CHAPTER 13
Multiple Integration

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C H A P T E R 13
Multiple Integration

Section 13.1 Iterated Integrals and Area in the Plane

Solutions to Odd-Numbered Exercises

1. $\int_0^x (2x - y)\,dy = \left[2xy - \frac{1}{2}y^2\right]_0^x = \frac{3}{2}x^2$

3. $\int_1^{2y} \frac{y}{x}\,dx = \left[y \ln x\right]_1^{2y} = y \ln 2y - 0 = y \ln 2y$

5. $\int_0^{\sqrt{4-x^2}} x^2 y\,dy = \left[\frac{1}{2}x^2 y^2\right]_0^{\sqrt{4-x^2}} = \frac{4x^2 - x^4}{2}$

7. $\int_{e^y}^{y} \frac{y \ln x}{x}\,dx = \left[\frac{1}{2}y \ln^2 x\right]_{e^y}^{y} = \frac{1}{2}y[\ln^2 y - \ln^2 e^y] = \frac{y}{2}[(\ln y)^2 - y^2]$

9. $\int_0^{x^3} ye^{-y/x}\,dy = \left[-xye^{-y/x}\right]_0^{x^3} + x\int_0^{x^3} e^{-y/x}\,dy = -x^4 e^{-x^2} - \left[x^2 e^{-y/x}\right]_0^{x^3} = x^2(1 - e^{-x^2} - x^2 e^{-x^2})$

$u = y,\ du = dy,\ dv = e^{-y/x}\,dy,\ v = -xe^{-y/x}$

11. $\int_0^1\int_0^2 (x + y)\,dy\,dx = \int_0^1 \left[xy + \frac{1}{2}y^2\right]_0^2 dx = \int_0^1 (2x + 2)\,dx = \left[x^2 + 2x\right]_0^1 = 3$

13. $\int_0^1\int_0^x \sqrt{1 - x^2}\,dy\,dx = \int_0^1 \left[y\sqrt{1 - x^2}\right]_0^x dx = \int_0^1 x\sqrt{1 - x^2}\,dx = \left[-\frac{1}{2}\left(\frac{2}{3}\right)(1 - x^2)^{3/2}\right]_0^1 = \frac{1}{3}$

15. $\int_1^2\int_0^4 (x^2 - 2y^2 + 1)\,dx\,dy = \int_1^2 \left[\frac{1}{3}x^3 - 2xy^2 + x\right]_0^4 dy$

$= \int_1^2 \left(\frac{64}{3} - 8y^2 + 4\right) dy = \frac{4}{3}\int_1^2 (19 - 6y^2)\,dy = \left[\frac{4}{3}(19y - 2y^3)\right]_1^2 = \frac{20}{3}$

17. $\int_0^1\int_0^{\sqrt{1-y^2}} (x + y)\,dx\,dy = \int_0^1 \left[\frac{1}{2}x^2 + xy\right]_0^{\sqrt{1-y^2}} dy$

$= \int_0^1 \left[\frac{1}{2}(1 - y^2) + y\sqrt{1 - y^2}\right] dy = \left[\frac{1}{2}y - \frac{1}{6}y^3 - \frac{1}{2}\left(\frac{2}{3}\right)(1 - y^2)^{3/2}\right]_0^1 = \frac{2}{3}$

19. $\int_0^2\int_0^{\sqrt{4-y^2}} \frac{2}{\sqrt{4 - y^2}}\,dx\,dy = \int_0^2 \left[\frac{2x}{\sqrt{4 - y^2}}\right]_0^{\sqrt{4-y^2}} dy = \int_0^2 2\,dy = \left[2y\right]_0^2 = 4$

21. $\int_0^{\pi/2}\int_0^{\sin\theta} \theta r\,dr\,d\theta = \int_0^{\pi/2} \left[\theta\frac{r^2}{2}\right]_0^{\sin\theta} d\theta = \int_0^{\pi/2} \frac{1}{2}\theta \sin^2\theta\,d\theta$

$= \frac{1}{4}\int_0^{\pi/2} (\theta - \theta\cos 2\theta)\,d\theta = \frac{1}{4}\left[\frac{\theta^2}{2} - \left(\frac{1}{4}\cos 2\theta + \frac{\theta}{2}\sin 2\theta\right)\right]_0^{\pi/2} = \frac{\pi^2}{32} + \frac{1}{8}$

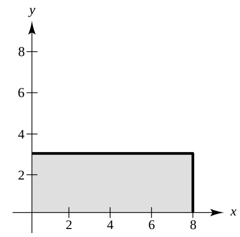
23. $\displaystyle\int_1^{\infty}\int_0^{1/x} y\,dy\,dx = \int_1^{\infty}\left[\frac{y^2}{2}\right]_0^{1/x} dx = \frac{1}{2}\int_1^{\infty}\frac{1}{x^2}\,dx = \left[-\frac{1}{2x}\right]_1^{\infty} = 0 + \frac{1}{2} = \frac{1}{2}$

25. $\displaystyle\int_1^{\infty}\int_1^{\infty}\frac{1}{xy}\,dx\,dy = \int_1^{\infty}\left[\frac{1}{y}\ln x\right]_1^{\infty} dy = \int_1^{\infty}\left[\frac{1}{y}(\infty) - \frac{1}{y}(0)\right] dy$

Diverges

27. $\displaystyle A = \int_0^8\int_0^3 dy\,dx = \int_0^8\Big[y\Big]_0^3 dx = \int_0^8 3\,dx = \Big[3x\Big]_0^8 = 24$

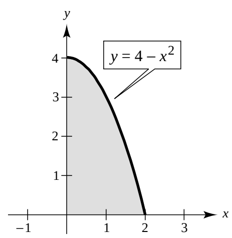
$\displaystyle A = \int_0^3\int_0^8 dx\,dy = \int_0^3\Big[x\Big]_0^8 dy = \int_0^3 8\,dy = \Big[8y\Big]_0^3 = 24$



29. $\displaystyle A = \int_0^2\int_0^{4-x^2} dy\,dx = \int_0^2\Big[y\Big]_0^{4-x^2} dx$

$\displaystyle = \int_0^2 (4 - x^2)\,dx$

$\displaystyle = \left[4x - \frac{x^3}{3}\right]_0^2 = \frac{16}{3}$



$\displaystyle A = \int_0^4\int_0^{\sqrt{4-y}} dx\,dy$

$\displaystyle = \int_0^4\Big[x\Big]_0^{\sqrt{4-y}} dy = \int_0^4 \sqrt{4-y}\,dy = -\int_0^4 (4-y)^{1/2}(-1)\,dy = \left[-\frac{2}{3}(4-y)^{3/2}\right]_0^4 = \frac{2}{3}(8) = \frac{16}{3}$

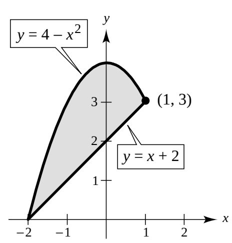
31. $\displaystyle A = \int_{-2}^1\int_{x+2}^{4-x^2} dy\,dx$

$\displaystyle = \int_{-2}^1\Big[y\Big]_{x+2}^{4-x^2} dx$

$\displaystyle = \int_{-2}^1 (4 - x^2 - x - 2)\,dx$

$\displaystyle = \int_{-2}^1 (2 - x - x^2)\,dx$

$\displaystyle = \left[2x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-2}^1 = \frac{9}{2}$



$\displaystyle A = \int_0^3\int_{-\sqrt{4-y}}^{y-2} dx\,dy + 2\int_3^4\int_0^{\sqrt{4-y}} dx\,dy$

$\displaystyle = \int_0^3\Big[x\Big]_{-\sqrt{4-y}}^{y-2} dy + 2\int_3^4\Big[x\Big]_0^{\sqrt{4-y}} dy$

$\displaystyle = \int_0^3\left(y - 2 + \sqrt{4-y}\right) dy + 2\int_3^4 \sqrt{4-y}\,dy$

$\displaystyle = \left[\frac{1}{2}y^2 - 2y - \frac{2}{3}(4-y)^{3/2}\right]_0^3 - \left[\frac{4}{3}(4-y)^{3/2}\right]_3^4 = \frac{9}{2}$

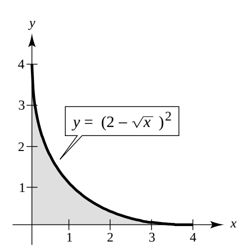
33. $\displaystyle\int_0^4\int_0^{(2-\sqrt{x})^2} dy\,dx = \int_0^4\Big[y\Big]_0^{(2-\sqrt{x})^2} dx$

$\displaystyle = \int_0^4 \left(4 - 4\sqrt{x} + x\right) dx$

$\displaystyle = \left[4x - \frac{8}{3}x\sqrt{x} + \frac{x^2}{2}\right]_0^4 = \frac{8}{3}$

$\displaystyle\int_0^4\int_0^{(2-\sqrt{y})^2} dx\,dy = \frac{8}{3}$

Integration steps are similar to those above.



35. $A = \int_0^3 \int_0^{2x/3} dy\, dx + \int_3^5 \int_0^{5-x} dy\, dx$

$= \int_0^3 \Big[y \Big]_0^{2x/3} dx + \int_3^5 \Big[y \Big]_0^{5-x} dx$

$= \int_0^3 \frac{2x}{3}\, dx + \int_3^5 (5 - x)\, dx$

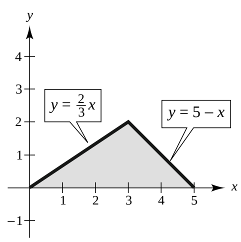
$= \left[\frac{1}{3}x^2\right]_0^3 + \left[5x - \frac{1}{2}x^2\right]_3^5 = 5$

$A = \int_0^2 \int_{3y/2}^{5-y} dx\, dy$

$= \int_0^2 \Big[x \Big]_{3y/2}^{5-y} dy$

$= \int_0^2 \left(5 - y - \frac{3y}{2}\right) dy$

$= \int_2^2 \left(5 - \frac{5y}{2}\right) dy = \left[5y - \frac{5}{4}y^2\right]_0^2 = 5$



37. $\frac{A}{4} = \int_0^a \int_0^{(b/a)\sqrt{a^2-x^2}} dy\, dx = \int_0^a \Big[y \Big]_0^{(b/a)\sqrt{a^2-x^2}} dx$

$= \frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\, dx = ab\int_0^{\pi/2} \cos^2\theta\, d\theta$

$(x = a\sin\theta, dx = a\cos\theta\, d\theta)$

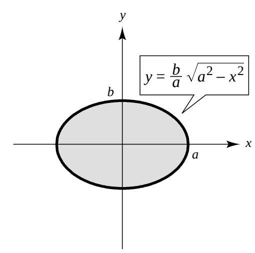
$= \frac{ab}{2}\int_0^{\pi/2} (1 + \cos 2\theta)\, d\theta = \left[\frac{ab}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_0^{\pi/2}$

$= \frac{\pi ab}{4}$

Therefore, $A = \pi ab$.

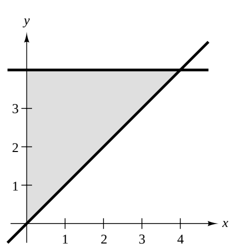
$\frac{A}{4} = \int_0^b \int_0^{(a/b)\sqrt{b^2-y^2}} dx\, dy = \frac{\pi ab}{4}$

Therefore, $A = \pi ab$. Integration steps are similar to those above.



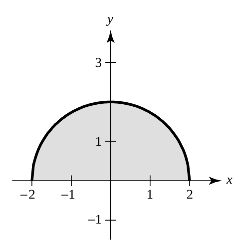
39. $\int_0^4 \int_0^y f(x, y)\, dx\, dy,\ 0 \le x \le y,\ 0 \le y \le 4$

$= \int_0^4 \int_x^4 f(x, y)\, dy\, dx$



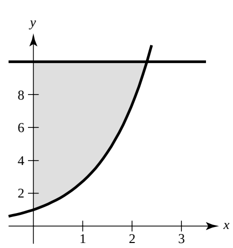
41. $\int_{-2}^2 \int_0^{\sqrt{4-x^2}} f(x, y) dy\, dx, 0 \le y \le \sqrt{4 - x^2}, -2 \le x \le 2$

$= \int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dx\, dy$



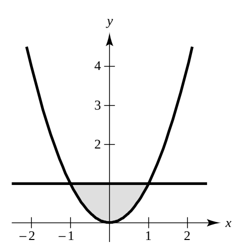
43. $\int_1^{10} \int_0^{\ln y} f(x, y) dx\, dy, 0 \le x \le \ln y, 1 \le y \le 10$

$= \int_0^{\ln 10} \int_{e^x}^{10} f(x, y) dy\, dx$

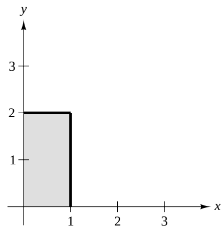


45. $\int_{-1}^1 \int_{x^2}^1 f(x, y)\, dy\, dx, x^2 \le y \le 1, 1 \le x \le 1$

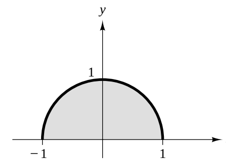
$= \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y)\, dx\, dy$



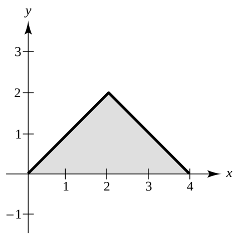
47. $\displaystyle\int_0^1\int_0^2 dy\,dx = \int_0^2\int_0^1 dx\,dy = 2$



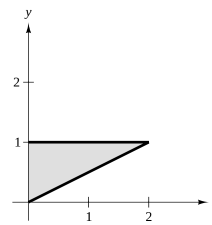
49. $\displaystyle\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\,dy = \int_{-1}^1\int_0^{\sqrt{1-x^2}} dy\,dx = \frac{\pi}{2}$



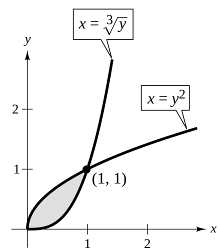
51. $\displaystyle\int_0^2\int_0^x dy\,dx + \int_2^4\int_0^{4-x} dy\,dx = \int_0^2\int_y^{4-y} dx\,dy = 4$



53. $\displaystyle\int_0^2\int_{x/2}^1 dy\,dx = \int_0^1\int_0^{2y} dx\,dy = 1$



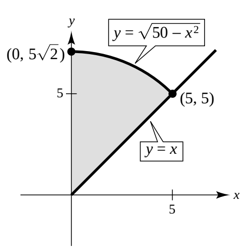
55. $\displaystyle\int_0^1\int_{y^2}^{\sqrt[3]{y}} dx\,dy = \int_0^1\int_{x^3}^{\sqrt{x}} dy\,dx = \frac{5}{12}$



57. The first integral arises using vertical representative rectangles. The second two integrals arise using horizontal representative rectangles.

$$\int_0^5\int_x^{\sqrt{50-x^2}} x^2y^2\,dy\,dx = \int_0^5\left[\frac{1}{3}x^2(50 - x^2)^{3/2} - \frac{1}{3}x^5\right]dx$$
$$= \frac{15625}{24}\pi$$

$$\int_0^5\int_0^y x^2y^2\,dx\,dy + \int_5^{5\sqrt{2}}\int_0^{\sqrt{50-y^2}} x^2y^2\,dx\,dy = \int_0^5 \frac{1}{3}y^5\,dy + \int_5^{5\sqrt{2}} \frac{1}{3}(50 - y^2)^{3/2}\,y^2\,dy = \frac{15625}{18} + \left(\frac{15625}{18}\pi - \frac{15625}{18}\right)$$
$$= \frac{15625}{24}\pi$$



59. $\int_0^2\int_x^2 x\sqrt{1+y^3}\,dy\,dx = \int_0^2\int_0^y x\sqrt{1+y^3}\,dx\,dy = \int_0^2 \left[\sqrt{1+y^3}\cdot\frac{x^2}{2}\right]_0^y dy$

$$= \frac{1}{2}\int_0^2 \sqrt{1+y^3}\,y^2\,dy = \left[\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{2}{3}(1+y^3)^{3/2}\right]_0^2 = \frac{1}{9}(27) - \frac{1}{9}(1) = \frac{26}{9}$$

61. $\int_0^1\int_y^1 \sin(x^2)\,dx\,dy = \int_0^1\int_0^x \sin(x^2)\,dy\,dx = \int_0^1\left[y\sin(x^2)\right]_0^x dx$

$$= \int_0^1 x\sin(x^2)\,dx = \left[-\frac{1}{2}\cos(x^2)\right]_0^1 = -\frac{1}{2}\cos 1 + \frac{1}{2}(1) = \frac{1}{2}(1-\cos 1) \approx 0.2298$$

63. $\int_0^2\int_{x^2}^{2x}(x^3+3y^2)\,dy\,dx = \frac{1664}{105} \approx 15.848$

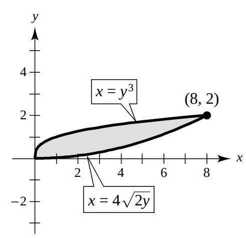
65. $\int_0^4\int_0^y \frac{2}{(x+1)(y+1)}\,dx\,dy = (\ln 5)^2 \approx 2.590$

67. (a) $x = y^3 \Leftrightarrow y = x^{1/3}$

$x = 4\sqrt{2y} \Leftrightarrow x^2 = 32y \Leftrightarrow y = \frac{x^2}{32}$

(b) $\int_0^8\int_{x^2/32}^{x^{1/3}}(x^2y - xy^2)\,dy\,dx$

(c) Both integrals equal $67520/693 \approx 97.43$



69. $\int_0^2\int_0^{4-x^2} e^{xy}\,dy\,dx \approx 20.5648$

71. $\int_0^{2\pi}\int_0^{1+\cos\theta} 6r^2\cos\theta\,dr\,d\theta = \frac{15\pi}{2}$

73. An iterated integral is a double integral of a function of two variables. First integrate with respect to one variable while holding the other variable constant. Then integrate with respect to the second variable.

