Review Exercises for Chapter 13

1. $\displaystyle\int_1^{x^2} x\ln y\,dy = \Big[xy(-1+\ln y)\Big]_1^{x^2} = x^3(-1+\ln x^2) + x = x - x^3 + x^3\ln x^2$

3. $\displaystyle\int_0^1\int_0^{1+x}(3x+2y)\,dy\,dx = \int_0^1\Big[3xy + y^2\Big]_0^{1+x}dx = \int_0^1(4x^2+5x+1)\,dx = \Big[\frac{4}{3}x^3 + \frac{5}{2}x^2 + x\Big]_0^1 = \frac{29}{6}$

5. $\displaystyle\int_0^3\int_0^{\sqrt{9-x^2}} 4x\,dy\,dx = \int_0^3 4x\sqrt{9-x^2}\,dx = \Big[-\frac{4}{3}(9-x^2)^{3/2}\Big]_0^3 = 36$

7. $\displaystyle\int_0^3\int_0^{(3-x)/3} dy\,dx = \int_0^1\int_0^{3-3y} dx\,dy$

$$A = \int_0^1\int_0^{3-3y} dx\,dy = \int_0^1(3-3y)\,dy = \Big[3y - \frac{3}{2}y^2\Big]_0^1 = \frac{3}{2}$$

9. $\displaystyle\int_{-5}^3\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} dy\,dx = \int_{-5}^{-4}\int_{-\sqrt{25-y^2}}^{\sqrt{25-y^2}} dx\,dy + \int_{-4}^4\int_{-\sqrt{25-y^2}}^{3} dx\,dy + \int_4^5\int_{-\sqrt{25-y^2}}^{\sqrt{25-y^2}} dx\,dy$

$$A = 2\int_{-5}^3\int_0^{\sqrt{25-x^2}} dy\,dx = 2\int_{-5}^3\sqrt{25-x^2}\,dx = \Big[x\sqrt{25-x^2} + 25\arcsin\frac{x}{5}\Big]_{-5}^3 = \frac{25\pi}{2} + 12 + 25\arcsin\frac{3}{5} \approx 67.36$$

11. $\displaystyle A = 4\int_0^1\int_0^{x\sqrt{1-x^2}} dy\,dx = 4\int_0^1 x\sqrt{1-x^2}\,dx = \Big[-\frac{4}{3}(1-x^2)^{3/2}\Big]_0^1 = \frac{4}{3}$

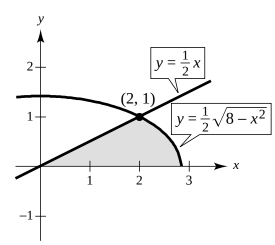
$$A = 4\int_0^{1/2}\int_{\sqrt{(1-\sqrt{1-4y^2})/2}}^{\sqrt{(1+\sqrt{1-4y^2})/2}} dx\,dy$$

13. $\displaystyle A = \int_2^5\int_{x-3}^{\sqrt{x-1}} dy\,dx + 2\int_1^2\int_0^{\sqrt{x-1}} dy\,dx = \int_{-1}^2\int_{y^2+1}^{y+3} dx\,dy = \frac{9}{2}$

15. Both integrations are over the common region R shown in the figure. Analytically,

$$\int_0^1\int_{2y}^{2\sqrt{2-y^2}}(x+y)\,dx\,dy = \frac{4}{3} + \frac{4}{3}\sqrt{2}$$

$$\int_0^2\int_0^{x/2}(x+y)\,dy\,dx + \int_2^{2\sqrt{2}}\int_0^{\sqrt{8-x^2}/2}(x+y)\,dy\,dx = \frac{5}{3} + \left(\frac{4}{3}\sqrt{2} - \frac{1}{3}\right) = \frac{4}{3} + \frac{4}{3}\sqrt{2}$$



17. $\displaystyle V = \int_0^4\int_0^{x^2+4}(x^2 - y + 4)\,dy\,dx$

$$= \int_0^4\Big[x^2y - \frac{1}{2}y^2 + 4y\Big]_0^{x^2+4}dx$$

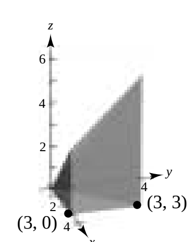
$$= \int_0^4\left(\frac{1}{2}x^4 + 4x^2 + 8\right)dx$$

$$= \Big[\frac{1}{10}x^5 + \frac{4}{3}x^3 + 8x\Big]_0^4 = \frac{3296}{15}$$

19. Volume $\approx$ (base)(height)

$$\approx \frac{9}{2}(3) = \frac{27}{2}$$

Matches (c)



