

# CHAPTER 4

## Integration

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# CHAPTER 4

## Integration

### Section 4.1 Antiderivatives and Indefinite Integration

Solutions to Odd-Numbered Exercises

$$1. \frac{d}{dx}\left(\frac{3}{x^3} + C\right) = \frac{d}{dx}(3x^{-3} + C) = -9x^{-4} = \frac{-9}{x^4}$$

$$3. \frac{d}{dx}\left(\frac{1}{3}x^3 - 4x + C\right) = x^2 - 4 = (x - 2)(x + 2)$$

$$5. \frac{dy}{dt} = 3t^2$$

$$y = t^3 + C$$

$$\text{Check: } \frac{d}{dt}[t^3 + C] = 3t^2$$

$$7. \frac{dy}{dx} = x^{3/2}$$

$$y = \frac{2}{5}x^{5/2} + C$$

$$\text{Check: } \frac{d}{dx}\left[\frac{2}{5}x^{5/2} + C\right] = x^{3/2}$$

Given

Rewrite

Integrate

Simplify

$$9. \int \sqrt[3]{x} dx$$

$$\int x^{1/3} dx$$

$$\frac{x^{4/3}}{4/3} + C$$

$$\frac{3}{4}x^{4/3} + C$$

$$11. \int \frac{1}{x\sqrt{x}} dx$$

$$\int x^{-3/2} dx$$

$$\frac{x^{-1/2}}{-1/2} + C$$

$$-\frac{2}{\sqrt{x}} + C$$

$$13. \int \frac{1}{2x^3} dx$$

$$\frac{1}{2} \int x^{-3} dx$$

$$\frac{1}{2} \left(\frac{x^{-2}}{-2}\right) + C$$

$$-\frac{1}{4x^2} + C$$

$$15. \int (x + 3) dx = \frac{x^2}{2} + 3x + C$$

$$\text{Check: } \frac{d}{dx}\left[\frac{x^2}{2} + 3x + C\right] = x + 3$$

$$17. \int (2x - 3x^2) dx = x^2 - x^3 + C$$

$$\text{Check: } \frac{d}{dx}[x^2 - x^3 + C] = 2x - 3x^2$$

$$19. \int (x^3 + 2) dx = \frac{1}{4}x^4 + 2x + C$$

$$\text{Check: } \frac{d}{dx}\left(\frac{1}{4}x^4 + 2x + C\right) = x^3 + 2$$

$$21. \int (x^{3/2} + 2x + 1) dx = \frac{2}{5}x^{5/2} + x^2 + x + C$$

$$\text{Check: } \frac{d}{dx}\left(\frac{2}{5}x^{5/2} + x^2 + x + C\right) = x^{3/2} + 2x + 1$$

$$23. \int \sqrt[3]{x^2} dx = \int x^{2/3} dx = \frac{x^{5/3}}{5/3} + C = \frac{3}{5}x^{5/3} + C$$

$$\text{Check: } \frac{d}{dx}\left(\frac{3}{5}x^{5/3} + C\right) = x^{2/3} = \sqrt[3]{x^2}$$

$$25. \int \frac{1}{x^3} dx = \int x^{-3} dx = \frac{x^{-2}}{-2} + C = -\frac{1}{2x^2} + C$$

$$\text{Check: } \frac{d}{dx}\left(-\frac{1}{2x^2} + C\right) = \frac{1}{x^3}$$

$$27. \int \frac{x^2 + x + 1}{\sqrt{x}} dx = \int (x^{3/2} + x^{1/2} + x^{-1/2}) dx = \frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} + 2x^{1/2} + C = \frac{2}{15}x^{1/2}(3x^2 + 5x + 15) + C$$

**Check:**  $\frac{d}{dx}\left(\frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} + 2x^{1/2} + C\right) = x^{3/2} + x^{1/2} + x^{-1/2} = \frac{x^2 + x + 1}{\sqrt{x}}$

$$29. \int (x + 1)(3x - 2) dx = \int (3x^2 + x - 2) dx = x^3 + \frac{1}{2}x^2 - 2x + C$$

**Check:**  $\frac{d}{dx}\left(x^3 + \frac{1}{2}x^2 - 2x + C\right) = 3x^2 + x - 2 = (x + 1)(3x - 2)$

$$31. \int y^2 \sqrt{y} dy = \int y^{5/2} dy = \frac{2}{7}y^{7/2} + C$$

**Check:**  $\frac{d}{dy}\left(\frac{2}{7}y^{7/2} + C\right) = y^{5/2} = y^2 \sqrt{y}$

$$33. \int dx = \int 1 dx = x + C$$

**Check:**  $\frac{d}{dx}(x + C) = 1$

$$35. \int (2 \sin x + 3 \cos x) dx = -2 \cos x + 3 \sin x + C$$

**Check:**  $\frac{d}{dx}(-2 \cos x + 3 \sin x + C) = 2 \sin x + 3 \cos x$

$$37. \int (1 - \csc t \cot t) dt = t + \csc t + C$$

**Check:**  $\frac{d}{dt}(t + \csc t + C) = 1 - \csc t \cot t$

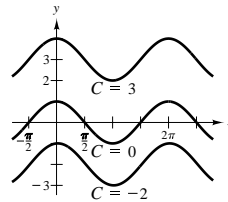
$$39. \int (\sec^2 \theta - \sin \theta) d\theta = \tan \theta + \cos \theta + C$$

**Check:**  $\frac{d}{d\theta}(\tan \theta + \cos \theta + C) = \sec^2 \theta - \sin \theta$

$$41. \int (\tan^2 y + 1) dy = \int \sec^2 y dy = \tan y + C$$

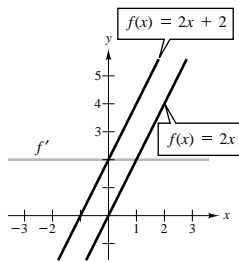
**Check:**  $\frac{d}{dy}(\tan y + C) = \sec^2 y = \tan^2 y + 1$

43.  $f(x) = \cos x$



45.  $f'(x) = 2$

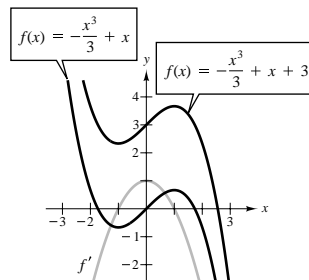
$f(x) = 2x + C$



Answers will vary.

47.  $f'(x) = 1 - x^2$

$f(x) = x - \frac{x^3}{3} + C$



Answers will vary.

49.  $\frac{dy}{dx} = 2x - 1, (1, 1)$

$y = \int (2x - 1) dx = x^2 - x + C$

$1 = (1)^2 - (1) + C \Rightarrow C = 1$

$y = x^2 - x + 1$

51.  $\frac{dy}{dx} = \cos x, (0, 4)$

$$y = \int \cos x \, dx = \sin x + C$$

$$4 = \sin 0 + C \Rightarrow C = 4$$

$$y = \sin x + 4$$

55.  $f'(x) = 4x, f(0) = 6$

$$f(x) = \int 4x \, dx = 2x^2 + C$$

$$f(0) = 6 = 2(0)^2 + C \Rightarrow C = 6$$

$$f(x) = 2x^2 + 6$$

59.  $f''(x) = 2$

$$f'(2) = 5$$

$$f(2) = 10$$

$$f'(x) = \int 2 \, dx = 2x + C_1$$

$$f'(2) = 4 + C_1 = 5 \Rightarrow C_1 = 1$$

$$f'(x) = 2x + 1$$

$$f(x) = \int (2x + 1) \, dx = x^2 + x + C_2$$

$$f(2) = 6 + C_2 = 10 \Rightarrow C_2 = 4$$

$$f(x) = x^2 + x + 4$$

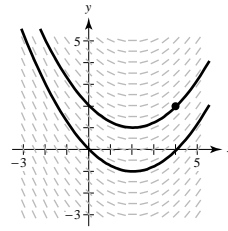
63. (a)  $h(t) = \int (1.5t + 5) \, dt = 0.75t^2 + 5t + C$

$$h(0) = 0 + 0 + C = 12 \Rightarrow C = 12$$

$$h(t) = 0.75t^2 + 5t + 12$$

(b)  $h(6) = 0.75(6)^2 + 5(6) + 12 = 69 \text{ cm}$

53. (a) Answers will vary.



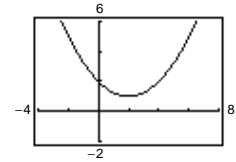
(b)  $\frac{dy}{dx} = \frac{1}{2}x - 1, (4, 2)$

$$y = \frac{x^2}{4} - x + C$$

$$2 = \frac{4^2}{4} - 4 + C$$

$$2 = C$$

$$y = \frac{x^2}{4} - x + 2$$



57.  $h'(t) = 8t^3 + 5, h(1) = -4$

$$h(t) = \int (8t^3 + 5) \, dt = 2t^4 + 5t + C$$

$$h(1) = -4 = 2 + 5 + C \Rightarrow C = -11$$

$$h(t) = 2t^4 + 5t - 11$$

61.  $f''(x) = x^{-3/2}$

$$f'(4) = 2$$

$$f(0) = 0$$

$$f'(x) = \int x^{-3/2} \, dx = -2x^{-1/2} + C_1 = -\frac{2}{\sqrt{x}} + C_1$$

$$f'(4) = -\frac{2}{2} + C_1 = 2 \Rightarrow C_1 = 3$$

$$f'(x) = -\frac{2}{\sqrt{x}} + 3$$

$$f(x) = \int \left(-2x^{-1/2} + 3\right) \, dx = -4x^{1/2} + 3x + C_2$$

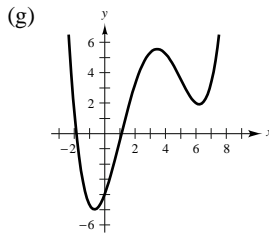
$$f(0) = 0 + 0 + C_2 = 0 \Rightarrow C_2 = 0$$

$$f(x) = -4x^{1/2} + 3x = -4\sqrt{x} + 3x$$

65.  $f(0) = -4$ . Graph of  $f'$  is given.

- (a)  $f'(4) \approx -1.0$   
 (b) No. The slopes of the tangent lines are greater than 2 on  $[0, 2]$ . Therefore,  $f$  must increase more than 4 units on  $[0, 4]$ .  
 (c) No,  $f(5) < f(4)$  because  $f$  is decreasing on  $[4, 5]$ .  
 (d)  $f$  is an maximum at  $x = 3.5$  because  $f'(3.5) \approx 0$  and the first derivative test.  
 (e)  $f$  is concave upward when  $f'$  is increasing on  $(-\infty, 1)$  and  $(5, \infty)$ .  $f$  is concave downward on  $(1, 5)$ . Points of inflection at  $x = 1, 5$ .

(f)  $f''$  is a minimum at  $x = 3$ .



