26. See Theorem 13.5.

28. $x = 4u - v,\ y = 4v - w,\ z = u + w$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 4 & -1 & 0 \\ 0 & 4 & -1 \\ 1 & 0 & 1 \end{vmatrix} = 17$$

30. $x = r\cos\theta,\ y = r\sin\theta,\ z = z$

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1[r\cos^2\theta + r\sin^2\theta] = r$$

Review Exercises for Chapter 13

2. $\displaystyle\int_y^{2y} (x^2 + y^2)\,dx = \left[\frac{x^3}{3} + xy^2\right]_y^{2y} = \frac{10y^3}{3}$

4. $\displaystyle\int_0^2\int_{x^2}^{2x} (x^2 + 2y)\,dy\,dx = \int_0^2 \Big[x^2y + y^2\Big]_{x^2}^{2x} dx = \int_0^2 (4x^2 + 2x^3 - 2x^4)\,dx = \left[\frac{4}{3}x^3 + \frac{1}{2}x^4 - \frac{2}{5}x^5\right]_0^2 = \frac{88}{15}$

6. $\displaystyle\int_0^{\sqrt{3}}\int_{2-\sqrt{4-y^2}}^{2+\sqrt{4-y^2}} dx\,dy = 2\int_0^{\sqrt{3}} \sqrt{4 - y^2}\,dy = \left[y\sqrt{4 - y^2} + 4\arcsin\frac{y}{2}\right]_0^{\sqrt{3}} = \sqrt{3} + \frac{4\pi}{3}$

8. $\displaystyle\int_0^2\int_0^x dy\,dx + \int_2^3\int_0^{6-2x} dy\,dx = \int_0^2\int_y^{(6-y)/2} dx\,dy$

$$A = \int_0^2\int_y^{(6-y)/2} dx\,dy$$

$$= \frac{1}{2}\int_0^2 (6 - 3y)\,dy = \left[\frac{1}{2}\left(6y - \frac{3}{2}y^2\right)\right]_0^2 = 3$$

10. $\displaystyle\int_0^4\int_{x^2-2x}^{6x-x^2} dy\,dx = \int_{-1}^0\int_{1-\sqrt{1+y}}^{1+\sqrt{1+y}} dy\,dx + \int_0^8\int_{3-\sqrt{9-y}}^{1+\sqrt{1+y}} dx\,dy + \int_8^9\int_{3-\sqrt{9-y}}^{3+\sqrt{9-y}} dx\,dy$

$$A = \int_0^4\int_{x^2-2x}^{6x-x^2} dy\,dx = \int_0^4 (8x - 2x^2)\,dx = \left[4x^2 - \frac{2}{3}x^3\right]_0^4 = \frac{64}{3}$$

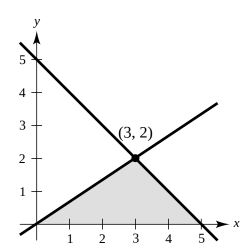
12. $\displaystyle A = \int_0^2\int_0^{y^2+1} dx\,dy = \int_0^1\int_0^2 dy\,dx + \int_1^5\int_{\sqrt{x-1}}^2 dy\,dx = \frac{14}{3}$

14. $\displaystyle A = \int_0^3\int_{-y}^{2y-y^2} dx\,dy = \int_{-3}^0\int_{-x}^{1+\sqrt{1-x}} dy\,dx + \int_0^1\int_{1-\sqrt{1-x}}^{1+\sqrt{1-x}} dy\,dx = \frac{9}{2}$

16. Both integrations are over the common region R shown in the figure. Analytically,

$$\int_0^2\int_{3y/2}^{5-y} e^{x+y}\,dx\,dy = \frac{2}{5} + \frac{8}{5}e^5$$

$$\int_0^3\int_0^{2x/3} e^{x+y}\,dy\,dx + \int_3^5\int_0^{5-x} e^{x+y}\,dy\,dx = \left(\frac{3}{5}e^5 - e^3 + \frac{2}{5}\right) + (e^5 + e^3) = \frac{8}{5}e^5 + \frac{2}{5}$$



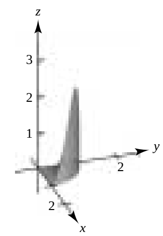
18. $V = \int_0^3 \int_0^x (x + y)\, dy\, dx$

$= \int_0^3 \left[xy + \frac{1}{2}y^2\right]_0^x dx$

$= \frac{3}{2}\int_0^3 x^2\, dx$

$= \left[\frac{1}{2}x^3\right]_0^3 = \frac{27}{2}$

20. Matches (c)



22. $\int_0^1 \int_0^x kxy\, dy\, dx = \int_0^1 \left[\frac{kxy^2}{2}\right]_0^x dx$

$= \int_0^1 \frac{kx^3}{2}\, dx$

$= \left[\frac{kx^4}{8}\right]_0^1 = \frac{k}{8}$

Since $k/8 = 1$, we have $k = 8$.

