CHAPTER 14
Vector Analysis

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Section 14.1 Vector Fields

Solutions to Odd-Numbered Exercises

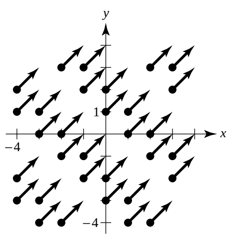
1. All vectors are parallel to y-axis.
Matches (c)

3. All vectors point outward.
Matches (b)

5. Vectors are parallel to x-axis for $y = n\pi$.
Matches (a)

7. $\mathbf{F}(x, y) = \mathbf{i} + \mathbf{j}$

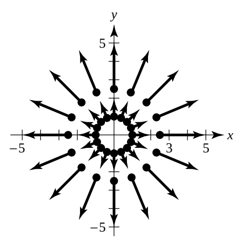
$\|\mathbf{F}\| = \sqrt{2}$



9. $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$

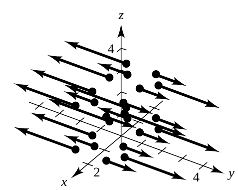
$\|\mathbf{F}\| = \sqrt{x^2 + y^2} = c$

$x^2 + y^2 = c^2$



11. $\mathbf{F}(x, y, z) = 3y\mathbf{j}$

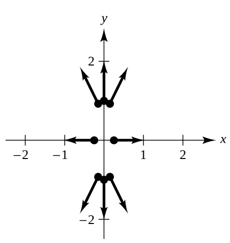
$\|\mathbf{F}\| = 3|y| = c$



13. $\mathbf{F}(x, y) = 4x\mathbf{i} + y\mathbf{j}$

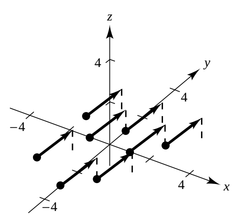
$\|\mathbf{F}\| = \sqrt{16x^2 + y^2} = c$

$\dfrac{x^2}{c^2/16} + \dfrac{y^2}{c^2} = 1$

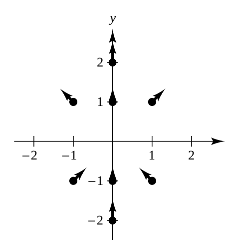


15. $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$

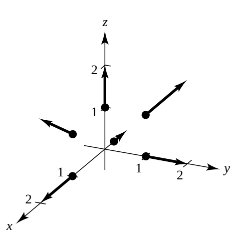
$\|\mathbf{F}\| = \sqrt{3}$



17.



19.



21. $f(x, y) = 5x^2 + 3xy + 10y^2$

$f_x(x, y) = 10x + 3y$

$f_y(x, y) = 3x + 20y$

$\mathbf{F}(x, y) = (10x + 3y)\mathbf{i} + (3x + 20y)\mathbf{j}$

23. $f(x, y, z) = z - ye^{x^2}$

$f_x(x, y, z) = -2xye^{x^2}$

$f_y(x, y, z) = -e^{x^2}$

$f_z = 1$

$\mathbf{F}(x, y, z) = -2xye^{x^2}\mathbf{i} - e^{x^2}\mathbf{j} + \mathbf{k}$

25. $g(x, y, z) = xy\ln(x + y)$

$$g_x(x, y, z) = y\ln(x + y) + \frac{xy}{x + y}$$
$$g_y(x, y, z) = x\ln(x + y) + \frac{xy}{x + y}$$
$$g_z(x, y, z) = 0$$
$$\mathbf{G}(x, y, z) = \left[\frac{xy}{x + y} + y\ln(x + y)\right]\mathbf{i} + \left[\frac{xy}{x + y} + x\ln(x + y)\right]\mathbf{j}$$

27. $\mathbf{F}(x, y) = 12xy\mathbf{i} + 6(x^2 + y)\mathbf{j}$

$M = 12xy$ and $N = 6(x^2 + y)$ have continuous first partial derivatives.

$$\frac{\partial N}{\partial x} = 12x = \frac{\partial M}{\partial y} \implies \mathbf{F} \text{ is conservative.}$$

29. $\mathbf{F}(x, y) = \sin y\,\mathbf{i} + x\cos y\,\mathbf{j}$

$M = \sin y$ and $N = x\cos y$ have continuous first partial derivatives.

$$\frac{\partial N}{\partial x} = \cos y = \frac{\partial M}{\partial y} \implies \mathbf{F} \text{ is conservative.}$$

31. $M = 15y^3, N = -5xy^2$

$$\frac{\partial N}{\partial x} = -5y^2 \neq \frac{\partial M}{\partial y} = 45y^2 \implies \text{Not conservative}$$

33. $M = \frac{2}{y}e^{2x/y}, N = \frac{-2x}{y^2}e^{2x/y}$

$$\frac{\partial N}{\partial x} = \frac{-2(y + 2x)}{y^3}e^{2x/y} = \frac{\partial M}{\partial y} \implies \text{Conservative}$$

35. $\mathbf{F}(x, y) = 2xy\mathbf{i} + x^2\mathbf{j}$

$$\frac{\partial}{\partial y}[2xy] = 2x$$
$$\frac{\partial}{\partial x}[x^2] = 2x$$

Conservative

$$f_x(x, y) = 2xy$$
$$f_y(x, y) = x^2$$
$$f(x, y) = x^2y + K$$

37. $\mathbf{F}(x, y) = xe^{x^2y}(2y\mathbf{i} + x\mathbf{j})$

$$\frac{\partial}{\partial y}[2xye^{x^2y}] = 2xe^{x^2y} + 2x^3ye^{x^2y}$$
$$\frac{\partial}{\partial x}[x^2e^{x^2y}] = 2xe^{x^2y} + 2x^3ye^{x^2y}$$

Conservative

$$f_x(x, y) = 2xye^{x^2y}$$
$$f_y(x, y) = x^2e^{x^2y}$$
$$f(x, y) = e^{x^2y} + K$$

39. $\mathbf{F}(x, y) = \frac{x}{x^2 + y^2}\mathbf{i} + \frac{y}{x^2 + y^2}\mathbf{j}$

$$\frac{\partial}{\partial y}\left[\frac{x}{x^2 + y^2}\right] = -\frac{2xy}{(x^2 + y^2)^2}$$
$$\frac{\partial}{\partial x}\left[\frac{y}{x^2 + y^2}\right] = -\frac{2xy}{(x^2 + y^2)^2}$$

Conservative

$$f_x(x, y) = \frac{x}{x^2 + y^2}$$
$$f_y(x, y) = \frac{y}{x^2 + y^2}$$
$$f(x, y) = \frac{1}{2}\ln(x^2 + y^2) + K$$

41. $\mathbf{F}(x, y) = e^x(\cos y\,\mathbf{i} + \sin y\,\mathbf{j})$

$$\frac{\partial}{\partial y}[e^x\cos y] = -e^x\sin y$$
$$\frac{\partial}{\partial x}[e^x\sin y] = e^x\sin y$$

Not conservative

43. $\mathbf{F}(x, y, z) = xyz\,\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \ (1, 2, 1)$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & y & z \end{vmatrix} = xy\mathbf{j} - xz\mathbf{k}$$

$$\textbf{curl F}\,(1, 2, 1) = 2\mathbf{j} - \mathbf{k}$$

45. $\mathbf{F}(x, y, z) = e^x\sin y\,\mathbf{i} - e^x\cos y\,\mathbf{j}, \ (0, 0, 3)$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x\sin y & -e^x\cos y & 0 \end{vmatrix} = -2e^x\cos y\mathbf{k}$$

$$\textbf{curl F}\,(0, 0, 3) = -2\mathbf{k}$$

47. $\mathbf{F}(x, y, z) = \arctan\left(\frac{x}{y}\right)\mathbf{i} + \ln\sqrt{x^2 + y^2}\,\mathbf{j} + \mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \arctan\left(\frac{x}{y}\right) & \frac{1}{2}\ln(x^2 + y^2) & 1 \end{vmatrix} = \left[\frac{x}{x^2 + y^2} - \frac{(-x/y^2)}{1 + (x/y)^2}\right]\mathbf{k} = \frac{2x}{x^2 + y^2}\mathbf{k}$$

49. $\mathbf{F}(x, y, z) = \sin(x - y)\mathbf{i} + \sin(y - z)\mathbf{j} + \sin(z - x)\mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin(x - y) & \sin(y - z) & \sin(z - x) \end{vmatrix} = \cos(y - z)\mathbf{i} + \cos(z - x)\mathbf{j} + \cos(x - y)\mathbf{k}$$

51. $\mathbf{F}(x, y, z) = \sin y\,\mathbf{i} - x\cos y\,\mathbf{j} + \mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y & -x\cos y & 1 \end{vmatrix} = -2\cos y\,\mathbf{k} \neq \mathbf{0}$$

Not conservative

53. $\mathbf{F}(x, y, z) = e^z(y\mathbf{i} + x\mathbf{j} + xy\mathbf{k})$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^z & xe^z & xye^z \end{vmatrix} = \mathbf{0}$$

Conservative

$$f_x(x, y, z) = ye^z$$

$$f_y(x, y, z) = xe^z$$

$$f_z(x, y, z) = xye^z$$

$$f(x, y, z) = xye^z + K$$

55. $\mathbf{F}(x, y, z) = \frac{1}{y}\mathbf{i} - \frac{x}{y^2}\mathbf{j} + (2z - 1)\mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{1}{y} & -\frac{x}{y^2} & 2z - 1 \end{vmatrix} = \mathbf{0}$$

Conservative

$$f_x(x, y, z) = \frac{1}{y}$$

$$f_y(x, y, z) = -\frac{x}{y^2}$$

$$f_z(x, y, z) = 2z - 1$$

$$f(x, y, z) = \int \frac{1}{y}\,dx = \frac{x}{y} + g(y, z) + K_1$$

$$f(x, y, z) = \int -\frac{x}{y^2}\,dy = \frac{x}{y} + h(x, z) + K_2$$

$$f(x, y, z) = \int (2z - 1)\,dz$$

$$= z^2 - z + p(x, y) + K_3$$

$$f(x, y, z) = \frac{x}{y} + z^2 - z + K$$

57. $\mathbf{F}(x, y) = 6x^2\mathbf{i} - xy^2\mathbf{j}$

$$\text{div}\,\mathbf{F}(x, y) = \frac{\partial}{\partial x}[6x^2] + \frac{\partial}{\partial y}[-xy^2]$$

$$= 12x - 2xy$$

59. $\mathbf{F}(x, y, z) = \sin x\,\mathbf{i} + \cos y\,\mathbf{j} + z^2\mathbf{k}$

$$\text{div}\,\mathbf{F}(x, y, z) = \frac{\partial}{\partial x}[\sin x] + \frac{\partial}{\partial y}[\cos y] + \frac{\partial}{\partial z}[z^2] = \cos x - \sin y + 2z$$

61. $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$\text{div}\,\mathbf{F}(x, y, z) = yz + 1 + 1 = yz + 2$

$\text{div}\,\mathbf{F}(1, 2, 1) = 4$

63. $\mathbf{F}(x, y, z) = e^x \sin y\mathbf{i} - e^x \cos y\mathbf{j}$

$\text{div}\,\mathbf{F}(x, y, z) = e^x \sin y + e^x \sin y$

$\text{div}\,\mathbf{F}(0, 0, 3) = 0$

65. See the definition, page 1008. Examples include velocity fields, gravitational fields and magnetic fields.

67. See the definition on page 1014.

69. $\mathbf{F}(x, y, z) = \mathbf{i} + 2x\mathbf{j} + 3y\mathbf{k}$

$\mathbf{G}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$

$$\mathbf{F} \times \mathbf{G}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 3y \\ x & -y & z \end{vmatrix} = (2xz + 3y^2)\mathbf{i} - (z - 3xy)\mathbf{j} + (-y - 2x^2)\mathbf{k}$$

$$\mathbf{curl}\,(\mathbf{F} \times \mathbf{G}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xz + 3y^2 & 3xy - z & -y - 2x^2 \end{vmatrix} = (-1 + 1)\mathbf{i} - (-4x - 2x)\mathbf{j} + (3y - 6y)\mathbf{k} = 6x\mathbf{j} - 3y\mathbf{k}$$

71. $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xyz & y & z \end{vmatrix} = xy\mathbf{j} - xz\mathbf{k}$$

$$\mathbf{curl}(\mathbf{curl}\,\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & xy & -xz \end{vmatrix} = z\mathbf{j} + y\mathbf{k}$$

73. $\mathbf{F}(x, y, z) = \mathbf{i} + 2x\mathbf{j} + 3y\mathbf{k}$

$\mathbf{G}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 3y \\ x & -y & z \end{vmatrix}$$

$$= (2xz + 3y^2)\mathbf{i} - (z - 3xy)\mathbf{j} + (-y - 2x^2)\mathbf{k}$$

$\text{div}(\mathbf{F} \times \mathbf{G}) = 2z + 3x$

75. $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xyz & y & z \end{vmatrix} = xy\mathbf{j} - xz\mathbf{k}$$

$\text{div}(\mathbf{curl}\,\mathbf{F}) = x - x = 0$

77. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ and $\mathbf{G} = Q\mathbf{i} + R\mathbf{j} + S\mathbf{k}$ where $M, N, P, Q, R,$ and S have continuous partial derivatives.

$\mathbf{F} + \mathbf{G} = (M + Q)\mathbf{i} + (N + R)\mathbf{j} + (P + S)\mathbf{k}$

$$\mathbf{curl}(\mathbf{F} + \mathbf{G}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M + Q & N + R & P + S \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(P + S) - \frac{\partial}{\partial z}(N + R)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(P + S) - \frac{\partial}{\partial z}(M + Q)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(N + R) - \frac{\partial}{\partial y}(M + Q)\right]\mathbf{k}$$

$$= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} + \left(\frac{\partial S}{\partial y} - \frac{\partial R}{\partial z}\right)\mathbf{i} - \left(\frac{\partial S}{\partial x} - \frac{\partial Q}{\partial z}\right)\mathbf{j} + \left(\frac{\partial R}{\partial x} - \frac{\partial Q}{\partial y}\right)\mathbf{k}$$

$$= \mathbf{curl}\,\mathbf{F} + \mathbf{curl}\,\mathbf{G}$$

79. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ and $\mathbf{G} = R\mathbf{i} + S\mathbf{j} + T\mathbf{k}$.

$$\begin{aligned}\operatorname{div}(\mathbf{F} + \mathbf{G}) = \frac{\partial}{\partial x}(M + R) + \frac{\partial}{\partial y}(N + S) + \frac{\partial}{\partial z}(P + T) &= \frac{\partial M}{\partial x} + \frac{\partial R}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial S}{\partial y} + \frac{\partial P}{\partial z} + \frac{\partial T}{\partial z} \\ &= \left[\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right] + \left[\frac{\partial R}{\partial x} + \frac{\partial S}{\partial y} + \frac{\partial T}{\partial z}\right] \\ &= \operatorname{div}\mathbf{F} + \operatorname{div}\mathbf{G}\end{aligned}$$

81. $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$

$$\begin{aligned}\nabla \times [\nabla f + (\nabla \times \mathbf{F})] &= \mathbf{curl}(\nabla f + (\nabla \times \mathbf{F})) \\ &= \mathbf{curl}(\nabla f) + \mathbf{curl}(\nabla \times \mathbf{F}) \qquad \text{(Exercise 77)} \\ &= \mathbf{curl}(\nabla \times \mathbf{F}) \qquad \text{(Exercise 78)} \\ &= \nabla \times (\nabla \times \mathbf{F})\end{aligned}$$

83. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$, then $f\mathbf{F} = fM\mathbf{i} + fN\mathbf{j} + fP\mathbf{k}$.

$$\begin{aligned}\operatorname{div}(f\mathbf{F}) = \frac{\partial}{\partial x}(fM) + \frac{\partial}{\partial y}(fN) + \frac{\partial}{\partial z}(fP) &= f\frac{\partial M}{\partial x} + M\frac{\partial f}{\partial x} + f\frac{\partial N}{\partial y} + N\frac{\partial f}{\partial y} + f\frac{\partial P}{\partial z} + P\frac{\partial f}{\partial z} \\ &= f\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial N}{\partial z}\right) + \left(\frac{\partial f}{\partial x}M + \frac{\partial f}{\partial y}N + \frac{\partial f}{\partial z}P\right) \\ &= f\operatorname{div}\mathbf{F} + \nabla f \cdot \mathbf{F}\end{aligned}$$

In Exercises 85 and 87, $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $f(x, y, z) = \|\mathbf{F}(x, y, z)\| = \sqrt{x^2 + y^2 + z^2}$.