21. $\int_0^\infty \int_0^\infty kxye^{-(x+y)}\, dy\, dx = \int_0^\infty \Big[-kxe^{-(x+y)}(y+1)\Big]_0^\infty dx = \int_0^\infty kxe^{-x}\, dx = \Big[-k(x+1)e^{-x}\Big]_0^\infty = k$

Therefore, $k = 1$.

$$P = \int_0^1 \int_0^1 xye^{-(x+y)}\, dy\, dx \approx 0.070$$

23. True

25. True

27. $\int_0^h \int_0^x \sqrt{x^2+y^2}\, dy\, dx = \int_0^{\pi/4} \int_0^{h \sec\theta} r^2\, dr\, d\theta$

$$= \frac{h^3}{3}\int_0^{\pi/4} \sec^3\theta\, d\theta = \frac{h^3}{6}\Big[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big]_0^{\pi/4} = \frac{h^3}{6}\Big[\sqrt{2} + \ln\big(\sqrt{2}+1\big)\Big]$$

29. $V = 4\int_0^h \int_0^{\pi/2} \int_1^{\sqrt{1+z^2}} r\, dr\, d\theta\, dz$

$$= 2\int_0^h \int_0^{\pi/2} (1 + z^2 - 1)\, d\theta\, dz$$

$$= \pi \int_0^h z^2\, dz$$

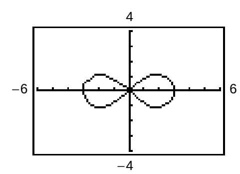
$$= \left[\pi\left(\frac{1}{3}z^3\right)\right]_0^h = \frac{\pi h^3}{3}$$

31. (a) $(x^2+y^2)^2 = 9(x^2-y^2)$

$$(r^2)^2 = 9(r^2\cos^2\theta - r^2\sin^2\theta)$$

$$r^2 = 9(\cos^2\theta - \sin^2\theta) = 9\cos 2\theta$$

$$r = 3\sqrt{\cos 2\theta}$$



(b) $A = 4\int_0^{\pi/4} \int_0^{3\sqrt{\cos 2\theta}} r\, dr\, d\theta = 9$

(c) $V = 4\int_0^{\pi/4} \int_0^{3\sqrt{\cos 2\theta}} \sqrt{9 - r^2}\, r\, dr\, d\theta \approx 20.392$

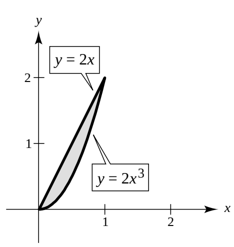
33. (a) $m = k\int_0^1 \int_{2x^3}^{2x} xy\, dy\, dx = \frac{k}{4}$

$$M_x = k\int_0^1 \int_{2x^3}^{2x} xy^2\, dy\, dx = \frac{16k}{55}$$

$$M_y = k\int_0^1 \int_{2x^3}^{2x} x^2y\, dy\, dx = \frac{8k}{45}$$

$$\bar{x} = \frac{M_y}{m} = \frac{32}{45}$$

$$\bar{y} = \frac{M_x}{m} = \frac{64}{55}$$



(b) $m = k\int_0^1 \int_{2x^3}^{2x} (x^2+y^2)\, dy\, dx = \frac{17k}{30}$

$$M_x = k\int_0^1 \int_{2x^3}^{2x} y(x^2+y^2)\, dy\, dx = \frac{392k}{585}$$

$$M_y = k\int_0^1 \int_{2x^3}^{2x} x(x^2+y^2)\, dy\, dx = \frac{156k}{385}$$

$$\bar{x} = \frac{M_y}{m} = \frac{936}{1309}$$

$$\bar{y} = \frac{M_x}{m} = \frac{784}{663}$$

35. $I_x = \int\int_R y^2\,\rho(x, y)dA = \int_0^a\int_0^b kxy^2\,dy\,dx = \frac{1}{6}kb^3a^2$

$I_y = \int\int_R x^2\,\rho(x, y)dA = \int_0^a\int_0^b kx^3\,dy\,dx = \frac{1}{4}kba^4$

