# CHAPTER 14
# Vector Analysis

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## Section 14.1 Vector Fields

Solutions to Even-Numbered Exercises

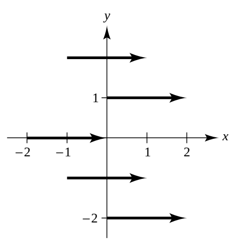
**2.** All vectors are parallel to $x$-axis.
Matches (d)

**4.** Vectors are in rotational pattern.
Matches (e)

**6.** Vectors along $x$-axis have no $x$-component.
Matches (f)

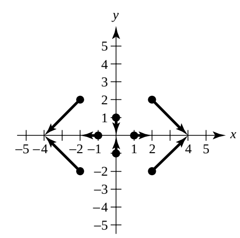
**8.** $\mathbf{F}(x, y) = 2\mathbf{i}$

$\|\mathbf{F}\| = 2$



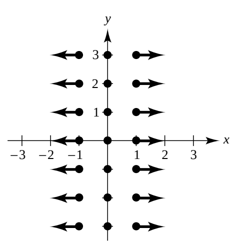

**10.** $\mathbf{F}(x, y) = x\mathbf{i} - y\mathbf{j}$

$\|\mathbf{F}\| = \sqrt{x^2 + y^2}$



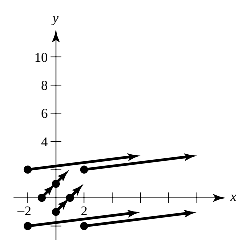

**12.** $\mathbf{F}(x, y) = x\mathbf{i}$

$\|\mathbf{F}\| = |x| = c$




**14.** $\mathbf{F}(x, y) = (x^2 + y^2)\mathbf{i} + \mathbf{j}$

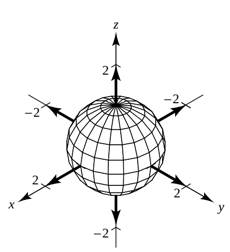
$\|\mathbf{F}\| = \sqrt{1 + (x^2 + y^2)^2}$



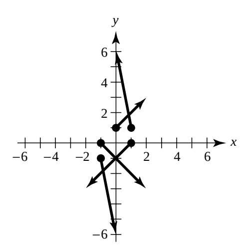

**16.** $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$\|\mathbf{F}\| = \sqrt{x^2 + y^2 + z^2} = c$

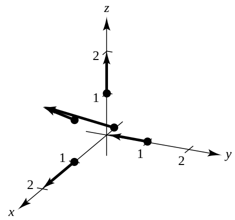
$x^2 + y^2 + z^2 = c^2$




**18.** $\mathbf{F}(x, y) = (2y - 3x)\mathbf{i} + (2y + 3x)\mathbf{j}$



**20.** $\mathbf{F}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$




**22.** $f(x, y) = \sin 3x \cos 4y$

$f_x(x, y) = 3 \cos 3x \cos 4y$

$f_y(x, y) = -4 \sin 3x \sin 4y$

$\mathbf{F}(x, y) = 3 \cos 3x \cos 4y\mathbf{i} - 4 \sin 3x \sin 4y\mathbf{j}$

**24.** $f(x, y, z) = \dfrac{y}{z} + \dfrac{z}{x} - \dfrac{xz}{y}$

$f_x(x, y, z) = -\dfrac{z}{x^2} - \dfrac{z}{y}$

$f_y(x, y, z) = \dfrac{1}{z} + \dfrac{xz}{y^2}$

$f_z(x, y, z) = -\dfrac{y}{z^2} + \dfrac{1}{x} - \dfrac{x}{y}$

$\mathbf{F}(x, y, z) = \left(-\dfrac{z}{x^2} - \dfrac{z}{y}\right)\mathbf{i} + \left(\dfrac{1}{z} + \dfrac{xz}{y^2}\right)\mathbf{j} + \left(-\dfrac{y}{z^2} + \dfrac{1}{x} - \dfrac{x}{y}\right)\mathbf{k}$

**26.** $g(x, y, z) = x \arcsin yz$

$g_x(x, y, z) = \arcsin yz$

$g_y(x, y, z) = \dfrac{xz}{\sqrt{1 - y^2z^2}}$

$g_z(x, y, z) = \dfrac{xy}{\sqrt{1 - y^2z^2}}$

$\mathbf{G}(x, y, z) = (\arcsin yz)\mathbf{i} + \dfrac{xz}{\sqrt{1 - y^2z^2}}\mathbf{j} + \dfrac{xy}{\sqrt{1 - y^2z^2}}\mathbf{k}$

**28.** $\mathbf{F}(x, y) = \dfrac{1}{x^2}(y\mathbf{i} - x\mathbf{j}) = \dfrac{y}{x^2}\mathbf{i} - \dfrac{1}{x}\mathbf{j}$

$M = y/x^2$ and $N = -(1/x)$ have continuous first partial derivatives for all $x \neq 0$.

$\dfrac{\partial N}{\partial x} = \dfrac{1}{x^2} = \dfrac{\partial M}{\partial y} \implies$ **F** is conservative.

**30.** $\mathbf{F}(x, y) = \dfrac{1}{xy}(y\mathbf{i} - x\mathbf{j}) = \dfrac{1}{x}\mathbf{i} - \dfrac{1}{y}\mathbf{j}$

$M = 1/x$ and $N = -1/y$ have continuous first partial derivatives for all $x, y \neq 0$.

$\dfrac{\partial N}{\partial x} = 0 = \dfrac{\partial M}{\partial y} \implies$ **F** is conservative.

**32.** $M = \dfrac{x}{\sqrt{x^2 + y^2}}, N = \dfrac{y}{\sqrt{x^2 + y^2}}$

$\dfrac{\partial N}{\partial x} = \; + \dfrac{-xy}{(x^2 + y^2)^{3/2}} = \dfrac{\partial M}{\partial y} \implies$ Conservative

**34.** $M = \dfrac{y}{\sqrt{1 - x^2y^2}}, N = \dfrac{-x}{\sqrt{1 - x^2y^2}}$

$\dfrac{\partial N}{\partial x} = \; + \dfrac{-1}{(1 - x^2y^2)^{3/2}} \neq \dfrac{\partial M}{\partial y} \; = \dfrac{1}{(1 - x^2y^2)^{3/2}}$

$\implies$ Not conservative

**36.** $\mathbf{F}(x, y) = \dfrac{1}{y^2}(y\mathbf{i} - 2x\mathbf{j})$

$= \dfrac{1}{y}\mathbf{i} - \dfrac{2x}{y^2}\mathbf{j}$

$\dfrac{\partial}{\partial y}\left[\dfrac{1}{y}\right] = -\dfrac{1}{y^2}$

$\dfrac{\partial}{\partial x}\left[-\dfrac{2x}{y^2}\right] = -\dfrac{2}{y^2}$

Not conservative

**38.** $\mathbf{F}(x, y) = 3x^2y^2\mathbf{i} + 2x^3y\mathbf{j}$

$\dfrac{\partial}{\partial y}[3x^2y^2] = 6x^2y$

$\dfrac{\partial}{\partial x}[2x^3y] = 6x^2y$

Conservative

$f_x(x, y) = 3x^2y^2$

$f_y(x, y) = 2x^3y$

$f(x, y) = x^3y^2 + K$

**40.** $\mathbf{F}(x, y) = \dfrac{2y}{x}\mathbf{i} - \dfrac{x^2}{y^2}\mathbf{j}$

$\dfrac{\partial}{\partial y}\left[\dfrac{2y}{x}\right] = \dfrac{2}{x}$

$\dfrac{\partial}{\partial x}\left[-\dfrac{x^2}{y^2}\right] = -\dfrac{2x}{y^2}$

Not conservative

**42.** $\mathbf{F}(x, y) = \dfrac{2x}{(x^2 + y^2)^2}\mathbf{i} + \dfrac{2y}{(x^2 + y^2)^2}\mathbf{j}$

$$\frac{\partial}{\partial y}\left[\frac{2x}{(x^2+y^2)^2}\right] = -\frac{8xy}{(x^2+y^2)^3}$$

$$\frac{\partial}{\partial x}\left[\frac{2y}{(x^2+y^2)^2}\right] = -\frac{8xy}{(x^2+y^2)^3}$$

Conservative

$$f_x(x, y) = \frac{2x}{(x^2+y^2)^2}$$

$$f_y(x, y) = \frac{2y}{(x^2+y^2)^2}$$

$$f(x, y) = -\frac{1}{x^2+y^2} + K$$

**44.** $\mathbf{F}(x, y, z) = x^2z\,\mathbf{i} - 2xz\,\mathbf{j} + yz\,\mathbf{k}$, $(2, -1, 3)$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2z & -2xz & yz \end{vmatrix}$$

$$= (z + 2x)\mathbf{i} - (0 - x^2)\mathbf{j} + (-2z - 0)\mathbf{k}$$

$$= (z + 2x)\mathbf{i} + x^2\mathbf{j} - 2z\mathbf{k}$$

$$\textbf{curl F}\,(2, -1, 3) = 7\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}$$

**46.** $\mathbf{F}(x, y, z) = e^{-xyz}(\mathbf{i} + \mathbf{j} + \mathbf{k})$, $(3, 2, 0)$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^{-xyz} & e^{-xyz} & e^{-xyz} \end{vmatrix} = (-xz + xy)e^{-xyz}\mathbf{i} - (-yz + xy)e^{-xyz}\mathbf{j} + (-yz + xz)e^{-xyz}\mathbf{k}$$

$$\textbf{curl F}\,(3, 2, 0) = 6\mathbf{i} - 6\mathbf{j}$$

**48.** $\mathbf{F}(x, y, z) = \dfrac{yz}{y - z}\mathbf{i} + \dfrac{xz}{x - z}\mathbf{j} + \dfrac{xy}{x - y}\mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{yz}{y-z} & \dfrac{xz}{x-z} & \dfrac{xy}{x-y} \end{vmatrix}$$

$$= \left[\frac{x^2}{(x-y)^2} - \frac{x^2}{(x-z)^2}\right]\mathbf{i} - \left[\frac{-y^2}{(x-y)^2} - \frac{y^2}{(y-z)^2}\right]\mathbf{j} + \left[\frac{-z^2}{(x-z)^2} - \frac{-z^2}{(y-z)^2}\right]\mathbf{k}$$

$$= x^2\left[\frac{1}{(x-y)^2} - \frac{1}{(x-z)^2}\right]\mathbf{i} + y^2\left[\frac{1}{(x-y)^2} + \frac{1}{(y-z)^2}\right]\mathbf{j} + z^2\left[\frac{1}{(y-z)^2} - \frac{1}{(x-z)^2}\right]\mathbf{k}$$

**50.** $\mathbf{F}(x, y, z) = \sqrt{x^2 + y^2 + z^2}(\mathbf{i} + \mathbf{j} + \mathbf{k})$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \sqrt{x^2+y^2+z^2} & \sqrt{x^2+y^2+z^2} & \sqrt{x^2+y^2+z^2} \end{vmatrix} = \frac{(y - z)\mathbf{i} + (z - x)\mathbf{j} + (x - y)\mathbf{k}}{\sqrt{x^2+y^2+z^2}}$$

**52.** $\mathbf{F}(x, y, z) = e^z(y\mathbf{i} + x\mathbf{j} + \mathbf{k})$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ ye^z & xe^z & e^z \end{vmatrix} = -xe^z\mathbf{i} + ye^z\mathbf{j} \neq \mathbf{0}$$

Not conservative

**54.** $\mathbf{F}(x, y, z) = y^2z^3\mathbf{i} + 2xyz^3\mathbf{j} + 3xy^2z^2\mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} = \mathbf{0}$$

Conservative

$$f(x, y, z) = xy^2z^3 + K$$

**56.** $\mathbf{F}(x, y, z) = \dfrac{x}{x^2 + y^2}\mathbf{i} + \dfrac{y}{x^2 + y^2}\mathbf{j} + \mathbf{k}$

$$\mathbf{curl\, F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{x}{x^2 + y^2} & \dfrac{y}{x^2 + y^2} & 1 \end{vmatrix} = \mathbf{0}$$

Conservative

$$f_x(x, y, z) = \frac{x}{x^2 + y^2}$$

$$f_y(x, y, z) = \frac{y}{x^2 + y^2}$$

$$f_z(x, y, z) = 1$$

$$f(x, y, z) = \int \frac{x}{x^2 + y^2}\, dx = \frac{1}{2}\ln(x^2 + y^2) + g(y, z) + K_1$$

$$f(x, y, z) = \int \frac{y}{x^2 + y^2}\, dy = \frac{1}{2}\ln(x^2 + y^2) + h(x, z) + K_2$$

$$f(x, y, z) = \int dz = z + p(x, y) + K_3$$

$$f(x, y, z) = \frac{1}{2}\ln(x^2 + y^2) + z + K$$

**58.** $\mathbf{F}(x, y) = xe^x\mathbf{i} + ye^y\mathbf{j}$

$$\text{div}\,\mathbf{F}(x, y) = \frac{\partial}{\partial x}[xe^x] + \frac{\partial}{\partial y}[ye^y] = xe^x + e^x + ye^y + e^y = e^x(x + 1) + e^y(y + 1)$$

**60.** $\mathbf{F}(x, y, z) = \ln(x^2 + y^2)\mathbf{i} + xy\mathbf{j} + \ln(y^2 + z^2)\mathbf{k}$

$$\text{div}\,\mathbf{F}(x, y, z) = \frac{\partial}{\partial x}[\ln(x^2 + y^2)] + \frac{\partial}{\partial y}[xy] + \frac{\partial}{\partial z}[\ln(y^2 + z^2)] = \frac{2x}{x^2 + y^2} + x + \frac{2z}{y^2 + z^2}$$

**62.** $\mathbf{F}(x, y, z) = x^2z\mathbf{i} - 2xz\mathbf{j} + yz\mathbf{k}$

$$\text{div}\,\mathbf{F}(x, y, z) = 2xz + y$$

$$\text{div}\,\mathbf{F}(2, -1, 3) = 11$$

**64.** $\mathbf{F}(x, y, z) = \ln(xyz)(\mathbf{i} + \mathbf{j} + \mathbf{k})$

$$\text{div}\,\mathbf{F}(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

$$\text{div}\,\mathbf{F}(3, 2, 1) = \frac{1}{3} + \frac{1}{2} + 1 = \frac{11}{6}$$

**66.** See the definition of Conservative Vector Field on page 1011. To test for a conservative vector field, see Theorem 14.1 and 14.2.

**68.** See the definition on page 1016.

**70.** $\mathbf{F}(x, y, z) = x\mathbf{i} - z\mathbf{k}$

$\mathbf{G}(x, y, z) = x^2\mathbf{i} + y\mathbf{j} + z^2\mathbf{k}$

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & 0 & -z \\ x^2 & y & z^2 \end{vmatrix} = yz\mathbf{i} - (xz^2 + x^2z)\mathbf{j} + xy\mathbf{k}$$

$$\mathbf{curl}(\mathbf{F} \times \mathbf{G}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & -xz^2 - x^2z & xy \end{vmatrix}$$

$$= (x + 2xz + x^2)\mathbf{i} - (y - y)\mathbf{j} + (-z^2 - 2xz - z)\mathbf{k}$$

$$= x(x + 2z + 1)\mathbf{i} - z(z + 2x + 1)\mathbf{k}$$

**72.** $\mathbf{F}(x, y, z) = x^2z\mathbf{i} - 2xz\,\mathbf{j} + yz\mathbf{k}$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z & -2xz & yz \end{vmatrix} = (z + 2x)\mathbf{i} + x^2\mathbf{j} - 2z\mathbf{k}$$

$$\mathbf{curl}(\mathbf{curl\ F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z + 2x & x^2 & -2z \end{vmatrix} = \mathbf{j} + 2x\mathbf{k}$$

**74.** $\mathbf{F}(x, y, z) = x\mathbf{i} - z\mathbf{k}$

$\mathbf{G}(x, y, z) = x^2\mathbf{i} + y\mathbf{j} + z^2\mathbf{k}$

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & 0 & -z \\ x^2 & y & z^2 \end{vmatrix} = yz\mathbf{i} - (xz^2 + x^2z)\mathbf{j} + xy\mathbf{k}$$

$\text{div}(\mathbf{F} \times \mathbf{G}) = 0$

**76.** $\mathbf{F}(x, y, z) = x^2z\mathbf{i} - 2xz\,\mathbf{j} + yz\mathbf{k}$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z & -2xz & yz \end{vmatrix} = (z + 2x)\mathbf{i} + x^2\mathbf{j} - 2z\mathbf{k}$$

$\text{div}(\mathbf{curl\ F}) = 2 - 2 = 0$

**78.** Let $f(x, y, z)$ be a scalar function whose second partial derivatives are continuous.

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

$$\mathbf{curl}(\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\mathbf{i} - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\mathbf{k} = \mathbf{0}$$

**80.** Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ and $\mathbf{G} = R\mathbf{i} + S\mathbf{j} + T\mathbf{k}$.

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ M & N & P \\ R & S & T \end{vmatrix} = (NT - PS)\mathbf{i} - (MT - PR)\mathbf{j} + (MS - NR)\mathbf{k}$$

$$\begin{aligned} \text{div}(\mathbf{F} \times \mathbf{G}) &= \frac{\partial}{\partial x}(NT - PS) + \frac{\partial}{\partial y}(PR - MT) + \frac{\partial}{\partial z}(MS - NR) \\ &= N\frac{\partial T}{\partial x} + T\frac{\partial N}{\partial x} - P\frac{\partial S}{\partial x} - S\frac{\partial P}{\partial x} + P\frac{\partial R}{\partial y} + R\frac{\partial P}{\partial y} - M\frac{\partial T}{\partial y} - T\frac{\partial M}{\partial y} + M\frac{\partial S}{\partial z} + S\frac{\partial M}{\partial z} - N\frac{\partial R}{\partial z} - R\frac{\partial N}{\partial z} \\ &= \left[\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)R + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)S + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)T\right] - \left[M\left(\frac{\partial T}{\partial y} - \frac{\partial S}{\partial z}\right) + N\left(\frac{\partial R}{\partial z} - \frac{\partial T}{\partial x}\right) + P\left(\frac{\partial S}{\partial x} - \frac{\partial R}{\partial y}\right)\right] \\ &= (\mathbf{curl\ F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\mathbf{curl\ G}) \end{aligned}$$

**82.** Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$.

$$\begin{aligned} \nabla \times (f\mathbf{F}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fM & fN & fP \end{vmatrix} \\ &= \left(\frac{\partial f}{\partial y}P + f\frac{\partial P}{\partial y} - \frac{\partial f}{\partial z}N - f\frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial f}{\partial x}P + f\frac{\partial P}{\partial x} - \frac{\partial f}{\partial z}M - f\frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial f}{\partial x}N + f\frac{\partial N}{\partial x} - \frac{\partial f}{\partial y}M - f\frac{\partial M}{\partial y}\right)\mathbf{k} \\ &= f\left[\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}\right] + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ M & N & P \end{vmatrix} = f[\nabla \times \mathbf{F}] + (\nabla f) \times \mathbf{F} \end{aligned}$$

**84.** Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$.

$$\textbf{curl } \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$$

$$\text{div}(\textbf{curl } \mathbf{F}) = \frac{\partial}{\partial x}\left[\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right] - \frac{\partial}{\partial y}\left[\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right] + \frac{\partial}{\partial z}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]$$

$$= \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z} - \frac{\partial^2 P}{\partial y \partial x} + \frac{\partial^2 M}{\partial y \partial z} + \frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 M}{\partial z \partial y} = 0 \quad \text{(since the mixed partials are equal)}$$

**In Exercises 86 and 88, $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $f(x, y, z) = \|\mathbf{F}(x, y, z)\| = \sqrt{x^2 + y^2 + z^2}$.**

**86.** $$\frac{1}{f} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

$$\nabla\left(\frac{1}{f}\right) = \frac{-x}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{i} + \frac{-y}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{j} + \frac{-z}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{k} = \frac{-(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{\left(\sqrt{x^2 + y^2 + z^2}\right)^3} = -\frac{\mathbf{F}}{f^3}$$

**88.** $$w = \frac{1}{f} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial w}{\partial x} = -\frac{x}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\frac{\partial w}{\partial y} = -\frac{y}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\frac{\partial w}{\partial z} = -\frac{z}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{2y^2 - x^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}$$

$$\frac{\partial^2 w}{\partial z^2} = \frac{2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}}$$

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0$$

Therefore $w = \dfrac{1}{f}$ is harmonic.

