

# CHAPTER 2

## Differentiation

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# CHAPTER 2

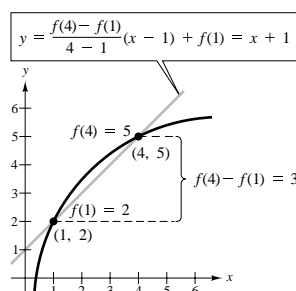
## Differentiation

### Section 2.1 The Derivative and the Tangent Line Problem

Solutions to Odd-Numbered Exercises

1. (a)  $m = 0$   
 (b)  $m = -3$

3. (a), (b)



$$\begin{aligned} \text{(c) } y &= \frac{f(4) - f(1)}{4 - 1}(x - 1) + f(1) \\ &= \frac{3}{3}(x - 1) + 2 \\ &= 1(x - 1) + 2 \\ &= x + 1 \end{aligned}$$

5.  $f(x) = 3 - 2x$  is a line. Slope =  $-2$

$$\begin{aligned} \text{7. Slope at } (1, -3) &= \lim_{\Delta x \rightarrow 0} \frac{g(1 + \Delta x) - g(1)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(1 + \Delta x)^2 - 4 - (-3)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{1 + 2(\Delta x) + (\Delta x)^2 - 1}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} [2 + 2(\Delta x)] = 2 \end{aligned}$$

$$\begin{aligned} \text{9. Slope at } (0, 0) &= \lim_{\Delta t \rightarrow 0} \frac{f(0 + \Delta t) - f(0)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{3(\Delta t) - (\Delta t)^2 - 0}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} (3 - \Delta t) = 3 \end{aligned}$$

$$\begin{aligned} \text{11. } f(x) &= 3 \\ f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{3 - 3}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} 0 = 0 \end{aligned}$$

13.  $f(x) = -5x$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-5(x + \Delta x) - (-5x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} -5 = -5 \end{aligned}$$

15.  $h(s) = 3 + \frac{2}{3}s$

$$\begin{aligned} h'(s) &= \lim_{\Delta s \rightarrow 0} \frac{h(s + \Delta s) - h(s)}{\Delta s} \\ &= \lim_{\Delta s \rightarrow 0} \frac{3 + \frac{2}{3}(s + \Delta s) - \left(3 + \frac{2}{3}s\right)}{\Delta s} \\ &= \lim_{\Delta s \rightarrow 0} \frac{\frac{2}{3}\Delta s}{\Delta s} = \frac{2}{3} \end{aligned}$$

17.  $f(x) = 2x^2 + x - 1$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{[2(x + \Delta x)^2 + (x + \Delta x) - 1] - [2x^2 + x - 1]}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(2x^2 + 4x\Delta x + 2(\Delta x)^2 + x + \Delta x - 1) - (2x^2 + x - 1)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{4x\Delta x + 2(\Delta x)^2 + \Delta x}{\Delta x} = \lim_{\Delta x \rightarrow 0} (4x + 2\Delta x + 1) = 4x + 1
 \end{aligned}$$

19.  $f(x) = x^3 - 12x$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^3 - 12(x + \Delta x)] - [x^3 - 12x]}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 12x - 12\Delta x - x^3 + 12x}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 12\Delta x}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} (3x^2 + 3x\Delta x + (\Delta x)^2 - 12) = 3x^2 - 12
 \end{aligned}$$

21.  $f(x) = \frac{1}{x - 1}$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{x + \Delta x - 1} - \frac{1}{x - 1}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(x - 1) - (x + \Delta x - 1)}{\Delta x(x + \Delta x - 1)(x - 1)} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{-\Delta x}{\Delta x(x + \Delta x - 1)(x - 1)} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{-1}{(x + \Delta x - 1)(x - 1)} \\
 &= -\frac{1}{(x - 1)^2}
 \end{aligned}$$

23.  $f(x) = \sqrt{x + 1}$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x + 1} - \sqrt{x + 1}}{\Delta x} \cdot \left( \frac{\sqrt{x + \Delta x + 1} + \sqrt{x + 1}}{\sqrt{x + \Delta x + 1} + \sqrt{x + 1}} \right) \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x + 1) - (x + 1)}{\Delta x[\sqrt{x + \Delta x + 1} + \sqrt{x + 1}]} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x + 1} + \sqrt{x + 1}} \\
 &= \frac{1}{\sqrt{x + 1} + \sqrt{x + 1}} = \frac{1}{2\sqrt{x + 1}}
 \end{aligned}$$

25. (a)  $f(x) = x^2 + 1$

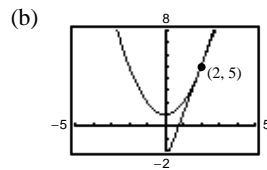
$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^2 + 1] - [x^2 + 1]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (2x + \Delta x) = 2x \end{aligned}$$

At  $(2, 5)$ , the slope of the tangent line is  $m = 2(2) = 4$ . The equation of the tangent line is

$$y - 5 = 4(x - 2)$$

$$y - 5 = 4x - 8$$

$$y = 4x - 3.$$



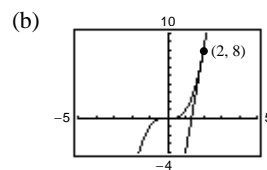
27. (a)  $f(x) = x^3$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (3x^2 + 3x\Delta x + (\Delta x)^2) = 3x^2 \end{aligned}$$

At  $(2, 8)$ , the slope of the tangent is  $m = 3(2)^2 = 12$ . The equation of the tangent line is

$$y - 8 = 12(x - 2)$$

$$y = 12x - 16.$$



29. (a)  $f(x) = \sqrt{x}$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x) - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

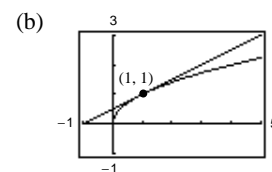
At  $(1, 1)$ , the slope of the tangent line is

$$m = \frac{1}{2\sqrt{1}} = \frac{1}{2}.$$

The equation of the tangent line is

$$y - 1 = \frac{1}{2}(x - 1)$$

$$y = \frac{1}{2}x + \frac{1}{2}.$$



31. (a)  $f(x) = \frac{4}{x}$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x) + \frac{4}{x + \Delta x} - \left(x + \frac{4}{x}\right)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x(x + \Delta x)(x + \Delta x) + 4x - x^2(x + \Delta x) - 4(x + \Delta x)}{x(\Delta x)(x + \Delta x)} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^3 + 2x^2(\Delta x) + x(\Delta x)^2 - x^3 - x^2(\Delta x) - 4(\Delta x)}{x(\Delta x)(x + \Delta x)} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^2(\Delta x) + x(\Delta x)^2 - 4(\Delta x)}{x(\Delta x)(x + \Delta x)} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^2 + x(\Delta x) - 4}{x(x + \Delta x)} \\ &= \frac{x^2 - 4}{x^2} = 1 - \frac{4}{x^2} \end{aligned}$$

At (4, 5), the slope of the tangent line is

$$m = 1 - \frac{4}{16} = \frac{3}{4}$$

The equation of the tangent line is

$$y - 5 = \frac{3}{4}(x - 4)$$

$$y = \frac{3}{4}x + 2$$

33. From Exercise 27 we know that  $f'(x) = 3x^2$ . Since the slope of the given line is 3, we have

$$3x^2 = 3$$

$$x = \pm 1.$$

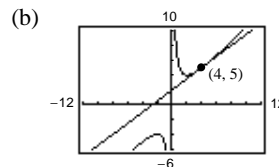
Therefore, at the points (1, 1) and (-1, -1) the tangent lines are parallel to  $3x - y + 1 = 0$ . These lines have equations

$$y - 1 = 3(x - 1) \quad \text{and} \quad y + 1 = 3(x + 1)$$

$$y = 3x - 2 \quad \quad \quad y = 3x + 2.$$

37.  $g(5) = 2$  because the tangent line passes through (5, 2)

$$g'(5) = \frac{2 - 0}{5 - 9} = \frac{2}{-4} = -\frac{1}{2}$$



35. Using the limit definition of derivative,

$$f'(x) = \frac{-1}{2x\sqrt{x}}.$$

Since the slope of the given line is  $-\frac{1}{2}$ , we have

$$-\frac{1}{2x\sqrt{x}} = -\frac{1}{2}$$

$$x = 1.$$

Therefore, at the point (1, 1) the tangent line is parallel to  $x + 2y - 6 = 0$ . The equation of this line is

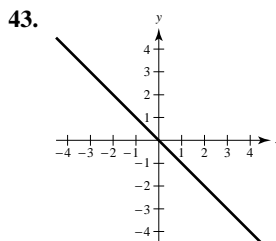
$$y - 1 = -\frac{1}{2}(x - 1)$$

$$y - 1 = -\frac{1}{2}x + \frac{1}{2}$$

$$y = -\frac{1}{2}x + \frac{3}{2}.$$

39.  $f(x) = x \Rightarrow f'(x) = 1$  (b)

41.  $f(x) = \sqrt{x} \Rightarrow f'(x)$  matches (a)  
(decreasing slope as  $x \rightarrow \infty$ )



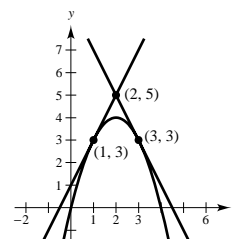
Answers will vary.

Sample answer:  $y = -x$

45. (a) If  $f'(c) = 3$  and  $f$  is odd, then  $f'(-c) = f'(c) = 3$   
(b) If  $f'(c) = 3$  and  $f$  is even, then  $f'(-c) = -f'(c) = -3$

47. Let  $(x_0, y_0)$  be a point of tangency on the graph of  $f$ . By the limit definition for the derivative,  $f'(x) = 4 - 2x$ . The slope of the line through  $(2, 5)$  and  $(x_0, y_0)$  equals the derivative of  $f$  at  $x_0$ :

$$\begin{aligned} \frac{5 - y_0}{2 - x_0} &= 4 - 2x_0 \\ 5 - y_0 &= (2 - x_0)(4 - 2x_0) \\ 5 - (4x_0 - x_0^2) &= 8 - 8x_0 + 2x_0^2 \\ 0 &= x_0^2 - 4x_0 + 3 \\ 0 &= (x_0 - 1)(x_0 - 3) \Rightarrow x_0 = 1, 3 \end{aligned}$$



Therefore, the points of tangency are  $(1, 3)$  and  $(3, 3)$ , and the corresponding slopes are 2 and  $-2$ . The equations of the tangent lines are

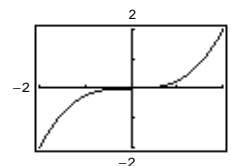
$$\begin{aligned} y - 5 &= 2(x - 2) & y - 5 &= -2(x - 2) \\ y &= 2x + 1 & y &= -2x + 9 \end{aligned}$$

49. (a)  $g'(0) = -3$   
(b)  $g'(3) = 0$   
(c) Because  $g'(1) = -\frac{8}{3}$ ,  $g$  is decreasing (falling) at  $x = 1$ .  
(d) Because  $g'(-4) = \frac{7}{3}$ ,  $g$  is increasing (rising) at  $x = -4$ .  
(e) Because  $g'(4)$  and  $g'(6)$  are both positive,  $g(6)$  is greater than  $g(4)$ , and  $g(6) - g(4) > 0$ .  
(f) No, it is not possible. All you can say is that  $g$  is decreasing (falling) at  $x = 2$ .

51.  $f(x) = \frac{1}{4}x^3$

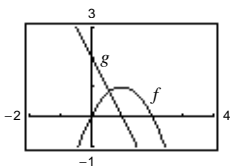
By the limit definition of the derivative we have  $f'(x) = \frac{3}{4}x^2$ .

$x$	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$f(x)$	-2	$-\frac{27}{32}$	$-\frac{1}{4}$	$-\frac{1}{32}$	0	$\frac{1}{32}$	$\frac{1}{4}$	$\frac{27}{32}$	2
$f'(x)$	3	$\frac{27}{16}$	$\frac{3}{4}$	$\frac{3}{16}$	0	$\frac{3}{16}$	$\frac{3}{4}$	$\frac{27}{16}$	3



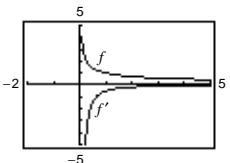
$$53. g(x) = \frac{f(x + 0.01) - f(x)}{0.01}$$

$$= (2(x + 0.01) - (x + 0.01)^2 - 2x + x^2) + 100$$



The graph of  $g(x)$  is approximately the graph of  $f'(x)$ .

$$57. f(x) = \frac{1}{\sqrt{x}} \text{ and } f'(x) = \frac{-1}{2x^{3/2}}.$$



As  $x \rightarrow \infty$ ,  $f$  is nearly horizontal and thus  $f' \approx 0$ .

$$59. f(x) = 4 - (x - 3)^2$$

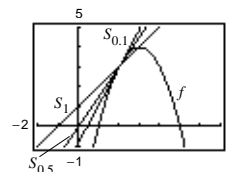
$$S_{\Delta x}(x) = \frac{f(2 + \Delta x) - f(2)}{\Delta x}(x - 2) + f(2)$$

$$= \frac{4 - (2 + \Delta x - 3)^2 - 3}{\Delta x}(x - 2) + 3 = \frac{1 - (\Delta x - 1)^2}{\Delta x}(x - 2) + 3 = (-\Delta x + 2)(x - 2) + 3$$

$$(a) \Delta x = 1: S_{\Delta x} = (x - 2) + 3 = x + 1$$

$$\Delta x = 0.5: S_{\Delta x} = \left(\frac{3}{2}\right)(x - 2) + 3 = \frac{3}{2}x$$

$$\Delta x = 0.1: S_{\Delta x} = \left(\frac{19}{10}\right)(x - 2) + 3 = \frac{19}{10}x - \frac{4}{5}$$



(b) As  $\Delta x \rightarrow 0$ , the line approaches the tangent line to  $f$  at  $(2, 3)$ .

$$61. f(x) = x^2 - 1, c = 2$$

$$f'(2) = \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2} \frac{(x^2 - 1) - 3}{x - 2} = \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \rightarrow 2} (x + 2) = 4$$

$$63. f(x) = x^3 + 2x^2 + 1, c = -2$$

$$f'(-2) = \lim_{x \rightarrow -2} \frac{f(x) - f(-2)}{x - (-2)} = \lim_{x \rightarrow -2} \frac{x^2(x + 2)}{x + 2} = \lim_{x \rightarrow -2} \frac{(x^3 + 2x^2 + 1) - 1}{x + 2} = \lim_{x \rightarrow -2} x^2 = 4$$

$$65. g(x) = \sqrt{|x|}, c = 0$$

$$g'(0) = \lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{\sqrt{|x|}}{x}. \text{ Does not exist.}$$

$$\text{As } x \rightarrow 0^-, \frac{\sqrt{|x|}}{x} = \frac{-1}{\sqrt{x}} \rightarrow -\infty$$

$$\text{As } x \rightarrow 0^+, \frac{\sqrt{|x|}}{x} = \frac{1}{\sqrt{x}} \rightarrow \infty$$

$$55. f(2) = 2(4 - 2) = 4, f(2.1) = 2.1(4 - 2.1) = 3.99$$

$$f'(2) \approx \frac{3.99 - 4}{2.1 - 2} = -0.1 \text{ [Exact: } f'(2) = 0 \text{]}$$

$$67. f(x) = (x - 6)^{2/3}, c = 6$$

$$f'(6) = \lim_{x \rightarrow 6} \frac{f(x) - f(6)}{x - 6}$$

$$= \lim_{x \rightarrow 6} \frac{(x - 6)^{2/3} - 0}{x - 6}$$

$$= \lim_{x \rightarrow 6} \frac{1}{(x - 6)^{1/3}}$$

Does not exist.

69.  $h(x) = (x + 5)$ ,  $c = -5$

$$\begin{aligned} h'(-5) &= \lim_{x \rightarrow -5} \frac{h(x) - h(-5)}{x - (-5)} \\ &= \lim_{x \rightarrow -5} \frac{|x + 5| - 0}{x + 5} \\ &= \lim_{x \rightarrow -5} \frac{|x + 5|}{x + 5} \end{aligned}$$

Does not exist.

73.  $f(x)$  is differentiable everywhere except at  $x = -1$ .  
(Discontinuity)

77.  $f(x)$  is differentiable on the interval  $(1, \infty)$ .  
(At  $x = 1$  the tangent line is vertical)

81.  $f(x) = |x - 1|$

The derivative from the left is

$$\lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^-} \frac{|x - 1| - 0}{x - 1} = -1.$$

The derivative from the right is

$$\lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^+} \frac{|x - 1| - 0}{x - 1} = 1.$$

The one-sided limits are not equal. Therefore,  $f$  is not differentiable at  $x = 1$ .

71.  $f(x)$  is differentiable everywhere except at  $x = -3$ .  
(Sharp turn in the graph.)

75.  $f(x)$  is differentiable everywhere except at  $x = 3$ .  
(Sharp turn in the graph)

79.  $f(x)$  is differentiable everywhere except at  $x = 0$ .  
(Discontinuity)

83.  $f(x) = \begin{cases} (x - 1)^3, & x \leq 1 \\ (x - 1)^2, & x > 1 \end{cases}$

The derivative from the left is

$$\begin{aligned} \lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} &= \lim_{x \rightarrow 1^-} \frac{(x - 1)^3 - 0}{x - 1} \\ &= \lim_{x \rightarrow 1^-} (x - 1)^2 = 0. \end{aligned}$$

The derivative from the right is

$$\begin{aligned} \lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} &= \lim_{x \rightarrow 1^+} \frac{(x - 1)^2 - 0}{x - 1} \\ &= \lim_{x \rightarrow 1^+} (x - 1) = 0. \end{aligned}$$

These one-sided limits are equal. Therefore,  $f$  is differentiable at  $x = 1$ . ( $f'(1) = 0$ )

85. Note that  $f$  is continuous at  $x = 2$ .  $f(x) = \begin{cases} x^2 + 1, & x \leq 2 \\ 4x - 3, & x > 2 \end{cases}$

The derivative from the left is  $\lim_{x \rightarrow 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^-} \frac{(x^2 + 1) - 5}{x - 2} = \lim_{x \rightarrow 2^-} (x + 2) = 4$ .

