k\pi a^4\left(\frac{1}{2}\phi - \frac{1}{4}\sin 2\phi\right)\right]_0^{\pi/2}$$

$$= k\pi a^4 \frac{\pi}{4} = \frac{1}{4}k\pi^2 a^4$$

**36.** $\bar{x} = \bar{y} = 0$ by symmetry

$$m = k\left(\frac{2}{3}\pi R^3 - \frac{2}{3}\pi r^3\right) = \frac{2}{3}k\pi(R^3 - r^3)$$

$$M_{xy} = 4k\int_0^{\pi/2}\int_0^{\pi/2}\int_r^R \rho^3 \cos\phi \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \frac{1}{2}k(R^4 - r^4)\int_0^{\pi/2}\int_0^{\pi/2} \sin 2\phi \, d\theta \, d\phi$$

$$= \frac{1}{4}k\pi(R^4 - r^4)\int_0^{\pi/2} \sin 2\phi \, d\phi$$

$$= \left[-\frac{1}{8}k\pi(R^4 - r^4)\cos 2\phi\right]_0^{\pi/2} = \frac{1}{4}k\pi(R^4 - r^4)$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{k\pi(R^4 - r^4)/4}{2k\pi(R^3 - r^3)/3} = \frac{3(R^4 - r^4)}{8(R^3 - r^3)}$$

**38.** $I_z = 4k\int_0^{\pi/2}\int_0^{\pi/2}\int_r^R \rho^4 \sin^3\phi \, d\rho \, d\theta \, d\phi$

$$= \frac{4k}{5}(R^5 - r^5)\int_0^{\pi/2}\int_0^{\pi/2} \sin^3\phi \, d\theta \, d\phi$$

$$= \frac{2k\pi}{5}(R^5 - r^5)\int_0^{\pi/2} \sin\phi(1 - \cos^2\phi) \, d\phi$$

$$= \left[\frac{2k\pi}{5}(R^5 - r^5)\left(-\cos\phi + \frac{\cos^3\phi}{3}\right)\right]_0^{\pi/2}$$

$$= \frac{4k\pi}{15}(R^5 - r^5)$$

**40.** $x = \rho \sin\phi \cos\theta$ $\qquad \rho^2 = x^2 + y^2 + z^2$

$y = \rho \sin\phi \sin\theta$ $\qquad \tan\theta = \dfrac{y}{x}$

$z = \rho \cos\phi$ $\qquad \cos\phi = \dfrac{z}{\sqrt{x^2 + y^2 + z^2}}$

**42.** $\displaystyle\int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_1}^{\rho_2} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

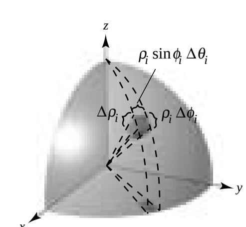
**44.** (a) You are integrating over a cylindrical wedge.

(b) You are integrating over a spherical block.

**46.** The volume of this spherical block can be determined as follows. One side is length $\Delta\rho$. Another side is $\rho\Delta\phi$. Finally, the third side is given by the length of an arc of angle $\Delta\theta$ in a circle of radius $\rho \sin\phi$. Thus:

$$\Delta V \approx (\Delta\rho)(\rho\Delta\phi)(\Delta\theta\rho\sin\phi)$$

$$= \rho^2 \sin\phi\Delta\rho\Delta\phi\Delta\theta$$



# Section 13.8 Change of Variables: Jacobians

**2.** $x = au + bv$

$y = cu + dv$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = ad - cb$$

**4.** $x = uv - 2u$

$y = uv$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = (v - 2)u - vu = -2u$$

**6.** $x = u + a$

$y = v + a$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = (1)(1) - (0)(0) = 1$$

**8.** $x = \dfrac{u}{v}$

$y = u + v$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \left(\frac{1}{v}\right)(1) - (1)\left(-\frac{u}{v^2}\right) = \frac{1}{v} + \frac{u}{v^2} = \frac{u + v}{v^2}$$

