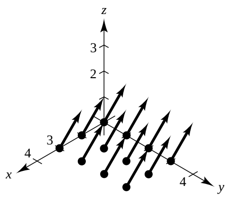
Review Exercises for Chapter 14

1. $\mathbf{F}(x, y, z) = x\mathbf{i} + \mathbf{j} + 2\mathbf{k}$



3. $f(x, y, z) = 8x^2 + xy + z^2$

$\mathbf{F}(x, y, z) = (16x + y)\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}$

5. Since $\partial M/\partial y = -1/y^2 \neq \partial N/\partial x$, $\mathbf{F}$ is not conservative.

7. Since $\partial M/\partial y = 12xy = \partial N/\partial x$, $\mathbf{F}$ is conservative. From $M = \partial U/\partial x = 6xy^2 - 3x^2$ and $N = \partial U/\partial y = 6x^2y + 3y^2 - 7$, partial integration yields $U = 3x^2y^2 - x^3 + h(y)$ and $U = 3x^2y^2 + y^3 - 7y + g(x)$ which suggests $h(y) = y^3 - 7y$, $g(x) = -x^3$, and $U(x, y) = 3x^2y^2 - x^3 + y^3 - 7y + C$.

9. Since

$$\frac{\partial M}{\partial y} = 4x = \frac{\partial N}{\partial x},$$

$$\frac{\partial M}{\partial z} = 1 \neq \frac{\partial P}{\partial x}.$$

$\mathbf{F}$ is not conservative.

11. Since

$$\frac{\partial M}{\partial y} = \frac{-1}{y^2z} = \frac{\partial N}{\partial x}, \quad \frac{\partial M}{\partial z} = \frac{-1}{yz^2} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial z} = \frac{x}{y^2z^2} = \frac{\partial P}{\partial y},$$

$\mathbf{F}$ is conservative. From

$$M = \frac{\partial U}{\partial x} = \frac{1}{yz}, \quad N = \frac{\partial U}{\partial y} = \frac{-x}{y^2z}, \quad P = \frac{\partial U}{\partial z} = \frac{-x}{yz^2}$$

we obtain

$$U = \frac{x}{yz} + f(y, z), \quad U = \frac{x}{yz} + g(x, z), \quad U = \frac{x}{yz} + h(x, y) \implies f(x, y, z) = \frac{x}{yz} + K$$

13. Since $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$:

(a) $\text{div}\,\mathbf{F} = 2x + 2y + 2z$

(b) $\mathbf{curl\,F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} = 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

15. Since $\mathbf{F} = (\cos y + y\cos x)\mathbf{i} + (\sin x - x\sin y)\mathbf{j} + xyz\mathbf{k}$:

(a) $\text{div}\,\mathbf{F} = -y\sin x - x\cos y + xy$

(b) $\mathbf{curl\,F} = xz\mathbf{i} - yz\mathbf{j} + (\cos x - \sin y + \sin y - \cos x)\mathbf{k} = xz\mathbf{i} - yz\mathbf{j}$

17. Since $\mathbf{F} = \arcsin x\mathbf{i} + xy^2\mathbf{j} + yz^2\mathbf{k}$:

(a) $\text{div}\,\mathbf{F} = \dfrac{1}{\sqrt{1-x^2}} + 2xy + 2yz$

(b) $\mathbf{curl}\,\mathbf{F} = z^2\mathbf{i} + y^2\mathbf{k}$

19. Since $\mathbf{F} = \ln(x^2+y^2)\mathbf{i} + \ln(x^2+y^2)\mathbf{j} + z\mathbf{k}$:

(a) $\text{div}\,\mathbf{F} = \dfrac{2x}{x^2+y^2} + \dfrac{2y}{x^2+y^2} + 1$

$= \dfrac{2x+2y}{x^2+y^2} + 1$

(b) $\mathbf{curl}\,\mathbf{F} = \dfrac{2x-2y}{x^2+y^2}\mathbf{k}$

21. (a) Let $x = t, y = t, -1 \le t \le 2$, then $ds = \sqrt{2}\,dt$.

$$\int_C (x^2+y^2)\,ds = \int_{-1}^{2} 2t^2\sqrt{2}\,dt = \left[2\sqrt{2}\left(\frac{t^3}{3}\right)\right]_{-1}^{2} = 6\sqrt{2}$$