## Section 14.2 Line Integrals

**2.** $$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

$$\cos^2 t + \sin^2 t = 1$$

$$\cos^2 t = \frac{x^2}{16}$$

$$\sin^2 t = \frac{y^2}{9}$$

$$x = 4\cos t$$

$$y = 3\sin t$$

$$\mathbf{r}(t) = 4\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$$

$$0 \le t \le 2\pi$$

**4.** $$\mathbf{r}(t) = \begin{cases} t\mathbf{i} + \frac{4}{5}t\mathbf{j}, & 0 \le t \le 5 \\ 5\mathbf{i} + (9 - t)\mathbf{j}, & 5 \le t \le 9 \\ (14 - t)\mathbf{i}, & 9 \le t \le 14 \end{cases}$$

**6.** $$\mathbf{r}(t) = \begin{cases} t\mathbf{i} + t^2\mathbf{j}, & 0 \le t \le 2 \\ (4 - t)\mathbf{i} + 4\mathbf{j}, & 2 \le t \le 4 \\ (8 - t)\mathbf{j}, & 4 \le t \le 8 \end{cases}$$

**8.** $\mathbf{r}(t) = t\mathbf{i} + (2 - t)\mathbf{j},\ 0 \le t \le 2;\ \mathbf{r}'(t) = \mathbf{i} - \mathbf{j}$

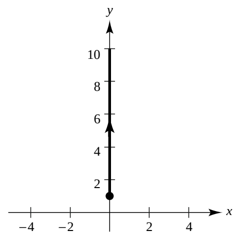
$$\int_C 4xy\, ds = \int_0^2 4t(2 - t)\sqrt{1 + 1}\, dt = 4\sqrt{2}\int_0^2 (2t - t^2)\, dt = 4\sqrt{2}\left[t^2 - \frac{t^3}{3}\right]_0^2 = 4\sqrt{2}\left(4 - \frac{8}{3}\right) = \frac{16\sqrt{2}}{3}$$

**10.** $\mathbf{r}(t) = 12t\mathbf{i} + 5t\mathbf{j} + 3\mathbf{k},\ 0 \le t \le 2;\ \mathbf{r}'(t) = 12\mathbf{i} + 5\mathbf{j}$

$$\int_C 8xyz\,ds = \int_0^2 8(12t)(5t)(3)\sqrt{12^2 + 5^2 + 0^2}\,dt = \int_0^2 18{,}720t^2\,dt = 18{,}720\left[\frac{t^3}{3}\right]_0^2 = 49{,}920$$

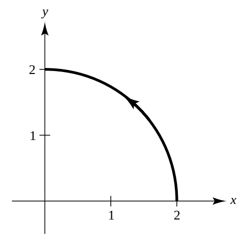
**12.** $\mathbf{r}(t) = t\mathbf{j},\ 1 \le t \le 10$

$$\int_C (x^2 + y^2)\,ds = \int_1^{10} [0 + t^2]\sqrt{0 + 1}\,dt$$
$$= \int_1^{10} t^2\,dt$$
$$= \left[\frac{1}{3}t^3\right]_1^{10} = 333$$



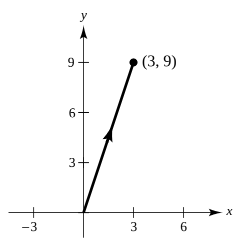

**14.** $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j},\ 0 \le t \le \dfrac{\pi}{2}$

$$\int_C (x^2 + y^2)\,ds = \int_0^{\pi/2} [4\cos^2 t + 4\sin^2 t]\sqrt{(-2\sin t)^2 + (2\cos t)^2}\,dt$$
$$= \int_0^{\pi/2} 8\,dt = 4\pi$$

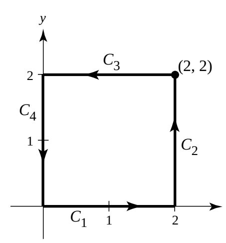



**16.** $\mathbf{r}(t) = t\mathbf{i} + 3t\mathbf{j},\ 0 \le t \le 3$

$$\int_C (x + 4\sqrt{y})\,ds = \int_0^3 \left(t + 4\sqrt{3t}\right)\sqrt{1 + 9}\,dt$$
$$= \left[\sqrt{10}\left(\frac{t^2}{2} + \frac{8\sqrt{3}}{3}t^{3/2}\right)\right]_0^3$$
$$= \frac{\sqrt{10}}{6}(27 + 144) = \frac{57\sqrt{10}}{2}$$




**18.** $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 2 \\ 2\mathbf{i} + (t-2)\mathbf{j}, & 2 \le t \le 4 \\ (6-t)\mathbf{i} + 2\mathbf{j}, & 4 \le t \le 6 \\ (8-t)\mathbf{j}, & 6 \le t \le 8 \end{cases}$




$$\int_{C_1} \left(x + 4\sqrt{y}\right)ds = \int_0^2 t\,dt = 2$$

$$\int_{C_2} \left(x + 4\sqrt{y}\right)ds = \int_2^4 \left(2 + 4\sqrt{t-2}\right)ds = 4 + \frac{16\sqrt{2}}{3}$$

$$\int_{C_3} \left(x + 4\sqrt{y}\right)ds = \int_4^6 \left((6-t) + 4\sqrt{2}\right)ds = 2 + 8\sqrt{2}$$

$$\int_{C_4} \left(x + 4\sqrt{y}\right)ds = \int_6^8 4\sqrt{8-t}\,ds = \frac{16\sqrt{2}}{3}$$

$$\int_C \left(x + 4\sqrt{y}\right)ds = 2 + 4 + \frac{16\sqrt{2}}{3} + 2 + 8\sqrt{2} + \frac{16\sqrt{2}}{3} = 8 + \frac{56}{3}\sqrt{2}$$

**20.** $\rho(x, y, z) = z$

$$\mathbf{r}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + 2t\mathbf{k},\ 0 \le t \le 4\pi$$

$$\mathbf{r}'(t) = -3\sin t\mathbf{i} + 3\cos t\mathbf{j} + 2\mathbf{k}$$

$$\|\mathbf{r}'(t)\| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + (2)^2} = \sqrt{13}$$

$$\text{Mass} = \int_C \rho(x, y, z)\, ds = \int_0^{4\pi} 2t\sqrt{13}\, dt = 16\pi^2\sqrt{13}$$

**22.** $\mathbf{F}(x, y) = xy\mathbf{i} + y\mathbf{j}$

$$C\colon \mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j},\ 0 \le t \le \frac{\pi}{2}$$

$$\mathbf{F}(t) = 16\sin t\cos t\mathbf{i} + 4\sin t\mathbf{j}$$

$$\mathbf{r}'(t) = -4\sin t\mathbf{i} + 4\cos t\mathbf{j}$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/2} (-64\sin^2 t\cos t + 16\sin t\cos t)\,dt$$

$$= \left[-\frac{64}{3}\sin^3 t + 8\sin^2 t\right]_0^{\pi/2} = -\frac{40}{3}$$

**24.** $\mathbf{F}(x, y) = 3x\mathbf{i} + 4y\mathbf{j}$

$$C\colon\ \mathbf{r}(t) = t\mathbf{i} + \sqrt{4 - t^2}\mathbf{j},\ -2 \le t \le 2$$

$$\mathbf{F}(t) = 3t\mathbf{i} + 4\sqrt{4 - t^2}\mathbf{j}$$

$$\mathbf{r}'(t) = \mathbf{i} - \frac{t}{\sqrt{4 - t^2}}\mathbf{j}$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{-2}^{2} (3t - 4t)\, dt = \left[-\frac{t^2}{2}\right]_{-2}^{2} = 0$$

**26.** $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

$$C\colon\ \mathbf{r}(t) = 2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \frac{1}{2}t^2\mathbf{k},\ 0 \le t \le \pi$$

$$\mathbf{F}(t) = 4\sin^2 t\mathbf{i} + 4\cos^2 t\mathbf{j} + \frac{1}{4}t^4\mathbf{k}$$

$$\mathbf{r}'(t) = 2\cos t\mathbf{i} - 2\sin t\mathbf{j} + t\mathbf{k}$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi}\left(8\sin^2 t\cos t - 8\cos^2 t\sin t + \frac{1}{4}t^5\right) dt$$

$$= \left[\frac{8}{3}\sin^3 t + \frac{8}{3}\cos^3 t + \frac{t^6}{24}\right]_0^{\pi}$$

$$= -\frac{8}{3} + \frac{\pi^6}{24} - \frac{8}{3} = \frac{\pi^6}{24} - \frac{16}{3}$$

**28.** $\mathbf{F}(x, y, z) = \dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$

$$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + e^t\mathbf{k},\ 0 \le t \le 2$$

$$\mathbf{F}(t) = \frac{t\mathbf{i} + t\mathbf{j} + e^t\mathbf{k}}{\sqrt{2t^2 + e^{2t}}}$$

$$d\mathbf{r} = (\mathbf{i} + \mathbf{j} + e^t\mathbf{k})\, dt$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^2 \frac{1}{\sqrt{2t^2 + e^{2t}}}(2t + e^{2t})\,dt \approx 6.91$$

**30.** $\mathbf{F}(x, y) = x^2\mathbf{i} - xy\mathbf{j}$

$C\colon\ x = \cos^3 t,\ y = \sin^3 t$ from $(1, 0)$ to $(0, 1)$

$$\mathbf{r}(t) = \cos^3 t\mathbf{i} + \sin^3 t\mathbf{j},\ 0 \le t \le \frac{\pi}{2}$$

$$\mathbf{r}'(t) = -3\cos^2 t\sin t\mathbf{i} + 3\sin^2 t\cos t\mathbf{j}$$

$$\mathbf{F}(t) = \cos^6 t\mathbf{i} - \cos^3 t\sin^3 t\mathbf{j}$$

$$\mathbf{F}\cdot\mathbf{r}' = -3\cos^8 t\sin t - 3\cos^4 t\sin^5 t$$

$$= -3\cos^4 t\sin t(\cos^4 t + \sin^4 t)$$

$$= -3\cos^4 t\sin t[\cos^4 t + (1 - \cos^2 t)^2]$$

$$= -3\cos^4 t\sin t(2\cos^4 t - 2\cos^2 t + 1)$$

$$= -6\cos^8 t\sin t + 6\cos^6 t\sin t - 3\cos^4 t\sin t$$

$$\text{Work} = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/2} [-6\cos^8 t\sin t + 6\cos^6 t\sin t - 3\cos^4 t\sin t]\, dt$$

$$= \left[\frac{2\cos^9 t}{3} - \frac{6\cos^7 t}{7} + \frac{3\cos^5 t}{5}\right]_0^{\pi/2} = -\frac{43}{105}$$

**32.** $\mathbf{F}(x, y) = -y\mathbf{i} - x\mathbf{j}$

$C$: counterclockwise along the semicircle $y = \sqrt{4 - x^2}$ from $(2, 0)$ to $(-2, 0)$

$$\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}, \quad 0 \le t \le \pi$$

$$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j}$$

$$\mathbf{F}(t) = -2\sin t\mathbf{i} - 2\cos t\mathbf{j}$$

$$\mathbf{F}\cdot\mathbf{r}' = 4\sin^2 t - 4\cos^2 t = -4\cos 2t$$

$$\text{Work} = \int_C \mathbf{F}\cdot d\mathbf{r} = -4\int_0^{\pi}\cos 2t\,dt = \Big[-2\sin 2t\Big]_0^{\pi} = 0$$

**34.** $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

$C$: line from $(0, 0, 0)$ to $(5, 3, 2)$

$$\mathbf{r}(t) = 5t\mathbf{i} + 3t\mathbf{j} + 2t\mathbf{k}, \quad 0 \le t \le 1$$

$$\mathbf{r}'(t) = 5\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{F}(t) = 6t^2\mathbf{i} + 10t^2\mathbf{j} + 15t^2\mathbf{k}$$

$$\mathbf{F}\cdot\mathbf{r}' = 90t^2$$

$$\text{Work} = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 90t^2\,dt = 30$$

**36.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j},\ 0 \le t \le 1$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} \approx \frac{1-0}{3(4)}[5 + 4(4) + 2(4) + 4(6) + 11] = \frac{16}{3}$$

| $(x, y)$ | $(0, 0)$ | $\left(\frac{1}{4}, \frac{1}{16}\right)$ | $\left(\frac{1}{2}, \frac{1}{4}\right)$ | $\left(\frac{3}{4}, \frac{9}{16}\right)$ | $(1, 1)$ |
|---|---|---|---|---|---|
| $\mathbf{F}(x, y)$ | $5\mathbf{i}$ | $3.5\mathbf{i} + \mathbf{j}$ | $2\mathbf{i} + 2\mathbf{j}$ | $1.5\mathbf{i} + 3\mathbf{j}$ | $\mathbf{i} + 5\mathbf{j}$ |
| $\mathbf{r}'(t)$ | $\mathbf{i}$ | $\mathbf{i} + 0.5\mathbf{j}$ | $\mathbf{i} + \mathbf{j}$ | $\mathbf{i} + 1.5\mathbf{j}$ | $\mathbf{i} + 2\mathbf{j}$ |
| $\mathbf{F}\cdot\mathbf{r}'$ | 5 | 4 | 4 | 6 | 11 |

**38.** $\mathbf{F}(x, y) = x^2y\mathbf{i} + xy^{3/2}\mathbf{j}$

(a) $\mathbf{r}_1(t) = (t + 1)\mathbf{i} + t^2\mathbf{j},\ 0 \le t \le 2$

$$\mathbf{r}_1'(t) = \mathbf{i} + 2t\mathbf{j}$$

$$\mathbf{F}(t) = (t + 1)^2t^2\mathbf{i} + (t + 1)t^3\mathbf{j}$$

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_0^2 \left[(t + 1)^2t^2 + 2t^4(t + 1)\right]dt = \frac{256}{3}$$

(b) $\mathbf{r}_2(t) = (1 + 2\cos t)\mathbf{i} + 4\cos^2 t\mathbf{j},\ 0 \le t \le \dfrac{\pi}{2}$

$$\mathbf{r}_2'(t) = -2\sin t\mathbf{i} - 8\cos t\sin t\mathbf{j}$$

$$\mathbf{F}(t) = (1 + 2\cos t)^2(4\cos^2 t)\mathbf{i} + (1 + 2\cos t)(8\cos^3 t)\mathbf{j}$$

$$\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/2}\Big[(1 + 2\cos t)^2(4\cos^2 t)(-2\sin t) - 8\cos t\sin t(1 + 2\cos t)(8\cos^3 t)\,dt = -\frac{256}{5}$$

Both paths join $(1, 0)$ and $(3, 4)$. The integrals are negatives of each other because the orientations are different.

**40.** $\mathbf{F}(x, y) = -3y\mathbf{i} + x\mathbf{j}$

$C$: $\mathbf{r}(t) = t\mathbf{i} - t^3\mathbf{j}$

$$\mathbf{r}'(t) = \mathbf{i} - 3t^2\mathbf{j}$$

$$\mathbf{F}(t) = 3t^3\mathbf{i} + t\mathbf{j}$$

$$\mathbf{F}\cdot\mathbf{r}' = 3t^3 - 3t^3 = 0$$

Thus, $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = 0.$

**42.** $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$

$C$: $\mathbf{r}(t) = 3\sin t\mathbf{i} + 3\cos t\mathbf{j}$

$$\mathbf{r}'(t) = 3\cos t\mathbf{i} - 3\sin t\mathbf{j}$$

$$\mathbf{F}(t) = 3\sin t\mathbf{i} + 3\cos t\mathbf{j}$$

$$\mathbf{F}\cdot\mathbf{r}' = 9\sin t\cos t - 9\sin t\cos t = 0$$

Thus, $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = 0.$

**44.** $x = 2t,\ y = 10t,\ 0 \le t \le 1 \implies y = 5x,\ 0 \le x \le 2$

$$\int_C (x + 3y^2)\,dx = \int_0^2 (x + 75x^2)\,dx = \left[\frac{x^2}{2} + 25x^3\right]_0^2 = 202$$

**46.** $x = 2t,\ y = 10t,\ 0 \le t \le 1 \Longrightarrow y = 5x,\ dy = 5\,dx,\ 0 \le x \le 2$

$$\int_C (3y - x)\,dx + y^2\,dy = \int_0^2 (3(5x) - x)\,dx + (5x)^2 5\,dx = \int_0^2 (14x + 125x^2)\,dx$$

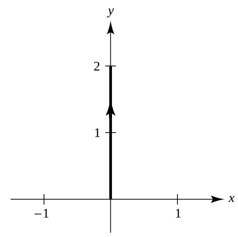
$$= \left[7x^2 + \frac{125}{3}x^3\right]_0^2 = 28 + \frac{125}{3}(8) = \frac{1084}{3}$$

**48.** $\mathbf{r}(t) = t\,\mathbf{j},\ 0 \le t \le 2$

$x(t) = 0,\ y(t) = t$

$dx = 0,\ dy = dt$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = \int_0^2 3t\,dt = \left[\frac{3}{2}t^2\right]_0^2 = 6$$




**50.** $\mathbf{r}(t) = \begin{cases} -t\,\mathbf{j}, & 0 \le t \le 3 \\ (t - 3)\mathbf{i} - 3\mathbf{j}, & 3 \le t \le 5 \end{cases}$

$C_1$: $x(t) = 0,\ y(t) = -t$

$dx = 0,\ dy = -dt$

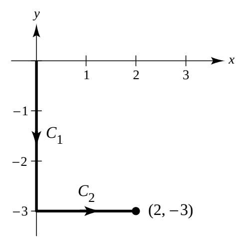
$$\int_{C_1} (2x - y)\,dx + (x + 3y)\,dy = \int_0^3 3t\,dt = \frac{27}{2}$$

$C_2$: $x(t) = t - 3,\ y(t) = -3$

$dx = dt,\ dy = 0$

$$\int_{C_2} (2x - y)\,dx + (x + 3y)\,dy = \int_3^5 [2(t - 3) + 3]\,dt = \left[(t - 3)^2 + 3t\right]_3^5 = 10$$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = \frac{27}{2} + 10 = \frac{47}{2}$$




**52.** $x(t) = t,\ y(t) = t^{3/2},\ 0 \le t \le 4,\ dx = dt,\ dy = \frac{3}{2}t^{1/2}\,dt$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = \int_0^4 \left[(2t - t^{3/2}) + (t + 3t^{3/2})\left(\frac{3}{2}t^{1/2}\right)\right] dt$$

$$= \int_0^4 \left(\frac{9}{2}t^2 + \frac{1}{2}t^{3/2} + 2t\right) dt = \left[\frac{3}{2}t^3 + \frac{1}{5}t^{5/2} + t^2\right]_0^4 = 96 + \frac{1}{5}(32) + 16 = \frac{592}{5}$$

**54.** $x(t) = 4\sin t,\ y(t) = 3\cos t,\ 0 \le t \le \frac{\pi}{2}$

$dx = 4\cos t\,dt,\ dy = -3\sin t\,dt$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = \int_0^{\pi/2} (8\sin t - 3\cos t)(4\cos t)\,dt + (4\sin t + 9\cos t)(-3\sin t)\,dt$$

$$= \int_0^{\pi/2} (5\sin t\cos t - 12\cos^2 t - 12\sin^2 t)\,dt$$

$$= \left[\frac{5}{2}\sin^2 t - 12t\right]_0^{\pi/2} = \frac{5}{2} - 6\pi$$

**56.** $f(x, y) = y$

$C$: line from (0, 0) to (4, 4)

$$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j},\ 0 \le t \le 4$$
$$\mathbf{r}'(t) = \mathbf{i} + \mathbf{j}$$
$$\|\mathbf{r}'(t)\| = \sqrt{2}$$