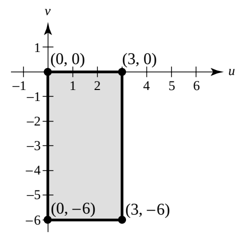
**10.** $x = \dfrac{1}{3}(4u - v)$

$y = \dfrac{1}{3}(u - v)$

$u = x - y$

$v = x - 4y$

| $(x, y)$ | $(u, v)$ |
|---|---|
| $(0, 0)$ | $(0, 0)$ |
| $(4, 1)$ | $(3, 0)$ |
| $(2, 2)$ | $(0, -6)$ |
| $(6, 3)$ | $(3, -6)$ |




**12.** $x = \dfrac{1}{2}(u + v), \qquad u = x - y$

$y = -\dfrac{1}{2}(u - v), \qquad v = x + y$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \frac{1}{2}\left(\frac{1}{2}\right) - \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{2}$$

| $(x, y)$ | $(u, v)$ |
|---|---|
| $(0, 1)$ | $(-1, 1)$ |
| $(2, 1)$ | $(1, 3)$ |
| $(1, 2)$ | $(-1, 3)$ |
| $(1, 0)$ | $(1, 1)$ |

$$\iint_R 60xy\, dA$$

$$= \int_{-1}^{1}\int_{1}^{3} 60\left(\frac{1}{2}(u + v)\right)\left(-\frac{1}{2}(u - v)\right)\left(\frac{1}{2}\right) dv\, du$$

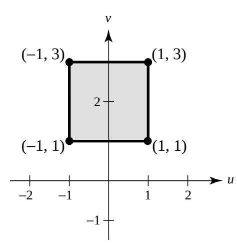
$$= \int_{-1}^{1}\int_{1}^{3} -\frac{15}{2}(v^2 - u^2)\, dv\, du$$

$$= \int_{-1}^{1}\left[-\frac{15}{2}\left(\frac{v^3}{3} - u^2v\right)\right]_1^3 du$$

$$= \int_{-1}^{1}\frac{15}{2}\left(2u^2 - \frac{26}{3}\right) du$$

$$= \left[\frac{15}{2}\left(\frac{2}{3}u^3 - \frac{26}{3}u\right)\right]_{-1}^{1}$$

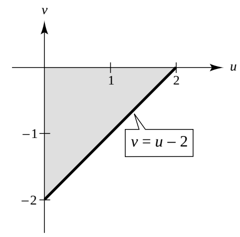
$$= 15\left(\frac{2}{3} - \frac{26}{3}\right) = -120$$



**14.** $x = \frac{1}{2}(u + v)$

$y = \frac{1}{2}(u - v)$

$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = -\frac{1}{2}$




$$\iint_R 4(x + y)e^{x-y}\,dA = \int_0^2\int_{u-2}^0 4ue^v\left(\frac{1}{2}\right)dv\,du$$

$$= \int_0^2 2u(1 - e^{u-2})\,du = 2\left[\frac{u^2}{2} - ue^{u-2} + e^{u-2}\right]_0^2 = 2(1 - e^{-2})$$

**16.** $x = \frac{u}{v}$

$y = v$

$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \frac{1}{v}$

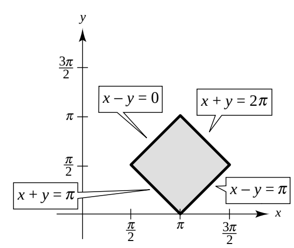
$$\iint_R y\sin xy\,dA = \int_1^4\int_1^4 v(\sin u)\frac{1}{v}\,dv\,du = \int_1^4 3\sin u\,du = \Big[-3\cos u\Big]_1^4 = 3(\cos 1 - \cos 4) \approx 3.5818$$

**18.** $u = x + y = \pi,\qquad v = x - y = 0$

$u = x + y = 2\pi,\qquad v = x - y = \pi$

$x = \frac{1}{2}(u + v),\qquad y = \frac{1}{2}(u - v)$

$\frac{\partial(x, y)}{\partial(u, v)} = -\frac{1}{2}$




$$\iint_R (x + y)^2\sin^2(x - y)\,dA = \int_0^\pi\int_\pi^{2\pi} u^2\sin^2 v\left(\frac{1}{2}\right)du\,dv$$