(b) Let $x = 4\cos t, y = 4\sin t, 0 \le t \le 2\pi$, then $ds = 4\,dt$.

$$\int_C (x^2+y^2)\,ds = \int_0^{2\pi} 16(4\,dt) = 128\pi$$

23. $x = \cos t + t\sin t, y = \sin t - t\cos t, 0 \le t \le 2\pi, \dfrac{dx}{dt} = t\cos t, \dfrac{dy}{dt} = t\sin t$

$$\int_C (x^2+y^2)\,ds = \int_0^{2\pi} [(\cos t + t\sin t)^2 + (\sin t - t\cos t)^2]\sqrt{t^2\cos^2 t + t^2\sin^2 t}\,dt = \int_0^{2\pi} [t^3 + t]\,dt$$

$$= 2\pi^2(1+2\pi^2)$$

25. (a) Let $x = 2t, y = -3t, 0 \le t \le 1$

$$\int_C (2x-y)\,dx + (x+3y)\,dy = \int_0^1 [7t(2) + (-7t)(-3)]\,dt = \int_0^1 35t\,dt = \frac{35}{2}$$

(b) $x = 3\cos t, y = 3\sin t, dx = -3\sin t\,dt, dy = 3\cos t\,dt, 0 \le t \le 2\pi$

$$\int_C (2x-y)\,dx + (x+3y)\,dy = \int_0^{2\pi} (9 + 9\sin t\cos t)\,dt = 18\pi$$

27. $\displaystyle\int_C (2x+y)\,ds, \mathbf{r}(t) = a\cos^3 t\,\mathbf{i} + a\sin^3 t\,\mathbf{j}, 0 \le t \le \frac{\pi}{2}$

$x'(t) = -3a \cdot \cos^2 t\sin t$

$y'(t) = 3a \cdot \sin^2 t\cos t$

$$\int_C (2x+y)\,ds = \int_0^{\pi/2} (2(a\cdot\cos^3 t) + a\cdot\sin^3 t)\sqrt{x'(t)^2 + y'(t)^2}\,dt = \frac{9a^2}{5}$$

29. $f(x, y) = 5 + \sin(x+y)$

C: $y = 3x$ from $(0, 0)$ to $(2, 6)$

$\mathbf{r}(t) = t\mathbf{i} + 3t\mathbf{j}, 0 \le t \le 2$

$\mathbf{r}'(t) = \mathbf{i} + 3\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{10}$

Lateral surface area:

$$\int_{C_2} f(x,y)\,ds = \int_0^2 [5 + \sin(t+3t)]\sqrt{10}\,dt = \sqrt{10}\int_0^2 (5 + \sin 4t)\,dt = \frac{\sqrt{10}}{4}(41 - \cos 8) \approx 32.528$$

31. $d\mathbf{r} = (2t\mathbf{i} + 3t^2\mathbf{j})\,dt$

$\mathbf{F} = t^5\mathbf{i} + t^4\mathbf{j},\ 0 \le t \le 1$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 5t^6\,dt = \frac{5}{7}$$

33. $d\mathbf{r} = [(-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} + \mathbf{k}]\,dt$

$\mathbf{F} = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + t\mathbf{k},\ 0 \le t \le 2\pi$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} t\,dt = 2\pi^2$$

35. Let $x = t, y = -t, z = 2t^2, -2 \le t \le 2, d\mathbf{r} = [\mathbf{i} - \mathbf{j} + 4t\mathbf{k}]\,dt$.

$\mathbf{F} = (-t - 2t^2)\mathbf{i} + (2t^2 - t)\mathbf{j} + (2t)\mathbf{k}$

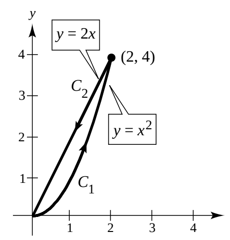
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{-2}^{2} 4t^2\,dt = \left[\frac{4t^3}{3}\right]_{-2}^{2} = \frac{64}{3}$$

37. For $y = x^2$, $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j},\ 0 \le t \le 2$

For $y = 2x$, $\mathbf{r}_2(t) = (2 - t)\mathbf{i} + (4 - 2t)\mathbf{j},\ 0 \le t \le 2$

$$\int_C xy\,dx + (x^2 + y^2)\,dy = \int_{C_1} xy\,dx + (x^2 + y^2)\,dy + \int_{C_2} xy\,dx + (x^2 + y^2)\,dy$$

$$= \frac{100}{3} + (-32) = \frac{4}{3}$$



39. $\mathbf{F} = x\mathbf{i} - \sqrt{y}\,\mathbf{j}$ is conservative.