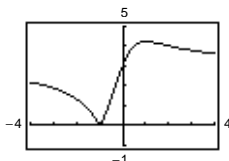
The derivative from the right is  $\lim_{x \rightarrow 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^+} \frac{(4x - 3) - 5}{x - 2} = \lim_{x \rightarrow 2^+} 4 = 4$ .

The one-sided limits are equal. Therefore,  $f$  is differentiable at  $x = 2$ . ( $f'(2) = 4$ )

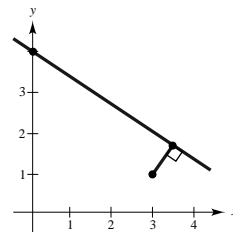
87. (a) The distance from  $(3, 1)$  to the line  $mx - y + 4 = 0$  is

$$\begin{aligned} d &= \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} \\ &= \frac{|m(3) - 1(1) + 4|}{\sqrt{m^2 + 1}} = \frac{|3m + 3|}{\sqrt{m^2 + 1}} \end{aligned}$$

(b)



The function  $d$  is not differentiable at  $m = -1$ . This corresponds to the line  $y = -x + 4$ , which passes through the point  $(3, 1)$ .





89. False. the slope is  $\lim_{\Delta x \rightarrow 0} \frac{f(2 + \Delta x) - f(2)}{\Delta x}$ .

91. False. If the derivative from the left of a point does not equal the derivative from the right of a point, then the derivative does not exist at that point. For example, if  $f(x) = |x|$ , then the derivative from the left at  $x = 0$  is  $-1$  and the derivative from the right at  $x = 0$  is  $1$ . At  $x = 0$ , the derivative does not exist.

$$93. f(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Using the Squeeze Theorem, we have  $-|x| \leq x \sin(1/x) \leq |x|$ ,  $x \neq 0$ . Thus,  $\lim_{x \rightarrow 0} x \sin(1/x) = 0 = f(0)$  and  $f$  is continuous at  $x = 0$ . Using the alternative form of the derivative we have

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x \sin(1/x) - 0}{x - 0} = \lim_{x \rightarrow 0} \left( \sin \frac{1}{x} \right).$$

Since this limit does not exist (it oscillates between  $-1$  and  $1$ ), the function is not differentiable at  $x = 0$ .

$$g(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Using the Squeeze Theorem again we have  $-x^2 \leq x^2 \sin(1/x) \leq x^2$ ,  $x \neq 0$ . Thus,  $\lim_{x \rightarrow 0} x^2 \sin(1/x) = 0 = f(0)$  and  $f$  is continuous at  $x = 0$ . Using the alternative form of the derivative again we have

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x^2 \sin(1/x) - 0}{x - 0} = \lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0.$$

Therefore,  $g$  is differentiable at  $x = 0$ ,  $g'(0) = 0$ .

## Section 2.2 Basic Differentiation Rules and Rates of Change

1. (a)  $y = x^{1/2}$   
 $y' = \frac{1}{2}x^{-1/2}$   
 $y'(1) = \frac{1}{2}$

(b)  $y = x^{3/2}$   
 $y' = \frac{3}{2}x^{1/2}$   
 $y'(1) = \frac{3}{2}$

(c)  $y = x^2$   
 $y' = 2x$   
 $y'(1) = 2$

(d)  $y = x^3$   
 $y' = 3x^2$   
 $y'(1) = 3$

3.  $y = 8$   
 $y' = 0$

5.  $y = x^6$   
 $y' = 6x^5$

7.  $y = \frac{1}{x^7} = x^{-7}$   
 $y' = -7x^{-8} = \frac{-7}{x^8}$

9.  $y = \sqrt[5]{x} = x^{1/5}$   
 $y' = \frac{1}{5}x^{-4/5} = \frac{1}{5x^{4/5}}$

11.  $f(x) = x + 1$   
 $f'(x) = 1$

13.  $f(t) = -2t^2 + 3t - 6$   
 $f'(t) = -4t + 3$

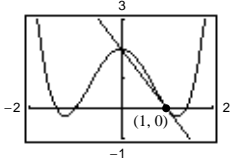
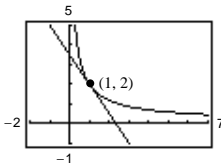
15.  $g(x) = x^2 + 4x^3$   
 $g'(x) = 2x + 12x^2$

17.  $s(t) = t^3 - 2t + 4$   
 $s'(t) = 3t^2 - 2$

19.  $y = \frac{\pi}{2} \sin \theta - \cos \theta$   
 $y' = \frac{\pi}{2} \cos \theta + \sin \theta$

21.  $y = x^2 - \frac{1}{2} \cos x$   
 $y' = 2x + \frac{1}{2} \sin x$

23.  $y = \frac{1}{x} - 3 \sin x$   
 $y' = -\frac{1}{x^2} - 3 \cos x$

- | <u>Function</u>              | <u>Rewrite</u>          | <u>Derivative</u>           | <u>Simplify</u>            |
|------------------------------|-------------------------|-----------------------------|----------------------------|
| 25. $y = \frac{5}{2x^2}$     | $y = \frac{5}{2}x^{-2}$ | $y' = -5x^{-3}$             | $y' = \frac{-5}{x^3}$      |
| 27. $y = \frac{3}{(2x)^3}$   | $y = \frac{3}{8}x^{-3}$ | $y' = \frac{-9}{8}x^{-4}$   | $y' = \frac{-9}{8x^4}$     |
| 29. $y = \frac{\sqrt{x}}{x}$ | $y = x^{-1/2}$          | $y' = -\frac{1}{2}x^{-3/2}$ | $y' = -\frac{1}{2x^{3/2}}$ |
- 
- |   |  |  |
|---|--|--|
| <p>31. <math>f(x) = \frac{3}{x^2} = 3x^{-2}, (1, 3)</math></p> <p><math>f'(x) = -6x^{-3} = \frac{-6}{x^3}</math></p> <p><math>f'(1) = -6</math></p>   | <p>33. <math>f(x) = -\frac{1}{2} + \frac{7}{5}x^3, \left(0, -\frac{1}{2}\right)</math></p> <p><math>f'(x) = \frac{21}{5}x^2</math></p> <p><math>f'(0) = 0</math></p>   | <p>35. <math>y = (2x + 1)^2, (0, 1)</math></p> <p><math>= 4x^2 + 4x + 1</math></p> <p><math>y' = 8x + 4</math></p> <p><math>y'(0) = 4</math></p> |
| <p>37. <math>f(\theta) = 4 \sin \theta - \theta, (0, 0)</math></p> <p><math>f'(\theta) = 4 \cos \theta - 1</math></p> <p><math>f'(0) = 4(1) - 1 = 3</math></p>  | <p>39. <math>f(x) = x^2 + 5 - 3x^{-2}</math></p> <p><math>f'(x) = 2x + 6x^{-3} = 2x + \frac{6}{x^3}</math></p>   | <p>41. <math>g(t) = t^2 - \frac{4}{t^3} = t^2 - 4t^{-3}</math></p> <p><math>g'(t) = 2t + 12t^{-4} = 2t + \frac{12}{t^4}</math></p>               |
| <p>43. <math>f(x) = \frac{x^3 - 3x^2 + 4}{x^2} = x - 3 + 4x^{-2}</math></p> <p><math>f'(x) = 1 - \frac{8}{x^3} = \frac{x^3 - 8}{x^3}</math></p>   | <p>45. <math>y = x(x^2 + 1) = x^3 + x</math></p> <p><math>y' = 3x^2 + 1</math></p>   |  |
| <p>47. <math>f(x) = \sqrt{x} - 6\sqrt[3]{x} = x^{1/2} - 6x^{1/3}</math></p> <p><math>f'(x) = \frac{1}{2}x^{-1/2} + 2x^{-2/3} = \frac{1}{2\sqrt{x}} + \frac{2}{x^{2/3}}</math></p>   | <p>49. <math>h(s) = s^{4/5} - s^{2/3}</math></p> <p><math>h'(s) = \frac{4}{5}s^{-4/5} - \frac{2}{3}s^{-1/3} = \frac{4}{5s^{4/5}} - \frac{2}{3s^{1/3}}</math></p>   |  |
| <p>51. <math>f(x) = 6\sqrt{x} + 5 \cos x = 6x^{1/2} + 5 \cos x</math></p> <p><math>f'(x) = 3x^{-1/2} - 5 \sin x = \frac{3}{\sqrt{x}} - 5 \sin x</math></p>  |  |  |
| <p>53. (a) <math>y = x^4 - 3x^2 + 2</math></p> <p><math>y' = 4x^3 - 6x</math></p> <p>At <math>(1, 0)</math>: <math>y' = 4(1)^3 - 6(1) = -2</math>.</p> <p>Tangent line: <math>y - 0 = -2(x - 1)</math></p> <p style="text-align: center;"><math>2x + y - 2 = 0</math></p> | <p>55. (a) <math>f(x) = \frac{2}{\sqrt[4]{x^3}} = 2x^{-3/4}</math></p> <p><math>f'(x) = \frac{-3}{2}x^{-7/4} = \frac{-3}{2x^{7/4}}</math></p> <p>At <math>(1, 2)</math>, <math>f'(1) = \frac{-3}{2}</math></p> <p>Tangent line: <math>y - 2 = -\frac{3}{2}(x - 1)</math></p> <p style="text-align: center;"><math>y = -\frac{3}{2}x + \frac{7}{2}</math></p> <p style="text-align: center;"><math>3x + 2y - 7 = 0</math></p> |  |
| <p>(b)</p>   | <p>(b)</p>   |  |

57.  $y = x^4 - 8x^2 + 2$

$$y' = 4x^3 - 16x$$

$$= 4x(x^2 - 4)$$

$$= 4x(x - 2)(x + 2)$$

$$y' = 0 \Rightarrow x = 0, \pm 2$$

Horizontal tangents:  $(0, 2)$ ,  $(2, -14)$ ,  $(-2, -14)$

61.  $y = x + \sin x$ ,  $0 \leq x < 2\pi$

$$y' = 1 + \cos x = 0$$

$$\cos x = -1 \Rightarrow x = \pi$$

At  $x = \pi$ ,  $y = \pi$ .

Horizontal tangent:  $(\pi, \pi)$

65.  $\frac{k}{x} = -\frac{3}{4}x + 3$  Equate functions

$$-\frac{k}{x^2} = -\frac{3}{4}$$
 Equate derivatives

$$\text{Hence, } k = \frac{3}{4}x^2 \text{ and } \frac{\frac{3}{4}x^2}{x} = -\frac{3}{4}x + 3 \Rightarrow \frac{3}{4}x = -\frac{3}{4}x + 3 \Rightarrow \frac{3}{2}x = 3 \Rightarrow x = 2 \Rightarrow k = 3.$$

67. (a) The slope appears to be steepest between  $A$  and  $B$ .

(b) The average rate of change between  $A$  and  $B$  is **greater** than the instantaneous rate of change at  $B$ .

59.  $y = \frac{1}{x^2} = x^{-2}$

$$y' = -2x^{-3} = \frac{-2}{x^3} \text{ cannot equal zero.}$$

Therefore, there are no horizontal tangents.

63.  $x^2 - kx = 4x - 9$  Equate functions

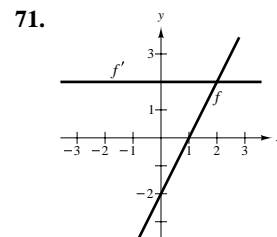
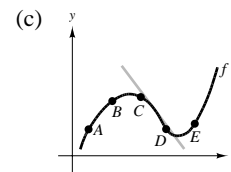
$$2x - k = 4 \quad \text{Equate derivatives}$$

Hence,  $k = 2x - 4$  and

$$x^2 - (2x - 4)x = 4x - 9 \Rightarrow -x^2 = -9 \Rightarrow x = \pm 3.$$

For  $x = 3$ ,  $k = 2$  and for  $x = -3$ ,  $k = -10$ .

69.  $g(x) = f(x) + 6 \Rightarrow g'(x) = f'(x)$



If  $f$  is linear then its derivative is a constant function.

$$f(x) = ax + b$$

$$f'(x) = a$$

73. Let  $(x_1, y_1)$  and  $(x_2, y_2)$  be the points of tangency on  $y = x^2$  and  $y = -x^2 + 6x - 5$ , respectively. The derivatives of these functions are

$$y' = 2x \Rightarrow m = 2x_1 \quad \text{and} \quad y' = -2x + 6 \Rightarrow m = -2x_2 + 6.$$

$$m = 2x_1 = -2x_2 + 6$$

$$x_1 = -x_2 + 3$$

Since  $y_1 = x_1^2$  and  $y_2 = -x_2^2 + 6x_2 - 5$ ,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-x_2^2 + 6x_2 - 5) - (-x_1^2)}{x_2 - x_1} = -2x_2 + 6.$$

$$\frac{(-x_2^2 + 6x_2 - 5) - (-x_2 + 3)^2}{x_2 - (-x_2 + 3)} = -2x_2 + 6$$

$$(-x_2^2 + 6x_2 - 5) - (x_2^2 - 6x_2 + 9) = (-2x_2 + 6)(2x_2 - 3)$$

$$-2x_2^2 + 12x_2 - 14 = -4x_2^2 + 18x_2 - 18$$

$$2x_2^2 - 6x_2 + 4 = 0$$

$$2(x_2 - 2)(x_2 - 1) = 0$$

$$x_2 = 1 \text{ or } 2$$

$$x_2 = 1 \Rightarrow y_2 = 0, x_1 = 2 \text{ and } y_1 = 4$$

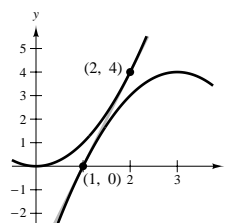
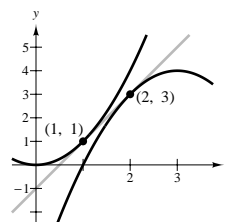
Thus, the tangent line through  $(1, 0)$  and  $(2, 4)$  is

$$y - 0 = \left(\frac{4 - 0}{2 - 1}\right)(x - 1) \Rightarrow y = 4x - 4.$$

$$x_2 = 2 \Rightarrow y_2 = 3, x_1 = 1 \text{ and } y_1 = 1$$

Thus, the tangent line through  $(2, 3)$  and  $(1, 1)$  is

$$y - 1 = \left(\frac{3 - 1}{2 - 1}\right)(x - 1) \Rightarrow y = 2x - 1.$$



75.  $f(x) = \sqrt{x}$ ,  $(-4, 0)$

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\frac{1}{2\sqrt{x}} = \frac{0 - y}{-4 - x}$$

$$4 + x = 2\sqrt{x}y$$

$$4 + x = 2\sqrt{x}\sqrt{x}$$

$$4 + x = 2x$$

$$x = 4, y = 2$$

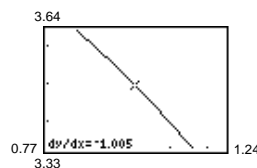
The point  $(4, 2)$  is on the graph of  $f$ .

$$\text{Tangent line: } y - 2 = \frac{0 - 2}{-4 - 4}(x - 4)$$

$$4y - 8 = x - 4$$

$$0 = x - 4y + 4$$

77.  $f'(1) = -1$

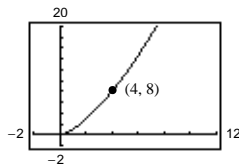


79. (a) One possible secant is between (3.9, 7.7019) and (4, 8):

$$y - 8 = \frac{8 - 7.7019}{4 - 3.9}(x - 4)$$

$$y - 8 = 2.981(x - 4)$$

$$y = S(x) = 2.981x - 3.924$$

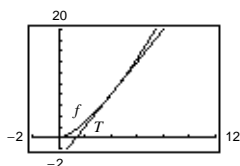


(b)  $f'(x) = \frac{3}{2}x^{1/2} \Rightarrow f'(4) = \frac{3}{2}(2) = 3$

$$T(x) = 3(x - 4) + 8 = 3x - 4$$

$S(x)$  is an approximation of the tangent line  $T(x)$ .

- (c) As you move further away from (4, 8), the accuracy of the approximation  $T$  gets worse.



(d)

$\Delta x$	-3	-2	-1	-0.5	-0.1	0	0.1	0.5	1	2	3
$f(4 + \Delta x)$	1	2.828	5.196	6.548	7.702	8	8.302	9.546	11.180	14.697	18.520
$T(4 + \Delta x)$	-1	2	5	6.5	7.7	8	8.3	9.5	11	14	17

81. False. Let  $f(x) = x^2$  and  $g(x) = x^2 + 4$ . Then  $f'(x) = g'(x) = 2x$ , but  $f(x) \neq g(x)$ .

83. False. If  $y = \pi^2$ , then  $dy/dx = 0$ . ( $\pi^2$  is a constant.)

85. True. If  $g(x) = 3f(x)$ , then  $g'(x) = 3f'(x)$ .

87.  $f(t) = 2t + 7, [1, 2]$

$$f'(t) = 2$$

Instantaneous rate of change is the constant 2.

Average rate of change:

$$\frac{f(2) - f(1)}{2 - 1} = \frac{[2(2) + 7] - [2(1) + 7]}{1} = 2$$

(These are the same because  $f$  is a line of slope 2.)

89.  $f(x) = -\frac{1}{x}, [1, 2]$

$$f'(x) = \frac{1}{x^2}$$

Instantaneous rate of change:

$$(1, -1) \Rightarrow f'(1) = 1$$

$$\left(2, -\frac{1}{2}\right) \Rightarrow f'(2) = \frac{1}{4}$$

Average rate of change:

$$\frac{f(2) - f(1)}{2 - 1} = \frac{(-1/2) - (-1)}{1} = \frac{1}{2}$$

91. (a)  $s(t) = -16t^2 + 1362$

$$v(t) = -32t$$

(b)  $\frac{s(2) - s(1)}{2 - 1} = 1298 - 1346 = -48 \text{ ft/sec}$

(c)  $v(t) = s'(t) = -32t$

When  $t = 1$ :  $v(1) = -32 \text{ ft/sec}$ .