$I_0 = I_x + I_y = \frac{1}{6}kb^3a^2 + \frac{1}{4}kba^4 = \frac{ka^2b}{12}(2b^2 + 3a^2)$

$m = \int\int_R \rho(x, y)dA = \int_0^a\int_0^b kx\,dy\,dx = \frac{1}{2}kba^2$

$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{(1/4)kba^4}{(1/2)kba^2}} = \sqrt{\frac{a^2}{2}} = \frac{a\sqrt{2}}{2}$

$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{(1/6)kb^3a^2}{(1/2)kba^2}} = \sqrt{\frac{b^2}{3}} = \frac{b\sqrt{3}}{3}$

37. $S = \int\int_R \sqrt{1 + (f_x)^2 + (f_y)^2}\,dA$

$= 4\int_0^4\int_0^{\sqrt{16-x^2}} \sqrt{1 + 4x^2 + 4y^2}\,dy\,dx$

$= 4\int_0^{\pi/2}\int_0^4 \sqrt{1 + 4r^2}\,r\,dr\,d\theta$

$= \left[\frac{1}{3}(65^{3/2} - 1)\theta\right]_0^{\pi/2} = \frac{\pi}{6}\left(65\sqrt{65} - 1\right)$

39. $f(x, y) = 9 - y^2$

$f_x = 0,\ f_y = -2y$

$S = \int\int_R \sqrt{1 + f_x^2 + f_y^2}\,dA$

$= \int_0^3\int_{-y}^{y} \sqrt{1 + 4y^2}\,dx\,dy$

$= \int_0^3\left[\sqrt{1 + 4y^2}\,x\right]_{-y}^{y} dy$

$= \int_0^3 2\sqrt{1 + 4y^2}\,dy = \frac{1}{4}\frac{2}{3}(1 + 4y^2)^{3/2}\Big]_0^3 = \frac{1}{6}[(37)^{3/2} - 1]$

41. $\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{x^2+y^2}^{9} \sqrt{x^2 + y^2}\,dz\,dy\,dx = \int_0^{2\pi}\int_0^3\int_{r^2}^{9} r^2\,dz\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^3 (9r^2 - r^4)\,dr\,d\theta = \int_0^{2\pi}\left[3r^3 - \frac{r^5}{5}\right]_0^3 d\theta = \frac{162}{5}\int_0^{2\pi} d\theta = \frac{324\pi}{5}$

43. $\int_0^a\int_0^b\int_0^c (x^2 + y^2 + z^2)\,dx\,dy\,dz = \int_0^a\int_0^b\left(\frac{1}{3}c^3 + cy^2 + cz^2\right)dy\,dz$

$= \int_0^a\left(\frac{1}{3}bc^3 + \frac{1}{3}b^3c + bcz^2\right)dz = \frac{1}{3}abc^3 + \frac{1}{3}ab^3c + \frac{1}{3}a^3bc = \frac{1}{3}abc(a^2 + b^2 + c^2)$

45. $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} (x^2 + y^2)\,dz\,dy\,dx = \int_0^{2\pi}\int_0^1\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r^3\,dz\,dr\,d\theta = \frac{8\pi}{15}$

47. $$V = 4\int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\int_0^{2\cos\theta} r\sqrt{4-r^2}\,dr\,d\theta$$

$$= -\int_0^{\pi/2}\left[\frac{4}{3}(4-r^2)^{3/2}\right]_0^{2\cos\theta} d\theta$$

$$= \frac{32}{3}\int_0^{\pi/2}(1-\sin^3\theta)\,d\theta$$

$$= \frac{32}{3}\left[\theta + \cos\theta - \frac{1}{3}\cos^3\theta\right]_0^{\pi/2} = \frac{32}{3}\left(\frac{\pi}{2} - \frac{2}{3}\right)$$