Lateral surface area:

$$\int_C f(x, y)\, ds = \int_0^4 t\left(\sqrt{2}\right) dt = 8\sqrt{2}$$

**58.** $f(x, y) = x + y$

$C$: $x^2 + y^2 = 1$ from (1, 0) to (0, 1)

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j},\ 0 \le t \le \frac{\pi}{2}$$
$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$$
$$\|\mathbf{r}'(t)\| = 1$$

Lateral surface area:

$$\int_C f(x, y)\, ds = \int_0^{\pi/2} (\cos t + \sin t)\, dt = \Big[\sin t - \cos t\Big]_0^{\pi/2} = 2$$

**60.** $f(x, y) = y + 1$

$C$: $y = 1 - x^2$ from (1, 0) to (0, 1)

$$\mathbf{r}(t) = (1 - t)\mathbf{i} + \left[1 - (1 - t)^2\right]\mathbf{j},\ 0 \le t \le 1$$
$$\mathbf{r}'(t) = -\mathbf{i} + 2(1 - t)\mathbf{j}$$
$$\|\mathbf{r}'(t)\| = \sqrt{1 + 4(1 - t)^2}$$

Lateral surface area:

$$\begin{aligned}
\int_C f(x, y)\, ds &= \int_0^1 \left[2 - (1 - t)^2\right]\sqrt{1 + 4(1 - t)^2}\, dt \\
&= 2\int_0^1 \sqrt{1 + 4(1 - t)^2}\, dt - \int_0^1 (1 - t)^2\sqrt{1 + 4(1 - t)^2}\, dt \\
&= -\frac{1}{2}\left[2(1 - t)\sqrt{1 + 4(1 - t)^2} + \ln\left|2(1 - t) + \sqrt{1 + 4(1 - t)^2}\right|\right]_0^1 \\
&\quad + \frac{1}{64}\left[2(1 - t)[2(4)(1 - t)^2 + 1]\sqrt{1 + 4(1 - t)^2} - \ln\left|2(1 - t) + \sqrt{1 + 4(1 - t)^2}\right|\right]_0^1 \\
&= \frac{1}{2}\left[2\sqrt{5} + \ln\left(2 + \sqrt{5}\right)\right] - \frac{1}{64}\left[18\sqrt{5} - \ln\left(2 + \sqrt{5}\right)\right] \\
&= \frac{23}{32}\sqrt{5} + \frac{33}{64}\ln\left(2 + \sqrt{5}\right) = \frac{1}{64}\left[46\sqrt{5} + 33\ln\left(2 + \sqrt{5}\right)\right] \approx 2.3515
\end{aligned}$$

**62.** $f(x, y) = x^2 - y^2 + 4$

$C$: $x^2 + y^2 = 4$

$$\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j},\ 0 \le t \le 2\pi$$
$$\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$$
$$\|\mathbf{r}'(t)\| = 2$$

Lateral surface area:

$$\int_C f(x, y)\, ds = \int_0^{2\pi} (4\cos^2 t - 4\sin^2 t + 4)(2)\, dt = 8\int_0^{2\pi} (1 + \cos 2t)\, dt = \left[8\left(t + \frac{1}{2}\sin 2t\right)\right]_0^{2\pi} = 16\pi$$

**64.** $f(x, y) = 20 + \frac{1}{4}x$

$C$: $y = x^{3/2}$, $0 \le x \le 40$

$\mathbf{r}(t) = t\mathbf{i} + t^{3/2}\mathbf{j}$, $0 \le t \le 40$

$\mathbf{r}'(t) = \mathbf{i} + \frac{3}{2}t^{1/2}\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{1 + \left(\frac{9}{4}\right)t}$

Lateral surface area: $\int_C f(x, y)\, ds = \int_0^{40} \left(20 + \frac{1}{4}t\right)\sqrt{1 + \left(\frac{9}{4}\right)t}\, dt$

Let $u = \sqrt{1 + \left(\frac{9}{4}\right)t}$, then $t = \frac{4}{9}(u^2 - 1)$ and $dt = \frac{8}{9}u\, du$.

$$\int_0^{40} \left(20 + \frac{1}{4}t\right)\sqrt{1 + \left(\frac{9}{4}\right)t}\, dt = \int_1^{\sqrt{91}} \left[20 + \frac{1}{9}(u^2 - 1)\right](u)\left(\frac{8}{9}u\right) du = \frac{8}{81}\int_1^{\sqrt{91}} (u^4 + 179u^2)\, du$$

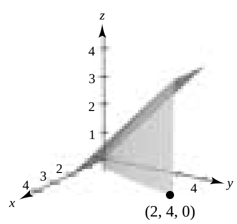
$$= \frac{8}{81}\left[\frac{u^5}{5} + \frac{179u^3}{3}\right]_1^{\sqrt{91}} = \frac{850{,}304\sqrt{91} - 7184}{1215} \approx 6670.12$$

**66.** $f(x, y) = y$

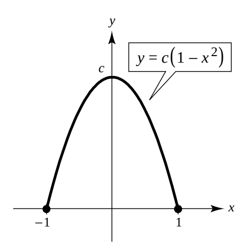
$C$: $y = x^2$ from $(0, 0)$ to $(2, 4)$

$S \approx 8$

Matches $c$.




**68.** $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M\, dx + N\, dy$




$M = 15(4 - x^2y) = 60 - 15x^2(c - cx^2)$

$N = -15xy = -15x(c - cx^2)$

$dx = dx$, $dy = -2cx\, dx$

$$W = \int_{-1}^{1} \left[60 - 15x^2(c - cx^2) + (-15x(c - cx^2))(-2\,cx)\right] dx$$

$= 120 - 4c + 8c^2$ (parabola)

$w' = 16c - 4 = 0 \implies c = \frac{1}{4}$ yields the minimum work, 119.5. Along the straight line path, $y = 0$, the work is 120.

**70.** See the definition, page 1024.

**72.** (a) Work $= 0$

(b) Work is negative, since against force field.

(c) Work is positive, since with force field.

**74.** False, the orientation of $C$ does not affect the form

$$\int_C f(x, y)\, ds.$$

**76.** False. For example, see Exercise 32.

## Section 14.3 Conservative Vector Fields and Independence of Path

**2.** $\mathbf{F}(x, y) = (x^2 + y^2)\mathbf{i} - x\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + \sqrt{t}\mathbf{j},\ 0 \le t \le 4$

$\mathbf{r}_1'(t) = \mathbf{i} + \frac{1}{2\sqrt{t}}\mathbf{j}$

$\mathbf{F}(t) = (t^2 + t)\mathbf{i} - t\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^4 \left(t^2 + t - \frac{1}{2}\sqrt{t}\right) dt$$

$$= \left[\frac{t^3}{3} + \frac{t^2}{2} - \frac{t^{3/2}}{3}\right]_0^4 = \frac{80}{3}$$

(b) $\mathbf{r}_2(w) = w^2\mathbf{i} + w\mathbf{j},\ 0 \le w \le 2$

$\mathbf{r}_2'(w) = 2w\mathbf{i} + \mathbf{j}$

$\mathbf{F}(w) = (w^4 + w^2)\mathbf{i} - w^2\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \left[2w(w^4 + w^2) - w^2\right] dw$$

$$= \left[\frac{w^6}{3} + \frac{w^4}{2} - \frac{w^3}{3}\right]_0^2 = \frac{80}{3}$$

**4.** $\mathbf{F}(x, y) = y\mathbf{i} + x^2\mathbf{j}$

(a) $\mathbf{r}_1(t) = (2 + t)\mathbf{i} + (3 - t)\mathbf{j},\ 0 \le t \le 3$

$\mathbf{r}_1'(t) = \mathbf{i} - \mathbf{j}$

$\mathbf{F}(t) = (3 - t)\mathbf{i} + (2 + t)^2\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 \left[(3 - t) - (2 + t)^2\right] dt = \left[-\frac{(3 - t)^2}{2} - \frac{(2 + t)^3}{3}\right]_0^3 = -\frac{69}{2}$$

(b) $\mathbf{r}_2(w) = (2 + \ln w)\mathbf{i} + (3 - \ln w)\mathbf{j},\ 1 \le w \le e^3$

$\mathbf{r}_2'(w) = \frac{1}{w}\mathbf{i} - \frac{1}{w}\mathbf{j}$

$\mathbf{F}(w) = (3 - \ln w)\mathbf{i} + (2 + \ln w)^2\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_1^{e^3} \left[(3 - \ln w)\left(\frac{1}{w}\right) - (2 + \ln w)^2\left(\frac{1}{w}\right)\right] dw = \left[-\frac{(3 - \ln w)^2}{2} - \frac{(2 + \ln w)^3}{3}\right]_1^{e^3} = -\frac{69}{2}$$

**6.** $\mathbf{F}(x, y) = 15x^2y^2\mathbf{i} + 10x^3y\mathbf{j}$

$\frac{\partial N}{\partial x} = 30x^2y \qquad \frac{\partial M}{\partial y} = 30x^2y$

Since $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$, $\mathbf{F}$ is conservative.

**8.** $\mathbf{F}(x, y, z) = y \ln z\,\mathbf{i} - x \ln z\,\mathbf{j} + \frac{xy}{z}\mathbf{k}$

**curl F** $\neq$ **0** so **F** is not conservative.

$\left(\frac{\partial P}{\partial y} = \frac{x}{z} \neq -\frac{x}{z} = \frac{\partial N}{\partial z}\right)$

**10.** $\mathbf{F}(x, y, z) = \sin(yz)\mathbf{i} + xz\cos(yz)\mathbf{j} + xy\sin(yz)\mathbf{k}$

**curl F** $\neq$ **0**, so **F** is not conservative.

**12.** $\mathbf{F}(x, y) = ye^{xy}\mathbf{i} + xe^{xy}\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} - (t - 3)\mathbf{j},\ 0 \le t \le 3$

$\mathbf{r}_1'(t) = \mathbf{i} - \mathbf{j}$

$\mathbf{F}(t) = -(t - 3)e^{3t - t^2}\mathbf{i} + te^{3t - t^2}\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 \left[-(t - 3)e^{3t - t^2} - te^{3t - t^2}\right] dt$$

$$= \int_0^3 e^{3t - t^2}(3 - 2t)\, dt$$

$$= \left[e^{3t - t^2}\right]_0^3 = e^0 - e^0 = 0$$

(b) $\mathbf{F}(x, y)$ is conservative since

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = xye^{xy} + e^{xy}.$$

The potential function is $f(x, y) = e^{xy} + k$.

**14.** $\mathbf{F}(x, y) = xy^2\mathbf{i} + 2x^2y\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j}, \quad 1 \le t \le 3$

$\mathbf{r}_1'(t) = \mathbf{i} - \frac{1}{t^2}\mathbf{j}$

$\mathbf{F}(t) = \frac{1}{t}\mathbf{i} + 2t\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_1^3 -\frac{1}{t}\,dt = \Big[-\ln|t|\Big]_1^3 = -\ln 3$$

(b) $\mathbf{r}_2(t) = (t+1)\mathbf{i} - \frac{1}{3}(t-3)\mathbf{j}, \quad 0 \le t \le 2$

$\mathbf{r}_2'(t) = \mathbf{i} - \frac{1}{3}\mathbf{j}$

$\mathbf{F}(t) = \frac{1}{9}(t+1)(t-3)^2\mathbf{i} - \frac{2}{3}(t+1)^2(t-3)\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \left[\frac{1}{9}(t+1)(t-3)^2 + \frac{2}{9}(t+1)^2(t-3)\right] dt = \frac{1}{9}\int_0^2 (3t^3 - 7t^2 - 7t + 3)\,dt = \frac{1}{9}\left[\frac{3t^4}{4} - \frac{7t^3}{3} - \frac{7t^2}{2} + 3t\right]_0^2 = -\frac{44}{27}$$

**16.** $\int_C (2x - 3y + 1)\,dx - (3x + y - 5)\,dy$

Since $\partial M/\partial y = \partial N/\partial x = -3$, $\mathbf{F}(x, y) = (2x - 3y + 1)\mathbf{i} - (3x + y - 5)\mathbf{j}$ is conservative. The potential function is $f(x, y) = x^2 - 3xy - (y^2/2) + x + 5y + k$.

(a) and (d) Since $C$ is a closed curve, $\int_C (2x - 3y + 1)\,dx - (3x + y - 5)\,dy = 0$.

(b) $$\int_C (2x - 3y + 1)\,dx - (3x + y - 5)\,dy = \left[x^2 - 3xy - \frac{y^2}{2} + x + 5y\right]_{(0,-1)}^{(0,1)} = 10$$

(c) $$\int_C (2x - 3y + 1)\,dx - (3x + y - 5)\,dy = \left[x^2 - 3xy - \frac{y^2}{2} + x + 5y\right]_{(0,1)}^{(2,e^2)} = \frac{1}{2}(3 - 2e^2 - e^4)$$

**18.** $\int_C (x^2 + y^2)\,dx + 2xy\,dy$

Since $\partial M/\partial y = \partial N/\partial x = 2y$,

$\mathbf{F}(x, y) = (x^2 + y^2)\mathbf{i} + 2xy\mathbf{j}$

is conservative. The potential function is

$f(x, y) = (x^3/3) + xy^2 + k$.

(a) $$\int_C (x^2 + y^2)\,dx + 2xy\,dy = \left[\frac{x^3}{3} + xy^2\right]_{(0,0)}^{(8,4)} = \frac{896}{3}$$

(b) $$\int_C (x^2 + y^2)\,dx + 2xy\,dy = \left[\frac{x^3}{3} + xy^2\right]_{(2,0)}^{(0,2)} = -\frac{8}{3}$$

**20.** $\mathbf{F}(x, y, z) = \mathbf{i} + z\mathbf{j} + y\mathbf{k}$

Since **curl F** $= \mathbf{0}$, $\mathbf{F}(x, y, z)$ is conservative. The potential function is $f(x, y, z) = x + yz + k$.

(a) $\mathbf{r}_1(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t^2\mathbf{k}, \ 0 \le t \le \pi$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \Big[x + yz\Big]_{(1,0,0)}^{(-1,0,\pi^2)} = -2$$

(b) $\mathbf{r}_2(t) = (1 - 2t)\mathbf{i} + \pi^2 t\mathbf{k}, \ 0 \le t \le 1$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \Big[x + yz\Big]_{(1,0,0)}^{(-1,0,\pi^2)} = -2$$

**22.** $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j} + 3xz^2\mathbf{k}$

$\mathbf{F}(x, y, z)$ is not conservative.

(a) $\mathbf{r}_1(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k}, \ 0 \le t \le \pi$

$\mathbf{r}_1'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$

$\mathbf{F}(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + 3t^2\cos t\,\mathbf{k}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi [\sin^2 t + \cos^2 t + 3t^2\cos t]\,dt = \int_0^\pi [1 + 3t^2\cos t]\,dt$$

$$= \Big[t\Big]_0^\pi + 3\Big[t^2\sin t\Big]_0^\pi - 6\int_0^\pi t\sin t\,dt = \Big[t + 3t^2\sin t - 6(\sin t - t\cos t)\Big]_0^\pi = -5\pi$$

**—CONTINUED—**

**22. —CONTINUED—**

(b) $\mathbf{r}_2(t) = (1 - 2t)\mathbf{i} + \pi t\mathbf{k},\ 0 \le t \le 1$

$\mathbf{r}_2'(t) = -2\mathbf{i} + \pi\mathbf{k}$

$\mathbf{F}(t) = (1 - 2t)\mathbf{j} + 3\pi^2 t^2(1 - 2t)\mathbf{k}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 3\pi^3 t^2(1 - 2t)\,dt = 3\pi^3 \int_0^1 (t^2 - 2t^3)\,dt = 3\pi^3\left[\frac{t^3}{3} - \frac{t^4}{2}\right]_0^1 = -\frac{\pi^3}{2}$$

**24.** $\mathbf{F}(x, y, z) = y \sin z\,\mathbf{i} + x \sin z\,\mathbf{j} + xy \cos x\,\mathbf{k}$

(a) $\mathbf{r}_1(t) = t^2\mathbf{i} + t^2\mathbf{j},\ 0 \le t \le 2$

$\mathbf{r}_1'(t) = 2t\mathbf{i} + 2t\mathbf{j}$

$\mathbf{F}(t) = t^4 \cos t^2\mathbf{k}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 0\,dt = 0$$

(b) $\mathbf{r}_2(t) = 4t\mathbf{i} + 4t\mathbf{j},\ 0 \le t \le 1$

$\mathbf{r}_2'(t) = 4\mathbf{i} + 4\mathbf{j}$

$\mathbf{F}(t) = 16t^2 \cos(4t)\mathbf{k}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 0\,dt = 0$$

**26.** $$\int_C [2(x + y)\mathbf{i} + 2(x + y)\mathbf{j}] \cdot d\mathbf{r} = \Big[(x + y)^2\Big]_{(-3, 2)}^{(4, 3)} = 49$$

**28.** $$\int_C \frac{y\,dx - x\,dy}{x^2 + y^2} = \left[\arctan\left(\frac{x}{y}\right)\right]_{(1, 1)}^{(2\sqrt{3}, 2)} = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

**30.** $$\int_C \frac{2x}{(x^2 + y^2)^2}\,dx + \frac{2y}{(x^2 + y^2)^2}\,dy = \left[-\frac{1}{x^2 + y^2}\right]_{(7, 5)}^{(1, 5)} = -\frac{1}{26} + \frac{1}{74} = \frac{-12}{481}$$

**32.** $$\int_C zy\,dx + xz\,dy + xy\,dz$$

**Note:** Since $\mathbf{F}(x, y, z) = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\mathbf{k}$ is conservative and the potential function is $f(x, y, z) = xyz + k$, the integral is independent of path as illustrated below.

(a) $\Big[xyz\Big]_{(0, 0, 0)}^{(1, 1, 1)} = 1$

(b) $\Big[xyz\Big]_{(0, 0, 0)}^{(0, 0, 1)} + \Big[xyz\Big]_{(0, 0, 1)}^{(1, 1, 1)} = 0 + 1 = 1$

(c) $\Big[xyz\Big]_{(0, 0, 0)}^{(1, 0, 0)} + \Big[xyz\Big]_{(1, 0, 0)}^{(1, 1, 0)} + \Big[xyz\Big]_{(1, 1, 0)}^{(1, 1, 1)} = 0 + 0 + 1 = 1$

**34.** $$\int_C 6x\,dx - 4z\,dy - (4y - 20z)\,dz = \Big[3x^2 - 4yz + 10z^2\Big]_{(0, 0, 0)}^{(4, 3, 1)} = 46$$

**36.** $\mathbf{F}(x, y) = \dfrac{2x}{y}\mathbf{i} - \dfrac{x^2}{y^2}\mathbf{j}$ is conservative.

$$\text{Work} = \left[\frac{x^2}{y}\right]_{(-3, 2)}^{(1, 4)} = \frac{1}{4} - \frac{9}{2} = -\frac{17}{4}$$

**38.** $\mathbf{F}(x, y, z) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

Since $\mathbf{F}(x, y, z)$ is conservative, the work done in moving a particle along any path from $P$ to $Q$ is

$$f(x, y, z) = \Big[a_1x + a_2y + a_3z\Big]_{P=(p_1,\, p_2,\, p_3)}^{Q=(q_1,\, q_2,\, q_3)}$$

$$= a_1(q_1 - p_1) + a_2(q_2 - p_2) + a_3(q_3 - p_3) = \mathbf{F}\cdot\overrightarrow{PQ}.$$

**40.** $\mathbf{F} = -150\mathbf{j}$

(a) $\mathbf{r}(t) = t\mathbf{i} + (50 - t)\mathbf{j},\ 0 \le t \le 50$

$d\mathbf{r} = (\mathbf{i} - \mathbf{j})\, dt$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{50} 150\, dt = 7500 \text{ ft}\cdot\text{lbs}$$

(b) $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{50}(50 - t)^2\mathbf{j}$

$d\mathbf{r} = \left(\mathbf{i} - \frac{1}{25}(50 - t)\mathbf{j}\right) dt$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = 6\int_0^{50} (50 - t)\, dt = 7500 \text{ ft}\cdot\text{lbs}$$

**42.** $\mathbf{F}(x, y) = \dfrac{y}{x^2 + y^2}\mathbf{i} - \dfrac{x}{x^2 + y^2}\mathbf{j}$

(a) $M = \dfrac{y}{x^2 + y^2}$

$$\frac{\partial M}{\partial y} = \frac{(x^2 + y^2)(1) - y(2y)}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$N = -\dfrac{x}{x^2 + y^2}$

$$\frac{\partial N}{\partial x} = \frac{(x^2 + y^2)(-1) + x(2x)}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

Thus, $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$.