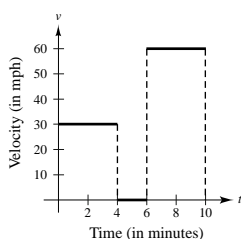
When  $t = 2$ :  $v(2) = -64 \text{ ft/sec}$ .

(d)  $-16t^2 + 1362 = 0$

$$t^2 = \frac{1362}{16} \Rightarrow t = \frac{\sqrt{1362}}{4} \approx 9.226 \text{ sec}$$

(e)  $v\left(\frac{\sqrt{1362}}{4}\right) = -32\left(\frac{\sqrt{1362}}{4}\right)$   
 $= -8\sqrt{1362} \approx -295.242 \text{ ft/sec}$

95.



(The velocity has been converted to miles per hour)

99. (a) Using a graphing utility, you obtain

$$R = 0.167v - 0.02.$$

(c)  $T = R + B = 0.00586v^2 + 0.1431v + 0.44$

(e)  $\frac{dT}{dv} = 0.01172v + 0.1431$

For  $v = 40$ ,  $T'(40) \approx 0.612$ .

For  $v = 80$ ,  $T'(80) \approx 1.081$ .

For  $v = 100$ ,  $T'(100) \approx 1.315$ .

101.  $A = s^2, \frac{dA}{ds} = 2s$

When  $s = 4 \text{ m}$ ,

$$\frac{dA}{ds} = 8 \text{ square meters per meter change in } s.$$

 105. (a)  $f'(1.47)$  is the rate of change of the amount of gasoline sold when the price is \$1.47 per gallon.

 (b)  $f'(1.47)$  is usually negative. As prices go up, sales go down.

93.  $s(t) = -4.9t^2 + v_0t + s_0$

$$= -4.9t^2 + 120t$$

$$v(t) = -9.8t + 120$$

$$v(5) = -9.8(5) + 120 = 71 \text{ m/sec}$$

$$v(10) = -9.8(10) + 120 = 22 \text{ m/sec}$$

97.  $v = 40 \text{ mph} = \frac{2}{3} \text{ mi/min}$

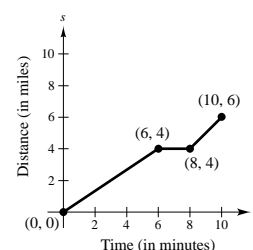
$$\left(\frac{2}{3} \text{ mi/min}\right)(6 \text{ min}) = 4 \text{ mi}$$

$$v = 0 \text{ mph} = 0 \text{ mi/min}$$

$$(0 \text{ mi/min})(2 \text{ min}) = 0 \text{ mi}$$

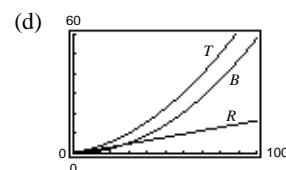
$$v = 60 \text{ mph} = 1 \text{ mi/min}$$

$$(1 \text{ mi/min})(2 \text{ min}) = 2 \text{ mi}$$



(b) Using a graphing utility, you obtain

$$B = 0.00586v^2 - 0.0239v + 0.46.$$



(f) For increasing speeds, the total stopping distance increases.

103. 
$$C = \frac{1,008,000}{Q} + 6.3Q$$

$$\frac{dC}{dQ} = -\frac{1,008,000}{Q^2} + 6.3$$

$$C(351) - C(350) \approx 5083.095 - 5085 \approx -\$1.91$$

When  $Q = 350$ ,  $\frac{dC}{dQ} \approx -\$1.93$ .

107.  $y = ax^2 + bx + c$

Since the parabola passes through (0, 1) and (1, 0), we have

$$(0, 1): 1 = a(0)^2 + b(0) + c \Rightarrow c = 1$$

$$(1, 0): 0 = a(1)^2 + b(1) + 1 \Rightarrow b = -a - 1.$$

Thus,  $y = ax^2 + (-a - 1)x + 1$ . From the tangent line  $y = x - 1$ , we know that the derivative is 1 at the point (1, 0).

$$y' = 2ax + (-a - 1)$$

$$1 = 2a(1) + (-a - 1)$$

$$1 = a - 1$$

$$a = 2$$

$$b = -a - 1 = -3$$

Therefore,  $y = 2x^2 - 3x + 1$ .

109.  $y = x^3 - 9x$

$$y' = 3x^2 - 9$$

Tangent lines through (1, -9):

$$y + 9 = (3x^2 - 9)(x - 1)$$

$$(x^3 - 9x) + 9 = 3x^3 - 3x^2 - 9x + 9$$

$$0 = 2x^3 - 3x^2 = x^2(2x - 3)$$

$$x = 0 \text{ or } x = \frac{3}{2}$$

The points of tangency are (0, 0) and  $(\frac{3}{2}, -\frac{81}{8})$ . At (0, 0) the slope is  $y'(0) = -9$ . At  $(\frac{3}{2}, -\frac{81}{8})$  the slope is  $y'(\frac{3}{2}) = -\frac{9}{4}$ .

Tangent lines:

$$y - 0 = -9(x - 0) \quad \text{and} \quad y + \frac{81}{8} = -\frac{9}{4}(x - \frac{3}{2})$$

$$y = -9x \quad \quad \quad y = -\frac{9}{4}x - \frac{27}{4}$$

$$9x + y = 0 \quad \quad \quad 9x + 4y + 27 = 0$$

111.  $f(x) = \begin{cases} ax^3, & x \leq 2 \\ x^2 + b, & x > 2 \end{cases}$

$f$  must be continuous at  $x = 2$  to be differentiable at  $x = 2$ .

$$\left. \begin{array}{l} \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} ax^3 = 8a \\ \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (x^2 + b) = 4 + b \end{array} \right\} \begin{array}{l} 8a = 4 + b \\ 8a - 4 = b \end{array}$$

$$f'(x) = \begin{cases} 3ax^2, & x < 2 \\ 2x, & x > 2 \end{cases}$$

For  $f$  to be differentiable at  $x = 2$ , the left derivative must equal the right derivative.

$$3a(2)^2 = 2(2)$$

$$12a = 4$$

$$a = \frac{1}{3}$$

$$b = 8a - 4 = -\frac{4}{3}$$

113. Let  $f(x) = \cos x$ .

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\cos x \cos \Delta x - \sin x \sin \Delta x - \cos x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\cos x(\cos \Delta x - 1)}{\Delta x} - \lim_{\Delta x \rightarrow 0} \sin x \left( \frac{\sin \Delta x}{\Delta x} \right) \\ &= 0 - \sin x(1) = -\sin x \end{aligned}$$

## Section 2.3 The Product and Quotient Rules and Higher-Order Derivatives

1.  $g(x) = (x^2 + 1)(x^2 - 2x)$

$$\begin{aligned} g'(x) &= (x^2 + 1)(2x - 2) + (x^2 - 2x)(2x) \\ &= 2x^3 - 2x^2 + 2x - 2 + 2x^3 - 4x^2 \\ &= 4x^3 - 6x^2 + 2x - 2 \end{aligned}$$

5.  $f(x) = x^3 \cos x$

$$\begin{aligned} f'(x) &= x^3(-\sin x) + \cos x(3x^2) \\ &= 3x^2 \cos x - x^3 \sin x \end{aligned}$$

9.  $h(x) = \frac{\sqrt[3]{x}}{x^3 + 1} = \frac{x^{1/3}}{x^3 + 1}$

$$\begin{aligned} h'(x) &= \frac{(x^3 + 1) \frac{1}{3} x^{-2/3} - x^{1/3}(3x^2)}{(x^3 + 1)^2} \\ &= \frac{(x^3 + 1) - x(9x^2)}{3x^{2/3}(x^3 + 1)^2} \\ &= \frac{1 - 8x^3}{3x^{2/3}(x^3 + 1)^2} \end{aligned}$$

13.  $f(x) = (x^3 - 3x)(2x^2 + 3x + 5)$

$$\begin{aligned} f'(x) &= (x^3 - 3x)(4x + 3) + (2x^2 + 3x + 5)(3x^2 - 3) \\ &= 10x^4 + 12x^3 - 3x^2 - 18x - 15 \\ f'(0) &= -15 \end{aligned}$$

17.  $f(x) = x \cos x$

$$f'(x) = (x)(-\sin x) + (\cos x)(1) = \cos x - x \sin x$$

$$f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} - \frac{\pi}{4} \left( \frac{\sqrt{2}}{2} \right) = \frac{\sqrt{2}}{8} (4 - \pi)$$

3.  $h(t) = \sqrt[3]{t}(t^2 + 4) = t^{1/3}(t^2 + 4)$

$$\begin{aligned} h'(t) &= t^{1/3}(2t) + (t^2 + 4) \frac{1}{3} t^{-2/3} \\ &= 2t^{4/3} + \frac{t^2 + 4}{3t^{2/3}} \\ &= \frac{7t^2 + 4}{3t^{2/3}} \end{aligned}$$

7.  $f(x) = \frac{x}{x^2 + 1}$

$$f'(x) = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$$

11.  $g(x) = \frac{\sin x}{x^2}$

$$g'(x) = \frac{x^2(\cos x) - \sin x(2x)}{(x^2)^2} = \frac{x \cos x - 2 \sin x}{x^3}$$

15.  $f(x) = \frac{x^2 - 4}{x - 3}$

$$f'(x) = \frac{(x - 3)(2x) - (x^2 - 4)(1)}{(x - 3)^2} = \frac{2x^2 - 6x - x^2 + 4}{(x - 3)^2}$$

$$= \frac{x^2 - 6x + 4}{(x - 3)^2}$$

$$f'(1) = \frac{1 - 6 + 4}{(1 - 3)^2} = -\frac{1}{4}$$



<i>Function</i>	<i>Rewrite</i>	<i>Derivative</i>	<i>Simplify</i>
19. $y = \frac{x^2 + 2x}{3}$	$y = \frac{1}{3}x^2 + \frac{2}{3}x$	$y' = \frac{2}{3}x + \frac{2}{3}$	$y' = \frac{2x + 2}{3}$
21. $y = \frac{7}{3x^3}$	$y = \frac{7}{3}x^{-3}$	$y' = -7x^{-4}$	$y' = -\frac{7}{x^4}$
23. $y = \frac{4x^{3/2}}{x}$	$y = 4\sqrt{x}, x > 0$	$y' = 2x^{-1/2}$	$y' = \frac{2}{\sqrt{x}}$
25. $f(x) = \frac{3 - 2x - x^2}{x^2 - 1}$			
	$f'(x) = \frac{(x^2 - 1)(-2 - 2x) - (3 - 2x - x^2)(2x)}{(x^2 - 1)^2}$		
	$= \frac{2x^2 - 4x + 2}{(x^2 - 1)^2} = \frac{2(x - 1)^2}{(x^2 - 1)^2}$		
	$= \frac{2}{(x + 1)^2}, x \neq 1$		
27. $f(x) = x\left(1 - \frac{4}{x + 3}\right) = x - \frac{4x}{x + 3}$			
	$f'(x) = 1 - \frac{(x + 3)4 - 4x(1)}{(x + 3)^2} = \frac{(x^2 + 6x + 9) - 12}{(x + 3)^2}$		
	$= \frac{x^2 + 6x - 3}{(x + 3)^2}$		
29. $f(x) = \frac{2x + 5}{\sqrt{x}} = 2x^{1/2} + 5x^{-1/2}$			
	$f'(x) = x^{-1/2} - \frac{5}{2}x^{-3/2} = x^{-3/2}\left[x - \frac{5}{2}\right]$		
	$= \frac{2x - 5}{2x\sqrt{x}} = \frac{2x - 5}{2x^{3/2}}$		
31. $h(s) = (s^3 - 2)^2 = s^6 - 4s^3 + 4$			
	$h'(s) = 6s^5 - 12s^2 = 6s^2(s^3 - 2)$		
33. $f(x) = \frac{2 - \frac{1}{x}}{x - 3} = \frac{2x - 1}{x(x - 3)} = \frac{2x - 1}{x^2 - 3x}$			
	$f'(x) = \frac{(x^2 - 3x)2 - (2x - 1)(2x - 3)}{(x^2 - 3x)^2} = \frac{2x^2 - 6x - 4x^2 + 8x - 3}{(x^2 - 3x)^2}$		
	$= \frac{-2x^2 + 2x - 3}{(x^2 - 3x)^2} = -\frac{2x^2 - 2x + 3}{x^2(x - 3)^2}$		
35. $f(x) = (3x^3 + 4x)(x - 5)(x + 1)$			
	$f'(x) = (9x^2 + 4)(x - 5)(x + 1) + (3x^3 + 4x)(1)(x + 1) + (3x^3 + 4x)(x - 5)(1)$		
	$= (9x^2 + 4)(x^2 - 4x - 5) + 3x^4 + 3x^3 + 4x^2 + 4x + 3x^4 - 15x^3 + 4x^2 - 20x$		
	$= 9x^4 - 36x^3 - 41x^2 - 16x - 20 + 6x^4 - 12x^3 + 8x^2 - 16x$		
	$= 15x^4 - 48x^3 - 33x^2 - 32x - 20$		
37. $f(x) = \frac{x^2 + c^2}{x^2 - c^2}$			
	$f'(x) = \frac{(x^2 - c^2)(2x) - (x^2 + c^2)(2x)}{(x^2 - c^2)^2}$		
	$= \frac{-4xc^2}{(x^2 - c^2)^2}$		
39. $f(x) = t^2 \sin t$			
	$f'(t) = t^2 \cos t + 2t \sin t$		
	$= t(t \cos t + 2 \sin t)$		

$$41. f(t) = \frac{\cos t}{t}$$

$$f'(t) = \frac{-t \sin t - \cos t}{t^2} = -\frac{t \sin t + \cos t}{t^2}$$

$$45. g(t) = \sqrt[4]{t} + 8 \sec t = t^{1/4} + 8 \sec t$$

$$g'(t) = \frac{1}{4}t^{-3/4} + 8 \sec t \tan t = \frac{1}{4t^{3/4}} + 8 \sec t \tan t$$

$$49. y = -\csc x - \sin x$$

$$y' = \csc x \cot x - \cos x$$

$$= \frac{\cos x}{\sin^2 x} - \cos x$$

$$= \cos x(\csc^2 x - 1)$$

$$= \cos x \cot^2 x$$

$$53. y = 2x \sin x + x^2 \cos x$$

$$y' = 2x \cos x + 2 \sin x + x^2(-\sin x) + 2x \cos x$$

$$= 4x \cos x + 2 \sin x - x^2 \sin x$$

$$57. g(\theta) = \frac{\theta}{1 - \sin \theta}$$

$$g'(\theta) = \frac{1 - \sin \theta + \theta \cos \theta}{(\sin \theta - 1)^2} \quad (\text{form of answer may vary})$$

$$59. y = \frac{1 + \csc x}{1 - \csc x}$$

$$y' = \frac{(1 - \csc x)(-\csc x \cot x) - (1 + \csc x)(\csc x \cot x)}{(1 - \csc x)^2} = \frac{-2 \csc x \cot x}{(1 - \csc x)^2}$$

$$y'\left(\frac{\pi}{6}\right) = \frac{-2(2)(\sqrt{3})}{(1-2)^2} = -4\sqrt{3}$$

$$61. h(t) = \frac{\sec t}{t}$$

$$h'(t) = \frac{t(\sec t \tan t) - (\sec t)(1)}{t^2}$$

$$= \frac{\sec t(t \tan t - 1)}{t^2}$$

$$h'(\pi) = \frac{\sec \pi(\pi \tan \pi - 1)}{\pi^2} = \frac{1}{\pi^2}$$

$$43. f(x) = -x + \tan x$$

$$f'(x) = -1 + \sec^2 x = \tan^2 x$$

$$47. y = \frac{3(1 - \sin x)}{2 \cos x} = \frac{3}{2}(\sec x - \tan x)$$

$$y' = \frac{3}{2}(\sec x \tan x - \sec^2 x) = \frac{3}{2}\sec x(\tan x - \sec x)$$

$$= \frac{3}{2}(\sec x \tan x - \tan^2 x - 1)$$

$$51. f(x) = x^2 \tan x$$

$$f'(x) = x^2 \sec^2 x + 2x \tan x$$

$$= x(x \sec^2 x + 2 \tan x)$$

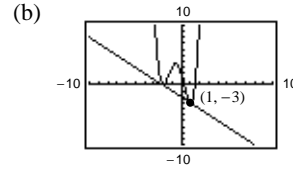
$$55. g(x) = \left(\frac{x+1}{x+2}\right)(2x-5)$$

$$g'(x) = \frac{2x^2 + 8x - 1}{(x+2)^2} \quad (\text{form of answer may vary})$$

63. (a)  $f(x) = (x^3 - 3x + 1)(x + 2)$ ,  $(1, -3)$   
 $f'(x) = (x^3 - 3x + 1)(1) + (x + 2)(3x^2 - 3)$   
 $= 4x^3 + 6x^2 - 6x - 5$

$$f'(1) = -1 = \text{slope at } (1, -3).$$

$$\text{Tangent line: } y + 3 = -1(x - 1) \Rightarrow y = -x - 2$$

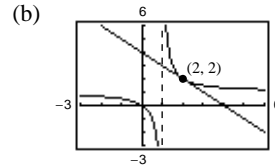


65. (a)  $f(x) = \frac{x}{x-1}$ ,  $(2, 2)$

$$f'(x) = \frac{(x-1)(1) - x(1)}{(x-1)^2} = \frac{-1}{(x-1)^2}$$

$$f'(2) = \frac{-1}{(2-1)^2} = -1 = \text{slope at } (2, 2).$$

$$\text{Tangent line: } y - 2 = -1(x - 2) \Rightarrow y = -x + 4$$



67. (a)  $f(x) = \tan x$ ,  $\left(\frac{\pi}{4}, 1\right)$

$$f'(x) = \sec^2 x$$

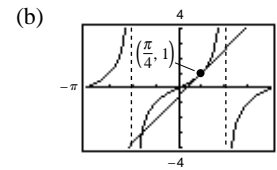
$$f'\left(\frac{\pi}{4}\right) = 2 = \text{slope at } \left(\frac{\pi}{4}, 1\right).$$

Tangent line:

$$y - 1 = 2\left(x - \frac{\pi}{4}\right)$$

$$y - 1 = 2x - \frac{\pi}{2}$$

$$4x - 2y - \pi + 2 = 0$$



69.  $f(x) = \frac{x^2}{x-1}$

$$f'(x) = \frac{(x-1)(2x) - x^2(1)}{(x-1)^2}$$

$$= \frac{x^2 - 2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$$

$$f'(x) = 0 \text{ when } x = 0 \text{ or } x = 2.$$

Horizontal tangents are at  $(0, 0)$  and  $(2, 4)$ .