$P = \int_0^{0.5} \int_0^{0.25} 8xy\, dy\, dx = 0.03125$

24. False, $\int_0^1 \int_0^1 x\, dy\, dx \neq \int_1^2 \int_1^2 x\, dy\, dx$

26. True, $\int_0^1 \int_0^1 \frac{1}{1 + x^2 + y^2}\, dx\, dy < \int_0^1 \int_0^1 \frac{1}{1 + x^2}\, dx\, dy = \frac{\pi}{4}$

28. $\int_0^4 \int_0^{\sqrt{16-y^2}} (x^2 + y^2)\, dx\, dy = \int_0^{\pi/2} \int_0^4 r^3\, dr\, d\theta = \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^4 d\theta = \int_0^{\pi/2} 64\, d\theta = 32\pi$

30. $V = 8\int_0^{\pi/2} \int_b^R \sqrt{R^2 - r^2}\, r\, dr\, d\theta$

$= -\frac{8}{3}\int_0^{\pi/2} \left[(R^2 - r^2)^{3/2}\right]_b^R d\theta$

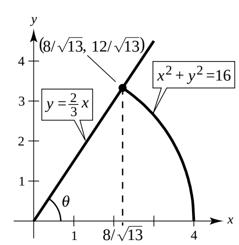
$= \frac{8}{3}(R^2 - b^2)^{3/2} \int_0^{\pi/2} d\theta$

$= \frac{4}{3}\pi(R^2 - b^2)^{3/2}$

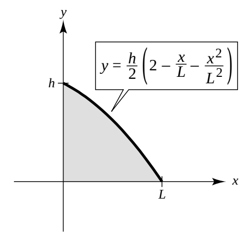
32. $\tan\theta = \frac{12\sqrt{13}}{8\sqrt{13}} = \frac{3}{2} \implies \theta \approx 0.9828$

The polar region is given by $0 \le r \le 4$ and $0 \le \theta \le 0.9828$. Hence,

$$\int_0^{\arctan(3/2)} \int_0^4 (r\cos\theta)(r\sin\theta) r\, dr\, d\theta = \frac{288}{13}$$



34. $m = k\int_0^L\int_0^{(h/2)[2-(x/L)-(x^2/L^2)]} dy\,dx = \frac{kh}{2}\int_0^L\left(2 - \frac{x}{L} - \frac{x^2}{L^2}\right)dx = \frac{7khL}{12}$



$$M_x = k\int_0^L\int_0^{(h/2)[2-(x/L)-(x^2/L^2)]} y\,dy\,dx$$

$$= \frac{kh^2}{8}\int_0^L\left(2 - \frac{x}{L} - \frac{x^2}{L^2}\right)^2 dx$$

$$= \frac{kh^2}{8}\int_0^L\left[4 - \frac{4x}{L} - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} + \frac{x^4}{L^4}\right]dx$$

$$= \frac{kh^2}{8}\left[4x - \frac{2x^2}{L} - \frac{x^3}{L^2} + \frac{x^4}{2L^3} + \frac{x^5}{5L^4}\right]_0^L = \frac{kh^2}{8}\cdot\frac{17L}{10} = \frac{17kh^2L}{80}$$

$$M_y = k\int_0^L\int_0^{(h/2)[2-(x/L)-(x^2/L^2)]} x\,dy\,dx$$

$$= \frac{kh}{2}\int_0^L\left(2x - \frac{x^2}{L} - \frac{x^3}{L^2}\right)dx = \frac{kh}{2}\left[x^2 - \frac{x^3}{3L} - \frac{x^4}{4L^2}\right]_0^L = \frac{kh}{2}\cdot\frac{5L^2}{12} = \frac{5khL^2}{24}$$

$$\bar{x} = \frac{M_y}{m} = \frac{5khL^2}{24}\cdot\frac{12}{7khL} = \frac{5L}{14}$$

$$\bar{y} = \frac{M_x}{m} = \frac{17kh^2L}{80}\cdot\frac{12}{7khL} = \frac{51h}{140}$$