$$\text{Work} = \left[\frac{1}{2}x^2 - \frac{2}{3}y^{3/2}\right]_{(0,\,0)}^{(4,\,8)} = \frac{1}{2}(16) - \left(\frac{2}{3}\right)8^{3/2} = \frac{8}{3}\left(3 - 4\sqrt{2}\right)$$

41. $$\int_C 2xyz\,dx + x^2z\,dy + x^2y\,dz = \Big[x^2yz\Big]_{(0,\,0,\,0)}^{(1,\,4,\,3)} = 12$$

43. (a) $$\int_C y^2\,dx + 2xy\,dy = \int_0^1 [(1 + t)^2(3) + 2(1 + 3t)(1 + t)]\,dt$$

$$= \int_0^1 3(t^2 + 2t + 1) + 2(3t^2 + 4t + 1)]\,dt$$

$$= \int_0^1 (9t^2 + 14t + 5)\,dt$$

$$= \Big[3t^3 + 7t^2 + 5t\Big]_0^1 = 15$$

(b) $$\int_C y^2\,dx + 2xy\,dy = \int_1^4 \left[t(1) + 2(t)\left(\sqrt{t}\right)\frac{1}{2\sqrt{t}}\right] dt$$

$$= \int_1^4 (t + t)\,dt$$

$$= \Big[t^2\Big]_1^4 = 15$$

(c) $\mathbf{F}(x, y) = y^2\mathbf{i} + 2xy\,\mathbf{j} = \nabla f$ where $f(x, y) = xy^2$.

Hence,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 4(2)^2 - 1(1)^2 = 15$$

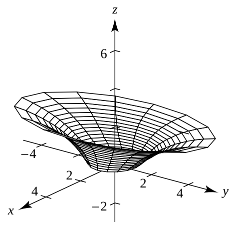
45. $$\int_C y\,dx + 2x\,dy = \int_0^2\int_0^2 (2 - 1)\,dy\,dx = \int_0^2 2\,dx = 4$$

47. $$\int_C xy^2\,dx + x^2y\,dy = \iint_R (2xy - 2xy)\,dA = 0$$

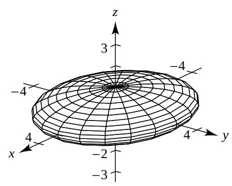
49. $\displaystyle\int_C xy\,dx + x^2\,dy = \int_0^1\int_{x^2}^{x} x\,dy\,dx = \int_0^1 (x^2 - x^3)\,dx = \frac{1}{12}$

51. $\mathbf{r}(u, v) = \sec u\cos v\,\mathbf{i} + (1 + 2\tan u)\sin v\,\mathbf{j} + 2u\,\mathbf{k}$

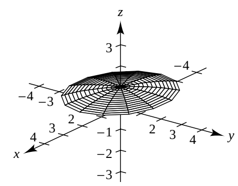
$0 \le u \le \dfrac{\pi}{3},\quad 0 \le v \le 2\pi$



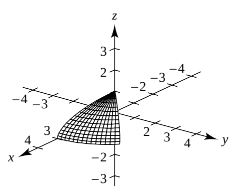
53. (a)



(b)



(c)



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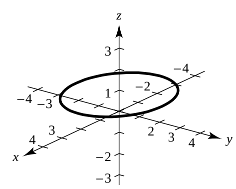


The space curve is a circle:

$$\mathbf{r}\left(u, \frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}\cos u\,\mathbf{i} + \frac{3\sqrt{2}}{2}\sin u\,\mathbf{j} + \frac{\sqrt{2}}{2}\mathbf{k}$$