71.  $f'(x) = \frac{(x+2)3 - 3x(1)}{(x+2)^2} = \frac{6}{(x+2)^2}$

$$g'(x) = \frac{(x+2)5 - (5x+4)(1)}{(x+2)^2} = \frac{6}{(x+2)^2}$$

$$g(x) = \frac{5x+4}{(x+2)} = \frac{3x}{(x+2)} + \frac{2x+4}{(x+2)} = f(x) + 2$$

$f$  and  $g$  differ by a constant.

73.  $f(x) = x^n \sin x$

$$f'(x) = x^n \cos x + nx^{n-1} \sin x$$

$$= x^{n-1}(x \cos x + n \sin x)$$

When  $n = 1$ :  $f'(x) = x \cos x + \sin x$ .

When  $n = 2$ :  $f'(x) = x(x \cos x + 2 \sin x)$ .

When  $n = 3$ :  $f'(x) = x^2(x \cos x + 3 \sin x)$ .

When  $n = 4$ :  $f'(x) = x^3(x \cos x + 4 \sin x)$ .

For general  $n$ ,  $f'(x) = x^{n-1}(x \cos x + n \sin x)$ .

75. Area =  $A(t) = (2t + 1)\sqrt{t} = 2t^{3/2} + t^{1/2}$

$$A'(t) = 2\left(\frac{3}{2}t^{1/2}\right) + \frac{1}{2}t^{-1/2}$$

$$= 3t^{1/2} + \frac{1}{2}t^{-1/2}$$

$$= \frac{6t + 1}{2\sqrt{t}} \text{ cm}^2/\text{sec}$$

$$77. C = 100\left(\frac{200}{x^2} + \frac{x}{x+30}\right), \quad 1 \leq x$$

$$\frac{dC}{dx} = 100\left(-\frac{400}{x^3} + \frac{30}{(x+30)^2}\right)$$

$$(a) \text{ When } x = 10: \frac{dC}{dx} = -\$38.13.$$

$$(b) \text{ When } x = 15: \frac{dC}{dx} = -\$10.37.$$

$$(c) \text{ When } x = 20: \frac{dC}{dx} = -\$3.80.$$

As the order size increases, the cost per item decreases.

$$81. (a) \quad \sec x = \frac{1}{\cos x}$$

$$\frac{d}{dx}[\sec x] = \frac{d}{dx}\left[\frac{1}{\cos x}\right] = \frac{(\cos x)(0) - (1)(-\sin x)}{(\cos x)^2} = \frac{\sin x}{\cos x \cos x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x$$

$$(b) \quad \csc x = \frac{1}{\sin x}$$

$$\frac{d}{dx}[\csc x] = \frac{d}{dx}\left[\frac{1}{\sin x}\right] = \frac{(\sin x)(0) - (1)(\cos x)}{(\sin x)^2} = -\frac{\cos x}{\sin x \sin x} = -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x$$

$$(c) \quad \cot x = \frac{\cos x}{\sin x}$$

$$\frac{d}{dx}[\cot x] = \frac{d}{dx}\left[\frac{\cos x}{\sin x}\right] = \frac{\sin x(-\sin x) - (\cos x)(\cos x)}{(\sin x)^2} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x$$

$$83. f(x) = 4x^{3/2}$$

$$f'(x) = 6x^{1/2}$$

$$f''(x) = 3x^{-1/2} = \frac{3}{\sqrt{x}}$$

$$85. f(x) = \frac{x}{x-1}$$

$$f'(x) = \frac{(x-1)(1) - x(1)}{(x-1)^2} = \frac{-1}{(x-1)^2}$$

$$f''(x) = \frac{2}{(x-1)^3}$$

$$87. f(x) = 3 \sin x$$

$$f'(x) = 3 \cos x$$

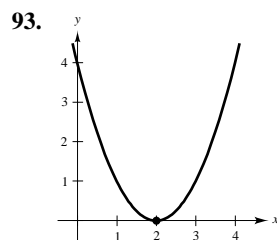
$$f''(x) = -3 \sin x$$

$$89. f'(x) = x^2$$

$$f''(x) = 2x$$

$$91. f'''(x) = 2\sqrt{x}$$

$$f^{(4)}(x) = \frac{1}{2}(2)x^{-1/2} = \frac{1}{\sqrt{x}}$$



$$f(2) = 0$$

One such function is  $f(x) = (x-2)^2$ .

$$95. f(x) = 2g(x) + h(x)$$

$$f'(x) = 2g'(x) + h'(x)$$

$$f'(2) = 2g'(2) + h'(2)$$

$$= 2(-2) + 4$$

$$= 0$$

$$97. f(x) = \frac{g(x)}{h(x)}$$

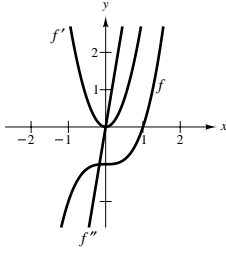
$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$$

$$f'(2) = \frac{h(2)g'(2) - g(2)h'(2)}{[h(2)]^2}$$

$$= \frac{(-1)(-2) - (3)(4)}{(-1)^2}$$

$$= -10$$

99.



It appears that  $f$  is cubic; so  $f'$  would be quadratic and  $f''$  would be linear.

101.  $v(t) = 36 - t^2, 0 \leq t \leq 6$

$a(t) = -2t$

$v(3) = 27 \text{ m/sec}$

$a(3) = -6 \text{ m/sec}$

The speed of the object is decreasing.

103.  $v(t) = \frac{100t}{2t + 15}$

$$a(t) = \frac{(2t + 15)(100) - (100t)(2)}{(2t + 15)^2}$$

$$= \frac{1500}{(2t + 15)^2}$$

(a)  $a(5) = \frac{1500}{[2(5) + 15]^2} = 2.4 \text{ ft/sec}^2$

(b)  $a(10) = \frac{1500}{[2(10) + 15]^2} \approx 1.2 \text{ ft/sec}^2$

(c)  $a(20) = \frac{1500}{[2(20) + 15]^2} \approx 0.5 \text{ ft/sec}^2$

105.  $f(x) = g(x)h(x)$

(a)  $f'(x) = g(x)h'(x) + h(x)g'(x)$

$$f''(x) = g(x)h''(x) + g'(x)h'(x) + h(x)g''(x) + h'(x)g'(x)$$

$$= g(x)h''(x) + 2g'(x)h'(x) + h(x)g''(x)$$

$$f'''(x) = g(x)h'''(x) + g''(x)h''(x) + 2g'(x)h''(x) + 2g''(x)h'(x) + h(x)g'''(x) + h'(x)g''(x)$$

$$= g(x)h'''(x) + 3g''(x)h''(x) + 3g'''(x)h'(x) + g'''(x)h(x)$$

$$f^{(4)}(x) = g(x)h^{(4)}(x) + g'''(x)h'''(x) + 3g''(x)h'''(x) + 3g'''(x)h''(x) + 3g^{(4)}(x)h'(x) + 3g'''(x)h'(x)$$

$$+ g^{(4)}(x)h(x)$$

$$= g(x)h^{(4)}(x) + 4g''(x)h'''(x) + 6g'''(x)h''(x) + 4g^{(4)}(x)h'(x) + g^{(4)}(x)h(x)$$

(b)  $f^{(n)}(x) = g(x)h^{(n)}(x) + \frac{n(n-1)(n-2) \cdots (2)(1)}{1[(n-1)(n-2) \cdots (2)(1)]} g'(x)h^{(n-1)}(x) + \frac{n(n-1)(n-2) \cdots (2)(1)}{(2)(1)[(n-2)(n-3) \cdots (2)(1)]} g''(x)h^{(n-2)}(x)$

$$+ \frac{n(n-1)(n-2) \cdots (2)(1)}{(3)(2)(1)[(n-3)(n-4) \cdots (2)(1)]} g'''(x)h^{(n-3)}(x) + \cdots$$

$$+ \frac{n(n-1)(n-2) \cdots (2)(1)}{[(n-1)(n-2) \cdots (2)(1)](1)} g^{(n-1)}(x)h'(x) + g^{(n)}(x)h(x)$$

$$= g(x)h^{(n)}(x) + \frac{n!}{1!(n-1)!} g'(x)h^{(n-1)}(x) + \frac{n!}{2!(n-2)!} g''(x)h^{(n-2)}(x) + \cdots$$

$$+ \frac{n!}{(n-1)!1!} g^{(n-1)}(x)h'(x) + g^{(n)}(x)h(x)$$

**Note:**  $n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$  (read “ $n$  factorial.”)

$$107. f(x) = \cos x \quad f\left(\frac{\pi}{3}\right) = \cos \frac{\pi}{3} = \frac{1}{2}$$

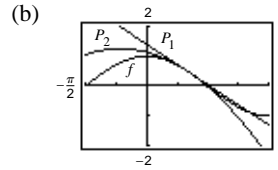
$$f'(x) = -\sin x \quad f'\left(\frac{\pi}{3}\right) = -\sin \frac{\pi}{3} = -\frac{\sqrt{3}}{2}$$

$$f''(x) = -\cos x \quad f''\left(\frac{\pi}{3}\right) = -\cos \frac{\pi}{3} = -\frac{1}{2}$$

$$(a) P_1(x) = f'(a)(x - a) + f(a) = -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) + \frac{1}{2}$$

$$\begin{aligned} P_2(x) &= \frac{1}{2}f''(a)(x - a)^2 + f'(a)(x - a) + f(a) \\ &= -\frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) + \frac{1}{2} \end{aligned}$$

(c)  $P_2$  is a better approximation.



(d) The accuracy worsens as you move farther away from  $x = a = (\pi/3)$ .

109. False. If  $y = f(x)g(x)$ , then

$$\frac{dy}{dx} = f(x)g'(x) + g(x)f'(x).$$

111. True

$$\begin{aligned} h'(c) &= f(c)g'(c) + g(c)f'(c) \\ &= f(c)(0) + g(c)(0) \\ &= 0 \end{aligned}$$

113. True

$$115. f(x) = x|x| = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 2x, & \text{if } x \geq 0 \\ -2x, & \text{if } x < 0 \end{cases} = 2|x|$$

$$f''(x) = \begin{cases} 2, & \text{if } x > 0 \\ -2, & \text{if } x < 0 \end{cases}$$

$f''(0)$  does not exist since the left and right derivatives are not equal.

## Section 2.4 The Chain Rule

$$\underline{y = f(g(x))} \quad \underline{u = g(x)} \quad \underline{y = f(u)}$$

$$1. y = (6x - 5)^4 \quad u = 6x - 5 \quad y = u^4$$

$$3. y = \sqrt{x^2 - 1} \quad u = x^2 - 1 \quad y = \sqrt{u}$$

$$5. y = \csc^3 x \quad u = \csc x \quad y = u^3$$

$$\begin{aligned} 7. y &= (2x - 7)^3 \\ y' &= 3(2x - 7)^2(2) = 6(2x - 7)^2 \end{aligned}$$

$$\begin{aligned} 9. g(x) &= 3(4 - 9x)^4 \\ g'(x) &= 12(4 - 9x)^3(-9) = -108(4 - 9x)^3 \end{aligned}$$

$$\begin{aligned} 11. f(x) &= (9 - x^2)^{2/3} \\ f'(x) &= \frac{2}{3}(9 - x^2)^{-1/3}(-2x) = -\frac{4x}{3(9 - x^2)^{1/3}} \end{aligned}$$

$$\begin{aligned} 13. f(t) &= (1 - t)^{1/2} \\ f'(t) &= \frac{1}{2}(1 - t)^{-1/2}(-1) = -\frac{1}{2\sqrt{1 - t}} \end{aligned}$$

15.  $y = (9x^2 + 4)^{1/3}$

$$y' = \frac{1}{3}(9x^2 + 4)^{-2/3}(18x) = \frac{6x}{(9x^2 + 4)^{2/3}}$$

19.  $y = (x - 2)^{-1}$

$$y' = -1(2 - x)^{-2}(1) = \frac{-1}{(x - 2)^2}$$

23.  $y = (x + 2)^{-1/2}$

$$\frac{dy}{dx} = -\frac{1}{2}(x + 2)^{-3/2} = -\frac{1}{2(x + 2)^{3/2}}$$

27.  $y = x\sqrt{1 - x^2} = x(1 - x^2)^{1/2}$

$$\begin{aligned} y' &= x\left[\frac{1}{2}(1 - x^2)^{-1/2}(-2x)\right] + (1 - x^2)^{1/2}(1) \\ &= -x^2(1 - x^2)^{-1/2} + (1 - x^2)^{1/2} \\ &= (1 - x^2)^{-1/2}[-x^2 + (1 - x^2)] \\ &= \frac{1 - 2x^2}{\sqrt{1 - x^2}} \end{aligned}$$

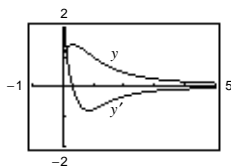
31.  $g(x) = \left(\frac{x + 5}{x^2 + 2}\right)^2$

$$\begin{aligned} g'(x) &= 2\left(\frac{x + 5}{x^2 + 2}\right)\left(\frac{(x^2 + 2) - (x + 5)(2x)}{(x^2 + 2)^2}\right) \\ &= \frac{2(x + 5)(2 - 10x - x^2)}{(x^2 + 2)^3} \end{aligned}$$

35.  $y = \frac{\sqrt{x} + 1}{x^2 + 1}$

$$y' = \frac{1 - 3x^2 - 4x^{3/2}}{2\sqrt{x}(x^2 + 1)^2}$$

The zero of  $y'$  corresponds to the point on the graph of  $y$  where the tangent line is horizontal.



17.  $y = 2(4 - x^2)^{1/4}$

$$\begin{aligned} y' &= 2\left(\frac{1}{4}\right)(4 - x^2)^{-3/4}(-2x) \\ &= \frac{-x}{\sqrt[4]{(4 - x^2)^3}} \end{aligned}$$

21.  $f(t) = (t - 3)^{-2}$

$$f'(t) = -2(t - 3)^{-3} = \frac{-2}{(t - 3)^3}$$

25.  $f(x) = x^2(x - 2)^4$

$$\begin{aligned} f'(x) &= x^2[4(x - 2)^3(1)] + (x - 2)^4(2x) \\ &= 2x(x - 2)^3[2x + (x - 2)] \\ &= 2x(x - 2)^3(3x - 2) \end{aligned}$$

29.  $y = \frac{x}{\sqrt{x^2 + 1}} = x(x^2 + 1)^{-1/2}$

$$\begin{aligned} y' &= x\left[-\frac{1}{2}(x^2 + 1)^{-3/2}(2x)\right] + (x^2 + 1)^{-1/2}(1) \\ &= -x^2(x^2 + 1)^{-3/2} + (x^2 + 1)^{-1/2} \\ &= (x^2 + 1)^{-3/2}[-x^2 + (x^2 + 1)] \\ &= \frac{1}{(x^2 + 1)^{3/2}} \end{aligned}$$

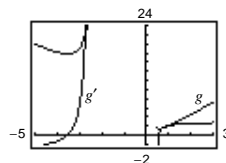
33.  $f(v) = \left(\frac{1 - 2v}{1 + v}\right)^3$

$$\begin{aligned} f'(v) &= 3\left(\frac{1 - 2v}{1 + v}\right)^2\left(\frac{(1 + v)(-2) - (1 - 2v)(1)}{(1 + v)^2}\right) \\ &= \frac{-9(1 - 2v)^2}{(1 + v)^4} \end{aligned}$$

37.  $g(t) = \frac{3t^2}{\sqrt{t^2 + 2t - 1}}$

$$g'(t) = \frac{3t(t^2 + 3t - 2)}{(t^2 + 2t - 1)^{3/2}}$$

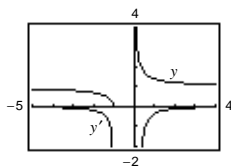
The zeros of  $g'$  correspond to the points on the graph of  $g$  where the tangent lines are horizontal.



$$39. y = \sqrt{\frac{x+1}{x}}$$

$$y' = -\frac{\sqrt{(x+1)/x}}{2x(x+1)}$$

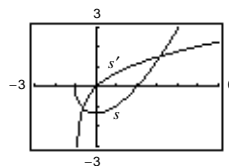
$y'$  has no zeros.



$$41. s(t) = \frac{-2(2-t)\sqrt{1+t}}{3}$$

$$s'(t) = \frac{t}{\sqrt{1+t}}$$

The zero of  $s'(t)$  corresponds to the point on the graph of  $s(t)$  where the tangent line is horizontal.