85. $\ln f = \frac{1}{2}\ln(x^2 + y^2 + z^2)$

$$\nabla(\ln f) = \frac{x}{x^2 + y^2 + z^2}\mathbf{i} + \frac{y}{x^2 + y^2 + z^2}\mathbf{j} + \frac{z}{x^2 + y^2 + z^2}\mathbf{k} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{x^2 + y^2 + z^2} = \frac{\mathbf{F}}{f^2}$$

87. $f^n = \left(\sqrt{x^2 + y^2 + z^2}\right)^n$

$$\begin{aligned}\nabla f^n &= n\left(\sqrt{x^2 + y^2 + z^2}\right)^{n-1}\frac{x}{\sqrt{x^2 + y^2 + z^2}}\mathbf{i} + n\left(\sqrt{x^2 + y^2 + z^2}\right)^{n-1}\frac{y}{\sqrt{x^2 + y^2 + z^2}}\mathbf{j} \\ &\quad + n\left(\sqrt{x^2 + y^2 + z^2}\right)^{n-1}\frac{z}{\sqrt{x^2 + y^2 + z^2}}\mathbf{k} \\ &= n\left(\sqrt{x^2 + y^2 + z^2}\right)^{n-2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = nf^{n-2}\mathbf{F}\end{aligned}$$

89. The winds are stronger over Phoenix. Although the winds over both cities are northeasterly, they are more towards the east over Atlanta.

Section 14.2 Line Integrals

1.
$$x^2 + y^2 = 9$$
$$\frac{x^2}{9} + \frac{y^2}{9} = 1$$
$$\cos^2 t + \sin^2 t = 1$$
$$\cos^2 t = \frac{x^2}{9}$$
$$\sin^2 t = \frac{y^2}{9}$$
$$x = 3\cos t$$
$$y = 3\sin t$$
$$\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$$
$$0 \le t \le 2\pi$$

3. $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 3 \\ 3\mathbf{i} + (t-3)\mathbf{j}, & 3 \le t \le 6 \\ (9-t)\mathbf{i} + 3\mathbf{j}, & 6 \le t \le 9 \\ (12-t)\mathbf{j}, & 9 \le t \le 12 \end{cases}$

5. $\mathbf{r}(t) = \begin{cases} t\mathbf{i} + \sqrt{t}\,\mathbf{j}, & 0 \le t \le 1 \\ (2-t)\mathbf{i} + (2-t)\mathbf{j}, & 1 \le t \le 2 \end{cases}$

7. $\mathbf{r}(t) = 4t\mathbf{i} + 3t\mathbf{j},\ 0 \le t \le 2;\ \mathbf{r}'(t) = 4\mathbf{i} + 3\mathbf{j}$

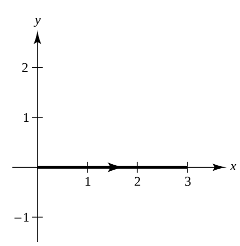
$$\int_C (x-y)\,ds = \int_0^2 (4t-3t)\sqrt{(4)^2 + (3)^2}\,dt = \int_0^2 5t\,dt = \left[\frac{5t^2}{2}\right]_0^2 = 10$$

9. $\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + 8t\mathbf{k},\ 0 \le t \le \frac{\pi}{2};\ \mathbf{r}'(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{j} + 8\mathbf{k}$

$$\int_C (x^2 + y^2 + z^2)\,ds = \int_0^{\pi/2} (\sin^2 t + \cos^2 t + 64t^2)\sqrt{(\cos t)^2 + (-\sin t)^2 + 64}\,dt$$
$$= \int_0^{\pi/2} \sqrt{65}(1 + 64t^2)\,dt = \left[\sqrt{65}\left(t + \frac{64t^3}{3}\right)\right]_0^{\pi/2} = \sqrt{65}\left(\frac{\pi}{2} + \frac{8\pi^3}{3}\right) = \frac{\sqrt{65}\pi}{6}(3 + 16\pi^2)$$

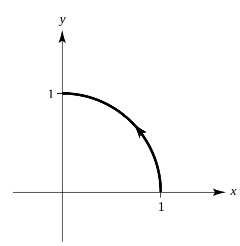
11. $\mathbf{r}(t) = t\mathbf{i},\ 0 \le t \le 3$

$$\int_C (x^2 + y^2)\,ds = \int_0^3 [t^2 + 0^2]\sqrt{1+0}\,dt$$
$$= \int_0^3 t^2\,dt$$
$$= \left[\frac{1}{3}t^3\right]_0^3 = 9$$



13. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j},\ 0 \le t \le \frac{\pi}{2}$

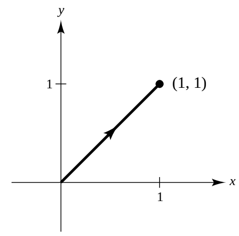
$$\int_C (x^2 + y^2)\,ds = \int_0^{\pi/2} [\cos^2 t + \sin^2 t]\sqrt{(-\sin t)^2 + (\cos t)^2}\,dt$$
$$= \int_0^{\pi/2} dt = \frac{\pi}{2}$$



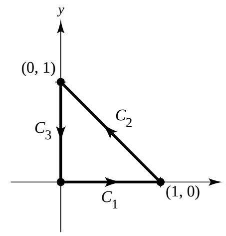
15. $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j},\ 0 \le t \le 1$

$$\int_C \left(x + 4\sqrt{y}\right) ds = \int_0^1 \left(t + 4\sqrt{t}\right)\sqrt{1+1}\, dt$$

$$= \left[\sqrt{2}\left(\frac{t^2}{2} + \frac{8}{3}t^{3/2}\right)\right]_0^1 = \frac{19\sqrt{2}}{6}$$



17. $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 1 \\ (2-t)\mathbf{i} + (t-1)\mathbf{j}, & 1 \le t \le 2 \\ (3-t)\mathbf{j}, & 2 \le t \le 3 \end{cases}$



$$\int_{C_1} \left(x + 4\sqrt{y}\right) ds = \int_0^1 t\, dt = \frac{1}{2}$$

$$\int_{C_2} \left(x + 4\sqrt{y}\right) ds = \int_1^2 \left[(2-t) + 4\sqrt{t-1}\right]\sqrt{1+1}\, dt$$

$$= \sqrt{2}\left[2t - \frac{t^2}{2} + \frac{8}{3}(t-1)^{3/2}\right]_1^2 = \frac{19\sqrt{2}}{6}$$

$$\int_{C_3} \left(x + 4\sqrt{y}\right) ds = \int_2^3 4\sqrt{3-t}\, dt = \left[-\frac{8}{3}(3-t)^{3/2}\right]_2^3 = \frac{8}{3}$$

$$\int_C \left(x + 4\sqrt{y}\right) ds = \frac{1}{2} + \frac{19\sqrt{2}}{6} + \frac{8}{3} = \frac{19 + 19\sqrt{2}}{6} = \frac{19\left(1+\sqrt{2}\right)}{6}$$

19. $\rho(x, y, z) = \frac{1}{2}(x^2 + y^2 + z^2)$

$$\mathbf{r}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + 2t\mathbf{k},\ 0 \le t \le 4\pi$$

$$\mathbf{r}'(t) = -3\sin t\mathbf{i} + 3\cos t\mathbf{j} + 2\mathbf{k}$$

$$\|\mathbf{r}'(t)\| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + (2)^2} = \sqrt{13}$$

$$\text{Mass} = \int_C \rho(x, y, z)\, ds = \int_0^{4\pi} \frac{1}{2}\left[(3\cos t)^2 + (3\sin t)^2 + (2t)^2\right]\sqrt{13}\, dt$$

$$= \frac{\sqrt{13}}{2}\int_0^{4\pi} (9 + 4t^2)\, dt = \left[\frac{\sqrt{13}}{2}\left(9t + \frac{4t^3}{3}\right)\right]_0^{4\pi}$$

$$= \frac{2\sqrt{13}\pi}{3}(27 + 64\pi^2) \approx 4973.8$$

21. $\mathbf{F}(x, y) = xy\mathbf{i} + y\mathbf{j}$

C: $\mathbf{r}(t) = 4t\mathbf{i} + t\mathbf{j},\ 0 \le t \le 1$

$\mathbf{F}(t) = 4t^2\mathbf{i} + t\mathbf{j}$

$\mathbf{r}'(t) = 4\mathbf{i} + \mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (16t^2 + t)\, dt$$

$$= \left[\frac{16}{3}t^3 + \frac{1}{2}t^2\right]_0^1 = \frac{35}{6}$$

23. $\mathbf{F}(x, y) = 3x\mathbf{i} + 4y\mathbf{i}$

C: $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j},\ 0 \le t \le \frac{\pi}{2}$

$\mathbf{F}(t) = 6\cos t\mathbf{i} + 8\sin t\mathbf{j}$

$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} (-12\sin t\cos t + 16\sin t\cos t)\, dt$$

$$= \left[2\sin^2 t\right]_0^{\pi/2} = 2$$

25. $\mathbf{F}(x, y, z) = x^2y\mathbf{i} + (x - z)\mathbf{j} + xyz\mathbf{k}$

C: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 2\mathbf{k},\ 0 \le t \le 1$

$\mathbf{F}(t) = t^4\mathbf{i} + (t - 2)\mathbf{j} + 2t^3\mathbf{k}$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left[t^4 + 2t(t - 2)\right] dt = \left[\frac{t^5}{5} + \frac{2t^3}{3} - 2t^2\right]_0^1 = -\frac{17}{15}$$

27. $\mathbf{F}(x, y, z) = x^2z\mathbf{i} + 6y\mathbf{j} + yz^2\mathbf{k}$

$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \ln t\mathbf{k},\ 1 \le t \le 3$

$\mathbf{F}(t) = t^2 \ln t\mathbf{i} + 6t^2\mathbf{j} + t^2 \ln^2 t\mathbf{k}$

$d\mathbf{r} = \left(\mathbf{i} + 2t\mathbf{j} + \frac{1}{t}\mathbf{k}\right) dt$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_1^3 \left[t^2 \ln t + 12t^3 + t(\ln t)^2\right] dt \approx 249.49$$

29. $\mathbf{F}(x, y) = -x\mathbf{i} - 2y\mathbf{j}$

C: $y = x^3$ from $(0, 0)$ to $(2, 8)$

$\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j},\ 0 \le t \le 2$

$\mathbf{r}'(t) = \mathbf{i} + 3t^2\mathbf{j}$

$\mathbf{F}(t) = -t\mathbf{i} - 2t^3\mathbf{j}$

$\mathbf{F} \cdot \mathbf{r}' = -t - 6t^5$

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 (-t - 6t^5)\, dt = \left[-\frac{1}{2}t^2 - t^6\right]_0^2 = -66$$

31. $\mathbf{F}(x, y) = 2x\mathbf{i} + y\mathbf{j}$

C: counterclockwise around the triangle whose vertices are $(0, 0)$, $(1, 0)$, $(1, 1)$

$$\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 1 \\ \mathbf{i} + (t - 1)\mathbf{j}, & 1 \le t \le 2 \\ (3 - t)\mathbf{i} + (3 - t)\mathbf{j}, & 2 \le t \le 3 \end{cases}$$

On C_1: $\mathbf{F}(t) = 2t\mathbf{i},\ \mathbf{r}'(t) = \mathbf{i}$

$$\text{Work} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 2t\, dt = 1$$

On C_2: $\mathbf{F}(t) = 2\mathbf{i} + (t - 1)\mathbf{j},\ \mathbf{r}'(t) = \mathbf{j}$

$$\text{Work} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_1^2 (t - 1)\, dt = \frac{1}{2}$$

On C_3: $\mathbf{F}(t) = 2(3 - t)\mathbf{i} + (3 - t)\mathbf{j},\ \mathbf{r}'(t) = -\mathbf{i} - \mathbf{j}$

$$\text{Work} = \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_2^3 \left[-2(3 - t) - (3 - t)\right] dt = -\frac{3}{2}$$

$$\text{Total work} = \int_C \mathbf{F} \cdot d\mathbf{r} = 1 + \frac{1}{2} - \frac{3}{2} = 0$$

33. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} - 5z\mathbf{k}$

C: $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k},\ 0 \le t \le 2\pi$

$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \mathbf{k}$

$\mathbf{F}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} - 5t\mathbf{k}$

$\mathbf{F} \cdot \mathbf{r}' = -5t$

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} -5t\, dt = -10\pi^2$$

35. $\mathbf{r}(t) = 3\sin t\mathbf{i} + 3\cos t\mathbf{j} + \frac{10}{2\pi}t\mathbf{k},\ 0 \le t \le 2\pi$

$\mathbf{F} = 150\mathbf{k}$

$d\mathbf{r} = \left(3\cos t\mathbf{i} - 3\sin t\mathbf{j} + \frac{10}{2\pi}\mathbf{k}\right) dt$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \frac{1500}{2\pi}\, dt = \left[\frac{1500}{2\pi}t\right]_0^{2\pi} = 1500 \text{ ft} \cdot \text{lb}$$

37. $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$

(a) $\mathbf{r}_1(t) = 2t\mathbf{i} + (t - 1)\mathbf{j},\ 1 \le t \le 3$

$\mathbf{r}_1'(t) = 2\mathbf{i} + \mathbf{j}$

$\mathbf{F}(t) = 4t^2\mathbf{i} + 2t(t - 1)\mathbf{j}$

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_1^3 (8t^2 + 2t(t - 1))\, dt = \frac{236}{3}$$

Both paths join (2, 0) and (6, 2). The integrals are negatives of each other because the orientations are different.

(b) $\mathbf{r}_2(t) = 2(3 - t)\mathbf{i} + (2 - t)\mathbf{j},\ 0 \le t \le 2$

$\mathbf{r}_2'(t) = -2\mathbf{i} - \mathbf{j}$

$\mathbf{F}(t) = 4(3 - t)^2\mathbf{i} + 2(3 - t)(2 - t)\mathbf{j}$

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^2 [-8(3 - t)^2 - 2(3 - t)(2 - t)]\, dt$$

$$= -\frac{236}{3}$$

39. $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

C: $\mathbf{r}(t) = t\mathbf{i} - 2t\mathbf{j}$

$\mathbf{r}'(t) = \mathbf{i} - 2\mathbf{j}$

$\mathbf{F}(t) = -2t\mathbf{i} - t\mathbf{j}$

$\mathbf{F} \cdot \mathbf{r}' = -2t + 2t = 0$

Thus, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0.$

41. $\mathbf{F}(x, y) = (x^3 - 2x^2)\mathbf{i} + \left(x - \frac{y}{2}\right)\mathbf{j}$

C: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{F}(t) = (t^3 - 2t^2)\mathbf{i} + \left(t - \frac{t^2}{2}\right)\mathbf{j}$

$\mathbf{F} \cdot \mathbf{r}' = (t^3 - 2t^2) + 2t\left(t - \frac{t^2}{2}\right) = 0$

Thus, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0.$

43. $x = 2t,\ y = 10t,\ 0 \le t \le 1 \implies y = 5x$ or $x = \frac{y}{5},\ 0 \le y \le 10$

$$\int_C (x + 3y^2)\, dy = \int_0^{10} \left(\frac{y}{5} + 3y^2\right) dy = \left[\frac{y^2}{10} + y^3\right]_0^{10} = 1010$$

45. $x = 2t,\ y = 10t,\ 0 \le t \le 1 \implies x = \frac{y}{5},\ 0 \le y \le 10,\ dx = \frac{1}{5}\, dy$

$$\int_C xy\, dx + y\, dy = \int_0^{10} \left(\frac{y^2}{25} + y\right) dy = \left[\frac{y^3}{75} + \frac{y^2}{2}\right]_0^{10} = \frac{190}{3} \quad \text{OR}$$

$y = 5x,\ dy = 5\, dx,\ \ 0 \le x \le 2$

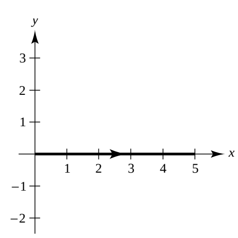
$$\int_C xy\, dx + y\, dy = \int_0^2 (5x^2 + 25x)\, dx = \left[\frac{5x^3}{3} + \frac{25x^2}{2}\right]_0^2 = \frac{190}{3}$$