(b) $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j},\ 0 \le t \le \pi$

$\mathbf{F} = \sin t\mathbf{i} - \cos t\mathbf{j}$

$d\mathbf{r} = (-\sin t\mathbf{i} + \cos t\mathbf{j})\, dt$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi} (-\sin^2 t - \cos^2 t)\, dt = \Big[-t\Big]_0^{\pi} = -\pi$$

(c) $\mathbf{r}(t) = \cos t\mathbf{i} - \sin t\mathbf{j},\ 0 \le t \le \pi$

$\mathbf{F} = -\sin t\mathbf{i} - \cos t\mathbf{j}$

$d\mathbf{r} = (-\sin t\mathbf{i} - \cos t\mathbf{j})\, dt$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi} (\sin^2 t + \cos^2 t)\, dt$$

$$= \Big[t\Big]_0^{\pi} = \pi$$

(d) $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j},\ 0 \le t \le 2\pi$

$\mathbf{F} = \sin t\mathbf{i} - \cos t\mathbf{j}$

$d\mathbf{r} = (-\sin t\mathbf{i} + \cos t\mathbf{j})\, dt$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} (-\sin^2 t - \cos^2 t)\, dt$$

$$= \Big[-t\Big]_0^{2\pi} = -2\pi$$

This does not contradict Theorem 14.7 since $\mathbf{F}$ is not continuous at $(0, 0)$ in $R$ enclosed by curve $C$.

(e) $$\nabla\left(\arctan\frac{x}{y}\right) = \frac{1/y}{1 + (x/y)^2}\mathbf{i} + \frac{-x/y^2}{1 + (x/y)^2}\mathbf{j}$$

$$= \frac{y}{x^2 + y^2}\mathbf{i} - \frac{x}{x^2 + y^2}\mathbf{j} = \mathbf{F}$$

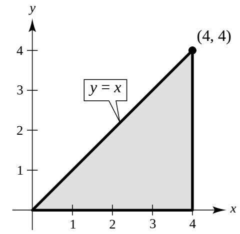
**44.** A line integral is independent of path if $\int_C \mathbf{F}\cdot d\mathbf{r}$ does not depend on the curve joining $P$ and $Q$. See Theorem 14.6

**46.** No, the amount of fuel required depends on the flight path. Fuel consumption is dependent on wind speed and direction. The vector field is not conservative.

**48.** True

**50.** False, the requirement is $\partial M/\partial y = \partial N/\partial x$.

## Section 14.4 Green's Theorem



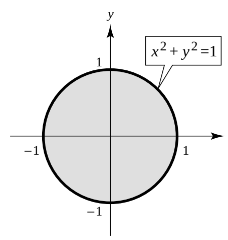

**2.** $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 4 \\ 4\mathbf{i} + (t-4)\mathbf{j}, & 4 \le t \le 8 \\ (12-t)\mathbf{i} + (12-t)\mathbf{j}, & 8 \le t \le 12 \end{cases}$

$$\int_C y^2\,dx + x^2\,dy = \int_0^4 [0\,dt + t^2(0)] + \int_4^8 [(t-4)^2(0) + 16\,dt]$$

$$+ \int_8^{12} [(12-t)^2(-dt) + (12-t)^2(-dt)] = 0 + 64 - \frac{128}{3} = \frac{64}{3}$$

By Green's Theorem, $\displaystyle\int_R\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_0^4\int_0^x (2x - 2y)\,dy\,dx = \int_0^4 x^2\,dx = \frac{64}{3}.$

**4.** $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j},\ 0 \le t \le 2\pi$




$$\int_C y^2\,dx + x^2\,dy = \int_0^{2\pi} [\sin^2 t(-\sin t\,dt) + \cos^2 t(\cos t\,dt)]$$

$$= \int_0^{2\pi} (\cos^3 t - \sin^3 t)\,dt$$

$$= \int_0^{2\pi} [\cos t(1 - \sin^2 t) - \sin t(1 - \cos^2 t)]\,dt$$

$$= \left[\sin t - \frac{\sin^3 t}{3} + \cos t - \frac{\cos^3 t}{3}\right]_0^{2\pi} = 0$$

By Green's Theorem,

$$\int_R\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2x - 2y)\,dy\,dx$$

$$= \int_0^{2\pi}\int_0^1 (2r\cos\theta - 2r\sin\theta)r\,dr\,d\theta = \frac{2}{3}\int_0^{2\pi} (\cos\theta - \sin\theta)\,d\theta = \frac{2}{3}(0) = 0.$$

**6.** $C$: boundary of the region lying between the graphs of $y = x$ and $y = x^3$

$$\int_C xe^y\,dx + e^x\,dy = \int_0^1 (xe^{x^3} + 3x^2e^x)\,dx + \int_1^0 (xe^x + e^x)\,dx \approx 2.936 - 2.718 \approx 0.22$$

$$\int_R\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_0^1\int_{x^3}^x (e^x - xe^y)\,dy\,dx = \int_0^1 (xe^{x^3} - x^3e^x)\,dx \approx 0.22$$

**In Exercises 8 and 10,** $\boldsymbol{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = 1}$**.**

**8.** Since $C$ is an ellipse with $a = 2$ and $b = 1$, then $R$ is an ellipse of area $\pi ab = 2\pi$. Thus, Green's Theorem yields

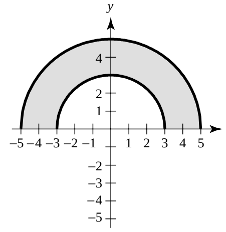
$$\int_C (y - x)\,dx + (2x - y)\,dy = \int_R\int 1\,dA = \text{Area of ellipse} = 2\pi.$$

**10.** $R$ is the shaded region of the accompanying figure.

$$\int_C (y - x)\,dx + (2x - y)\,dy = \int_R\int 1\,dA$$

$$= \text{Area of shaded region}$$

$$= \frac{1}{2}\pi[25 - 9] = 8\pi$$



**12.** The given curves intersect at (0, 0) and (9, 3). Thus, Green's Theorem yields

$$\int_C y^2\,dx + xy\,dy = \iint_R (y - 2y)\,dA$$
$$= \int_0^9 \int_0^{\sqrt{x}} -y\,dy\,dx = \int_0^9 \left[\frac{-y^2}{2}\right]_0^{\sqrt{x}} dx = \int_0^9 \frac{-x}{2}\,dx = \left[\frac{-x^2}{4}\right]_0^9 = -\frac{81}{4}$$

**14.** In this case, let $y = r\sin\theta$, $x = r\cos\theta$. Then $dA = r\,dr\,d\theta$ and Green's Theorem yields

$$\int_C (x^2 - y^2)\,dx + 2xy\,dy = \iint_R 4y\,dA = 4\int_0^{2\pi}\int_0^{1+\cos\theta} r\sin\theta r\,dr\,d\theta$$
$$= 4\int_0^{2\pi}\int_0^{1+\cos\theta} r^2 \sin\theta\,dr\,d\theta$$
$$= \frac{4}{3}\int_0^{2\pi} \sin\theta(1 + \cos\theta)^3\,d\theta$$
$$= \left[-\frac{(1 + \cos\theta)^4}{3}\right]_0^{2\pi} = 0.$$

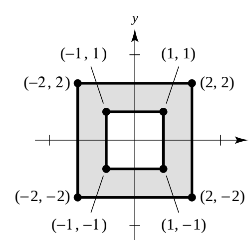
**16.** Since $\dfrac{\partial M}{\partial y} = -2e^x \sin 2y = \dfrac{\partial N}{\partial x}$ we have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = 0.$$

**18.** By Green's Theorem,

$$\int_C (e^{-x^2/2} - y)\,dx + (e^{-y^2/2} + x)\,dy = \iint_R 2\,dA = 2(\text{Area of } R) = 2[\pi(6)^2 - \pi(2)(3)] = 60\pi.$$

**20.** By Green's Theorem,




$$\int_C 3x^2 e^y\,dx + e^y\,dy = \iint_R -3x^2e^y\,dA$$
$$= \int_1^2\int_{-2}^2 -3x^2e^y\,dy\,dx + \int_{-1}^1\int_1^2 -3x^2e^y\,dy\,dx$$
$$+ \int_{-2}^{-1}\int_{-2}^2 -3x^2e^y\,dy\,dx + \int_{-1}^1\int_{-2}^{-1} -3x^2e^y\,dy\,dx$$
$$= -7(e^2 - e^{-2}) - 2(e^2 - e) - 7(e^2 - e^{-2}) - 2(e^{-1} - e^{-2})$$
$$= -16e^2 + 16e^{-2} + 2e - 2e^{-1}.$$

**22.** $\mathbf{F}(x, y) = (e^x - 3y)\mathbf{i} + (e^y + 6x)\mathbf{j}$

$C$: $r = 2\cos\theta$

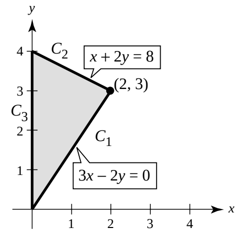
Work $= \displaystyle\int_C (e^x - 3y)\,dx + (e^y + 6x)\,dy = \iint_R 9\,dA = 9\pi$ since $r = 2\cos\theta$ is a circle with a radius of one.

**24.** $\mathbf{F}(x, y) = (3x^2 + y)\mathbf{i} + 4xy^2\mathbf{j}$

$C$: boundary of the region bounded by the graphs of $y = \sqrt{x}$, $y = 0$, $x = 4$

Work $= \displaystyle\int_C (3x^2 + y)\,dx + 4xy^2\,dy = \int_0^4\int_0^{\sqrt{x}} (4y^2 - 1)\,dy\,dx = \int_0^4 \left(\tfrac{4}{3}x^{3/2} - x^{1/2}\right) dx = \tfrac{176}{15}$

**26.** From the figure we see that




$$C_1: \; y = \frac{3}{2}x, \; dy = \frac{3}{2}\,dx, \; 0 \le x \le 2$$

$$C_2: \; y = -\frac{x}{2} + 4, \; dy = -\frac{1}{2}\,dx$$

$$C_3: \; x = 0, \; dx = 0.$$

$$A = \frac{1}{2}\int_0^2 \left(\frac{3}{2}x - \frac{3}{2}x\right) dx + \frac{1}{2}\int_2^0 \left(-\frac{1}{2}x + \frac{x}{2} - 4\right) dx + \frac{1}{2}(0)$$

$$= \frac{1}{2}\int_2^0 (-4)\,dx = 2\int_0^2 dx = 4$$

**28.** Since the loop of the folium is formed on the interval $0 \le t \le \infty$,

$$dx = \frac{3(1 - 2t^3)}{(t^3 + 1)^2}\,dt \text{ and } dy = \frac{3(2t - t^4)}{(t^3 + 1)^2}\,dt,$$

we have

$$A = \frac{1}{2}\int_0^\infty \left[\left(\frac{3t}{t^3 + 1}\right)\frac{3(2t - t^4)}{(t^3 + 1)^2} - \left(\frac{3t^2}{t^3 + 1}\right)\frac{3(1 - 2t^3)}{(t^3 + 1)^2}\right] dt$$

$$= \frac{9}{2}\int_0^\infty \frac{t^5 + t^2}{(t^3 + 1)^3}\,dt = \frac{9}{2}\int_0^\infty \frac{t^2(t^3 + 1)}{(t^3 + 1)^3}\,dt = \frac{3}{2}\int_0^\infty 3t^2(t^3 + 1)^{-2}\,dt = \left[\frac{-3}{2(t^3 + 1)}\right]_0^\infty = \frac{3}{2}.$$

**30.** See Theorem 14.9: $A = \frac{1}{2}\int_C x\,dy - y\,dx.$

**32.** (a) For the moment about the $x$-axis, $M_x = \iint_R y\,dA$. Let $N = 0$ and $M = -y^2/2$. By Green's Theorem,

$$M_x = \int_C -\frac{y^2}{2}\,dx = -\frac{1}{2}\int_C y^2\,dx \text{ and } \bar{y} = \frac{M_x}{2A} = -\frac{1}{2A}\int_C y^2\,dx.$$

For the moment about the $y$-axis, $M_y = \iint_R x\,dA$. Let $N = x^2/2$ and $M = 0$. By Green's Theorem,

$$M_y = \int_C \frac{x^2}{2}\,dy = \frac{1}{2}\int_C x^2\,dy \text{ and } \bar{x} = \frac{M_y}{2A} = \frac{1}{2A}\int_C x^2\,dy.$$

(b) By Theorem 14.9 and the fact that $x = r\cos\theta, \; y = r\sin\theta$, we have

$$A = \frac{1}{2}\int x\,dy - y\,dx = \frac{1}{2}\int (r\cos\theta)(r\cos\theta)\,d\theta - (r\sin\theta)(-r\sin\theta)\,d\theta = \frac{1}{2}\int_C r^2\,d\theta.$$

**34.** Since $A =$ area of semicircle $= \frac{\pi a^2}{2}$, we have $\frac{1}{2A} = \frac{1}{\pi a^2}$. Note that $y = 0$ and $dy = 0$ along the boundary $y = 0$. Let $x = a\cos t, \; y = a\sin t, \; 0 \le t \le \pi$, then

$$\bar{x} = \frac{1}{\pi a^2}\int_0^\pi a^2\cos^2 t(a\cos t)\,dt = \frac{a}{\pi}\int_0^\pi \cos^3 t\,dt = \frac{a}{\pi}\int_0^\pi (1 - \sin^2 t)\cos t\,dt = \frac{a}{\pi}\left[\sin t - \frac{\sin^3 t}{3}\right]_0^\pi = 0$$

$$\bar{y} = \frac{-1}{\pi a^2}\int_0^\pi a^2\sin^2 t(-a\sin t\,dt) = \frac{a}{\pi}\int_0^\pi \sin^3 t\,dt = \frac{a}{\pi}\left[-\cos t + \frac{\cos^3 t}{3}\right]_0^\pi = \frac{4a}{3\pi}.$$

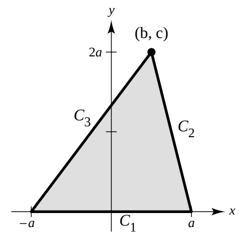
$$(\bar{x}, \bar{y}) = \left(0, \frac{4a}{3\pi}\right)$$

**36.** Since $A = \frac{1}{2}(2a)(c) = ac$, we have $\frac{1}{2A} = \frac{1}{2ac}$,

$C_1$: $y = 0,\ dy = 0$

$C_2$: $y = \frac{c}{b-a}(x-a),\ dy = \frac{c}{b-a}\,dx$

$C_3$: $y = \frac{c}{b+a}(x+a),\ dy = \frac{c}{b+a}\,dx.$




Thus,

$$\bar{x} = \frac{1}{2ac}\int_C x^2\,dy = \frac{1}{2ac}\left[\int_{-a}^{a} 0 + \int_a^b x^2\frac{c}{b-a}\,dx + \int_b^{-a} x^2\frac{c}{b+a}\,dx\right] = \frac{1}{2ac}\left[0 + \frac{2abc}{3}\right] = \frac{b}{3}$$

$$\bar{y} = \frac{-1}{2ac}\int_C y^2\,dx = \frac{-1}{2ac}\left[0 + \int_a^b\left(\frac{c}{b-a}\right)^2(x-a)^2\,dx + \int_b^{-a}\left(\frac{c}{b+a}\right)^2(x+a)^2\,dx\right]$$

$$= \frac{-1}{2ac}\left[\frac{c^2(b-a)}{3} - \frac{c^2(b+a)}{3}\right] = \frac{c}{3}.$$

$$(\bar{x}, \bar{y}) = \left(\frac{b}{3}, \frac{c}{3}\right)$$

**38.** $A = \frac{1}{2}\int_0^{\pi} a^2\cos^2 3\theta\,d\theta = \frac{a^2}{2}\int_0^{\pi}\frac{1+\cos 6\theta}{2}\,d\theta = \frac{a^2}{4}\left[\theta + \frac{\sin 6\theta}{6}\right]_0^{\pi} = \frac{\pi a^2}{4}$

Note: In this case $R$ is enclosed by $r = a\cos 3\theta$ where $0 \le \theta \le \pi$.

**40.** In this case, $0 \le \theta \le 2\pi$ and we let

$$u = \frac{\sin\theta}{1+\cos\theta},\ \cos\theta = \frac{1-u^2}{1+u^2},\ d\theta = \frac{2\,du}{1+u^2}.$$

Now $u \Longrightarrow \infty$ as $\theta \Longrightarrow \pi$ and we have

$$A = 2\left(\frac{1}{2}\right)\int_0^{\pi}\frac{9}{(2-\cos\theta)^2}\,d\theta = 9\int_0^{\infty}\frac{\frac{2du}{1+u^2}}{4 - 4\left(\frac{1-u^2}{1+u^2}\right) + \frac{(1-u^2)^2}{(1+u^2)^2}} = 18\int_0^{\infty}\frac{1+u^2}{(1+3u^2)^2}\,du$$

$$= 18\int_0^{\infty}\frac{1/3}{1+3u^2}\,du + 18\int_0^{\infty}\frac{2/3}{(1+3u^2)^2}\,du = \left[\frac{6}{\sqrt{3}}\arctan\sqrt{3}\,u\right]_0^{\infty} + \frac{12}{\sqrt{3}}\left(\frac{1}{2}\right)\left[\frac{u}{1+3u^2} + \int\frac{\sqrt{3}}{1+3u^2}\,du\right]_0^{\infty}$$

$$= \frac{6}{\sqrt{3}}\left(\frac{\pi}{2}\right) + \frac{6}{\sqrt{3}}\left[\frac{u}{1+3u^2}\right]_0^{\infty} + \left[\frac{6}{\sqrt{3}}\arctan\sqrt{3}u\right]_0^{\infty} = \frac{3\pi}{\sqrt{3}} + 0 + \frac{3\pi}{\sqrt{3}} = 2\sqrt{3}\pi.$$

**42.** (a) Let $C$ be the line segment joining $(x_1, y_1)$ and $(x_2, y_2)$.

$$y = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1$$

$$dy = \frac{y_2 - y_1}{x_2 - x_1}\,dx$$

$$\int_C -y\,dx + x\,dy = \int_{x_1}^{x_2}\left[-\frac{y_2-y_1}{x_2-x_1}(x-x_1) - y_1 + x\left(\frac{y_2-y_1}{x_2-x_1}\right)\right]dx = \int_{x_1}^{x_2}\left[x_1\left(\frac{y_2-y_1}{x_2-x_1}\right) - y_1\right]dx$$

$$= \left[\left[x_1\left(\frac{y_2-y_1}{x_2-x_1}\right) - y_1\right]x\right]_{x_1}^{x_2} = \left[x_1\left(\frac{y_2-y_1}{x_2-x_1}\right) - y_1\right](x_2 - x_1)$$

$$= x_1(y_2 - y_1) - y_1(x_2 - x_1) = x_1y_2 - x_2y_1$$

**—CONTINUED—**

**42. —CONTINUED—**

(b) Let $C$ be the boundary of the region $A = \frac{1}{2}\int_C -y\,dx + x\,dy = \frac{1}{2}\int\int_R (1 - (-1))\,dA = \int\int_R dA.$

Therefore,

$$\int\int_R dA = \frac{1}{2}\left[\int_{C_1} -y\,dx + x\,dy + \int_{C_2} -y\,dx + x\,dy + \cdots + \int_{C_n} -y\,dx + x\,dy\right]$$

where $C_1$ is the line segment joining $(x_1, y_1)$ and $(x_2, y_2)$, $C_2$ is the line segment joining $(x_2, y_2)$ and $(x_3, y_3)$, . . . , and $C_n$ is the line segment joining $(x_n, y_n)$ and $(x_1, y_1)$. Thus,

$$\int\int_R dA = \frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n)].$$

**44.** Hexagon: (0, 0), (2, 0), (3, 2), (2, 4), (0, 3), (−1, 1)

$A = \frac{1}{2}[(0 - 0) + (4 - 0) + (12 - 4) + (6 - 0) + (0 + 3) + (0 - 0)] = \frac{21}{2}$

**46.** Since $\int_C \mathbf{F} \cdot \mathbf{N}\,ds = \int\int_R \text{div}\,\mathbf{F}\,dA$, then

$$\int_C fD_{\mathbf{N}}g\,ds = \int_C f\nabla g \cdot \mathbf{N}\,ds$$

$$= \int\int_R \text{div}(f\nabla g)\,dA = \int\int_R (f\,\text{div}\,(\nabla g) + \nabla f \cdot \nabla g)\,dA = \int\int_R (f\nabla^2 g + \nabla f \cdot \nabla g)\,dA.$$

**48.** $\int_C f(x)\,dx + g(y)\,dy = \int\int_R \left[\frac{\partial}{\partial x}g(y) - \frac{\partial}{\partial y}f(x)\right] dA = \int\int_R (0 - 0)\,dA = 0$

## Section 14.5 Parametric Surfaces

**2.** $\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + u\mathbf{k}$

$x^2 + y^2 = z^2$

Matches d.