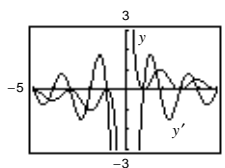


$$43. y = \frac{\cos \pi x + 1}{x}$$

$$\frac{dy}{dx} = \frac{-\pi x \sin \pi x - \cos \pi x - 1}{x^2}$$

$$= -\frac{\pi x \sin \pi x + \cos \pi x + 1}{x^2}$$

The zeros of  $y'$  correspond to the points on the graph of  $y$  where the tangent lines are horizontal.



$$45. (a) y = \sin x$$

$$y' = \cos x$$

$$y'(0) = 1$$

1 cycle in  $[0, 2\pi]$

$$(b) y = \sin 2x$$

$$y' = 2 \cos 2x$$

$$y'(0) = 2$$

2 cycles in  $[0, 2\pi]$

The slope of  $\sin ax$  at the origin is  $a$ .

$$47. y = \cos 3x$$

$$\frac{dy}{dx} = -3 \sin 3x$$

$$49. g(x) = 3 \tan 4x$$

$$g'(x) = 12 \sec^2 4x$$

$$51. y = \sin(\pi x)^2 = \sin(\pi^2 x^2)$$

$$y' = \cos(\pi x^2)[2\pi^2 x] = 2\pi^2 x \cos(\pi^2 x^2)$$

$$53. h(x) = \sin 2x \cos 2x$$

$$h'(x) = \sin 2x(-2 \sin 2x) + \cos 2x(2 \cos 2x)$$

$$= 2 \cos^2 2x - 2 \sin^2 2x$$

$$= 2 \cos 4x.$$

Alternate solution:  $h(x) = \frac{1}{2} \sin 4x$

$$h'(x) = \frac{1}{2} \cos 4x(4) = 2 \cos 4x$$

$$55. f(x) = \frac{\cot x}{\sin x} = \frac{\cos x}{\sin^2 x}$$

$$f'(x) = \frac{\sin^2 x(-\sin x) - \cos x(2 \sin x \cos x)}{\sin^4 x}$$

$$= \frac{-\sin^2 x - 2 \cos^2 x}{\sin^3 x} = \frac{-1 - \cos^2 x}{\sin^3 x}$$



57.  $y = 4 \sec^2 x$

$$y' = 8 \sec x \cdot \sec x \tan x = 8 \sec^2 x \tan x$$

61.  $f(x) = 3 \sec^2(\pi t - 1)$

$$\begin{aligned} f'(t) &= 6 \sec(\pi t - 1) \sec(\pi t - 1) \tan(\pi t - 1)(\pi) \\ &= 6\pi \sec^2(\pi t - 1) \tan(\pi t - 1) = \frac{6\pi \sin(\pi t - 1)}{\cos^3(\pi t - 1)} \end{aligned}$$

65.  $y = \sin(\cos x)$

$$\begin{aligned} \frac{dy}{dx} &= \cos(\cos x) \cdot (-\sin x) \\ &= -\sin x \cos(\cos x) \end{aligned}$$

69.  $f(x) = \frac{3}{x^3 - 4} = 3(x^3 - 4)^{-1}, \left(-1, -\frac{3}{5}\right)$

$$f'(x) = -3(x^3 - 4)^{-2}(3x^2) = -\frac{9x^2}{(x^3 - 4)^2}$$

$$f'(-1) = -\frac{9}{25}$$

73.  $y = 37 - \sec^3(2x), (0, 36)$

$$y' = -3 \sec^2(2x)[2 \sec(2x) \tan(2x)]$$

$$= -6 \sec^3(2x) \tan(2x)$$

$$y'(0) = 0$$

75. (a)  $f(x) = \sqrt{3x^2 - 2}, (3, 5)$

$$f'(x) = \frac{1}{2}(3x^2 - 2)^{-1/2}(6x)$$

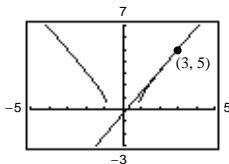
$$= \frac{3x}{\sqrt{3x^2 - 2}}$$

$$f'(3) = \frac{9}{5}$$

Tangent line:

$$y - 5 = \frac{9}{5}(x - 3) \Rightarrow 9x - 5y - 2 = 0$$

(b)



59.  $f(\theta) = \frac{1}{4} \sin^2 2\theta = \frac{1}{4}(\sin 2\theta)^2$

$$\begin{aligned} f'(\theta) &= 2\left(\frac{1}{4}\right)(\sin 2\theta)(\cos 2\theta)(2) \\ &= \sin 2\theta \cos 2\theta = \frac{1}{2} \sin 4\theta \end{aligned}$$

63.  $y = \sqrt{x} + \frac{1}{4} \sin(2x)^2$

$$= \sqrt{x} + \frac{1}{4} \sin(4x^2)$$

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} + \frac{1}{4} \cos(4x^2)(8x)$$

$$= \frac{1}{2\sqrt{x}} + 2x \cos(2x)^2$$

67.  $s(t) = (t^2 + 2t + 8)^{1/2}, (2, 4)$

$$s'(t) = \frac{1}{2}(t^2 + 2t + 8)^{-1/2}(2t + 2)$$

$$= \frac{t + 1}{\sqrt{t^2 + 2t + 8}}$$

$$s'(2) = \frac{3}{4}$$

71.  $f(t) = \frac{3t + 2}{t - 1}, (0, -2)$

$$f'(t) = \frac{(t - 1)(3) - (3t + 2)(1)}{(t - 1)^2} = \frac{-5}{(t - 1)^2}$$

$$f'(0) = -5$$

77. (a)  $f(x) = \sin 2x, (\pi, 0)$

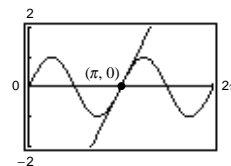
$$f'(x) = 2 \cos 2x$$

$$f'(\pi) = 2$$

Tangent line:

$$y = 2(x - \pi) \Rightarrow 2x - y - 2\pi = 0$$

(b)



$$79. f(x) = 2(x^2 - 1)^3$$

$$f'(x) = 6(x^2 - 1)^2(2x)$$

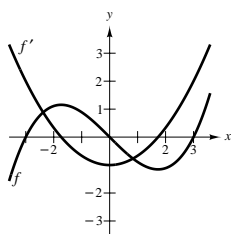
$$= 12x(x^4 - 2x^2 + 1)$$

$$= 12x^5 - 24x^3 + 12x$$

$$f''(x) = 60x^4 - 72x^2 + 12$$

$$= 12(5x^2 - 1)(x^2 - 1)$$

83.



The zeros of  $f'$  correspond to the points where the graph of  $f$  has horizontal tangents.

$$87. g(x) = f(3x)$$

$$g'(x) = f'(3x)(3) \Rightarrow g'(x) = 3f'(3x)$$

$$89. (a) f(x) = g(x)h(x)$$

$$f'(x) = g(x)h'(x) + g'(x)h(x)$$

$$f'(5) = (-3)(-2) + (6)(3) = 24$$

$$(c) f(x) = \frac{g(x)}{h(x)}$$

$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$$

$$f'(5) = \frac{(3)(6) - (-3)(-2)}{(3)^2} = \frac{12}{9} = \frac{4}{3}$$

$$91. (a) f = 132,400(331 - v)^{-1}$$

$$f' = (-1)(132,400)(331 - v)^{-2}(-1)$$

$$= \frac{132,400}{(331 - v)^2}$$

$$\text{When } v = 30, f' \approx 1.461.$$

$$93. \theta = 0.2 \cos 8t$$

The maximum angular displacement is  $\theta = 0.2$  (since  $-1 \leq \cos 8t \leq 1$ ).

$$\frac{d\theta}{dt} = 0.2[-8 \sin 8t] = -1.6 \sin 8t$$

When  $t = 3$ ,  $d\theta/dt = -1.6 \sin 24 \approx 1.4489$  radians per second.

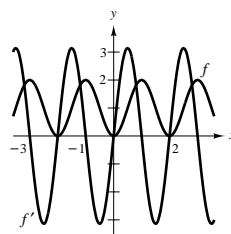
$$81. f(x) = \sin x^2$$

$$f'(x) = 2x \cos x^2$$

$$f''(x) = 2x[2x(-\sin x^2)] + 2 \cos x^2$$

$$= 2[\cos x^2 - 2x^2 \sin x^2]$$

85.



The zeros of  $f'$  correspond to the points where the graph of  $f$  has horizontal tangents.

$$(b) f(x) = g(h(x))$$

$$f'(x) = g'(h(x))h'(x)$$

$$f'(5) = g'(3)(-2) = -2g'(3)$$

Need  $g'(3)$  to find  $f'(5)$ .

$$(d) f(x) = [g(x)]^3$$

$$f'(x) = 3[g(x)]^2 g'(x)$$

$$f'(5) = 3(-3)^2(6) = 162$$

$$(b) f = 132,400(331 + v)^{-1}$$

$$f' = (-1)(132,400)(331 + v)^{-2}(1)$$

$$= \frac{-132,400}{(331 + v)^2}$$

$$\text{When } v = 30, f' \approx -1.016.$$

$$95. S = C(R^2 - r^2)$$

$$\frac{dS}{dt} = C\left(2R \frac{dR}{dt} - 2r \frac{dr}{dt}\right)$$

Since  $r$  is constant, we have  $dr/dt = 0$  and

$$\frac{dS}{dt} = (1.76 \times 10^5)(2)(1.2 \times 10^{-2})(10^{-5})$$

$$= 4.224 \times 10^{-2} = 0.04224.$$

97. (a)  $x = -1.6372t^3 + 19.3120t^2 - 0.5082t - 0.6161$

(b)  $C = 60x + 1350$

$$= 60(-1.6372t^3 + 19.3120t^2 - 0.5082t - 0.6161) + 1350$$

$$\frac{dC}{dt} = 60(-4.9116t^2 + 38.624t - 0.5082)$$

$$= -294.696t^2 + 2317.44t - 30.492$$

The function  $\frac{dC}{dt}$  is quadratic, not linear. The cost function levels off at the end of the day, perhaps due to fatigue.

99.  $f(x) = \sin \beta x$

(a)  $f'(x) = \beta \cos \beta x$

$$f''(x) = -\beta^2 \sin \beta x$$

$$f'''(x) = -\beta^3 \cos \beta x$$

$$f^{(4)} = \beta^4 \sin \beta x$$

(b)  $f''(x) + \beta^2 f(x) = -\beta^2 \sin \beta x + \beta^2(\sin \beta x) = 0$

(c)  $f^{(2k)}(x) = (-1)^k \beta^{2k} \sin \beta x$

$$f^{(2k-1)}(x) = (-1)^{k+1} \beta^{2k-1} \cos \beta x$$

101. (a)  $r'(x) = f'(g(x))g'(x)$

$$r'(1) = f'(g(1))g'(1)$$

Note that  $g(1) = 4$  and  $f'(4) = \frac{5-0}{6-2} = \frac{5}{4}$

Also,  $g'(1) = 0$ . Thus,  $r'(1) = 0$

(b)  $s'(x) = g'(f(x))f'(x)$

$$s'(4) = g'(f(4))f'(4)$$

Note that  $f(4) = \frac{5}{2}$ ,  $g'\left(\frac{5}{2}\right) = \frac{6-4}{6-2} = \frac{1}{2}$  and

$$f'(4) = \frac{5}{4}$$

Thus,  $s'(4) = \frac{1}{2}\left(\frac{5}{4}\right) = \frac{5}{8}$ .

103.  $g = \sqrt{x(x+n)}$

$$= \sqrt{x^2 + nx}$$

$$\frac{dg}{dx} = \frac{1}{2}(x^2 + nx)^{-1/2}(2x + n)$$

$$= \frac{2x + n}{2\sqrt{x^2 + nx}}$$

$$= \frac{(2x + n)/2}{\sqrt{x(x+n)}}$$

$$= \frac{[x + (x+n)]/2}{\sqrt{x(x+n)}}$$

$$= \frac{a}{g}$$

105.  $g(x) = |2x - 3|$

$$g'(x) = 2\left(\frac{2x-3}{|2x-3|}\right), \quad x \neq \frac{3}{2}$$

107.  $h(x) = |x|\cos x$

$$h'(x) = -|x|\sin x + \frac{x}{|x|}\cos x, \quad x \neq 0$$

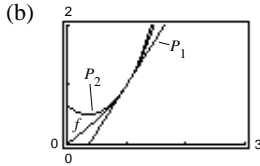
109. (a)  $f(x) = \tan \frac{\pi x}{4}$   $f(1) = 1$

$f'(x) = \frac{\pi}{4} \sec^2 \frac{\pi x}{4}$   $f'(1) = \frac{\pi}{4}(2) = \frac{\pi}{2}$

$f''(x) = \frac{\pi}{2} \sec^2 \frac{\pi x}{4} \cdot \tan \frac{\pi x}{4} \left(\frac{\pi}{4}\right)$   $f''(1) = \frac{\pi}{8}(2)(1) = \frac{\pi}{4}$

$P_1(x) = f'(1)(x - 1) + f(1) = \frac{\pi}{2}(x - 1) + 1.$

$P_2(x) = \frac{1}{2} \left(\frac{\pi}{4}\right)(x - 1)^2 + f'(1)(x - 1) + f(1) = \frac{\pi}{8}(x - 1)^2 + \frac{\pi}{2}(x - 1) + 1$



- (c)  $P_2$  is a better approximation than  $P_1$
- (d) The accuracy worsens as you move away from  $x = c = 1$ .

111. False. If  $y = (1 - x)^{1/2}$ , then  $y' = \frac{1}{2}(1 - x)^{-1/2}(-1)$ .

113. True

## Section 2.5 Implicit Differentiation

1.  $x^2 + y^2 = 36$

$$2x + 2yy' = 0$$

$$y' = \frac{-x}{y}$$

3.  $x^{1/2} + y^{1/2} = 9$

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}y' = 0$$

$$y' = -\frac{x^{-1/2}}{y^{-1/2}} = -\sqrt{\frac{y}{x}}$$

5.  $x^3 - xy + y^2 = 4$

$$3x^2 - xy' - y + 2yy' = 0$$

$$(2y - x)y' = y - 3x^2$$

$$y' = \frac{y - 3x^2}{2y - x}$$

7.  $x^3y^3 - y - x = 0$

$$3x^3y^2y' + 3x^2y^3 - y' - 1 = 0$$

$$(3x^3y^2 - 1)y' = 1 - 3x^2y^3$$

$$y' = \frac{1 - 3x^2y^3}{3x^3y^2 - 1}$$

9.  $x^3 - 3x^2 + 2xy^2 = 12$

$$3x^2 - 3x^2y' - 6xy + 4xyy' + 2y^2 = 0$$

$$(4xy - 3x^2)y' = 6xy - 3x^2 - 2y^2$$

$$y' = \frac{6xy - 3x^2 - 2y^2}{4xy - 3x^2}$$

11.  $\sin x + 2\cos 2y = 1$

$$\cos x - 4(\sin 2y)y' = 0$$

$$y' = \frac{\cos x}{4 \sin 2y}$$

13.  $\sin x = x(1 + \tan y)$

$$\cos x = x(\sec^2 y)y' + (1 + \tan y)(1)$$

$$y' = \frac{\cos x - \tan y - 1}{x \sec^2 y}$$

17. (a)  $x^2 + y^2 = 16$

$$y^2 = 16 - x^2$$

$$y = \pm \sqrt{16 - x^2}$$

(c) Explicitly:

$$\begin{aligned} \frac{dy}{dx} &= \pm \frac{1}{2}(16 - x^2)^{-1/2}(-2x) \\ &= \frac{\mp x}{\sqrt{16 - x^2}} = \frac{-x}{\pm \sqrt{16 - x^2}} = \frac{-x}{y} \end{aligned}$$

19. (a)  $16y^2 = 144 - 9x^2$

$$y^2 = \frac{1}{16}(144 - 9x^2) = \frac{9}{16}(16 - x^2)$$

$$y = \pm \frac{3}{4}\sqrt{16 - x^2}$$

(c) Explicitly:

$$\begin{aligned} \frac{dy}{dx} &= \pm \frac{3}{8}(16 - x^2)^{-1/2}(-2x) \\ &= \mp \frac{3x}{4\sqrt{16 - x^2}} = \frac{-3x}{4(4/3)y} = \frac{-9x}{16y} \end{aligned}$$

21.  $xy = 4$

$$xy' + y(1) = 0$$

$$xy' = -y$$

$$y' = \frac{-y}{x}$$

$$\text{At } (-4, -1): y' = -\frac{1}{4}$$

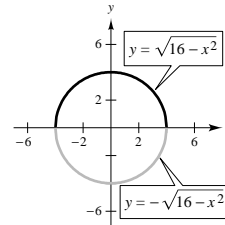
15.  $y = \sin(xy)$

$$y' = [xy' + y] \cos(xy)$$

$$y' - x \cos(xy)y' = y \cos(xy)$$

$$y' = \frac{y \cos(xy)}{1 - x \cos(xy)}$$

(b)

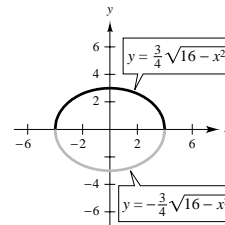


(d) Implicitly:

$$2x + 2yy' = 0$$

$$y' = -\frac{x}{y}$$

(b)



(d) Implicitly:

$$18x + 32yy' = 0$$

$$y' = \frac{-9x}{16y}$$

23.  $y^2 = \frac{x^2 - 4}{x^2 + 4}$

$$2yy' = \frac{(x^2 + 4)(2x) - (x^2 - 4)(2x)}{(x^2 + 4)^2}$$

$$2yy' = \frac{16x}{(x^2 + 4)^2}$$

$$y' = \frac{8x}{y(x^2 + 4)^2}$$

At  $(2, 0)$ ,  $y'$  is undefined.

25.  $x^{2/3} + y^{2/3} = 5$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$

$$y' = \frac{-x^{-1/3}}{y^{-1/3}} = -\sqrt[3]{\frac{y}{x}}$$

At (8, 1):  $y' = -\frac{1}{2}$ .

29.  $(x^2 + 4)y = 8$

$$(x^2 + 4)y' + y(2x) = 0$$

$$\begin{aligned} y' &= \frac{-2xy}{x^2 + 4} \\ &= \frac{-2x[8/(x^2 + 4)]}{x^2 + 4} \\ &= \frac{-16x}{(x^2 + 4)^2} \end{aligned}$$

At (2, 1):  $y' = \frac{-32}{64} = -\frac{1}{2}$

(Or, you could just solve for  $y$ :  $y = \frac{8}{x^2 + 4}$ )

33.  $\tan y = x$

$$y' \sec^2 y = 1$$

$$y' = \frac{1}{\sec^2 y} = \cos^2 y, \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$\sec^2 y = 1 + \tan^2 y = 1 + x^2$$

$$y' = \frac{1}{1 + x^2}$$

37.  $x^2 - y^2 = 16$

$$2x - 2yy' = 0$$

$$y' = \frac{x}{y}$$

$$x - yy' = 0$$

$$1 - yy'' - (y')^2 = 0$$

$$1 - yy'' - \left(\frac{x}{y}\right)^2 = 0$$

$$y^2 - y^3y'' = x^2$$

$$y'' = \frac{y^2 - x^2}{y^3} = \frac{-16}{y^3}$$

27.  $\tan(x + y) = x$

$$(1 + y') \sec^2(x + y) = 1$$

$$\begin{aligned} y' &= \frac{1 - \sec^2(x + y)}{\sec^2(x + y)} \\ &= \frac{-\tan^2(x + y)}{\tan^2(x + y) + 1} = -\sin^2(x + y) \\ &= -\frac{x^2}{x^2 + 1} \end{aligned}$$

At (0, 0):  $y' = 0$ .