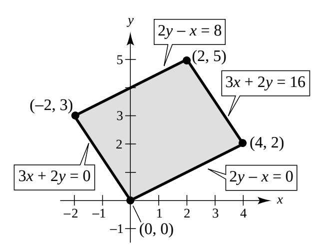
$$= \int_0^\pi\left[\frac{1}{2}\left(\frac{u^3}{3}\right)\frac{1 - \cos 2v}{2}\right]_\pi^{2\pi} dv = \left[\frac{7\pi^3}{12}\left(v - \frac{1}{2}\sin 2v\right)\right]_0^\pi = \frac{7\pi^4}{12}$$

**20.** $u = 3x + 2y = 0,\qquad v = 2y - x = 0$

$u = 3x + 2y = 16,\qquad v = 2y - x = 8$

$x = \frac{1}{4}(u - v),\qquad y = \frac{1}{8}(u + 3v)$

$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \frac{1}{4}\left(\frac{3}{8}\right) - \frac{1}{8}\left(-\frac{1}{4}\right) = \frac{1}{8}$




$$\iint_R (3x + 2y)(2y - x)^{3/2}\,dA = \int_0^8\int_0^{16} uv^{3/2}\left(\frac{1}{8}\right)du\,dv$$

$$= \int_0^8 16v^{3/2}\,dv = \left(\frac{2}{5}\right)16v^{5/2}\Big]_0^8 = \frac{4096}{5}\sqrt{2}$$

**22.** $u = x = 1, \quad v = xy = 1$

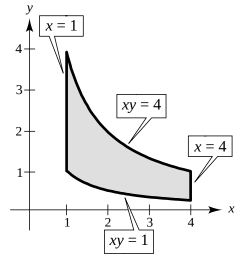
$u = x = 4, \quad v = xy = 4$

$x = u, \quad y = \dfrac{v}{u}$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \frac{1}{u}$$

$$\iint_R \frac{xy}{1 + x^2y^2}\, dA = \int_1^4\int_1^4 \frac{v}{1 + v^2}\left(\frac{1}{u}\right) dv\, du$$

$$= \int_1^4 \left[\frac{1}{2}\ln(1 + v^2)\right]_1^4 \frac{1}{u}\, du = \left[\frac{1}{2}[\ln 17 - \ln 2]\ln u\right]_1^4 = \frac{1}{2}\left(\ln\frac{17}{2}\right)(\ln 4)$$




**24.** (a) $f(x, y) = 16 - x^2 - y^2$

$R: \dfrac{x^2}{16} + \dfrac{y^2}{9} \le 1$

$$V = \iint_R f(x, y)\, dA$$

Let $x = 4u$ and $y = 3v$.

$$\iint_R (16 - x^2 - y^2)\, dA = \int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} (16 - 16u^2 - 9v^2)\, 12 dv\, du \qquad (\text{Let } u = r\cos\theta,\ v = r\sin\theta.)$$

$$= \int_0^{2\pi}\int_0^1 (16 - 16r^2\cos^2\theta - 9r^2\sin^2\theta)\, 12r\, dr\, d\theta$$

$$= 12\int_0^{2\pi}\left[8r^2 - 4r^4\cos^2\theta - \frac{9}{4}r^4\sin^2\theta\right]_0^1 d\theta = 12\int_0^{2\pi}\left[8 - 4\cos^2\theta - \frac{9}{4}\sin^2\theta\right] d\theta$$

$$= 12\int_0^{2\pi}\left[8 - 4\left(\frac{1 + \cos 2\theta}{2}\right) - \frac{9}{4}\left(\frac{1 - \cos 2\theta}{2}\right)\right] d\theta = 12\int_0^{2\pi}\left[\frac{39}{8} - \frac{7}{8}\cos 2\theta\right] d\theta$$

$$= 12\left[\frac{39}{8}\theta - \frac{7}{16}\sin 2\theta\right]_0^{2\pi} = 12\left[\frac{39\pi}{4}\right] = 117\pi$$