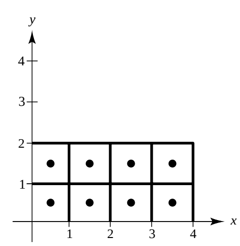
75. The region is a rectangle.

77. True

Section 13.2 Double Integrals and Volume

For Exercise 1–3, $\Delta x_i = \Delta y_i = 1$ and the midpoints of the squares are

$$\left(\frac{1}{2},\frac{1}{2}\right), \left(\frac{3}{2},\frac{1}{2}\right), \left(\frac{5}{2},\frac{1}{2}\right), \left(\frac{7}{2},\frac{1}{2}\right), \left(\frac{1}{2},\frac{3}{2}\right), \left(\frac{3}{2},\frac{3}{2}\right), \left(\frac{5}{2},\frac{3}{2}\right), \left(\frac{7}{2},\frac{3}{2}\right).$$



1. $f(x, y) = x + y$

$$\sum_{i=1}^{8} f(x_i, y_i)\,\Delta x_i\,\Delta y_i = 1 + 2 + 3 + 4 + 2 + 3 + 4 + 5 = 24$$

$$\int_0^4\int_0^2 (x+y)\,dy\,dx = \int_0^4\left[xy + \frac{y^2}{2}\right]_0^2 dx = \int_0^4 (2x+2)\,dx = \left[x^2+2x\right]_0^4 = 24$$

3. $f(x, y) = x^2 + y^2$

$$\sum_{i=1}^{8} f(x_i, y_i)\,\Delta x_i\,\Delta y_i = \frac{2}{4} + \frac{10}{4} + \frac{26}{4} + \frac{50}{4} + \frac{10}{4} + \frac{18}{4} + \frac{34}{4} + \frac{58}{4} = 52$$

$$\int_0^4 \int_0^2 (x^2 + y^2)\,dy\,dx = \int_0^4 \left[x^2y + \frac{y^3}{3}\right]_0^2 dx = \int_0^4 \left(2x^2 + \frac{8}{3}\right) dx = \left[\frac{2x^3}{3} + \frac{8x}{3}\right]_0^4 = \frac{160}{3}$$

5. $$\int_0^4 \int_0^4 f(x, y)dy\,dx \approx (32 + 31 + 28 + 23) + (31 + 30 + 27 + 22) + (28 + 27 + 24 + 19) + (23 + 22 + 19 + 14)$$

$$= 400$$

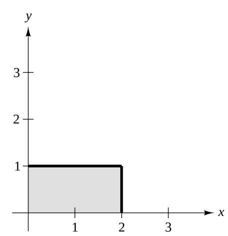
Using the corner of the ith square furthest from the origin, you obtain 272.

7. $$\int_0^2 \int_0^1 (1 + 2x + 2y)\,dy\,dx = \int_0^2 \left[y + 2xy + y^2\right]_0^1 dx$$

$$= \int_0^2 (2 + 2x)\,dx$$

$$= \left[2x + x^2\right]_0^2$$

$$= 8$$

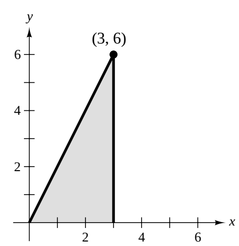


9. $$\int_0^6 \int_{y/2}^3 (x + y)\,dx\,dy = \int_0^6 \left[\frac{1}{2}x^2 + xy\right]_{y/2}^3 dy$$

$$= \int_0^6 \left(\frac{9}{2} + 3y - \frac{5}{8}y^2\right) dy$$

$$= \left[\frac{9}{2}y + \frac{3}{2}y^2 - \frac{5}{24}y^3\right]_0^6$$

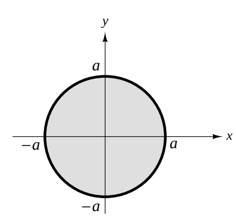
$$= 36$$



11. $$\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (x + y)\,dy\,dx = \int_{-a}^{a} \left[xy + \frac{1}{2}y^2\right]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx$$

$$= \int_{-a}^{a} 2x\sqrt{a^2 - x^2}\,dx$$

$$= \left[-\frac{2}{3}(a^2 - x^2)^{3/2}\right]_{-a}^{a} = 0$$

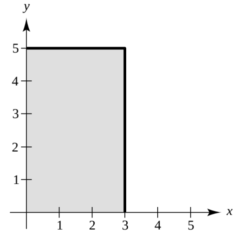


13. $$\int_0^5 \int_0^3 xy\,dx\,dy = \int_0^3 \int_0^5 x\,y\,dy\,dx$$

$$= \int_0^3 \left[\frac{1}{2}x\,y^2\right]_0^5 dx$$

$$= \frac{25}{2}\int_0^3 x\,dx$$

$$= \left[\frac{25}{4}x^2\right]_0^3 = \frac{225}{4}$$



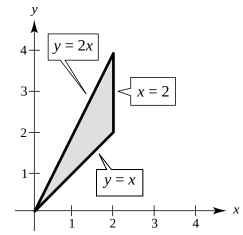
15. $\displaystyle\int_0^2\int_{y/2}^{y}\frac{y}{x^2+y^2}\,dx\,dy + \int_2^4\int_{y/2}^{2}\frac{y}{x^2+y^2}\,dx\,dy = \int_0^2\int_x^{2x}\frac{y}{x^2+y^2}\,dy\,dx$

$$= \frac{1}{2}\int_0^2\Big[\ln(x^2+y^2)\Big]_x^{2x}\,dx$$

$$= \frac{1}{2}\int_0^2(\ln 5x^2 - \ln 2x^2)\,dx$$

$$= \frac{1}{2}\ln\frac{5}{2}\int_0^2 dx$$

$$= \left[\frac{1}{2}\left(\ln\frac{5}{2}\right)x\right]_0^2 = \ln\frac{5}{2}$$

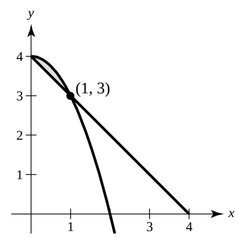


17. $\displaystyle\int_3^4\int_{4-y}^{\sqrt{4-y}} -2y\ln x\,dx\,dy = \int_0^1\int_{4-x}^{4-x^2} -2y\ln x\,dy\,dx$

$$= -\int_0^1\Big[\ln x\cdot y^2\Big]_{4-x}^{4-x^2}\,dx$$

$$= -\int_0^1[\ln x[(4-x^2)^2-(4-x)^2]]\,dx$$

$$= \frac{26}{25}$$

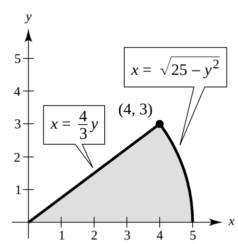


19. $\displaystyle\int_0^4\int_0^{3x/4} x\,dy\,dx + \int_4^5\int_0^{\sqrt{25-x^2}} x\,dy\,dx = \int_0^3\int_{4y/3}^{\sqrt{25-y^2}} x\,dx\,dy$

$$= \int_0^3\left[\frac{1}{2}x^2\right]_{4y/3}^{\sqrt{25-y^2}}\,dy$$

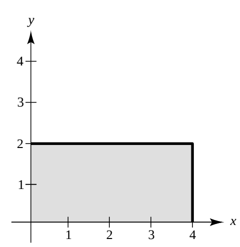
$$= \frac{25}{18}\int_0^3(9-y^2)\,dy$$

$$= \left[\frac{25}{18}\left(9y-\frac{1}{3}y^3\right)\right]_0^3 = 25$$



21. $\displaystyle\int_0^4\int_0^2\frac{y}{2}\,dy\,dx = \int_0^4\left[\frac{y^2}{4}\right]_0^2 dx$

$$= \int_0^4 dx = 4$$

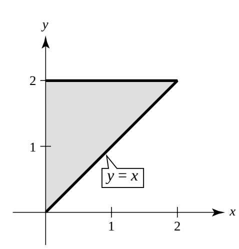


23. $\displaystyle\int_0^2\int_0^y(4-x-y)dx\,dy = \int_0^2\left[4x-\frac{x^2}{2}-xy\right]_0^y dy$

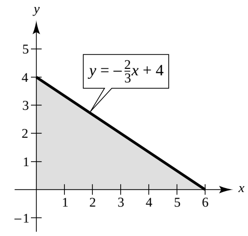
$$= \int_0^2\left(4y-\frac{y^2}{2}-y^2\right)dy$$

$$= \left[2y^2-\frac{y^3}{6}-\frac{y^3}{3}\right]_0^2$$

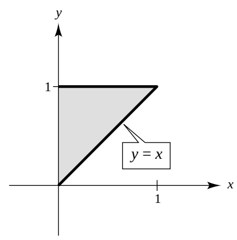
$$= 8-\frac{8}{6}-\frac{8}{3} = 4$$



25. $$\int_0^6 \int_0^{(-2/3)x+4} \left(\frac{12 - 2x - 3y}{4}\right) dy\,dx = \int_0^6 \left[\frac{1}{4}\left(12y - 2xy - \frac{3}{2}y^2\right)\right]_0^{(-2/3)x+4} dx$$
$$= \int_0^6 \left(\frac{1}{6}x^2 - 2x + 6\right) dx$$
$$= \left[\frac{1}{18}x^3 - x^2 + 6x\right]_0^6$$
$$= 12$$



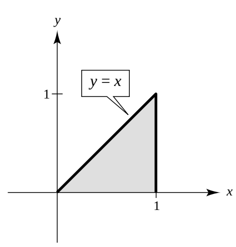
27. $$\int_0^1 \int_0^y (1 - xy)\,dx\,dy = \int_0^1 \left[x - \frac{x^2y}{2}\right]_0^y dy$$
$$= \int_0^1 \left(y - \frac{y^3}{2}\right) dy$$
$$= \left[\frac{y^2}{2} - \frac{y^4}{8}\right]_0^1$$
$$= \frac{3}{8}$$



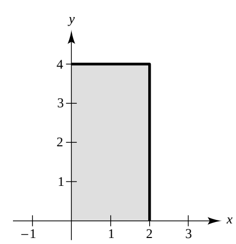
29. $$\int_0^\infty \int_0^\infty \frac{1}{(x+1)^2(y+1)^2}\,dy\,dx = \int_0^\infty \left[-\frac{1}{(x+1)^2(y+1)}\right]_0^\infty dx = \int_0^\infty \frac{1}{(x+1)^2}\,dx = \left[-\frac{1}{(x+1)}\right]_0^\infty = 1$$

31. $$4\int_0^2 \int_0^{\sqrt{4-x^2}} (4 - x^2 - y^2)\,dy\,dx = 8\pi$$

33. $$V = \int_0^1 \int_0^x xy\,dy\,dx$$
$$= \int_0^1 \left[\frac{1}{2}xy^2\right]_0^x dx = \frac{1}{2}\int_0^1 x^3\,dx$$
$$= \left[\frac{1}{8}x^4\right]_0^1 = \frac{1}{8}$$

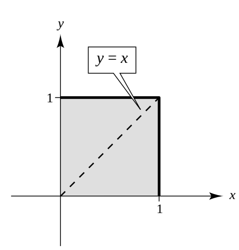


35. $$V = \int_0^2 \int_0^4 x^2\,dy\,dx$$
$$= \int_0^2 \left[x^2y\right]_0^4 dx = \int_0^2 4x^2\,dx$$
$$= \left[\frac{4x^3}{3}\right]_0^2 = \frac{32}{3}$$



37. Divide the solid into two equal parts.

$$V = 2\int_0^1 \int_0^x \sqrt{1 - x^2}\,dy\,dx$$
$$= 2\int_0^1 \left[y\sqrt{1 - x^2}\right]_0^x dx$$
$$= 2\int_0^1 x\sqrt{1 - x^2}\,dx$$
$$= \left[-\frac{2}{3}(1 - x^2)^{3/2}\right]_0^1 = \frac{2}{3}$$

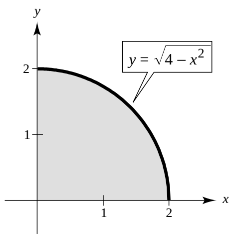


39. $V = \int_0^2 \int_0^{\sqrt{4-x^2}} (x + y)\, dy\, dx$

$= \int_0^2 \left[xy + \frac{1}{2}y^2\right]_0^{\sqrt{4-x^2}} dx$

$= \int_0^2 \left(x\sqrt{4 - x^2} + 2 - \frac{1}{2}x^2\right) dx$

$= \left[-\frac{1}{3}(4 - x^2)^{3/2} + 2x - \frac{1}{6}x^3\right]_0^2 = \frac{16}{3}$



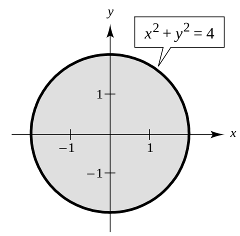
41. $V = 4\int_0^2 \int_0^{\sqrt{4-x^2}} (x^2 + y^2)\, dy\, dx$

$= 4\int_0^2 \left[x^2\sqrt{4 - x^2} + \frac{1}{3}(4 - x^2)^{3/2}\right] dx, \quad x = 2 \sin \theta$

$= 4\int_0^{\pi/2} \left(16 \cos^2 \theta - \frac{32}{3} \cos^4 \theta\right) d\theta$

$= 4\left[16\left(\frac{\pi}{4}\right) - \frac{32}{3}\left(\frac{3\pi}{16}\right)\right]$

$= 8\pi$



43. $V = 4\int_0^2 \int_0^{\sqrt{4-x^2}} (4 - x^2 - y^2)\, dy\, dx = 8\pi$

45. $V = \int_0^2 \int_0^{-0.5x+1} \frac{2}{1 + x^2 + y^2}\, dy\, dx \approx 1.2315$

47. f is a continuous function such that $0 \le f(x, y) \le 1$ over a region R of area 1. Let $f(m, n) =$ the minimum value of f over R and $f(M, N) =$ the maximum value of f over R. Then

$$f(m, n)\iint_R dA \le \iint_R f(x, y)\, dA \le f(M, N)\iint_R dA.$$

Since $\iint_R dA = 1$ and $0 \le f(m, n) \le f(M, N) \le 1$, we have $0 \le f(m, n)(1) \le \iint_R f(x, y)\, dA \le f(M, N)(1) \le 1$.

Therefore, $0 \le \iint_R f(x, y)\, dA \le 1$.