67.  $a(t) = -32 \text{ ft/sec}^2$

$$v(t) = \int -32 \, dt = -32t + C_1$$

$$v(0) = 60 = C_1$$

$$s(t) = \int (-32t + 60) \, dt = -16t^2 + 60t + C_2$$

$$s(0) = 6 = C_2$$

$$s(t) = -16t^2 + 60t + 6 \text{ Position function}$$

The ball reaches its maximum height when

$$v(t) = -32t + 60 = 0$$

$$32t = 60$$

$$t = \frac{15}{8} \text{ seconds}$$

$$s\left(\frac{15}{8}\right) = -16\left(\frac{15}{8}\right)^2 + 60\left(\frac{15}{8}\right) + 6 \approx 62.26 \text{ feet}$$

71.  $a(t) = -9.8$

$$v(t) = \int -9.8 \, dt = -9.8t + C_1$$

$$v(0) = v_0 = C_1 \Rightarrow v(t) = -9.8t + v_0$$

$$f(t) = \int (-9.8t + v_0) \, dt = -4.9t^2 + v_0t + C_2$$

$$f(0) = s_0 = C_2 \Rightarrow f(t) = -4.9t^2 + v_0t + s_0$$

75.  $a = -1.6$

$$v(t) = \int -1.6 \, dt = -1.6t + v_0 = -1.6t, \text{ since the stone was dropped, } v_0 = 0.$$

$$s(t) = \int (-1.6t) \, dt = -0.8t^2 + s_0$$

$$s(20) = 0 \Rightarrow -0.8(20)^2 + s_0 = 0$$

$$s_0 = 320$$

Thus, the height of the cliff is 320 meters.

$$v(t) = -1.6t$$

$$v(20) = -32 \text{ m/sec}$$

69. From Exercise 68, we have:

$$s(t) = -16t^2 + v_0t$$

$$s'(t) = -32t + v_0 = 0 \text{ when } t = \frac{v_0}{32} = \text{time to reach maximum height.}$$

$$s\left(\frac{v_0}{32}\right) = -16\left(\frac{v_0}{32}\right)^2 + v_0\left(\frac{v_0}{32}\right) = 550$$

$$-\frac{v_0^2}{64} + \frac{v_0^2}{32} = 550$$

$$v_0^2 = 35,200$$

$$v_0 \approx 187.617 \text{ ft/sec}$$

73. From Exercise 71,  $f(t) = -4.9t^2 + 10t + 2$ .

$$v(t) = -9.8t + 10 = 0 \text{ (Maximum height when } v = 0.)$$

$$9.8t = 10$$

$$t = \frac{10}{9.8}$$

$$f\left(\frac{10}{9.8}\right) \approx 7.1 \text{ m}$$

$$77. x(t) = t^3 - 6t^2 + 9t - 2 \quad 0 \leq t \leq 5$$

$$(a) v(t) = x'(t) = 3t^2 - 12t + 9 \\ = 3(t^2 - 4t + 3) = 3(t-1)(t-3)$$

$$a(t) = v'(t) = 6t - 12 = 6(t-2)$$

$$(b) v(t) > 0 \text{ when } 0 < t < 1 \text{ or } 3 < t < 5.$$

$$(c) a(t) = 6(t-2) = 0 \text{ when } t = 2.$$

$$v(2) = 3(1)(-1) = -3$$

$$81. (a) v(0) = 25 \text{ km/hr} = 25 \cdot \frac{1000}{3600} = \frac{250}{36} \text{ m/sec}$$

$$v(13) = 80 \text{ km/hr} = 80 \cdot \frac{1000}{3600} = \frac{800}{36} \text{ m/sec}$$

$$a(t) = a \text{ (constant acceleration)}$$

$$v(t) = at + C$$

$$v(0) = \frac{250}{36} \Rightarrow v(t) = at + \frac{250}{36}$$

$$v(13) = \frac{800}{36} = 13a + \frac{250}{36}$$

$$\frac{550}{36} = 13a$$

$$a = \frac{550}{468} = \frac{275}{234} \approx 1.175 \text{ m/sec}^2$$

$$(b) s(t) = a \frac{t^2}{2} + \frac{250}{36}t \quad (s(0) = 0)$$

$$s(13) = \frac{275}{234} \frac{(13)^2}{2} + \frac{250}{36}(13) \approx 189.58 \text{ m}$$

$$85. \frac{(1 \text{ mi/hr})(5280 \text{ ft/mi})}{(3600 \text{ sec/hr})} = \frac{22}{15} \text{ ft/sec}$$

$t$	0	5	10	15	20	25	30
$V_1$ (ft/sec)	0	3.67	10.27	23.47	42.53	66	95.33
$V_2$ (ft/sec)	0	30.8	55.73	74.8	88	93.87	95.33

$$(c) S_1(t) = \int V_1(t) dt = \frac{0.1068}{3}t^3 - \frac{0.0416}{2}t^2 + 0.3679t$$

$$S_2(t) = \int V_2(t) dt = -\frac{0.1208t^3}{3} + \frac{6.7991t^2}{2} - 0.0707t$$

[In both cases, the constant of integration is 0 because  $S_1(0) = S_2(0) = 0$ ]

$$S_1(30) \approx 953.5 \text{ feet}$$

$$S_2(30) \approx 1970.3 \text{ feet}$$

The second car was going faster than the first until the end.

$$79. v(t) = \frac{1}{\sqrt{t}} = t^{-1/2} \quad t > 0$$

$$x(t) = \int v(t) dt = 2t^{1/2} + C$$

$$x(1) = 4 = 2(1) + C \Rightarrow C = 2$$

$$x(t) = 2t^{1/2} + 2 \text{ position function}$$

$$a(t) = v'(t) = -\frac{1}{2}t^{-3/2} = \frac{-1}{2t^{3/2}} \text{ acceleration}$$

$$83. \text{ Truck: } v(t) = 30$$

$$s(t) = 30t \text{ (Let } s(0) = 0.)$$

$$\text{Automobile: } a(t) = 6$$

$$v(t) = 6t \text{ (Let } v(0) = 0.)$$

$$s(t) = 3t^2 \text{ (Let } s(0) = 0.)$$

At the point where the automobile overtakes the truck:

$$30t = 3t^2$$

$$0 = 3t^2 - 30t$$

$$0 = 3t(t - 10) \text{ when } t = 10 \text{ sec.}$$

$$(a) s(10) = 3(10)^2 = 300 \text{ ft}$$

$$(b) v(10) = 6(10) = 60 \text{ ft/sec} \approx 41 \text{ mph}$$

$$(b) V_1(t) = 0.1068t^2 - 0.0416t + 0.3679$$

$$V_2(t) = -0.1208t^2 + 6.7991t - 0.0707$$

87.  $a(t) = k$

$v(t) = kt$

$s(t) = \frac{k}{2}t^2$  since  $v(0) = s(0) = 0$ .

At the time of lift-off,  $kt = 160$  and  $(k/2)t^2 = 0.7$ . Since  $(k/2)t^2 = 0.7$ ,

$$t = \sqrt{\frac{1.4}{k}}$$

$$v\left(\sqrt{\frac{1.4}{k}}\right) = k\sqrt{\frac{1.4}{k}} = 160$$

$$1.4k = 160^2 \Rightarrow k = \frac{160^2}{1.4}$$

$$\approx 18,285.714 \text{ mi/hr}^2$$

$$\approx 7.45 \text{ ft/sec}^2.$$

89. True

91. True

93. False. For example,  $\int x \cdot x \, dx \neq \int x \, dx \cdot \int x \, dx$  because  $\frac{x^3}{3} + C \neq \left(\frac{x^2}{2} + C_1\right)\left(\frac{x^2}{2} + C_2\right)$

$$95. f'(x) = \begin{cases} 1, & 0 \leq x < 2 \\ 3x, & 2 \leq x \leq 5 \end{cases}$$

$$f(x) = \begin{cases} x + C_1, & 0 \leq x < 2 \\ \frac{3x^2}{2} + C_2, & 2 \leq x \leq 5 \end{cases}$$

$$f(1) = 3 \Rightarrow 1 + C_1 = 3 \Rightarrow C_1 = 2$$

$f$  is continuous: Values must agree at  $x = 2$ :

$$4 = 6 + C_2 \Rightarrow C_2 = -2$$

$$f(x) = \begin{cases} x + 2, & 0 \leq x < 2 \\ \frac{3x^2}{2} - 2, & 2 \leq x \leq 5 \end{cases}$$

The left and right hand derivatives at  $x = 2$  do not agree. Hence  $f$  is not differentiable at  $x = 2$ .

## Section 4.2 Area

$$1. \sum_{i=1}^5 (2i + 1) = 2 \sum_{i=1}^5 i + \sum_{i=1}^5 1 = 2(1 + 2 + 3 + 4 + 5) + 5 = 35$$

$$3. \sum_{k=0}^4 \frac{1}{k^2 + 1} = 1 + \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} = \frac{158}{85}$$

$$7. \sum_{i=1}^9 \frac{1}{3i}$$

$$9. \sum_{j=1}^8 \left[ 5\left(\frac{j}{8}\right) + 3 \right]$$

$$5. \sum_{k=1}^4 c = c + c + c + c = 4c$$

$$11. \frac{2}{n} \sum_{i=1}^n \left[ \left(\frac{2i}{n}\right)^3 - \left(\frac{2i}{n}\right) \right] \quad 13. \frac{3}{n} \sum_{i=1}^n \left[ 2\left(1 + \frac{3i}{n}\right)^2 \right]$$

$$15. \sum_{i=1}^{20} 2i = 2 \sum_{i=1}^{20} i = 2 \left[ \frac{20(21)}{2} \right] = 420$$

$$17. \sum_{i=1}^{20} (i - 1)^2 = \sum_{i=1}^{19} i^2$$

$$= \left[ \frac{19(20)(39)}{6} \right] = 2470$$

$$\begin{aligned}
 19. \sum_{i=1}^{15} i(i-1)^2 &= \sum_{i=1}^{15} i^3 - 2 \sum_{i=1}^{15} i^2 + \sum_{i=1}^{15} i \\
 &= \frac{15^2(16)^2}{4} - 2 \frac{15(16)(31)}{6} + \frac{15(16)}{2} \\
 &= 14,400 - 2,480 + 120 \\
 &= 12,040
 \end{aligned}$$

$$21. \text{sum seq}(x \square 2 + 3, x, 1, 20, 1) = 2930 \quad (TI-82)$$

$$\begin{aligned}
 \sum_{i=1}^{20} (i^2 + 3) &= \frac{20(20+1)(2(20)+1)}{6} + 3(20) \\
 &= \frac{(20)(21)(41)}{6} + 60 = 2930
 \end{aligned}$$

$$\begin{aligned}
 23. S &= [3 + 4 + \frac{9}{2} + 5](1) = \frac{33}{2} = 16.5 \\
 s &= [1 + 3 + 4 + \frac{9}{2}](1) = \frac{25}{2} = 12.5
 \end{aligned}$$

$$\begin{aligned}
 25. S &= [3 + 3 + 5](1) = 11 \\
 s &= [2 + 2 + 3](1) = 7
 \end{aligned}$$