47. $\mathbf{r}(t) = t\mathbf{i},\ 0 \le t \le 5$

$x(t) = t, \quad y(t) = 0$

$dx = dt, \quad dy = 0$

$$\int_C (2x - y)\, dx + (x + 3y)\, dy = \int_0^5 2t\, dt = 25$$



49. $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 3 \\ 3\mathbf{i} + (t-3)\mathbf{j}, & 3 \le t \le 6 \end{cases}$

C_1: $x(t) = t,\ y(t) = 0,$

$dx = dt,\ dy = 0$

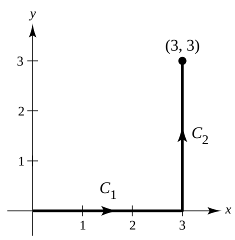
$$\int_{C_1} (2x - y)\,dx + (x + 3y)\,dy = \int_0^3 2t\,dt = 9$$

C_2: $x(t) = 3,\ y(t) = t - 3$

$dx = 0,\ dy = dt$

$$\int_{C_2} (2x - y)\,dx + (x + 3y)\,dy = \int_3^6 [3 + 3(t-3)]\,dt = \left[\frac{3t^2}{2} - 6t\right]_3^6 = \frac{45}{2}$$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = 9 + \frac{45}{2} = \frac{63}{2}$$



51. $x(t) = t,\ y(t) = 1 - t^2,\ 0 \le t \le 1,\ dx = dt,\ dy = -2t\,dt$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = \int_0^1 [(2t - 1 + t^2) + (t + 3 - 3t^2)(-2t)]\,dt$$

$$= \int_0^1 (6t^3 - t^2 - 4t - 1)\,dt = \left[\frac{3t^4}{2} - \frac{t^3}{3} - 2t^2 - t\right]_0^1 = -\frac{11}{6}$$

53. $x(t) = t,\ y(t) = 2t^2,\ 0 \le t \le 2$

$dx = dt,\ dy = 4t\,dt$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = \int_0^2 (2t - 2t^2)\,dt + (t + 6t^2)4t\,dt$$

$$= \int_0^2 (24t^3 + 2t^2 + 2t)\,dt = \left[6t^4 + \frac{2}{3}t^3 + t^2\right]_0^2 = \frac{316}{3}$$

55. $f(x, y) = h$

C: line from $(0, 0)$ to $(3, 4)$

$\mathbf{r} = 3t\mathbf{i} + 4t\mathbf{j},\ 0 \le t \le 1$

$\mathbf{r}'(t) = 3\mathbf{i} + 4\mathbf{j}$

$\|\mathbf{r}'(t)\| = 5$

Lateral surface area:

$$\int_C f(x, y)\,ds = \int_0^1 5h\,dt = 5h$$

57. $f(x, y) = xy$

C: $x^2 + y^2 = 1$ from $(1, 0)$ to $(0, 1)$

$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j},\ 0 \le t \le \frac{\pi}{2}$

$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$

$\|\mathbf{r}'(t)\| = 1$

Lateral surface area:

$$\int_C f(x, y)\,ds = \int_0^{\pi/2} \cos t \sin t\,dt$$

$$= \left[\frac{\sin^2 t}{2}\right]_0^{\pi/2} = \frac{1}{2}$$

59. $f(x, y) = h$

C: $y = 1 - x^2$ from $(1, 0)$ to $(0, 1)$

$$\mathbf{r}(t) = (1 - t)\mathbf{i} + [1 - (1 - t)^2]\mathbf{j}, \quad 0 \le t \le 1$$

$$\mathbf{r}'(t) = -\mathbf{i} + 2(1 - t)\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{1 + 4(1 - t)^2}$$

Lateral surface area:

$$\int_C f(x, y)\, ds = \int_0^1 h\sqrt{1 + 4(1 - t)^2}\, dt$$

$$= -\frac{h}{4}\left[2(1 - t)\sqrt{1 + 4(1 - t)^2} + \ln\left|2(1 - t) + \sqrt{1 + 4(1 - t)^2}\right|\right]_0^1$$

$$= \frac{h}{4}\left[2\sqrt{5} + \ln\left(2 + \sqrt{5}\right)\right] \approx 1.4789h$$

61. $f(x, y) = xy$

C: $y = 1 - x^2$ from $(1, 0)$ to $(0, 1)$

You could parameterize the curve C as in Exercises 59 and 60. Alternatively, let $x = \cos t$, then:

$$y = 1 - \cos^2 t = \sin^2 t$$

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin^2 t\,\mathbf{j}, \quad 0 \le t \le \frac{\pi}{2}$$

$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + 2\sin t\cos t\,\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + 4\sin^2 t\cos^2 t} = \sin t\sqrt{1 + 4\cos^2 t}$$

Lateral surface area:

$$\int_C f(x, y)\, ds = \int_0^{\pi/2} \cos t\sin^2 t\left(\sin t\sqrt{1 + 4\cos^2 t}\right) dt = \int_0^{\pi/2} \sin^2 t\left[(1 + 4\cos^2 t)^{1/2}\sin t\cos t\right] dt$$

Let $u = \sin^2 t$ and $dv = (1 + 4\cos^2 t)^{1/2}\sin t\cos t$, then $du = 2\sin t\cos t\, dt$ and $v = -\frac{1}{12}(1 + 4\cos^2 t)^{3/2}$.

$$\int_C f(x, y)\, ds = \left[-\frac{1}{12}\sin^2 t(1 + 4\cos^2 t)^{3/2}\right]_0^{\pi/2} + \frac{1}{6}\int_0^{\pi/2} (1 + 4\cos^2 t)^{3/2}\sin t\cos t\, dt$$

$$= \left[-\frac{1}{12}\sin^2 t(1 + 4\cos^2 t)^{3/2} - \frac{1}{120}(1 + 4\cos^2 t)^{5/2}\right]_0^{\pi/2}$$

$$= \left(-\frac{1}{12} - \frac{1}{120}\right) + \frac{1}{120}(5)^{5/2} = \frac{1}{120}\left(25\sqrt{5} - 11\right) \approx 0.3742$$

63. (a) $f(x, y) = 1 + y^2$

$$\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}, \quad 0 \le t \le 2\pi$$

$$\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$$

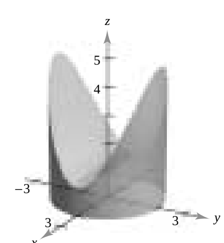
$$\|\mathbf{r}'(t)\| = 2$$

$$S = \int_C f(x, y)\, ds = \int_0^{2\pi} (1 + 4\sin^2 t)(2)\, dt$$

$$= \Big[2t + 4(t - \sin t\cos t)\Big]_0^{2\pi} = 12\pi \approx 37.70 \text{ cm}^2$$

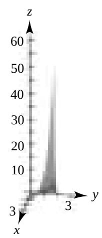
(b) $0.2(12\pi) = \dfrac{12\pi}{5} \approx 7.54 \text{ cm}^3$

(c)

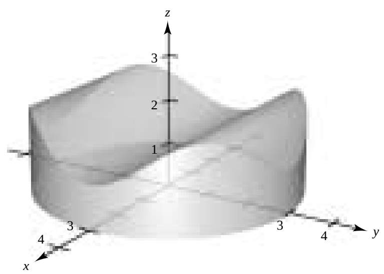


65. $S \approx 25$

Matches b



67. (a) Graph of: $\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j} + (1 + \sin^2 2t)\mathbf{k}\ \ 0 \le t \le 2\pi$



(b) Consider the portion of the surface in the first quadrant. The curve $z = 1 + \sin^2 2t$ is over the curve $\mathbf{r}_1(t) = 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j},\ 0 \le t \le \pi/2$. Hence, the total lateral surface area is

$$4\int_C f(x, y)\,ds = 4\int_0^{\pi/2} (1 + \sin^2 2t)3\,dt = 12\left(\frac{3\pi}{4}\right) = 9\pi \text{ sq. cm}$$

(c) The cross sections parallel to the xz-plane are rectangles of height $1 + 4(y/3)^2(1 - y^2/9)$ and base $2\sqrt{9 - y^2}$. Hence,

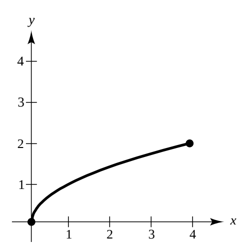
$$\text{Volume} = 2\int_0^3 2\sqrt{9 - y^2}\left(1 + 4\frac{y^2}{9}\left(1 - \frac{y^2}{9}\right)\right)dy \approx 42.412 \text{ cm}^3$$

69. See the definition of Line Integral, page 1020.

See Theorem 14.4.

71. The greater the height of the surface over the curve, the greater the lateral surface area. Hence,

$$z_3 < z_1 < z_2 < z_4.$$



73. False

$$\int_C xy\,ds = \sqrt{2}\int_0^1 t^2\,dt$$

75. False, the orientations are different.

Section 14.3 Conservative Vector Fields and Independence of Path

1. $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j},\ 0 \le t \le 1$

$\mathbf{r}_1'(t) = \mathbf{i} + 2t\,\mathbf{j}$

$\mathbf{F}(t) = t^2\mathbf{i} + t^3\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (t^2 + 2t^4)\,dt = \frac{11}{15}$$

(b) $\mathbf{r}_2(\theta) = \sin\theta\mathbf{i} + \sin^2\theta\mathbf{j},\ 0 \le \theta \le \frac{\pi}{2}$

$\mathbf{r}_2'(\theta) = \cos\theta\mathbf{i} + 2\sin\theta\cos\theta\mathbf{j}$

$\mathbf{F}(t) = \sin^2\theta\mathbf{i} + \sin^3\theta\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} (\sin^2\theta\cos\theta + 2\sin^4\theta\cos\theta)\,d\theta$$

$$= \left[\frac{\sin^3\theta}{3} + \frac{2\sin^5\theta}{5}\right]_0^{\pi/2} = \frac{11}{15}$$

3. $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

(a) $\mathbf{r}_1(\theta) = \sec\theta\mathbf{i} + \tan\theta\mathbf{j},\ 0 \le \theta \le \frac{\pi}{3}$

$\mathbf{r}_1'(\theta) = \sec\theta\tan\theta\mathbf{i} + \sec^2\theta\mathbf{j}$

$\mathbf{F}(\theta) = \tan\theta\mathbf{i} - \sec\theta\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/3} (\sec\theta\tan^2\theta - \sec^3\theta)\,d\theta = \int_0^{\pi/3} [\sec\theta(\sec^2\theta - 1) - \sec^3\theta]\,d\theta$$

$$= -\int_0^{\pi/3} \sec\theta\,d\theta = \Big[-\ln|\sec\theta + \tan\theta|\Big]_0^{\pi/3} = -\ln\left(2 + \sqrt{3}\right) \approx -1.317$$

(b) $\mathbf{r}_2(t) = \sqrt{t+1}\,\mathbf{i} + \sqrt{t}\,\mathbf{j},\quad 0 \le t \le 3$

$$\mathbf{r}_2'(t) = \frac{1}{2\sqrt{t+1}}\mathbf{i} + \frac{1}{2\sqrt{t}}\mathbf{j}$$

$\mathbf{F}(t) = \sqrt{t}\,\mathbf{i} - \sqrt{t+1}\,\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 \left[\frac{\sqrt{t}}{2\sqrt{t+1}} - \frac{\sqrt{t+1}}{2\sqrt{t}}\right] dt = -\frac{1}{2}\int_0^3 \frac{1}{\sqrt{t}\sqrt{t+1}}\,dt = -\frac{1}{2}\int_0^3 \frac{1}{\sqrt{t^2 + t + (1/4) - (1/4)}}\,dt$$

$$= -\frac{1}{2}\int_0^3 \frac{1}{\sqrt{[t + (1/2)]^2 - (1/4)}}\,dt = \left[-\frac{1}{2}\ln\left|\left(t + \frac{1}{2}\right) + \sqrt{t^2 + t}\right|\right]_0^3$$

$$= -\frac{1}{2}\left[\ln\left(\frac{7}{2} + 2\sqrt{3}\right) - \ln\left(\frac{1}{2}\right)\right] = -\frac{1}{2}\ln\left(7 + 4\sqrt{3}\right) \approx -1.317$$

5. $\mathbf{F}(x, y) = e^x\sin y\mathbf{i} + e^x\cos y\mathbf{j}$

$$\frac{\partial N}{\partial x} = e^x\cos y \qquad \frac{\partial M}{\partial y} = e^x\cos y$$

Since $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$, $\mathbf{F}$ is conservative.

7. $\mathbf{F}(x, y) = \dfrac{1}{y}\mathbf{i} + \dfrac{x}{y^2}\mathbf{j}$

$$\frac{\partial N}{\partial x} = \frac{1}{y^2} \qquad \frac{\partial M}{\partial y} = -\frac{1}{y^2}$$

Since $\dfrac{\partial N}{\partial x} \ne \dfrac{\partial M}{\partial y}$, $\mathbf{F}$ is not conservative.

9. $\mathbf{F}(x, y, z) = y^2z\mathbf{i} + 2xyz\mathbf{j} + xy^2\mathbf{k}$

$\textbf{curl}\,\mathbf{F} = \mathbf{0} \implies \mathbf{F}$ is conservative.

11. $\mathbf{F}(x, y) = 2xy\mathbf{i} + x^2\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j},\ 0 \le t \le 1$

$\mathbf{r}_1'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{F}(t) = 2t^3\mathbf{i} + t^2\mathbf{j}$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 4t^3\, dt = 1$

(b) $\mathbf{r}_2(t) = t\mathbf{i} + t^3\mathbf{j},\ 0 \le t \le 1$

$\mathbf{r}_2'(t) = \mathbf{i} + 3t^2\mathbf{j}$

$\mathbf{F}(t) = 2t^4\mathbf{i} + t^2\mathbf{j}$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 5t^4\, dt = 1$

13. $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + t\mathbf{j},\quad 0 \le t \le 1$

$\mathbf{r}_1'(t) = \mathbf{i} + \mathbf{j}$

$\mathbf{F}(t) = t\mathbf{i} - t\mathbf{j}$

$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$

(b) $\mathbf{r}_2(t) = t\mathbf{i} + t^2\mathbf{j},\quad 0 \le t \le 1$

$\mathbf{r}_2'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{F}(t) = t^2\mathbf{i} - t\mathbf{j}$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 -t^2\, dt = -\frac{1}{3}$

(c) $\mathbf{r}_3(t) = t\mathbf{i} + t^3\mathbf{j},\quad 0 \le t \le 1$

$\mathbf{r}_3'(t) = \mathbf{i} + 3t^2\mathbf{j}$

$\mathbf{F}(t) = t^3\mathbf{i} - t\mathbf{j}$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 -2t^3\, dt = -\frac{1}{2}$

15. $\int_C y^2\, dx + 2xy\, dy$

Since $\partial M/\partial y = \partial N/\partial x = 2y$, $\mathbf{F}(x, y) = y^2\mathbf{i} + 2xy\mathbf{j}$ is conservative. The potential function is $f(x, y) = xy^2 + k$. Therefore, we can use the Fundamental Theorem of Line Integrals.

(a) $\int_C y^2\, dx + 2xy\, dy = \Big[x^2y\Big]_{(0,\,0)}^{(4,\,4)} = 64$

(b) $\int_C y^2\, dx + 2xy\, dy = \Big[x^2y\Big]_{(-1,\,0)}^{(1,\,0)} = 0$

(c) and (d) Since C is a closed curve, $\int_C y^2\, dx + 2xy\, dy = 0$.

17. $\int_C 2xy\, dx + (x^2 + y^2)\, dy$

Since $\partial M/\partial y = \partial N/\partial x = 2x$,

$\mathbf{F}(x, y) = 2xy\mathbf{i} + (x^2 + y^2)\mathbf{j}$ is conservative.

The potential function is $f(x, y) = x^2y + \frac{y^3}{3} + k$.

(a) $\int_C 2xy\, dx + (x^2 + y^2)\, dy = \left[x^2y + \frac{y^3}{3}\right]_{(5,\,0)}^{(0,\,4)} = \frac{64}{3}$

(b) $\int_C 2xy\, dx + (x^2 + y^2)\, dy = \left[x^2y + \frac{y^3}{3}\right]_{(2,\,0)}^{(0,\,4)} = \frac{64}{3}$

19. $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

Since **curl F** $= \mathbf{0}$, $\mathbf{F}(x, y, z)$ is conservative. The potential function is $f(x, y, z) = xyz + k$.