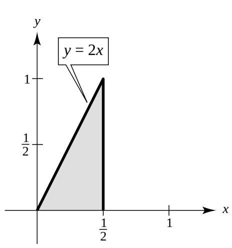
49. $\int_0^1 \int_{y/2}^{1/2} e^{-x^2}\, dx\, dy = \int_0^{1/2} \int_0^{2x} e^{-x^2}\, dy\, dx$

$= \int_0^{1/2} 2xe^{-x^2}\, dx$

$= \left[-e^{-x^2}\right]_0^{1/2}$

$= -e^{-1/4} + 1$

$= 1 - e^{-1/4} \approx 0.221$

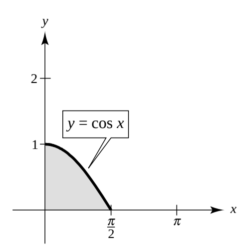


51. $\int_0^1 \int_0^{\arccos y} \sin x\sqrt{1 + \sin^2 x}\, dx\, dy$

$= \int_0^{\pi/2} \int_0^{\cos x} \sin x\sqrt{1 + \sin^2 x}\, dy\, dx$

$= \int_0^{\pi/2} (1 + \sin^2 x)^{1/2} \sin x \cos x\, dx$

$= \left[\frac{1}{2} \cdot \frac{2}{3}(1 + \sin^2 x)^{3/2}\right]_0^{\pi/2} = \frac{1}{3}\left[2\sqrt{2} - 1\right]$



53. Average $= \frac{1}{8}\int_0^4\int_0^2 x\,dy\,dx = \frac{1}{8}\int_0^4 2x\,dx = \left[\frac{x^2}{8}\right]_0^4 = 2$

55. Average $= \frac{1}{4}\int_0^2\int_0^2 (x^2 + y^2)\,dx\,dy$

$$= \frac{1}{4}\int_0^2 \left[\frac{x^3}{3} + xy^2\right]_0^2 dy = \frac{1}{4}\int_0^2 \left(\frac{8}{3} + 2y^2\right) dy$$

$$= \left[\frac{1}{4}\left(\frac{8}{3}y + \frac{2}{3}y^3\right)\right]_0^2 = \frac{8}{3}$$

57. See the definition on page 946.

59. The value of $\int_R\int f(x, y)\,dA$ would be kB.

61. Average $= \frac{1}{1250}\int_{300}^{325}\int_{200}^{250} 100x^{0.6}y^{0.4}\,dx\,dy$

$$= \frac{1}{1250}\int_{300}^{325}\left[(100y^{0.4})\frac{x^{1.6}}{1.6}\right]_{200}^{250} dy = \frac{128{,}844.1}{1250}\int_{300}^{325} y^{0.4}\,dy = 103.0753\left[\frac{y^{1.4}}{1.4}\right]_{300}^{325} \approx 25{,}645.24$$

63. $f(x, y) \geq 0$ for all (x, y) and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dA = \int_0^5\int_0^2 \frac{1}{10}\,dy\,dx = \int_0^5 \frac{1}{5}\,dx = 1$$

$$P(0 \leq x \leq 2, 1 \leq y \leq 2) = \int_0^2\int_1^2 \frac{1}{10}\,dy\,dx = \int_0^2 \frac{1}{10}\,dx = \frac{1}{5}.$$

65. $f(x, y) \geq 0$ for all (x, y) and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dA = \int_0^3\int_3^6 \frac{1}{27}(9 - x - y)\,dy\,dx$$

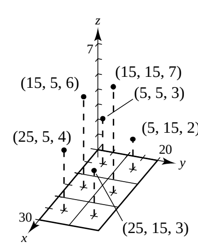
$$= \int_0^3 \frac{1}{27}\left[9y - xy - \frac{y^2}{2}\right]_3^6 dx = \int_0^3 \left(\frac{1}{2} - \frac{1}{9}x\right) dx = \left[\frac{x}{2} - \frac{x^2}{18}\right]_0^3 = 1$$

$$P(0 \leq x \leq 1, 4 \leq y \leq 6) = \int_0^1\int_4^6 \frac{1}{27}(9 - x - y)\,dy\,dx = \int_0^1 \frac{2}{27}(4 - x)\,dx = \frac{7}{27}.$$

67. Divide the base into six squares, and assume the height at the center of each square is the height of the entire square.

Thus,

$$V \approx (4 + 3 + 6 + 7 + 3 + 2)(100) = 2500m^3.$$



69. $\int_0^1\int_0^2 \sin\sqrt{x + y}\,dy\,dx \qquad m = 4, n = 8$

(a) 1.78435

(b) 1.7879

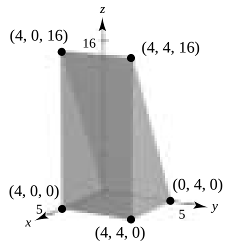
71. $\int_4^6\int_0^2 y\cos\sqrt{x}\,dx\,dy \qquad m = 4, n = 8$

(a) 11.0571

(b) 11.0414

73. $V \approx 125$

Matches d.



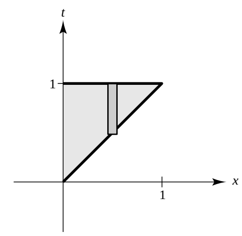
75. False

$$V = 8\int_0^1 \int_0^{\sqrt{1-y^2}} \sqrt{1 - x^2 - y^2}\, dx\, dy$$

77. $$\text{Average} = \int_0^1 f(x)\, dx = \int_0^1 \int_1^x e^{t^2}\, dt\, dx = -\int_0^1 \int_x^1 e^{t^2}\, dt\, dx$$

$$= -\int_0^1 \int_0^t e^{t^2}\, dx\, dt = -\int_0^1 te^{t^2}\, dt$$

$$= \left[-\frac{1}{2}e^{t^2}\right]_0^1 = -\frac{1}{2}(e - 1) = \frac{1}{2}(1 - e)$$



Section 13.3 Change of Variables: Polar Coordinates

1. Rectangular coordinates

3. Polar coordinates

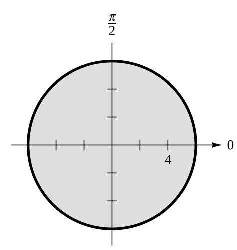
5. $R = \{(r, \theta)\colon 0 \le r \le 8, 0 \le \theta \le \pi\}$

7. $R = \{(r, \theta)\colon 0 \le r \le 3 + 3\sin\theta, 0 \le \theta \le 2\pi\}$ Cardioid

9. $$\int_0^{2\pi}\int_0^6 3r^2 \sin\theta\, dr\, d\theta = \int_0^{2\pi} \Big[r^3 \sin\theta\Big]_0^6\, d\theta$$

$$= \int_0^{2\pi} 216 \sin\theta\, d\theta$$

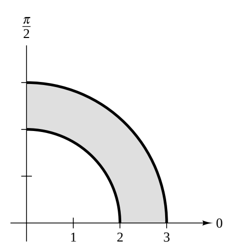
$$= \Big[-216\cos\theta\Big]_0^{2\pi} = 0$$



11. $$\int_0^{\pi/2}\int_2^3 \sqrt{9 - r^2}\, r\, dr\, d\theta = \int_0^{\pi/2} \left[-\frac{1}{3}(9 - r^2)^{3/2}\right]_2^3 d\theta$$

$$= \left[\frac{5\sqrt{5}}{3}\theta\right]_0^{\pi/2}$$

$$= \frac{5\sqrt{5}\pi}{6}$$

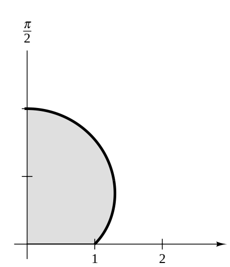


13. $$\int_0^{\pi/2}\int_0^{1+\sin\theta} \theta r\, dr\, d\theta = \int_0^{\pi/2} \left[\frac{\theta r^2}{2}\right]_0^{1+\sin\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}\theta(1 + \sin\theta)^2\, d\theta$$

$$= \left[\frac{1}{8}\theta^2 + \sin\theta - \theta\cos\theta + \frac{1}{2}\theta\left(-\frac{1}{2}\cos\theta\cdot\sin\theta + \frac{1}{2}\theta\right) + \frac{1}{8}\sin^2\theta\right]_0^{\pi/2}$$

$$= \frac{3}{32}\pi^2 + \frac{9}{8}$$



15. $\int_0^a \int_0^{\sqrt{a^2-y^2}} y\,dx\,dy = \int_0^{\pi/2}\int_0^a r^2 \sin\theta\,dr\,d\theta = \frac{a^3}{3}\int_0^{\pi/2} \sin\theta\,d\theta = \left[\frac{a^3}{3}(-\cos\theta)\right]_0^{\pi/2} = \frac{a^3}{3}$

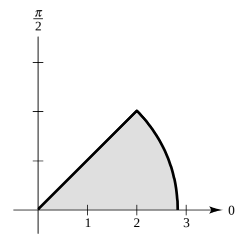
17. $\int_0^3 \int_0^{\sqrt{9-x^2}} (x^2+y^2)^{3/2}\,dy\,dx = \int_0^{\pi/2}\int_0^3 r^4\,dr\,d\theta = \frac{243}{5}\int_0^{\pi/2} d\theta = \frac{243\pi}{10}$

19. $\int_0^2 \int_0^{\sqrt{2x-x^2}} xy\,dy\,dx = \int_0^{\pi/2}\int_0^{2\cos\theta} r^3\cos\theta\sin\theta\,dr\,d\theta = 4\int_0^{\pi/2}\cos^5\theta\sin d\theta = \left[-\frac{4\cos^6\theta}{6}\right]_0^{\pi/2} = \frac{2}{3}$

21. $\int_0^2 \int_0^x \sqrt{x^2+y^2}\,dy\,dx + \int_2^{2\sqrt{2}}\int_0^{\sqrt{8-x^2}} \sqrt{x^2+y^2}\,dy\,dx = \int_0^{\pi/4}\int_0^{2\sqrt{2}} r^2\,dr\,d\theta$

$= \int_0^{\pi/4} \frac{16\sqrt{2}}{3}\,d\theta$

$= \frac{4\sqrt{2}\pi}{3}$



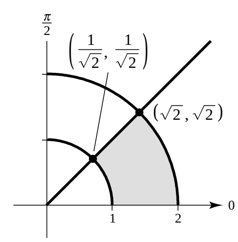
23. $\int_0^2 \int_0^{\sqrt{4-x^2}} (x+y)\,dy\,dx = \int_0^{\pi/2}\int_0^2 (r\cos\theta + r\sin\theta)r\,dr\,d\theta = \int_0^{\pi/2}\int_0^2 (\cos\theta+\sin\theta)r^2\,dr\,d\theta$

$= \frac{8}{3}\int_0^{\pi/2}(\cos\theta+\sin\theta)\,d\theta = \left[\frac{8}{3}(\sin\theta - \cos\theta)\right]_0^{\pi/2} = \frac{16}{3}$

25. $\int_0^{1/\sqrt{2}} \int_{\sqrt{1-y^2}}^{\sqrt{4-y^2}} \arctan\frac{y}{x}\,dx\,dy + \int_{1/\sqrt{2}}^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} \arctan\frac{y}{x}\,dx\,dy$

$= \int_0^{\pi/4}\int_1^2 \theta r\,dr\,d\theta$

$= \int_0^{\pi/4} \frac{3}{2}\theta\,d\theta = \left[\frac{3\theta^2}{4}\right]_0^{\pi/4} = \frac{3\pi^2}{64}$



27. $V = \int_0^{\pi/2}\int_0^1 (r\cos\theta)(r\sin\theta)r\,dr\,d\theta$

$= \frac{1}{2}\int_0^{\pi/2}\int_0^1 r^3\sin 2\theta\,dr\,d\theta = \frac{1}{8}\int_0^{\pi/2}\sin 2\theta\,d\theta = \left[-\frac{1}{16}\cos 2\theta\right]_0^{\pi/2} = \frac{1}{8}$

29. $V = \int_0^{2\pi}\int_0^5 r^2\,dr\,d\theta = \frac{250\pi}{3}$

31. $V = 2\int_0^{\pi/2}\int_0^{4\cos\theta} \sqrt{16-r^2}\,r\,dr\,d\theta = 2\int_0^{\pi/2}\left[-\frac{1}{3}\left(\sqrt{16-r^2}\right)^3\right]_0^{4\cos\theta} d\theta = -\frac{2}{3}\int_0^{\pi/2}(64\sin^3\theta - 64)\,d\theta$

$= \frac{128}{3}\int_0^{\pi/2}[1 - \sin\theta(1-\cos^2\theta)]\,d\theta = \frac{128}{3}\left[\theta + \cos\theta - \frac{\cos^3\theta}{3}\right]_0^{\pi/2} = \frac{64}{9}(3\pi - 4)$

33. $V = \int_0^{2\pi}\int_a^4 \sqrt{16 - r^2}\, r\, dr\, d\theta = \int_0^{2\pi}\left[-\frac{1}{3}\left(\sqrt{16 - r^2}\right)^3\right]_a^4 d\theta = \frac{1}{3}\left(\sqrt{16 - a^2}\right)^3(2\pi)$

One-half the volume of the hemisphere is $(64\pi)/3$.

$$\frac{2\pi}{3}(16 - a^2)^{3/2} = \frac{64\pi}{3}$$
$$(16 - a^2)^{3/2} = 32$$
$$16 - a^2 = 32^{2/3}$$
$$a^2 = 16 - 32^{2/3} = 16 - 8\sqrt[3]{2}$$
$$a = \sqrt{4\left(4 - 2\sqrt[3]{2}\right)} = 2\sqrt{4 - 2\sqrt[3]{2}} \approx 2.4332$$

35. Total Volume $= V = \int_0^{2\pi}\int_0^4 25e^{-r^2/4}\, r\, dr\, d\theta$

$$= \int_0^{2\pi}\left[-50e^{-r^2/4}\right]_0^4 d\theta$$
$$= \int_0^{2\pi} -50(e^{-4} - 1)\, d\theta$$
$$= (1 - e^{-4})\, 100\pi \approx 308.40524$$

Let c be the radius of the hole that is removed.

$$\frac{1}{10}V = \int_0^{2\pi}\int_0^c 25e^{-r^2/4}\, r\, dr\, d\theta = \int_0^{2\pi}\left[-50e^{-r^2/4}\right]_0^c d\theta$$
$$= \int_0^{2\pi} -50(e^{-c^2/4} - 1)\, d\theta \Longrightarrow 30.84052 = 100\pi(1 - e^{-c^2/4})$$
$$\Longrightarrow e^{-c^2/4} = 0.90183$$
$$-\frac{c^2}{4} = -0.10333$$
$$c^2 = 0.41331$$
$$c = 0.6429$$
$$\Longrightarrow \text{diameter} = 2c = 1.2858$$

37. $A = \int_0^{\pi}\int_0^{6\cos\theta} r\, dr\, d\theta = \int_0^{\pi} 18\cos^2\theta\, d\theta = 9\int_0^{\pi}(1 + \cos 2\theta)\, d\theta = \left[9\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_0^{\pi} = 9\pi$

39. $\int_0^{2\pi}\int_0^{1+\cos\theta} r\, dr\, d\theta = \frac{1}{2}\int_0^{2\pi}(1 + 2\cos\theta + \cos^2\theta)\, d\theta$

$$= \frac{1}{2}\int_0^{2\pi}\left(1 + 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right) d\theta = \frac{1}{2}\left[\theta + 2\sin\theta + \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_0^{2\pi} = \frac{3\pi}{2}$$

41. $3\int_0^{\pi/3}\int_0^{2\sin 3\theta} r\, dr\, d\theta = \frac{3}{2}\int_0^{\pi/3} 4\sin^2 3\theta\, d\theta = 3\int_0^{\pi/3}(1 - \cos 6\theta)\, d\theta = 3\left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi/3} = \pi$

43. Let R be a region bounded by the graphs of $r = g_1(\theta)$ and $r = g_2(\theta)$, and the lines $\theta = a$ and $\theta = b$.

When using polar coordinates to evaluate a double integral over R, R can be partitioned into small polar sectors.

45. r-simple regions have fixed bounds for θ.

θ-simple regions have fixed bounds for r.

47. You would need to insert a factor of r because of the $r\,dr\,d\theta$ nature of polar coordinate integrals. The plane regions would be sectors of circles.