(e) $\mathbf{r}_u = -3\cos v\sin u\,\mathbf{i} + 3\cos v\cos u\,\mathbf{j}$

$\mathbf{r}_v = -3\sin v\cos u\,\mathbf{i} - 3\sin v\sin u\,\mathbf{j} + \cos v\,\mathbf{k}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3\cos v\sin u & 3\cos v\cos u & 0 \\ -3\sin v\cos u & -3\sin v\sin u & \cos v \end{vmatrix}$$

$$= (3\cos^2 v\cos u)\mathbf{i} + (3\cos^2 v\sin u)\mathbf{j} + (9\cos v\sin v\sin^2 u + 9\cos v\sin v\cos^2 u)\mathbf{k}$$

$$= (3\cos^2 v\cos u)\mathbf{i} + (3\cos^2 v\sin u)\mathbf{j} + (9\cos v\sin v)\mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{9\cos^4 v\cos^2 u + 9\cos^4 v\sin^2 u + 81\cos^2 v\sin^2 v}$$

$$= \sqrt{9\cos^4 v + 81\cos^2 v\sin^2 v}$$

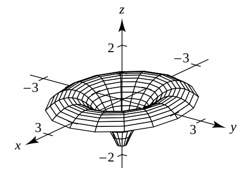
Using a Symbolic integration utility,

$$\int_{\pi/4}^{\pi/2}\int_0^{2\pi} \|\mathbf{r}_u \times \mathbf{r}_v\|\,du\,dv \approx 14.44$$

(f) Similarly,

$$\int_0^{\pi/4}\int_0^{\pi/2} \|\mathbf{r}_u \times \mathbf{r}_v\|\,dv\,du \approx 4.27$$

55. $S: \mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + (u - 1)(2 - u)\mathbf{k}, \quad 0 \le u \le 2, 0 \le v \le 2\pi$



$\mathbf{r}_u(u, v) = \cos v\,\mathbf{i} + \sin v\,\mathbf{j} + (3 - 2u)\mathbf{k}$

$\mathbf{r}_v(u, v) = -u\sin v\,\mathbf{i} + u\cos v\,\mathbf{j}$

$$\mathbf{r}_u \times \mathbf{r}_u = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 3 - 2u \\ -u\sin v & u\cos v & 0 \end{vmatrix} = (2u - 3)u\cos v\,\mathbf{i} + (2u - 3)u\sin v\,\mathbf{j} + u\,\mathbf{k}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = u\sqrt{(2u - 3)^2 + 1}$

$$\iint_S (x + y)\,dS = \int_0^{2\pi}\int_0^2 (u\cos v + u\sin v)\,u\sqrt{(2u - 3)^2 + 1}\,du\,dv$$

$$= \int_0^2\int_0^{2\pi} (\cos v + \sin v)u^2\sqrt{(2u - 3)^2 + 1}\,dv\,du = 0$$

57. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + xy\mathbf{j} + z\mathbf{k}$

Q: solid region bounded by the coordinates planes and the plane $2x + 3y + 4z = 12$

Surface Integral: There are four surfaces for this solid.

$z = 0 \quad \mathbf{N} = -\mathbf{k}, \quad \mathbf{F}\cdot\mathbf{N} = -z, \quad \iint_{S_1} 0\,dS = 0$

$y = 0, \quad \mathbf{N} = -\mathbf{j}, \quad \mathbf{F}\cdot\mathbf{N} = -xy, \quad \iint_{S_2} 0\,dS = 0$

$x = 0, \quad \mathbf{N} = -\mathbf{i}, \quad \mathbf{F}\cdot\mathbf{N} = -x^2, \quad \iint_{S_3} 0\,dS = 0$

$$2x + 3y + 4z = 12,\ \mathbf{N} = \frac{2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}}{\sqrt{29}},\ dS = \sqrt{1 + \left(\frac{1}{4}\right) + \left(\frac{9}{16}\right)}dA = \frac{\sqrt{29}}{4}\,dA$$

$$\iint_{S_4} \mathbf{F}\cdot\mathbf{N}\,dS = \frac{1}{4}\iint_R (2x^2 + 3xy + 4z)\,dA$$

$$= \frac{1}{4}\int_0^6\int_0^{4-(2x/3)} (2x^2 + 3xy + 12 - 2x - 3y)\,dy\,dx$$

$$= \frac{1}{4}\int_0^6\left[2x^2\left(\frac{12 - 2x}{3}\right) + \frac{3x}{2}\left(\frac{12 - 2x}{3}\right)^2 + 12\left(\frac{12 - 2x}{3}\right) - 2x\left(\frac{12 - 2x}{3}\right) - \frac{3}{2}\left(\frac{12 - 2x}{3}\right)^2\right]dx$$

$$= \frac{1}{6}\int_0^6 (-x^3 + x^2 + 24x + 36)\,dx = \frac{1}{6}\left[-\frac{x^4}{4} + \frac{x^3}{3} + 12x^2 + 36x\right]_0^6 = 66$$