(a) $\mathbf{r}_1(t) = t\mathbf{i} + 2\mathbf{j} + t\mathbf{k},\ 0 \le t \le 4$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \Big[xyz\Big]_{(0,\,2,\,0)}^{(4,\,2,\,4)} = 32$

(b) $\mathbf{r}_2(t) = t^2\mathbf{i} + t\mathbf{j} + t^2\mathbf{k},\ 0 \le t \le 2$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \Big[xyz\Big]_{(0,\,0,\,0)}^{(4,\,2,\,4)} = 32$

21. $\mathbf{F}(x, y, z) = (2y + x)\mathbf{i} + (x^2 - z)\mathbf{j} + (2y - 4z)\mathbf{k}$

$\mathbf{F}(x, y, z)$ is not conservative.

(a) $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j} + \mathbf{k},\ 0 \le t \le 1$

$\mathbf{r}_1'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{F}(t) = (2t^2 + t)\mathbf{i} + (t^2 - 1)\mathbf{j} + (2t^2 - 4)\mathbf{k}$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (2t^3 + 2t^2 - t)\, dt = \frac{2}{3}$

—CONTINUED—

21. —CONTINUED—

(b) $\mathbf{r}_2(t) = t\mathbf{i} + t\mathbf{j} + (2t-1)^2\mathbf{k},\ 0 \le t \le 1$

$\mathbf{r}_2'(t) = \mathbf{i} + \mathbf{j} + 4(2t-1)\mathbf{k}$

$\mathbf{F}(t) = 3t\mathbf{i} + [t^2 - (2t-1)^2]\mathbf{j} + [2t - 4(2t-1)^2]\mathbf{k}$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 [3t + t^2 - (2t-1)^2 + 8t(2t-1) - 16(2t-1)^3]\,dt$$

$$= \int_0^1 [17t^2 - 5t - (2t-1)^2 - 16(2t-1)^3]\,dt = \left[\frac{17t^3}{3} - \frac{5t^2}{2} - \frac{(2t-1)^3}{6} - 2(2t-1)^4\right]_0^1 = \frac{17}{6}$$

23. $\mathbf{F}(x, y, z) = e^z(y\mathbf{i} + x\mathbf{j} + xy\mathbf{k})$

$\mathbf{F}(x, y, z)$ is conservative. The potential function is $f(x, y, z) = xye^z + k$.

(a) $\mathbf{r}_1(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j} + 3\mathbf{k},\ 0 \le t \le \pi$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \Big[xye^z\Big]_{(4,0,3)}^{(-4,0,3)} = 0$$

(b) $\mathbf{r}_2(t) = (4-8t)\mathbf{i} + 3\mathbf{k},\ 0 \le t \le 1$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \Big[xye^z\Big]_{(4,0,3)}^{(-4,0,3)} = 0$$

25. $\displaystyle\int_C (y\mathbf{i} + x\mathbf{j})\cdot d\mathbf{r} = \Big[xy\Big]_{(0,0)}^{(3,8)} = 24$

27. $\displaystyle\int_C \cos x\sin y\,dx + \sin x\cos y\,dy = \Big[\sin x\sin y\Big]_{(0,-\pi)}^{(3\pi/2,\,\pi/2)} = -1$

29. $\displaystyle\int_C e^x\sin y\,dx + e^x\cos y\,dy = \Big[e^x\sin y\Big]_{(0,0)}^{(2\pi,0)} = 0$

31. $\displaystyle\int_C (y+2z)\,dx + (x-3z)\,dy + (2x-3y)\,dz$

$\mathbf{F}(x, y, z)$ is conservative and the potential function is $f(x, y, z) = xy - 3yz + 2xz$.

(a) $\Big[xy - 3yz + 2xz\Big]_{(0,0,0)}^{(1,1,1)} = 0 - 0 = 0$

(b) $\Big[xy - 3yz + 2xz\Big]_{(0,0,0)}^{(0,0,1)} + \Big[xy - 3yz + 2xz\Big]_{(0,0,1)}^{(1,1,1)} = 0 + 0 = 0$

(c) $\Big[xy - 3yz + 2xz\Big]_{(0,0,0)}^{(1,0,0)} + \Big[xy - 3yz + 2xz\Big]_{(1,0,0)}^{(1,1,0)} + \Big[xy - 3yz + 2xz\Big]_{(1,1,0)}^{(1,1,1)} = 0 + 1 + (-1) = 0$

33. $\displaystyle\int_C -\sin x\,dx + z\,dy + y\,dz = \Big[\cos x + yz\Big]_{(0,0,0)}^{(\pi/2,3,4)} = 12 - 1 = 11$

35. $\mathbf{F}(x, y) = 9x^2y^2\mathbf{i} + (6x^3y - 1)\mathbf{j}$ is conservative.

Work $= \Big[3x^3y^2 - y\Big]_{(0,0)}^{(5,9)} = 30{,}366$

37. $\mathbf{r}(t) = 2\cos 2\pi t\,\mathbf{i} + 2\sin 2\pi t\,\mathbf{j}$

$\mathbf{r}'(t) = -4\pi\sin 2\pi t\,\mathbf{i} + 4\pi\cos 2\pi t\,\mathbf{j}$

$\mathbf{a}(t) = -8\pi^2\cos 2\pi t\,\mathbf{i} - 8\pi^2\sin 2\pi t\,\mathbf{j}$

$\mathbf{F}(t) = m\cdot\mathbf{a}(t) = \frac{1}{32}\mathbf{a}(t) = -\frac{\pi^2}{4}(\cos 2\pi t\,\mathbf{i} + \sin 2\pi t\,\mathbf{j})$

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C -\frac{\pi^2}{4}(\cos 2\pi t\,\mathbf{i} + \sin 2\pi t\,\mathbf{j})\cdot 4\pi(-\sin 2\pi t\,\mathbf{i} + \cos 2\pi t\,\mathbf{j})\,dt = -\pi^3\int_C 0\,dt = 0$$

39. Since the sum of the potential and kinetic energies remains constant from point to point, if the kinetic energy is decreasing at a rate of 10 units per minute, then the potential energy is increasing at a rate of 10 units per minute.

41. No. The force field is conservative.

43. See Theorem 14.5, page 1033.

45. (a) The direct path along the line segment joining $(-4, 0)$ to $(3, 4)$ requires less work than the path going from $(-4, 0)$ to $(-4, 4)$ and then to $(3, 4)$.

(b) The closed curve given by the line segments joining $(-4, 0)$, $(-4, 4)$, $(3, 4)$, and $(-4, 0)$ satisfies $\int_C \mathbf{F}\cdot d\mathbf{r} \neq 0$.

47. False, it would be true if **F** were conservative.

49. True

51. Let

$$\mathbf{F} = M\mathbf{i} + N\mathbf{j} = \frac{\partial f}{\partial y}\mathbf{i} - \frac{\partial f}{\partial x}\mathbf{j}.$$

Then $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{\partial f}{\partial x}\right) = -\frac{\partial^2 f}{\partial x^2}$. Since

$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ we have $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

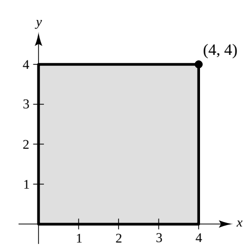
Thus, **F** is conservative. Therefore, by Theorem 14.7, we have

$$\int_C \left(\frac{\partial f}{\partial y}dx - \frac{\partial f}{\partial x}dy\right) = \int_C (M\,dx + N\,dy) = \int_C \mathbf{F}\cdot d\mathbf{r} = 0$$

for every closed curve in the plane.

Section 14.4 Green's Theorem

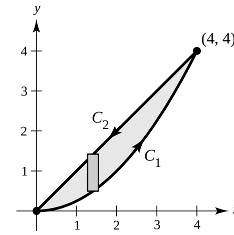
1. $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 4 \\ 4\mathbf{i} + (t-4)\mathbf{j}, & 4 \le t \le 8 \\ (12-t)\mathbf{i} + 4\mathbf{j}, & 8 \le t \le 12 \\ (16-t)\mathbf{j}, & 12 \le t \le 16 \end{cases}$



$$\int_C y^2\,dx + x^2\,dy = \int_0^4 [0\,dt + t^2(0)] + \int_4^8 [(t-4)^2(0) + 16\,dt]$$
$$+ \int_8^{12} [16(-dt) + (12-t)^2(0)] + \int_{12}^{16} [(16-t)^2(0) + 0(-dt)]$$
$$= 0 + 64 - 64 + 0 = 0$$

By Green's Theorem, $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_0^4\int_0^4 (2x - 2y)\,dy\,dx = \int_0^4 (8x - 16)\,dx = 0.$

3. $\mathbf{r}(t) = \begin{cases} t\mathbf{i} + t^2/4\mathbf{j}, & 0 \le t \le 4 \\ (8-t)\mathbf{i} + (8-t)\mathbf{j}, & 4 \le t \le 8 \end{cases}$



$$\int_C y^2\,dx + x^2\,dy = \int_0^4 \left[\frac{t^4}{16}(dt) + t^2\left(\frac{t}{2}\,dt\right)\right] + \int_4^8 \left[(8-t)^2(-dt) + (8-t)^2(-dt)\right]$$

$$= \int_0^4 \left[\frac{t^4}{16} + \frac{t^3}{2}\right] dt + \int_4^8 -2(8-t)^2\,dt = \frac{224}{5} - \frac{128}{3} = \frac{32}{15}$$

By Green's Theorem,

$$\int_R\!\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_0^4 \int_{x^2/4}^{x} (2x - 2y)\,dy\,dx = \int_0^4 \left(x^2 - \frac{x^3}{2} + \frac{x^4}{16}\right) dx = \frac{32}{15}.$$

5. C: $x^2 + y^2 = 4$

Let $x = 2\cos t$ and $y = 2\sin t$, $0 \le t \le 2\pi$.

$$\int_C xe^y\,dx + e^x\,dy = \int_0^{2\pi} \left[2\cos t e^{2\sin t}(-2\sin t) + e^{2\cos t}(2\cos t)\right] dt \approx 19.99$$

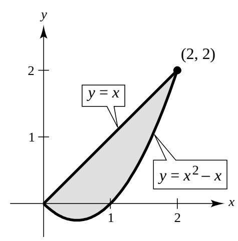
$$\int_R\!\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (e^x - xe^y)\,dy\,dx = \int_{-2}^{2} \left[2\sqrt{4-x^2}\,e^x - xe^{\sqrt{4-x^2}} + xe^{-\sqrt{4-x^2}}\right] dx \approx 19.99$$

In Exercises 7 and 9, $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = 1$.

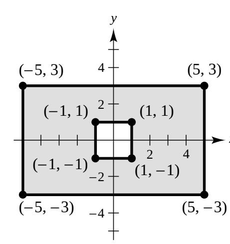
7. $$\int_C (y - x)\,dx + (2x - y)\,dy = \int_0^2 \int_{x^2-x}^{x} dy\,dx$$

$$= \int_0^2 (2x - x^2)\,dx$$

$$= \frac{4}{3}$$



9. From the accompanying figure, we see that R is the shaded region. Thus, Green's Theorem yields



$$\int_C (y - x)\,dx + (2x - y)\,dy = \int_R\!\int 1\,dA$$

$$= \text{Area of } R$$

$$= 6(10) - 2(2)$$

$$= 56.$$

11. Since the curves $y = 0$ and $y = 4 - x^2$ intersect at $(-2, 0)$ and $(2, 0)$, Green's Theorem yields

$$\int_C 2xy\,dx + (x + y)\,dy = \int_R\!\int (1 - 2x)\,dA = \int_{-2}^{2} \int_0^{4-x^2} (1 - 2x)\,dy\,dx$$

$$= \int_{-2}^{2} \Big[y - 2xy\Big]_0^{4-x^2} dx$$

$$= \int_{-2}^{2} (4 - 8x - x^2 + 2x^3)\,dx$$

$$= \left[4x - 4x^2 - \frac{x^3}{3} + \frac{x^4}{2}\right]_{-2}^{2}$$

$$= -\frac{8}{3} - \frac{8}{3} + 16 = \frac{32}{3}.$$

13. Since R is the interior of the circle $x^2 + y^2 = a^2$, Green's Theorem yields

$$\int_C (x^2 - y^2)\,dx + 2xy\,dy = \int_R\int (2y + 2y)\,dA$$

$$= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 4y\,dy\,dx = 4\int_{-a}^{a} 0\,dx = 0.$$

15. Since $\dfrac{\partial M}{\partial y} = \dfrac{2x}{x^2 + y^2} = \dfrac{\partial N}{\partial x}$,

we have path independence and

$$\int_R\int\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = 0.$$

17. By Green's Theorem,

$$\int_C \sin x \cos y\,dx + (xy + \cos x \sin y)\,dy = \int_R\int [(y - \sin x \sin y) - (-\sin x \sin y)]\,dA$$

$$= \int_0^1\int_x^{\sqrt{x}} y\,dy\,dx = \frac{1}{2}\int_0^1 (x - x^2)\,dx = \frac{1}{2}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{12}.$$

19. By Green's Theorem,

$$\int_C xy\,dx + (x + y)\,dy = \int_R\int (1 - x)\,dA$$

$$= \int_0^{2\pi}\int_1^3 (1 - r\cos\theta)r\,dr\,d\theta = \int_0^{2\pi}\left(4 - \frac{26}{3}\cos\theta\right) d\theta = 8\pi.$$

21. $\mathbf{F}(x, y) = xy\mathbf{i} + (x + y)\mathbf{j}$

C: $x^2 + y^2 = 4$

$$\text{Work} = \int_C xy\,dx + (x + y)\,dy = \int_R\int (1 - x)\,dA = \int_0^{2\pi}\int_0^2 (1 - r\cos\theta)r\,dr\,d\theta = \int_0^{2\pi}\left(2 - \frac{8}{3}\cos\theta\right) d\theta = 4\pi$$

23. $\mathbf{F}(x, y) = (x^{3/2} - 3y)\mathbf{i} + (6x + 5\sqrt{y})\mathbf{j}$

C: boundary of the triangle with vertices $(0, 0)$, $(5, 0)$, $(0, 5)$

$$\text{Work} = \int_C (x^{3/2} - 3y)\,dx + \left(6x + 5\sqrt{y}\right) dy = \int_R\int 9\,dA = 9\left(\tfrac{1}{2}\right)(5)(5) = \tfrac{225}{2}$$

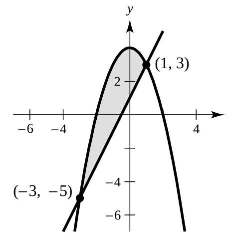
25. C: let $x = a\cos t$, $y = a\sin t$, $0 \le t \le 2\pi$. By Theorem 14.9, we have

$$A = \frac{1}{2}\int_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} [a\cos t(a\cos t) - a\sin t(-a\sin t)]\,dt = \frac{1}{2}\int_0^{2\pi} a^2\,dt = \left[\frac{a^2}{2}t\right]_0^{2\pi} = \pi a^2.$$

27. From the accompanying figure we see that

C_1: $y = 2x + 1$, $dy = 2\,dx$

C_2: $y = 4 - x^2$, $dy = -2x\,dx$.



Thus, by Theorem 14.9, we have

$$A = \frac{1}{2}\int_{-3}^{1}[x(2) - (2x + 1)]\,dx + \frac{1}{2}\int_{1}^{-3}[x(-2x) - (4 - x^2)]\,dx$$

$$= \frac{1}{2}\int_{-3}^{1}(-1)\,dx + \frac{1}{2}\int_{1}^{-3}(-x^2 - 4)\,dx$$

$$= \frac{1}{2}\int_{-3}^{1}(-1)\,dx + \frac{1}{2}\int_{-3}^{1}(x^2 + 4)\,dx = \frac{1}{2}\int_{-3}^{1}(3 + x^2)\,dx = \frac{1}{2}\left[3x + \frac{x^3}{3}\right]_{-3}^{1} = \frac{32}{3}.$$

29. See Theorem 14.8, page 1042.

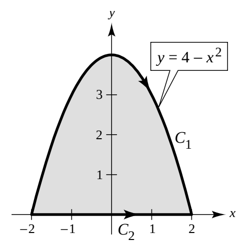
31. Answers will vary.