(b) $f(x, y) = A\cos\left[\dfrac{\pi}{2}\sqrt{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}}\right]$

$R: \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1$

Let $x = au$ and $y = bv$.

$$\iint_R f(x, y)\, dA = \int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} A\cos\left[\frac{\pi}{2}\sqrt{u^2 + v^2}\right] ab\, dv\, du$$

Let $u = r\cos\theta$, $v = r\sin\theta$.

$$Aab\int_0^{2\pi}\int_0^1 \cos\left[\frac{\pi}{2}r\right] r\, dr\, d\theta = Aab\left[\frac{2r}{\pi}\sin\left(\frac{\pi r}{2}\right) + \frac{4}{\pi^2}\cos\left(\frac{\pi r}{2}\right)\right]_0^1 (2\pi)$$

$$= 2\pi Aab\left[\left(\frac{2}{\pi} + 0\right) - \left(0 + \frac{4}{\pi^2}\right)\right] = \frac{4(\pi - 2)Aab}{\pi}$$

**26.** See Theorem 13.5.

**28.** $x = 4u - v,\ y = 4v - w,\ z = u + w$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 4 & -1 & 0 \\ 0 & 4 & -1 \\ 1 & 0 & 1 \end{vmatrix} = 17$$

**30.** $x = r\cos\theta,\ y = r\sin\theta,\ z = z$

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1[r\cos^2\theta + r\sin^2\theta] = r$$

# Review Exercises for Chapter 13

**2.** $\displaystyle\int_y^{2y} (x^2 + y^2)\,dx = \left[\frac{x^3}{3} + xy^2\right]_y^{2y} = \frac{10y^3}{3}$

**4.** $\displaystyle\int_0^2\int_{x^2}^{2x} (x^2 + 2y)\,dy\,dx = \int_0^2 \left[x^2y + y^2\right]_{x^2}^{2x} dx = \int_0^2 (4x^2 + 2x^3 - 2x^4)\,dx = \left[\frac{4}{3}x^3 + \frac{1}{2}x^4 - \frac{2}{5}x^5\right]_0^2 = \frac{88}{15}$

**6.** $\displaystyle\int_0^{\sqrt{3}}\int_{2-\sqrt{4-y^2}}^{2+\sqrt{4-y^2}} dx\,dy = 2\int_0^{\sqrt{3}} \sqrt{4 - y^2}\,dy = \left[y\sqrt{4 - y^2} + 4\arcsin\frac{y}{2}\right]_0^{\sqrt{3}} = \sqrt{3} + \frac{4\pi}{3}$

**8.** $\displaystyle\int_0^2\int_0^x dy\,dx + \int_2^3\int_0^{6-2x} dy\,dx = \int_0^2\int_y^{(6-y)/2} dx\,dy$

$$A = \int_0^2\int_y^{(6-y)/2} dx\,dy$$
$$= \frac{1}{2}\int_0^2 (6 - 3y)\,dy = \left[\frac{1}{2}\left(6y - \frac{3}{2}y^2\right)\right]_0^2 = 3$$

**10.** $\displaystyle\int_0^4\int_{x^2-2x}^{6x-x^2} dy\,dx = \int_{-1}^0\int_{1-\sqrt{1+y}}^{1+\sqrt{1+y}} dy\,dx + \int_0^8\int_{3-\sqrt{9-y}}^{1+\sqrt{1+y}} dx\,dy + \int_8^9\int_{3-\sqrt{9-y}}^{3+\sqrt{9-y}} dx\,dy$

$$A = \int_0^4\int_{x^2-2x}^{6x-x^2} dy\,dx = \int_0^4 (8x - 2x^2)\,dx = \left[4x^2 - \frac{2}{3}x^3\right]_0^4 = \frac{64}{3}$$

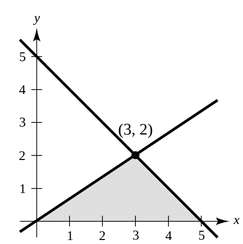
**12.** $\displaystyle A = \int_0^2\int_0^{y^2+1} dx\,dy = \int_0^1\int_0^2 dy\,dx + \int_1^5\int_{\sqrt{x-1}}^2 dy\,dx = \frac{14}{3}$