31.  $(x^2 + y^2)^2 = 4x^2y$

$$2(x^2 + y^2)(2x + 2yy') = 4x^2y' + y(8x)$$

$$4x^3 + 4x^2yy' + 4xy^2 + 4y^3y' = 4x^2y' + 8xy$$

$$4x^2yy' + 4y^3y' - 4x^2y' = 8xy - 4x^3 - 4xy^2$$

$$4y'(x^2y + y^3 - x^2) = 4(2xy - x^3 - xy^2)$$

$$y' = \frac{2xy - x^3 - xy^2}{x^2y + y^3 - x^2}$$

At (1, 1):  $y' = 0$ .

35.  $x^2 + y^2 = 36$

$$2x + 2yy' = 0$$

$$y' = \frac{-x}{y}$$

$$y'' = \frac{y(-1) + xy'}{y^2} = \frac{-y + x\left(\frac{-x}{y}\right)}{y^2} = \frac{-y^2 - x^2}{y^3} = \frac{-36}{y^3}$$

39.  $y^2 = x^3$

$$2yy' = 3x^2$$

$$y' = \frac{3x^2}{2y} = \frac{3x^2}{2y} \cdot \frac{xy}{xy} = \frac{3y}{2x} \cdot \frac{x^3}{y^2} = \frac{3y}{2x}$$

$$y'' = \frac{2x(3y') - 3y(2)}{4x^2}$$

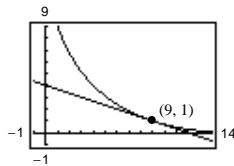
$$= \frac{2x[3 \cdot (3y/2x)] - 6y}{4x^2}$$

$$= \frac{3y}{4x^2} = \frac{3x}{4y}$$

41.  $\sqrt{x} + \sqrt{y} = 4$

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}y' = 0$$

$$y' = -\frac{\sqrt{y}}{\sqrt{x}}$$



At  $(9, 1)$ ,  $y' = -\frac{1}{3}$

Tangent line:  $y - 1 = -\frac{1}{3}(x - 9)$

$$y = -\frac{1}{3}x + 4$$

$$x + 3y - 12 = 0$$

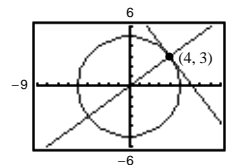
43.  $x^2 + y^2 = 25$

$$y' = \frac{-x}{y}$$

At  $(4, 3)$ :

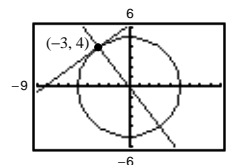
Tangent line:  $y - 3 = \frac{-4}{3}(x - 4) \Rightarrow 4x + 3y - 25 = 0$

Normal line:  $y - 3 = \frac{3}{4}(x - 4) \Rightarrow 3x - 4y = 0$

At  $(-3, 4)$ :

Tangent line:  $y - 4 = \frac{3}{4}(x + 3) \Rightarrow 3x - 4y + 25 = 0$

Normal line:  $y - 4 = \frac{-4}{3}(x + 3) \Rightarrow 4x + 3y = 0$



45.  $x^2 + y^2 = r^2$

$$2x + 2yy' = 0$$

$$y' = \frac{-x}{y} = \text{slope of tangent line}$$

$$\frac{y}{x} = \text{slope of normal line}$$

Let  $(x_0, y_0)$  be a point on the circle. If  $x_0 = 0$ , then the tangent line is horizontal, the normal line is vertical and, hence, passes through the origin. If  $x_0 \neq 0$ , then the equation of the normal line is

$$y - y_0 = \frac{y_0}{x_0}(x - x_0)$$

$$y = \frac{y_0}{x_0}x$$

which passes through the origin.

47.  $25x^2 + 16y^2 + 200x - 160y + 400 = 0$

$$50x + 32yy' + 200 - 160y' = 0$$

$$y' = \frac{200 + 50x}{160 - 32y}$$

Horizontal tangents occur when  $x = -4$ :

$$25(16) + 16y^2 + 200(-4) - 160y + 400 = 0$$

$$y(y - 10) = 0 \Rightarrow y = 0, 10$$

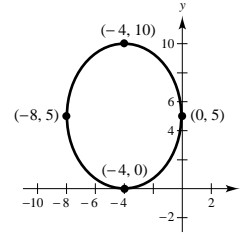
Horizontal tangents:  $(-4, 0), (-4, 10)$ .

Vertical tangents occur when  $y = 5$ :

$$25x^2 + 400 + 200x - 800 + 400 = 0$$

$$25x(x + 8) = 0 \Rightarrow x = 0, -8$$

Vertical tangents:  $(0, 5), (-8, 5)$ .



49. Find the points of intersection by letting  $y^2 = 4x$  in the equation  $2x^2 + y^2 = 6$ .

$$2x^2 + 4x = 6 \quad \text{and} \quad (x + 3)(x - 1) = 0$$

The curves intersect at  $(1, \pm 2)$ .

Ellipse:

$$4x + 2yy' = 0$$

$$y' = -\frac{2x}{y}$$

At  $(1, 2)$ , the slopes are:

$$y' = -1$$

At  $(1, -2)$ , the slopes are:

$$y' = 1$$

Parabola:

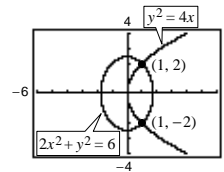
$$2yy' = 4$$

$$y' = \frac{2}{y}$$

$$y' = 1$$

$$y' = -1$$

Tangents are perpendicular.



51.  $y = -x$  and  $x = \sin y$

Point of intersection:  $(0, 0)$

$$y = -x:$$

$$y' = -1$$

$$x = \sin y:$$

$$1 = y' \cos y$$

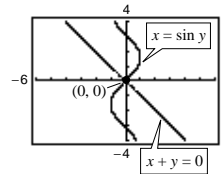
$$y' = \sec y$$

At  $(0, 0)$ , the slopes are:

$$y' = -1$$

$$y' = 1$$

Tangents are perpendicular.



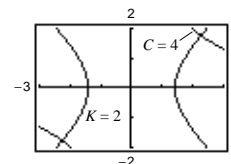
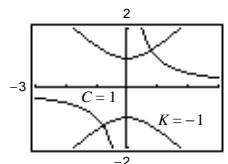
53.  $xy = C$        $x^2 - y^2 = K$

$$xy' + y = 0 \quad 2x - 2yy' = 0$$

$$y' = -\frac{y}{x}$$

$$y' = \frac{x}{y}$$

At any point of intersection  $(x, y)$  the product of the slopes is  $(-y/x)(x/y) = -1$ . The curves are orthogonal.





55.  $2y^2 - 3x^4 = 0$

(a)  $4yy' - 12x^3 = 0$

$$4yy' = 12x^3$$

$$y' = \frac{12x^3}{4y} = \frac{3x^3}{y}$$

(b)  $4y \frac{dy}{dt} - 12x^3 \frac{dx}{dt} = 0$

$$y \frac{dy}{dt} = 3x^3 \frac{dx}{dt}$$

57.  $\cos \pi y - 3 \sin \pi x = 1$

(a)  $-\pi \sin(\pi y)y' - 3\pi \cos \pi x = 0$

$$y' = \frac{-3 \cos \pi x}{\sin \pi y}$$

(b)  $-\pi \sin(\pi y) \frac{dy}{dt} - 3\pi \cos(\pi x) \frac{dx}{dt} = 0$

$$-\sin(\pi y) \frac{dy}{dt} = 3 \cos(\pi x) \frac{dx}{dt}$$

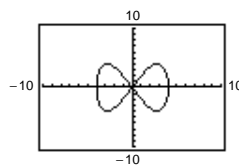
59. A function is in explicit form if  $y$  is written as a function of  $x$ :  $y = f(x)$ . For example,  $y = x^3$ . An implicit equation is not in the form  $y = f(x)$ . For example,  $x^2 + y^2 = 5$ .

61. (a)  $x^4 = 4(4x^2 - y^2)$

$$4y^2 = 16x^2 - x^4$$

$$y^2 = 4x^2 - \frac{1}{4}x^4$$

$$y = \pm \sqrt{4x^2 - \frac{1}{4}x^4}$$



(b)  $y = 3 \Rightarrow 9 = 4x^2 - \frac{1}{4}x^4$

$$36 = 16x^2 - x^4$$

$$x^4 - 16x^2 + 36 = 0$$

$$x^2 = \frac{16 \pm \sqrt{256 - 144}}{2} = 8 \pm \sqrt{28}$$

**Note** that  $x^2 = 8 \pm \sqrt{28} = 8 \pm 2\sqrt{7} = (1 \pm \sqrt{7})^2$ .

Hence, there are four values of  $x$ :

$$-1 - \sqrt{7}, 1 - \sqrt{7}, -1 + \sqrt{7}, 1 + \sqrt{7}$$

To find the slope,  $2yy' = 8x - x^3 \Rightarrow y' = \frac{x(8 - x^2)}{2(3)}$ .

For  $x = -1 - \sqrt{7}$ ,  $y' = \frac{1}{3}(\sqrt{7} + 7)$ , and the line is

$$y_1 = \frac{1}{3}(\sqrt{7} + 7)(x + 1 + \sqrt{7}) + 3 = \frac{1}{3}[(\sqrt{7} + 7)x + 8\sqrt{7} + 23].$$

For  $x = 1 - \sqrt{7}$ ,  $y' = \frac{1}{3}(\sqrt{7} - 7)$ , and the line is

$$y_2 = \frac{1}{3}(\sqrt{7} - 7)(x - 1 + \sqrt{7}) + 3 = \frac{1}{3}[(\sqrt{7} - 7)x + 23 - 8\sqrt{7}].$$

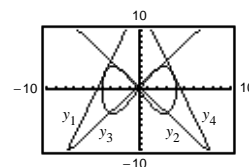
For  $x = -1 + \sqrt{7}$ ,  $y' = -\frac{1}{3}(\sqrt{7} - 7)$ , and the line is

$$y_3 = -\frac{1}{3}(\sqrt{7} - 7)(x + 1 - \sqrt{7}) + 3 = -\frac{1}{3}[(\sqrt{7} - 7)x - (23 - 8\sqrt{7})].$$

For  $x = 1 + \sqrt{7}$ ,  $y' = -\frac{1}{3}(\sqrt{7} + 7)$ , and the line is

$$y_4 = -\frac{1}{3}(\sqrt{7} + 7)(x - 1 - \sqrt{7}) + 3 = -\frac{1}{3}[(\sqrt{7} + 7)x - (8\sqrt{7} + 23)].$$

—CONTINUED—



## 61. —CONTINUED—

(c) Equating  $y_3$  and  $y_4$ ,

$$\begin{aligned} -\frac{1}{3}(\sqrt{7} - 7)(x + 1 - \sqrt{7}) + 3 &= -\frac{1}{3}(\sqrt{7} + 7)(x - 1 - \sqrt{7}) + 3 \\ (\sqrt{7} - 7)(x + 1 - \sqrt{7}) &= (\sqrt{7} + 7)(x - 1 - \sqrt{7}) \\ \sqrt{7}x + \sqrt{7} - 7 - 7x - 7 + 7\sqrt{7} &= \sqrt{7}x - \sqrt{7} - 7 + 7x - 7 - 7\sqrt{7} \\ 16\sqrt{7} &= 14x \\ x &= \frac{8\sqrt{7}}{7} \end{aligned}$$

If  $x = \frac{8\sqrt{7}}{7}$ , then  $y = 5$  and the lines intersect at  $\left(\frac{8\sqrt{7}}{7}, 5\right)$ .63. Let  $f(x) = x^n = x^{p/q}$ , where  $p$  and  $q$  are nonzero integers and  $q > 0$ . First consider the case where  $p = 1$ . The derivative of  $f(x) = x^{1/q}$  is given by

$$\frac{d}{dx}[x^{1/q}] = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x}$$

where  $t = x + \Delta x$ . Observe that

$$\begin{aligned} \frac{f(t) - f(x)}{t - x} &= \frac{t^{1/q} - x^{1/q}}{t - x} = \frac{t^{1/q} - x^{1/q}}{(t^{1/q})^q - (x^{1/q})^q} \\ &= \frac{t^{1/q} - x^{1/q}}{(t^{1/q} - x^{1/q})(t^{1-1/q} + t^{1-(2/q)}x^{1/q} + \cdots + t^{1/q}x^{1-(2/q)} + x^{1-(1/q)})} \\ &= \frac{1}{t^{1-(1/q)} + t^{1-(2/q)}x^{1/q} + \cdots + t^{1/q}x^{1-(2/q)} + x^{1-(1/q)}}. \end{aligned}$$

As  $t \rightarrow x$ , the denominator approaches  $qx^{1-(1/q)}$ . That is,

$$\frac{d}{dx}[x^{1/q}] = \frac{1}{qx^{1-(1/q)}} = \frac{1}{q}x^{(1/q)-1}.$$

Now consider  $f(x) = x^{p/q} = (x^p)^{1/q}$ . From the Chain Rule,

$$f'(x) = \frac{1}{q}(x^p)^{(1/q)-1} \frac{d}{dx}[x^p] = \frac{1}{q}(x^p)^{(1/q)-1} px^{p-1} = \frac{p}{q}x^{[(p/q)-p]+(p-1)} = \frac{p}{q}x^{(p/q)-1} = nx^{n-1} \left(n = \frac{p}{q}\right).$$

## Section 2.6 Related Rates

1.  $y = \sqrt{x}$

$$\frac{dy}{dt} = \left(\frac{1}{2\sqrt{x}}\right) \frac{dx}{dt}$$

$$\frac{dx}{dt} = 2\sqrt{x} \frac{dy}{dt}$$

(a) When  $x = 4$  and  $dx/dt = 3$ ,

$$\frac{dy}{dt} = \frac{1}{2\sqrt{4}}(3) = \frac{3}{4}.$$

(b) When  $x = 25$  and  $dy/dt = 2$ ,

$$\frac{dx}{dt} = 2\sqrt{25}(2) = 20.$$

3.  $xy = 4$

$$x \frac{dy}{dt} + y \frac{dx}{dt} = 0$$

$$\frac{dy}{dt} = \left(-\frac{y}{x}\right) \frac{dx}{dt}$$

$$\frac{dx}{dt} = \left(-\frac{x}{y}\right) \frac{dy}{dt}$$

(a) When  $x = 8$ ,  $y = 1/2$ , and  $dx/dt = 10$ ,

$$\frac{dy}{dt} = -\frac{1/2}{8}(10) = -\frac{5}{8}.$$

(b) When  $x = 1$ ,  $y = 4$ , and  $dy/dt = -6$ ,

$$\frac{dx}{dt} = -\frac{1}{4}(-6) = \frac{3}{2}.$$

5.  $y = x^2 + 1$

$$\frac{dx}{dt} = 2$$

$$\frac{dy}{dt} = 2x \frac{dx}{dt}$$

(a) When  $x = -1$ ,

$$\frac{dy}{dt} = 2(-1)(2) = -4 \text{ cm/sec.}$$

(b) When  $x = 0$ ,

$$\frac{dy}{dt} = 2(0)(2) = 0 \text{ cm/sec.}$$

(c) When  $x = 1$ ,

$$\frac{dy}{dt} = 2(1)(2) = 4 \text{ cm/sec.}$$

9. (a)  $\frac{dx}{dt}$  negative  $\Rightarrow \frac{dy}{dt}$  positive

(b)  $\frac{dy}{dt}$  positive  $\Rightarrow \frac{dx}{dt}$  negative

13.  $D = \sqrt{x^2 + y^2} = \sqrt{x^2 + (x^2 + 1)^2} = \sqrt{x^4 + 3x^2 + 1}$

$$\frac{dx}{dt} = 2$$

$$\frac{dD}{dt} = \frac{1}{2}(x^4 + 3x^2 + 1)^{-1/2}(4x^3 + 6x) \frac{dx}{dt} = \frac{2x^3 + 3x}{\sqrt{x^4 + 3x^2 + 1}} \frac{dx}{dt} = \frac{4x^3 + 6x}{\sqrt{x^4 + 3x^2 + 1}}$$

15.  $A = \pi r^2$

$$\frac{dr}{dt} = 3$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

(a) When  $r = 6$ ,

$$\frac{dA}{dt} = 2\pi(6)(3) = 36\pi \text{ cm}^2/\text{min.}$$

(b) When  $r = 24$ ,

$$\frac{dA}{dt} = 2\pi(24)(3) = 144\pi \text{ cm}^2/\text{min.}$$

7.  $y = \tan x$

$$\frac{dx}{dt} = 2$$

$$\frac{dy}{dt} = \sec^2 x \frac{dx}{dt}$$

(a) When  $x = -\pi/3$ ,

$$\frac{dy}{dt} = (2)^2(2) = 8 \text{ cm/sec.}$$

(b) When  $x = -\pi/4$ ,

$$\frac{dy}{dt} = (\sqrt{2})^2(2) = 4 \text{ cm/sec.}$$

(c) When  $x = 0$ ,

$$\frac{dy}{dt} = (1)^2(2) = 2 \text{ cm/sec.}$$

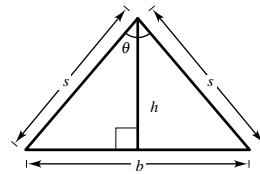
11. Yes,  $y$  changes at a constant rate:  $\frac{dy}{dt} = a \cdot \frac{dx}{dt}$ .

No, the rate  $\frac{dy}{dt}$  is a multiple of  $\frac{dx}{dt}$ .