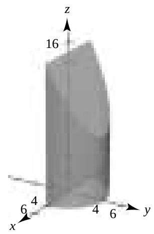
49. $\displaystyle\int_{\pi/4}^{\pi/2}\int_0^5 r\sqrt{1+r^3}\sin\sqrt{\theta}\,dr\,d\theta \approx 56.051$

$\left[\textbf{Note:}\text{ This integral equals }\left(\displaystyle\int_{\pi/4}^{\pi/2}\sin\sqrt{\theta}\,d\theta\right)\left(\displaystyle\int_0^5 r\sqrt{1+r^3}\,dr\right)\right]$

51. Volume = base × height

$\approx 8\pi \times 12 \approx 300$

Answer (c)



53. False

Let $f(r, \theta) = r - 1$ where R is the circular sector $0 \le r \le 6$ and $0 \le \theta \le \pi$. Then,

$$\int_R\int (r-1)\,dA > 0 \quad \text{but} \quad r - 1 \not> 0 \text{ for all } r.$$

55. (a) $I^2 = \displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\,dA = 4\int_0^{\pi/2}\int_0^{\infty} e^{-r^2/2}\,r\,dr\,d\theta = 4\int_0^{\pi/2}\Big[-e^{-r^2/2}\Big]_0^{\infty}\,d\theta = 4\int_0^{\pi/2} d\theta = 2\pi$

(b) Therefore, $I = \sqrt{2\pi}$.

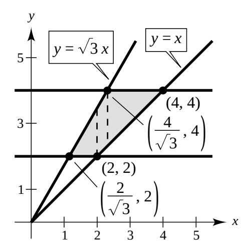
57. $\displaystyle\int_{-7}^{7}\int_{-\sqrt{49-x^2}}^{\sqrt{49-x^2}} 4000e^{-0.01(x^2+y^2)}\,dy\,dx = \int_0^{2\pi}\int_0^7 4000e^{-0.01r^2}\,r\,dr\,d\theta = \int_0^{2\pi}\Big[-200{,}000e^{-0.01r^2}\Big]_0^7 d\theta$

$= 2\pi(-200{,}000)(e^{-0.49} - 1) = 400{,}000\pi(1 - e^{-0.49}) \approx 486{,}788$

59. (a) $\displaystyle\int_2^4\int_{y/\sqrt{3}}^{y} f\,dx\,dy$

(b) $\displaystyle\int_{2/\sqrt{3}}^{2}\int_2^{\sqrt{3}x} f\,dy\,dx + \int_2^{4/\sqrt{3}}\int_x^{\sqrt{3}x} f\,dy\,dx + \int_{4/\sqrt{3}}^{4}\int_x^4 f\,dy\,dx$

(c) $\displaystyle\int_{\pi/4}^{\pi/3}\int_{2\csc\theta}^{4\csc\theta} fr\,dr\,d\theta$



61. $A = \dfrac{\Delta\theta r_2^{\,2}}{2} - \dfrac{\Delta\theta r_1^{\,2}}{2} = \Delta\theta\left(\dfrac{r_1 + r_2}{2}\right)(r_2 - r_1) = r\Delta r\Delta\theta$

Section 13.4 Center of Mass and Moments of Inertia

1. $m = \displaystyle\int_0^4\int_0^3 xy\,dy\,dx = \int_0^4\left[\frac{xy^2}{2}\right]_0^3 dx = \int_0^4 \frac{9}{2}x\,dx = \left[\frac{9x^2}{4}\right]_0^4 = 36$

3. $m = \displaystyle\int_0^{\pi/2}\int_0^2 (r\cos\theta)(r\sin\theta)r\,dr\,d\theta = \int_0^{\pi/2}\int_0^2 \cos\theta\sin\theta \cdot r^3\,dr\,d\theta$

$= \displaystyle\int_0^{\pi/2} 4\cos\theta\sin\theta\,d\theta$

$= \left[4\dfrac{\sin^2\theta}{2}\right]_0^{\pi/2} = 2$

5. (a) $m = \int_0^a \int_0^b k\,dy\,dx = kab$

$M_x = \int_0^a \int_0^b ky\,dy\,dx = \dfrac{kab^2}{2}$

$M_y = \int_0^a \int_0^b kx\,dy\,dx = \dfrac{ka^2b}{2}$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{ka^2b/2}{kab} = \dfrac{a}{2}$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{kab^2/2}{kab} = \dfrac{b}{2}$

$(\bar{x}, \bar{y}) = \left(\dfrac{a}{2}, \dfrac{b}{2}\right)$

(b) $m = \int_0^a \int_0^b ky\,dy\,dx = \dfrac{kab^2}{2}$

$M_x = \int_0^a \int_0^b ky^2\,dy\,dx = \dfrac{kab^3}{3}$

$M_y = \int_0^a \int_0^b kxy\,dy\,dx = \dfrac{ka^2b^2}{4}$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{ka^2b^2/4}{kab^2/2} = \dfrac{a}{2}$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{kab^3/3}{kab^2/2} = \dfrac{2}{3}b$

$(\bar{x}, \bar{y}) = \left(\dfrac{a}{2}, \dfrac{2}{3}b\right)$

(c) $m = \int_0^a \int_0^b kx\,dy\,dx = k\int_0^a xb\,dx = \dfrac{1}{2}ka^2b$

$M_x = \int_0^a \int_0^b kxy\,dy\,dx = \dfrac{ka^2b^2}{4}$

$M_y = \int_0^a \int_0^b kx^2\,dy\,dx = \dfrac{ka^3b}{3}$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{ka^3b/3}{ka^2b/2} = \dfrac{2}{3}a$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{ka^2b^2/4}{ka^2b/2} = \dfrac{b}{2}$

$(\bar{x}, \bar{y}) = \left(\dfrac{2}{3}a, \dfrac{b}{2}\right)$

7. (a) $m = \dfrac{k}{2}bh$

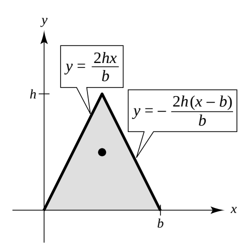
$\bar{x} = \dfrac{b}{2}$ by symmetry

$M_x = \int_0^{b/2} \int_0^{2hx/b} ky\,dy\,dx + \int_{b/2}^{b} \int_0^{-2h(x-b)/b} ky\,dy\,dx$

$= \dfrac{kbh^2}{12} + \dfrac{kbh^2}{12} = \dfrac{kbh^2}{6}$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{kbh^2/6}{kbh/2} = \dfrac{h}{3}$

$(\bar{x}, \bar{y}) = \left(\dfrac{b}{2}, \dfrac{h}{3}\right)$



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7. —CONTINUED—

(b) $m = \int_0^{b/2}\int_0^{2hx/b} ky\,dy\,dx + \int_{b/2}^{b}\int_0^{-2h(x-b)/b} ky\,dy\,dx = \frac{kbh^2}{6}$

$M_x = \int_0^{b/2}\int_0^{2hx/b} ky^2\,dy\,dx + \int_{b/2}^{b}\int_0^{-2h(x-b)/b} ky^2\,dy\,dx = \frac{kbh^3}{12}$

$M_y = \int_0^{b/2}\int_0^{2hx/b} kxy\,dy\,dx + \int_{b/2}^{b}\int_0^{-2h(x-b)/b} kxy\,dy\,dx = \frac{kb^2h^2}{12}$

$\bar{x} = \frac{M_y}{m} = \frac{kb^2h^2/12}{kbh^2/6} = \frac{b}{2}$

$\bar{y} = \frac{M_x}{m} = \frac{kbh^3/12}{kbh^2/6} = \frac{h}{2}$

(c) $m = \int_0^{b/2}\int_0^{2hx/b} kx\,dy\,dx + \int_{b/2}^{b}\int_0^{-2h(x-b)/b} kx\,dy\,dx$

$= \frac{1}{12}kb^2h + \frac{1}{6}kb^2h = \frac{1}{4}kb^2h$

$M_x = \int_0^{b/2}\int_0^{2hx/b} kxy\,dy\,dx + \int_{b/2}^{b}\int_0^{-2h(x-b)/b} kxy\,dy\,dx$

$= \frac{1}{32}kh^2b^2 + \frac{5}{96}kh^2b^2 = \frac{1}{12}kh^2b^2$

$M_y = \int_0^{b/2}\int_0^{2hx/b} kx^2\,dy\,dx + \int_{b/2}^{b}\int_0^{-2h(x-b)/b} kx^2\,dy\,dx$

$= \frac{1}{32}kb^3h + \frac{11}{96}kb^3h = \frac{7}{48}kb^3h$

$\bar{x} = \frac{M_y}{m} = \frac{7kb^3h/48}{kb^2h/4} = \frac{7}{12}b$

$\bar{y} = \frac{M_x}{m} = \frac{kh^2b^2/12}{kb^2h/4} = \frac{h}{3}$

9. (a) The x-coordinate changes by 5: $(\bar{x}, \bar{y}) = \left(\frac{a}{2} + 5, \frac{b}{2}\right)$

(b) The x-coordinate changes by 5: $(\bar{x}, \bar{y}) = \left(\frac{a}{2} + 5, \frac{2b}{3}\right)$

(c) $m = \int_5^{a+5}\int_0^{b} kx\,dy\,dx = \frac{1}{2}k(a+5)^2b - \frac{25}{2}kb$

$M_x = \int_5^{a+5}\int_0^{b} kxy\,dy\,dx = \frac{1}{4}k(a+5)^2b^2 - \frac{25}{4}kb^2$

$M_y = \int_5^{a+5}\int_0^{b} kx^2\,dy\,dx = \frac{1}{3}k(a+5)^3\,b - \frac{125}{3}kb$

$\bar{x} = \frac{M_y}{m} = \frac{2(a^2 + 15a + 75)}{3(a + 10)}$

$\bar{y} = \frac{M_x}{m} = \frac{b}{2}$

11. (a) $\bar{x} = 0$ by symmetry

$m = \frac{\pi a^2 k}{2}$

$M_x = \int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} yk\,dy\,dx = \frac{2a^3k}{3}$

$\bar{y} = \frac{M_x}{m} = \frac{2a^3k}{3} \cdot \frac{2}{\pi a^2 k} = \frac{4a}{3\pi}$

(b) $m = \int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} k(a-y)\,y\,dy\,dx = \frac{a^4k}{24}(16 - 3\pi)$

$M_x = \int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} k(a-y)\,y^2\,dy\,dx = \frac{a^5k}{120}(15\pi - 32)$

$M_y = \int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} kx(a-y)\,y\,dy\,dx = 0$

$\bar{x} = \frac{M_y}{m} = 0$

$\bar{y} = \frac{M_x}{m} = \frac{a}{5}\left[\frac{15\pi - 32}{16 - 3\pi}\right]$

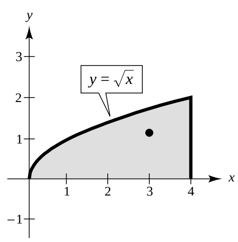
13. $m = \int_0^4 \int_0^{\sqrt{x}} kxy\,dy\,dx = \dfrac{32k}{3}$

$M_x = \int_0^4 \int_0^{\sqrt{x}} kxy^2\,dy\,dx = \dfrac{256k}{21}$

$M_y = \int_0^4 \int_0^{\sqrt{x}} kx^2y\,dy\,dx = 32k$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{32k}{1} \cdot \dfrac{3}{32k} = 3$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{256k}{21} \cdot \dfrac{3}{32k} = \dfrac{8}{7}$

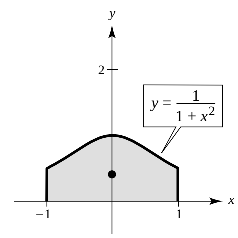


15. $\bar{x} = 0$ by symmetry

$m = \int_{-1}^1 \int_0^{1/(1+x^2)} k\,dy\,dx = \dfrac{k\pi}{2}$

$M_x = \int_{-1}^1 \int_0^{1/(1+x^2)} ky\,dy\,dx = \dfrac{k}{8}(2 + \pi)$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{k}{8}(2 + \pi) \cdot \dfrac{2}{k\pi} = \dfrac{2 + \pi}{4\pi}$

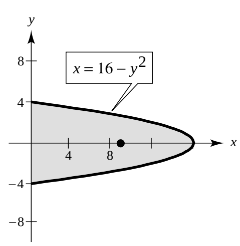


17. $\bar{y} = 0$ by symmetry

$m = \int_{-4}^4 \int_0^{16-y^2} kx\,dx\,dy = \dfrac{8192k}{15}$

$M_y = \int_{-4}^4 \int_0^{16-y^2} kx^2\,dx\,dy = \dfrac{524{,}288k}{105}$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{524{,}288k}{105} \cdot \dfrac{15}{8192k} = \dfrac{64}{7}$

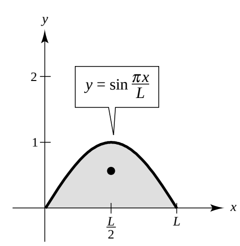


19. $\bar{x} = \dfrac{L}{2}$ by symmetry

$m = \int_0^L \int_0^{\sin \pi x/L} ky\,dy\,dx = \dfrac{kL}{4}$

$M_x = \int_0^L \int_0^{\sin \pi x/L} ky^2\,dy\,dx = \dfrac{4kL}{9\pi}$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{4kL}{9\pi} \cdot \dfrac{4}{kL} = \dfrac{16}{9\pi}$



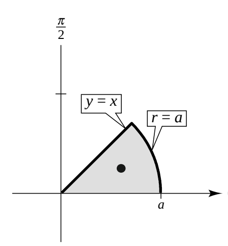
21. $m = \dfrac{\pi a^2 k}{8}$

$M_x = \iint_R ky\,dA = \int_0^{\pi/4} \int_0^a kr^2 \sin\theta\,dr\,d\theta = \dfrac{ka^3\left(2 - \sqrt{2}\right)}{6}$

$M_y = \iint_R kx\,dA = \int_0^{\pi/4} \int_0^a kr^2 \cos\theta\,dr\,d\theta = \dfrac{ka^3\sqrt{2}}{6}$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{ka^3\sqrt{2}}{6} \cdot \dfrac{8}{\pi a^2 k} = \dfrac{4a\sqrt{2}}{3\pi}$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{ka^3\left(2 - \sqrt{2}\right)}{6} \cdot \dfrac{8}{\pi a^2 k} = \dfrac{4a\left(2 - \sqrt{2}\right)}{3\pi}$



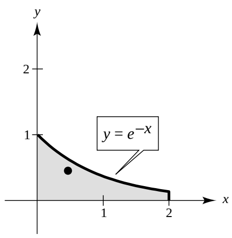
23. $m = \int_0^2 \int_0^{e^{-x}} ky\,dy\,dx = \frac{k}{4}(1 - e^{-4})$

$M_x = \int_0^2 \int_0^{e^{-x}} ky^2\,dy\,dx = \frac{k}{9}(1 - e^{-6})$

$M_y = \int_0^2 \int_0^{e^{-x}} kxy\,dy\,dx = \frac{k(1 - 5e^{-4})}{8}$

$\bar{x} = \frac{M_y}{m} = \frac{k(e^4 - 5)}{8e^4} \cdot \frac{4e^4}{k(e^4 - 1)} = \frac{e^4 - 5}{2(e^4 - 1)} \approx 0.46$

$\bar{y} = \frac{M_x}{m} = \frac{k(e^6 - 1)}{9e^6} \cdot \frac{4e^4}{k(e^4 - 1)} = \frac{4}{9}\left[\frac{e^6 - 1}{e^6 - e^2}\right] \approx 0.45$



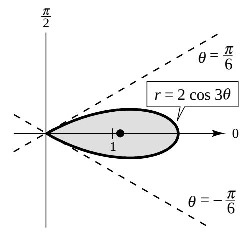
25. $\bar{y} = 0$ by symmetry

$m = \iint_R k\,dA = \int_{-\pi/6}^{\pi/6} \int_0^{2\cos 3\theta} kr\,dr\,d\theta = \frac{k\pi}{3}$

$M_y = \iint_R kx\,dA$

$= \int_{-\pi/6}^{\pi/6} \int_0^{2\cos 3\theta} kr^2 \cos\theta\,dr\,d\theta \approx 1.17k$

$\bar{x} = \frac{M_y}{m} \approx 1.17k\left(\frac{3}{\pi k}\right) \approx 1.12$



27. $m = bh$

$I_x = \int_0^b \int_0^h y^2\,dy\,dx = \frac{bh^3}{3}$

$I_y = \int_0^b \int_0^h x^2 dy\,dx = \frac{b^3h}{3}$

$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{b^3h}{3} \cdot \frac{1}{bh}} = \sqrt{\frac{b^2}{3}} = \frac{b}{\sqrt{3}} = \frac{\sqrt{3}}{3}b$