Divergence Theorem: Since div $\mathbf{F} = 2x + x + 1 = 3x + 1$, Divergence Theorem yields

$$\iiint_Q \text{div}\,\mathbf{F}\,dV = \int_0^6\int_0^{(12-2x)/3}\int_0^{(12-2x-3y)/4} (3x + 1)\,dz\,dy\,dx$$

$$= \int_0^6\int_0^{(12-2x)/3} (3x + 1)\left(\frac{12 - 2x - 3y}{4}\right)dy\,dx$$

$$= \frac{1}{4}\int_0^6 (3x + 1)\left[12y - 2xy - \frac{3}{2}y^2\right]_0^{(12-2x)/3} dx$$

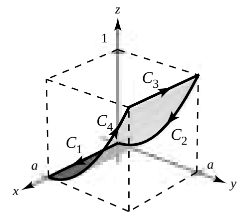
$$= \frac{1}{4}\int_0^6 (3x + 1)\left[4(12 - 2x) - 2x\left(\frac{12 - 2x}{3}\right) - \frac{3}{2}\left(\frac{12 - 2x}{3}\right)^2\right]dx$$

$$= \frac{1}{4}\int_0^6 \frac{2}{3}(3x^3 - 35x^2 + 96x + 36)\,dx = \frac{1}{6}\left[\frac{3x^4}{4} - \frac{35x^3}{3} + 48x^2 + 36x\right]_0^6 = 66.$$

59. $\mathbf{F}(x, y, z) = (\cos y + y \cos x)\mathbf{i} + (\sin x - x \sin y)\mathbf{j} + xyz\mathbf{k}$

S: portion of $z = y^2$ over the square in the xy-plane with vertices $(0, 0)$, $(a, 0)$, (a, a), $(0, a)$

Line Integral: Using the line integral we have:



C_1: $y = 0, \quad dy = 0$

C_2: $x = 0, \quad dx = 0, \quad z = y^2, \quad dz = 2y\,dy$

C_3: $y = a, \quad dy = 0, \quad z = a^2, \quad dz = 0$

C_4: $x = a, \quad dx = 0, \quad z = y^2, \quad dz = 2y\,dy$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (\cos y + y \cos x)\,dx + (\sin x - x \sin y)\,dy + xyz\,dz$$

$$= \int_{C_1} dx + \int_{C_2} 0 + \int_{C_3} (\cos a + a \cos x)\,dx + \int_{C_4} (\sin a - a \sin y)\,dy + ay^3(2y\,dy)$$

$$= \int_0^a dx + \int_a^0 (\cos a + a \cos x)\,dx + \int_0^a (\sin a - a \sin y)\,dy + \int_0^a 2ay^4\,dy$$

$$= a + \Big[x \cos a + a \sin x\Big]_a^0 + \Big[y \sin a + a \cos y\Big]_0^a + \Big[2a\frac{y^5}{5}\Big]_0^a$$

$$= a - a \cos a - a \sin a + a \sin a + a \cos a - a + \frac{2a^6}{5} = \frac{2a^6}{5}$$

Double Integral: Considering $f(x, y, z) = z - y^2$, we have:

$$\mathbf{N} = \frac{\nabla f}{\|\nabla f\|} = \frac{-2y\mathbf{j} + \mathbf{k}}{\sqrt{1 + 4y^2}}, \; dS = \sqrt{1 + 4y^2}\,dA, \text{ and } \mathbf{curl\ F} = xz\mathbf{i} - yz\mathbf{j}.$$

Hence,

$$\int\!\!\int_S (\mathbf{curl\ F}) \cdot \mathbf{N}\,dS = \int_0^a\!\int_0^a 2y^2 z\,dy\,dx = \int_0^a\!\int_0^a 2y^4\,dy\,dx = \int_0^a \frac{2a^5}{5}\,dx = \frac{2a^6}{5}.$$