49. $$m = 4k\int_{\pi/4}^{\pi/2}\int_0^{\pi/2}\int_0^{\cos\phi}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \frac{4}{3}k\int_{\pi/4}^{\pi/2}\int_0^{\pi/2}\cos^3\phi\sin\phi\,d\theta\,d\phi = \frac{2}{3}k\pi\int_{\pi/4}^{\pi/2}\cos^3\phi\sin\phi\,d\phi = \left[-\frac{2}{3}k\pi\left(\frac{1}{4}\cos^4\phi\right)\right]_{\pi/4}^{\pi/2} = \frac{k\pi}{24}$$

$$M_{xy} = 4k\int_{\pi/4}^{\pi/2}\int_0^{\pi/2}\int_0^{\cos\phi}\rho^3\cos\phi\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= k\int_{\pi/4}^{\pi/2}\int_0^{\pi/2}\cos^5\phi\sin\phi\,d\theta\,d\phi = \frac{1}{2}k\pi\int_{\pi/4}^{\pi/2}\cos^5\phi\sin\phi\,d\phi = \left[-\frac{1}{12}k\pi\cos^6\phi\right]_{\pi/4}^{\pi/2} = \frac{k\pi}{96}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{k\pi/96}{k\pi/24} = \frac{1}{4}$$

$\bar{x} = \bar{y} = 0$ by symmetry

51. $$m = k\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \frac{k\pi a^3}{6}$$

$$M_{xy} = k\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a(\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \frac{k\pi a^4}{16}$$

$$\bar{x} = \bar{y} = \bar{z} = \frac{M_{xy}}{m} = \frac{k\pi a^4}{16}\left(\frac{6}{k\pi a^3}\right) = \frac{3a}{8}$$

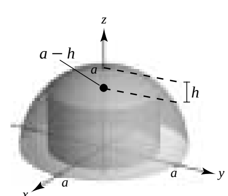
53. $$I_z = 4k\int_0^{\pi/2}\int_3^4\int_0^{16-r^2} r^3\,dz\,dr\,d\theta$$

$$= 4k\int_0^{\pi/2}\int_3^4(16r^3 - r^5)\,dr\,d\theta = \frac{833\pi k}{3}$$

55. $$z = f(x, y) = \sqrt{a^2 - x^2 - y^2}$$

$$= \sqrt{a^2 - r^2}$$

$$0 \le r \le \sqrt{2ah - h^2}$$

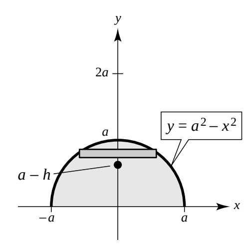


(a) Disc Method

$$V = \pi\int_{a-h}^a (a^2 - y^2)dy$$

$$= \pi\left[a^2y - \frac{y^3}{3}\right]_{a-h}^a = \pi\left[\left(a^3 - \frac{a^3}{3}\right) - \left(a^2(a-h) - \frac{(a-h)^3}{3}\right)\right]$$

$$= \pi\left[a^3 - \frac{a^3}{3} - a^3 + a^2h + \frac{a^3}{3} - a^2h + ah^2 - \frac{h^3}{3}\right] = \pi\left[ah^2 - \frac{h^3}{3}\right] = \frac{1}{3}\pi h^2[3a - h]$$



Equivalently, use spherical coordinates

$$V = \int_0^{2\pi}\int_0^{\cos^{-1}(a-h/a)}\int_{(a-h)\sec\phi}^a \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

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55. —CONTINUED—

(b) $M_{xy} = \int_0^{2\pi}\int_0^{\cos^{-1}(a-h/a)}\int_{(a-h)\sec\phi}^{a} (\rho\cos\phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$= \frac{1}{4}h^2\pi(2a - h)^2$

$\bar{z} = \frac{M_{xy}}{V} = \frac{\frac{1}{4}h^2\pi(2a-h)^2}{\frac{1}{3}h^2\pi(3a-h)} = \frac{3}{4}\frac{(2a-h)^2}{3a-h}$

centroid: $\left(0, 0, \frac{3(2a-h)^2}{4(3a-h)}\right)$

(c) If $h = a$, $\bar{z} = \frac{3(a)^2}{4(2a)} = \frac{3}{8}a$

centroid of hemisphere: $\left(0, 0, \frac{3}{8}a\right)$

(d) $\lim_{h\to 0} \bar{z} = \lim_{h\to 0} \frac{3(2a-h)^2}{4(3a-h)} = \frac{3(4a^2)}{12a} = a$

(e) $x^2 + y^2 = \rho^2 \sin^2\phi$

$I_z = \int_0^{2\pi}\int_0^{\cos^{-1}(a-h/a)}\int_{(a-h)\sec\phi}^{a} (\rho^2\sin^2\phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$= \frac{h^3}{30}(20a^2 - 15ah + 3h^2)\pi$