$$\begin{aligned}
 27. S(4) &= \sqrt{\frac{1}{4}}\left(\frac{1}{4}\right) + \sqrt{\frac{1}{2}}\left(\frac{1}{4}\right) + \sqrt{\frac{3}{4}}\left(\frac{1}{4}\right) + \sqrt{1}\left(\frac{1}{4}\right) = \frac{1 + \sqrt{2} + \sqrt{3} + 2}{8} \approx 0.768 \\
 s(4) &= 0\left(\frac{1}{4}\right) + \sqrt{\frac{1}{4}}\left(\frac{1}{4}\right) + \sqrt{\frac{1}{2}}\left(\frac{1}{4}\right) + \sqrt{\frac{3}{4}}\left(\frac{1}{4}\right) = \frac{1 + \sqrt{2} + \sqrt{3}}{8} \approx 0.518
 \end{aligned}$$

$$\begin{aligned}
 29. S(5) &= 1\left(\frac{1}{5}\right) + \frac{1}{6/5}\left(\frac{1}{5}\right) + \frac{1}{7/5}\left(\frac{1}{5}\right) + \frac{1}{8/5}\left(\frac{1}{5}\right) + \frac{1}{9/5}\left(\frac{1}{5}\right) = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} \approx 0.746 \\
 s(5) &= \frac{1}{6/5}\left(\frac{1}{5}\right) + \frac{1}{7/5}\left(\frac{1}{5}\right) + \frac{1}{8/5}\left(\frac{1}{5}\right) + \frac{1}{9/5}\left(\frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5}\right) = \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \approx 0.646
 \end{aligned}$$

$$31. \lim_{n \rightarrow \infty} \left[ \left( \frac{81}{n^4} \right) \frac{n^2(n+1)^2}{4} \right] = \frac{81}{4} \lim_{n \rightarrow \infty} \left[ \frac{n^4 + 2n^3 + n^2}{n^4} \right] = \frac{81}{4}(1) = \frac{81}{4}$$

$$33. \lim_{n \rightarrow \infty} \left[ \left( \frac{18}{n^2} \right) \frac{n(n+1)}{2} \right] = \frac{18}{2} \lim_{n \rightarrow \infty} \left[ \frac{n^2 + n}{n^2} \right] = \frac{18}{2}(1) = 9$$

$$\begin{aligned}
 35. \sum_{i=1}^n \frac{2i+1}{n^2} &= \frac{1}{n^2} \sum_{i=1}^n (2i+1) = \frac{1}{n^2} \left[ 2 \frac{n(n+1)}{2} + n \right] = \frac{n+2}{n} = S(n) \\
 S(10) &= \frac{12}{10} = 1.2
 \end{aligned}$$

$$S(100) = 1.02$$

$$S(1000) = 1.002$$

$$S(10,000) = 1.0002$$

$$\begin{aligned}
 37. \sum_{k=1}^n \frac{6k(k-1)}{n^3} &= \frac{6}{n^3} \sum_{k=1}^n (k^2 - k) = \frac{6}{n^3} \left[ \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} \right] \\
 &= \frac{6}{n^3} \left[ \frac{2n^2 + 3n + 1 - 3n - 3}{6} \right] = \frac{1}{n^2} [2n^2 - 2] = S(n)
 \end{aligned}$$

$$S(10) = 1.98$$

$$S(100) = 1.9998$$

$$S(1000) = 1.999998$$

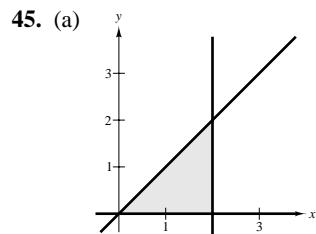
$$S(10,000) = 1.99999998$$

$$39. \lim_{n \rightarrow \infty} \sum_{i=1}^n \left( \frac{16i}{n^2} \right) = \lim_{n \rightarrow \infty} \frac{16}{n^2} \sum_{i=1}^n i = \lim_{n \rightarrow \infty} \frac{16}{n^2} \left( \frac{n(n+1)}{2} \right) = \lim_{n \rightarrow \infty} \left[ 8 \left( \frac{n^2 + n}{n^2} \right) \right] = 8 \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right) = 8$$



$$\begin{aligned}
 41. \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n^3} (i-1)^2 &= \lim_{n \rightarrow \infty} \frac{1}{n^3} \sum_{i=1}^{n-1} i^2 = \lim_{n \rightarrow \infty} \frac{1}{n^3} \left[ \frac{(n-1)(n)(2n-1)}{6} \right] \\
 &= \lim_{n \rightarrow \infty} \frac{1}{6} \left[ \frac{2n^3 - 3n^2 + n}{n^3} \right] = \lim_{n \rightarrow \infty} \left[ \frac{1}{6} \left( \frac{2 - (3/n) + (1/n^2)}{1} \right) \right] = \frac{1}{3}
 \end{aligned}$$

$$43. \lim_{n \rightarrow \infty} \sum_{i=1}^n \left( 1 + \frac{i}{n} \right) \left( \frac{2}{n} \right) = 2 \lim_{n \rightarrow \infty} \frac{1}{n} \left[ \sum_{i=1}^n 1 + \frac{1}{n} \sum_{i=1}^n i \right] = 2 \lim_{n \rightarrow \infty} \frac{1}{n} \left[ n + \frac{1}{n} \left( \frac{n(n+1)}{2} \right) \right] = 2 \lim_{n \rightarrow \infty} \left[ 1 + \frac{n^2 + n}{2n^2} \right] = 2 \left( 1 + \frac{1}{2} \right) = 3$$



(b)  $\Delta x = \frac{2-0}{n} = \frac{2}{n}$

Endpoints:

$$0 < 1 \left( \frac{2}{n} \right) < 2 \left( \frac{2}{n} \right) < \dots < (n-1) \left( \frac{2}{n} \right) < n \left( \frac{2}{n} \right) = 2$$

(c) Since  $y = x$  is increasing,  $f(m_i) = f(x_{i-1})$  on  $[x_{i-1}, x_i]$ .

$$\begin{aligned}
 s(n) &= \sum_{i=1}^n f(x_{i-1}) \Delta x \\
 &= \sum_{i=1}^n f \left( \frac{2i-2}{n} \right) \left( \frac{2}{n} \right) = \sum_{i=1}^n \left[ (i-1) \left( \frac{2}{n} \right) \right] \left( \frac{2}{n} \right)
 \end{aligned}$$

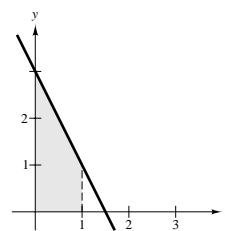
(d)  $f(M_i) = f(x_i)$  on  $[x_{i-1}, x_i]$

$$S(n) = \sum_{i=1}^n f(x_i) \Delta x = \sum_{i=1}^n f \left( \frac{2i}{n} \right) \frac{2}{n} = \sum_{i=1}^n \left[ i \left( \frac{2}{n} \right) \right] \left( \frac{2}{n} \right)$$

47.  $y = -2x + 3$  on  $[0, 1]$ . (Note:  $\Delta x = \frac{1-0}{n} = \frac{1}{n}$ )

$$\begin{aligned}
 s(n) &= \sum_{i=1}^n f \left( \frac{i}{n} \right) \left( \frac{1}{n} \right) = \sum_{i=1}^n \left[ -2 \left( \frac{i}{n} \right) + 3 \right] \left( \frac{1}{n} \right) \\
 &= 3 - \frac{2}{n^2} \sum_{i=1}^n i = 3 - \frac{2(n+1)n}{2n^2} = 2 - \frac{1}{n}
 \end{aligned}$$

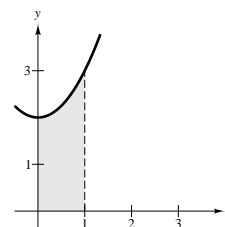
Area =  $\lim_{n \rightarrow \infty} s(n) = 2$



49.  $y = x^2 + 2$  on  $[0, 1]$ . (Note:  $\Delta x = \frac{1}{n}$ )

$$\begin{aligned}
 S(n) &= \sum_{i=1}^n f \left( \frac{i}{n} \right) \left( \frac{1}{n} \right) = \sum_{i=1}^n \left[ \left( \frac{i}{n} \right)^2 + 2 \right] \left( \frac{1}{n} \right) \\
 &= \left[ \frac{1}{n^3} \sum_{i=1}^n i^2 \right] + 2 = \frac{n(n+1)(2n+1)}{6n^3} + 2 = \frac{1}{6} \left( 2 + \frac{3}{n} + \frac{1}{n^2} \right) + 2
 \end{aligned}$$

Area =  $\lim_{n \rightarrow \infty} S(n) = \frac{7}{3}$



(e)

$x$	5	10	50	100
$s(n)$	1.6	1.8	1.96	1.98
$S(n)$	2.4	2.2	2.04	2.02

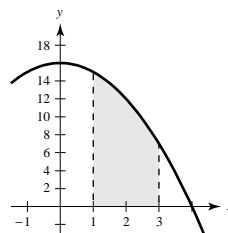
$$\begin{aligned}
 (f) \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[ (i-1) \left( \frac{2}{n} \right) \right] \left( \frac{2}{n} \right) &= \lim_{n \rightarrow \infty} \frac{4}{n^2} \sum_{i=1}^n (i-1) \\
 &= \lim_{n \rightarrow \infty} \frac{4}{n^2} \left[ \frac{n(n+1)}{2} - n \right] \\
 &= \lim_{n \rightarrow \infty} \left[ \frac{2(n+1)}{n} - \frac{4}{n} \right] = 2
 \end{aligned}$$

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[ i \left( \frac{2}{n} \right) \right] \left( \frac{2}{n} \right) &= \lim_{n \rightarrow \infty} \frac{4}{n^2} \sum_{i=1}^n i \\
 &= \lim_{n \rightarrow \infty} \left( \frac{4}{n^2} \right) \frac{n(n+1)}{2} \\
 &= \lim_{n \rightarrow \infty} \frac{2(n+1)}{n} = 2
 \end{aligned}$$

51.  $y = 16 - x^2$  on  $[1, 3]$ . (Note:  $\Delta x = \frac{2}{n}$ )

$$\begin{aligned} s(n) &= \sum_{i=1}^n f\left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^n \left[16 - \left(1 + \frac{2i}{n}\right)^2\right]\left(\frac{2}{n}\right) \\ &= \frac{2}{n} \sum_{i=1}^n \left[15 - \frac{4i^2}{n^2} - \frac{4i}{n}\right] \\ &= \frac{2}{n} \left[15n - \frac{4}{n^2} \frac{n(n+1)(2n+1)}{6} - \frac{4}{n} \frac{n(n+1)}{2}\right] \\ &= 30 - \frac{8}{6n^2}(n+1)(2n+1) - \frac{4}{n}(n+1) \end{aligned}$$

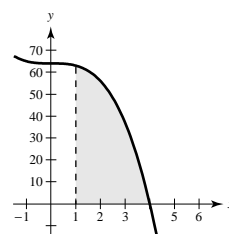
$$\text{Area} = \lim_{n \rightarrow \infty} s(n) = 30 - \frac{8}{3} - 4 = \frac{70}{3} = 23\frac{1}{3}$$