$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{bh^3}{3} \cdot \frac{1}{bh}} = \sqrt{\frac{h^2}{3}} = \frac{h}{\sqrt{3}} = \frac{\sqrt{3}}{3}h$

29. $m = \pi a^2$

$I_x = \iint_R y^2\,dA = \int_0^{2\pi} \int_0^a r^3 \sin^2\theta\,dr\,d\theta = \frac{a^4\pi}{4}$

$I_y = \iint_R x^2\,dA = \int_0^{2\pi} \int_0^a r^3 \cos^2\theta\,dr\,d\theta = \frac{a^4\pi}{4}$

$I_0 = I_x + I_y = \frac{a^4\pi}{4} + \frac{a^4\pi}{4} = \frac{a^4\pi}{2}$

$\bar{\bar{x}} = \bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{a^4\pi}{4} \cdot \frac{1}{\pi a^2}} = \frac{a}{2}$

31. $m = \frac{\pi a^2}{4}$

$I_x = \iint_R y^2\,dA = \int_0^{\pi/2} \int_0^a r^3 \sin^2\theta\,dr\,d\theta = \frac{\pi a^4}{16}$

$I_y = \iint_R x^2\,dA = \int_0^{\pi/2} \int_0^a r^3 \cos^2\theta\,dr\,d\theta = \frac{\pi a^4}{16}$

$I_0 = I_x + I_y = \frac{\pi a^4}{16} + \frac{\pi a^4}{16} = \frac{\pi a^4}{8}$

$\bar{\bar{x}} = \bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{\pi a^4}{16} \cdot \frac{4}{\pi a^2}} = \frac{a}{2}$

33. $\rho = ky$

$m = k\int_0^a \int_0^b y\,dy\,dx = \frac{kab^2}{2}$

$I_x = k\int_0^a \int_0^b y^3\,dy\,dx = \frac{kab^4}{4}$

$I_y = k\int_0^a \int_0^b x^2 y\,ydy\,dx = \frac{ka^3b^2}{6}$

$I_0 = I_x + I_y = \frac{3kab^4 + 2kb^2a^3}{12}$

$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{ka^3b^2/6}{kab^2/2}} = \sqrt{\frac{a^2}{3}} = \frac{a}{\sqrt{3}} = \frac{\sqrt{3}}{3}a$

$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{kab^4/4}{kab^2/2}} = \sqrt{\frac{b^2}{2}} = \frac{b}{\sqrt{2}} = \frac{\sqrt{2}}{2}b$

35. $\rho = kx$

$$m = k\int_0^2\int_0^{4-x^2} x\,dy\,dx = 4k$$

$$I_x = k\int_0^2\int_0^{4-x^2} xy^2\,dy\,dx = \frac{32k}{3}$$

$$I_y = k\int_0^2\int_0^{4-x^2} x^3\,dy\,dx = \frac{16k}{3}$$

$$I_0 = I_x + I_y = 16k$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{16k/3}{4k}} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{32k/3}{4k}} = \sqrt{\frac{8}{3}} = \frac{4}{\sqrt{6}} = \frac{2\sqrt{6}}{3}$$

37. $\rho = kxy$

$$m = \int_0^4\int_0^{\sqrt{x}} kxy\,dy\,dx = \frac{32k}{3}$$

$$I_x = \int_0^4\int_0^{\sqrt{x}} kxy^3\,dy\,dx = 16k$$

$$I_y = \int_0^4\int_0^{\sqrt{x}} kx^3y\,dy\,dx = \frac{512k}{5}$$

$$I_0 = I_x + I_y = \frac{592k}{5}$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{512k}{5}\cdot\frac{3}{32k}} = \sqrt{\frac{48}{5}} = \frac{4\sqrt{15}}{5}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{16k}{1}\cdot\frac{3}{32k}} = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}$$

39. $\rho = kx$

$$m = \int_0^1\int_{x^2}^{\sqrt{x}} kx\,dy\,dx = \frac{3k}{20}$$

$$I_x = \int_0^1\int_{x^2}^{\sqrt{x}} kxy^2\,dy\,dx = \frac{3k}{56}$$

$$I_y = \int_0^1\int_{x^2}^{\sqrt{x}} kx^3\,dy\,dx = \frac{k}{18}$$

$$I_0 = I_x + I_y = \frac{55k}{504}$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{k}{18}\cdot\frac{20}{3k}} = \frac{\sqrt{30}}{9}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{3k}{56}\cdot\frac{20}{3k}} = \frac{\sqrt{70}}{14}$$

41. $$I = 2k\int_{-b}^{b}\int_0^{\sqrt{b^2-x^2}} (x-a)^2\,dy\,dx = 2k\int_{-b}^{b} (x-a)^2\sqrt{b^2-x^2}\,dx$$

$$= 2k\left[\int_{-b}^{b} x^2\sqrt{b^2-x^2}\,dx - 2a\int_{-b}^{b} x\sqrt{b^2-x^2}\,dx + a^2\int_{-b}^{b}\sqrt{b^2-x^2}\,dx\right]$$

$$= 2k\left[\frac{\pi b^4}{8} + 0 + \frac{\pi a^2b^2}{2}\right] = \frac{k\pi b^2}{4}(b^2 + 4a^2)$$

43. $$I = \int_0^4\int_0^{\sqrt{x}} kx(x-6)^2\,dy\,dx = \int_0^4 kx\sqrt{x}(x^2 - 12x + 36)\,dx = k\left[\frac{2}{9}x^{9/2} - \frac{24}{7}x^{7/2} + \frac{72}{5}x^{5/2}\right]_0^4 = \frac{42{,}752k}{315}$$

45. $I = \int_0^a \int_0^{\sqrt{a^2-x^2}} k(a-y)(y-a)^2\,dy\,dx = \int_0^a\int_0^{\sqrt{a^2-x^2}} k(a-y)^3\,dy\,dx = \int_0^a \left[-\frac{k}{4}(a-y)^4\right]_0^{\sqrt{a^2-x^2}} dx$

$$= -\frac{k}{4}\int_0^a \left[a^4 - 4a^3y + 6a^2y^2 - 4ay^3 + y^4\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= -\frac{k}{4}\int_0^a \left[a^4 - 4a^3\sqrt{a^2-x^2} + 6a^2(a^2-x^2) - 4a(a^2-x^2)\sqrt{a^2-x^2} + (a^4 - 2a^2x^2 + x^4) - a^4\right] dx$$

$$= -\frac{k}{4}\int_0^a \left[7a^4 - 8a^2x^2 + x^4 - 8a^3\sqrt{a^2-x^2} + 4ax^2\sqrt{a^2-x^2}\right] dx$$

$$= -\frac{k}{4}\left[7a^4x - \frac{8a^2}{3}x^3 + \frac{x^5}{5} - 4a^3\left(x\sqrt{a^2-x^2} + a^2\arcsin\frac{x}{a}\right) + \frac{a}{2}\left(x(2x^2-a^2)\sqrt{a^2-x^2} + a^4\arcsin\frac{x}{a}\right)\right]_0^a$$

$$= -\frac{k}{4}\left(7a^5 - \frac{8}{3}a^5 + \frac{1}{5}a^5 - 2a^5\pi + \frac{1}{4}a^5\pi\right) = a^5k\left(\frac{7\pi}{16} - \frac{17}{15}\right)$$

47. $\rho(x, y) = ky$. $\bar{y}$ will increase

49. $\rho(x, y) = kxy$.

Both $\bar{x}$ and $\bar{y}$ will increase

51. Let $\rho(x, y)$ be a continuous density function on the planar lamina R.

The movements of mass with respect to the x- and y-axes are

$$M_x = \int_R\int y\,\rho(x, y)dA \text{ and } M_y = \int_R\int x\,\rho(x, y)\,dA.$$

If m is the mass of the lamina, then the center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right).$$

53. See the definition on page 968

55. $\bar{y} = \frac{L}{2}, A = bL, h = \frac{L}{2}$

$$I_{\bar{y}} = \int_0^b\int_0^L \left(y - \frac{L}{2}\right)^2 dy\,dx$$

$$= \int_0^b \left[\frac{[y - (L/2)]^3}{3}\right]_0^L dx = \frac{L^3b}{12}$$

$$y_a = \bar{y} - \frac{I_{\bar{y}}}{hA} = \frac{L}{2} - \frac{L^3b/12}{(L/2)(bL)} = \frac{L}{3}$$

57. $\bar{y} = \frac{2L}{3}. A = \frac{bL}{2}, h = \frac{L}{3}$

$$I_{\bar{y}} = 2\int_0^{b/2}\int_{2Lx/b}^L \left(y - \frac{2L}{3}\right)^2 dy\,dx$$

$$= \frac{2}{3}\int_0^{b/2} \left[\left(y - \frac{2L}{3}\right)^3\right]_{2Lx/b}^L dx$$

$$= \frac{2}{3}\int_0^{b/2} \left[\frac{L}{27} - \left(\frac{2Lx}{b} - \frac{2L}{3}\right)^3\right] dx$$

$$= \frac{2}{3}\left[\frac{L^3x}{27} - \frac{b}{8L}\left(\frac{2Lx}{b} - \frac{2L}{3}\right)^4\right]_0^{b/2} = \frac{L^3b}{36}$$

$$y_a = \frac{2L}{3} - \frac{L^3b/36}{L^2b/6} = \frac{L}{2}$$

Section 13.5 Surface Area

1. $f(x, y) = 2x + 2y$

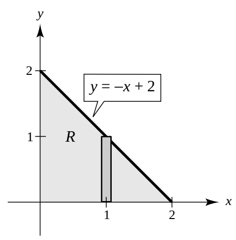
$R =$ triangle with vertices $(0, 0)$, $(2, 0)$, $(0, 2)$

$f_x = 2, f_y = 2$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = 3$

$$S = \int_0^2 \int_0^{2-x} 3\, dy\, dx = 3\int_0^2 (2 - x)\, dx$$

$$= \left[3\left(2x - \frac{x^2}{2}\right)\right]_0^2 = 6$$



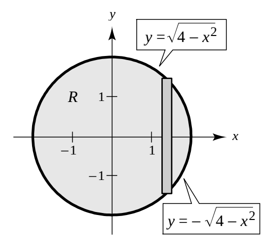
3. $f(x, y) = 8 + 2x + 2y$

$R = \{(x, y): x^2 + y^2 \le 4\}$

$f_x = 2, f_y = 2$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = 3$

$$S = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 3\, dy\, dx = \int_0^{2\pi}\int_0^2 3r\, dr\, d\theta = 12\pi$$



5. $f(x, y) = 9 - x^2$

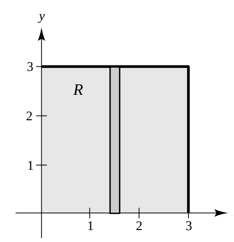
$R =$ square with vertices, $(0, 0)$, $(3, 0)$, $(0, 3)$, $(3, 3)$

$f_x = -2x,\ f_y = 0$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2}$

$$S = \int_0^3\int_0^3 \sqrt{1 + 4x^2}\, dy\, dx = \int_0^3 3\sqrt{1 + 4x^2}\, dx$$

$$= \left[\frac{3}{4}\left(2x\sqrt{1 + 4x^2} + \ln\left|2x + \sqrt{1 + 4x^2}\right|\right)\right]_0^3 = \frac{3}{4}\left(6\sqrt{37} + \ln\left|6 + \sqrt{37}\right|\right)$$



7. $f(x, y) = 2 + x^{3/2}$

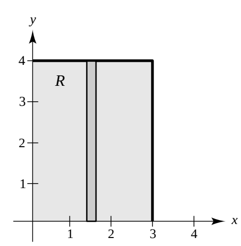
$R =$ rectangle with vertices $(0, 0)$, $(0, 4)$, $(3, 4)$, $(3, 0)$

$f_x = \dfrac{3}{2}x^{1/2},\ f_y = 0$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \left(\frac{9}{4}\right)x} = \frac{\sqrt{4 + 9x}}{2}$$

$$S = \int_0^3\int_0^4 \frac{\sqrt{4 + 9x}}{2}\, dy\, dx = \int_0^3 4\left(\frac{\sqrt{4 + 9x}}{2}\right) dx$$

$$= \left[\frac{4}{27}(4 + 9x)^{3/2}\right]_0^3 = \frac{4}{27}\left(31\sqrt{31} - 8\right)$$



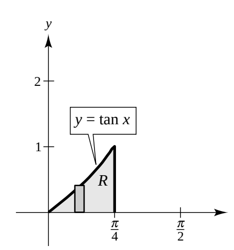
9. $f(x, y) = \ln|\sec x|$

$$R = \left\{(x, y): 0 \le x \le \frac{\pi}{4},\ 0 \le y \le \tan x\right\}$$

$f_x = \tan x,\ f_y = 0$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \tan^2 x} = \sec x$

$$S = \int_0^{\pi/4}\int_0^{\tan x} \sec x\, dy\, dx = \int_0^{\pi/4} \sec x \tan x\, dx = \Big[\sec x\Big]_0^{\pi/4} = \sqrt{2} - 1$$



11. $f(x, y) = \sqrt{x^2 + y^2}$

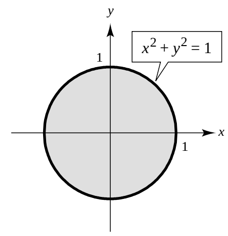
$R = \{(x, y): 0 \le f(x, y) \le 1\}$

$0 \le \sqrt{x^2 + y^2} \le 1,\ x^2 + y^2 \le 1$

$$f_x = \frac{x}{\sqrt{x^2 + y^2}},\ f_y = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}} = \sqrt{2}$$

$$S = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{2}\, dy\, dx = \int_0^{2\pi}\int_0^1 \sqrt{2}\, r\, dr\, d\theta = \sqrt{2}\pi$$



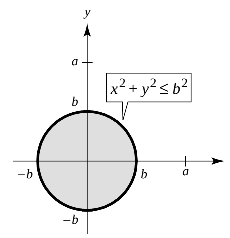
13. $f(x, y) = \sqrt{a^2 - x^2 - y^2}$

$R = \{(x, y): x^2 + y^2 \le b^2, b < a\}$

$$f_x = \frac{-x}{\sqrt{a^2 - x^2 - y^2}},\ f_y = \frac{-y}{\sqrt{a^2 - x^2 - y^2}}$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2}} = \frac{a}{\sqrt{a^2 - x^2 - y^2}}$$

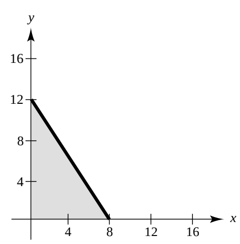
$$S = \int_{-b}^{b}\int_{-\sqrt{b^2-x^2}}^{\sqrt{b^2-x^2}} \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dy\, dx = \int_0^{2\pi}\int_0^b \frac{a}{\sqrt{a^2 - r^2}}\, r\, dr\, d\theta = 2\pi a\left(a - \sqrt{a^2 - b^2}\right)$$



15. $z = 24 - 3x - 2y$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{14}$

$$S = \int_0^8\int_0^{-(3/2)x+12} \sqrt{14}\, dy\, dx = 48\sqrt{14}$$

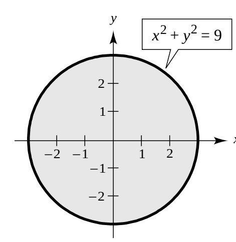


17. $z = \sqrt{25 - x^2 - y^2}$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \frac{x^2}{25 - x^2 - y^2} + \frac{y^2}{25 - x^2 - y^2}} = \frac{5}{\sqrt{25 - x^2 - y^2}}$$

$$S = 2\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \frac{5}{\sqrt{25 - (x^2 + y^2)}}\, dy\, dx$$

$$= 2\int_0^{2\pi}\int_0^3 \frac{5}{\sqrt{25 - r^2}}\, r\, dr\, d\theta = 20\pi$$

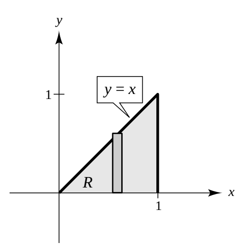


19. $f(x, y) = 2y + x^2$

R = triangle with vertices (0, 0), (1, 0), (1, 1)

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{5 + 4x^2}$

$$S = \int_0^1\int_0^x \sqrt{5 + 4x^2}\, dy\, dx = \frac{1}{12}\left(27 - 5\sqrt{5}\right)$$