17. (a)  $\sin \frac{\theta}{2} = \frac{(1/2)b}{s} \Rightarrow b = 2s \sin \frac{\theta}{2}$

$$\cos \frac{\theta}{2} = \frac{h}{s} \Rightarrow h = s \cos \frac{\theta}{2}$$

$$\begin{aligned} A &= \frac{1}{2}bh = \frac{1}{2}\left(2s \sin \frac{\theta}{2}\right)\left(s \cos \frac{\theta}{2}\right) \\ &= \frac{s^2}{2}\left(2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}\right) = \frac{s^2}{2} \sin \theta \end{aligned}$$



(b)  $\frac{dA}{dt} = \frac{s^2}{2} \cos \theta \frac{d\theta}{dt}$  where  $\frac{d\theta}{dt} = \frac{1}{2}$  rad/min.

$$\text{When } \theta = \frac{\pi}{6}, \frac{dA}{dt} = \frac{s^2}{2} \left(\frac{\sqrt{3}}{2}\right) \left(\frac{1}{2}\right) = \frac{\sqrt{3}s^2}{8}$$

$$\text{When } \theta = \frac{\pi}{3}, \frac{dA}{dt} = \frac{s^2}{2} \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) = \frac{s^2}{8}$$

(c) If  $d\theta/dt$  is constant,  $dA/dt$  is proportional to  $\cos \theta$ .

$$19. \quad V = \frac{4}{3}\pi r^3, \quad \frac{dV}{dt} = 800$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{4\pi r^2} \left( \frac{dV}{dt} \right) = \frac{1}{4\pi r^2} (800)$$

$$(a) \quad \text{When } r = 30, \quad \frac{dr}{dt} = \frac{1}{4\pi(30)^2} (800) = \frac{2}{9\pi} \text{ cm/min.}$$

$$(b) \quad \text{When } r = 60, \quad \frac{dr}{dt} = \frac{1}{4\pi(60)^2} (800) = \frac{1}{18\pi} \text{ cm/min.}$$

$$23. \quad V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left( \frac{9}{4}h^2 \right) h \quad [\text{since } 2r = 3h]$$

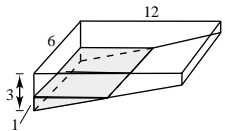
$$= \frac{3\pi}{4} h^3$$

$$\frac{dV}{dt} = 10$$

$$\frac{dV}{dt} = \frac{9\pi}{4} h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{4(dV/dt)}{9\pi h^2}$$

$$\text{When } h = 15, \quad \frac{dh}{dt} = \frac{4(10)}{9\pi(15)^2} = \frac{8}{405\pi} \text{ ft/min.}$$

25.



$$(a) \quad \text{Total volume of pool} = \frac{1}{2}(2)(12)(6) + (1)(6)(12) = 144 \text{ m}^3$$

$$\text{Volume of 1m. of water} = \frac{1}{2}(1)(6)(6) = 18 \text{ m}^3$$

(see similar triangle diagram)

$$\% \text{ pool filled} = \frac{18}{144}(100\%) = 12.5\%$$

(b) Since for  $0 \leq h \leq 2$ ,  $b = 6h$ , you have

$$V = \frac{1}{2}bh(6) = 3bh = 3(6h)h = 18h^2$$

$$\frac{dV}{dt} = 36h \frac{dh}{dt} = \frac{1}{4} \Rightarrow \frac{dh}{dt} = \frac{1}{144h} = \frac{1}{144(1)} = \frac{1}{144} \text{ m/min.}$$

$$21. \quad s = 6x^2$$

$$\frac{dx}{dt} = 3$$

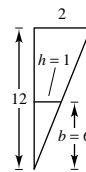
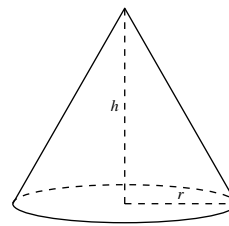
$$\frac{ds}{dt} = 12x \frac{dx}{dt}$$

(a) When  $x = 1$ ,

$$\frac{ds}{dt} = 12(1)(3) = 36 \text{ cm}^2/\text{sec.}$$

(b) When  $x = 10$ ,

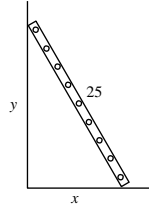
$$\frac{ds}{dt} = 12(10)(3) = 360 \text{ cm}^2/\text{sec.}$$



27.  $x^2 + y^2 = 25^2$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{-x}{y} \cdot \frac{dx}{dt} = \frac{-2x}{y} \text{ since } \frac{dx}{dt} = 2.$$



(a) When  $x = 7$ ,  $y = \sqrt{576} = 24$ ,  $\frac{dy}{dt} = \frac{-2(7)}{24} = \frac{-7}{12}$  ft/sec.

When  $x = 15$ ,  $y = \sqrt{400} = 20$ ,  $\frac{dy}{dt} = \frac{-2(15)}{20} = \frac{-3}{2}$  ft/sec.

When  $x = 24$ ,  $y = 7$ ,  $\frac{dy}{dt} = \frac{-2(24)}{7} = \frac{-48}{7}$  ft/sec.

(b)  $A = \frac{1}{2}xy$

$$\frac{dA}{dt} = \frac{1}{2} \left( x \frac{dy}{dt} + y \frac{dx}{dt} \right)$$

From part (a) we have  $x = 7$ ,  $y = 24$ ,  $\frac{dx}{dt} = 2$ ,

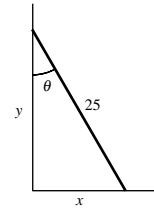
and  $\frac{dy}{dt} = -\frac{7}{12}$ .

$$\begin{aligned} \text{Thus, } \frac{dA}{dt} &= \frac{1}{2} \left[ 7 \left( -\frac{7}{12} \right) + 24(2) \right] \\ &= \frac{527}{24} \approx 21.96 \text{ ft}^2/\text{sec}. \end{aligned}$$

(c)  $\tan \theta = \frac{x}{y}$

$$\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{y} \cdot \frac{dx}{dt} - \frac{x}{y^2} \cdot \frac{dy}{dt}$$

$$\frac{d\theta}{dt} = \cos^2 \theta \left[ \frac{1}{y} \cdot \frac{dx}{dt} - \frac{x}{y^2} \cdot \frac{dy}{dt} \right]$$



Using  $x = 7$ ,  $y = 24$ ,  $\frac{dx}{dt} = 2$ ,  $\frac{dy}{dt} = -\frac{7}{12}$  and  $\cos \theta = \frac{24}{25}$ , we have  $\frac{d\theta}{dt} = \left( \frac{24}{25} \right)^2 \left[ \frac{1}{24}(2) - \frac{7}{(24)^2} \left( -\frac{7}{12} \right) \right] = \frac{1}{12}$  rad/sec.

29. When  $y = 6$ ,  $x = \sqrt{12^2 - 6^2} = 6\sqrt{3}$ , and

$$\begin{aligned} s &= \sqrt{x^2 + (12 - y)^2} \\ &= \sqrt{108 + 36} = 12. \end{aligned}$$

$$x^2 + (12 - y)^2 = s^2$$

$$2x \frac{dx}{dt} + 2(12 - y)(-1) \frac{dy}{dt} = 2s \frac{ds}{dt}$$

$$x \frac{dx}{dt} + (y - 12) \frac{dy}{dt} = s \frac{ds}{dt}$$

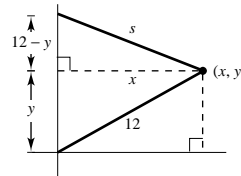
Also,  $x^2 + y^2 = 12^2$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{-x}{y} \frac{dx}{dt}$$

Thus,  $x \frac{dx}{dt} + (y - 12) \left( \frac{-x}{y} \frac{dx}{dt} \right) = s \frac{ds}{dt}$

$$\frac{dx}{dt} \left[ x - x + \frac{12x}{y} \right] = s \frac{ds}{dt} \Rightarrow \frac{dx}{dt} = \frac{sy}{12x} \cdot \frac{ds}{dt} = \frac{(12)(6)}{(12)(6\sqrt{3})} (-0.2) = \frac{-1}{5\sqrt{3}} = \frac{-\sqrt{3}}{15} \text{ m/sec (horizontal)}$$

$$\frac{dy}{dt} = \frac{-x}{y} \frac{dx}{dt} = \frac{-6\sqrt{3}}{6} \cdot \frac{(-\sqrt{3})}{15} = \frac{1}{5} \text{ m/sec (vertical).}$$



31. (a)  $s^2 = x^2 + y^2$

$$\frac{dx}{dt} = -450$$

$$\frac{dy}{dt} = -600$$

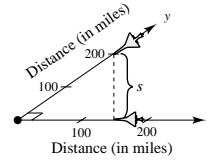
$$2s \frac{ds}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{x(dx/dt) + y(dy/dt)}{s}$$

When  $x = 150$  and  $y = 200$ ,  $s = 250$  and

$$\frac{ds}{dt} = \frac{150(-450) + 200(-600)}{250} = -750 \text{ mph.}$$

(b)  $t = \frac{250}{750} = \frac{1}{3} \text{ hr} = 20 \text{ min}$



33.  $s^2 = 90^2 + x^2$

$$x = 30$$

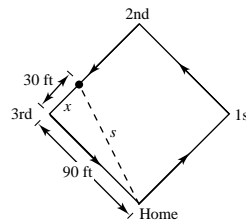
$$\frac{dx}{dt} = -28$$

$$2s \frac{ds}{dt} = 2x \frac{dx}{dt} \Rightarrow \frac{ds}{dt} = \frac{x}{s} \cdot \frac{dx}{dt}$$

When  $x = 30$ ,

$$s = \sqrt{90^2 + 30^2} = 30\sqrt{10}$$

$$\frac{ds}{dt} = \frac{30}{30\sqrt{10}}(-28) = \frac{-28}{\sqrt{10}} \approx -8.85 \text{ ft/sec.}$$



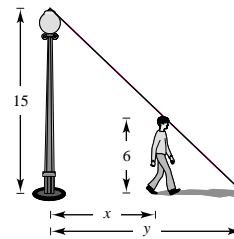
35. (a)  $\frac{15}{6} = \frac{y}{y-x} \Rightarrow 15y - 15x = 6y$

$$y = \frac{5}{3}x$$

$$\frac{dx}{dt} = 5$$

$$\frac{dy}{dt} = \frac{5}{3} \cdot \frac{dx}{dt} = \frac{5}{3}(5) = \frac{25}{3} \text{ ft/sec}$$

(b)  $\frac{d(y-x)}{dt} = \frac{dy}{dt} - \frac{dx}{dt} = \frac{25}{3} - 5 = \frac{10}{3} \text{ ft/sec}$



$$37. x(t) = \frac{1}{2} \sin \frac{\pi t}{6}, x^2 + y^2 = 1$$

$$(a) \text{ Period: } \frac{2\pi}{\pi/6} = 12 \text{ seconds}$$

$$(b) \text{ When } x = \frac{1}{2}, y = \sqrt{1^2 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2} \text{ m.}$$

$$\text{Lowest point: } \left(0, \frac{\sqrt{3}}{2}\right)$$

$$(c) \text{ When } x = \frac{1}{4}, y = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4} \text{ and } t = 1$$

$$\frac{dx}{dt} = \frac{1}{2} \left(\frac{\pi}{6}\right) \cos \frac{\pi t}{6} = \frac{\pi}{12} \cos \frac{\pi t}{6}$$

$$x^2 + y^2 = 1$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{-x}{y} \frac{dx}{dt}$$

Thus,

$$\begin{aligned} \frac{dy}{dt} &= -\frac{1/4}{\sqrt{15}/4} \cdot \frac{\pi}{12} \cos\left(\frac{\pi}{6}\right) \\ &= \frac{-\pi}{\sqrt{15}} \left(\frac{1}{12}\right) \frac{\sqrt{3}}{2} = \frac{-\pi}{24} \frac{1}{\sqrt{5}} = \frac{-\sqrt{5}\pi}{120} \end{aligned}$$

$$\text{Speed} = \left| \frac{-\sqrt{5}\pi}{120} \right| = \frac{\sqrt{5}\pi}{120} \text{ m/sec}$$

$$41. pV^{1.3} = k$$

$$1.3 pV^{0.3} \frac{dV}{dt} + V^{1.3} \frac{dp}{dt} = 0$$

$$V^{0.3} \left( 1.3p \frac{dV}{dt} + V \frac{dp}{dt} \right) = 0$$

$$1.3p \frac{dV}{dt} = -V \frac{dp}{dt}$$

$$43. \tan \theta = \frac{y}{30}$$

$$\frac{dy}{dt} = 3 \text{ m/sec.}$$

$$\sec^2 \theta \cdot \frac{d\theta}{dt} = \frac{1}{30} \frac{dy}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{30} \cos^2 \theta \cdot \frac{dy}{dt}$$

When  $y = 30$ ,  $\theta = \pi/4$  and  $\cos \theta = \sqrt{2}/2$ . Thus,

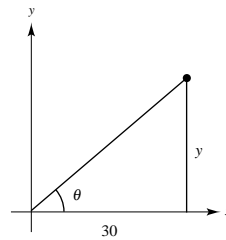
$$\frac{d\theta}{dt} = \frac{1}{30} \left(\frac{1}{2}\right) (3) = \frac{1}{20} \text{ rad/sec.}$$

39. Since the evaporation rate is proportional to the surface area,  $dV/dt = k(4\pi r^2)$ . However, since  $V = (4/3)\pi r^3$ , we have

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Therefore,

$$k(4\pi r^2) = 4\pi r^2 \frac{dr}{dt} \Rightarrow k = \frac{dr}{dt}$$



45.  $\tan \theta = \frac{y}{x}, y = 5$

$$\frac{dx}{dt} = -600 \text{ mi/hr}$$

$$(\sec^2 \theta) \frac{d\theta}{dt} = -\frac{5}{x^2} \cdot \frac{dx}{dt}$$

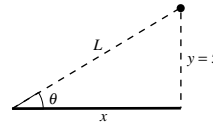
$$\frac{d\theta}{dt} = \cos^2 \theta \left( -\frac{5}{x^2} \right) \frac{dx}{dt} = \frac{x^2}{L^2} \left( -\frac{5}{x^2} \right) \frac{dx}{dt}$$

$$= \left( -\frac{5^2}{L^2} \right) \left( \frac{1}{5} \right) \frac{dx}{dt} = (-\sin^2 \theta) \left( \frac{1}{5} \right) (-600) = 120 \sin^2 \theta$$

(a) When  $\theta = 30^\circ$ ,  $\frac{d\theta}{dt} = \frac{120}{4} = 30 \text{ rad/hr} = \frac{1}{2} \text{ rad/min}$ .

(b) When  $\theta = 60^\circ$ ,  $\frac{d\theta}{dt} = 120 \left( \frac{3}{4} \right) = 90 \text{ rad/hr} = \frac{3}{2} \text{ rad/min}$ .

(c) When  $\theta = 75^\circ$ ,  $\frac{d\theta}{dt} = 120 \sin^2 75^\circ \approx 111.96 \text{ rad/hr} \approx 1.87 \text{ rad/min}$ .

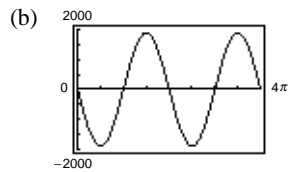
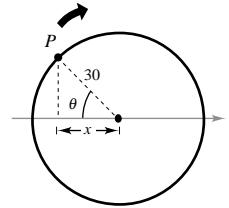


47.  $\frac{d\theta}{dt} = (10 \text{ rev/sec})(2\pi \text{ rad/rev}) = 20\pi \text{ rad/sec}$

(a)  $\cos \theta = \frac{x}{30}$

$$-\sin \theta \frac{d\theta}{dt} = \frac{1}{30} \frac{dx}{dt}$$

$$\frac{dx}{dt} = -30 \sin \theta \frac{d\theta}{dt} = -30 \sin \theta (20\pi) = -600\pi \sin \theta$$



(c)  $|dx/dt| = |-600\pi \sin \theta|$  is greatest when  $\sin \theta = 1 \Rightarrow \theta = (\pi/2) + n\pi$  (or  $90^\circ + n \cdot 180^\circ$ )

$|dx/dt|$  is least when  $\theta = n\pi$  (or  $n \cdot 180^\circ$ ).

(d) For  $\theta = 30^\circ$ ,  $\frac{dx}{dt} = -600\pi \sin(30^\circ) = -600\pi \frac{1}{2} = -300\pi \text{ cm/sec}$ .

For  $\theta = 60^\circ$ ,  $\frac{dx}{dt} = -600\pi \sin(60^\circ) = -600\pi \frac{\sqrt{3}}{2} = -300\sqrt{3} \pi \text{ cm/sec}$

49.  $\tan \theta = \frac{x}{50} \Rightarrow x = 50 \tan \theta$

$$\frac{dx}{dt} = 50 \sec^2 \theta \frac{d\theta}{dt}$$

$$2 = 50 \sec^2 \theta \frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{25} \cos^2 \theta, -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$$



51.  $x^2 + y^2 = 25$ ; acceleration of the top of the ladder  $= \frac{d^2y}{dt^2}$

$$\text{First derivative: } 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$x \frac{dx}{dt} + y \frac{dy}{dt} = 0$$

$$\text{Second derivative: } x \frac{d^2x}{dt^2} + \frac{dx}{dt} \cdot \frac{dx}{dt} + y \frac{d^2y}{dt^2} + \frac{dy}{dt} \cdot \frac{dy}{dt} = 0$$

$$\frac{d^2y}{dt^2} = \left(\frac{1}{y}\right) \left[ -x \frac{d^2x}{dt^2} - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 \right]$$

When  $x = 7$ ,  $y = 24$ ,  $\frac{dy}{dt} = -\frac{7}{12}$ , and  $\frac{dx}{dt} = 2$  (see Exercise 27). Since  $\frac{dx}{dt}$  is constant,  $\frac{d^2x}{dt^2} = 0$ .

$$\frac{d^2y}{dt^2} = \frac{1}{24} \left[ -7(0) - (2)^2 - \left(-\frac{7}{12}\right)^2 \right] = \frac{1}{24} \left[ -4 - \frac{49}{144} \right] = \frac{1}{24} \left[ -\frac{625}{144} \right] \approx -0.1808 \text{ ft/sec}^2$$

53. (a) Using a graphing utility, you obtain  $m(s) = -0.881s^2 + 29.10s - 206.2$

$$(b) \frac{dm}{dt} = \frac{dm}{ds} \frac{ds}{dt} = (-1.762s + 29.10) \frac{ds}{dt}$$

(c) If  $t = s$  (1995), then  $s = 15.5$  and  $\frac{ds}{dt} = 1.2$ .

$$\text{Thus, } \frac{dm}{dt} = (-1.762(15.5) + 29.10)(1.2) \approx 2.15 \text{ million.}$$

## Review Exercises for Chapter 2

1.  $f(x) = x^2 - 2x + 3$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^2 - 2(x + \Delta x) + 3] - [x^2 - 2x + 3]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x^2 + 2x(\Delta x) + (\Delta x)^2 - 2x - 2(\Delta x) + 3) - (x^2 - 2x + 3)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{2x(\Delta x) + (\Delta x)^2 - 2(\Delta x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} (2x + \Delta x - 2) = 2x - 2 \end{aligned}$$

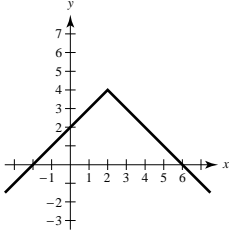
3.  $f(x) = \sqrt{x} + 1$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(\sqrt{x + \Delta x} + 1) - (\sqrt{x} + 1)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x) - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

5.  $f$  is differentiable for all  $x \neq -1$ .