$\mathbf{F}_1(x, y) = y\mathbf{i} + x\mathbf{j}$

$\mathbf{F}_2(x, y) = x^2\mathbf{i} + y^2\mathbf{j}$

$\mathbf{F}_3(x, y) = 2xy\mathbf{i} + x^2\mathbf{j}$

33. $A = \displaystyle\int_{-2}^{2}(4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{32}{3}$



$$\bar{x} = \frac{1}{2A}\int_{C_1} x^2\,dy + \frac{1}{2A}\int_{C_2} x^2\,dy$$

For C_1, $dy = -2x\,dx$ and for C_2, $dy = 0$. Thus,

$$\bar{x} = \frac{1}{2(32/3)}\int_{2}^{-2} x^2(-2x\,dx) = \left[\frac{3}{64}\left(-\frac{x^4}{2}\right)\right]_{2}^{-2} = 0.$$

To calculate $\bar{y}$, note that $y = 0$ along C_2. Thus,

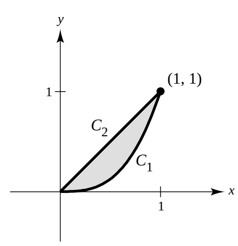
$$\bar{y} = \frac{-1}{2(32/3)}\int_{2}^{-2}(4 - x^2)^2\,dx = \frac{3}{64}\int_{-2}^{2}(16 - 8x^2 + x^4)\,dx = \frac{3}{64}\left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_{-2}^{2} = \frac{8}{5}.$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{8}{5}\right)$$

35. Since $A = \displaystyle\int_{0}^{1}(x - x^3)\,dx = \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1} = \frac{1}{4}$, we have $\dfrac{1}{2A} = 2$. On C_1 we have $y = x^3$, $dy = 3x^2\,dx$ and on C_2 we have $y = x$, $dy = dx$. Thus,

$$\bar{x} = 2\int_{C} x^2\,dy = 2\int_{C_1} x^2(3x^2\,dx) + 2\int_{C_2} x^2\,dx$$

$$= 6\int_{0}^{1} x^4\,dx + 2\int_{1}^{0} x^2\,dx = \frac{6}{5} - \frac{2}{3} = \frac{8}{15}$$



$$\bar{y} = -2\int_{C} y^2\,dx$$

$$= -2\int_{0}^{1} x^6\,dx - 2\int_{1}^{0} x^2\,dx = -\frac{2}{7} + \frac{2}{3} = \frac{8}{21}.$$

$$(\bar{x}, \bar{y}) = \left(\frac{8}{15}, \frac{8}{21}\right)$$

37. $A = \frac{1}{2}\int_0^{2\pi} a^2(1 - \cos\theta)^2\, d\theta$

$$= \frac{a^2}{2}\int_0^{2\pi}\left(1 - 2\cos\theta + \frac{1}{2} + \frac{\cos 2\theta}{2}\right)d\theta = \frac{a^2}{2}\left[\frac{3\theta}{2} - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{a^2}{2}(3\pi) = \frac{3\pi a^2}{2}$$

39. In this case the inner loop has domain $\frac{2\pi}{3} \le \theta \le \frac{4\pi}{3}$. Thus,

$$A = \frac{1}{2}\int_{2\pi/3}^{4\pi/3}(1 + 4\cos\theta + 4\cos^2\theta)\,d\theta$$
$$= \frac{1}{2}\int_{2\pi/3}^{4\pi/3}(3 + 4\cos\theta + 2\cos 2\theta)\,d\theta = \frac{1}{2}\Big[3\theta + 4\sin\theta + \sin 2\theta\Big]_{2\pi/3}^{4\pi/3} = \pi - \frac{3\sqrt{3}}{2}.$$

41. $I = \int_C \frac{y\,dx - x\,dy}{x^2 + y^2}$

(a) Let $\mathbf{F} = \frac{y}{x^2 + y^2}\mathbf{i} - \frac{x}{x^2 + y^2}\mathbf{j}$.

$\mathbf{F}$ is conservative since $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$.

$\mathbf{F}$ is defined and has continuous first partials everywhere except at the origin. If C is a circle (a closed path) that does not contain the origin, then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C M\,dx + N\,dy = \int_R\int\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA = 0.$$

(b) Let $\mathbf{r} = a\cos t\,\mathbf{i} - a\sin t\,\mathbf{j},\ 0 \le t \le 2\pi$ be a circle C_1 oriented clockwise inside C (see figure). Introduce line segments C_2 and C_3 as illustrated in Example 6 of this section in the text. For the region inside C and outside C_1, Green's Theorem applies. Note that since C_2 and C_3 have opposite orientations, the line integrals over them cancel. Thus, $C_4 = C_1 + C_2 + C + C_3$ and

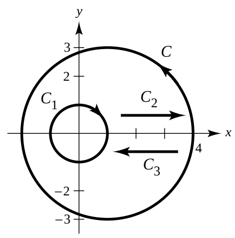
$$\int_{C_4}\mathbf{F}\cdot d\mathbf{r} = \int_{C_1}\mathbf{F}\cdot d\mathbf{r} + \int_C \mathbf{F}\cdot d\mathbf{r} = 0.$$

But,

$$\int_{C_1}\mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\left[\frac{(-a\sin t)(-a\sin t)}{a^2\cos^2 t + a^2\sin^2 t} + \frac{(-a\cos t)(-a\cos t)}{a^2\cos^2 t + a^2\sin^2 t}\right]dt$$
$$= \int_0^{2\pi}(\sin^2 t + \cos^2 t)\,dt = \Big[t\Big]_0^{2\pi} = 2\pi.$$

Finally, $\int_C \mathbf{F}\cdot d\mathbf{r} = -\int_{C_1}\mathbf{F}\cdot d\mathbf{r} = -2\pi$.

Note: If C were orientated clockwise, then the answer would have been 2π.



43. Pentagon: $(0, 0), (2, 0), (3, 2), (1, 4), (-1, 1)$

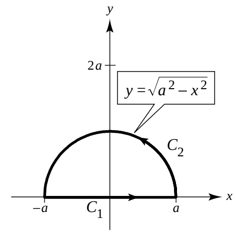
$A = \frac{1}{2}[(0 - 0) + (4 - 0) + (12 - 2) + (1 + 4) + (0 - 0)] = \frac{19}{2}$

45. $\int_C y^n\,dx + x^n\,dy = \int_R\int\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA$

For the line integral, use the two paths

C_1: $\mathbf{r}_1(x) = x\mathbf{i},\ -a \le x \le a$

C_2: $\mathbf{r}_2(x) = x\mathbf{i} + \sqrt{a^2 - x^2}\,\mathbf{j},\ \ x = a$ to $x = -a$



$$\int_{C_1} y^n\,dx + x^n\,dy = 0$$

$$\int_{C_2} y^n\,dx + x^n\,dy = \int_a^{-a}\left[(a^2 - x^2)^{n/2} + x^n\frac{-x}{\sqrt{a^2 - x^2}}\right]dx$$

$$\int_R\int\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA = \int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}}\left[nx^{n-1} - ny^{n-1}\right]dy\,dx$$

(a) For $n = 1, 3, 5, 7$, both integrals give 0.

(b) For n even, you obtain

$n = 2 : -\frac{4}{3}a^3$ $\quad n = 4 : -\frac{16}{15}a^5$ $\quad n = 6 : -\frac{32}{35}a^7$ $\quad n = 8 : -\frac{256}{315}a^9$

(c) If n is odd and $0 < a < 1$, then the integral equals 0.

47. $$\int_C (fD_{\mathbf{N}}g - gD_{\mathbf{N}}f)\,ds = \int_C fD_{\mathbf{N}}g\,ds - \int_C gD_{\mathbf{N}}f\,ds$$

$$= \int_R\int (f\nabla^2 g + \nabla f\cdot\nabla g)\,dA - \int_R\int(g\nabla^2 f + \nabla g\cdot\nabla f)\,dA = \int_R\int (f\nabla^2 g - g\nabla^2 f)\,dA$$

49. $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = 0 \implies \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0.$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C M\,dx + N\,dy = \int_R\int\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA = \int_R\int (0)\,dA = 0$$

Section 14.5 Parametric Surfaces

1. $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + uv\mathbf{k}$

$z = xy$

Matches c.

3. $\mathbf{r}(u, v) = 2\cos v\cos u\mathbf{i} + 2\cos v\sin u\mathbf{j} + 2\sin v\mathbf{k}$

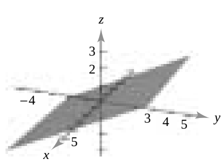
$x^2 + y^2 + z^2 = 4$

Matches b.

5. $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \frac{v}{2}\mathbf{k}$

$y - 2z = 0$

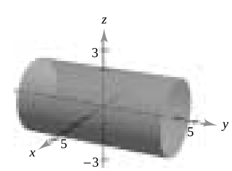
Plane



7. $\mathbf{r}(u, v) = 2\cos u\mathbf{i} + v\mathbf{j} + 2\sin u\mathbf{k}$

$x^2 + z^2 = 4$

Cylinder

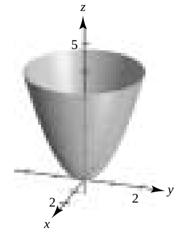


For Exercises 9 and 11,

$\mathbf{r}(u, v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + u^2\mathbf{k},\ 0 \le u \le 2,\ 0 \le v \le 2\pi.$

Eliminating the parameter yields

$z = x^2 + y^2,\ 0 \le z \le 4.$



9. $\mathbf{s}(u, v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} - u^2\mathbf{k},\ 0 \le u \le 2,\ 0 \le v \le 2\pi$

$z = -(x^2 + y^2)$

The paraboloid is reflected (inverted) through the xy-plane.

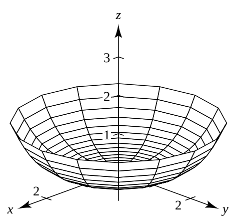
11. $\mathbf{s}(u, v) = u\cos v\mathbf{i} - u\sin v\mathbf{j} + u^2\mathbf{k},\ 0 \le u \le 3,\ 0 \le v \le 2\pi$

The height of the paraboloid is increased from 4 to 9.

13. $\mathbf{r}(u, v) = 2u\cos v\mathbf{i} + 2u\sin v\mathbf{j} + u^4\mathbf{k},$

$0 \le u \le 1,\ 0 \le v \le 2\pi$

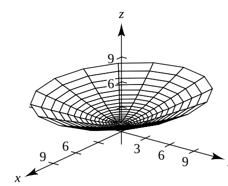
$z = \dfrac{(x^2 + y^2)^2}{16}$



15. $\mathbf{r}(u, v) = 2\sinh u\cos v\mathbf{i} + \sinh u\sin v\mathbf{j} + \cosh u\mathbf{k},$

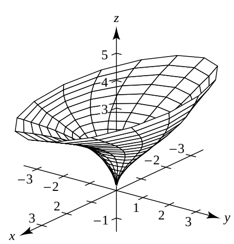
$0 \le u \le 2,\ 0 \le v \le 2\pi$

$\dfrac{z^2}{1} - \dfrac{x^2}{4} - \dfrac{y^2}{1} = 1$



17. $\mathbf{r}(u, v) = (u - \sin u)\cos v\mathbf{i} + (1 - \cos u)\sin v\mathbf{j} + u\mathbf{k},$

$0 \le u \le \pi,\ 0 \le v \le 2\pi$



19. $z = y$

$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + v\mathbf{k}$

21. $x^2 + y^2 = 16$

$\mathbf{r}(u, v) = 4\cos u\mathbf{i} + 4\sin u\mathbf{j} + v\mathbf{k}$

23. $z = x^2$

$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + u^2\mathbf{k}$

25. $z = 4$ inside $x^2 + y^2 = 9$.

$\mathbf{r}(u, v) = v\cos u\mathbf{i} + v\sin u\mathbf{j} + 4\mathbf{k},\ 0 \le v \le 3$

27. Function: $y = \dfrac{x}{2},\ 0 \le x \le 6$

Axis of revolution: x-axis

$x = u,\ y = \dfrac{u}{2}\cos v,\ z = \dfrac{u}{2}\sin v$

$0 \le u \le 6,\ 0 \le v \le 2\pi$

29. Function: $x = \sin z,\ 0 \le z \le \pi$

Axis of revolution: z-axis

$x = \sin u\cos v,\ y = \sin u\sin v,\ z = u$

$0 \le u \le \pi,\ 0 \le v \le 2\pi$

31. $\mathbf{r}(u, v) = (u + v)\mathbf{i} + (u - v)\mathbf{j} + v\mathbf{k},\ (1, -1, 1)$

$\mathbf{r}_u(u, v) = \mathbf{i} + \mathbf{j},\ \mathbf{r}_v(u, v) = \mathbf{i} - \mathbf{j} + \mathbf{k}$

At $(1, -1, 1)$, $u = 0$ and $v = 1$.

$\mathbf{r}_u(0, 1) = \mathbf{i} + \mathbf{j},\ \mathbf{r}_v(0, 1) = \mathbf{i} - \mathbf{j} + \mathbf{k}$

$$\mathbf{N} = \mathbf{r}_u(0, 1) \times \mathbf{r}_v(0, 1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j} - 2\mathbf{k}$$

Tangent plane: $(x - 1) - (y + 1) - 2(z - 1) = 0$

$$x - y - 2z = 0$$

(The original plane!)

33. $\mathbf{r}(u, v) = 2u \cos v\mathbf{i} + 3u \sin v\mathbf{j} + u^2\mathbf{k},\ (0, 6, 4)$

$\mathbf{r}_u(u, v) = 2 \cos v\mathbf{i} + 3 \sin v\mathbf{j} + 2u\mathbf{k}$

$\mathbf{r}_v(u, v) = -2u \sin v\mathbf{i} + 3u \cos v\mathbf{j}$

At $(0, 6, 4)$, $u = 2$ and $v = \pi/2$.

$$\mathbf{r}_u\left(2, \frac{\pi}{2}\right) = 3\mathbf{j} + 4\mathbf{k},\ \mathbf{r}_v\left(2, \frac{\pi}{2}\right) = -4\mathbf{i}$$

$$\mathbf{N} = \mathbf{r}_u\left(2, \frac{\pi}{2}\right) \times \mathbf{r}_v\left(2, \frac{\pi}{2}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & 4 \\ -4 & 0 & 0 \end{vmatrix} = -16\mathbf{j} + 12\mathbf{k}$$

Direction numbers: $0, 4, -3$

Tangent plane: $4(y - 6) - 3(z - 4) = 0$

$$4y - 3z = 12$$

35. $\mathbf{r}(u, v) = 2u\mathbf{i} - \dfrac{v}{2}\mathbf{j} + \dfrac{v}{2}\mathbf{k},\ 0 \le u \le 2,\ 0 \le v \le 1$

$\mathbf{r}_u(u, v) = 2\mathbf{i},\ \mathbf{r}_v(u, v) = -\dfrac{1}{2}\mathbf{j} + \dfrac{1}{2}\mathbf{k}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\mathbf{j} - \mathbf{k}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{2}$

$$A = \int_0^1 \int_0^2 \sqrt{2}\, du\, dv = 2\sqrt{2}$$

37. $\mathbf{r}(u, v) = a \cos u\mathbf{i} + a \sin u\mathbf{j} + v\mathbf{k},\ 0 \le u \le 2\pi,\ 0 \le v \le b$

$\mathbf{r}_u(u, v) = -a \sin u\mathbf{i} + a \cos u\mathbf{j}$

$\mathbf{r}_v(u, v) = \mathbf{k}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin u & a \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = a \cos u\mathbf{i} + a \sin u\mathbf{j}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = a$

$$A = \int_0^b \int_0^{2\pi} a\, du\, dv = 2\pi ab$$

39. $\mathbf{r}(u, v) = au \cos v\mathbf{i} + au \sin v\mathbf{j} + u\mathbf{k},\ 0 \le u \le b,\ 0 \le v \le 2\pi$

$\mathbf{r}_u(u, v) = a \cos v\mathbf{i} + a \sin v\mathbf{j} + \mathbf{k}$

$\mathbf{r}_v(u, v) = -au \sin v\mathbf{i} + au \cos v\mathbf{j}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a \cos v & a \sin v & 1 \\ -au \sin v & au \cos v & 0 \end{vmatrix} = -au \cos v\mathbf{i} - au \sin v\mathbf{j} + a^2u\mathbf{k}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = au\sqrt{1 + a^2}$

$$A = \int_0^{2\pi} \int_0^b a\sqrt{1 + a^2}\, u\, du\, dv = \pi ab^2\sqrt{1 + a^2}$$