21. $f(x, y) = 4 - x^2 - y^2$

$R = \{(x, y): 0 \le f(x, y)\}$

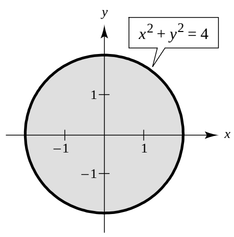
$0 \le 4 - x^2 - y^2,\ x^2 + y^2 \le 4$

$f_x = -2x,\ f_y = -2y$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}$

$$S = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{1 + 4x^2 + 4y^2}\, dy\, dx$$

$$= \int_0^{2\pi}\int_0^2 \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \frac{\left(17\sqrt{17} - 1\right)\pi}{6}$$



23. $f(x, y) = 4 - x^2 - y^2$

$R = \{(x, y): 0 \le x \le 1,\ 0 \le y \le 1\}$

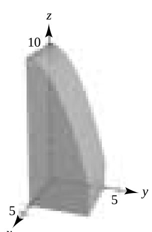
$f_x = -2x,\ f_y = -2y$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}$

$$S = \int_0^1\int_0^1 \sqrt{(1 + 4x^2) + 4y^2}\, dy\, dx \approx 1.8616$$

25. Surface area $> (4) \cdot (6) = 24$.

Matches (e)



27. $f(x, y) = e^x$

$R = \{(x, y): 0 \le x \le 1,\ 0 \le y \le 1\}$

$f_x = e^x,\ f_y = 0$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + e^{2x}}$

$$S = \int_0^1\int_0^1 \sqrt{1 + e^{2x}}\, dy\, dx$$

$$= \int_0^1 \sqrt{1 + e^{2x}} \approx 2.0035$$

29. $f(x, y) = x^3 - 3xy + y^3$

$R =$ square with vertices $(1, 1), (-1, 1), (-1, -1), (1, -1)$

$f_x = 3x^2 - 3y = 3(x^2 - y),\ f_y = -3x + 3y^2 = 3(y^2 - x)$

$$S = \int_{-1}^{1}\int_{-1}^{1} \sqrt{1 + 9(x^2 - y)^2 + 9(y^2 - x)^2}\, dy\, dx$$

31. $f(x, y) = e^{-x}\sin y$

$f_x = -e^{-x}\sin y,\ f_y = e^{-x}\cos y$

$$\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + e^{-2x}\sin^2 y + e^{-2x}\cos^2 y} = \sqrt{1 + e^{-2x}}$$

$$S = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{1 + e^{-2x}}\, dy\, dx$$

33. $f(x, y) = e^{xy}$

$R = \{(x, y): 0 \le x \le 4,\ 0 \le y \le 10\}$

$f_x = ye^{xy},\ f_y = xe^{xy}$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + y^2e^{2xy} + x^2e^{2xy}} = \sqrt{1 + e^{2xy}(x^2 + y^2)}$

$$S = \int_0^4\int_0^{10} \sqrt{1 + e^{2xy}(x^2 + y^2)}\, dy\, dx$$

35. See the definition on page 972.

37. $f(x, y) = \sqrt{1 - x^2}$; $f_x = \dfrac{-x}{\sqrt{1^2 - x^2}}$, $f_y = 0$

$$S = \iint_R \sqrt{1 + f_x^{\,2} + f_y^{\,2}}\, dA$$
$$= 16\int_0^1 \int_0^x \frac{1}{\sqrt{1 - x^2}}\, dy\, dx$$
$$= 16\int_0^1 \frac{x}{\sqrt{1 - x^2}}\, dx = \Big[-16(1 - x^2)^{1/2}\Big]_0^1 = 16$$

39. (a) $V = \displaystyle\int_0^{50} \int_0^{\sqrt{50^2 - x^2}} \left(20 + \frac{xy}{100} - \frac{x + y}{5}\right) dy\, dx$

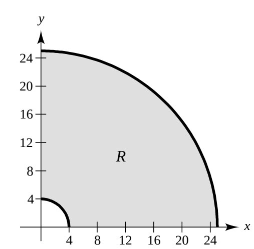
$$= \int_0^{50} \left[20\sqrt{50^2 - x^2} + \frac{x}{200}(50^2 - x^2) - \frac{x}{5}\sqrt{50^2 - x^2} - \frac{50^2 - x^2}{10}\right] dy$$
$$= \left[10\left(x\sqrt{50 - x^2} + 50^2 \arcsin\frac{x}{50}\right) + \frac{25}{4}x^2 - \frac{x^4}{800} + \frac{1}{15}(50^2 - x^2)^{3/2} - 250x + \frac{x^3}{30}\right]_0^{50}$$
$$\approx 30{,}415.74 \text{ ft}^3$$

(b) $z = 20 + \dfrac{xy}{100}$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \frac{y^2}{100^2} + \frac{x^2}{100^2}} = \frac{\sqrt{100^2 + x^2 + y^2}}{100}$$
$$S = \frac{1}{100}\int_0^{50} \int_0^{\sqrt{50^2 - x^2}} \sqrt{100^2 + x^2 + y^2}\, dy\, dx$$
$$= \frac{1}{100}\int_0^{\pi/2} \int_0^{50} \sqrt{100^2 + r^2}\, r\, dr\, d\theta \approx 2081.53 \text{ ft}^2$$

41. (a) $V = \displaystyle\iint_R f(x, y)$

$$= 8\iint_R \sqrt{625 - x^2 - y^2}\, dA \qquad \text{where } R \text{ is the region in the first quadrant}$$
$$= 8\int_0^{\pi/2} \int_4^{25} \sqrt{625 - r^2}\, r\, dr\, d\theta$$
$$= -4\int_0^{\pi/2} \left[\frac{2}{3}(625 - r^2)^{3/2}\right]_4^{25} d\theta$$
$$= -\frac{8}{3}\left[0 - 609\sqrt{609}\right] \cdot \frac{\pi}{2}$$
$$= 812\pi\sqrt{609} \text{ cm}^3$$



(b) $A = \displaystyle\iint_R \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA = 8\iint_R \sqrt{1 + \frac{x^2}{625 - x^2 - y^2} + \frac{y^2}{625 - x^2 - y^2}}\, dA$

$$= 8\iint_R \frac{25}{\sqrt{625 - x^2 - y^2}}\, dA = 8\int_0^{\pi/2} \int_4^{25} \frac{25}{\sqrt{625 - r^2}}\, r\, dr\, d\theta$$
$$= \lim_{b \to 25^-}\left[-200\sqrt{625 - r^2}\right]_4^b \cdot \frac{\pi}{2} = 100\pi\sqrt{609} \text{ cm}^2$$

Section 13.6 Triple Integrals and Applications

1. $$\int_0^3\int_0^2\int_0^1 (x+y+z)\,dx\,dy\,dx = \int_0^3\int_0^2\left[\frac{1}{2}x^2+xy+xz\right]_0^1 dy\,dx$$
$$= \int_0^3\int_0^2\left(\frac{1}{2}+y+z\right)dy\,dz = \int_0^3\left[\frac{1}{2}y+\frac{1}{2}y^2+yz\right]_0^2 dz = \left[3z+z^2\right]_0^3 = 18$$

3. $$\int_0^1\int_0^x\int_0^{xy} x\,dz\,dy\,dx = \int_0^1\int_0^x\left[xz\right]_0^{xy} dy\,dx$$
$$= \int_0^1\int_0^x x^2y\,dy\,dx = \int_0^1\left[\frac{x^2y^2}{2}\right]_0^x dx = \int_0^1\frac{x^4}{2}\,dx = \left[\frac{x^5}{10}\right]_0^1 = \frac{1}{10}$$

5. $$\int_1^4\int_0^1\int_0^x 2ze^{-x^2}\,dy\,dx\,dz = \int_1^4\int_0^1\left[(2ze^{-x^2})y\right]_0^x dx\,dz = \int_1^4\int_0^1 2zxe^{-x^2}\,dx\,dz$$
$$= \int_1^4\left[-ze^{-x^2}\right]_0^1 dz = \int_1^4 z(1-e^{-1})\,dz = \left[(1-e^{-1})\frac{z^2}{2}\right]_1^4 = \frac{15}{2}\left(1-\frac{1}{e}\right)$$

7. $$\int_0^4\int_0^{\pi/2}\int_0^{1-x} x\cos y\,dz\,dy\,dx = \int_0^4\int_0^{\pi/2}\left[(x\cos y)z\right]_0^{1-x} dy\,dx = \int_0^4\int_0^{\pi/2} x(1-x)\cos y\,dy\,dx$$
$$= \int_0^4\left[x(1-x)\sin y\right]_0^{\pi/2} dx = \int_0^4 x(1-x)dx = \left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^4 = 8-\frac{64}{3} = \frac{-40}{3}$$

9. $$\int_0^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{x^2} x\,dz\,dy\,dx = \int_0^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} x^3\,dy\,dx = \frac{128}{15}$$

11. $$\int_0^2\int_0^{\sqrt{4-x^2}}\int_1^4 \frac{x^2\sin y}{z}\,dz\,dy\,dx = \int_0^2\int_0^{\sqrt{4-x^2}}\left[x^2\sin y\ln|z|\right]_1^4 dy\,dx$$
$$= \int_0^2\left[x^2\ln 4(-\cos y)\right]_0^{\sqrt{4-x^2}} dx = \int_0^2 x^2\ln 4\left[1-\cos\sqrt{4-x^2}\right]dx \approx 2.44167$$

13. $$\int_0^4\int_0^{4-x}\int_0^{4-x-y} dz\,dy\,dx$$

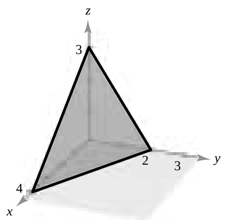
15. $$\int_{-3}^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2} dz\,dy\,dx$$

17. $$\int_{-2}^2\int_0^{4-y^2}\int_0^x dz\,dx\,dy = \int_{-2}^2\int_0^{4-y^2} x\,dx\,dy$$
$$= \frac{1}{2}\int_{-2}^2 (4-y^2)^2\,dy = \int_0^2 (16-8y^2+y^4)\,dy = \left[16y-\frac{8}{3}y^3+\frac{1}{5}y^5\right]_0^2 = \frac{256}{15}$$

19. $$8\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx = 8\int_0^a\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx$$
$$= 4\int_0^a\left[y\sqrt{a^2-x^2-y^2}+(a^2-x^2)\arcsin\left(\frac{y}{\sqrt{a^2-x^2}}\right)\right]_0^{\sqrt{a^2-x^2}} dx$$
$$= 4\left(\frac{\pi}{2}\right)\int_0^a (a^2-x^2)\,dx = \left[2\pi\left(a^2x-\frac{1}{3}x^3\right)\right]_0^a = \frac{4}{3}\pi a^3$$

21. $\displaystyle\int_0^2\int_0^{4-x^2}\int_0^{4-x^2} dz\,dy\,dx = \int_0^2 (4 - x^2)^2\,dx = \int_0^2 (16 - 8x^2 + x^4)\,dx = \left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_0^2 = \frac{256}{15}$

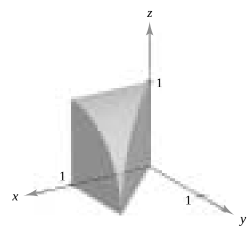
23. Plane: $3x + 6y + 4z = 12$



$$\int_0^3\int_0^{(12-4z)/3}\int_0^{(12-4z-3x)/6} dy\,dx\,dz$$

25. Top cylinder: $y^2 + z^2 = 1$

Side plane: $x = y$



$$\int_0^1\int_0^x\int_0^{\sqrt{1-y^2}} dz\,dy\,dx$$

27. $Q = \{(x, y, z): 0 \le x \le 1, 0 \le y \le x, 0 \le z \le 3\}$

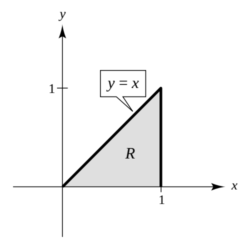
$$\iiint_Q xyz\,dV = \int_0^3\int_0^1\int_y^1 xyz\,dx\,dy\,dz = \int_0^3\int_0^1\int_0^x xyz\,dy\,dx\,dz$$

$$= \int_0^1\int_0^3\int_y^1 xyz\,dx\,dz\,dy$$

$$= \int_0^1\int_0^3\int_0^x xyz\,dy\,dz\,dx$$

$$= \int_0^1\int_y^1\int_0^3 xyz\,dz\,dx\,dy$$

$$= \int_0^1\int_0^x\int_0^3 xyz\,dz\,dy\,dx\left(= \frac{9}{16}\right)$$



29. $Q = \{(x, y, z): x^2 + y^2 \le 9, 0 \le z \le 4\}$

$$\iiint_Q xyz\,dV = \int_0^4\int_{-3}^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} xyz\,dy\,dx\,dz$$

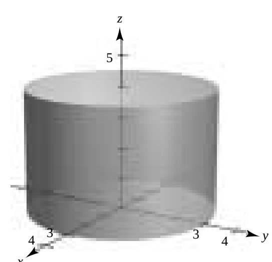
$$= \int_0^4\int_{-3}^3\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} xyz\,dx\,dy\,dz$$

$$= \int_{-3}^3\int_0^4\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} xyz\,dx\,dz\,dy$$

$$= \int_{-3}^3\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_0^4 xyz\,dz\,dx\,dy$$

$$= \int_{-3}^3\int_0^4\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} xyz\,dy\,dz\,dx$$

$$= \int_{-3}^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^4 xyz\,dz\,dy\,dx\;\;(= 0)$$



31. $$m = k\int_0^6\int_0^{4-(2x/3)}\int_0^{2-(y/2)-(x/3)} dz\,dy\,dx = 8k$$

$$M_{yz} = k\int_0^6\int_0^{4-(2x/3)}\int_0^{2-(y/2)-(x/3)} x\,dz\,dy\,dx = 12k$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{12k}{8k} = \frac{3}{2}$$

33. $$m = k\int_0^4\int_0^4\int_0^{4-x} x\,dz\,dy\,dx = k\int_0^4\int_0^4 x(4-x)\,dy\,dx = 4k\int_0^4 (4x - x^2)\,dx = \frac{128k}{3}$$

$$M_{xy} = k\int_0^4\int_0^4\int_0^{4-x} xz\,dz\,dy\,dx = k\int_0^4\int_0^4 x\frac{(4-x)^2}{2}\,dy\,dx = 2k\int_0^4 (16x - 8x^2 + x^3)\,dx = \frac{128k}{3}$$

$$\bar{z} = \frac{M_{xy}}{m} = 1$$

35. $$m = k\int_0^b\int_0^b\int_0^b xy\,dz\,dy\,dx = \frac{kb^5}{4}$$

$$M_{yz} = k\int_0^b\int_0^b\int_0^b x^2y\,dz\,dy\,dx = \frac{kb^6}{6}$$

$$M_{xz} = k\int_0^b\int_0^b\int_0^b xy^2\,dz\,dy\,dx = \frac{kb^6}{6}$$

$$M_{xy} = k\int_0^b\int_0^b\int_0^b xyz\,dz\,dy\,dx = \frac{kb^6}{8}$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{kb^6/6}{kb^5/4} = \frac{2b}{3}$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{kb^6/6}{kb^5/4} = \frac{2b}{3}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{kb^6/8}{kb^5/4} = \frac{b}{2}$$

37. $\bar{x}$ will be greater than 2, whereas $\bar{y}$ and $\bar{z}$ will be unchanged.

39. $\bar{y}$ will be greater than 0, whereas $\bar{x}$ and $\bar{z}$ will be unchanged.