53.  $y = 64 - x^3$  on  $[1, 4]$ . (Note:  $\Delta x = \frac{4-1}{n} = \frac{3}{n}$ )

$$\begin{aligned} s(n) &= \sum_{i=1}^n f\left(1 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) = \sum_{i=1}^n \left[64 - \left(1 + \frac{3i}{n}\right)^3\right]\left(\frac{3}{n}\right) \\ &= \frac{3}{n} \sum_{i=1}^n \left[63 - \frac{27i^3}{n^3} - \frac{27i^2}{n^2} - \frac{9i}{n}\right] \\ &= \frac{3}{n} \left[63n - \frac{27}{n^3} \frac{n^2(n+1)^2}{4} - \frac{27}{n^2} \frac{n(n+1)(2n+1)}{6} - \frac{9}{n} \frac{n(n+1)}{2}\right] \\ &= 189 - \frac{81}{4n^2}(n+1)^2 - \frac{81}{6n^2}(n+1)(2n+1) - \frac{27}{2} \frac{n+1}{n} \end{aligned}$$

$$\text{Area} = \lim_{n \rightarrow \infty} s(n) = 189 - \frac{81}{4} - 27 - \frac{27}{2} = \frac{513}{4} = 128.25$$

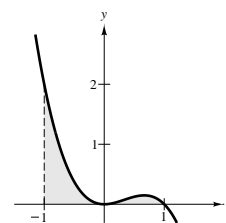


55.  $y = x^2 - x^3$  on  $[-1, 1]$ . (Note:  $\Delta x = \frac{1 - (-1)}{n} = \frac{2}{n}$ )

Again,  $T(n)$  is neither an upper nor a lower sum.

$$\begin{aligned} T(n) &= \sum_{i=1}^n f\left(-1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^n \left[\left(-1 + \frac{2i}{n}\right)^2 - \left(-1 + \frac{2i}{n}\right)^3\right]\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^n \left[\left(1 - \frac{4i}{n} + \frac{4i^2}{n^2}\right) - \left(-1 + \frac{6i}{n} - \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right)\right]\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^n \left[2 - \frac{10i}{n} + \frac{16i^2}{n^2} - \frac{8i^3}{n^3}\right]\left(\frac{2}{n}\right) = \frac{4}{n} \sum_{i=1}^n 1 - \frac{20}{n^2} \sum_{i=1}^n i + \frac{32}{n^3} \sum_{i=1}^n i^2 - \frac{16}{n^4} \sum_{i=1}^n i^3 \\ &= \frac{4}{n}(n) - \frac{20}{n^2} \cdot \frac{n(n+1)}{2} + \frac{32}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{16}{n^4} \cdot \frac{n^2(n+1)^2}{4} \\ &= 4 - 10\left(1 + \frac{1}{n}\right) + \frac{16}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) - 4\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) \end{aligned}$$

$$\text{Area} = \lim_{n \rightarrow \infty} T(n) = 4 - 10 + \frac{32}{3} - 4 = \frac{2}{3}$$

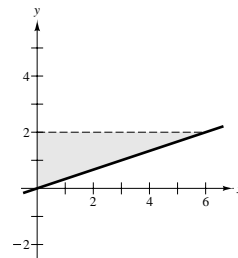


57.  $f(y) = 3y, 0 \leq y \leq 2$  (Note:  $\Delta y = \frac{2-0}{n} = \frac{2}{n}$ )

$$S(n) = \sum_{i=1}^n f(m_i) \Delta y = \sum_{i=1}^n f\left(\frac{2i}{n}\right) \left(\frac{2}{n}\right) = \sum_{i=1}^n 3\left(\frac{2i}{n}\right) \left(\frac{2}{n}\right)$$

$$= \frac{12}{n^2} \sum_{i=1}^n i = \left(\frac{12}{n^2}\right) \cdot \frac{n(n+1)}{2} = \frac{6(n+1)}{n} = 6 + \frac{6}{n}$$

$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = \lim_{n \rightarrow \infty} \left(6 + \frac{6}{n}\right) = 6$$

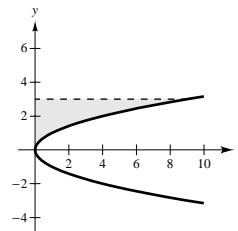


59.  $f(y) = y^2, 0 \leq y \leq 3$  (Note:  $\Delta y = \frac{3-0}{n} = \frac{3}{n}$ )

$$S(n) = \sum_{i=1}^n f\left(\frac{3i}{n}\right) \left(\frac{3}{n}\right) = \sum_{i=1}^n \left(\frac{3i}{n}\right)^2 \left(\frac{3}{n}\right) = \frac{27}{n^3} \sum_{i=1}^n i^2$$

$$= \frac{27}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{9}{n^2} \left(\frac{2n^2+3n+1}{2}\right) = 9 + \frac{27}{2n} + \frac{9}{2n^2}$$

$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = \lim_{n \rightarrow \infty} \left(9 + \frac{27}{2n} + \frac{9}{2n^2}\right) = 9$$



61.  $g(y) = 4y^2 - y^3, 1 \leq y \leq 3$ . (Note:  $\Delta y = \frac{3-1}{n} = \frac{2}{n}$ )

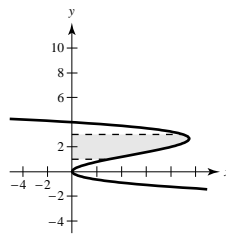
$$S(n) = \sum_{i=1}^n g\left(1 + \frac{2i}{n}\right) \left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^n \left[4\left(1 + \frac{2i}{n}\right)^2 - \left(1 + \frac{2i}{n}\right)^3\right] \frac{2}{n}$$

$$= \frac{2}{n} \sum_{i=1}^n \left[4\left(1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right) - \left(1 + \frac{6i}{n} + \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right)\right]$$

$$= \frac{2}{n} \sum_{i=1}^n \left[3 + \frac{10i}{n} + \frac{4i^2}{n^2} - \frac{8i^3}{n^3}\right] = \frac{2}{n} \left[3n + \frac{10}{n} \frac{n(n+1)}{2} + \frac{4}{n^2} \frac{n(n+1)(2n+1)}{6} - \frac{8}{n^2} \frac{n^2(n+1)^2}{4}\right]$$

$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = 6 + 10 + \frac{8}{3} - 4 = \frac{44}{3}$$



63.  $f(x) = x^2 + 3, 0 \leq x \leq 2, n = 4$

$$\text{Let } c_i = \frac{x_i + x_{i-1}}{2}.$$

$$\Delta x = \frac{1}{2}, c_1 = \frac{1}{4}, c_2 = \frac{3}{4}, c_3 = \frac{5}{4}, c_4 = \frac{7}{4}$$

$$\text{Area} \approx \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^4 [c_i^2 + 3] \left(\frac{1}{2}\right)$$

$$= \frac{1}{2} \left[ \left(\frac{1}{16} + 3\right) + \left(\frac{9}{16} + 3\right) + \left(\frac{25}{16} + 3\right) + \left(\frac{49}{16} + 3\right) \right]$$

$$= \frac{69}{8}$$

65.  $f(x) = \tan x, 0 \leq x \leq \frac{\pi}{4}, n = 4$

$$\text{Let } c_i = \frac{x_i + x_{i-1}}{2}.$$

$$\Delta x = \frac{\pi}{16}, c_1 = \frac{\pi}{32}, c_2 = \frac{3\pi}{32}, c_3 = \frac{5\pi}{32}, c_4 = \frac{7\pi}{32}$$

$$\text{Area} \approx \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^4 (\tan c_i) \left(\frac{\pi}{16}\right)$$

$$= \frac{\pi}{16} \left( \tan \frac{\pi}{32} + \tan \frac{3\pi}{32} + \tan \frac{5\pi}{32} + \tan \frac{7\pi}{32} \right) \approx 0.345$$

67.  $f(x) = \sqrt{x}$  on  $[0, 4]$ .

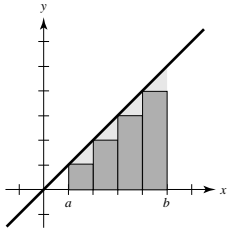
$n$	4	8	12	16	20
Approximate area	5.3838	5.3523	5.3439	5.3403	5.3384

(Exact value is  $16/3$ )

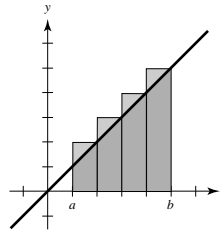
69.  $f(x) = \tan\left(\frac{\pi x}{8}\right)$  on  $[1, 3]$ .

$n$	4	8	12	16	20
Approximate area	2.2223	2.2387	2.2418	2.2430	2.2435

71. We can use the line  $y = x$  bounded by  $x = a$  and  $x = b$ . The sum of the areas of these inscribed rectangles is the lower sum.



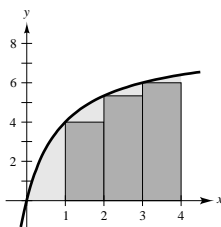
The sum of the areas of these circumscribed rectangles is the upper sum.



We can see that the rectangles do not contain all of the area in the first graph and the rectangles in the second graph cover more than the area of the region.

The exact value of the area lies between these two sums.

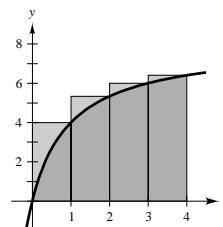
73. (a)



Lower sum:

$$s(4) = 0 + 4 + 5\frac{1}{3} + 6 = 15\frac{1}{3} = \frac{46}{3} \approx 15.333$$

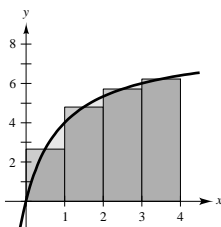
(b)



Upper sum:

$$S(4) = 4 + 5\frac{1}{3} + 6 + 6\frac{2}{5} = 21\frac{11}{15} = \frac{326}{15} \approx 21.733$$

(c)



Midpoint Rule:

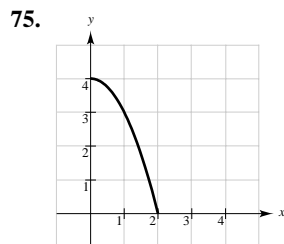
$$M(4) = 2\frac{2}{3} + 4\frac{4}{5} + 5\frac{5}{7} + 6\frac{2}{9} = \frac{6112}{315} \approx 19.403$$

(d) In each case,  $\Delta x = 4/n$ . The lower sum uses left endpoints,  $(i - 1)(4/n)$ . The upper sum uses right endpoints,  $(i)(4/n)$ . The Midpoint Rule uses midpoints,  $(i - \frac{1}{2})(4/n)$ .