**4.** $\mathbf{r}(u, v) = 4\cos u\,\mathbf{i} + 4\sin u\,\mathbf{j} + v\mathbf{k}$

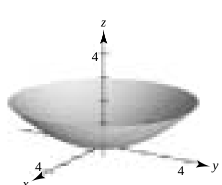
$x^2 + y^2 = 16$

Matches a.

**6.** $\mathbf{r}(u, v) = 2u\cos v\,\mathbf{i} + 2u\sin v\,\mathbf{j} + \frac{1}{2}u^2\mathbf{k}$

$z = \frac{1}{2}u^2,\ x^2 + y^2 = 4u^2 \implies z = \frac{1}{8}(x^2 + y^2)$

Paraboloid



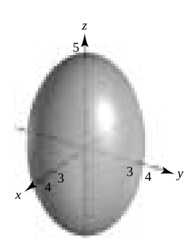

**8.** $\mathbf{r}(u, v) = 3\cos v\cos u\,\mathbf{i} + 3\cos v\sin u\,\mathbf{j} + 5\sin v\mathbf{k}$

$x^2 + y^2 = 9\cos^2 v\cos^2 u + 9\cos^2 v\sin^2 u = 9\cos^2 v$

$\frac{x^2 + y^2}{9} + \frac{z^2}{25} = \cos^2 v + \sin^2 v = 1$

$\frac{x^2}{9} + \frac{y^2}{9} + \frac{z^2}{25} = 1$

Ellipsoid

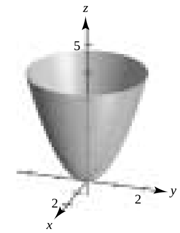


**For Exercises 10 and 12,**

$$\mathbf{r}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + u^2\mathbf{k},\ 0 \le u \le 2,\ 0 \le v \le 2\pi.$$

**Eliminating the parameter yields**

$$z = x^2 + y^2,\ 0 \le z \le 4.$$




**10.** $\mathbf{s}(u, v) = u \cos v\mathbf{i} + u^2\mathbf{j} + u \sin v\mathbf{k},\ 0 \le u \le 2,\ 0 \le v \le 2\pi$

$y = x^2 + z^2$

The paraboloid opens along the $y$-axis instead of the $z$-axis.

**12.** $\mathbf{s}(u, v) = 4u \cos v\mathbf{i} + 4u \sin v\mathbf{j} + u^2\mathbf{k},\ 0 \le u \le 2,\ 0 \le v \le 2\pi$

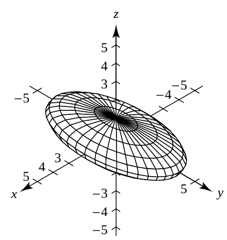
$z = \dfrac{x^2 + y^2}{16}$

The paraboloid is "wider." The top is now the circle $x^2 + y^2 = 64$. It was $x^2 + y^2 = 4$.

**14.** $\mathbf{r}(u, v) = 2 \cos v \cos u\mathbf{i} + 4 \cos v \sin u\mathbf{j} + \sin v\mathbf{k},$

$0 \le u \le 2\pi,\ 0 \le v \le 2\pi$

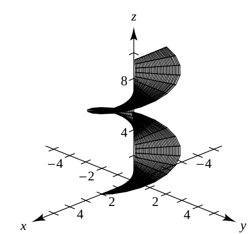
$\dfrac{x^2}{4} + \dfrac{y^2}{16} + \dfrac{z^2}{1} = 1$




**16.** $\mathbf{r}(u, v) = 2u \cos v\mathbf{i} + 2u \sin v\mathbf{j} + v\mathbf{k},$

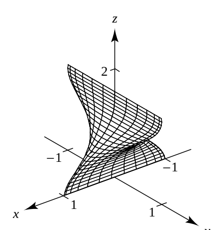
$0 \le u \le 1,\ 0 \le v \le 3\pi$

$z = \arctan\left(\dfrac{y}{x}\right)$




**18.** $\mathbf{r}(u, v) = \cos^3 u \cos v\mathbf{i} + \sin^3 u \sin v\mathbf{j} + u\mathbf{k},$

$0 \le u \le \dfrac{\pi}{2},\ 0 \le v \le 2\pi$




**20.** $z = 6 - x - y$

$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + (6 - u - v)\mathbf{k}$

**22.** $4x^2 + y^2 = 16$

$\mathbf{r}(u, v) = 2 \cos u\mathbf{i} + 4 \sin u\mathbf{j} + v\mathbf{k}$

**24.** $\dfrac{x^2}{9} + \dfrac{y^2}{4} + \dfrac{z^2}{1} = 1$

$\mathbf{r}(u, v) = 3 \cos v \cos u\mathbf{i} + 2 \cos v \sin u\mathbf{j} + \sin v\mathbf{k}$

**26.** $z = x^2 + y^2$ inside $x^2 + y^2 = 9$.

$\mathbf{r}(u, v) = v \cos u\mathbf{i} + v \sin u\mathbf{j} + v^2\mathbf{k},\ 0 \le v \le 3$

**28.** Function: $y = x^{3/2},\ 0 \le x \le 4$

Axis of revolution: $x$-axis

$x = u,\ y = u^{3/2} \cos v,\ z = u^{3/2} \sin v$

$0 \le u \le 4,\ 0 \le v \le 2\pi$

**30.** Function: $z = 4 - y^2,\ 0 \le y \le 2$

Axis of revolution: $y$-axis

$x = (4 - u^2) \cos v,\ y = u,\ z = (4 - u^2) \sin v$

$0 \le u \le 2,\ 0 \le v \le 2\pi$

**32.** $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \sqrt{uv}\,\mathbf{k},\ (1, 1, 1)$

$\mathbf{r}_u(u, v) = \mathbf{i} + \dfrac{v}{2\sqrt{uv}}\mathbf{k},\ \mathbf{r}_v(u, v) = \mathbf{j} + \dfrac{u}{2\sqrt{uv}}\mathbf{k}$

At $(1, 1, 1)$, $u = 1$ and $v = 1$.

$\mathbf{r}_u(1, 1) = \mathbf{i} + \dfrac{1}{2}\mathbf{k},\ \mathbf{r}_v(1, 1) = \mathbf{j} + \dfrac{1}{2}\mathbf{k}$

$$\mathbf{N} = \mathbf{r}_u(1,1) \times \mathbf{r}_v(1,1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{vmatrix} = -\tfrac{1}{2}\mathbf{i} - \tfrac{1}{2}\mathbf{j} + \mathbf{k}$$

Direction numbers: $1, 1, -2$

Tangent plane: $(x - 1) + (y - 1) - 2(z - 1) = 0$

$$x + y - 2z = 0$$

**34.** $\mathbf{r}(u, v) = 2u\cosh v\,\mathbf{i} + 2u\sinh v\,\mathbf{j} + \dfrac{1}{2}u^2\mathbf{k},$

$\mathbf{r}_u(u, v) = 2\cosh v\,\mathbf{i} + 2\sinh v\,\mathbf{j} + u\mathbf{k}$

$\mathbf{r}_v(u, v) = 2u\sinh v\,\mathbf{i} + 2u\cosh v\,\mathbf{j}$

At $(-4, 0, 2)$, $u = -2$ and $v = 0$.

$\mathbf{r}_u(-2, 0) = 2\mathbf{i} - 2\mathbf{k},\ \mathbf{r}_v(-2, 0) = -4\mathbf{j}$

$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = -8\mathbf{i} - 8\mathbf{k}$

Direction numbers: $1, 0, 1$

Tangent plane: $(x + 4) + (z - 2) = 0$

$$x + z = -2$$

**36.** $\mathbf{r}(u, v) = 4u\cos v\,\mathbf{i} + 4u\sin v\,\mathbf{j} + u^2\mathbf{k},\ 0 \le u \le 2,\ 0 \le v \le 2\pi$

$\mathbf{r}_u(u, v) = 4\cos v\,\mathbf{i} + 4\sin v\,\mathbf{j} + 2u\mathbf{k}$

$\mathbf{r}_v(u, v) = -4u\sin v\,\mathbf{i} + 4u\cos v\,\mathbf{j}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4\cos v & 4\sin v & 2u \\ -4u\sin v & 4u\cos v & 0 \end{vmatrix} = -8u^2\cos v\,\mathbf{i} - 8u^2\sin v\,\mathbf{j} + 16u\mathbf{k}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{64u^4 + 256u^2} = 8u\sqrt{u^2 + 4}$

$$A = \int_0^{2\pi}\int_0^2 8u\sqrt{u^2 + 4}\,du\,dv = \int_0^{2\pi}\left(\frac{128\sqrt{2}}{3} - \frac{64}{3}\right)dv = \frac{128\pi}{3}\left(2\sqrt{2} - 1\right)$$

**38.** $\mathbf{r}(u, v) = a\sin u\cos v\,\mathbf{i} + a\sin u\sin v\,\mathbf{j} + a\cos u\,\mathbf{k},\ 0 \le u \le \pi,\ 0 \le v \le 2\pi$

$\mathbf{r}_u(u, v) = a\cos u\cos v\,\mathbf{i} + a\cos u\sin v\,\mathbf{j} - a\sin u\,\mathbf{k}$

$\mathbf{r}_v(u, v) = -a\sin u\sin v\,\mathbf{i} + a\sin u\cos v\,\mathbf{j}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos u\cos v & a\cos u\sin v & -a\sin u \\ -a\sin u\sin v & a\sin u\cos v & 0 \end{vmatrix} = a^2\sin^2 u\cos v\,\mathbf{i} + a^2\sin^2 u\sin v\,\mathbf{j} + a^2\sin u\cos u\,\mathbf{k}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = a^2\sin u$

$$A = \int_0^{2\pi}\int_0^{\pi} a^2\sin u\,du\,dv = 4\pi a^2$$

**40.** $\mathbf{r}(u, v) = (a + b\cos v)\cos u\,\mathbf{i} + (a + b\cos v)\sin u\,\mathbf{j} + b\sin v\,\mathbf{k},\ a > b,\ 0 \le u \le 2\pi,\ 0 \le v \le 2\pi$

$\mathbf{r}_u(u, v) = -(a + b\cos v)\sin u\,\mathbf{i} + (a + b\cos v)\cos u\,\mathbf{j}$

$\mathbf{r}_v(u, v) = -b\sin v\cos u\,\mathbf{i} - b\sin v\sin u\,\mathbf{j} + b\cos v\,\mathbf{k}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -(a + b\cos v)\sin u & (a + b\cos v)\cos u & 0 \\ -b\sin v\cos u & -b\sin v\sin u & b\cos v \end{vmatrix}$$

$$= b\cos u\cos v(a + b\cos v)\mathbf{i} + b\sin u\cos v(a + b\cos v)\mathbf{j} + b\sin v(a + b\cos v)\mathbf{k}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = b(a + b\cos v)$

$$A = \int_0^{2\pi}\int_0^{2\pi} b(a + b\cos v)\,du\,dv = 4\pi^2 ab$$

**42.** $\mathbf{r}(u, v) = \sin u \cos v\mathbf{i} + u\mathbf{j} + \sin u \sin v\mathbf{k},\ 0 \le u \le \pi,\ 0 \le v \le 2\pi$

$\mathbf{r}_u(u, v) = \cos u \cos v\mathbf{i} + \mathbf{j} + \cos u \sin v\mathbf{k}$

$\mathbf{r}_v(u, v) = -\sin u \sin v\mathbf{i} + \sin u \cos v\mathbf{k}$

$\mathbf{r}_u \times \mathbf{r}_v = \sin u \cos v\mathbf{i} - \cos u \sin u\mathbf{j} + \sin u \sin v\mathbf{k}$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sin u\sqrt{1 + \cos^2 u}$

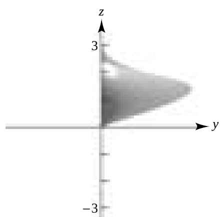
$$A = \int_0^{2\pi}\int_0^{\pi} \sin u\sqrt{1 + \cos^2 u}\, du\, dv = \pi\left[2\sqrt{2} + \ln\left|\frac{\sqrt{2} + 1}{\sqrt{2} - 1}\right|\right]$$

**44.** See the definition, page 1055.

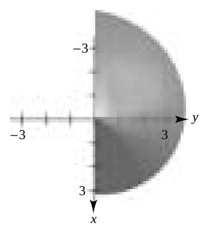
**46.** Graph of $\mathbf{r}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + v\mathbf{k}$

$0 \le u \le \pi,\ 0 \le v \le \pi$ from

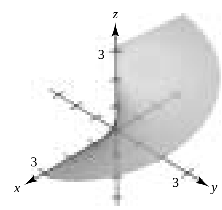
(a) $(10, 0, 0)$




(b) $(0, 0, 10)$




(c) $(10, 10, 10)$




**48.** $\mathbf{r}(u, v) = 2u \cos v\mathbf{i} + 2u \sin v\mathbf{j} + v\mathbf{k},\ 0 \le u \le 1, 0 \le v \le 3\pi$

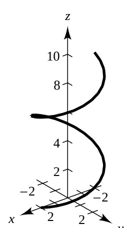
(a) If $u = 1$:

$\mathbf{r}(1, v) = 2 \cos v\mathbf{i} + 2 \sin v\mathbf{j} + v\mathbf{k}$

$x^2 + y^2 = 4$

$0 \le z \le 3\pi$

Helix



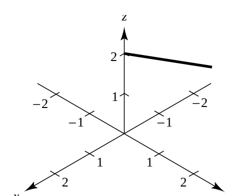

(b) If $v = \dfrac{2\pi}{3}$:

$\mathbf{r}\left(u, \dfrac{2\pi}{3}\right) = -u\mathbf{i} + \sqrt{3}u\mathbf{j} + \dfrac{2\pi}{3}\mathbf{k}$

$y = -\sqrt{3}x$

$z = \dfrac{2\pi}{3}$

Line




(c) If one parameter is held constant, the result is a **curve** in 3-space.

**50.** $x^2 + y^2 - z^2 = 1$

Let $x = u \cos v$, $y = u \sin v$, and $z = \sqrt{u^2 - 1}$. Then,

$\mathbf{r}_u(u, v) = \cos v\mathbf{i} + \sin v\mathbf{j} + \dfrac{u}{\sqrt{u^2 - 1}}\mathbf{k}$

$\mathbf{r}_v(u, v) = -u \sin v\mathbf{i} + u \cos v\mathbf{j}$.

At $(1, 0, 0)$, $u = 1$ and $v = 0$. $\mathbf{r}_u(1, 0)$ is undefined and $\mathbf{r}_v(1, 0) = \mathbf{j}$. The tangent plane at $(1, 0, 0)$ is $x = 1$.

**52.** $\mathbf{r}(u, v) = u\mathbf{i} + f(u)\cos v\,\mathbf{j} + f(u)\sin v\,\mathbf{k}, \ a \le u \le b, \ 0 \le v \le 2\pi$

$\mathbf{r}_u(u, v) = \mathbf{i} + f'(u)\cos v\,\mathbf{j} + f'(u)\sin v\,\mathbf{k}$

$\mathbf{r}_v(u, v) = -f(u)\sin v\,\mathbf{j} + f(u)\cos v\,\mathbf{k}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(u)\cos v & f'(u)\sin v \\ 0 & -f(u)\sin v & f(u)\cos v \end{vmatrix} = f(u)f'(u)\mathbf{i} - f(u)\cos v\,\mathbf{j} - f(u)\sin v\,\mathbf{k}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = f(u)\sqrt{1 + [f'(u)]^2}$

$$A = \int_0^{2\pi}\int_a^b f(u)\sqrt{1 + [f'(u)]^2}\,du\,dv$$

$$= 2\pi\int_a^b f(x)\sqrt{1 + [f'(x)]^2}\,dx \qquad (\text{since } u = x)$$

## Section 14.6 Surface Integrals

**2.** $S\colon\ z = 15 - 2x + 3y, \ 0 \le x \le 2, \ 0 \le y \le 4, \ \dfrac{\partial z}{\partial x} = -2, \ \dfrac{\partial z}{\partial y} = 3, dS = \sqrt{1 + 4 + 9}\,dy\,dx = \sqrt{14}\,dy\,dx$

$$\iint_S (x - 2y + z)\,dS = \int_0^2\int_0^4 (x - 2y + 15 - 2x + 3y)\sqrt{14}\,dy\,dx$$

$$= \sqrt{14}\int_0^2\int_0^4 (15 - x + y)\,dy\,dx = 128\sqrt{14}$$

**4.** $S\colon\ z = \dfrac{2}{3}x^{3/2}, \ 0 \le x \le 1, \ 0 \le y \le x, \ \dfrac{\partial z}{\partial x} = x^{1/2}, \ \dfrac{\partial z}{\partial y} = 0$

$$\iint_S (x - 2y + z)\,dS = \int_0^1\int_0^x \left(x - 2y + \frac{2}{3}x^{3/2}\right)\sqrt{1 + (x^{1/2})^2 + (0)^2}\,dy\,dx$$

$$= \int_0^1\int_0^x \left(x - 2y + \frac{2}{3}x^{3/2}\right)\sqrt{1 + x}\,dy\,dx$$

$$= \frac{2}{3}\int_0^1 x^{5/2}\sqrt{x + 1}\,dx$$

$$= \frac{2}{3}\left[\frac{1}{4}x^{5/2}(1 + x)^{3/2}\right]_0^1 - \frac{5}{12}\int_0^1 x^{3/2}\sqrt{1 + x}\,dx$$

$$= \left[\frac{1}{6}x^{5/2}(1 + x)^{3/2}\right]_0^1 - \frac{5}{12}\left(\frac{1}{3}\right)\left[x^{3/2}(1 + x)^{3/2}\right]_0^1 + \frac{5}{24}\int_0^1 x^{1/2}\sqrt{1 + x}\,dx$$

$$= \frac{\sqrt{2}}{3} - \frac{5\sqrt{2}}{18} + \frac{5}{24}\int_0^1 \sqrt{x + x^2}\,dx$$

$$= \frac{\sqrt{2}}{18} + \frac{5}{24}\int_0^1 \sqrt{\left(x + \frac{1}{2}\right)^2 - \frac{1}{4}}\,dx$$

$$= \frac{\sqrt{2}}{18} + \frac{5}{24}\left(\frac{1}{2}\right)\left[\left(x + \frac{1}{2}\right)\sqrt{x^2 + x} - \frac{1}{4}\ln\left|\left(x + \frac{1}{2}\right) + \sqrt{x^2 + x}\right|\right]_0^1$$

$$= \frac{\sqrt{2}}{18} + \frac{5}{48}\left[\frac{3}{2}\sqrt{2} - \frac{1}{4}\ln\left|\frac{3}{2} + \sqrt{2}\right| + \frac{1}{4}\ln\left|\frac{1}{2}\right|\right]$$

$$= \frac{\sqrt{2}}{18} + \frac{15\sqrt{2}}{96} + \frac{5}{192}\ln\left|\frac{1}{3 + 2\sqrt{2}}\right| = \frac{61\sqrt{2}}{288} - \frac{5}{192}\ln\left|3 + 2\sqrt{2}\right| \approx 0.2536$$