7.  $f(x) = 4 - |x - 2|$

 (a) Continuous at  $x = 2$ .

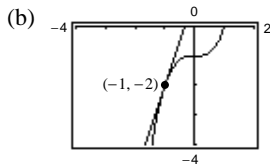
 (b) Not differentiable at  $x = 2$  because of the sharp turn in the graph.


11. (a) Using the limit definition,  $f'(x) = 3x^2$ .

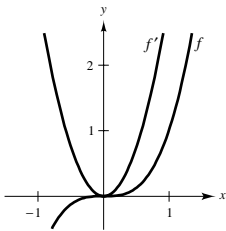
 At  $x = -1$ ,  $f'(-1) = 3$ . The tangent line is

$$y - (-2) = 3(x - (-1))$$

$$y = 3x + 1$$



15.



17.  $y = 25$

$$y' = 0$$

19.  $f(x) = x^8$

$$f'(x) = 8x^7$$

21.  $h(t) = 3t^4$

$$h'(t) = 12t^3$$

25.  $h(x) = 6\sqrt{x} + 3\sqrt[3]{x} = 6x^{1/2} + 3x^{1/3}$

$$h'(x) = 3x^{-1/2} + x^{-2/3} = \frac{3}{\sqrt{x}} + \frac{1}{\sqrt[3]{x^2}}$$

29.  $f(\theta) = 2\theta - 3\sin\theta$

$$f'(\theta) = 2 - 3\cos\theta$$

9. Using the limit definition, you obtain  $g'(x) = \frac{4}{3}x - \frac{1}{6}$ .

At  $x = -1$ ,  $g'(-1) = -\frac{4}{3} - \frac{1}{6} = -\frac{3}{2}$

13.  $g'(2) = \lim_{x \rightarrow 2} \frac{g(x) - g(2)}{x - 2}$

$$= \lim_{x \rightarrow 2} \frac{x^2(x - 1) - 4}{x - 2}$$

$$= \lim_{x \rightarrow 2} \frac{x^3 - x^2 - 4}{x - 2}$$

$$= \lim_{x \rightarrow 2} \frac{(x - 2)(x^2 + x + 2)}{x - 2}$$

$$= \lim_{x \rightarrow 2} (x^2 + x + 2) = 8$$

23.  $f(x) = x^3 - 3x^2$

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

27.  $g(t) = \frac{2}{3}t^{-2}$

$$g'(t) = \frac{-4}{3}t^{-3} = -\frac{4}{3t^3}$$

31.  $f(\theta) = 3\cos\theta - \frac{\sin\theta}{4}$

$$f'(\theta) = -3\sin\theta - \frac{\cos\theta}{4}$$

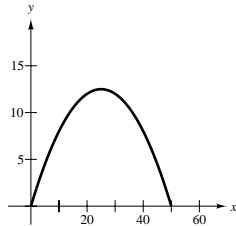
33.  $F = 200\sqrt{T}$

$$F'(t) = \frac{100}{\sqrt{T}}$$

(a) When  $T = 4$ ,  $F'(4) = 50$  vibrations/sec/lb.

(b) When  $T = 9$ ,  $F'(9) = 33\frac{1}{3}$  vibrations/sec/lb.

37. (a)



Total horizontal distance: 50

(b)  $0 = x - 0.02x^2$

$$0 = x\left(1 - \frac{x}{50}\right) \text{ implies } x = 50.$$

39.  $x(t) = t^2 - 3t + 2 = (t - 2)(t - 1)$

(a)  $v(t) = x'(t) = 2t - 3$

$$a(t) = v'(t) = 2$$

(c)  $v(t) = 0$  for  $t = \frac{3}{2}$ .

$$x = \left(\frac{3}{2} - 2\right)\left(\frac{3}{2} - 1\right) = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{4}$$

41.  $f(x) = (3x^2 + 7)(x^2 - 2x + 3)$

$$\begin{aligned} f'(x) &= (3x^2 + 7)(2x - 2) + (x^2 - 2x + 3)(6x) \\ &= 2(6x^3 - 9x^2 + 16x - 7) \end{aligned}$$

45.  $f(x) = 2x - x^{-2}$

$$\begin{aligned} f'(x) &= 2 + 2x^{-3} = 2\left(1 + \frac{1}{x^3}\right) \\ &= \frac{2(x^3 + 1)}{x^3} \end{aligned}$$

49.  $f(x) = (4 - 3x^2)^{-1}$

$$f'(x) = -(4 - 3x^2)^{-2}(-6x) = \frac{6x}{(4 - 3x^2)^2}$$

53.  $y = 3x^2 \sec x$

$$y' = 3x^2 \sec x \tan x + 6x \sec x$$

35.  $s(t) = -16t^2 + s_0$

$$s(9.2) = -16(9.2)^2 + s_0 = 0$$

$$s_0 = 1354.24$$

The building is approximately 1354 feet high (or 415 m).

(c) Ball reaches maximum height when  $x = 25$ .

(d)  $y = x - 0.02x^2$

$$y' = 1 - 0.04x$$

$$y'(0) = 1$$

$$y'(10) = 0.6$$

$$y'(25) = 0$$

$$y'(30) = -0.2$$

$$y'(50) = -1$$

(e)  $y'(25) = 0$

(b)  $v(t) < 0$  for  $t < \frac{3}{2}$ .

(d)  $x(t) = 0$  for  $t = 1, 2$ .

$$|v(1)| = |2(1) - 3| = 1$$

$$|v(2)| = |2(2) - 3| = 1$$

The speed is 1 when the position is 0.

43.  $h(x) = \sqrt{x} \sin x = x^{1/2} \sin x$

$$h'(x) = \frac{1}{2\sqrt{x}} \sin x + \sqrt{x} \cos x$$

47.  $f(x) = \frac{x^2 + x - 1}{x^2 - 1}$

$$\begin{aligned} f'(x) &= \frac{(x^2 - 1)(2x + 1) - (x^2 + x - 1)(2x)}{(x^2 - 1)^2} \\ &= \frac{-(x^2 + 1)}{(x^2 - 1)^2} \end{aligned}$$

51.  $y = \frac{x^2}{\cos x}$

$$y' = \frac{\cos x (2x) - x^2(-\sin x)}{\cos^2 x} = \frac{2x \cos x + x^2 \sin x}{\cos^2 x}$$

55.  $y = -x \tan x$

$$y' = -x \sec^2 x - \tan x$$

57.  $y = x \cos x - \sin x$

$$y' = -x \sin x + \cos x - \cos x = -x \sin x$$

61.  $f(\theta) = 3 \tan \theta$

$$f'(\theta) = 3 \sec^2 \theta$$

$$f''(\theta) = 6 \sec \theta (\sec \theta \tan \theta) = 6 \sec^2 \theta \tan \theta$$

65.  $f(x) = (1 - x^3)^{1/2}$

$$\begin{aligned} f'(x) &= \frac{1}{2}(1 - x^3)^{-1/2}(-3x^2) \\ &= -\frac{3x^2}{2\sqrt{1 - x^3}} \end{aligned}$$

69.  $f(s) = (s^2 - 1)^{5/2}(s^3 + 5)$

$$\begin{aligned} f'(s) &= (s^2 - 1)^{5/2}(3s^2) + (s^3 + 5)\left(\frac{5}{2}\right)(s^2 - 1)^{3/2}(2s) \\ &= s(s^2 - 1)^{3/2}[3s(s^2 - 1) + 5(s^3 + 5)] \\ &= s(s^2 - 1)^{3/2}(8s^3 - 3s + 25) \end{aligned}$$

73.  $y = \frac{1}{2} \csc 2x$

$$\begin{aligned} y' &= \frac{1}{2}(-\csc 2x \cot 2x)(2) \\ &= -\csc 2x \cot 2x \end{aligned}$$

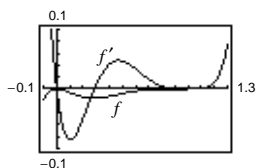
77.  $y = \frac{2}{3} \sin^{3/2} x - \frac{2}{7} \sin^{7/2} x$

$$\begin{aligned} y' &= \sin^{1/2} x \cos x - \sin^{5/2} x \cos x \\ &= (\cos x) \sqrt{\sin x} (1 - \sin^2 x) \\ &= (\cos^3 x) \sqrt{\sin x} \end{aligned}$$

81.  $f(t) = t^2(t - 1)^5$

$$f'(t) = t(t - 1)^4(7t - 2)$$

The zeros of  $f'$  correspond to the points on the graph of  $f$  where the tangent line is horizontal.



59.  $g(t) = t^3 - 3t + 2$

$$g'(t) = 3t^2 - 3$$

$$g''(t) = 6t$$

63.  $y = 2 \sin x + 3 \cos x$

$$y' = 2 \cos x - 3 \sin x$$

$$y'' = -2 \sin x - 3 \cos x$$

$$\begin{aligned} y'' + y &= -(2 \sin x + 3 \cos x) + (2 \sin x + 3 \cos x) \\ &= 0 \end{aligned}$$

67.  $h(x) = \left(\frac{x-3}{x^2+1}\right)^2$

$$\begin{aligned} h'(x) &= 2\left(\frac{x-3}{x^2+1}\right)\left(\frac{(x^2+1)(1) - (x-3)(2x)}{(x^2+1)^2}\right) \\ &= \frac{2(x-3)(-x^2+6x+1)}{(x^2+1)^3} \end{aligned}$$

71.  $y = 3 \cos(3x + 1)$

$$y' = -9 \sin(3x + 1)$$

75.  $y = \frac{x}{2} - \frac{\sin 2x}{4}$

$$\begin{aligned} y' &= \frac{1}{2} - \frac{1}{4} \cos 2x(2) \\ &= \frac{1}{2}(1 - \cos 2x) = \sin^2 x \end{aligned}$$

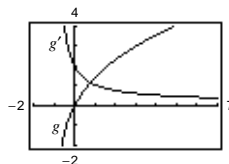
79.  $y = \frac{\sin \pi x}{x + 2}$

$$y' = \frac{(x+2)\pi \cos \pi x - \sin \pi x}{(x+2)^2}$$

83.  $g(x) = 2x(x+1)^{-1/2}$

$$g'(x) = \frac{x+2}{(x+1)^{3/2}}$$

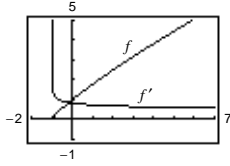
$g'$  does not equal zero for any value of  $x$  in the domain. The graph of  $g$  has no horizontal tangent lines.



85.  $f(t) = (t + 1)^{1/2}(t + 1)^{1/3} = (t + 1)^{5/6}$

$$f'(t) = \frac{5}{6(t + 1)^{1/6}}$$

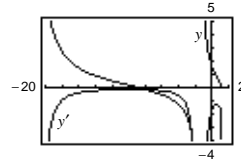
$f'$  does not equal zero for any  $x$  in the domain. The graph of  $f$  has no horizontal tangent lines.



87.  $y = \tan \sqrt{1 - x}$

$$y' = -\frac{\sec^2 \sqrt{1 - x}}{2\sqrt{1 - x}}$$

$y'$  does not equal zero for any  $x$  in the domain. The graph has no horizontal tangent lines.



89.  $y = 2x^2 + \sin 2x$

$$y' = 4x + 2 \cos 2x$$

$$y'' = 4 - 4 \sin 2x$$

91.  $f(x) = \cot x$

$$f'(x) = -\csc^2 x$$

$$f'' = -2 \csc x (-\csc x \cdot \cot x)$$

$$= 2 \csc^2 x \cot x$$

93.  $f(t) = \frac{t}{(1 - t)^2}$

$$f'(t) = \frac{t + 1}{(1 - t)^3}$$

$$f''(t) = \frac{2(t + 2)}{(1 - t)^4}$$

95.  $g(\theta) = \tan 3\theta - \sin(\theta - 1)$

$$g'(\theta) = 3 \sec^2 3\theta - \cos(\theta - 1)$$

$$g''(\theta) = 18 \sec^2 3\theta \tan 3\theta + \sin(\theta - 1)$$

97.  $T = 700(t^2 + 4t + 10)^{-1}$

$$T' = \frac{-1400(t + 2)}{(t^2 + 4t + 10)^2}$$

(a) When  $t = 1$ ,

$$T' = \frac{-1400(1 + 2)}{(1 + 4 + 10)^2} \approx -18.667 \text{ deg/hr.}$$

(c) When  $t = 5$ ,

$$T' = \frac{-1400(5 + 2)}{(25 + 30 + 10)^2} \approx -3.240 \text{ deg/hr.}$$

(b) When  $t = 3$ ,

$$T' = \frac{-1400(3 + 2)}{(9 + 12 + 10)^2} \approx -7.284 \text{ deg/hr.}$$

(d) When  $t = 10$ ,

$$T' = \frac{-1400(10 + 2)}{(100 + 40 + 10)^2} \approx -0.747 \text{ deg/hr.}$$

99.  $x^2 + 3xy + y^3 = 10$

$$2x + 3xy' + 3y + 3y^2y' = 0$$

$$3(x + y^2)y' = -(2x + 3y)$$

$$y' = \frac{-(2x + 3y)}{3(x + y^2)}$$

101.  $y\sqrt{x} - x\sqrt{y} = 16$

$$y\left(\frac{1}{2}x^{-1/2}\right) + x^{1/2}y' - x\left(\frac{1}{2}y^{-1/2}y'\right) - y^{1/2} = 0$$

$$\left(\sqrt{x} - \frac{x}{2\sqrt{y}}\right)y' = \sqrt{y} - \frac{y}{2\sqrt{x}}$$

$$\frac{2\sqrt{xy} - x}{2\sqrt{y}}y' = \frac{2\sqrt{xy} - y}{2\sqrt{x}}$$

$$y' = \frac{2\sqrt{xy} - y}{2\sqrt{x}} \cdot \frac{2\sqrt{y}}{2\sqrt{xy} - x} = \frac{2y\sqrt{x} - y\sqrt{y}}{2x\sqrt{y} - x\sqrt{x}}$$

103.  $x \sin y = y \cos x$

$$(x \cos y)y' + \sin y = -y \sin x + y' \cos x$$

$$y'(x \cos y - \cos x) = -y \sin x - \sin y$$

$$y' = \frac{y \sin x + \sin y}{\cos x - x \cos y}$$

107.  $y = \sqrt{x}$

$$\frac{dy}{dt} = 2 \text{ units/sec}$$

$$\frac{dy}{dt} = \frac{1}{2\sqrt{x}} \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = 2\sqrt{x} \frac{dy}{dt} = 4\sqrt{x}$$

(a) When  $x = \frac{1}{2}$ ,  $\frac{dx}{dt} = 2\sqrt{2}$  units/sec.

(b) When  $x = 1$ ,  $\frac{dx}{dt} = 4$  units/sec.

(c) When  $x = 4$ ,  $\frac{dx}{dt} = 8$  units/sec.

109.  $\frac{s}{h} = \frac{1/2}{2}$

$$s = \frac{1}{4}h$$

$$\frac{dV}{dt} = 1$$

Width of water at depth  $h$ :

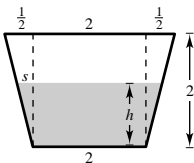
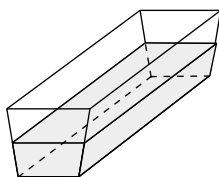
$$w = 2 + 2s = 2 + 2\left(\frac{1}{4}h\right) = \frac{4 + h}{2}$$

$$V = \frac{5}{2}\left(2 + \frac{4 + h}{2}\right)h = \frac{5}{4}(8 + h)h$$

$$\frac{dV}{dt} = \frac{5}{2}(4 + h)\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{2(dV/dt)}{5(4 + h)}$$

When  $h = 1$ ,  $\frac{dh}{dt} = \frac{2}{25}$  m/min.



105.  $x^2 + y^2 = 20$

$$2x + 2yy' = 0$$

$$y' = -\frac{x}{y}$$

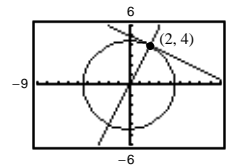
At  $(2, 4)$ :  $y' = -\frac{1}{2}$

Tangent line:  $y - 4 = -\frac{1}{2}(x - 2)$

$$x + 2y - 10 = 0$$

Normal line:  $y - 4 = 2(x - 2)$

$$2x - y = 0$$



111.  $s(t) = 60 - 4.9t^2$

$$s'(t) = -9.8t$$

$$s = 35 = 60 - 4.9t^2$$

$$4.9t^2 = 25$$

$$t = \frac{5}{\sqrt{4.9}}$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{s(t)}{x(t)}$$

$$x(t) = \sqrt{3}s(t)$$

$$\frac{dx}{dt} = \sqrt{3} \frac{ds}{dt} = \sqrt{3}(-9.8) \frac{5}{\sqrt{4.9}}$$

$$\approx -38.34 \text{ m/sec}$$