(e)

$n$	4	8	20	100	200
$s(n)$	15.333	17.368	18.459	18.995	19.06
$S(n)$	21.733	20.568	19.739	19.251	19.188
$M(n)$	19.403	19.201	19.137	19.125	19.125

(f)  $s(n)$  increases because the lower sum approaches the exact value as  $n$  increases.  $S(n)$  decreases because the upper sum approaches the exact value as  $n$  increases. Because of the shape of the graph, the lower sum is always smaller than the exact value, whereas the upper sum is always larger.



b.  $A \approx 6$  square units

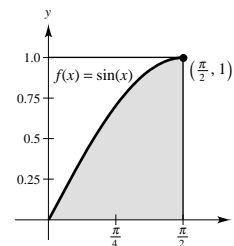
79.  $f(x) = \sin x, \left[0, \frac{\pi}{2}\right]$

Let  $A_1$  = area bounded by  $f(x) = \sin x$ , the  $x$ -axis,  $x = 0$  and  $x = \pi/2$ . Let  $A_2$  = area of the rectangle bounded by  $y = 1$ ,  $y = 0$ ,  $x = 0$ , and  $x = \pi/2$ . Thus,  $A_2 = (\pi/2)(1) \approx 1.570796$ .

In this program, the computer is generating  $N_2$  pairs of random points in the rectangle whose area is represented by  $A_2$ . It is keeping track of how many of these points,  $N_1$ , lie in the region whose area is represented by  $A_1$ . Since the points are randomly generated, we assume that

$$\frac{A_1}{A_2} \approx \frac{N_1}{N_2} \Rightarrow A_1 \approx \frac{N_1}{N_2} A_2.$$

The larger  $N_2$  is the better the approximation to  $A_1$ .



81. Suppose there are  $n$  rows in the figure. The stars on the left total  $1 + 2 + \dots + n$ , as do the stars on the right. There are  $n(n + 1)$  stars in total, hence

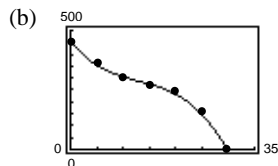
$$2[1 + 2 + \dots + n] = n(n + 1)$$

$$1 + 2 + \dots + n = \frac{1}{2}n(n + 1).$$

83. (a)  $y = (-4.09 \times 10^{-5})x^3 + 0.016x^2 - 2.67x + 452.9$

(c) Using the integration capability of a graphing utility, you obtain

$$A \approx 76,897.5 \text{ ft}^2.$$



### Section 4.3 Riemann Sums and Definite Integrals

1.  $f(x) = \sqrt{x}, y = 0, x = 0, x = 3, c_i = \frac{3i^2}{n^2}$

$$\Delta x_i = \frac{3i^2}{n^2} - \frac{3(i-1)^2}{n^2} = \frac{3}{n^2}(2i-1)$$

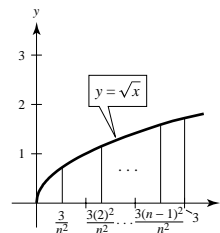
$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x_i = \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{\frac{3i^2}{n^2}} \frac{3}{n^2} (2i-1)$$

$$= \lim_{n \rightarrow \infty} \frac{3\sqrt{3}}{n^3} \sum_{i=1}^n (2i^2 - i)$$

$$= \lim_{n \rightarrow \infty} \frac{3\sqrt{3}}{n^3} \left[ 2 \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} \right]$$

$$= \lim_{n \rightarrow \infty} 3\sqrt{3} \left[ \frac{(n+1)(2n+1)}{3n^2} - \frac{n+1}{2n^2} \right]$$

$$= 3\sqrt{3} \left[ \frac{2}{3} - 0 \right] = 2\sqrt{3} \approx 3.464$$



3.  $y = 6$  on  $[4, 10]$ . (Note:  $\Delta x = \frac{10-4}{n} = \frac{6}{n}$ ,  $\|\Delta\| \rightarrow 0$  as  $n \rightarrow \infty$ )

$$\sum_{i=1}^n f(c_i) \Delta x_i = \sum_{i=1}^n f\left(4 + \frac{6i}{n}\right) \left(\frac{6}{n}\right) = \sum_{i=1}^n 6 \left(\frac{6}{n}\right) = \sum_{i=1}^n \frac{36}{n} = 36$$

$$\int_4^{10} 6 \, dx = \lim_{n \rightarrow \infty} 36 = 36$$

5.  $y = x^3$  on  $[-1, 1]$ . (Note:  $\Delta x = \frac{1 - (-1)}{n} = \frac{2}{n}$ ,  $\|\Delta\| \rightarrow 0$  as  $n \rightarrow \infty$ )

$$\sum_{i=1}^n f(c_i) \Delta x_i = \sum_{i=1}^n f\left(-1 + \frac{2i}{n}\right) \left(\frac{2}{n}\right) = \sum_{i=1}^n \left(-1 + \frac{2i}{n}\right)^3 \left(\frac{2}{n}\right) = \sum_{i=1}^n \left[-1 + \frac{6i}{n} - \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right] \left(\frac{2}{n}\right)$$

$$= -2 + \frac{12}{n^2} \sum_{i=1}^n i - \frac{24}{n^3} \sum_{i=1}^n i^2 + \frac{16}{n^4} \sum_{i=1}^n i^3$$

$$= -2 + 6\left(1 + \frac{1}{n}\right) - 4\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 4\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) = \frac{2}{n}$$

$$\int_{-1}^1 x^3 \, dx = \lim_{n \rightarrow \infty} \frac{2}{n} = 0$$

7.  $y = x^2 + 1$  on  $[1, 2]$ . (Note:  $\Delta x = \frac{2-1}{n} = \frac{1}{n}$ ,  $\|\Delta\| \rightarrow 0$  as  $n \rightarrow \infty$ )

$$\sum_{i=1}^n f(c_i) \Delta x_i = \sum_{i=1}^n f\left(1 + \frac{i}{n}\right) \left(\frac{1}{n}\right) = \sum_{i=1}^n \left[\left(1 + \frac{i}{n}\right)^2 + 1\right] \left(\frac{1}{n}\right) = \sum_{i=1}^n \left[1 + \frac{2i}{n} + \frac{i^2}{n^2} + 1\right] \left(\frac{1}{n}\right)$$

$$= 2 + \frac{2}{n^2} \sum_{i=1}^n i + \frac{1}{n^3} \sum_{i=1}^n i^2 = 2 + \left(1 + \frac{1}{n}\right) + \frac{1}{6} \left(2 + \frac{3}{n} + \frac{1}{n^2}\right) = \frac{10}{3} + \frac{3}{2n} + \frac{1}{6n^2}$$

$$\int_1^2 (x^2 + 1) \, dx = \lim_{n \rightarrow \infty} \left(\frac{10}{3} + \frac{3}{2n} + \frac{1}{6n^2}\right) = \frac{10}{3}$$

9.  $\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n (3c_i + 10) \Delta x_i = \int_{-1}^5 (3x + 10) \, dx$

on the interval  $[-1, 5]$ .

11.  $\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \sqrt{c_i^2 + 4} \Delta x_i = \int_0^3 \sqrt{x^2 + 4} \, dx$

on the interval  $[0, 3]$ .

13.  $\int_0^5 3 \, dx$

15.  $\int_{-4}^4 (4 - |x|) \, dx$

17.  $\int_{-2}^2 (4 - x^2) \, dx$

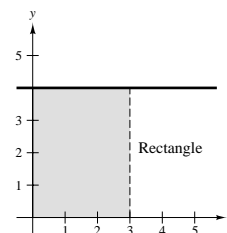
19.  $\int_0^\pi \sin x \, dx$

21.  $\int_0^2 y^3 \, dy$

23. Rectangle

$$A = bh = 3(4)$$

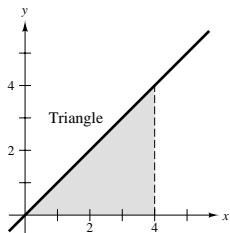
$$A = \int_0^3 4 \, dx = 12$$



25. Triangle

$$A = \frac{1}{2}bh = \frac{1}{2}(4)(4)$$

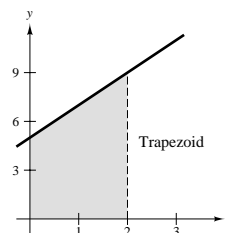
$$A = \int_0^4 x \, dx = 8$$



27. Trapezoid

$$A = \frac{b_1 + b_2}{2}h = \left(\frac{5 + 9}{2}\right)2$$

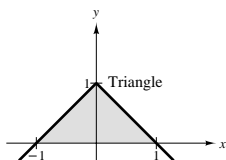
$$A = \int_0^2 (2x + 5) \, dx = 14$$



29. Triangle

$$A = \frac{1}{2}bh = \frac{1}{2}(2)(1)$$

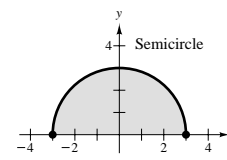
$$A = \int_{-1}^1 (1 - |x|) \, dx = 1$$



31. Semicircle

$$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(3)^2$$

$$A = \int_{-3}^3 \sqrt{9 - x^2} \, dx = \frac{9\pi}{2}$$



In Exercises 33–39,  $\int_2^4 x^3 \, dx = 60$ ,  $\int_2^4 x \, dx = 6$ ,  $\int_2^4 dx = 2$

$$33. \int_4^2 x \, dx = -\int_2^4 x \, dx = -6$$

$$37. \int_2^4 (x - 8) \, dx = \int_2^4 x \, dx - 8 \int_2^4 dx = 6 - 8(2) = -10$$

$$41. (a) \int_0^7 f(x) \, dx = \int_0^5 f(x) \, dx + \int_5^7 f(x) \, dx = 10 + 3 = 13$$

$$(b) \int_5^0 f(x) \, dx = -\int_0^5 f(x) \, dx = -10$$

$$(c) \int_5^5 f(x) \, dx = 0$$

$$(d) \int_0^5 3f(x) \, dx = 3 \int_0^5 f(x) \, dx = 3(10) = 30$$

$$35. \int_2^4 4x \, dx = 4 \int_2^4 x \, dx = 4(6) = 24$$

$$39. \int_2^4 \left(\frac{1}{2}x^3 - 3x + 2\right) \, dx = \frac{1}{2} \int_2^4 x^3 \, dx - 3 \int_2^4 x \, dx + 2 \int_2^4 dx$$

$$= \frac{1}{2}(60) - 3(6) + 2(2) = 16$$

$$43. (a) \int_2^6 [f(x) + g(x)] \, dx = \int_2^6 f(x) \, dx + \int_2^6 g(x) \, dx$$

$$= 10 + (-2) = 8$$

$$(b) \int_2^6 [g(x) - f(x)] \, dx = \int_2^6 g(x) \, dx - \int_2^6 f(x) \, dx$$

$$= -2 - 10 = -12$$

$$(c) \int_2^6 2g(x) \, dx = 2 \int_2^6 g(x) \, dx = 2(-2) = -4$$

$$(d) \int_2^6 3f(x) \, dx = 3 \int_2^6 f(x) \, dx = 3(10) = 30$$

$$45. (a) \text{ Quarter circle below } x\text{-axis: } -\frac{1}{4}\pi r^2 = -\frac{1}{4}\pi(2)^2 = -\pi$$

$$(b) \text{ Triangle: } \frac{1}{2}bh = \frac{1}{2}(4)(2) = 4$$

$$(c) \text{ Triangle + Semicircle below } x\text{-axis: } -\frac{1}{2}(2)(1) - \frac{1}{2}\pi(2)^2 = -(1 + 2\pi)$$

$$(d) \text{ Sum of parts (b) and (c): } 4 - (1 + 2\pi) = 3 - 2\pi$$

$$(e) \text{ Sum of absolute values of (b) and (c): } 4 + (1 + 2\pi) = 5 + 2\pi$$

$$(f) \text{ Answer to (d) plus } 2(10) = 20: (3 - 2\pi) + 20 = 23 - 2\pi$$

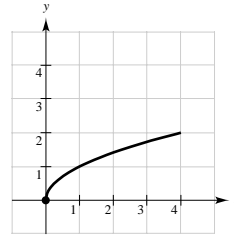
47. The left endpoint approximation will be greater than the actual area:  $>$

49. Because the curve is concave upward, the midpoint approximation will be less than the actual area:  $<$

51.  $f(x) = \frac{1}{x-4}$

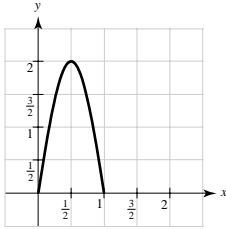
is not integrable on the interval  $[3, 5]$  and  $f$  has a discontinuity at  $x = 4$ .