41. $\mathbf{r}(u, v) = \sqrt{u}\cos v\,\mathbf{i} + \sqrt{u}\sin v\,\mathbf{j} + u\mathbf{k},\ 0 \le u \le 4,\ 0 \le v \le 2\pi$

$$\mathbf{r}_u(u, v) = \frac{\cos v}{2\sqrt{u}}\mathbf{i} + \frac{\sin v}{2\sqrt{u}}\mathbf{j} + \mathbf{k}$$

$$\mathbf{r}_v(u, v) = -\sqrt{u}\sin v\,\mathbf{i} + \sqrt{u}\cos v\,\mathbf{j}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\cos v}{2\sqrt{u}} & \dfrac{\sin v}{2\sqrt{u}} & 1 \\ -\sqrt{u}\sin v & \sqrt{u}\cos v & 0 \end{vmatrix} = -\sqrt{u}\cos v\,\mathbf{i} - \sqrt{u}\sin v\,\mathbf{j} + \frac{1}{2}\mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{u + \frac{1}{4}}$$

$$A = \int_0^{2\pi}\int_0^4 \sqrt{u + \frac{1}{4}}\,du\,dv = \frac{\pi}{6}\left(17\sqrt{17} - 1\right) \approx 36.177$$

43. See the definition, page 1051.

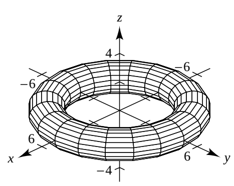
45. (a) From $(-10, 10, 0)$

(b) From $(10, 10, 10)$

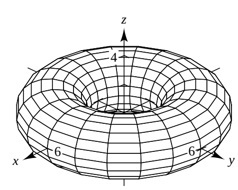
(c) From $(0, 10, 0)$

(d) From $(10, 0, 0)$

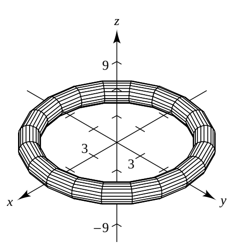
47. (a) $\mathbf{r}(u, v) = (4 + \cos v)\cos u\,\mathbf{i} + (4 + \cos v)\sin u\,\mathbf{j} + \sin v\,\mathbf{k},$
$0 \le u \le 2\pi,\ 0 \le v \le 2\pi$



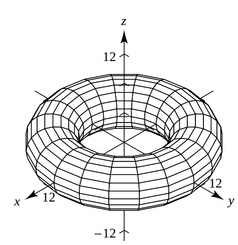
(b) $\mathbf{r}(u, v) = (4 + 2\cos v)\cos u\,\mathbf{i} + (4 + 2\cos v)\sin u\,\mathbf{j} + 2\sin v\,\mathbf{k},$
$0 \le u \le 2\pi, 0 \le v \le 2\pi$



(c) $\mathbf{r}(u, v) = (8 + \cos v)\cos u\,\mathbf{i} + (8 + \cos v)\sin u\,\mathbf{j} + \sin v\,\mathbf{k},$
$0 \le u \le 2\pi,\ 0 \le v \le 2\pi$



(d) $\mathbf{r}(u, v) = (8 + 3\cos v)\cos u\,\mathbf{i} + (8 + 3\cos v)\sin u\,\mathbf{j} + 3\sin v\,\mathbf{k},$
$0 \le u \le 2\pi, 0 \le v \le 2\pi$



The radius of the generating circle that is revolved about the z-axis is b, and its center is a units from the axis of revolution.

49. $\mathbf{r}(u, v) = 20 \sin u \cos v\mathbf{i} + 20 \sin u \sin v\mathbf{j} + 20 \cos u\mathbf{k}$ $0 \le u \le \pi/3,\ 0 \le v \le 2\pi$

$\mathbf{r}_u = 20 \cos u \cos v\mathbf{i} + 20 \cos u \sin v\mathbf{j} - 20 \sin u\mathbf{k}$

$\mathbf{r}_v = -20 \sin u \sin v\mathbf{i} + 20 \sin u \cos v\mathbf{j}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 20 \cos u \cos v & 20 \cos u \sin v & -20 \sin u \\ -20 \sin u \sin v & 20 \sin u \cos v & 0 \end{vmatrix}$$

$$= 400 \sin^2 u \cos v\mathbf{i} + 400 \sin^2 u \sin v\mathbf{j} + 400(\cos u \sin u \cos^2 v + \cos u \sin u \sin^2 v)\mathbf{k}$$

$$= 400[\sin^2 u \cos v\mathbf{i} + \sin^2 u \sin v\mathbf{j} + \cos u \sin u\mathbf{k}]$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = 400\sqrt{\sin^4 u \cos^2 v + \sin^4 u \sin^2 v + \cos^2 u \sin^2 u}$$

$$= 400\sqrt{\sin^4 u + \cos^2 u \sin^2 u}$$

$$= 400\sqrt{\sin^2 u} = 400 \sin u$$

$$S = \iint_S dS = \int_0^{2\pi}\int_0^{\pi/3} 400 \sin u\, du\, dv = \int_0^{2\pi} \Big[-400 \cos u\Big]_0^{\pi/3} dv$$

$$= \int_0^{2\pi} 200\, dv = 400\pi \text{ m}^2$$

51. $\mathbf{r}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + 2v\mathbf{k},\ 0 \le u \le 3,\ 0 \le v \le 2\pi$

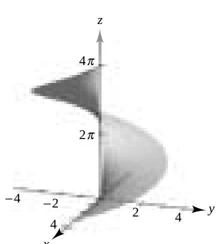
$\mathbf{r}_u(u, v) = \cos v\mathbf{i} + \sin v\mathbf{j}$

$\mathbf{r}_v(u, v) = -u \sin v\mathbf{i} + u \cos v\mathbf{j} + 2\mathbf{k}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 2 \end{vmatrix} = 2 \sin v\mathbf{i} - 2 \cos v\mathbf{j} + u\mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{4 + u^2}$$

$$A = \int_0^{2\pi}\int_0^3 \sqrt{4 + u^2}\, du\, dv = \pi\left[3\sqrt{13} + 4 \ln\left(\frac{3 + \sqrt{13}}{2}\right)\right]$$



53. Essay

Section 14.6 Surface Integrals

1. S: $z = 4 - x,\ 0 \le x \le 4,\ 0 \le y \le 4,\ \dfrac{\partial z}{\partial x} = -1,\ \dfrac{\partial z}{\partial y} = 0$

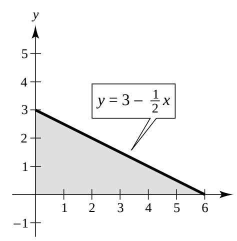
$$\iint_S (x - 2y + z)\, dS = \int_0^4\int_0^4 (x - 2y + 4 - x)\sqrt{1 + (-1)^2 + (0)^2}\, dy\, dx$$

$$= \sqrt{2}\int_0^4\int_0^4 (4 - 2y)\, dy\, dx = 0$$

3. S: $z = 10$, $x^2 + y^2 \le 1$, $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial y} = 0$

$$\int\!\!\int_S (x - 2y + z)\, dS = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x - 2y + 10)\sqrt{1 + (0)^2 + (0)^2}\, dy\, dx$$
$$= \int_0^{2\pi}\int_0^1 (r\cos\theta - 2r\sin\theta + 10)r\, dr\, d\theta$$
$$= \int_0^{2\pi}\left(\frac{1}{3}\cos\theta - \frac{2}{3}\sin\theta + 5\right) d\theta$$
$$= \left[\frac{1}{3}\sin\theta + \frac{2}{3}\cos\theta + 5\theta\right]_0^{2\pi} = 10\pi$$

5. S: $z = 6 - x - 2y$, (first octant) $\dfrac{\partial z}{\partial x} = -1$, $\dfrac{\partial z}{\partial y} = -2$



$$\int\!\!\int_S xy\, dS = \int_0^6\int_0^{3-(x/2)} xy\sqrt{1 + (-1)^2 + (-2)^2}\, dy\, dx$$
$$= \sqrt{6}\int_0^6 \left[\frac{xy^2}{2}\right]_0^{3-(x/2)} dx$$
$$= \frac{\sqrt{6}}{2}\int_0^6 x\left(9 - 3x + \frac{1}{4}x^2\right) dx$$
$$= \frac{\sqrt{6}}{2}\left[\frac{9x^2}{2} - x^3 + \frac{x^4}{16}\right]_0^6 = \frac{27\sqrt{6}}{2}$$

7. S: $z = 9 - x^2$, $0 \le x \le 2$, $0 \le y \le x$,

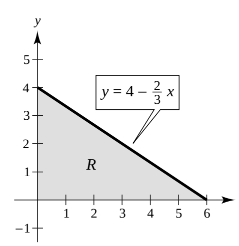
$$\frac{\partial z}{\partial x} = -2x,\ \frac{\partial z}{\partial y} = 0$$
$$\int\!\!\int_S xy\, dS = \int_0^2\int_y^2 xy\sqrt{1 + 4x^2}\, dx\, dy = \frac{391\sqrt{17} + 1}{240}$$

9. S: $z = 10 - x^2 - y^2$, $0 \le x \le 2$, $0 \le y \le 2$

$$\int\!\!\int_S (x^2 - 2xy)\, dS = \int_0^2\int_0^2 (x^2 - 2xy)\sqrt{1 + 4x^2 + 4y^2}\, dy\, dx \approx -11.47$$

11. S: $2x + 3y + 6z = 12$ (first octant) $\Rightarrow z = 2 - \dfrac{1}{3}x - \dfrac{1}{2}y$

$\rho(x, y, z) = x^2 + y^2$



$$m = \int\!\!\int_R (x^2 + y^2)\sqrt{1 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{2}\right)^2}\, dA$$
$$= \frac{7}{6}\int_0^6\int_0^{4-(2x/3)} (x^2 + y^2)\, dy\, dx$$
$$= \frac{7}{6}\int_0^6\left[x^2\left(4 - \frac{2}{3}x\right) + \frac{1}{3}\left(4 - \frac{2}{3}x\right)^3\right] dx = \frac{7}{6}\left[\frac{4}{3}x^3 - \frac{1}{6}x^4 - \frac{1}{8}\left(4 - \frac{2}{3}x\right)^4\right]_0^6 = \frac{364}{3}$$

13. $S\colon \mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \frac{v}{2}\mathbf{k},\ 0 \le u \le 1,\ 0 \le v \le 2$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \left\| -\frac{1}{2}\mathbf{j} + \mathbf{k} \right\| = \frac{\sqrt{5}}{2}$$

$$\iint_S (y + 5)\, dS = \int_0^2 \int_0^1 (v + 5)\frac{\sqrt{5}}{2}\, du\, dv = 6\sqrt{5}$$

15. $S\colon \mathbf{r}(u, v) = 2\cos u\mathbf{i} + 2\sin u\mathbf{j} + v\mathbf{k},\ 0 \le u \le \frac{\pi}{2},\ 0 \le v \le 2$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = \|2\cos u\mathbf{i} + 2\sin u\mathbf{j}\| = 2$

$$\iint_S xy\, dS = \int_0^2 \int_0^{\pi/2} 8\cos u \sin u\, du\, dv = 8$$

17. $f(x, y, z) = x^2 + y^2 + z^2$

$S\colon z = x + 2,\ x^2 + y^2 \le 1$

$$\begin{aligned}
\iint_S f(x, y, z)\, dS &= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} [x^2 + y^2 + (x + 2)^2]\sqrt{1 + (1)^2 + (0)^2}\, dy\, dx \\
&= \sqrt{2}\int_0^{2\pi}\int_0^1 [r^2 + (r\cos\theta + 2)^2] r\, dr\, d\theta \\
&= \sqrt{2}\int_0^{2\pi}\int_0^1 [r^2 + r^2\cos^2\theta + 4r\cos\theta + 4] r\, dr\, d\theta \\
&= \sqrt{2}\int_0^{2\pi} \left[\frac{r^4}{4} + \frac{r^4}{4}\cos^2\theta + \frac{4r^3}{3}\cos\theta + 2r^2\right]_0^1 d\theta \\
&= \sqrt{2}\int_0^{2\pi} \left[\frac{9}{4} + \left(\frac{1}{4}\right)\frac{1 + \cos 2\theta}{2} + \frac{4}{3}\cos\theta\right] d\theta \\
&= \sqrt{2}\left[\frac{9}{4}\theta + \frac{1}{8}\left(\theta + \frac{1}{2}\sin 2\theta\right) + \frac{4}{3}\sin\theta\right]_0^{2\pi} = \sqrt{2}\left[\frac{18\pi}{4} + \frac{\pi}{4}\right] = \frac{19\sqrt{2}\pi}{4}
\end{aligned}$$

19. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

$S\colon z = \sqrt{x^2 + y^2},\ x^2 + y^2 \le 4$

$$\begin{aligned}
\iint_S f(x, y, z)\, dS &= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2 + y^2 + \left(\sqrt{x^2 + y^2}\right)^2}\sqrt{1 + \left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}}\right)^2}\, dy\, dx \\
&= \sqrt{2}\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2 + y^2}\sqrt{\frac{x^2 + y^2 + x^2 + y^2}{x^2 + y^2}}\, dy\, dx \\
&= 2\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2 + y^2}\, dy\, dx \\
&= 2\int_0^{2\pi}\int_0^2 r^2\, dr\, d\theta = 2\int_0^{2\pi}\left[\frac{r^3}{3}\right]_0^2 d\theta = \left[\frac{16}{3}\theta\right]_0^{2\pi} = \frac{32\pi}{3}
\end{aligned}$$

21. $f(x, y, z) = x^2 + y^2 + z^2$

S: $x^2 + y^2 = 9$, $0 \le x \le 3$, $0 \le y \le 3$, $0 \le z \le 9$

Project the solid onto the yz-plane; $x = \sqrt{9 - y^2}$, $0 \le y \le 3$, $0 \le z \le 9$.

$$\iint_S f(x, y, z)\, dS = \int_0^3 \int_0^9 [(9 - y^2) + y^2 + z^2] \sqrt{1 + \left(\frac{y}{\sqrt{9 - y^2}}\right)^2 + (0)^2}\, dz\, dy$$

$$= \int_0^3 \int_0^9 (9 + z^2) \frac{3}{\sqrt{9 - y^2}}\, dz\, dy = \int_0^3 \left[\frac{3}{\sqrt{9 - y^2}}\left(9z + \frac{z^3}{3}\right)\right]_0^9 dy$$

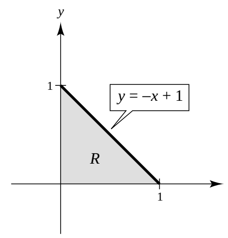
$$= 324 \int_0^3 \frac{3}{\sqrt{9 - y^2}}\, dy = \left[972 \arcsin\left(\frac{y}{3}\right)\right]_0^3 = 972\left(\frac{\pi}{2} - 0\right) = 486\pi$$

23. $\mathbf{F}(x, y, z) = 3z\mathbf{i} - 4\mathbf{j} + y\mathbf{k}$

S: $x + y + z = 1$ (first octant)

$G(x, y, z) = x + y + z - 1$

$\nabla G(x, y, z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$



$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = \iint_R \mathbf{F} \cdot \nabla G\, dA = \int_0^1 \int_0^{1-x} (3z - 4 + y)\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} [3(1 - x - y) - 4 + y]\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} (-1 - 3x - 2y)\, dy\, dx$$

$$= \int_0^1 \left[-y - 3xy - y^2\right]_0^{1-x} dx$$

$$= -\int_0^1 [(1 - x) + 3x(1 - x) + (1 - x)^2]\, dx$$

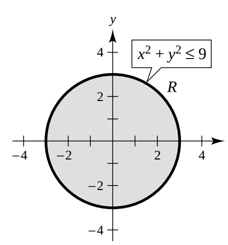
$$= -\int_0^1 (2 - 2x^2)\, dx = -\frac{4}{3}$$

25. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

S: $z = 9 - x^2 - y^2$, $0 \le z$

$G(x, y, z) = x^2 + y^2 + z - 9$

$\nabla G(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$



$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = \iint_R \mathbf{F} \cdot \nabla G\, dA = \iint_R (2x^2 + 2y^2 + z)\, dA$$

$$= \iint_R [2x^2 + 2y^2 + (9 - x^2 - y^2)]\, dA$$

$$= \iint_R (x^2 + y^2 + 9)\, dA$$

$$= \int_0^{2\pi} \int_0^3 (r^2 + 9) r\, dr\, d\theta$$

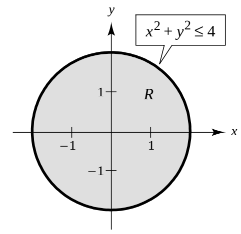
$$= \int_0^{2\pi} \left[\frac{r^4}{4} + \frac{9r^2}{2}\right]_0^3 d\theta = \frac{243\pi}{2}$$

27. $\mathbf{F}(x, y, z) = 4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$

$S: \ z = x^2 + y^2, \ x^2 + y^2 \leq 4$

$G(x, y, z) = -x^2 - y^2 + z$

$\nabla G(x, y, z) = -2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$



$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = \iint_R \mathbf{F} \cdot \nabla G\, dA = \iint_R (-8x + 6y + 5)\, dA$$

$$= \int_0^{2\pi}\int_0^2 [-8r\cos\theta + 6r\sin\theta + 5]r\, dr\, d\theta$$

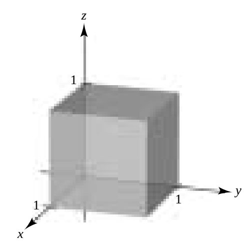
$$= \int_0^{2\pi}\left[-\frac{8}{3}r^3\cos\theta + 2r^3\sin\theta + \frac{5}{2}r^2\right]_0^2 d\theta$$

$$= \int_0^{2\pi}\left[-\frac{64}{3}\cos\theta + 16\sin\theta + 10\right] d\theta$$

$$= \left[-\frac{64}{3}\sin\theta - 16\cos\theta + 10\theta\right]_0^{2\pi} = 20\pi$$

29. $\mathbf{F}(x, y, z) = 4xy\mathbf{i} + z^2\mathbf{j} + yz\mathbf{k}$

S: unit cube bounded by $x = 0, \ x = 1, \ y = 0, \ y = 1, \ z = 0, \ z = 1$



S_1: The top of the cube

$\mathbf{N} = \mathbf{k}, \ z = 1$

$$\iint_{S_1} \mathbf{F} \cdot \mathbf{N}\, dS = \int_0^1\int_0^1 y(1)\, dy\, dx = \frac{1}{2}$$

S_2: The bottom of the cube

$\mathbf{N} = -\mathbf{k}, \ z = 0$

$$\iint_{S_2} \mathbf{F} \cdot \mathbf{N}\, dS = \int_0^1\int_0^1 -y(0)\, dy\, dx = 0$$

S_3: The front of the cube

$\mathbf{N} = \mathbf{i}, \ x = 1$

$$\iint_{S_3} \mathbf{F} \cdot \mathbf{N}\, dS = \int_0^1\int_0^1 4(1)y\, dy\, dz = 2$$

S_4: The back of the cube

$\mathbf{N} = -\mathbf{i}, \ x = 0$

$$\iint_{S_4} \mathbf{F} \cdot \mathbf{N}\, dS = \int_0^1\int_0^1 -4(0)y\, dy\, dx = 0$$

S_5: The right side of the cube

$\mathbf{N} = \mathbf{j}, \ y = 1$

$$\iint_{S_5} \mathbf{F} \cdot \mathbf{N}\, dS = \int_0^1\int_0^1 z^2\, dz\, dx = \frac{1}{3}$$

S_6: The left side of the cube

$\mathbf{N} = -\mathbf{j}, \ y = 0$

$$\iint_{S_6} \mathbf{F} \cdot \mathbf{N}\, dS = \int_0^1\int_0^1 -z^2\, dz\, dx = -\frac{1}{3}$$

Therefore,

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = \frac{1}{2} + 0 + 2 + 0 + \frac{1}{3} - \frac{1}{3} = \frac{5}{2}.$$

31. The surface integral of f over a surface S, where S is given by $z = g(x, y)$, is defined as

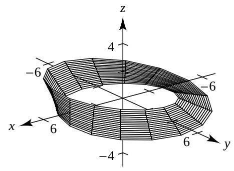
$$\iint_S f(x, y, z)\, dS = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta S_i. \ \text{(page 1061)}$$

See Theorem 14.10, page 1061.

33. See the definition, page 1067.

See Theorem 14.11, page 1067.

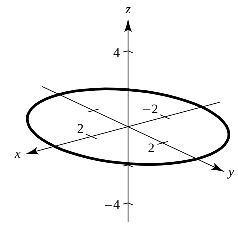
35. (a)



(b) If a normal vector at a point P on the surface is moved around the Möbius strip once, it will point in the opposite direction.

(c) $\mathbf{r}(u, 0) = 4\cos(2u)\mathbf{i} + 4\sin(2u)\mathbf{j}$

This is a circle.



(d) (construction)

(e) You obtain a strip with a double twist and twice as long as the original Möbius strip.

37. $z = \sqrt{x^2 + y^2},\ 0 \le z \le a$

$$m = \iint_S k\,dS = k\iint_R \sqrt{1 + \left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2}\,dA = k\iint_R \sqrt{2}\,dA = \sqrt{2}\,k\pi a^2$$

$$I_z = \iint_S k(x^2 + y^2)\,dS = \iint_R k(x^2 + y^2)\sqrt{2}\,dA$$

$$= \sqrt{2}k\int_0^{2\pi}\int_0^a r^3\,dr\,d\theta = \frac{\sqrt{2}ka^4}{4}(2\pi)$$

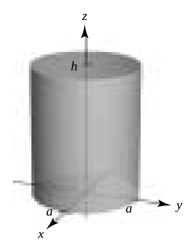
$$= \frac{\sqrt{2}k\pi a^4}{2} = \frac{a^2}{2}\left(\sqrt{2}k\pi a^2\right) = \frac{a^2 m}{2}$$

39. $x^2 + y^2 = a^2,\ 0 \le z \le h$

$\rho(x, y, z) = 1$

$y = \pm\sqrt{a^2 - x^2}$

Project the solid onto the xz-plane.



$$I_z = 4\iint_S (x^2 + y^2)(1)\,dS$$

$$= 4\int_0^h\int_0^a [x^2 + (a^2 - x^2)]\sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2}}\right)^2 + (0)^2}\,dx\,dz$$

$$= 4a^3\int_0^h\int_0^a \frac{1}{\sqrt{a^2 - x^2}}\,dx\,dz$$

$$= 4a^3\int_0^h\left[\arcsin\frac{x}{a}\right]_0^a dz = 4a^3\left(\frac{\pi}{2}\right)(h) = 2\pi a^3 h$$

41. $S\colon\ z = 16 - x^2 - y^2,\ z \ge 0$

$\mathbf{F}(x, y, z) = 0.5z\mathbf{k}$

$$\iint_S \rho\mathbf{F}\cdot\mathbf{N}\,dS = \iint_R \rho\mathbf{F}\cdot\left(-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k}\right)dA = \iint_R 0.5\rho z\mathbf{k}\cdot(2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})\,dA$$

$$= \iint_R 0.5\rho z\,dA = \iint_R 0.5\rho(16 - x^2 - y^2)\,dA$$

$$= 0.5\rho\int_0^{2\pi}\int_0^4 (16 - r^2)r\,dr\,d\theta = 0.5\rho\int_0^{2\pi} 64\,d\theta = 64\pi\rho$$

Section 14.7 Divergence Theorem

1. **Surface Integral:** There are six surfaces to the cube, each with $dS = \sqrt{1}\, dA$.

$$z = 0, \quad \mathbf{N} = -\mathbf{k}, \quad \mathbf{F}\cdot\mathbf{N} = -z^2, \quad \int_{S_1}\int 0\, dA = 0$$

$$z = a, \quad \mathbf{N} = \mathbf{k}, \quad \mathbf{F}\cdot\mathbf{N} = z^2, \quad \int_{S_2}\int a^2\, dA = \int_0^a\int_0^a a^2\, dx\, dy = a^4$$

$$x = 0, \quad \mathbf{N} = -\mathbf{i}, \quad \mathbf{F}\cdot\mathbf{N} = -2x, \quad \int_{S_3}\int 0\, dA = 0$$

$$x = a, \quad \mathbf{N} = \mathbf{i}, \quad \mathbf{F}\cdot\mathbf{N} = 2x, \quad \int_{S_4}\int 2a\, dy\, dz = \int_0^a\int_0^a 2a\, dy\, dz = 2a^3$$

$$y = 0, \quad \mathbf{N} = -\mathbf{j}, \quad \mathbf{F}\cdot\mathbf{N} = 2y, \quad \int_{S_5}\int 0\, dA = 0$$

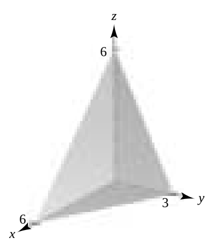
$$y = a, \quad \mathbf{N} = \mathbf{j}, \quad \mathbf{F}\cdot\mathbf{N} = -2y, \quad \int_{S_6}\int -2a\, dA = \int_0^a\int_0^a -2a\, dz\, dx = -2a^3$$

Therefore, $\int_S\int \mathbf{F}\cdot\mathbf{N}\, dS = a^4 + 2a^3 - 2a^3 = a^4$.

Divergence Theorem: Since div $\mathbf{F} = 2z$, the Divergence Theorem yields

$$\iiint_Q \text{div}\,\mathbf{F}\, dV = \int_0^a\int_0^a\int_0^a 2z\, dz\, dy\, dx = \int_0^a\int_0^a a^2\, dy\, dx = a^4.$$

3. **Surface Integral:** There are four surfaces to this solid.



$z = 0$, $\mathbf{N} = -\mathbf{k}$, $\mathbf{F}\cdot\mathbf{N} = -z$

$$\int_{S_1}\int 0\, dS = 0$$

$y = 0$, $\mathbf{N} = -\mathbf{j}$, $\mathbf{F}\cdot\mathbf{N} = 2y - z$, $dS = dA = dx\, dz$

$$\int_{S_2}\int -z\, dS = \int_0^6\int_0^{6-z} -z\, dx\, dz = \int_0^6 (z^2 - 6z)\, dz = -36$$

$x = 0$, $\mathbf{N} = -\mathbf{i}$, $\mathbf{F}\cdot\mathbf{N} = y - 2x$, $dS = dA = dz\, dy$

$$\int_{S_3}\int y\, dS = \int_0^3\int_0^{6-2y} y\, dz\, dy = \int_0^3 (6y - 2y^2)\, dy = 9$$

$x + 2y + z = 6$, $\mathbf{N} = \dfrac{\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{6}}$, $\mathbf{F}\cdot\mathbf{N} = \dfrac{2x - 5y + 3z}{\sqrt{6}}$, $dS = \sqrt{6}\, dA$

$$\int_{S_4}\int (2x - 5y + 3z)\, dz\, dy = \int_0^3\int_0^{6-2y} (18 - x - 11y)\, dx\, dy = \int_0^3 (90 - 90y + 20y^2)\, dy = 45$$

Therefore, $\int_S\int \mathbf{F}\cdot\mathbf{N}\, dS = 0 - 36 + 9 + 45 = 18$.

Divergence Theorem: Since div $\mathbf{F} = 1$, we have

$$\iiint_Q dV = (\text{Volume of solid}) = \frac{1}{3}(\text{Area of base}) \times (\text{Height}) = \frac{1}{3}(9)(6) = 18.$$

5. Since div $\mathbf{F} = 2x + 2y + 2z$, we have

$$\iiint_Q \text{div}\,\mathbf{F}\,dV = \int_0^a\int_0^a\int_0^a (2x + 2y + 2z)\,dz\,dy\,dx$$
$$= \int_0^a\int_0^a (2ax + 2ay + a^2)\,dy\,dx = \int_0^a (2a^2x + 2a^3)\,dx = \Big[a^2x^2 + 2a^3x\Big]_0^a = 3a^4.$$

7. Since div $\mathbf{F} = 2x - 2x + 2xyz = 2xyz$

$$\iiint_Q \text{div}\,\mathbf{F}\,dV = \iiint_Q 2xyz\,dV = \int_0^a\int_0^{2\pi}\int_0^{\pi/2} 2(\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)\rho^2\sin\phi\,d\phi\,d\theta\,d\rho$$
$$= \int_0^a\int_0^{2\pi}\int_0^{\pi/2} 2\rho^5(\sin\theta\cos\theta)(\sin^3\phi\cos\phi)\,d\phi\,d\theta\,d\rho$$
$$= \int_0^a\int_0^{2\pi} \frac{1}{2}\rho^5\sin\theta\cos\theta\,d\theta\,d\rho = \int_0^a\bigg[\left(\frac{\rho^5}{2}\right)\frac{\sin^2\theta}{2}\bigg]_0^{2\pi} d\rho = 0.$$

9. Since div $\mathbf{F} = 3$, we have

$$\iiint_Q 3\,dV = 3(\text{Volume of sphere}) = 3\bigg[\frac{4}{3}\pi(2)^3\bigg] = 32\pi.$$

11. Since div $\mathbf{F} = 1 + 2y - 1 = 2y$, we have

$$\iiint_Q 2y\,dV = \int_0^4\int_{-3}^3\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} 2y\,dx\,dy\,dz = \int_0^4\int_{-3}^3 4y\sqrt{9-y^2}\,dy\,dz = \int_0^4\bigg[-\frac{4}{3}(9-y^2)^{3/2}\bigg]_{-3}^3 dz = 0.$$

13. Since div $\mathbf{F} = 3x^2 + x^2 + 0 = 4x^2$, we have

$$\iiint_Q 4x^2\,dV = \int_0^6\int_0^4\int_0^{4-y} 4x^2\,dz\,dy\,dx = \int_0^6\int_0^4 4x^2(4-y)\,dy\,dx = \int_0^6 32x^2\,dx = 2304.$$

15. $\mathbf{F}(x, y, z) = xy\mathbf{i} + 4y\mathbf{j} + xz\mathbf{k}$

div $\mathbf{F} = y + 4 + x$

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,dS = \iiint_Q \text{div}\,\mathbf{F}\,dV = \iiint_Q (y + x + 4)\,dV$$
$$= \int_0^3\int_0^\pi\int_0^{2\pi} (\rho\sin\phi\sin\theta + \rho\sin\phi\cos\theta + 4)\rho^2\sin\phi\,d\theta\,d\phi\,d\rho$$
$$= \int_0^3\int_0^\pi\int_0^{2\pi} [\rho^3\sin^2\phi\sin\theta + \rho^3\sin^2\phi\cos\theta + 4\rho^2\sin\phi]\,d\theta\,d\phi\,d\rho$$
$$= \int_0^3\int_0^\pi\Big[-\rho^3\sin^2\phi\cos\theta + \rho^3\sin^2\phi\sin\theta + 4\rho^2\sin\phi\cdot\theta\Big]_0^{2\pi} d\phi\,d\rho$$
$$= \int_0^3\int_0^\pi 8\pi\rho^2\sin\phi\,d\phi\,d\rho$$
$$= \int_0^3\Big[-8\pi\rho^2\cos\phi\Big]_0^\pi d\rho$$
$$= \int_0^3 16\pi\rho^2\,d\rho = \bigg[\frac{16\pi\rho^3}{3}\bigg]_0^3 = 144\pi.$$

17. Using the Divergence Theorem, we have

$$\int_S\!\!\int \mathbf{curl\ F}\cdot\mathbf{N}\,dS = \iiint_Q \operatorname{div}(\mathbf{curlF})\,dV$$

$$\mathbf{curl\ F}(x, y, z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 4xy + z^2 & 2x^2 + 6yz & 2xz \end{vmatrix} = -6y\mathbf{i} - (2z - 2z)\mathbf{j} + (4x - 4x)\mathbf{k} = -6y\mathbf{i}$$

$$\operatorname{div}(\mathbf{curl\ F}) = 0.$$

Therefore, $\displaystyle\iiint_Q \operatorname{div}(\mathbf{curl\ F})\,dV = 0.$

19. See Theorem 14.12, page 1073.

21. Using the triple integral to find volume, we need $\mathbf{F}$ so that

$$\operatorname{div}\mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = 1.$$

Hence, we could have $\mathbf{F} = x\mathbf{i}$, $\mathbf{F} = y\mathbf{j}$, or $\mathbf{F} = z\mathbf{k}$.