**14.** $\displaystyle A = \int_0^3\int_{-y}^{2y-y^2} dx\,dy = \int_{-3}^0\int_{-x}^{1+\sqrt{1-x}} dy\,dx + \int_0^1\int_{1-\sqrt{1-x}}^{1+\sqrt{1-x}} dy\,dx = \frac{9}{2}$

**16.** Both integrations are over the common region $R$ shown in the figure. Analytically,

$$\int_0^2\int_{3y/2}^{5-y} e^{x+y}\,dx\,dy = \frac{2}{5} + \frac{8}{5}e^5$$

$$\int_0^3\int_0^{2x/3} e^{x+y}\,dy\,dx + \int_3^5\int_0^{5-x} e^{x+y}\,dy\,dx = \left(\frac{3}{5}e^5 - e^3 + \frac{2}{5}\right) + (e^5 + e^3) = \frac{8}{5}e^5 + \frac{2}{5}$$



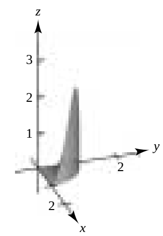
**18.** $V = \int_0^3 \int_0^x (x + y)\, dy\, dx$

$= \int_0^3 \left[ xy + \frac{1}{2}y^2 \right]_0^x dx$

$= \frac{3}{2}\int_0^3 x^2\, dx$

$= \left[ \frac{1}{2}x^3 \right]_0^3 = \frac{27}{2}$

**20.** Matches (c)




**22.** $\int_0^1 \int_0^x kxy\, dy\, dx = \int_0^1 \left[ \frac{kxy^2}{2} \right]_0^x dx$

$= \int_0^1 \frac{kx^3}{2}\, dx$

$= \left[ \frac{kx^4}{8} \right]_0^1 = \frac{k}{8}$

Since $k/8 = 1$, we have $k = 8$.

$P = \int_0^{0.5} \int_0^{0.25} 8xy\, dy\, dx = 0.03125$

**24.** False, $\int_0^1 \int_0^1 x\, dy\, dx \neq \int_1^2 \int_1^2 x\, dy\, dx$

**26.** True, $\int_0^1 \int_0^1 \frac{1}{1 + x^2 + y^2}\, dx\, dy < \int_0^1 \int_0^1 \frac{1}{1 + x^2}\, dx\, dy = \frac{\pi}{4}$

**28.** $\int_0^4 \int_0^{\sqrt{16-y^2}} (x^2 + y^2)\, dx\, dy = \int_0^{\pi/2} \int_0^4 r^3\, dr\, d\theta = \int_0^{\pi/2} \left[ \frac{r^4}{4} \right]_0^4 d\theta = \int_0^{\pi/2} 64\, d\theta = 32\pi$

**30.** $V = 8\int_0^{\pi/2} \int_b^R \sqrt{R^2 - r^2}\, r\, dr\, d\theta$

$= -\frac{8}{3}\int_0^{\pi/2} \left[ (R^2 - r^2)^{3/2} \right]_b^R d\theta$

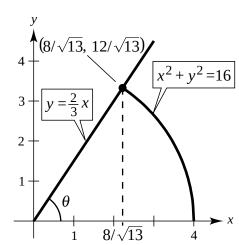
$= \frac{8}{3}(R^2 - b^2)^{3/2} \int_0^{\pi/2} d\theta$

$= \frac{4}{3}\pi(R^2 - b^2)^{3/2}$

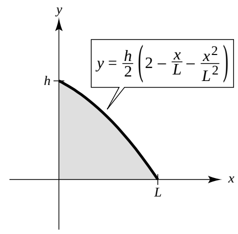
**32.** $\tan\theta = \dfrac{12\sqrt{13}}{8\sqrt{13}} = \dfrac{3}{2} \implies \theta \approx 0.9828$

The polar region is given by $0 \le r \le 4$ and $0 \le \theta \le 0.9828$. Hence,

$$\int_0^{\arctan(3/2)} \int_0^4 (r\cos\theta)(r\sin\theta) r\, dr\, d\theta = \frac{288}{13}$$