53.



a.  $A \approx 5$  square units

55.



d.  $\int_0^1 2 \sin \pi x \, dx \approx \frac{1}{2}(1)(2) \approx 1$

57.  $\int_0^3 x\sqrt{3-x} \, dx$

$n$	4	8	12	16	20
$L(n)$	3.6830	3.9956	4.0707	4.1016	4.1177
$M(n)$	4.3082	4.2076	4.1838	4.1740	4.1690
$R(n)$	3.6830	3.9956	4.0707	4.1016	4.1177

59.  $\int_0^{\pi/2} \sin^2 x \, dx$

$n$	4	8	12	16	20
$L(n)$	0.5890	0.6872	0.7199	0.7363	0.7461
$M(n)$	0.7854	0.7854	0.7854	0.7854	0.7854
$R(n)$	0.9817	0.8836	0.8508	0.8345	0.8247

61. True

63. True

65. False

$$\int_0^2 (-x) \, dx = -2$$

67.  $f(x) = x^2 + 3x, [0, 8]$

$$x_0 = 0, x_1 = 1, x_2 = 3, x_3 = 7, x_4 = 8$$

$$\Delta x_1 = 1, \Delta x_2 = 2, \Delta x_3 = 4, \Delta x_4 = 1$$

$$c_1 = 1, c_2 = 2, c_3 = 5, c_4 = 8$$

$$\begin{aligned} \sum_{i=1}^4 f(c_i) \Delta x &= f(1) \Delta x_1 + f(2) \Delta x_2 + f(5) \Delta x_3 + f(8) \Delta x_4 \\ &= (4)(1) + (10)(2) + (40)(4) + (88)(1) = 272 \end{aligned}$$



$$69. f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$$

is not integrable on the interval  $[0, 1]$ . As  $\|\Delta\| \rightarrow 0$ ,  $f(c_i) = 1$  or  $f(c_i) = 0$  in each subinterval since there are an infinite number of both rational and irrational numbers in any interval, no matter how small.

71. Let  $f(x) = x^2$ ,  $0 \leq x \leq 1$ , and  $\Delta x_i = 1/n$ . The appropriate Riemann Sum is

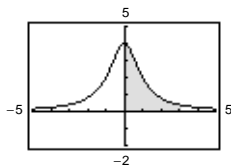
$$\sum_{i=1}^n f(c_i) \Delta x_i = \sum_{i=1}^n \left(\frac{i}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3} \sum_{i=1}^n i^2.$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n^3} [1^2 + 2^2 + 3^2 + \cdots + n^2] &= \lim_{n \rightarrow \infty} \frac{1}{n^3} \cdot \frac{n(2n+1)(n+1)}{6} \\ &= \lim_{n \rightarrow \infty} \frac{2n^2 + 3n + 1}{6n^2} = \lim_{n \rightarrow \infty} \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) = \frac{1}{3} \end{aligned}$$

## Section 4.4 The Fundamental Theorem of Calculus

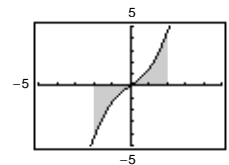
$$1. f(x) = \frac{4}{x^2 + 1}$$

$\int_0^\pi \frac{4}{x^2 + 1} dx$  is positive.



$$3. f(x) = x\sqrt{x^2 + 1}$$

$$\int_{-2}^2 x\sqrt{x^2 + 1} dx = 0$$



$$5. \int_0^1 2x dx = [x^2]_0^1 = 1 - 0 = 1$$

$$7. \int_{-1}^0 (x - 2) dx = \left[\frac{x^2}{2} - 2x\right]_{-1}^0 = 0 - \left(\frac{1}{2} + 2\right) = -\frac{5}{2}$$

$$9. \int_{-1}^1 (t^2 - 2) dt = \left[\frac{t^3}{3} - 2t\right]_{-1}^1 = \left(\frac{1}{3} - 2\right) - \left(-\frac{1}{3} + 2\right) = -\frac{10}{3}$$

$$11. \int_0^1 (2t - 1)^2 dt = \int_0^1 (4t^2 - 4t + 1) dt = \left[\frac{4}{3}t^3 - 2t^2 + t\right]_0^1 = \frac{4}{3} - 2 + 1 = \frac{1}{3}$$

$$13. \int_1^2 \left(\frac{3}{x^2} - 1\right) dx = \left[-\frac{3}{x} - x\right]_1^2 = \left(-\frac{3}{2} - 2\right) - (-3 - 1) = \frac{1}{2}$$

$$15. \int_1^4 \frac{u - 2}{\sqrt{u}} du = \int_1^4 (u^{1/2} - 2u^{-1/2}) du = \left[\frac{2}{3}u^{3/2} - 4u^{1/2}\right]_1^4 = \left[\frac{2}{3}(\sqrt{4})^3 - 4\sqrt{4}\right] - \left[\frac{2}{3} - 4\right] = \frac{2}{3}$$

$$17. \int_{-1}^1 (\sqrt[3]{t} - 2) dt = \left[\frac{3}{4}t^{4/3} - 2t\right]_{-1}^1 = \left(\frac{3}{4} - 2\right) - \left(\frac{3}{4} + 2\right) = -4$$

$$19. \int_0^1 \frac{x - \sqrt{x}}{3} dx = \frac{1}{3} \int_0^1 (x - x^{1/2}) dx = \frac{1}{3} \left[\frac{x^2}{2} - \frac{2}{3}x^{3/2}\right]_0^1 = \frac{1}{3} \left(\frac{1}{2} - \frac{2}{3}\right) = -\frac{1}{18}$$

$$21. \int_{-1}^0 (t^{1/3} - t^{2/3}) dt = \left[\frac{3}{4}t^{4/3} - \frac{3}{5}t^{5/3}\right]_{-1}^0 = 0 - \left(\frac{3}{4} + \frac{3}{5}\right) = -\frac{27}{20}$$

$$\begin{aligned} 23. \int_0^3 |2x - 3| dx &= \int_0^{3/2} (3 - 2x) dx + \int_{3/2}^3 (2x - 3) dx \quad \left(\text{split up the integral at the zero } x = \frac{3}{2}\right) \\ &= \left[3x - x^2\right]_0^{3/2} + \left[x^2 - 3x\right]_{3/2}^3 = \left(\frac{9}{2} - \frac{9}{4}\right) - 0 + (9 - 9) - \left(\frac{9}{4} - \frac{9}{2}\right) = 2\left(\frac{9}{2} - \frac{9}{4}\right) = \frac{9}{2} \end{aligned}$$

$$\begin{aligned}
 25. \int_0^3 |x^2 - 4| dx &= \int_0^2 (4 - x^2) dx + \int_2^3 (x^2 - 4) dx \\
 &= \left[ 4x - \frac{x^3}{3} \right]_0^2 + \left[ \frac{x^3}{3} - 4x \right]_2^3 \\
 &= \left( 8 - \frac{8}{3} \right) + (9 - 12) - \left( \frac{8}{3} - 8 \right) \\
 &= \frac{23}{3}
 \end{aligned}$$

$$27. \int_0^\pi (1 + \sin x) dx = \left[ x - \cos x \right]_0^\pi = (\pi + 1) - (0 - 1) = 2 + \pi$$

$$29. \int_{-\pi/6}^{\pi/6} \sec^2 x dx = \left[ \tan x \right]_{-\pi/6}^{\pi/6} = \frac{\sqrt{3}}{3} - \left( -\frac{\sqrt{3}}{3} \right) = \frac{2\sqrt{3}}{3}$$

$$31. \int_{-\pi/3}^{\pi/3} 4 \sec \theta \tan \theta d\theta = \left[ 4 \sec \theta \right]_{-\pi/3}^{\pi/3} = 4(2) - 4(2) = 0$$

$$33. \int_0^3 10,000(t - 6) dt = 10,000 \left[ \frac{t^2}{2} - 6t \right]_0^3 = -\$135,000 \qquad 35. A = \int_0^1 (x - x^2) dx = \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}$$

$$37. A = \int_0^3 (3 - x)\sqrt{x} dx = \int_0^3 (3x^{1/2} - x^{3/2}) dx = \left[ 2x^{3/2} - \frac{2}{5}x^{5/2} \right]_0^3 = \left[ \frac{x\sqrt{x}}{5}(10 - 2x) \right]_0^3 = \frac{12\sqrt{3}}{5}$$

$$39. A = \int_0^{\pi/2} \cos x dx = \left[ \sin x \right]_0^{\pi/2} = 1$$

41. Since  $y \geq 0$  on  $[0, 2]$ ,

$$A = \int_0^2 (3x^2 + 1) dx = \left[ x^3 + x \right]_0^2 = 8 + 2 = 10.$$

43. Since  $y \geq 0$  on  $[0, 2]$ ,

$$A = \int_0^2 (x^3 + x) dx = \left[ \frac{x^4}{4} + \frac{x^2}{2} \right]_0^2 = 4 + 2 = 6.$$

$$45. \int_0^2 (x - 2\sqrt{x}) dx = \left[ \frac{x^2}{2} - \frac{4x^{3/2}}{3} \right]_0^2 = 2 - \frac{8\sqrt{2}}{3}$$