36. $I_x = \iint_R y^2\,\rho(x, y)\,dA = \int_0^2\int_0^{4-x^2} ky^3\,dy\,dx = \frac{16{,}384}{315}k$

$$I_y = \iint_R x^2\,\rho(x, y)\,dA = \int_0^2\int_0^{4-x^2} kx^2y\,dy\,dx = \frac{512}{105}k$$

$$I_0 = I_x + I_y = \frac{16{,}384k}{315} + \frac{512k}{105} = \frac{17{,}920}{315}k = \frac{512}{9}k$$

$$m = \iint_R \rho(x, y)\,dA = \int_0^2\int_0^{4-x^2} ky\,dy\,dx = \frac{128}{15}k$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{512k/105}{128k/15}} = \sqrt{\frac{4}{7}}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{16{,}384k/315}{128k/15}} = \sqrt{\frac{128}{21}}$$

38. $f(x, y) = 16 - x - y^2$

$R = \{(x, y): 0 \le x \le 2,\ 0 \le y \le x\}$

$f_x = -1,\ f_y = -2y$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{2 + 4y^2}$

$$S = \int_0^2\int_y^2 \sqrt{2 + 4y^2}\,dx\,dy = \int_0^2\left[2\sqrt{2 + 4y^2} - y\sqrt{2 + 4y^2}\right]dy$$

$$= \left[\frac{1}{2}\left(2y\sqrt{2 + 4y^2} + 2\ln\left|2y + \sqrt{2 + 4y^2}\right|\right) - \frac{1}{12}(2 + 4y^2)^{3/2}\right]_0^2$$

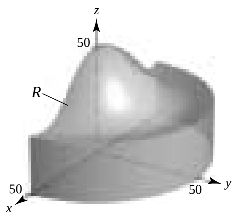
$$= \left[\frac{1}{2}\left(4\sqrt{18} + 2\ln\left|4 + \sqrt{18}\right|\right) - \frac{1}{12}\left(18\sqrt{18}\right)\right] - \left[\ln\sqrt{2} - \frac{2\sqrt{2}}{12}\right]$$

$$= 6\sqrt{2} + \ln\left|4 + 3\sqrt{2}\right| - \frac{9\sqrt{2}}{2} - \ln\sqrt{2} + \frac{\sqrt{2}}{6} = \frac{5\sqrt{2}}{3} + \ln\left|2\sqrt{2} + 3\right|$$

40. (a) Graph of

$$f(x, y) = z = 25\left[1 + e^{-(x^2+y^2)/1000}\cos^2\left(\frac{x^2+y^2}{1000}\right)\right]$$

over region R



(b) Surface area $= \int\int_R \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2}\, dA$

Using a symbolic computer program, you obtain surface area $\approx$ 4,540 sq. ft.

42. $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{(x^2+y^2)/2}(x^2+y^2)\,dz\,dy\,dx = \int_0^{2\pi}\int_0^2\int_0^{r^2/2} r^3\,dz\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^2 r^5\,dr\,d\theta = \frac{16}{3}\int_0^{2\pi}d\theta = \frac{32\pi}{3}$

44.
$$\int_0^5\int_0^{\sqrt{25-x^2}}\int_0^{\sqrt{25-x^2-y^2}}\frac{1}{1+x^2+y^2+z^2}\,dz\,dy\,dx = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^5\frac{\rho^2}{1+\rho^2}\sin\phi\,d\rho\,d\phi\,d\theta$$
$$= \int_0^{\pi/2}\int_0^{\pi/2}\Big[\rho - \arctan\rho\Big]_0^5\sin\phi\,d\phi\,d\theta$$
$$= \int_0^{\pi/2}\Big[(5 - \arctan 5)(-\cos\phi)\Big]_0^{\pi/2}d\theta = \frac{\pi}{2}(5 - \arctan 5)$$

46. $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} xyz\,dz\,dy\,dx = \frac{4}{3}$

48.
$$V = 2\int_0^{\pi/2}\int_0^{2\sin\theta}\int_0^{16-r^2} r\,dz\,dr\,d\theta = 2\int_0^{\pi/2}\int_0^{2\sin\theta} r(16 - r^2)\,dr\,d\theta$$
$$= 2\int_0^{\pi/2}(32\sin^2\theta - 4\sin^4\theta)\,d\theta = 8\int_0^{\pi/2}(8\sin^2\theta - \sin^4\theta)\,d\theta$$
$$= 8\left[4\theta - 2\sin 2\theta + \frac{1}{4}\sin^3\theta\cos\theta - \frac{3}{4}\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\right]_0^{\pi/2} = \frac{29\pi}{2}$$