(f) If $h = a$, $I_z = \frac{a^3\pi}{30}(20a^2 - 15a^2 + 3a^2) = \frac{4}{15}a^5\pi$

57. $\int_0^{2\pi}\int_0^{\pi}\int_0^{6\sin\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

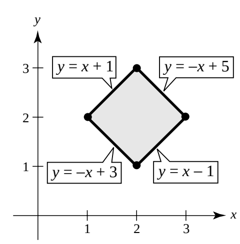
Since $\rho = 6\sin\phi$ represents (in the yz-plane) a circle of radius 3 centered at (0, 3, 0), the integral represents the volume of the torus formed by revolving $(0 < \theta < 2\pi)$ this circle about the z-axis.

59. $\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$

$= 1(-3) - 2(3) = -9$

61. $\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \frac{1}{2}\left(-\frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{2}\right) = -\frac{1}{2}$

$x = \frac{1}{2}(u + v), y = \frac{1}{2}(u - v) \implies u = x + y, v = x - y$



Boundaries in xy-plane	Boundaries in uv-plane
$x + y = 3$	$u = 3$
$x + y = 5$	$u = 5$
$x - y = -1$	$v = -1$
$x - y = 1$	$v = 1$

$\int\int_R \ln(x + y)dA = \int_3^5\int_{-1}^{1} \ln\left(\frac{1}{2}(u + v) + \frac{1}{2}(u - v)\right)\left(\frac{1}{2}\right) dv \, du = \int_3^5\int_{-1}^{1}\frac{1}{2}\ln u \, dv \, du = \int_3^5 \ln u \, du = \Big[u \ln u - u\Big]_3^5$

$= (5\ln 5 - 5) - (3\ln u - 3) = 5\ln 5 - 3\ln 3 - 2 \approx 2.751$

Problem Solving for Chapter 13

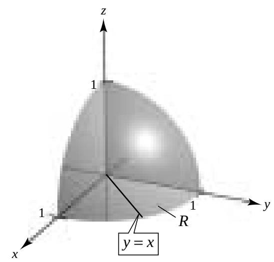
1. (a) $V = 16\iint_R \sqrt{1 - x^2}\, dA$

$= 16\int_0^{\pi/4}\int_0^1 \sqrt{1 - r^2\cos^2\theta}\, r\, dr\, d\theta$

$= -\frac{16}{3}\int_0^{\pi/4} \frac{1}{\cos^2\theta}\left[(1 - \cos^2\theta)^{3/2} - 1\right] d\theta$

$= -\frac{16}{3}\Big[\sec\theta + \cos\theta - \tan\theta\Big]_0^{\pi/4}$

$= 8\left(2 - \sqrt{2}\right) \approx 4.6863$



(b) Programs will vary.

3. (a) $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\frac{u}{a} + c$. Let $a^2 = 2 - u^2$, $u = v$.

Then $\int \frac{1}{(2 - u^2) + v^2}\, dv = \frac{1}{\sqrt{2 - u^2}}\arctan\frac{v}{\sqrt{2 - u^2}} + C$.

(b) $I_1 = \int_0^{\sqrt{2}/2}\left[\frac{2}{\sqrt{2 - u^2}}\arctan\frac{v}{\sqrt{2 - u^2}}\right]_{-u}^{u} du$

$= \int_0^{\sqrt{2}/2} \frac{2}{\sqrt{2 - u^2}}\left(\arctan\frac{u}{\sqrt{2 - u^2}} - \arctan\frac{-u}{\sqrt{2 - u^2}}\right) du$

$= \int_0^{\sqrt{2}/2} \frac{4}{\sqrt{2 - u^2}}\arctan\frac{u}{\sqrt{2 - u^2}}\, du$

Let $u = \sqrt{2}\sin\theta$, $du = \sqrt{2}\cos\theta\, d\theta$, $2 - u^2 = 2 - 2\sin^2\theta = 2\cos^2\theta$.