Problem Solving for Chapter 14

1. (a) $\nabla T = \dfrac{-25}{(x^2 + y^2 + z^2)^{3/2}}[x\mathbf{i} + y\mathbf{i} + z\mathbf{k}]$

$$\mathbf{N} = x\mathbf{i} + \sqrt{1 - x^2}\,\mathbf{k}$$

$$dS = \frac{1}{\sqrt{1 - x^2}}\,dy\,dx$$

$$\text{Flux} = \int\!\!\int_S -k\nabla T \cdot N\,dS$$

$$= 25k\int\!\!\int_R \left[\frac{x^2}{(x^2 + y^2 + z^2)^{3/2}(1 - x^2)^{1/2}} + \frac{z}{(x^2 + y^2 + z^2)^{3/2}}\right] dA$$

$$= 25k\int_{-1/2}^{1/2}\!\int_0^1 \left[\frac{x^2}{(x^2 + y^2 + z^2)^{3/2}(1 - x^2)^{1/2}} + \frac{1 - x^2}{(x^2 + y^2 + z^2)^{3/2}(1 - x^2)^{1/2}}\right] dy\,dx$$

$$= 25k\int_{-1/2}^{1/2}\!\int_0^1 \frac{1}{(1 + y^2)^{3/2}(1 - x^2)^{1/2}}\,dy\,dx$$

$$= 25k\int_0^1 \frac{1}{(1 + y^2)^{3/2}}\,dy \int_{-1/2}^{1/2} \frac{1}{(1 - x^2)^{1/2}}\,dx$$

$$= 25k\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\pi}{3}\right) = 25k\frac{\sqrt{2}\pi}{6}$$

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(b) $\mathbf{r}(u, v) = \langle \cos u, v, \sin u \rangle$

$\mathbf{r}_u = \langle -\sin u, 0, \cos u \rangle$, $\mathbf{r}_v = \langle 0, 1, 0 \rangle$

$\mathbf{r}_u \times \mathbf{r}_v = \langle -\cos u, 0, -\sin u \rangle$

$$\nabla T = \frac{-25}{(x^2 + y^2 + z^2)^{3/2}}[x\mathbf{i} + y\mathbf{j} + z\mathbf{k}]$$

$$= \frac{-25}{(v^2 + 1)^{3/2}}[\cos u\mathbf{i} + v\mathbf{j} + \sin u\mathbf{k}]$$

$$\nabla T \cdot (\mathbf{r}_u \times \mathbf{r}_v) = \frac{-25}{(v^2 + 1)^{3/2}}(-\cos^2 u - \sin^2 u) = \frac{25}{(v^2 + 1)^{3/2}}$$

$$\text{Flux} = \int_0^1 \int_{\pi/3}^{2\pi/3} \frac{25k}{(v^2 + 1)^{3/2}}\, du\, dv = 25k\frac{\sqrt{2}\pi}{6}$$

3. $\mathbf{r}(t) = \langle 3\cos t, 3\sin t, 2t \rangle$

$\mathbf{r}'(t) = \langle -3\sin t, 3\cos t, 2 \rangle$, $\|\mathbf{r}'(t)\| = \sqrt{13}$

$$I_x = \int_C (y^2 + z^2)\rho\, ds = \int_0^{2\pi} (9\sin^2 t + 4t^2)\sqrt{13}\, dt = \frac{1}{3}\sqrt{13}\pi(32\pi^2 + 27)$$

$$I_y = \int_C (x^2 + z^2)\rho\, ds = \int_0^{2\pi} (9\cos^2 t + 4t^2)\sqrt{13}\, dt = \frac{1}{3}\sqrt{13}\pi(32\pi^2 + 27)$$

$$I_z = \int_C (x^2 + y^2)\rho\, ds = \int_0^{2\pi} (9\cos^2 t + 9\sin^2 t)\sqrt{13}\, dt = 18\pi\sqrt{13}$$

5. $$\frac{1}{2}\int_C x\, dy - y\, dx = \frac{1}{2}\int_0^{2\pi} [a(\theta - \sin\theta)(a\sin\theta)\, d\theta - a(1 - \cos\theta)(a(1 - \cos\theta))\, d\theta]$$

$$= \frac{1}{2}a^2\int_0^{2\pi} [\theta\sin\theta - \sin^2\theta - 1 + 2\cos\theta - \cos^2\theta]\, d\theta$$

$$= \frac{1}{2}a^2\int_0^{2\pi} (\theta\sin\theta + 2\cos\theta - 2)\, d\theta$$

$$= -3\pi a^2$$

Hence, the area is $3\pi a^2$.