41. $$m = \frac{1}{3}k\pi r^2 h$$

$\bar{x} = \bar{y} = 0$ by symmetry

$$M_{xy} = 4k\int_0^r\int_0^{\sqrt{r^2-x^2}}\int_{h\sqrt{x^2+y^2}/r}^{h} z\,dz\,dy\,dx = \frac{3kh^2}{r^2}\int_0^r\int_0^{\sqrt{r^2-x^2}} (r^2 - x^2 - y^2)\,dy\,dx = \frac{4kh^2}{3r^2}\int_0^r (r^2 - x^2)^{3/2}\,dx = \frac{k\pi r^2h^2}{4}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{k\pi r^2h^2/4}{k\pi r^2h/3} = \frac{3h}{4}$$

43. $m = \dfrac{128k\pi}{3}$

$\bar{x} = \bar{y} = 0$ by symmetry

$z = \sqrt{4^2 - x^2 - y^2}$

$$M_{xy} = 4k\int_0^4\int_0^{\sqrt{4^2-x^2}}\int_0^{\sqrt{4^2-x^2-y^2}} z\,dz\,dy\,dx$$

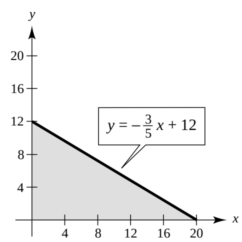
$$= 2k\int_0^4\int_0^{\sqrt{4^2-x^2}} (4^2 - x^2 - y^2)\,dy\,dx = 2k\int_0^4 \left[16y - x^2y - \frac{1}{3}y^3\right]_0^{\sqrt{4^2-x^2}} dx = \frac{4k}{3}\int_0^4 (4^2 - x^2)^{3/2}\,dx$$

$$= \frac{1024k}{3}\int_0^{\pi/2} \cos^4\theta\,d\theta \qquad (\text{let } x = 4\sin\theta)$$

$$= 64\pi k \qquad \text{by Wallis's Formula}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{64k\pi}{1}\cdot\frac{3}{128k\pi} = \frac{3}{2}$$

45. $f(x, y) = \dfrac{5}{12}y$



$$m = k\int_0^{20}\int_0^{-(3/5)x+12}\int_0^{(5/12)y} dz\,dy\,dx = 200k$$

$$M_{yz} = k\int_0^{20}\int_0^{-(3/5)x+12}\int_0^{(5/12)y} x\,dz\,dy\,dx = 1000k$$

$$M_{xz} = k\int_0^{20}\int_0^{-(3/5)+x+12}\int_0^{(5/12)y} y\,dz\,dy\,dx = 1200k$$

$$M_{xy} = k\int_0^{20}\int_0^{-(3/5)x+12}\int_0^{(5/12)y} z\,dz\,dy\,dx = 250k$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{1000k}{200k} = 5$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{1200k}{200k} = 6$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{250k}{200k} = \frac{5}{4}$$

47. (a) $I_x = k\int_0^a\int_0^a\int_0^a (y^2 + z^2)\,dx\,dy\,dz = ka\int_0^a\int_0^a (y^2 + z^2)\,dy\,dz$

$$= ka\int_0^a \left[\frac{1}{3}y^3 + z^2y\right]_0^a dz = ka\int_0^a \left(\frac{1}{3}a^3 + az^2\right) dz = \left[ka\left(\frac{1}{3}a^3z + \frac{1}{3}az^3\right)\right]_0^a = \frac{2ka^5}{3}$$

$I_x = I_y = I_z = \dfrac{2ka^5}{3}$ by symmetry

(b) $I_x = k\int_0^a\int_0^a\int_0^a (y^2 + z^2)xyz\,dx\,dy\,dz = \dfrac{ka^2}{2}\int_0^a\int_0^a (y^3z + yz^3)\,dy\,dz$

$$= \frac{ka^2}{2}\int_0^a \left[\frac{y^4z}{4} + \frac{y^2z^3}{2}\right]_0^a dz = \frac{ka^4}{8}\int_0^a (a^2z + 2z^3)\,dz = \left[\frac{ka^4}{8}\left(\frac{a^2z^2}{2} + \frac{2z^4}{4}\right)\right]_0^a = \frac{ka^8}{8}$$

$I_x = I_y = I_z = \dfrac{ka^8}{8}$ by symmetry

49. (a) $I_x = k\int_0^4\int_0^4\int_0^{4-x}(y^2+z^2)\,dz\,dy\,dx = k\int_0^4\int_0^4\left[y^2(4-x)+\frac{1}{3}(4-x)^3\right]dy\,dx$

$$= k\int_0^4\left[\frac{y^3}{3}(4-x)+\frac{y}{3}(4-x)^3\right]_0^4 dx = k\int_0^4\left[\frac{64}{3}(4-x)+\frac{4}{3}(4-x)^3\right]dx$$

$$= k\left[-\frac{32}{3}(4-x)^2-\frac{1}{3}(4-x)^4\right]_0^4 = 256k$$

$$I_y = k\int_0^4\int_0^4\int_0^{4-x}(x^2+z^2)\,dz\,dy\,dx = k\int_0^4\int_0^4\left[x^2(4-x)+\frac{1}{3}(4-x)^3\right]dy\,dx$$

$$= 4k\int_0^4\left[4x^2-x^3+\frac{1}{3}(4-x)^3\right]dx = 4k\left[\frac{4}{3}x^3-\frac{1}{4}x^4-\frac{1}{12}(4-x)^4\right]_0^4 = \frac{512k}{3}$$

$$I_z = k\int_0^4\int_0^4\int_0^{4-x}(x^2+y^2)\,dz\,dy\,dx = k\int_0^4\int_0^4(x^2+y^2)(4-x)\,dy\,dx$$

$$= k\int_0^4\left[\left(x^2y+\frac{y^3}{3}\right)(4-x)\right]_0^4 dx = k\int_0^4\left(4x^2+\frac{64}{3}\right)(4-x)\,dx = 256k$$

(b) $I_x = k\int_0^4\int_0^4\int_0^{4-x}y(y^2+z^2)\,dz\,dy\,dx = k\int_0^4\int_0^4\left[y^3(4-x)+\frac{1}{3}y(4-x)^3\right]dy\,dx$

$$= k\int_0^4\left[\frac{y^4}{4}(4-x)+\frac{y^2}{6}(4-x)^3\right]_0^4 dx = k\int_0^4\left[64(4-x)+\frac{8}{3}(4-x)^3\right]dx$$

$$= k\left[-32(4-x)^2-\frac{2}{3}(4-x)^4\right]_0^4 = \frac{2048k}{3}$$

$$I_y = k\int_0^4\int_0^4\int_0^{4-x}y(x^2+z^2)\,dz\,dy\,dx = k\int_0^4\int_0^4\left[x^2y(4-x)+\frac{1}{3}y(4-x)^3\right]dy\,dx$$

$$= 8k\int_0^4\left[4x^2-x^3+\frac{1}{3}(4-x)^3\right]dx = 8k\left[\frac{4}{3}x^3-\frac{1}{4}x^4-\frac{1}{12}(4-x)^4\right]_0^4 = \frac{1024k}{3}$$

$$I_z = k\int_0^4\int_0^4\int_0^{4-x}y(x^2+y^2)\,dz\,dy\,dx = k\int_0^4\int_0^4(x^2y+y^3)(4-x)\,dx$$

$$= k\int_0^4\left[\left(\frac{x^2y^2}{2}+\frac{y^4}{4}\right)(4-x)\right]_0^4 dx = k\int_0^4(8x^2+64)(4-x)\,dx$$

$$= 8k\int_0^4(32-8x+4x^2-x^3)\,dx = \left[8k\left(32x-4x^2+\frac{4}{3}x^3-\frac{1}{4}x^4\right)\right]_0^4 = \frac{2048k}{3}$$

51. $I_{xy} = k\int_{-L/2}^{L/2}\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} z^2\,dz\,dx\,dy = k\int_{-L/2}^{L/2}\int_{-a}^{a}\frac{2}{3}(a^2-x^2)\sqrt{a^2-x^2}\,dx\,dy$

$$= \frac{2}{3}\int_{-L/2}^{L/2}k\left[\frac{a^2}{2}\left(x\sqrt{a^2-x^2}+a^2\arcsin\frac{x}{a}\right)-\frac{1}{8}\left(x(2x^2-a^2)\sqrt{x^2-a^2}+a^4\arcsin\frac{x}{a}\right)\right]_{-a}^{a}dy$$

$$= \frac{2k}{3}\int_{-L/2}^{L/2}2\left(\frac{a^4\pi}{4}-\frac{a^4\pi}{16}\right)dy = \frac{a^4\pi Lk}{4}$$

Since $m = \pi a^2Lk$, $I_{xy} = ma^2/4$.

—CONTINUED—

51. —CONTINUED—

$$I_{xz} = k\int_{-L/2}^{L/2}\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y^2\,dz\,dx\,dy = 2k\int_{-L/2}^{L/2}\int_{-a}^{a} y^2\sqrt{a^2-x^2}\,dx\,dy$$

$$= 2k\int_{-L/2}^{L/2}\left[\frac{y^2}{2}\left(x\sqrt{a^2-x^2} + a^2\arcsin\frac{x}{a}\right)\right]_{-a}^{a} dy = k\pi a^2\int_{-L/2}^{L/2} y^2\,dy = \frac{2k\pi a^2}{3}\left(\frac{L^3}{8}\right) = \frac{1}{12}mL^2$$

$$I_{yz} = k\int_{-L/2}^{L/2}\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} x^2\,dz\,dx\,dy = 2k\int_{-L/2}^{L/2}\int_{-a}^{a} x^2\sqrt{a^2-x^2}\,dx\,dy$$

$$= 2k\int_{-L/2}^{L/2}\frac{1}{8}\left[x(2x^2-a^2)\sqrt{a^2-x^2} + a^4\arcsin\frac{x}{a}\right]_{-a}^{a} dy = \frac{ka^4\pi}{4}\int_{-L/2}^{L/2} dy = \frac{ka^4\pi L}{4} = \frac{ma^2}{4}$$

$$I_x = I_{xy} + I_{xz} = \frac{ma^2}{4} + \frac{mL^2}{12} = \frac{m}{12}(3a^2 + L^2)$$

$$I_y = I_{xy} + I_{yz} = \frac{ma^2}{4} + \frac{ma^2}{4} = \frac{ma^2}{2}$$

$$I_z = I_{xz} + I_{yz} = \frac{mL^2}{12} + \frac{ma^2}{4} = \frac{m}{12}(3a^2 + L^2)$$

53. $\displaystyle\int_{-1}^{1}\int_{-1}^{1}\int_{0}^{1-x} (x^2+y^2)\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$

55. See the definition, page 978.

See Theorem 13.4, page 979.

57. (a) The annular solid on the right has the greater density.

(b) The annular solid on the right has the greater movement of inertia.

(c) The solid on the left will reach the bottom first. The solid on the right has a greater resistance to rotational motion.

Section 13.7 Triple Integrals in Cylindrical and Spherical Coordinates

1. $\displaystyle\int_0^4\int_0^{\pi/2}\int_0^2 r\cos\theta\,dr\,d\theta\,dz = \int_0^4\int_0^{\pi/2}\left[\frac{r^2}{2}\cos\theta\right]_0^2 d\theta\,dz$

$$= \int_0^4\int_0^{\pi/2} 2\cos\theta\,d\theta\,dz = \int_0^4\Big[2\sin\theta\Big]_0^{\pi/2} dz = \int_0^4 2dz = 8$$

3. $\displaystyle\int_0^{\pi/2}\int_0^{2\cos^2\theta}\int_0^{4-r^2} r\sin\theta\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^{2\cos^2\theta} r(4-r^2)\sin\theta\,dr\,d\theta = \int_0^{\pi/2}\left[\left(2r^2 - \frac{r^4}{4}\right)\sin\theta\right]_0^{2\cos^2\theta} d\theta$

$$= \int_0^{\pi/2}[8\cos^4\theta - 4\cos^8\theta]\sin\theta\,d\theta = \left[-\frac{8\cos^5\theta}{5} + \frac{4\cos^9\theta}{9}\right]_0^{\pi/2} = \frac{52}{45}$$

5. $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/4}\cos^3\phi\sin\phi\,d\phi\,d\theta = -\frac{1}{12}\int_0^{2\pi}\Big[\cos^4\phi\Big]_0^{\pi/4} d\theta = \frac{\pi}{8}$

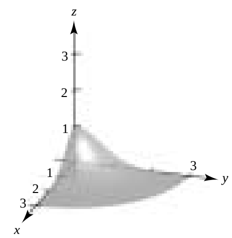
7. $\displaystyle\int_0^4\int_0^z\int_0^{\pi/2} re^r\,d\theta\,dr\,dz = \pi(e^4+3)$

9. $\displaystyle\int_0^{\pi/2}\int_0^3\int_0^{e^{-r^2}} r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^3 re^{-r^2}\,dr\,d\theta$

$\displaystyle= \int_0^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_0^3 d\theta$

$\displaystyle= \int_0^{\pi/2}\frac{1}{2}(1 - e^{-9})d\theta$

$\displaystyle= \frac{\pi}{4}(1 - e^{-9})$

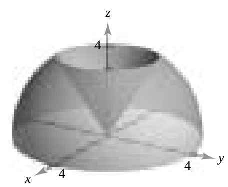


11. $\displaystyle\int_0^{2\pi}\int_{\pi/6}^{\pi/2}\int_0^4 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{64}{3}\int_0^{2\pi}\int_{\pi/6}^{\pi/2}\sin\phi\,d\phi\,d\theta$

$\displaystyle= \frac{64}{3}\int_0^{2\pi}\Big[-\cos\phi\Big]_{\pi/6}^{\pi/2}d\theta$

$\displaystyle= \frac{32\sqrt{3}}{3}\int_0^{2\pi}d\theta$

$\displaystyle= \frac{64\sqrt{3}\pi}{3}$



13. (a) $\displaystyle\int_0^{2\pi}\int_0^2\int_{r^2}^4 r^2\cos\theta\,dz\,dr\,d\theta = 0$

(b) $\displaystyle\int_0^{2\pi}\int_0^{\arctan(1/2)}\int_0^{4\sec\phi}\rho^3\sin^2\phi\cos\theta\,d\rho\,d\phi\,d\theta + \int_0^{2\pi}\int_{\arctan(1/2)}^{\pi/2}\int_0^{\cot\phi\csc\phi}\rho^3\sin^2\phi\cos\theta\,d\rho\,d\phi\,d\theta = 0$

15. (a) $\displaystyle\int_0^{2\pi}\int_0^a\int_a^{a+\sqrt{a^2-r^2}} r^2\cos\theta\,dz\,dr\,d\theta = 0$

(b) $\displaystyle\int_0^{\pi/4}\int_0^{2\pi}\int_{a\sec\phi}^{2a\cos\phi}\rho^3\sin^2\phi\cos\theta\,d\rho\,d\theta\,d\phi = 0$

17. $\displaystyle V = 4\int_0^{\pi/2}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta = 4\int_0^{\pi/2}\int_0^{a\cos\theta} r\sqrt{a^2 - r^2}\,dr\,d\theta$

$\displaystyle= \frac{4}{3}a^3\int_0^{\pi/2}(1 - \sin^3\theta)\,d\theta = \frac{4}{3}a^3\left[\theta + \frac{1}{3}\cos\theta(\sin^2\theta + 2)\right]_0^{\pi/2} = \frac{4}{3}a^3\left(\frac{\pi}{2} - \frac{2}{3}\right) = \frac{2a^3}{9}(3\pi - 4)$

19. $\displaystyle V = 2\int_0^{\pi}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$

$\displaystyle= 2\int_0^{\pi}\int_0^{a\cos\theta} r\sqrt{a^2 - r^2}\,dr\,d\theta$

$\displaystyle= 2\int_0^{\pi}\left[-\frac{1}{3}(a^2 - r^2)^{3/2}\right]_0^{a\cos\theta} d\theta$

$\displaystyle= \frac{2a^3}{3}\int_0^{\pi}(1 - \sin^3\theta)\,d\theta$

$\displaystyle= \frac{2a^3}{3}\left[\theta + \cos\theta - \frac{\cos^3\theta}{3}\right]_0^{\pi}$

$\displaystyle= \frac{2a^3}{9}(3\pi - 4)$

21. $\displaystyle m = \int_0^{2\pi}\int_0^2\int_0^{9-r\cos\theta-2r\sin\theta}(kr)r\,dz\,dr\,d\theta$

$\displaystyle= \int_0^{2\pi}\int_0^2 kr^2(9 - r\cos\theta - 2r\sin\theta)dr\,d\theta$

$\displaystyle= \int_0^{2\pi}k\left[3r^3 - \frac{r^4}{4}\cos\theta - \frac{r^4}{2}\sin\theta\right]_0^2 d\theta$