**34.** $m = k\int_0^L\int_0^{(h/2)[2-(x/L)-(x^2/L^2)]} dy\,dx = \frac{kh}{2}\int_0^L\left(2 - \frac{x}{L} - \frac{x^2}{L^2}\right)dx = \frac{7khL}{12}$




$$M_x = k\int_0^L\int_0^{(h/2)[2-(x/L)-(x^2/L^2)]} y\,dy\,dx$$

$$= \frac{kh^2}{8}\int_0^L\left(2 - \frac{x}{L} - \frac{x^2}{L^2}\right)^2 dx$$

$$= \frac{kh^2}{8}\int_0^L\left[4 - \frac{4x}{L} - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} + \frac{x^4}{L^4}\right]dx$$

$$= \frac{kh^2}{8}\left[4x - \frac{2x^2}{L} - \frac{x^3}{L^2} + \frac{x^4}{2L^3} + \frac{x^5}{5L^4}\right]_0^L = \frac{kh^2}{8}\cdot\frac{17L}{10} = \frac{17kh^2L}{80}$$

$$M_y = k\int_0^L\int_0^{(h/2)[2-(x/L)-(x^2/L^2)]} x\,dy\,dx$$

$$= \frac{kh}{2}\int_0^L\left(2x - \frac{x^2}{L} - \frac{x^3}{L^2}\right)dx = \frac{kh}{2}\left[x^2 - \frac{x^3}{3L} - \frac{x^4}{4L^2}\right]_0^L = \frac{kh}{2}\cdot\frac{5L^2}{12} = \frac{5khL^2}{24}$$

$$\bar{x} = \frac{M_y}{m} = \frac{5khL^2}{24}\cdot\frac{12}{7khL} = \frac{5L}{14}$$

$$\bar{y} = \frac{M_x}{m} = \frac{17kh^2L}{80}\cdot\frac{12}{7khL} = \frac{51h}{140}$$

**36.** $I_x = \iint_R y^2\,\rho(x, y)\,dA = \int_0^2\int_0^{4-x^2} ky^3\,dy\,dx = \frac{16{,}384}{315}k$

$$I_y = \iint_R x^2\,\rho(x, y)\,dA = \int_0^2\int_0^{4-x^2} kx^2y\,dy\,dx = \frac{512}{105}k$$

$$I_0 = I_x + I_y = \frac{16{,}384k}{315} + \frac{512k}{105} = \frac{17{,}920}{315}k = \frac{512}{9}k$$

$$m = \iint_R \rho(x, y)\,dA = \int_0^2\int_0^{4-x^2} ky\,dy\,dx = \frac{128}{15}k$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{512k/105}{128k/15}} = \sqrt{\frac{4}{7}}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{16{,}384k/315}{128k/15}} = \sqrt{\frac{128}{21}}$$

**38.** $f(x, y) = 16 - x - y^2$

$R = \{(x, y): 0 \le x \le 2,\ 0 \le y \le x\}$

$f_x = -1, f_y = -2y$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{2 + 4y^2}$

$$S = \int_0^2\int_y^2 \sqrt{2 + 4y^2}\,dx\,dy = \int_0^2\left[2\sqrt{2 + 4y^2} - y\sqrt{2 + 4y^2}\right]dy$$

$$= \left[\frac{1}{2}\left(2y\sqrt{2 + 4y^2} + 2\ln\left|2y + \sqrt{2 + 4y^2}\right|\right) - \frac{1}{12}(2 + 4y^2)^{3/2}\right]_0^2$$

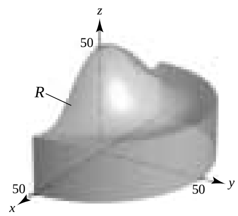
$$= \left[\frac{1}{2}\left(4\sqrt{18} + 2\ln\left|4 + \sqrt{18}\right|\right) - \frac{1}{12}\left(18\sqrt{18}\right)\right] - \left[\ln\sqrt{2} - \frac{2\sqrt{2}}{12}\right]$$

$$= 6\sqrt{2} + \ln\left|4 + 3\sqrt{2}\right| - \frac{9\sqrt{2}}{2} - \ln\sqrt{2} + \frac{\sqrt{2}}{6} = \frac{5\sqrt{2}}{3} + \ln\left|2\sqrt{2} + 3\right|$$