$I_1 = 4\int_0^{\pi/6} \frac{1}{\sqrt{2}\cos\theta}\arctan\left(\frac{\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta}\right) \cdot \sqrt{2}\cos\theta\, d\theta$

$= 4\int_0^{\pi/6} \arctan(\tan\theta) d\theta = \frac{4\theta^2}{2}\Big]_0^{\pi/6} = 2\left(\frac{\pi}{6}\right)^2 = \frac{\pi^2}{18}$

(c) $I_2 = \int_{\sqrt{2}/2}^{\sqrt{2}}\left[\frac{2}{\sqrt{2 - u^2}}\arctan\frac{v}{\sqrt{2 - u^2}}\right]_{u - \sqrt{2}}^{-u + \sqrt{2}} du$

$= \int_{\sqrt{2}/2}^{\sqrt{2}} \frac{2}{\sqrt{2 - u^2}}\left[\arctan\left(\frac{-u + \sqrt{2}}{\sqrt{2 - u^2}}\right) - \arctan\left(\frac{u - \sqrt{2}}{\sqrt{2 - u^2}}\right)\right] du$

$= \int_{\sqrt{2}/2}^{\sqrt{2}} \frac{4}{\sqrt{2 - u^2}}\arctan\left(\frac{\sqrt{2} - u}{\sqrt{2 - u^2}}\right) du$

Let $u = \sqrt{2}\sin\theta$.

$I_2 = 4\int_{\pi/6}^{\pi/2} \frac{1}{\sqrt{2}\cos\theta}\arctan\left(\frac{\sqrt{2} - \sqrt{2}\sin\theta}{\sqrt{2}\cos\theta}\right) \cdot \sqrt{2}\cos\theta\, d\theta$

$= 4\int_{\pi/6}^{\pi/2} \arctan\left(\frac{1 - \sin\theta}{\cos\theta}\right) d\theta$

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3. —CONTINUED—

(d) $$\tan\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right) = \sqrt{\frac{1-\cos((\pi/2)-\theta)}{1+\cos((\pi/2)-\theta)}} = \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}$$

$$= \sqrt{\frac{(1-\sin\theta)^2}{(1+\sin\theta)(1-\sin\theta)}} = \sqrt{\frac{(1-\sin\theta)^2}{\cos^2\theta}} = \frac{1-\sin\theta}{\cos\theta}$$

(e) $$I_2 = 4\int_{\pi/6}^{\pi/2}\arctan\left(\frac{1-\sin\theta}{\cos\theta}\right)d\theta = 4\int_{\pi/6}^{\pi/2}\arctan\left(\tan\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)\right)d\theta$$

$$= 4\int_{\pi/6}^{\pi/2}\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)d\theta = 2\int_{\pi/6}^{\pi/2}\left(\frac{\pi}{2}-\theta\right)d\theta$$

$$= 2\left[\frac{\pi}{2}\theta - \frac{\theta^2}{2}\right]_{\pi/6}^{\pi/2} = 2\left[\left(\frac{\pi^2}{4}-\frac{\pi^2}{8}\right)-\left(\frac{\pi^2}{12}-\frac{\pi^2}{72}\right)\right]$$

$$= 2\left[\frac{18-9-6+1}{72}\pi^2\right] = \frac{4}{36}\pi^2 = \frac{\pi^2}{9}$$

(f) $$\frac{1}{1-xy} = 1 + (xy) + (xy)^2 + \cdots \qquad |xy| < 1$$

$$\int_0^1\int_0^1 \frac{1}{1-xy}\,dx\,dy = \int_0^1\int_0^1 [1 + (xy) + (xy)^2 + \cdots]\,dx\,dy$$

$$= \int_0^1\int_0^1 \sum_{K=0}^{\infty}(xy)^K\,dx\,dy = \sum_{K=0}^{\infty}\int_0^1 \left.\frac{x^{K+1}y^K}{K+1}\right|_0^1 dy$$

$$= \sum_{K=0}^{\infty}\int_0^1 \frac{y^K}{K+1}\,dy = \sum_{K=0}^{\infty}\left.\frac{y^{K+1}}{(K+1)^2}\right|_0^1$$

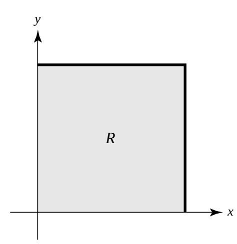
$$= \sum_{K=0}^{\infty}\frac{1}{(K+1)^2} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