**6.** $S\colon z = h,\ 0 \le x \le 2,\ 0 \le y \le \sqrt{4 - x^2},\ \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial y} = 0$

$$\iint_S dx\, dS = \int_0^2 \int_0^{\sqrt{4-x^2}} xy\, dy\, dx = \frac{1}{2}\int_0^2 x(4 - x^2)\, dx = \frac{1}{2}\left[2x^2 - \frac{x^4}{4}\right]_0^2 = 2$$

**8.** $S\colon z = \dfrac{1}{2}xy,\ 0 \le x \le 4, 0 \le y \le 4,\ \dfrac{\partial z}{\partial x} = \dfrac{1}{2}y,\ \dfrac{\partial z}{\partial y} = \dfrac{1}{2}x$

$$\iint_S xy\, dS = \int_0^4 \int_0^4 xy\sqrt{1 + \frac{y^2}{4} + \frac{x^2}{4}}\, dy\, dx = \frac{3904}{15} - \frac{160\sqrt{5}}{3}$$

**10.** $S\colon z = \cos x,\ 0 \le x \le \dfrac{\pi}{2},\ 0 \le y \le \dfrac{x}{2}$

$$\iint_S (x^2 - 2xy)\, dS = \int_0^{\pi/2} \int_0^{x/2} (x^2 - 2xy)\sqrt{1 + \sin^2 x}\, dy\, dx = \int_0^{\pi/2} \frac{x^3}{4}\sqrt{1 + \sin^2 x}\, dx \approx 0.52$$

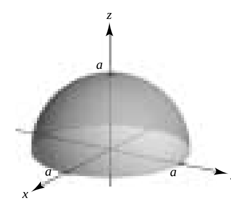
**12.** $S\colon z = \sqrt{a^2 - x^2 - y^2}$

$\rho(x, y, z) = kz$

$$m = \iint_S kz\, dS = \iint_R k\sqrt{a^2 - x^2 - y^2}\sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{a^2 - x^2 - y^2}}\right)^2}\, dA$$

$$= \iint_R k\sqrt{a^2 - x^2 - y^2}\left(\frac{a}{\sqrt{a^2 - x^2 - y^2}}\right) dA$$

$$= \iint_R ka\, dA = ka\iint_R dA = ka(2\pi a^2) = 2ka^3\pi$$




**14.** $S\colon \mathbf{r}(u, v) = 2\cos u\,\mathbf{i} + 2\sin u\,\mathbf{j} + v\mathbf{k},\ 0 \le u \le \dfrac{\pi}{2},$
$0 \le v \le 2$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = \|2\cos u\,\mathbf{i} + 2\sin u\,\mathbf{j}\| = 2$

$$\iint_S (x + y)\, dS = \int_0^2 \int_0^{\pi/2} (2\cos u + 2\sin u)2\, du\, dv = 16$$

**16.** $S\colon \mathbf{r}(u, v) = 4u\cos v\,\mathbf{i} + 4u\sin v\,\mathbf{j} + 3u\mathbf{k},\ 0 \le u \le 4,\ 0 \le v \le \pi$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = \|-12u\cos v\,\mathbf{i} - 12u\sin v\,\mathbf{j} + 16u\mathbf{k}\| = 20u$

$$\iint_S (x + y)\, dS = \int_0^{\pi} \int_0^4 (4u\cos v + 4u\sin v)20u\, du\, dv = \frac{10{,}240}{3}$$

**18.** $f(x, y, z) = \dfrac{xy}{z}$

$S\colon z = x^2 + y^2,\ 4 \le x^2 + y^2 \le 16$

$$\iint_S f(x, y, z)\, dS = \iint_S \frac{xy}{x^2 + y^2}\sqrt{1 + 4x^2 + 4y^2}\, dy\, dx = \int_0^{2\pi}\int_2^4 \frac{r^2\sin\theta\cos\theta}{r^2}\sqrt{1 + 4r^2}\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_2^4 r\sqrt{1 + 4r^2}\sin\theta\cos\theta\, dr\, d\theta = \int_0^{2\pi}\left[\frac{1}{12}(1 + 4r^2)^{3/2}\right]_2^4 \sin\theta\cos\theta\, d\theta$$

$$= \left[\frac{65\sqrt{65} - 17\sqrt{17}}{12}\left(\frac{\sin^2\theta}{2}\right)\right]_0^{2\pi} = 0$$

**20.** $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

$S$: $z = \sqrt{x^2 + y^2}$, $(x - 1)^2 + y^2 \le 1$

$$\iint_S f(x, y, z)\, dS = \iint_S \sqrt{x^2 + y^2 + \left(\sqrt{x^2 + y^2}\right)^2} \sqrt{1 + \left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}}\right)^2}\, dy\, dx$$

$$= \iint_S \sqrt{2(x^2 + y^2)} \sqrt{\frac{2(x^2 + y^2)}{x^2 + y^2}}\, dy\, dx$$

$$= 2\iint_S \sqrt{x^2 + y^2}\, dy\, dx = 2\int_0^{\pi} \int_0^{2\cos\theta} r^2\, dr\, d\theta$$

$$= \frac{16}{3}\int_0^{\pi} \cos^3\theta\, d\theta = \frac{16}{3}\int_0^{\pi} (1 - \sin^2\theta)\cos\theta\, d\theta$$

$$= \left[\frac{16}{3}\left(\sin\theta - \frac{\sin^3\theta}{3}\right)\right]_0^{\pi} = 0$$

**22.** $f(x, y, z) = x^2 + y^2 + z^2$

$S$: $x^2 + y^2 = 9$, $0 \le x \le 3$, $0 \le z \le x$

Project the solid onto the $xz$-plane; $y = \sqrt{9 - x^2}$.

$$\iint_S f(x, y, z)\, dS = \int_0^3 \int_0^x \left[x^2 + (9 - x^2) + z^2\right] \sqrt{1 + \left(\frac{-x}{\sqrt{9 - x^2}}\right)^2 + (0)^2}\, dz\, dx$$

$$= \int_0^3 \int_0^x (9 + z^2) \frac{3}{\sqrt{9 - x^2}}\, dz\, dx = \int_0^3 \left[\frac{3}{\sqrt{9 - x^2}}\left(9z + \frac{z^3}{3}\right)\right]_0^x dx$$

$$= \int_0^3 \frac{3}{\sqrt{9 - x^2}}\left(9x + \frac{x^3}{3}\right) dx = \int_0^3 27x(9 - x^2)^{-1/2}\, dx + \int_0^3 x^3(9 - x^2)^{-1/2}\, dx$$

Let $u = x^2$, $dv = x(9 - x^2)^{-1/2}\, dx$, then $du = 2x\, dx$, $v = -\sqrt{9 - x^2}$.

$$= \left[-27\sqrt{9 - x^2}\right]_0^3 + \left[\left[-x^2\sqrt{9 - x^2}\right]_0^3 + \int_0^3 2x\sqrt{9 - x^2}\, dx\right]$$

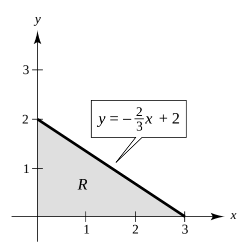
$$= \left[81 - \frac{2}{3}(9 - x^2)^{3/2}\right]_0^3 = 81 + 18 = 99$$

**24.** $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j}$

$S$: $2x + 3y + z = 6$ (first octant)

$G(x, y, z) = 2x + 3y + z - 6$

$\nabla G(x, y, z) = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

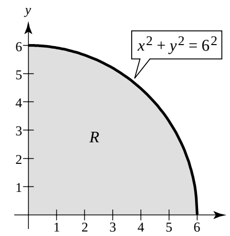



$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = \iint_R \mathbf{F} \cdot \nabla G\, dA = \int_0^3 \int_0^{-(2x/3)+2} (2x + 3y)\, dy\, dx$$

$$= \int_0^3 \left[-\frac{4}{3}x^2 + 4x + \frac{3}{2}\left(-\frac{2}{3}x + 2\right)^2\right] dx$$

$$= \left[-\frac{4}{9}x^3 + 2x^2 - \frac{3}{4}\left(-\frac{2}{3}x + 2\right)^3\right]_0^3 = 12$$

**26.** $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$




$S$: $x^2 + y^2 + z^2 = 36$ (first octant)

$z = \sqrt{36 - x^2 - y^2}$

$G(x, y, z) = z - \sqrt{36 - x^2 - y^2}$

$$\nabla G(x, y, z) = \frac{x}{\sqrt{36 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{36 - x^2 - y^2}}\mathbf{j} + \mathbf{k}$$

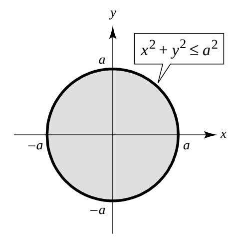
$$\mathbf{F} \cdot \nabla G = \frac{x^2}{\sqrt{36 - x^2 - y^2}} + \frac{y^2}{\sqrt{36 - x^2 - y^2}} + z = \frac{36}{\sqrt{36 - x^2 - y^2}}$$

$$\int_S\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_R\!\!\int \mathbf{F} \cdot \nabla G\, dA = \int_R\!\!\int \frac{36}{\sqrt{36 - x^2 - y^2}}\, dA$$

$$= \int_0^{\pi/2}\int_0^6 \frac{36}{\sqrt{36 - r^2}}\, r\, dr\, d\theta \quad \text{(improper)}$$

$$= 108\pi$$

**28.** $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k}$




$S$: $z = \sqrt{a^2 - x^2 - y^2}$

$G(x, y, z) = z - \sqrt{a^2 - x^2 - y^2}$

$$\nabla G(x, y, z) = \frac{x}{\sqrt{a^2 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{a^2 - x^2 - y^2}}\mathbf{j} + \mathbf{k}$$

$$\mathbf{F} \cdot \nabla G = \frac{x^2}{\sqrt{a^2 - x^2 - y^2}} + \frac{y^2}{\sqrt{a^2 - x^2 - y^2}} - 2\sqrt{a^2 - x^2 - y^2} = \frac{3x^2 + 3y^2 - 2a^2}{\sqrt{a^2 - x^2 - y^2}}$$

$$\int_S\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_R\!\!\int \mathbf{F} \cdot \nabla G\, dA = \int_R\!\!\int \frac{3x^2 + 3y^2 - 2a^2}{\sqrt{a^2 - x^2 - y^2}}\, dA$$

$$= \int_0^{2\pi}\int_0^a \frac{3r^2 - 2a^2}{\sqrt{a^2 - r^2}}\, r\, dr\, d\theta$$

$$= 3\int_0^{2\pi}\int_0^a \frac{r^3}{\sqrt{a^2 - r^2}}\, dr\, d\theta - 2a^2\int_0^{2\pi}\int_0^a \frac{r}{\sqrt{a^2 - r^2}}\, dr\, d\theta$$

$$= 3\left[\int_0^{2\pi}\left[-r^2\sqrt{a^2 - r^2} - \frac{2}{3}(a^2 - r^2)^{3/2}\right]_0^a d\theta\right] - 2a^2\int_0^{2\pi}\left[-\sqrt{a^2 - r^2}\right]_0^a d\theta$$

$$= 3\int_0^{2\pi}\frac{2}{3}a^3\, d\theta - 2a^2\int_0^{2\pi} a\, d\theta = 0$$

**30.** $\mathbf{F}(x, y, z) = (x + y)\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$S$: $z = 1 - x^2 - y^2,\ z = 0$

$G(x, y, z) = z + x^2 + y^2 - 1$

$\nabla G(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$

$\mathbf{F} \cdot \nabla G = 2x(x + y) + 2y(y) + (1 - x^2 - y^2) = x^2 + 2xy + y^2 + 1$

$$\int_S\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_R\!\!\int \mathbf{F} \cdot \nabla G\, dA = \int_R\!\!\int (x^2 + 2xy + y^2 + 1)\, dA$$

$$= \int_0^{2\pi}\int_0^1 (r^2 + 2r^2\cos\theta\sin\theta + 1)r\, dr\, d\theta$$

$$= \int_0^{2\pi}\left(\frac{3}{4} + \frac{1}{2}\sin\theta\cos\theta\right)d\theta = \left[\frac{3}{4}\theta + \frac{\sin^2\theta}{4}\right]_0^{2\pi} = \frac{3\pi}{2}$$

The flux across the bottom $z = 0$ is zero.

**32.** A surface is orientable if a unit normal vector $N$ can be defined at every nonboundary point of $S$ in such a way that the normal vectors vary continuously over the surface $S$.

**34.** Orientable

**36.** $\mathbf{E} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

$S$: $z = \sqrt{1 - x^2 - y^2}$

$$\int_S\int \mathbf{E} \cdot \mathbf{N}\, dS = \int_R\int \mathbf{E} \cdot (-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k})\, dA$$

$$= \int_R\int (yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}) \cdot \left(\frac{x}{\sqrt{1 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{1 - x^2 - y^2}}\mathbf{j} + \mathbf{k}\right) dA$$

$$= \int_R\int \left(\frac{2xyz}{\sqrt{1 - x^2 - y^2}} + xy\right) dA = \int_R\int 3xy\, dA = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 3xy\, dy\, dx = 0$$

**38.** $x^2 + y^2 + z^2 = a^2$

$z = \pm\sqrt{a^2 - x^2 - y^2}$

$$m = 2\int_S\int k\, dS = 2k\int_R\int \sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{a^2 - x^2 - y^2}}\right)^2}\, dA$$

$$= 2k\int_R\int \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dA = 2ka\int_0^{2\pi}\int_0^a \frac{r}{\sqrt{a^2 - r^2}}\, dr\, d\theta$$

$$= 2ka\Big[-\sqrt{a^2 - r^2}\Big]_0^a (2\pi) = 4\pi ka^2$$

$$I_z = 2\int_S\int k(x^2 + y^2)\, dS$$

$$= 2k\int_R\int (x^2 + y^2)\frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dA = 2ka\int_0^{2\pi}\int_0^a \frac{r^3}{\sqrt{a^2 - r^2}}\, dr\, d\theta \text{ (use integration by parts)}$$

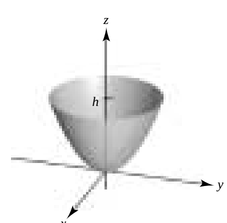
$$= 2ka\left[-r^2\sqrt{a^2 - r^2} - \frac{2}{3}(a^2 - r^2)^{3/2}\right]_0^a (2\pi)$$

$$= 2ka\left(\frac{2}{3}a^3\right)(2\pi) = \frac{2}{3}a^2(4\pi ka^2) = \frac{2}{3}a^2 m$$

Let $u = r^2,\ dv = r(a^2 - r^2)^{-1/2}\, dr,\ du = 2r\, dr,\ v = -\sqrt{a^2 - r^2}$.

**40.** $z = x^2 + y^2,\ 0 \le z \le h$

Project the solid onto the $xy$-plane.




$$I_z = \int_S\int (x^2 + y^2)(1)\, dS$$

$$= \int_{-\sqrt{h}}^{\sqrt{h}}\int_{-\sqrt{h-x^2}}^{\sqrt{h-x^2}} (x^2 + y^2)\sqrt{1 + 4x^2 + 4y^2}\, dy\, dx$$

$$= \int_0^{2\pi}\int_0^{\sqrt{h}} r^2\sqrt{1 + 4r^2}\, r\, dr\, d\theta$$

$$= 2\pi\left[\frac{h}{12}(1 + 4h)^{3/2} - \frac{1}{120}(1 + 4h)^{5/2}\right] + \frac{2\pi}{120}$$

$$= \frac{(1 + 4h)^{3/2}\pi}{60}[10h - (1 + 4h)] + \frac{\pi}{60} = \frac{\pi}{60}[(1 + 4h)^{3/2}(6h - 1) + 1]$$

**42.** $S$: $z = \sqrt{16 - x^2 - y^2}$

$\mathbf{F}(x, y, z) = 0.5z\mathbf{k}$

$$\int_S\int \rho\mathbf{F}\cdot\mathbf{N}\,dS = \int_R\int \rho\mathbf{F}\cdot(-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k})\,dA$$

$$= \int_R\int 0.5\rho z\mathbf{k}\cdot\left[\frac{x}{\sqrt{16 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{16 - x^2 - y^2}}\mathbf{j} + \mathbf{k}\right]dA$$

$$= \int_R\int 0.5\,\rho z\,dA = \int_R\int 0.5\rho\sqrt{16 - x^2 - y^2}\,dA$$

$$= 0.5\rho\int_0^{2\pi}\int_0^4 \sqrt{16 - r^2}\,r\,dr\,d\theta = 0.5\rho\int_0^{2\pi}\frac{64}{3}\,d\theta = \frac{64\pi\rho}{3}$$

## Section 14.7 Divergence Theorem

**2.** **Surface Integral:** There are three surfaces to the cylinder.

Bottom: $z = 0$, $\mathbf{N} = -\mathbf{k}$, $\mathbf{F}\cdot\mathbf{N} = -z^2$

$$\int_{S_1}\int 0\,dS = 0$$

Top: $z = h$, $\mathbf{N} = \mathbf{k}$, $\mathbf{F}\cdot\mathbf{N} = z^2$

$$\int_{S_2}\int h^2\,dS = h^2\,(\text{Area of circle}) = 4\pi h^2$$

Side: $\mathbf{r}(u, v) = 2\cos u\mathbf{i} + 2\sin u\mathbf{j} + v\mathbf{k}$, $0 \le u \le 2\pi$, $0 \le v \le h$

$\mathbf{r}_u = -2\sin u\mathbf{i} + 2\cos u\mathbf{j}$, $r_v = \mathbf{k}$

$\mathbf{r}_u \times \mathbf{r}_v = 2\cos u\mathbf{i} + 2\sin u\mathbf{j}$

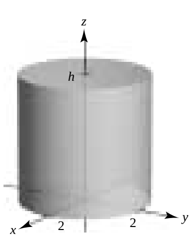
$\mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v) = 8\cos^2 u - 8\sin^2 u$

$$\int_{S_3}\int \mathbf{F}\cdot\mathbf{N}\,dS = \int_0^h\int_0^{2\pi}(8\cos^2 u - 8\sin^2 u)\,du\,dv = 0$$

Therefore, $\displaystyle\int_S\int \mathbf{F}\cdot\mathbf{N}\,dS = 0 + 4\pi h^2 + 0 = 4\pi h^2$.