**40.** (a) Graph of

$$f(x, y) = z$$
$$= 25\left[1 + e^{-(x^2+y^2)/1000}\cos^2\left(\frac{x^2+y^2}{1000}\right)\right]$$

over region $R$




(b) Surface area $= \iint_R \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2}\, dA$

Using a symbolic computer program, you obtain surface area $\approx$ 4,540 sq. ft.

**42.** $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{(x^2+y^2)/2}(x^2+y^2)\,dz\,dy\,dx = \int_0^{2\pi}\int_0^2\int_0^{r^2/2} r^3\,dz\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^2 r^5\,dr\,d\theta = \frac{16}{3}\int_0^{2\pi}d\theta = \frac{32\pi}{3}$

**44.** $\displaystyle\int_0^5\int_0^{\sqrt{25-x^2}}\int_0^{\sqrt{25-x^2-y^2}}\frac{1}{1+x^2+y^2+z^2}\,dz\,dy\,dx = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^5\frac{\rho^2}{1+\rho^2}\sin\phi\,d\rho\,d\phi\,d\theta$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\Big[\rho - \arctan\rho\Big]_0^5\sin\phi\,d\phi\,d\theta$$

$$= \int_0^{\pi/2}\Big[(5 - \arctan 5)(-\cos\phi)\Big]_0^{\pi/2}d\theta = \frac{\pi}{2}(5 - \arctan 5)$$

**46.** $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} xyz\,dz\,dy\,dx = \frac{4}{3}$

**48.** $\displaystyle V = 2\int_0^{\pi/2}\int_0^{2\sin\theta}\int_0^{16-r^2} r\,dz\,dr\,d\theta = 2\int_0^{\pi/2}\int_0^{2\sin\theta} r(16 - r^2)\,dr\,d\theta$

$$= 2\int_0^{\pi/2}(32\sin^2\theta - 4\sin^4\theta)\,d\theta = 8\int_0^{\pi/2}(8\sin^2\theta - \sin^4\theta)\,d\theta$$

$$= 8\left[4\theta - 2\sin 2\theta + \frac{1}{4}\sin^3\theta\cos\theta - \frac{3}{4}\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\right]_0^{\pi/2} = \frac{29\pi}{2}$$

**50.** $\displaystyle m = 2k\int_0^{\pi/2}\int_0^a\int_0^{cr\sin\theta} r\,dz\,dr\,d\theta = 2kc\int_0^{\pi/2}\int_0^a r^2\sin\theta\,dr\,d\theta = \frac{2}{3}kca^3\int_0^{\pi/2}\sin\theta\,d\theta = \frac{2}{3}kca^3$

$$M_{xz} = 2k\int_0^{\pi/2}\int_0^a\int_0^{cr\sin\theta} r^2\sin\theta\,dz\,dr\,d\theta = 2kc\int_0^{\pi/2}\int_0^a r^3\sin^2\theta\,dr\,d\theta = \frac{1}{2}kca^4\int_0^{\pi/2}\sin^2\theta\,d\theta = \frac{1}{8}\pi kca^4$$

$$M_{xy} = 2k\int_0^{\pi/2}\int_0^a\int_0^{cr\sin\theta} rz\,dz\,dr\,d\theta = kc^2\int_0^{\pi/2}\int_0^a r^3\sin^2\theta\,dr\,d\theta = \frac{1}{4}kc^2a^4\int_0^{\pi/2}\sin^2\theta\,d\theta = \frac{1}{16}\pi kc^2a^4$$

$$\bar{x} = 0$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{\pi kca^4/8}{2kca^3/3} = \frac{3\pi a}{16}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{\pi kc^2a^4/16}{2kca^3/3} = \frac{3\pi ca}{32}$$