For $dA = dy\,dz$ consider $\mathbf{F} = x\mathbf{i}$, $x = f(y, z)$, then $\mathbf{N} = \dfrac{\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}}{\sqrt{1 + f_y^2 + f_z^2}}$ and $dS = \sqrt{1 + f_y^2 + f_z^2}\,dy\,dz.$

For $dA = dz\,dx$ consider $\mathbf{F} = y\mathbf{j}$, $y = f(x, z)$, then $\mathbf{N} = \dfrac{f_x\mathbf{i} + \mathbf{j} + f_z\mathbf{k}}{\sqrt{1 + f_x^2 + f_z^2}}$ and $dS = \sqrt{1 + f_x^2 + f_z^2}\,dz\,dx.$

For $dA = dx\,dy$ consider $\mathbf{F} = z\mathbf{k}$, $z = f(x, y)$, then $\mathbf{N} = \dfrac{f_x\mathbf{i} + f_y\mathbf{j} + \mathbf{k}}{\sqrt{1 + f_x^2 + f_y^2}}$ and $dS = \sqrt{1 + f_x^2 + f_y^2}\,dx\,dy.$

Correspondingly, we then have $\displaystyle V = \int_S\!\!\int \mathbf{F}\cdot\mathbf{N}\,dS = \int_S\!\!\int x\,dy\,dz = \int_S\!\!\int y\,dz\,dx = \int_S\!\!\int z\,dx\,dy.$

23. Using the Divergence Theorem, we have $\displaystyle\int_S\!\!\int \mathbf{curl\ F}\cdot\mathbf{N}\,dS = \iiint_Q \operatorname{div}(\mathbf{curl\ F})\,dV.$ Let

$$\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$$

$$\mathbf{curl\ F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$$

$$\operatorname{div}(\mathbf{curl\ F}) = \frac{\partial^2 P}{\partial x\partial y} - \frac{\partial^2 N}{\partial x\partial z} - \frac{\partial^2 P}{\partial y\partial x} + \frac{\partial^2 M}{\partial y\partial z} + \frac{\partial^2 N}{\partial z\partial x} - \frac{\partial^2 M}{\partial z\partial y} = 0.$$

Therefore, $\displaystyle\int_S\!\!\int \mathbf{curl\ F}\cdot\mathbf{N}\,dS = \iiint_Q 0\,dV = 0.$

25. If $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then $\operatorname{div}\mathbf{F} = 3$.

$$\int_S\!\!\int \mathbf{F}\cdot\mathbf{N}\,dS = \iiint_Q \operatorname{div}\mathbf{F}\,dV = \iiint_Q 3\,dV = 3V.$$

27. $\displaystyle\int_S\!\!\int f D_{\mathbf{N}}g\,dS = \int_S\!\!\int f\nabla g\cdot\mathbf{N}\,dS$

$$= \iiint_Q \operatorname{div}(f\nabla g)\,dV = \iiint_Q (f\operatorname{div}\nabla g + \nabla f\cdot\nabla g)\,dV = \iiint_Q (f\nabla^2 g + \nabla f\cdot\nabla g)\,dV$$

Section 14.8 Stokes's Theorem

1. $\mathbf{F}(x, y, z) = (2y - z)\mathbf{i} + xyz\mathbf{j} + e^z\mathbf{k}$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2y - z & xyz & e^z \end{vmatrix} = -xy\mathbf{i} - \mathbf{j} + (yz - 2)\mathbf{k}$$

3. $\mathbf{F}(x, y, z) = 2z\mathbf{i} - 4x^2\mathbf{j} + \arctan x\mathbf{k}$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2z & -4x^2 & \arctan x \end{vmatrix} = \left(2 - \frac{1}{1 + x^2}\right)\mathbf{j} - 8x\mathbf{k}$$

5. $\mathbf{F}(x, y, z) = e^{x^2+y^2}\mathbf{i} + e^{y^2+z^2}\mathbf{j} + xyz\mathbf{k}$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^{x^2+y^2} & e^{y^2+z^2} & xyz \end{vmatrix}$$

$$= (xz - 2ze^{y^2+z^2})\mathbf{i} - yz\mathbf{j} - 2ye^{x^2+y^2}\mathbf{k}$$

$$= z(x - 2e^{y^2+z^2})\mathbf{i} - yz\mathbf{j} - 2ye^{x^2+y^2}\mathbf{k}$$

7. In this case, $M = -y + z,\ N = x - z,\ P = x - y$ and C is the circle $x^2 + y^2 = 1,\ z = 0,\ dz = 0$.

Line Integral: $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C -y\,dx + x\,dy$

Letting $x = \cos t,\ y = \sin t,$ we have $dx = -\sin t\,dt,\ dy = \cos t\,dt$ and

$$\int_C -y\,dx + x\,dy = \int_0^{2\pi} (\sin^2 t + \cos^2 t)dt = 2\pi.$$

Double Integral: Consider $F(x, y, z) = x^2 + y^2 + z^2 - 1$.

Then

$$\mathbf{N} = \frac{\nabla F}{\|\nabla F\|} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{x^2 + y^2 + z^2}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

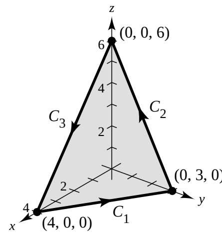
Since

$$z^2 = 1 - x^2 - y^2,\ z_x = \frac{-2x}{2z} = \frac{-x}{z}, \text{ and } z_y = \frac{-y}{z},\ dS = \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}}dA = \frac{1}{z}dA.$$

Now, since $\mathbf{curl\ F} = 2\mathbf{k}$, we have

$$\int_S\int (\mathbf{curl\ F}) \cdot \mathbf{N}\,dS = \int_R\int 2z\left(\frac{1}{z}\right) dA = \int_R\int 2\,dA = 2(\text{Area of circle of radius } 1) = 2\pi.$$

9. Line Integral: From the accompanying figure we see that for



C_1: $z = 0$, $dz = 0$

C_2: $x = 0$, $dx = 0$

C_3: $y = 0$, $dy = 0$.

Hence, $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C xyz\,dx + y\,dy + z\,dz$

$$= \int_{C_1} y\,dy + \int_{C_2} y\,dy + z\,dz + \int_{C_3} z\,dz$$

$$= \int_0^3 y\,dy + \int_3^0 y\,dy + \int_0^6 z\,dz + \int_6^0 z\,dz = 0.$$

Double Integral: $\textbf{curl F} = xy\mathbf{j} - xz\mathbf{k}$

Considering $F(x, y, z) = 3x + 4y + 2z - 12$, then

$$\mathbf{N} = \frac{\nabla F}{\|\nabla F\|} = \frac{3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}}{\sqrt{29}} \text{ and } dS = \sqrt{29}\,dA.$$

Thus,

$$\int_S\int (\textbf{curl F}) \cdot \mathbf{N}\,dS = \int_R\int (4xy - 2xz)\,dy\,dx$$

$$= \int_0^4 \int_0^{(-3x+12)/4} \left[4xy - 2x\left(6 - 2y - \frac{3}{2}x\right)\right] dy\,dx$$

$$= \int_0^4 \int_0^{(12-3x)/4} (8xy + 3x^2 - 12x)\,dy\,dx$$

$$= \int_0^4 0\,dx = 0.$$

11. Let $A = (0, 0, 0)$, $B = (1, 1, 1)$ and $C = (0, 2, 0)$. Then $\mathbf{U} = \overrightarrow{AB} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{V} = \overrightarrow{AC} = 2\mathbf{j}$. Thus,

$$\mathbf{N} = \frac{\mathbf{U} \times \mathbf{V}}{\|\mathbf{U} \times \mathbf{V}\|} = \frac{-2\mathbf{i} + 2\mathbf{k}}{2\sqrt{2}} = \frac{-\mathbf{i} + \mathbf{k}}{\sqrt{2}}.$$

Surface S has direction numbers $-1, 0, 1$, with equation $z - x = 0$ and $dS = \sqrt{2}\,dA$. Since $\textbf{curl F} = -3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, we have

$$\int_S\int (\textbf{curl F}) \cdot \mathbf{N}\,dS = \int_R\int \frac{1}{\sqrt{2}}(\sqrt{2})\,dA = \int_R\int dA = (\text{Area of triangle with } a = 1, b = 2) = 1.$$

13. $\mathbf{F}(x, y, z) = z^2\mathbf{i} + x^2\mathbf{j} + y^2\mathbf{k}$, S: $z = 4 - x^2 - y^2$, $0 \le z$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & x^2 & y^2 \end{vmatrix} = 2y\mathbf{i} + 2z\mathbf{j} + 2x\mathbf{k}$$

$G(x, y, z) = x^2 + y^2 + z - 4$

$\nabla G(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$

$$\int_S\int (\textbf{curl F}) \cdot \mathbf{N}\,dS = \int_R\int (4xy + 4yz + 2x)\,dA = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [4xy + 4y(4 - x^2 - y^2) + 2x]\,dy\,dx$$

$$= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [4xy + 16y - 4x^2y - 4y^3 + 2x]\,dy\,dx$$

$$= \int_{-2}^2 4x\sqrt{4 - x^2}\,dx = 0$$

15. $\mathbf{F}(x, y, z) = z^2\mathbf{i} + y\mathbf{j} + xz\mathbf{k}, \; S: z = \sqrt{4 - x^2 - y^2}$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z^2 & y & xz \end{vmatrix} = z\mathbf{j}$$

$$G(x, y, z) = z - \sqrt{4 - x^2 - y^2}$$

$$\nabla G(x, y, z) = \frac{x}{\sqrt{4 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{4 - x^2 - y^2}}\mathbf{j} + \mathbf{k}$$

$$\int_S\!\!\int (\mathbf{curl\ F}) \cdot \mathbf{F}\, dS = \int_R\!\!\int \frac{yz}{\sqrt{4 - x^2 - y^2}}\, dA = \int_R\!\!\int \frac{y\sqrt{4 - x^2 - y^2}}{\sqrt{4 - x^2 - y^2}}\, dA = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} y\, dy\, dx = 0$$

17. $\mathbf{F}(x, y, z) = -\ln\sqrt{x^2 + y^2}\,\mathbf{i} + \arctan\dfrac{x}{y}\,\mathbf{j} + \mathbf{k}$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -1/2\ln(x^2 + y^2) & \arctan x/y & 1 \end{vmatrix} = \left[\frac{(1/y)}{1 + (x^2/y^2)} + \frac{y}{x^2 + y^2}\right]\mathbf{k} = \left[\frac{2y}{x^2 + y^2}\right]\mathbf{k}$$

$S: z = 9 - 2x - 3y$ over one petal of $r = 2\sin 2\theta$ in the first octant.

$$G(x, y, z) = 2x + 3y + z - 9$$

$$\nabla G(x, y, z) = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

$$\begin{aligned} \int_S\!\!\int (\mathbf{curl\ F}) \cdot \mathbf{N}\, dS &= \int_R\!\!\int \frac{2y}{x^2 + y^2}\, dA \\ &= \int_0^{\pi/2}\int_0^{2\sin 2\theta} \frac{2r\sin\theta}{r^2} r\, dr\, d\theta \\ &= \int_0^{\pi/2}\int_0^{4\sin\theta\cos\theta} 2\sin\theta\, dr\, d\theta \\ &= \int_0^{\pi/2} 8\sin^2\theta\cos\theta\, d\theta = \left[\frac{8\sin^3\theta}{3}\right]_0^{\pi/2} = \frac{8}{3} \end{aligned}$$

19. From Exercise 10, we have $\mathbf{N} = \dfrac{2x\mathbf{i} - \mathbf{k}}{\sqrt{1 + 4x^2}}$ and $dS = \sqrt{1 + 4x^2}\, dA$. Since $\mathbf{curl\ F} = xy\mathbf{j} - xz\mathbf{k}$, we have

$$\int_S\!\!\int (\mathbf{curl\ F}) \cdot \mathbf{N}\, dS = \int_R\!\!\int xz\, dA = \int_0^a\int_0^a x^3\, dy\, dx = \int_0^a ax^3\, dx = \left[\frac{ax^4}{4}\right]_0^a = \frac{a^5}{4}.$$

21. $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 1 & 1 & -2 \end{vmatrix} = \mathbf{0}$$

Letting $\mathbf{N} = \mathbf{k}$, we have $\displaystyle\int_S\!\!\int (\mathbf{curl\ F}) \cdot \mathbf{N}\, dS = 0$.

23. See Theorem 14.13, page 1081.

25. (a) $\int_C f\nabla g \cdot d\mathbf{r} = \int_S\int \mathbf{curl}[f\nabla g] \cdot \mathbf{N}\, dS$ (Stoke's Theorem)

$$f\nabla g = f\frac{\partial g}{\partial x}\mathbf{i} + f\frac{\partial g}{\partial y}\mathbf{j} + f\frac{\partial g}{\partial z}\mathbf{k}$$

$$\mathbf{curl}\,(f\nabla g) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(\partial g/\partial x) & f(\partial g/\partial y) & f(\partial g/\partial z) \end{vmatrix}$$

$$= \left[\left[f\left(\frac{\partial^2 g}{\partial y \partial z}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial g}{\partial z}\right)\right] - \left[f\left(\frac{\partial^2 g}{\partial z \partial y}\right) + \left(\frac{\partial f}{\partial z}\right)\left(\frac{\partial g}{\partial y}\right)\right]\right]\mathbf{i}$$

$$- \left[\left[f\left(\frac{\partial^2 g}{\partial x \partial z}\right) + \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial g}{\partial z}\right)\right] - \left[f\left(\frac{\partial^2 g}{\partial z \partial x}\right) + \left(\frac{\partial f}{\partial z}\right)\left(\frac{\partial g}{\partial x}\right)\right]\right]\mathbf{j}$$

$$+ \left[\left[f\left(\frac{\partial^2 g}{\partial x \partial y}\right) + \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial g}{\partial y}\right)\right] - \left[f\left(\frac{\partial^2 g}{\partial y \partial x}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial g}{\partial x}\right)\right]\right]\mathbf{k}$$

$$= \left[\left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial g}{\partial z}\right) - \left(\frac{\partial f}{\partial z}\right)\left(\frac{\partial g}{\partial y}\right)\right]\mathbf{i} - \left[\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial g}{\partial z}\right) - \left(\frac{\partial f}{\partial z}\right)\left(\frac{\partial g}{\partial x}\right)\right]\mathbf{j} + \left[\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial g}{\partial y}\right) - \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial g}{\partial x}\right)\right]\mathbf{k}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{vmatrix} = \nabla f \times \nabla g$$

Therefore, $\int_C f\nabla g \cdot d\mathbf{r} = \int_S\int \mathbf{curl}[f\nabla g] \cdot \mathbf{N}\, dS = \int_S\int [\nabla f \times \nabla g] \cdot \mathbf{N}\, dS.$

(b) $\int_C (f\nabla f) \cdot d\mathbf{r} = \int_S\int (\nabla f \times \nabla f) \cdot \mathbf{N}\, dS$ (using part a.)

$= 0$ since $\nabla f \times \nabla f = 0$.

(c) $\int_C (f\nabla g + g\nabla f) \cdot d\mathbf{r} = \int_C (f\nabla g) \cdot d\mathbf{r} + \int_C (g\nabla f) \cdot d\mathbf{r}$

$= \int_S\int (\nabla f \times \nabla g) \cdot \mathbf{N}\, dS + \int_S\int (\nabla g \times \nabla f) \cdot \mathbf{N}\, dS$ (using part a.)

$= \int_S\int (\nabla f \times \nabla g) \cdot \mathbf{N}\, dS + \int_S\int -(\nabla f \times \nabla g) \cdot \mathbf{N}\, dS = 0$

27. Let $\mathbf{C} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then

$$\frac{1}{2}\int_C (\mathbf{C} \times \mathbf{r}) \cdot d\mathbf{r} = \frac{1}{2}\int_S\int \mathbf{curl}\,(\mathbf{C} \times \mathbf{r}) \cdot \mathbf{N}\, dS = \frac{1}{2}\int_S\int 2\mathbf{C} \cdot \mathbf{N}\, dS = \int_S\int \mathbf{C} \cdot \mathbf{N}\, dS$$

since

$$\mathbf{C} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ x & y & z \end{vmatrix} = (bz - cy)\mathbf{i} - (az - cx)\mathbf{j} + (ay - bx)\mathbf{k}$$

and

$$\mathbf{curl}(\mathbf{C} \times \mathbf{r}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ bz - cy & cx - az & ay - bx \end{vmatrix} = 2(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) = 2\mathbf{C}.$$