$\displaystyle= \int_0^{2\pi}k[24 - 4\cos\theta - 8\sin\theta]d\theta$

$\displaystyle= k\Big[24\theta - 4\sin\theta + 8\cos\theta\Big]_0^{2\pi}$

$= k[48\pi + 8 - 8] = 48k\pi$

23. $z = h - \frac{h}{r_0}\sqrt{x^2 + y^2} = \frac{h}{r_0}(r_0 - r)$

$$V = 4\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} r\,dz\,dr\,d\theta$$

$$= \frac{4h}{r_0}\int_0^{\pi/2}\int_0^{r_0}(r_0 r - r^2)\,dr\,d\theta$$

$$= \frac{4h}{r_0}\int_0^{\pi/2}\frac{r_0^3}{6}\,d\theta$$

$$= \frac{4h}{r_0}\left(\frac{r_0^3}{6}\right)\left(\frac{\pi}{2}\right) = \frac{1}{3}\pi r_0^2 h$$

25. $\rho = k\sqrt{x^2 + y^2} = kr$

$\bar{x} = \bar{y} = 0$ by symmetry

$$m = 4k\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} r^2\,dz\,dr\,d\theta$$

$$= \frac{1}{6}k\pi r_0^3 h$$

$$M_{xy} = 4k\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} r^2 z\,dz\,dr\,d\theta$$

$$= \frac{1}{30}k\pi r_0^3 h^2$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{k\pi r_0^3 h^2/30}{k\pi r_0^3 h/6} = \frac{h}{5}$$

27. $I_z = 4k\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} r^3\,dz\,dr\,d\theta$

$$= \frac{4kh}{r_0}\int_0^{\pi/2}\int_0^{r_0}(r_0 r^3 - r^4)\,dr\,d\theta$$

$$= \frac{4kh}{r_0}\left(\frac{r_0^5}{20}\right)\left(\frac{\pi}{2}\right)$$

$$= \frac{1}{10}k\pi r_0^4 h$$

Since the mass of the core is $m = kV = k\left(\frac{1}{3}\pi r_0^2 h\right)$ from Exercise 23, we have $k = 3m/\pi r_0^2 h$. Thus,

$$I_z = \frac{1}{10}k\pi r_0^4 h$$

$$= \frac{1}{10}\left(\frac{3m}{\pi r_0^2 h}\right)\pi r_0^4 h$$

$$= \frac{3}{10}m r_0^2$$

29. $m = k(\pi b^2 h - \pi a^2 h) = k\pi h(b^2 - a^2)$

$$I_z = 4k\int_0^{\pi/2}\int_a^b\int_0^h r^3\,dz\,dr\,d\theta$$

$$= 4kh\int_0^{\pi/2}\int_a^b r^3\,dr\,d\theta$$

$$= kh\int_0^{\pi/2}(b^4 - a^4)\,d\theta$$

$$= \frac{k\pi(b^4 - a^4)h}{2}$$

$$= \frac{k\pi(b^2 - a^2)(b^2 + a^2)h}{2}$$

$$= \frac{1}{2}m(a^2 + b^2)$$

31. $V = \int_0^{2\pi}\int_0^{\pi}\int_0^{4\sin\phi} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = 16\pi^2$

33. $m = 8k\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \rho^3 \sin\phi\,d\rho\,d\theta\,d\phi$

$$= 2ka^4\int_0^{\pi/2}\int_0^{\pi/2}\sin\phi\,d\theta\,d\phi$$

$$= k\pi a^4\int_0^{\pi/2}\sin\phi\,d\phi$$

$$= \Big[k\pi a^4(-\cos\phi)\Big]_0^{\pi/2}$$

$$= k\pi a^4$$

35. $m = \frac{2}{3}k\pi r^3$

$\bar{x} = \bar{y} = 0$ by symmetry

$$M_{xy} = 4k\int_0^{\pi/2}\int_0^{\pi/2}\int_0^r \rho^3 \cos\phi \sin\phi \, d\rho \, d\theta \, d\phi$$
$$= \frac{1}{2}kr^4\int_0^{\pi/2}\int_0^{\pi/2} \sin 2\phi \, d\theta \, d\phi$$
$$= \frac{kr^4\pi}{4}\int_0^{\pi/2} \sin 2\phi \, d\phi$$
$$= \left[-\frac{1}{8}k\pi r^4 \cos 2\phi\right]_0^{\pi/2} = \frac{1}{4}k\pi r^4$$
$$\bar{z} = \frac{M_{xy}}{m} = \frac{k\pi r^4/4}{2k\pi r^3/3} = \frac{3r}{8}$$

37. $I_z = 4k\int_{\pi/4}^{\pi/2}\int_0^{\pi/2}\int_0^{\cos\phi} \rho^4 \sin^3\phi \, d\rho \, d\theta \, d\phi$

$$= \frac{4}{5}k\int_{\pi/4}^{\pi/2}\int_0^{\pi/2} \cos^5\phi \sin^3\phi \, d\theta \, d\phi$$
$$= \frac{2}{5}k\pi\int_{\pi/4}^{\pi/2} \cos^5\phi(1 - \cos^2\phi)\sin\phi \, d\phi$$
$$= \left[\frac{2}{5}k\pi\left(-\frac{1}{6}\cos^6\phi + \frac{1}{8}\cos^8\phi\right)\right]_{\pi/4}^{\pi/2}$$
$$= \frac{k\pi}{192}$$

39. $x = r\cos\theta$ $\qquad x^2 + y^2 = r^2$

$y = r\sin\theta$ $\qquad \tan\theta = \frac{y}{x}$

$z = z$ $\qquad z = z$

41. $\displaystyle\int_{\theta_1}^{\theta_2}\int_{g_1(\theta)}^{g_2(\theta)}\int_{h_1(r\cos\theta,\, r\sin\theta)}^{h_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)r \, dz \, dr \, d\theta$

43. (a) $r = r_0$: right circular cylinder about z-axis

$\theta = \theta_0$: plane parallel to z-axis

$z = z_0$: plane parallel to xy-plane

(b) $\rho = \rho_0$: sphere of radius ρ_0

$\theta = \theta_0$: plane parallel to z-axis

$\phi = \phi_0$: cone

45. $\displaystyle 16\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}}\int_0^{\sqrt{a^2-x^2-y^2-z^2}} dw \, dz \, dy \, dx$

$$= 16\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} \sqrt{a^2 - x^2 - y^2 - z^2} \, dz \, dy \, dx$$
$$= 16\int_0^{\pi/2}\int_0^a\int_0^{\sqrt{a^2-r^2}} \sqrt{(a^2 - r^2) - z^2} \, dz(r \, dr \, d\theta)$$
$$= 16\int_0^{\pi/2}\int_0^a \frac{1}{2}\left[z\sqrt{(a^2 - r^2) - z^2} + (a^2 - r^2)\arcsin\frac{z}{\sqrt{a^2 - r^2}}\right]_0^{\sqrt{a^2-r^2}} r \, dr \, d\theta$$
$$= 8\int_0^{\pi/2}\int_0^a \frac{\pi}{2}(a^2 - r^2)r \, dr \, d\theta$$
$$= 4\pi\int_0^{\pi/2}\left[\frac{a^2r^2}{2} - \frac{r^4}{4}\right]_0^a d\theta$$
$$= a^4\pi\int_0^{\pi/2} d\theta = \frac{a^4\pi^2}{2}$$

Section 13.8 Change of Variables: Jacobians

1. $x = -\frac{1}{2}(u - v)$

$y = \frac{1}{2}(u + v)$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{2}$$

3. $x = u - v^2$

$y = u + v$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = (1)(1) - (1)(-2v) = 1 + 2v$$

5. $x = u\cos\theta - v\sin\theta$

$y = u\sin\theta + v\cos\theta$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \cos^2\theta + \sin^2\theta = 1$$

7. $x = e^u \sin v$

$y = e^u \cos v$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = (e^u\sin v)(-e^u\sin v) - (e^u\cos v)(e^u\cos v) = -e^{2u}$$

9. $x = 3u + 2v$

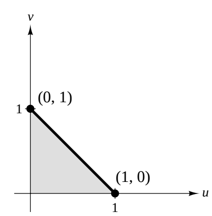
$y = 3v$

$v = \frac{y}{3}$

$u = \frac{x - 2v}{3} = \frac{x - 2(y/3)}{3}$

$= \frac{x}{3} - \frac{2y}{9}$

(x, y)	(u, v)
(0, 0)	(0, 0)
(3, 0)	(1, 0)
(2, 3)	(0, 1)



11. $x = \frac{1}{2}(u + v)$

$y = \frac{1}{2}(u - v)$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{2}$$

$$\iint_R 4(x^2 + y^2)\,dA = \int_{-1}^{1}\int_{-1}^{1} 4\left[\frac{1}{4}(u + v)^2 + \frac{1}{4}(u - v)^2\right]\left(\frac{1}{2}\right)dv\,du$$

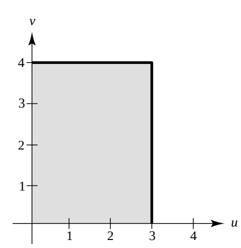
$$= \int_{-1}^{1}\int_{-1}^{1}(u^2 + v^2)\,dv\,du = \int_{-1}^{1} 2\left(u^2 + \frac{1}{3}\right)du = \left[2\left(\frac{u^3}{3} + \frac{u}{3}\right)\right]_{-1}^{1} = \frac{8}{3}$$

13. $x = u + v$

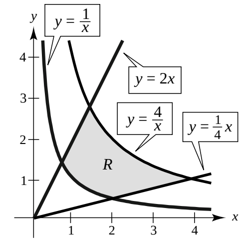
$y = u$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = (1)(0) - (1)(1) = -1$$

$$\iint_R y(x - y)\,dA = \int_0^3\int_0^4 uv(1)\,dv\,du = \int_0^3 8u\,du = 36$$



15. $\displaystyle\int_R\int e^{-xy/2}\,dA$

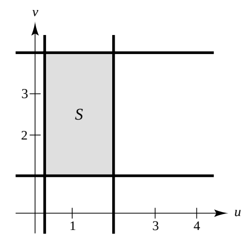


R: $y = \dfrac{x}{4}, y = 2x, y = \dfrac{1}{x}, y = \dfrac{4}{x}$

$x = \sqrt{v/u}, y = \sqrt{uv} \implies u = \dfrac{y}{x}, \ v = xy$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2}\frac{v^{1/2}}{u^{3/2}} & \frac{1}{2}\frac{1}{u^{1/2}v^{1/2}} \\ \frac{1}{2}\frac{v^{1/2}}{u^{1/2}} & \frac{1}{2}\frac{u^{1/2}}{v^{1/2}} \end{vmatrix} = -\frac{1}{4}\left(\frac{1}{u} + \frac{1}{u}\right) = -\frac{1}{2u}$$

Transformed Region:



$y = \dfrac{1}{x} \implies yx = 1 \implies v = 1$

$y = \dfrac{4}{x} \implies yx = 4 \implies v = 4$

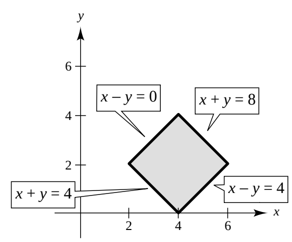
$y = 2x \implies \dfrac{y}{x} = 2 \implies u = 2$

$y = \dfrac{x}{4} \implies \dfrac{y}{x} = \dfrac{1}{4} \implies u = \dfrac{1}{4}$

$$\int_R\int e^{-xy/2}\,dA = \int_{1/4}^{2}\int_{1}^{4} e^{-v/2}\left(\frac{1}{2u}\right)dv\,du = -\int_{1/4}^{2}\left[\frac{e^{-v/2}}{u}\right]_1^4 du = -\int_{1/4}^{2}(e^{-2} - e^{-1/2})\frac{1}{u}\,du$$

$$= -\Big[(e^{-2} - e^{-1/2})\ln u\Big]_{1/4}^{2} = -(e^{-2} - e^{-1/2})\left(\ln 2 - \ln\frac{1}{4}\right) = (e^{-1/2} - e^{-2})\ln 8 \approx 0.9798$$

17. $u = x + y = 4, \quad v = x - y = 0$



$u = x + y = 8, \quad v = x - y = 4$

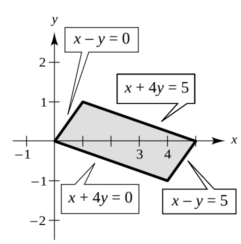
$x = \dfrac{1}{2}(u + v) \qquad y = \dfrac{1}{2}(u - v)$

$\dfrac{\partial(x, y)}{\partial(u, v)} = -\dfrac{1}{2}$

$$\int_R\int (x + y)e^{x-y}\,dA = \int_4^8\int_0^4 ue^v\left(\frac{1}{2}\right)dv\,du$$

$$= \frac{1}{2}\int_4^8 u(e^4 - 1)\,du = \left[\frac{1}{4}u^2(e^4 - 1)\right]_4^8 = 12(e^4 - 1)$$

19. $u = x + 4y = 0, \quad v = x - y = 0$



$u = x + 4y = 5, \quad v = x - y = 5$

$x = \dfrac{1}{5}(u + 4v), \qquad y = \dfrac{1}{5}(u - v)$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \left(\frac{1}{5}\right)\left(-\frac{1}{5}\right) - \left(\frac{1}{5}\right)\left(\frac{4}{5}\right) = -\frac{1}{5}$$

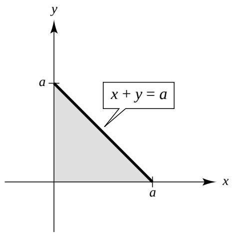
$$\int_R\int \sqrt{(x - y)(x + 4y)}\,dA = \int_0^5\int_0^5 \sqrt{uv}\left(\frac{1}{5}\right)du\,dv$$

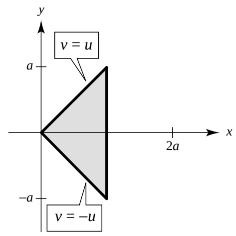
$$= \int_0^5\left[\frac{1}{5}\left(\frac{2}{3}\right)u^{3/2}\sqrt{v}\right]_0^5 dv = \left[\frac{2\sqrt{5}}{3}\left(\frac{2}{3}\right)v^{3/2}\right]_0^5 = \frac{100}{9}$$

21. $u = x + y, v = x - y, x = \frac{1}{2}(u + v), y = \frac{1}{2}(u - v)$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = -\frac{1}{2}$$

$$\int_R\int \sqrt{x + y}\, dA = \int_0^a \int_{-u}^{u} \sqrt{u}\left(\frac{1}{2}\right) dv\, du = \int_0^a u\sqrt{u}\, du = \left[\frac{2}{5}u^{5/2}\right]_0^a = \frac{2}{5}a^{5/2}$$



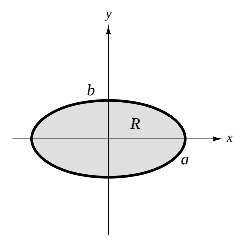


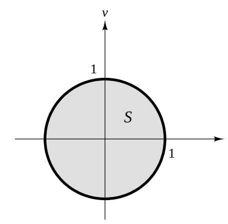
23. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, x = au, y = bv$

$$\frac{(au)^2}{a^2} + \frac{(bv)^2}{b^2} = 1$$

$$u^2 + v^2 = 1$$

(a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\qquad$ $u^2 + v^2 = 1$





(b) $\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$

$= (a)(b) - (0)(0) = ab$

(c) $A = \int_S\int ab\, dS$

$= ab(\pi(1)^2) = \pi ab$

25. Jacobian $= \frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$

27. $x = u(1 - v),\ y = uv(1 - w),\ z = uvw$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 1 - v & -u & 0 \\ v(1 - w) & u(1 - w) & -uv \\ vw & uw & uv \end{vmatrix} = (1 - v)[u^2v(1 - w) + u^2vw] + u[uv^2(1 - w) + uv^2w]$$

$$= (1 - v)(u^2v) + u(uv^2)$$

$$= u^2v$$

29. $x = \rho \sin\phi \cos\theta,\ y = \rho \sin\phi \sin\theta,\ z = \rho\cos\phi$

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\ \cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

$$= \cos\phi[-\rho^2\sin\phi\cos\phi\sin^2\theta - \rho^2\sin\phi\cos\phi\cos^2\theta] - \rho\sin\phi[\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta]$$

$$= \cos\phi[-\rho^2\sin\phi\cos\phi(\sin^2\theta + \cos^2\theta)] - \rho\sin\phi[\rho\sin^2\phi(\cos^2\theta + \sin^2\theta)]$$

$$= -\rho^2\sin\phi\cos^2\phi - \rho^2\sin^3\phi$$

$$= -\rho^2\sin\phi(\cos^2\phi + \sin^2\phi)$$

$$= -\rho^2\sin\phi$$