**52.** $m = \frac{500\pi}{3} - \int_0^3 \int_0^{2\pi} \int_4^{\sqrt{25-r^2}} r\,dz\,d\theta\,dr = \frac{500\pi}{3} - \int_0^3 \int_0^{2\pi} \left(r\sqrt{25 - r^2} - 4r\right) d\theta\,dr$

$= \frac{500\pi}{3} - 2\pi\left[-\frac{1}{3}(25 - r^2)^{3/2} - 2r^2\right]_0^3 = \frac{500\pi}{3} - 2\pi\left[-\frac{64}{3} - 18 + \frac{125}{3}\right] = \frac{500\pi}{3} - \frac{14\pi}{3} = 162\pi$

$\bar{x} = \bar{y} = 0$ by symmetry

$M_{xy} = \int_0^{2\pi} \int_0^3 \int_{-\sqrt{25-r^2}}^4 zr\,dz\,dr\,d\theta + \int_0^{2\pi} \int_3^5 \int_{-\sqrt{25-r^2}}^{\sqrt{25-r^2}} zr\,dz\,dr\,d\theta = \int_0^{2\pi} \int_0^3 \left[8 - \frac{1}{2}(25 - r^2)\right] r\,dr\,d\theta + 0$

$= \int_0^{2\pi} \int_0^3 \left[\frac{1}{2}r^3 - \frac{9}{2}r\right] dr\,d\theta = \int_0^{2\pi} \left[\frac{1}{8}r^4 - \frac{9}{4}r^2\right]_0^3 d\theta = \left[-\frac{81}{8}\theta\right]_0^{2\pi} = -\frac{81}{4}\pi$

$\bar{z} = \frac{M_{xy}}{m} = -\frac{81\pi}{4}\frac{1}{162\pi} = -\frac{1}{8}$

**54.** $I_z = k\int_0^{\pi} \int_0^{2\pi} \int_0^a \rho^2 \sin^2 \phi(\rho)\rho^2 \sin \phi\,d\rho\,d\theta\,d\phi$

$= \frac{4k\pi a^6}{9}$

**56.** $x^2 + y^2 + \frac{z^2}{a^2} = 1$

$I_z = \iiint_Q (x^2 + y^2)\,dV$

$= \int_{-a}^{a} \int_{-\sqrt{1-z^2-a^2}}^{\sqrt{1-z^2-a^2}} \int_{-\sqrt{1-y^2-z^2-a^2}}^{\sqrt{1-y^2-z^2-a^2}} (x^2 + y^2)\,dx\,dy\,dz$

$= \frac{8}{15}\pi a$

**58.** $\int_0^{\pi} \int_0^2 \int_0^{1+r^2} r\,dz\,dr\,d\theta$

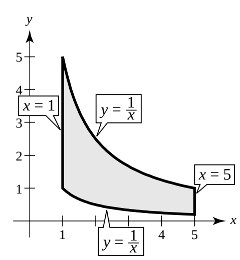
Since $z = 1 + r^2$ represents a paraboloid with vertex (0, 0, 1), this integral represents the volume of the solid below the paraboloid and above the semi-circle $y = \sqrt{4 - x^2}$ in the $xy$-plane.

**60.** $\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$

$= (2u)(-2v) - (2u)(2v) = -8uv$

**62.** $\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = 1\left(\frac{1}{u}\right) - 0 = \frac{1}{u}$

$x = u, y = \frac{v}{u} \Rightarrow u = x, v = xy$




| Boundary in $xy$-plane | Boundary in $uv$-plane |
|---|---|
| $x = 1$ | $u = 1$ |
| $x = 5$ | $u = 5$ |
| $xy = 1$ | $v = 1$ |
| $xy = 5$ | $v = 5$ |

$\iint_R \frac{x}{1 + x^2y^2}\,dA = \int_1^5 \int_1^5 \frac{u}{1 + u^2(v/u)^2}\left(\frac{1}{u}\right) du\,dv = \int_1^5 \int_1^5 \frac{1}{1 + v^2}\,du\,dv = \int_1^5 \frac{4}{1 + v^2}\,dv$

$= 4 \arctan v\Big]_1^5 = 4 \arctan 5 - \pi$