$$f(c)(2 - 0) = \frac{6 - 8\sqrt{2}}{3}$$

$$c - 2\sqrt{c} = \frac{3 - 4\sqrt{2}}{3}$$

$$c - 2\sqrt{c} + 1 = \frac{3 - 4\sqrt{2}}{3} + 1$$

$$(\sqrt{c} - 1)^2 = \frac{6 - 4\sqrt{2}}{3}$$

$$\sqrt{c} - 1 = \pm \sqrt{\frac{6 - 4\sqrt{2}}{3}}$$

$$c = \left[ 1 \pm \sqrt{\frac{6 - 4\sqrt{2}}{3}} \right]^2$$

$$c \approx 0.4380 \text{ or } c \approx 1.7908$$

$$47. \int_{-\pi/4}^{\pi/4} 2 \sec^2 x \, dx = \left[ 2 \tan x \right]_{-\pi/4}^{\pi/4} = 2(1) - 2(-1) = 4$$

$$f(c) \left[ \frac{\pi}{4} - \left( -\frac{\pi}{4} \right) \right] = 4$$

$$2 \sec^2 c = \frac{8}{\pi}$$

$$\sec^2 c = \frac{4}{\pi}$$

$$\sec c = \pm \frac{2}{\sqrt{\pi}}$$

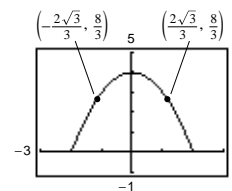
$$c = \pm \operatorname{arcsec} \left( \frac{2}{\sqrt{\pi}} \right)$$

$$= \pm \arccos \frac{\sqrt{\pi}}{2} \approx \pm 0.4817$$

$$49. \frac{1}{2 - (-2)} \int_{-2}^2 (4 - x^2) \, dx = \frac{1}{4} \left[ 4x - \frac{1}{3}x^3 \right]_{-2}^2 = \frac{1}{4} \left[ \left( 8 - \frac{8}{3} \right) - \left( -8 + \frac{8}{3} \right) \right] = \frac{8}{3}$$

$$\text{Average value} = \frac{8}{3}$$

$$4 - x^2 = \frac{8}{3} \text{ when } x^2 = 4 - \frac{8}{3} \text{ or } x = \pm \frac{2\sqrt{3}}{3} \approx \pm 1.155.$$

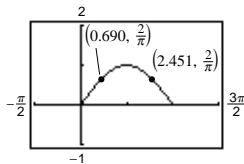


$$51. \frac{1}{\pi - 0} \int_0^{\pi} \sin x \, dx = \left[ -\frac{1}{\pi} \cos x \right]_0^{\pi} = \frac{2}{\pi}$$

$$\text{Average value} = \frac{2}{\pi}$$

$$\sin x = \frac{2}{\pi}$$

$$x \approx 0.690, 2.451$$



53. If  $f$  is continuous on  $[a, b]$  and  $F'(x) = f(x)$  on  $[a, b]$ ,

then  $\int_a^b f(x) \, dx = F(b) - F(a)$ .

$$55. \int_0^2 f(x) \, dx = -(\text{area of region A}) = -1.5$$

$$57. \int_0^6 |f(x)| \, dx = -\int_0^2 f(x) \, dx + \int_2^6 f(x) \, dx = 1.5 + 5.0 = 6.5$$

$$59. \int_0^6 [2 + f(x)] \, dx = \int_0^6 2 \, dx + \int_0^6 f(x) \, dx \\ = 12 + 3.5 = 15.5$$

$$61. (a) F(x) = k \sec^2 x$$

$$F(0) = k = 500$$

$$F(x) = 500 \sec^2 x$$

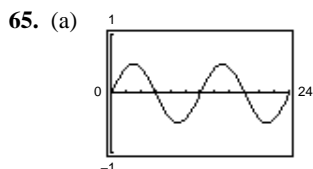
$$(b) \frac{1}{\pi/3 - 0} \int_0^{\pi/3} 500 \sec^2 x \, dx = \frac{1500}{\pi} \left[ \tan x \right]_0^{\pi/3}$$

$$= \frac{1500}{\pi} (\sqrt{3} - 0)$$

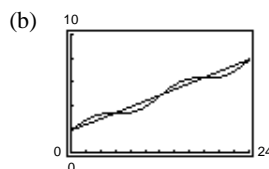
$$\approx 826.99 \text{ newtons}$$

$$\approx 827 \text{ newtons}$$

$$63. \frac{1}{5 - 0} \int_0^5 (0.1729t + 0.1552t^2 - 0.0374t^3) \, dt \approx \frac{1}{5} \left[ 0.08645t^2 + 0.05073t^3 - 0.00935t^4 \right]_0^5 \approx 0.5318 \text{ liter}$$

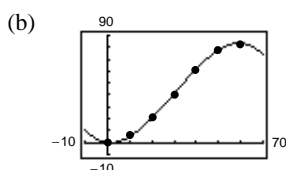


The area above the  $x$ -axis equals the area below the  $x$ -axis. Thus, the average value is zero.



The average value of  $S$  appears to be  $g$ .

67. (a)  $v = -8.61 \times 10^{-4}t^3 + 0.0782t^2 - 0.208t + 0.0952$



(c)  $\int_0^{60} v(t) dt = \left[ \frac{-8.61 \times 10^{-4}t^4}{4} + \frac{0.0782t^3}{3} - \frac{0.208t^2}{2} + 0.0952t \right]_0^{60} \approx 2476$  meters

69.  $F(x) = \int_0^x (t - 5) dt = \left[ \frac{t^2}{2} - 5t \right]_0^x = \frac{x^2}{2} - 5x$

$$F(2) = \frac{4}{2} - 5(2) = -8$$

$$F(5) = \frac{25}{2} - 5(5) = -\frac{25}{2}$$

$$F(8) = \frac{64}{2} - 5(8) = -8$$

71.  $F(x) = \int_1^x \frac{10}{v^2} dv = \int_1^x 10v^{-2} dv = \left[ \frac{-10}{v} \right]_1^x$

$$= -\frac{10}{x} + 10 = 10 \left( 1 - \frac{1}{x} \right)$$

$$F(2) = 10 \left( \frac{1}{2} \right) = 5$$

$$F(5) = 10 \left( \frac{4}{5} \right) = 8$$

$$F(8) = 10 \left( \frac{7}{8} \right) = \frac{35}{4}$$

73.  $F(x) = \int_1^x \cos \theta d\theta = \left[ \sin \theta \right]_1^x = \sin x - \sin 1$

$$F(2) = \sin 2 - \sin 1 = 0.0678$$

$$F(5) = \sin 5 - \sin 1 \approx -1.8004$$

$$F(8) = \sin 8 - \sin 1 \approx 0.1479$$

75. (a)  $\int_0^x (t + 2) dt = \left[ \frac{t^2}{2} + 2t \right]_0^x = \frac{1}{2}x^2 + 2x$

(b)  $\frac{d}{dx} \left[ \frac{1}{2}x^2 + 2x \right] = x + 2$

77. (a)  $\int_8^x \sqrt[3]{t} dt = \left[ \frac{3}{4}t^{4/3} \right]_8^x = \frac{3}{4}(x^{4/3} - 16) = \frac{3}{4}x^{4/3} - 12$

(b)  $\frac{d}{dx} \left[ \frac{3}{4}x^{4/3} - 12 \right] = x^{1/3} = \sqrt[3]{x}$

79. (a)  $\int_{x/4}^x \sec^2 t dt = \left[ \tan t \right]_{x/4}^x = \tan x - 1$

(b)  $\frac{d}{dx} [\tan x - 1] = \sec^2 x$

81.  $F(x) = \int_{-2}^x (t^2 - 2t) dt$

$$F'(x) = x^2 - 2x$$

83.  $F(x) = \int_{-1}^x \sqrt{t^4 + 1} dt$

$$F'(x) = \sqrt{x^4 + 1}$$

85.  $F(x) = \int_0^x t \cos t dt$

$$F'(x) = x \cos x$$

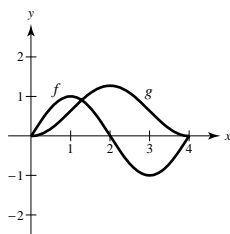
$$\begin{aligned}
 87. \quad F(x) &= \int_x^{x+2} (4t+1) dt \\
 &= \left[ 2t^2 + t \right]_x^{x+2} \\
 &= [2(x+2)^2 + (x+2)] - [2x^2 + x] \\
 &= 8x + 10 \\
 F'(x) &= 8
 \end{aligned}$$

$$\begin{aligned}
 89. \quad F(x) &= \int_0^{\sin x} \sqrt{t} dt = \left[ \frac{2}{3} t^{3/2} \right]_0^{\sin x} = \frac{2}{3} (\sin x)^{3/2} \\
 F'(x) &= (\sin x)^{1/2} \cos x = \cos x \sqrt{\sin x}
 \end{aligned}$$

**Alternate solution**

$$\begin{aligned}
 F(x) &= \int_0^{\sin x} \sqrt{t} dt \\
 F'(x) &= \sqrt{\sin x} \frac{d}{dx}(\sin x) = \sqrt{\sin x}(\cos x)
 \end{aligned}$$

$$\begin{aligned}
 93. \quad g(x) &= \int_0^x f(t) dt \\
 g(0) &= 0, g(1) \approx \frac{1}{2}, g(2) \approx 1, g(3) \approx \frac{1}{2}, g(4) = 0
 \end{aligned}$$



$g$  has a relative maximum at  $x = 2$ .

97. True

$$101. \quad f(x) = \int_0^{1/x} \frac{1}{t^2+1} dt + \int_0^x \frac{1}{t^2+1} dt$$

By the Second Fundamental Theorem of Calculus, we have

$$\begin{aligned}
 f'(x) &= \frac{1}{(1/x)^2+1} \left( -\frac{1}{x^2} \right) + \frac{1}{x^2+1} \\
 &= -\frac{1}{1+x^2} + \frac{1}{x^2+1} = 0.
 \end{aligned}$$

Since  $f'(x) = 0$ ,  $f(x)$  must be constant.