50.
$$m = 2k\int_0^{\pi/2}\int_0^a\int_0^{cr\sin\theta} r\,dz\,dr\,d\theta = 2kc\int_0^{\pi/2}\int_0^a r^2\sin\theta\,dr\,d\theta = \frac{2}{3}kca^3\int_0^{\pi/2}\sin\theta\,d\theta = \frac{2}{3}kca^3$$
$$M_{xz} = 2k\int_0^{\pi/2}\int_0^a\int_0^{cr\sin\theta} r^2\sin\theta\,dz\,dr\,d\theta = 2kc\int_0^{\pi/2}\int_0^a r^3\sin^2\theta\,dr\,d\theta = \frac{1}{2}kca^4\int_0^{\pi/2}\sin^2\theta\,d\theta = \frac{1}{8}\pi kca^4$$
$$M_{xy} = 2k\int_0^{\pi/2}\int_0^a\int_0^{cr\sin\theta} rz\,dz\,dr\,d\theta = kc^2\int_0^{\pi/2}\int_0^a r^3\sin^2\theta\,dr\,d\theta = \frac{1}{4}kc^2a^4\int_0^{\pi/2}\sin^2\theta\,d\theta = \frac{1}{16}\pi kc^2a^4$$
$$\bar{x} = 0$$
$$\bar{y} = \frac{M_{xz}}{m} = \frac{\pi kca^4/8}{2kca^3/3} = \frac{3\pi a}{16}$$
$$\bar{z} = \frac{M_{xy}}{m} = \frac{\pi kc^2a^4/16}{2kca^3/3} = \frac{3\pi ca}{32}$$

52. $m = \frac{500\pi}{3} - \int_0^3 \int_0^{2\pi} \int_4^{\sqrt{25-r^2}} r\,dz\,d\theta\,dr = \frac{500\pi}{3} - \int_0^3 \int_0^{2\pi} \left(r\sqrt{25-r^2} - 4r\right) d\theta\,dr$

$= \frac{500\pi}{3} - 2\pi\left[-\frac{1}{3}(25-r^2)^{3/2} - 2r^2\right]_0^3 = \frac{500\pi}{3} - 2\pi\left[-\frac{64}{3} - 18 + \frac{125}{3}\right] = \frac{500\pi}{3} - \frac{14\pi}{3} = 162\pi$

$\bar{x} = \bar{y} = 0$ by symmetry

$M_{xy} = \int_0^{2\pi} \int_0^3 \int_{-\sqrt{25-r^2}}^4 zr\,dz\,dr\,d\theta + \int_0^{2\pi} \int_3^5 \int_{-\sqrt{25-r^2}}^{\sqrt{25-r^2}} zr\,dz\,dr\,d\theta = \int_0^{2\pi} \int_0^3 \left[8 - \frac{1}{2}(25-r^2)\right] r\,dr\,d\theta + 0$

$= \int_0^{2\pi} \int_0^3 \left[\frac{1}{2}r^3 - \frac{9}{2}r\right] dr\,d\theta = \int_0^{2\pi} \left[\frac{1}{8}r^4 - \frac{9}{4}r^2\right]_0^3 d\theta = \left[-\frac{81}{8}\theta\right]_0^{2\pi} = -\frac{81}{4}\pi$

$\bar{z} = \frac{M_{xy}}{m} = -\frac{81\pi}{4}\frac{1}{162\pi} = -\frac{1}{8}$

54. $I_z = k\int_0^{\pi} \int_0^{2\pi} \int_0^a \rho^2 \sin^2 \phi(\rho)\rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$

$= \frac{4k\pi a^6}{9}$

56. $x^2 + y^2 + \frac{z^2}{a^2} = 1$

$I_z = \iiint_Q (x^2 + y^2)\,dV$

$= \int_{-a}^{a} \int_{-\sqrt{1-z^2-a^2}}^{\sqrt{1-z^2-a^2}} \int_{-\sqrt{1-y^2-z^2-a^2}}^{\sqrt{1-y^2-z^2-a^2}} (x^2 + y^2)\,dx\,dy\,dz$

$= \frac{8}{15}\pi a$

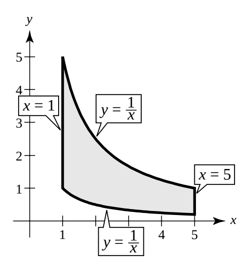
58. $\int_0^{\pi} \int_0^2 \int_0^{1+r^2} r\,dz\,dr\,d\theta$

Since $z = 1 + r^2$ represents a paraboloid with vertex (0, 0, 1), this integral represents the volume of the solid below the paraboloid and above the semi-circle $y = \sqrt{4 - x^2}$ in the xy-plane.