**Divergence Theorem:** $\operatorname{div}\mathbf{F} = 2 - 2 + 2z = 2z$

$$\iiint_Q 2z\,dV = \int_0^{2\pi}\int_0^2\int_0^h 2zr\,dz\,dr\,d\theta = 4\pi h^2.$$



**4.** $\mathbf{F}(x, y, z) = xy\mathbf{i} + z\mathbf{j} + (x + y)\mathbf{k}$

$S$: surface bounded by the planes $y = 4$, $z = 4 - x$ and the coordinate planes

**Surface Integral:** There are five surfaces to this solid.

$z = 0$, $\mathbf{N} = -\mathbf{k}$, $\mathbf{F} \cdot \mathbf{N} = -(x + y)$

$$\int_{S_1}\int -(x + y)\, dS = \int_0^4\int_0^4 -(x + y)\, dy\, dx = -\int_0^4 (4x + 8)\, dx = -64$$

$y = 0$, $\mathbf{N} = -\mathbf{j}$, $\mathbf{F} \cdot \mathbf{N} = -z$

$$\int_{S_2}\int -z\, dS = \int_0^4\int_0^{4-x} -z\, dz\, dx = -\int_0^4 \frac{(4 - x)^2}{2}\, dx = -\frac{32}{3}$$

$y = 4$, $\mathbf{N} = \mathbf{j}$, $\mathbf{F} \cdot \mathbf{N} = z$

$$\int_{S_3}\int z\, dS = \int_0^4\int_0^{4-x} z\, dz\, dx = \int_0^4 \frac{(4 - x)^2}{2}\, dx = \frac{32}{3}$$

$x = 0$, $\mathbf{N} = -\mathbf{i}$, $\mathbf{F} \cdot \mathbf{N} = -xy$

$$\int_{S_4}\int -xy\, dS = \int_0^4\int_0^4 0\, dS = 0$$

$x + z = 4$, $\mathbf{N} = \dfrac{\mathbf{i} + \mathbf{k}}{\sqrt{2}}$, $\mathbf{F} \cdot \mathbf{N} = \dfrac{1}{\sqrt{2}}[xy + x + y]$, $dS = \sqrt{2}\, dA$

$$\int_{S_5}\int \frac{1}{\sqrt{2}}[xy + x + y]\sqrt{2}\, dA = \int_0^4\int_0^4 (xy + x + y)\, dy\, dx = 128$$

Therefore, $\displaystyle\int_S\int \mathbf{F} \cdot \mathbf{N}\, dS = -64 - \tfrac{32}{3} + \tfrac{32}{3} + 0 + 128 = 64.$

**Divergence Theorem:** Since div $\mathbf{F} = y$, we have

$$\iiint_Q \text{div}\, \mathbf{F}\, dV = \int_0^4\int_0^4\int_0^{4-x} y\, dz\, dy\, dx = 64.$$

**6.** Since div $\mathbf{F} = 2xz^2 - 2 + 3xy$ we have

$$\iiint_Q \text{div}\, \mathbf{F}\, dV = \int_0^a\int_0^a\int_0^a (2xz^2 - 2 + 3xy)\, dz\, dy\, dx = \int_0^a\int_0^a \left(\frac{2}{3}xa^3 - 2a + 3xya\right) dy\, dx$$

$$= \int_0^a \left(\frac{2}{3}xa^4 - 2a^2 + \frac{3}{2}xa^3\right) dx$$

$$= \frac{1}{3}a^6 - 2a^3 + \frac{3}{4}a^5.$$

**8.** Since div $\mathbf{F} = y + z - y = z$, we have

$$\iiint_Q \text{div}\, \mathbf{F}\, dV = \int_{-a}^a\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} z\, dz\, dy\, dx = \int_0^{2\pi}\int_0^a\int_0^{\sqrt{a^2-r^2}} zr\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^a \left[\frac{a^2r}{2} - \frac{r^3}{2}\right] dr\, d\theta = \int_0^{2\pi}\left[\frac{a^2r^2}{4} - \frac{r^4}{8}\right]_0^a d\theta = \int_0^{2\pi} \frac{a^4}{8}\, d\theta = \frac{\pi a^4}{4}.$$

**10.** Since div $\mathbf{F} = xz$, we have

$$\iiint_Q xz\, dV = \int_0^4\int_{-3}^3\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} xz\, dx\, dy\, dz = \int_0^4\int_{-3}^3 \frac{z}{2}(0)\, dy\, dz = 0.$$

**12.** Since div $\mathbf{F} = y^2 + x^2 + e^z$, we have

$$\iiint_Q (x^2 + y^2 + e^z)\, dV = \int_0^{16} \int_{-\sqrt{256-x^2}}^{\sqrt{256-x^2}} \int_{(1/2)\sqrt{x^2+y^2}}^{8} (x^2 + y^2 + e^z)\, dz\, dy\, dx$$

$$= \int_0^{2\pi} \int_0^{16} \int_{r/2}^{8} (r^2 + e^z) r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^{16} \left(8r^3 + re^8 - \frac{1}{2}r^4 - re^{r/2}\right) dr\, d\theta$$

$$= \int_0^{2\pi} \left(\frac{131{,}052}{5} + 100e^8\right) d\theta = \frac{262{,}104}{5}\pi + 200e^8\pi$$

**14.** Since div $\mathbf{F} = e^z + e^z + e^z = 3e^z$, we have

$$\iiint_Q 3e^z\, dV = \int_0^6 \int_0^4 \int_0^{4-y} 3e^z\, dz\, dy\, dx = \int_0^6 \int_0^4 3[e^{4-y} - 1]\, dy\, dx = \int_0^6 3(e^4 - 5)\, dx = 18(e^4 - 5).$$

**16.** div $\mathbf{F} = 2$

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = \iiint_Q \text{div}\, \mathbf{F}\, dV = \iiint_Q 2\, dV.$$

The surface $S$ is the upper half of a hemisphere of radius 2. Since the volume is $\frac{1}{2}\left(\frac{4}{3}\pi(2^3)\right) = 16\pi/3$, you have

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = 2(\text{Volume}) = \frac{32\pi}{3}.$$

**18.** Using the Divergence Theorem, we have

$$\iint_S \textbf{curl}\, \mathbf{F} \cdot \mathbf{N}\, dS = \iiint_Q \text{div}\, (\textbf{curl}\mathbf{F})\, dV$$

$$\textbf{curl}\, \mathbf{F}(x, y, z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy\cos z & yz\sin x & xyz \end{vmatrix} = (xz - y\sin x)\mathbf{i} - (yz + xy\sin z)\mathbf{j} + (yz\cos x - x\cos z)\mathbf{k}.$$

Now, div $\textbf{curl}\, \mathbf{F}(x, y, z) = (z - y\cos x) - (z + x\sin z) + (y\cos x + x\sin z) = 0$. Therefore,

$$\iint_S \textbf{curl}\, \mathbf{F} \cdot \mathbf{N}\, dS = \iiint_Q \text{div}\, (\textbf{curl}\, \mathbf{F})\, dV = 0.$$

**20.** If div $\mathbf{F}(x, y, z) > 0$, then source.

If div $\mathbf{F}(x, y, z) < 0$, then sink.

If div $\mathbf{F}(x, y, z) = 0$, then incompressible.

**22.** $v = \int_0^a \int_0^a x\, dy\, dz = \int_0^a \int_0^a a\, dy\, dz = \int_0^a a^2\, dz = a^3$

Similarly, $\int_0^a \int_0^a y\, dz\, dx = \int_0^a \int_0^a z\, dx\, dy = a^3$.

**24.** If $\mathbf{F}(x, y, z) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, then div $\mathbf{F} = 0$.
Therefore,

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = \iiint_Q \text{div}\, \mathbf{F}\, dV = \iiint_Q 0\, dV = 0.$$

**26.** If $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then div $\mathbf{F} = 3$.

$$\frac{1}{\|\mathbf{F}\|}\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = \frac{1}{\|\mathbf{F}\|}\iiint_Q \text{div}\, \mathbf{F}\, dV = \frac{1}{\|\mathbf{F}\|}\iiint_Q 3\, dV = \frac{3}{\|\mathbf{F}\|}\iiint_Q dV$$

**28.** $$\iint_S (fD_{\mathbf{N}}g - gD_{\mathbf{N}}f)\, dS = \iint_S fD_{\mathbf{N}}g\, dS - \iint_S gD_{\mathbf{N}}f\, dS$$

$$= \iiint_Q (f\nabla^2 g + \nabla f \cdot \nabla g)\, dV - \iiint_Q (g\nabla^2 f + \nabla g \cdot \nabla f)\, dV = \iiint_Q (f\nabla^2 g - g\nabla^2 f)\, dV$$

## Section 14.8 Stokes's Theorem

**2.** $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + x^2\mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & y^2 & x^2 \end{vmatrix} = -2x\mathbf{j}$$

**4.** $\mathbf{F}(x, y, z) = x \sin y\mathbf{i} - y \cos x\mathbf{j} + yz^2\mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x \sin y & -y \cos x & yz^2 \end{vmatrix}$$

$$= z^2\mathbf{i} + (y \sin x - x \cos y)\mathbf{k}$$

**6.** $\mathbf{F}(x, y, z) = \arcsin y\mathbf{i} + \sqrt{1 - x^2}\,\mathbf{j} + y^2\mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \arcsin y & \sqrt{1 - x^2} & y^2 \end{vmatrix}$$

$$= 2y\mathbf{i} + \left[\frac{-x}{\sqrt{1 - x^2}} - \frac{1}{\sqrt{1 - y^2}}\right]\mathbf{k}$$

$$= 2y\mathbf{i} - \left[\frac{x}{\sqrt{1 - x^2}} + \frac{1}{\sqrt{1 - y^2}}\right]\mathbf{k}$$

**8.** In this case $C$ is the circle $x^2 + y^2 = 4,\ z = 0,\ dz = 0$.

**Line Integral:** $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C -y\,dx + x\,dy$

Let $x = 2 \cos t,\ y = 2 \sin t,$ then $dx = -2 \sin t\,dt,\ dy = 2 \cos t\,dt$, and $\displaystyle\int_C -y\,dx + x\,dy = \int_0^{2\pi} 4\,dt = 8\pi.$

**Double Integral:** $F(x, y, z) = z + x^2 + y^2 - 4,\ \mathbf{N} = \dfrac{\nabla F}{\|\nabla F\|} = \dfrac{2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}}{\sqrt{1 + 4x^2 + 4y^2}},\ dS = \sqrt{1 + 4x^2 + 4y^2}\,dA$

$\textbf{curl F} = 2\mathbf{k}$, therefore

$$\iint (\textbf{curl F}) \cdot \mathbf{N}dS = \int_R\int 2\,dA = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 2\,dy\,dx = 2\int_{-2}^{2} 2\sqrt{4 - x^2}\,dx$$

$$= 4\int_{-2}^{2} \sqrt{4 - x^2}\,dx = 2\left[x\sqrt{4 - x^2} + 4 \arcsin\frac{x}{2}\right]_{-2}^{2} = 8\pi.$$

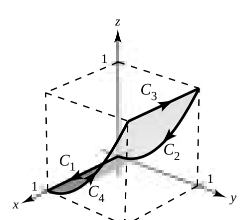
**10. Line Integral:** From the accompanying figure we see that for

$C_1$: $y = 0,\ z = 0,\ dy = dz = 0$

$C_2$: $z = y^2,\ x = 0,\ dx = 0,\ dz = 2y\,dy$

$C_3$: $y = a,\ z = a^2,\ dy = dz = 0$

$C_4$: $z = y^2,\ x = a,\ dx = 0,\ dz = 2y\,dy.$




Hence,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C z^2\,dx + x^2\,dy + y^2\,dz$$

$$= \int_{C_1} 0 + \int_{C_2} 2y^3\,dy + \int_{C_3} a^4\,dx + \int_{C_4} a^2\,dy + y^2(2y)\,dy$$

$$= \int_0^a 2y^3\,dy + \int_0^a a^4\,dx + \int_a^0 a^2\,dy + \int_a^0 2y^3\,dy = \Big[a^4x\Big]_0^a + \Big[a^2y\Big]_a^0 = a^5 - a^3 = a^3(a^2 - 1).$$

**—CONTINUED—**

**10. —CONTINUED—**

**Double Integral:** Since $\mathbf{F}(x, y, z) = y^2 - z$, we have

$$\mathbf{N} = \frac{2y\mathbf{j} - \mathbf{k}}{\sqrt{1 + 4y^2}} \text{ and } dS = \sqrt{1 + 4y^2}\, dA.$$

Furthermore, $\mathbf{curl\ F} = 2y\mathbf{i} + 2z\mathbf{j} + 2x\mathbf{k}$. Therefore,

$$\int_S\int (\mathbf{curl\ F}) \cdot \mathbf{N}\, dS = \int_R\int (4yz - 2x)\, dA = \int_0^a\int_0^a (4y^2 - 2x)\, dy\, dx = \int_0^a (a^4 - 2ax)\, dx = \Big[a^4x - ax^2\Big]_0^a = a^3(a^2 - 1).$$

**12.** Let $A = (0, 0, 0)$, $B = (1, 1, 1)$, and $C = (0, 0, 2)$. Then $\mathbf{U} = \overrightarrow{AB} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, and $\mathbf{V} = \overrightarrow{AC} = 2\mathbf{k}$, and

$$\mathbf{N} = \frac{\mathbf{U} \times \mathbf{V}}{\|\mathbf{U} \times \mathbf{V}\|} = \frac{2\mathbf{i} - 2\mathbf{j}}{2\sqrt{2}} = \frac{\mathbf{i} - \mathbf{j}}{\sqrt{2}}.$$

Hence, $F(x, y, z) = x - y$ and $dS = \sqrt{2}\, dA$. Since $\mathbf{curl\ F} = \dfrac{2x}{x^2 + y^2}\mathbf{k}$, we have $\displaystyle\int_S\int (\mathbf{curl\ F}) \cdot \mathbf{N}\, dS = \int_R\int 0\, dS = 0.$

**14.** $\mathbf{F}(x, y, z) = 4xz\mathbf{i} + y\mathbf{j} + 4xy\mathbf{k}$, $S$: $9 - x^2 - y^2$, $z \le 0$

$\mathbf{curl\ F} = 4x\mathbf{i} + (4x - 4y)\mathbf{j}$

$G(x, y, z) = x^2 + y^2 + z - 9$

$\nabla G(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$

$$\begin{aligned}\int_S\int (\mathbf{curl\ F}) \cdot \mathbf{N}\, dS &= \int_R\int \left[8x^2 + 2y(4x - 4y)\right] dA\\ &= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \left[8x^2 + 8xy - 8y^2\right] dy\, dx\\ &= \int_{-3}^{3}\left(16x^2\sqrt{9 - x^2} - \frac{16}{3}(9 - x^2)^{3/2}\right) dx = 0\end{aligned}$$

**16.** $\mathbf{F}(x, y, z) = x^2\mathbf{i} + z^2\mathbf{j} - xyz\mathbf{k}$, $S$: $z = \sqrt{4 - x^2 - y^2}$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & z^2 & -xyz \end{vmatrix} = (-xz - 2z)\mathbf{i} + yz\mathbf{j}$$

$G(x, y, z) = z - \sqrt{4 - x^2 - y^2}$

$$\nabla G(x, y, z) = \frac{x}{\sqrt{4 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{4 - x^2 - y^2}}\mathbf{j} + \mathbf{k}$$

$$\begin{aligned}\int_S\int (\mathbf{curl\ F}) \cdot \mathbf{N}\, dS &= \int_R\int \left[\frac{-z(x + 2)x}{\sqrt{4 - x^2 - y^2}} + \frac{y^2z}{\sqrt{4 - x^2 - y^2}}\right] dA\\ &= \int_R\int \left[-x(x + 2) + y^2\right] dA = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (-x^2 - 2x + y^2)\, dy\, dx\\ &= \int_{-2}^{2}\left[-x^2y - 2xy + \frac{y^3}{3}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx\\ &= \int_{-2}^{2}\left[-2x^2\sqrt{4 - x^2} - 4x\sqrt{4 - x^2} + \frac{2}{3}(4 - x^2)\sqrt{4 - x^2}\right] dx\\ &= \int_{-2}^{2}\left[-\frac{8}{3}x^2\sqrt{4 - x^2} - 4x\sqrt{4 - x^2} + \frac{8}{3}\sqrt{4 - x^2}\right] dx\\ &= \left[-\frac{8}{3}\left(\frac{1}{8}\right)\left[x(2x^2 - 4)\sqrt{4 - x^2} + 16\arcsin\frac{x}{2}\right] + \frac{4}{3}(4 - x^2)^{3/2} + \frac{8}{3}\left(\frac{1}{2}\right)\left[x\sqrt{4 - x^2} + 4\arcsin\frac{x}{2}\right]\right]_{-2}^{2}\\ &= \left[\left(-\frac{1}{3}\right)(8\pi) + \frac{4}{3}(2\pi) + \frac{1}{3}(-8\pi) - \frac{4}{3}(-2\pi)\right] = 0\end{aligned}$$

**18.** $\mathbf{F}(x, y, z) = yz\mathbf{i} + (2 - 3y)\mathbf{j} + (x^2 + y^2)\mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & 2 - 3y & x^2 + y^2 \end{vmatrix} = 2y\mathbf{i} + (y - 2x)\mathbf{j} - z\mathbf{k}$$

$S$: the first octant portion of $x^2 + z^2 = 16$ over $x^2 + y^2 = 16$

$G(x, y, z) = z - \sqrt{16 - x^2}$

$$\nabla G(x, y, z) = \frac{x}{\sqrt{16 - x^2}}\mathbf{i} + \mathbf{k}$$

$$\begin{aligned}
\int_S\int (\textbf{curl F}) \cdot \mathbf{N}\, dS &= \int_R\int \left[\frac{2xy}{\sqrt{16 - x^2}} - z\right] dA \\
&= \int_R\int \left[\frac{2xy}{\sqrt{16 - x^2}} - \sqrt{16 - x^2}\right] dA \\
&= \int_0^4 \int_0^{\sqrt{16 - x^2}} \left[\frac{2xy}{\sqrt{16 - x^2}} - \sqrt{16 - x^2}\right] dy\, dx \\
&= \int_0^4 \left[\frac{x}{\sqrt{16 - x^2}}y^2 - \sqrt{16 - x^2}\,y\right]_0^{\sqrt{16 - x^2}} dx \\
&= \int_0^4 \left[x\sqrt{16 - x^2} - (16 - x^2)\right] dx \\
&= \left[-\frac{1}{3}(16 - x^2)^{3/2} - 16x + \frac{x^3}{3}\right]_0^4 \\
&= \left(-64 + \frac{64}{3}\right) - \left(-\frac{64}{3}\right) = -\frac{64}{3}
\end{aligned}$$

**20.** $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xyz & y & z \end{vmatrix} = xy\mathbf{j} - xz\mathbf{k}$$

$S$: the first octant portion of $z = x^2$ over $x^2 + y^2 = a^2$. We have

$$\mathbf{N} = \frac{2x\mathbf{i} - \mathbf{k}}{\sqrt{1 + 4x^2}} \text{ and } dS = \sqrt{1 + 4x^2}\, dA.$$

$$\begin{aligned}
\int_S\int (\textbf{curl F}) \cdot \mathbf{N}\, dS &= \int_R\int xz\, dA = \int_R\int x^3\, dA \\
&= \int_0^a \int_0^{\sqrt{a^2 - x^2}} x^3\, dy\, dx \\
&= \int_0^a x^3\sqrt{a^2 - x^2}\, dx \\
&= \left[-\frac{1}{3}x^2(a^2 - x^2)^{3/2} - \frac{2}{15}(a^2 - x^2)^{5/2}\right]_0^a \\
&= \frac{2}{15}a^5
\end{aligned}$$

**22.** $\mathbf{F}(x, y, z) = -z\mathbf{i} + y\mathbf{k}$

$S: x^2 + y^2 = 1$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z & 0 & y \end{vmatrix} = \mathbf{i} - \mathbf{j}$$

Letting $\mathbf{N} = \mathbf{k}$, $\mathbf{curl\ F} \cdot \mathbf{N} = 0$ and $\displaystyle\int_S\int (\mathbf{curl\ F}) \cdot \mathbf{N}\, dS = 0.$

**24.** **curl F** measures the rotational tendency.

See page 1084.