7. (a) $\mathbf{r}(t) = t\mathbf{j}$, $0 \le t \le 1$

$\mathbf{r}'(t) = \mathbf{j}$

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (t\mathbf{i} + \mathbf{j}) \cdot \mathbf{j}\, dt = \int_0^1 dt = 1$$

(b) $\mathbf{r}(t) = (t - t^2)\mathbf{i} + t\mathbf{j}$, $0 \le t \le 1$

$\mathbf{r}'(t) = (1 - 2t)\mathbf{i} + \mathbf{j}$

$$W = \mathbf{F} \cdot d\mathbf{r} = \int_0^1 ((2t - t^2)\mathbf{i} + [(t - t^2)^2 + 1]\mathbf{j}) \cdot ((1 - 2t)\mathbf{i} + \mathbf{j})\, dt$$

$$= \int_0^1 [(1 - 2t)(2t - t^2) + (t^4 - 2t^3 + t^2 + 1)]\, dt$$

$$= \int_0^1 (t^4 - 4t^2 + 2t + 1)\, dt = \frac{13}{15}$$

—CONTINUED—

7. —CONTINUED—

(c) $\mathbf{r}(t) = c(t - t^2)\mathbf{i} + t\mathbf{j},\ 0 \le t \le 1$

$\mathbf{r}'(t) = c(1 - 2t)\mathbf{i} + \mathbf{j}$

$$\mathbf{F} \cdot d\mathbf{r} = (c(t - t^2) + t)(c(1 - 2t)) + (c^2(t - t^2)^2 + 1)(1)$$
$$= c^2t^4 - 2c^2t^2 + c^2t - 2ct^2 + ct + 1$$

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{30}c^2 - \frac{1}{6}c + 1$$

$$\frac{dW}{dc} = \frac{1}{15}c - \frac{1}{6} = 0 \implies c = \frac{5}{2}$$

$$\frac{d^2W}{dc^2} = \frac{1}{15} > 0 \quad c = \frac{5}{2} \text{ minimum.}$$

9. $\mathbf{v} \times \mathbf{r} = \langle a_1, a_2, a_3 \rangle \times \langle x, y, z \rangle$

$$= \langle a_2 z - a_3 y, -a_1 z + a_3 x, a_1 y - a_2 x \rangle$$

$\mathbf{curl}(\mathbf{v} \times \mathbf{r}) = \langle 2a_1, 2a_2, 2a_3 \rangle = 2\mathbf{v}$

By Stoke's Theorem,

$$\int_C (\mathbf{v} \times \mathbf{r})\, d\mathbf{r} = \int_S\int \mathbf{curl}(\mathbf{v} \times \mathbf{r}) \cdot \mathbf{N}\, dS$$
$$= \int_S\int 2\mathbf{v} \cdot N\, dS.$$

11. $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j} = \dfrac{m}{(x^2 + y^2)^{5/2}}[3xy\mathbf{i} + (2y^2 - x^2)\mathbf{j}]$

$$M = \frac{3mxy}{(x^2 + y^2)^{5/2}} = 3mxy(x^2 + y^2)^{-5/2}$$

$$\frac{\partial M}{\partial y} = 3mxy\left[-\frac{5}{2}(x^2 + y^2)^{-7/2}(2y)\right] + (x^2 + y^2)^{-5/2}(3mx)$$
$$= 3mx(x^2 + y^2)^{-7/2}[-5y^2 + (x^2 + y^2)] = \frac{3mx(x^2 - 4y^2)}{(x^2 + y^2)^{7/2}}$$

$$N = \frac{m(2y^2 - x^2)}{(x^2 + y^2)^{5/2}} = m(2y^2 - x^2)(x^2 + y^2)^{-5/2}$$

$$\frac{\partial N}{\partial x} = m(2y^2 - x^2)\left[-\frac{5}{2}(x^2 + y^2)^{-7/2}(2x)\right] + (x^2 + y^2)^{-5/2}(-2mx)$$
$$= mx(x^2 + y^2)^{-7/2}[(2y^2 - x^2)(-5) + (x^2 + y^2)(-2)]$$
$$= mx(x^2 + y^2)^{-7/2}(3x^2 - 12y^2) = \frac{3mx(x^2 - 4y^2)}{(x^2 + y^2)^{7/2}}$$

Therefore, $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$ and $\mathbf{F}$ is conservative.