60. $\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$

$= (2u)(-2v) - (2u)(2v) = -8uv$

62. $\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = 1\left(\frac{1}{u}\right) - 0 = \frac{1}{u}$



$x = u, y = \frac{v}{u} \implies u = x, v = xy$

Boundary in xy-plane	Boundary in uv-plane
$x = 1$	$u = 1$
$x = 5$	$u = 5$
$xy = 1$	$v = 1$
$xy = 5$	$v = 5$

$\iint_R \frac{x}{1 + x^2y^2}\,dA = \int_1^5 \int_1^5 \frac{u}{1 + u^2(v/u)^2}\left(\frac{1}{u}\right) du\,dv = \int_1^5 \int_1^5 \frac{1}{1 + v^2}\,du\,dv = \int_1^5 \frac{4}{1 + v^2}\,dv$

$= 4\arctan v\Big]_1^5 = 4\arctan 5 - \pi$

Problem Solving for Chapter 13

2. $z = \frac{1}{c}(d - ax - by)$ Plane

$f_x = -\frac{a}{c},\ f_y = -\frac{b}{c}$

$\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + \frac{a^2}{c^2} + \frac{b^2}{c^2}}$

$$S = \iint_R \sqrt{1 + \frac{a^2}{c^2} + \frac{b^2}{c^2}}\, dA$$

$$= \frac{\sqrt{a^2 + b^2 + c^2}}{c} \iint_R dA$$

$$= \frac{\sqrt{a^2 + b^2 + c^2}}{c} A(R)$$

4. A: $\displaystyle\int_0^{2\pi}\int_4^5 \left(\frac{r}{16} - \frac{r^2}{160}\right) r\, dr\, d\theta = \frac{1333\pi}{960} \approx 4.36 \text{ ft}^3$

$B = \displaystyle\int_0^{2\pi}\int_9^{10} \left(\frac{r}{16} - \frac{r^2}{160}\right) r\, dr\, d\theta = \frac{523\pi}{960} \approx 1.71 \text{ ft}^3$

The distribution is not uniform. Less water in region of greater area.

In one hour, the entire lawn receives

$$\int_0^{2\pi}\int_0^{10} \left(\frac{r}{16} - \frac{r^2}{160}\right) r\, dr\, d\theta = \frac{125\pi}{12} \approx 32.72 \text{ ft}^3.$$

6. (a) $V = \displaystyle\int_0^{2\pi}\int_0^{2}\int_2^{\sqrt{8-r^2}} r\, dz\, dr\, d\theta = \frac{8\pi}{3}\left(4\sqrt{2} - 5\right)$

(b) $V = \displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_{2\sec\phi}^{2\sqrt{2}} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \frac{8\pi}{3}\left(4\sqrt{2} - 5\right)$

8. Volume $\approx [5 + 6 + 5 + 5]4 = 84 \text{ m}^3$

10. Let $v = \ln\left(\frac{1}{x}\right),\ dv = -\frac{dx}{x}$.

$e^v = \frac{1}{x},\ x = e^{-v},\ dx = -e^{-v}\, dv$

$$\int_0^1 \sqrt{\ln(1/x)}\, dx = \int_\infty^0 \sqrt{v}\,(-e^{-v})\, dv = \int_0^\infty \sqrt{v}\, e^{-v}\, dv$$

Let $u = \sqrt{v},\ u^2 = v,\ 2u\, du = dv$.

$$\int_0^1 \sqrt{\ln(1/x)}\, dx = \int_0^\infty u\, e^{-u^2}(2u\, du) = 2\int_0^\infty u^2 e^{-u^2}\, du = 2\left(\frac{\sqrt{\pi}}{4}\right) = \frac{\sqrt{\pi}}{2} \quad \text{(PS \#9)}$$

12. Essay

14. The greater the angle between the given plane and the xy-plane, the greater the surface area. Hence:

$$z_2 < z_1 < z_4 < z_3$$