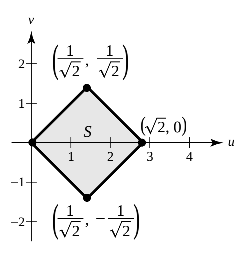
(g) $$u = \frac{x+y}{\sqrt{2}},\ v = \frac{y-x}{\sqrt{2}}$$

$$u - v = \frac{2x}{\sqrt{2}} \implies x = \frac{u-v}{\sqrt{2}}$$

$$u + v = \frac{2y}{\sqrt{2}} \implies y = \frac{u+v}{\sqrt{2}}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{vmatrix} = 1$$





<u>R</u>		<u>S</u>
(0, 0)	$\leftrightarrow$	(0, 0)
(1, 0)	$\leftrightarrow$	$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$
(0, 1)	$\leftrightarrow$	$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
(1, 1)	$\leftrightarrow$	$(\sqrt{2}, 0)$

$$\int_0^1\int_0^1\frac{1}{1-xy}\,dx\,dy = \int_0^{\sqrt{2}/2}\int_{-u}^{u}\frac{1}{1-\frac{u^2}{2}+\frac{v^2}{2}}\,dv\,du + \int_{\sqrt{2}/2}^{\sqrt{2}}\int_{u-\sqrt{2}}^{-u+\sqrt{2}}\frac{1}{1-\frac{u^2}{2}+\frac{v^2}{2}}\,dv\,du$$

$$= I_1 + I_2 = \frac{\pi^2}{18} + \frac{\pi^2}{9} = \frac{\pi^2}{6}$$

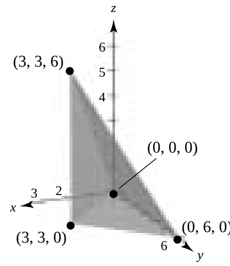
5. Boundary in *xy*-plane | Boundary in *uv*-plane

Boundary in *xy*-plane	Boundary in *uv*-plane
$y = \sqrt{x}$	$u = 1$
$y = \sqrt{2x}$	$u = 2$
$y = \frac{1}{3}x^2$	$v = 3$
$y = \frac{1}{4}x^2$	$v = 4$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{1}{3}\left(\frac{v}{u}\right)^{2/3} & \frac{2}{3}\left(\frac{u}{v}\right)^{1/3} \\ \frac{2}{3}\left(\frac{v}{u}\right)^{1/3} & \frac{1}{3}\left(\frac{u}{v}\right)^{2/3} \end{vmatrix} = -\frac{1}{3}$$

$$A = \int_R\int 1\,dA = \int_S\int 1\left|\frac{\partial(x, y)}{\partial(u, v)}\right| dA = \frac{1}{3}$$

7.



$$V = \int_0^3\int_0^{2x}\int_x^{6-x} dy\,dz\,dx = 18$$

9. From Exercise 55, Section 13.3,

$$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$$

Thus, $\displaystyle\int_0^{\infty} e^{-x^2/2}\,dx = \frac{\sqrt{2\pi}}{2}$ and $\displaystyle\int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$

$$\int_0^{\infty} x^2e^{-x^2}\,dx = \left[-\frac{1}{2}xe^{-x^2}\right]_0^{\infty} + \frac{1}{2}\int_0^{\infty} e^{-x^2}\,dx = \frac{1}{2}\frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{4}$$

11. $f(x, y) = \begin{cases} ke^{-(x+y)/a} & x \ge 0, y \ge 0 \\ 0 & \text{elsewhere} \end{cases}$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dA = \int_0^{\infty}\int_0^{\infty} ke^{-(x+y)/a}\,dx\,dy$$

$$= k\int_0^{\infty} e^{-x/a}\,dx \cdot \int_0^{\infty} e^{-y/a}\,dy$$

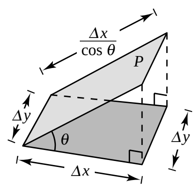
These two integrals are equal to

$$\int_0^{\infty} e^{-x/a}\,dx = \lim_{b\to\infty}\left[(-a)e^{-x/a}\right]_0^b = a.$$

Hence, assuming $a, k > 0$, you obtain

$$1 = ka^2 \quad \text{or} \quad a = \frac{1}{\sqrt{k}}.$$

13. $A = l \cdot w = \left(\frac{\Delta x}{\cos\theta}\right)\Delta y = \sec\theta\,\Delta x\,\Delta y$



Area in *xy*-plane: $\Delta x\,\Delta y$