**26.** $f(x, y, z) = xyz,\ g(x, y, z) = z,\ S: z = \sqrt{4 - x^2 - y^2}$

(a) $\nabla g(x, y, z) = \mathbf{k}$

$f(x, y, z)\nabla g(x, y, z) = xyz\mathbf{k}$

$\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 0\mathbf{k},\quad 0 \le t \le 2\pi$

$$\int_C [f(x, y, z)\nabla g(x, y, z)] \cdot d\mathbf{r} = 0$$

(b) $\nabla f(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

$\nabla g(x, y, z) = \mathbf{k}$

$$\nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ yz & xz & xy \\ 0 & 0 & 1 \end{vmatrix} = xz\mathbf{i} - yz\mathbf{j}$$

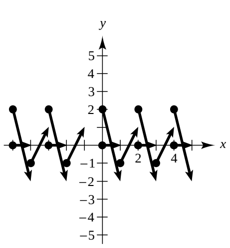
$$\mathbf{N} = \frac{x}{\sqrt{4 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{4 - x^2 - y^2}}\mathbf{j} + \mathbf{k}$$

$$dS = \sqrt{1 + \left(\frac{-x}{\sqrt{4 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{4 - x^2 - y^2}}\right)^2}\, dA = \frac{2}{\sqrt{4 - x^2 - y^2}}\, dA$$

$$\begin{aligned}\int_S\int [\nabla f(x, y, z) \times \nabla g(x, y, z)] \cdot \mathbf{N}\, dS &= \int_S\int \left[\frac{x^2 z}{\sqrt{4 - x^2 - y^2}} - \frac{y^2 z}{\sqrt{4 - x^2 - y^2}}\right] \frac{2}{\sqrt{4 - x^2 - y^2}} dA \\ &= \int_S\int \frac{2(x^2 - y^2)}{\sqrt{4 - x^2 - y^2}}\, dA \\ &= \int_0^2\int_0^{2\pi} \frac{2r^2(\cos^2\theta - \sin^2\theta)}{\sqrt{4 - r^2}} r\, d\theta\, dr \\ &= \int_0^2 \left[\frac{2r^3}{\sqrt{4 - r^2}}\left(\frac{1}{2}\sin 2\theta\right)\right]_0^{2\pi} dr = 0\end{aligned}$$

## Review Exercises for Chapter 14

**2.** $\mathbf{F}(x, y) = \mathbf{i} - 2y\mathbf{j}$




**4.** $f(x, y, z) = x^2 e^{yz}$

$$\begin{aligned}\mathbf{F}(x, y, z) &= 2xe^{yz}\,\mathbf{i} + x^2 z e^{yz}\,\mathbf{j} + x^2 y e^{yz}\,\mathbf{k} \\ &= xe^{yz}(2\mathbf{i} + xz\mathbf{j} + xy\mathbf{k})\end{aligned}$$

**6.** Since $\partial M/\partial y = -1/x^2 = \partial N/\partial x$, $\mathbf{F}$ is conservative. From $M = \partial U/\partial x = -y/x^2$ and $N = \partial U/\partial y = 1/x$, partial integration yields $U = (y/x) + h(y)$ and $U = (y/x) + g(x)$ which suggests that $U(x, y) = (y/x) + C$.

**8.** Since $\partial M/\partial y = -6y^2 \sin 2x = \partial N/\partial x$, $\mathbf{F}$ is conservative. From $M = \partial U/\partial x = -2y^3 \sin 2x$ and $N = \partial U/\partial y = 3y^2(1 + \cos 2x)$, we obtain $U = y^3 \cos 2x + h(y)$ and $U = y^3(1 + \cos 2x) + g(x)$ which suggests that $h(y) = y^3$, $g(x) = C$, and $U(x, y) = y^3(1 + \cos 2x) + C$.

**10.** Since

$$\frac{\partial M}{\partial y} = 4x = \frac{\partial N}{\partial x},$$

$$\frac{\partial M}{\partial z} = 2z = \frac{\partial P}{\partial x},$$

$$\frac{\partial N}{\partial z} = 6y \neq \frac{\partial P}{\partial y}$$

$\mathbf{F}$ is not conservative.

**12.** Since

$$\frac{\partial M}{\partial y} = \sin z = \frac{\partial N}{\partial x}, \quad \frac{\partial M}{\partial z} = y \cos z \neq \frac{\partial P}{\partial x},$$

$\mathbf{F}$ is not conservative.

**14.** Since $\mathbf{F} = xy^2\mathbf{j} - zx^2\mathbf{k}$;

(a) $\text{div}\, \mathbf{F} = 2xy - x^2$

(b) $\mathbf{curl\, F} = 2xz\,\mathbf{j} + y^2\mathbf{k}$

**16.** Since $\mathbf{F} = (3x - y)\mathbf{i} + (y - 2z)\mathbf{j} + (z - 3x)\mathbf{k}$:

(a) $\text{div}\, \mathbf{F} = 3 + 1 + 1 = 5$

(b) $\mathbf{curl\, F} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

**18.** Since $\mathbf{F} = (x^2 - y)\mathbf{i} - (x + \sin^2 y)\mathbf{j}$:

(a) $\text{div}\, \mathbf{F} = 2x - 2 \sin y \cos y$

(b) $\mathbf{curl\, F} = \mathbf{0}$

**20.** Since $\mathbf{F} = \dfrac{z}{x}\mathbf{i} + \dfrac{z}{y}\mathbf{j} + z^2\mathbf{k}$:

(a) $\text{div}\, \mathbf{F} = -\dfrac{z}{x^2} - \dfrac{z}{y^2} + 2z = z\left(2 - \dfrac{1}{x^2} - \dfrac{1}{y^2}\right)$

(b) $\mathbf{curl\, F} = -\dfrac{1}{y}\mathbf{i} + \dfrac{1}{x}\mathbf{j}$

**22.** (a) Let $x = 5t$, $y = 4t$, $0 \leq t \leq 1$, then $ds = \sqrt{41}\, dt$.

$$\int_C xy\, ds = \int_0^1 20t^2 \sqrt{41}\, dt = \frac{20\sqrt{41}}{3}$$

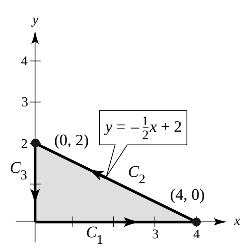
(b) $C_1$: $x = t$, $y = 0$, $0 \leq t \leq 4$, $ds = dt$

$C_2$: $x = 4 - 4t$, $y = 2t$, $0 \leq t \leq 1$, $ds = 2\sqrt{5}\, dt$

$C_3$: $x = 0$, $y = 2 - t$, $0 \leq t \leq 2$, $ds = dt$

Therefore, $\displaystyle\int_C xy\, ds = \int_0^4 0\, dt = \int_0^1 (8t - 8t^2)2\sqrt{5}\, dt + \int_0^2 0\, dt$

$$= 16\sqrt{5}\left[\frac{t^2}{2} - \frac{t^3}{3}\right]_0^1 = \frac{8\sqrt{5}}{3}.$$




**24.** $x = t - \sin t$, $y = 1 - \cos t$, $0 \leq t \leq 2\pi$, $\dfrac{dx}{dt} = 1 - \cos t$, $\dfrac{dy}{dt} = \sin t$

$$\int_C x\, ds = \int_0^{2\pi} (t - \sin t)\sqrt{(1 - \cos t)^2 + (\sin t)^2}\, dt = \int_0^{2\pi} (t - \sin t)\sqrt{2 - 2\cos t}\, dt$$

$$= \sqrt{2}\int_0^{2\pi} \left[t\sqrt{1 - \cos t} - \sin t\sqrt{1 - \cos t}\right] dt = \sqrt{2}\left[-\frac{2}{3}(1 - \cos t)^{3/2}\right]_0^{2\pi} + \sqrt{2}\int_0^{2\pi} t\sqrt{1 - \cos t}\, dt$$

$$= \sqrt{2}\int_0^{2\pi} t\sqrt{1 - \cos t}\, dt$$

$$= 8\pi$$

**26.** $x = \cos t + t\sin t,\ y = \sin t - t\sin t,\ 0 \le t \le \frac{\pi}{2},\ dx = t\cos t\,dt,\ dy = (\cos t - t\cos t - \sin t)\,dt$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = \int_0^{\pi/2} \left[\sin t\cos t(5t^2 - 6t + 2) + \cos^2 t(t + 1) + \sin^2 t(2t - 3)\right] dt \approx 1.01$$

**28.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^{3/2}\mathbf{k},\ 0 \le t \le 4$

$x'(t) = 1,\ y'(t) = 2t,\ z'(t) = \frac{3}{2}t^{1/2}$

$$\int_C (x^2 + y^2 + z^2)\,ds = \int_0^4 (t^2 + t^4 + t^3)\sqrt{1 + 4t^2 + \frac{9}{4}t}\ dt \approx 2080.59$$

**30.** $f(x, y) = 12 - x - y$

$C$: $y = x^2$ from $(0, 0)$ to $(2, 4)$

$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j},\ 0 \le t \le 2$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{1 + 4t^2}$

Lateral surface area:

$$\int_C f(x, y)\,ds = \int_0^2 (12 - t - t^2)\sqrt{1 + 4t^2}\,dt \approx 41.532$$

**32.** $d\mathbf{r} = [(-4\sin t)\mathbf{i} + 3\cos t\,\mathbf{j}]\,dt$

$\mathbf{F} = (4\cos t - 3\sin t)\mathbf{i} + (4\cos t + 3\sin t)\mathbf{j},\ 0 \le t \le 2\pi$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} (12 - 7\sin t\cos t)\,dt = \left[12t - \frac{7\sin^2 t}{2}\right]_0^{2\pi} = 24\pi$$

**34.** $x = 2 - t,\ y = 2 - t,\ z = \sqrt{4t - t^2},\ 0 \le t \le 2$

$$d\mathbf{r} = \left[-\mathbf{i} - \mathbf{j} + \frac{2 - t}{\sqrt{4t - t^2}}\mathbf{k}\right] dt$$

$\mathbf{F} = \left(4 - 2t - \sqrt{4t - t^2}\right)\mathbf{i} + \left(\sqrt{4t - t^2} - 2 + t\right)\mathbf{j} + 0\mathbf{k}$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^2 (t - 2)\,dt = \left[\frac{t^2}{2} - 2t\right]_0^2 = -2$$

**36.** Let $x = 2\sin t,\ y = -2\cos t,\ z = 4\sin^2 t,\ 0 \le t \le \pi.$

$d\mathbf{r} = [(2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + (8\sin t\cos t)\mathbf{k}]\,dt$

$\mathbf{F} = 0\mathbf{i} + 4\mathbf{j} + (2\sin t)\mathbf{k}$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi} (8\sin t + 16\sin^2 t\cos t)\,dt = \left[-8\cos t + \frac{16}{3}\sin^3 t\right]_0^{\pi} = 16$$

**38.** $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (2x - y)\,dx + (2y - x)\,dy$

$\mathbf{r}(t) = (2\cos t + 2t\sin t)\mathbf{i} + (2\sin t - 2t\cos t)\mathbf{j},\ 0 \le t \le \pi$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = 4\pi^2 + 4\pi$$

**40.** $\mathbf{r}(t) = 10\sin t\mathbf{i} + 10\cos t\mathbf{j} + \dfrac{2000/5280}{\pi/2}t\mathbf{k}, \quad 0 \le t \le \dfrac{\pi}{2}$

$= 10\sin t\mathbf{i} + 10\cos t\mathbf{j} + \dfrac{25}{33\pi}t\mathbf{k}$

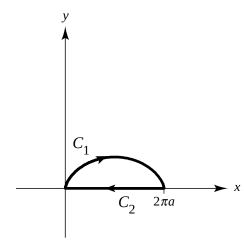
$\mathbf{F} = 20\mathbf{k}$

$d\mathbf{r} = \left(10\cos t\mathbf{i} - 10\sin t\mathbf{j} + \dfrac{25}{33\pi}\mathbf{k}\right)$

$\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/2} \frac{500}{33\pi}\,dt = \frac{250}{33}$ mi · ton

**42.** $\displaystyle\int_C y\,dx + x\,dy + \frac{1}{z}\,dz = \Big[xy + \ln|z|\Big]_{(0,0,1)}^{(4,4,4)} = 16 + \ln 4$

**44.** $x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta),\ 0 \le \theta \le 2\pi$




(a) $A = \dfrac{1}{2}\displaystyle\int_C x\,dy - y\,dx.$

Since these equations orient the curve backwards, we will use

$$A = \frac{1}{2}\int (y\,dx - x\,dy)$$

$$= \frac{1}{2}\int_0^{2\pi} \left[a^2(1-\cos\theta)(1-\cos\theta) - a^2(\theta - \sin\theta)(\sin\theta)\right]d\theta + \frac{1}{2}\int_0^{2\pi a}(0-0)\,d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi}\left[1 - 2\cos\theta + \cos^2\theta - \theta\sin\theta + \sin^2\theta\right]d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi}(2 - 2\cos\theta - \theta\sin\theta)\,d\theta = \frac{a^2}{2}(6\pi) = 3\pi a^2.$$

(b) By symmetry, $\bar{x} = \pi a$. From Section 14.4,

$$\bar{y} = -\frac{1}{2A}\int_C y^2\,dx = \frac{1}{2A}\int_0^{2\pi} a^3(1-\cos\theta)^2(1-\cos\theta)\,d\theta = \frac{1}{2(3\pi a^2)}a^3(5\pi) = \frac{5}{6}a$$

**46.** $\displaystyle\int_C xy\,dx + (x^2 + y^2)\,dy = \int_0^2\int_0^2 (2x - x)\,dy\,dx$

$\displaystyle= \int_0^2 2x\,dx = 4$

**48.** $\displaystyle\int_C (x^2 - y^2)\,dx + 2xy\,dy = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 4y\,dy\,dx$

$\displaystyle= \int_{-a}^{a} 0\,dx = 0$

**50.** $\displaystyle\int_C y^2\,dx + x^{4/3}\,dy = \int_{-1}^{1}\int_{-(1-x^{2/3})^{3/2}}^{(1-x^{2/3})^{3/2}} \left(\frac{4}{3}x^{1/3} - 2y\right)dy\,dx$

$$= \int_{-1}^{1}\left[\frac{4}{3}x^{1/3}y - y^2\right]_{-(1-x^{2/3})^{3/2}}^{(1-x^{2/3})^{3/2}} dx$$

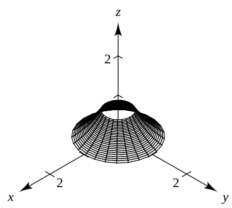
$$= \int_{-1}^{1}\frac{8}{3}x^{1/3}(1 - x^{2/3})^{3/2}\,dx$$

$$= \left[-\frac{8}{7}x^{2/3}(1 - x^{2/3})^{5/2} - \frac{16}{35}(1 - x^{2/3})^{5/2}\right]_{-1}^{1}$$

$$= 0$$

**52.** $\mathbf{r}(u, v) = e^{-u/4}\cos v\,\mathbf{i} + e^{-u/4}\sin v\,\mathbf{j} + \frac{u}{6}\mathbf{k}$

$0 \le u \le 4, \quad 0 \le v \le 2\pi$




**54.** $S$: $\mathbf{r}(u, v) = (u + v)\mathbf{i} + (u - v)\mathbf{j} + \sin v\,\mathbf{k}, \quad 0 \le u \le 2, 0 \le v \le \pi$

$\mathbf{r}_u(u, v) = \mathbf{i} + \mathbf{j}$

$\mathbf{r}_u(u, v) = \mathbf{i} - \mathbf{j} + \cos v\,\mathbf{k}$

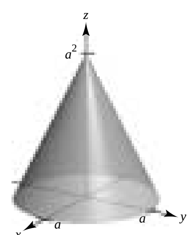
$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & \cos v \end{vmatrix} = \cos v\,\mathbf{i} - \cos v\,\mathbf{j} - 2\mathbf{k}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{2\cos^2 v + 4}$

$$\iint_S z\,dS = \int_0^{\pi}\int_0^2 \sin v\sqrt{2\cos^2 v + 4}\,du\,dv = 2\left[\sqrt{6} + \sqrt{2}\ln\left(\frac{\sqrt{6} + \sqrt{2}}{\sqrt{6} - \sqrt{2}}\right)\right]$$

**56.** (a) $z = a\left(a - \sqrt{x^2 + y^2}\right), 0 \le z \le a^2$

$z = 0 \Longrightarrow x^2 + y^2 = a^2$




(b) $S$: $g(x, y) = z = a^2 - a\sqrt{x^2 + y^2}$

$\rho(x, y) = k\sqrt{x^2 + y^2}$

$$m = \iint_S e(x, y, z)\,dS$$

$$= \iint_R k\sqrt{x^2 + y^2}\,\sqrt{1 + g_x^{\,2} + g_y^{\,2}}\,dA$$

$$= k\iint_R \sqrt{x^2 + y^2}\,\sqrt{1 + \frac{a^2x^2}{x^2 + y^2} + \frac{a^2y^2}{x^2 + y^2}}\,dA$$

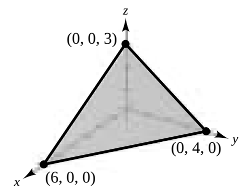
$$= k\iint_R \sqrt{a^2 + 1}\left(\sqrt{x^2 + y^2}\right)dA$$

$$= k\sqrt{a^2 + 1}\int_0^{2\pi}\int_0^a r^2\,dr\,d\theta$$

$$= k\sqrt{a^2 + 1}\int_0^{2\pi}\frac{a^3}{3}\,d\theta$$

$$= \frac{2}{3}k\sqrt{a^2 + 1}\,a^3\pi$$

**58.** $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$




$Q$: solid region bounded by the coordinate planes and the plane $2x + 3y + 4z = 12$

**Surface Integral**: There are four surfaces for this solid.

$$z = 0 \qquad \mathbf{N} = -\mathbf{k}, \qquad \mathbf{F}\cdot\mathbf{N} = -z, \qquad \int_{S_1}\!\!\int 0\, dS = 0$$

$$y = 0, \qquad \mathbf{N} = -\mathbf{j}, \qquad \mathbf{F}\cdot\mathbf{N} = -y, \qquad \int_{S_2}\!\!\int 0\, dS = 0$$

$$x = 0, \qquad \mathbf{N} = -\mathbf{i}, \qquad \mathbf{F}\cdot\mathbf{N} = -x, \qquad \int_{S_3}\!\!\int 0\, dS = 0$$

$$2x + 3y + 4z = 12, \mathbf{N} = \frac{2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}}{\sqrt{29}}, dS = \sqrt{1 + \left(\frac{1}{4}\right) + \left(\frac{9}{16}\right)}dA = \frac{\sqrt{29}}{4}\, dA$$

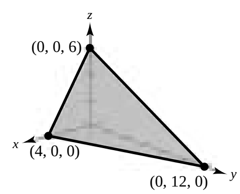
$$\int_{S_4}\!\!\int \mathbf{N}\cdot\mathbf{F}\, dS = \frac{1}{4}\int_R\!\!\int (2x + 3y + 4z)\, dy\, dx$$

$$= \frac{1}{4}\int_0^6\int_0^{(12-2x)/3} 12\, dy\, dx = 3\int_0^6 \left(4 - \frac{2x}{3}\right) dx = 3\left[4x - \frac{x^2}{3}\right]_0^6 = 36$$

**Triple Integral:** Since div $\mathbf{F} = 3$, the Divergence Theorem yields.

$$\iiint_Q \text{div}\,\mathbf{F}\, dV = \iiint_Q 3\, dV = 3(\text{Volume of solid}) = 3\left[\frac{1}{3}(\text{Area of base})(\text{Height})\right] = \frac{1}{2}(6)(4)(3) = 36.$$

**60.** $\mathbf{F}(x, y, z) = (x - z)\mathbf{i} + (y - z)\mathbf{j} + x^2\mathbf{k}$




$S$: first octant portion of the plane $3x + y + 2z = 12$

**Line Integral:**

$$C_1: y = 0, \qquad dy = 0, \qquad z = \frac{12 - 3x}{2}, \qquad dz = -\frac{3}{2}\, dx$$

$$C_2: x = 0, \qquad dx = 0, \qquad z = \frac{12 - y}{2}, \qquad dz = -\frac{1}{2}\, dy$$

$$C_3: z = 0, \qquad dz = 0, \qquad y = 12 - 3x, \qquad dy = -3\, dx$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (x - z)\, dx + (y - z)\, dy + x^2\, dz$$

$$= \int_{C_1}\left[x - \frac{12 - 3x}{2} + x^2\left(-\frac{3}{2}\right)\right] dx + \int_{C_2}\left[y - \frac{12 - y}{2}\right] dy + \int_{C_3}[x + (12 - 3x)(-3)]\, dx$$

$$= \int_4^0 \left(-\frac{3}{2}x^2 + \frac{5}{2}x - 6\right) dx + \int_0^{12}\left(\frac{3}{2}y - 6\right) dy + \int_0^4 (10x - 36)\, dx = 8$$

**Double Integral:** $G(x, y, z) = \dfrac{12 - 3x - y}{2} - z$

$$\nabla G(x, y, z) = -\frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} - \mathbf{k}$$

$$\textbf{curl F} = \mathbf{i} - (2x + 1)\mathbf{j}$$

$$\int_S\!\!\int (\textbf{curl F})\cdot\mathbf{N}\, dS = \int_0^4\int_0^{12-3x} (x - 1)\, dy\, dx = \int_0^4 (-3x^2 + 15x - 12)\, dx = 8$$