**Alternate solution:**

$$\begin{aligned}
 F(x) &= \int_x^{x+2} (4t+1) dt \\
 &= \int_x^0 (4t+1) dt + \int_0^{x+2} (4t+1) dt \\
 &= -\int_0^x (4t+1) dt + \int_0^{x+2} (4t+1) dt \\
 F'(x) &= -(4x+1) + 4(x+2) + 1 = 8
 \end{aligned}$$

$$\begin{aligned}
 91. \quad F(x) &= \int_0^{x^3} \sin t^2 dt \\
 F'(x) &= \sin(x^3)^2 \cdot 3x^2 = 3x^2 \sin x^6
 \end{aligned}$$

$$\begin{aligned}
 95. \quad (a) \quad C(x) &= 5000 \left( 25 + 3 \int_0^x t^{1/4} dt \right) \\
 &= 5000 \left( 25 + 3 \left[ \frac{4}{5} t^{5/4} \right]_0^x \right) \\
 &= 5000 \left( 25 + \frac{12}{5} x^{5/4} \right) = 1000(125 + 12x^{5/4})
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad C(1) &= 1000(125 + 12(1)) = \$137,000 \\
 C(5) &= 1000(125 + 12(5)^{5/4}) \approx \$214,721 \\
 C(10) &= 1000(125 + 12(10)^{5/4}) \approx \$338,394
 \end{aligned}$$

$$99. \quad \text{False; } \int_{-1}^1 x^{-2} dx = \int_{-1}^0 x^{-2} dx + \int_0^1 x^{-2} dx$$

Each of these integrals is infinite.  $f(x) = x^{-2}$  has a nonremovable discontinuity at  $x = 0$ .

$$103. x(t) = t^3 - 6t^2 + 9t - 2$$

$$x'(t) = 3t^2 - 12t + 9$$

$$= 3(t^2 - 4t + 3)$$

$$= 3(t - 3)(t - 1)$$

$$\begin{aligned} \text{Total distance} &= \int_0^5 |x'(t)| dt \\ &= \int_0^5 3|(t - 3)(t - 1)| dt \\ &= 3 \int_0^1 (t^2 - 4t + 3) dt - 3 \int_1^3 (t^2 - 4t + 3) dt + 3 \int_3^5 (t^2 - 4t + 3) dt \\ &= 4 + 4 + 20 \\ &= 28 \text{ units} \end{aligned}$$

$$\begin{aligned} 105. \text{ Total distance} &= \int_1^4 |x'(t)| dt \\ &= \int_1^4 |v(t)| dt \\ &= \int_1^4 \frac{1}{\sqrt{t}} dt \\ &= 2t^{1/2} \Big|_1^4 \\ &= 2(2 - 1) = 2 \text{ units} \end{aligned}$$

## Section 4.5 Integration by Substitution

$$\int \underline{f(g(x))g'(x) dx} \quad \underline{u = g(x)} \quad \underline{du = g'(x) dx}$$

$$1. \int (5x^2 + 1)^2(10x) dx \quad 5x^2 + 1 \quad 10x dx$$

$$3. \int \frac{x}{\sqrt{x^2 + 1}} dx \quad x^2 + 1 \quad 2x dx$$

$$5. \int \tan^2 x \sec^2 x dx \quad \tan x \quad \sec^2 x dx$$

$$7. \int (1 + 2x)^4 2 dx = \frac{(1 + 2x)^5}{5} + C$$

$$\text{Check: } \frac{d}{dx} \left[ \frac{(1 + 2x)^5}{5} + C \right] = 2(1 + 2x)^4$$

$$9. \int (9 - x^2)^{1/2}(-2x) dx = \frac{(9 - x^2)^{3/2}}{3/2} + C = \frac{2}{3}(9 - x^2)^{3/2} + C$$

$$\text{Check: } \frac{d}{dx} \left[ \frac{2}{3}(9 - x^2)^{3/2} + C \right] = \frac{2}{3} \cdot \frac{3}{2}(9 - x^2)^{1/2}(-2x) = \sqrt{9 - x^2}(-2x)$$

$$11. \int x^3(x^4 + 3)^2 dx = \frac{1}{4} \int (x^4 + 3)^2(4x^3) dx = \frac{1}{4} \frac{(x^4 + 3)^3}{3} + C = \frac{(x^4 + 3)^3}{12} + C$$

$$\text{Check: } \frac{d}{dx} \left[ \frac{(x^4 + 3)^3}{12} + C \right] = \frac{3(x^4 + 3)^2}{12} (4x^3) = (x^4 + 3)^2(x^3)$$

$$13. \int x^2(x^3 - 1)^4 dx = \frac{1}{3} \int (x^3 - 1)^4(3x^2) dx = \frac{1}{3} \left[ \frac{(x^3 - 1)^5}{5} \right] + C = \frac{(x^3 - 1)^5}{15} + C$$

$$\text{Check: } \frac{d}{dx} \left[ \frac{(x^3 - 1)^5}{15} + C \right] = \frac{5(x^3 - 1)^4(3x^2)}{15} = x^2(x^3 - 1)^4$$

$$15. \int t\sqrt{t^2 + 2} dt = \frac{1}{2} \int (t^2 + 2)^{1/2}(2t) dt = \frac{1}{2} \frac{(t^2 + 2)^{3/2}}{3/2} + C = \frac{(t^2 + 2)^{3/2}}{3} + C$$

$$\text{Check: } \frac{d}{dt} \left[ \frac{(t^2 + 2)^{3/2}}{3} + C \right] = \frac{3/2(t^2 + 2)^{1/2}(2t)}{3} = (t^2 + 2)^{1/2}t$$

$$17. \int 5x(1 - x^2)^{1/3} dx = -\frac{5}{2} \int (1 - x^2)^{1/3}(-2x) dx = -\frac{5}{2} \cdot \frac{(1 - x^2)^{4/3}}{4/3} + C = -\frac{15}{8}(1 - x^2)^{4/3} + C$$

$$\text{Check: } \frac{d}{dx} \left[ -\frac{15}{8}(1 - x^2)^{4/3} + C \right] = -\frac{15}{8} \cdot \frac{4}{3}(1 - x^2)^{1/3}(-2x) = 5x(1 - x^2)^{1/3} = 5x\sqrt[3]{1 - x^2}$$

$$19. \int \frac{x}{(1 - x^2)^3} dx = -\frac{1}{2} \int (1 - x^2)^{-3}(-2x) dx = -\frac{1}{2} \frac{(1 - x^2)^{-2}}{-2} + C = \frac{1}{4(1 - x^2)^2} + C$$

$$\text{Check: } \frac{d}{dx} \left[ \frac{1}{4(1 - x^2)^2} + C \right] = \frac{1}{4}(-2)(1 - x^2)^{-3}(-2x) = \frac{x}{(1 - x^2)^3}$$

$$21. \int \frac{x^2}{(1 + x^3)^2} dx = \frac{1}{3} \int (1 + x^3)^{-2}(3x^2) dx = \frac{1}{3} \left[ \frac{(1 + x^3)^{-1}}{-1} \right] + C = -\frac{1}{3(1 + x^3)} + C$$

$$\text{Check: } \frac{d}{dx} \left[ -\frac{1}{3(1 + x^3)} + C \right] = -\frac{1}{3}(-1)(1 + x^3)^{-2}(3x^2) = \frac{x^2}{(1 + x^3)^2}$$

$$23. \int \frac{x}{\sqrt{1 - x^2}} dx = -\frac{1}{2} \int (1 - x^2)^{-1/2}(-2x) dx = -\frac{1}{2} \frac{(1 - x^2)^{1/2}}{1/2} + C = -\sqrt{1 - x^2} + C$$

$$\text{Check: } \frac{d}{dx} [-\sqrt{1 - x^2} + C] = -\frac{1}{2}(1 - x^2)^{-1/2}(-2x) = \frac{x}{\sqrt{1 - x^2}}$$

$$25. \int \left(1 + \frac{1}{t}\right)^3 \left(\frac{1}{t^2}\right) dt = -\int \left(1 + \frac{1}{t}\right)^3 \left(-\frac{1}{t^2}\right) dt = -\frac{[1 + (1/t)]^4}{4} + C$$

$$\text{Check: } \frac{d}{dt} \left[ -\frac{[1 + (1/t)]^4}{4} + C \right] = -\frac{1}{4}(4) \left(1 + \frac{1}{t}\right)^3 \left(-\frac{1}{t^2}\right) = \frac{1}{t^2} \left(1 + \frac{1}{t}\right)^3$$

$$27. \int \frac{1}{\sqrt{2x}} dx = \frac{1}{2} \int (2x)^{-1/2} 2 dx = \frac{1}{2} \left[ \frac{(2x)^{1/2}}{1/2} \right] + C = \sqrt{2x} + C$$

$$\text{Check: } \frac{d}{dx} [\sqrt{2x} + C] = \frac{1}{2}(2x)^{-1/2}(2) = \frac{1}{\sqrt{2x}}$$

$$29. \int \frac{x^2 + 3x + 7}{\sqrt{x}} dx = \int (x^{3/2} + 3x^{1/2} + 7x^{-1/2}) dx = \frac{2}{5}x^{5/2} + 2x^{3/2} + 14x^{1/2} + C = \frac{2}{5}\sqrt{x}(x^2 + 5x + 35) + C$$

$$\text{Check: } \frac{d}{dx} \left[ \frac{2}{5}x^{5/2} + 2x^{3/2} + 14x^{1/2} + C \right] = \frac{x^2 + 3x + 7}{\sqrt{x}}$$

$$31. \int t^2 \left( t - \frac{2}{t} \right) dt = \int (t^3 - 2t) dt = \frac{1}{4}t^4 - t^2 + C$$

$$\text{Check: } \frac{d}{dt} \left[ \frac{1}{4}t^4 - t^2 + C \right] = t^3 - 2t = t^2 \left( t - \frac{2}{t} \right)$$

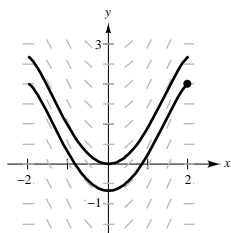
$$33. \int (9 - y)\sqrt{y} dy = \int (9y^{1/2} - y^{3/2}) dy = 9 \left( \frac{2}{3}y^{3/2} \right) - \frac{2}{5}y^{5/2} + C = \frac{2}{5}y^{3/2}(15 - y) + C$$

$$\text{Check: } \frac{d}{dy} \left[ \frac{2}{5}y^{3/2}(15 - y) + C \right] = \frac{d}{dy} \left[ 6y^{3/2} - \frac{2}{5}y^{5/2} + C \right] = 9y^{1/2} - y^{3/2} = (9 - y)\sqrt{y}$$

$$\begin{aligned} 35. y &= \int \left[ 4x + \frac{4x}{\sqrt{16 - x^2}} \right] dx \\ &= 4 \int x dx - 2 \int (16 - x^2)^{-1/2} (-2x) dx \\ &= 4 \left( \frac{x^2}{2} \right) - 2 \left[ \frac{(16 - x^2)^{1/2}}{1/2} \right] + C \\ &= 2x^2 - 4\sqrt{16 - x^2} + C \end{aligned}$$

$$\begin{aligned} 37. y &= \int \frac{x + 1}{(x^2 + 2x - 3)^2} dx \\ &= \frac{1}{2} \int (x^2 + 2x - 3)^{-2} (2x + 2) dx \\ &= \frac{1}{2} \left[ \frac{(x^2 + 2x - 3)^{-1}}{-1} \right] + C \\ &= -\frac{1}{2(x^2 + 2x - 3)} + C \end{aligned}$$

39. (a)

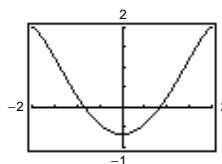


$$(b) \frac{dy}{dx} = x\sqrt{4 - x^2}, (2, 2)$$

$$\begin{aligned} y &= \int x\sqrt{4 - x^2} dx = -\frac{1}{2} \int (4 - x^2)^{1/2} (-2x) dx \\ &= -\frac{1}{2} \cdot \frac{2}{3} (4 - x^2)^{3/2} + C = -\frac{1}{3} (4 - x^2)^{3/2} + C \end{aligned}$$

$$(2, 2): 2 = -\frac{1}{3}(4 - 2^2)^{3/2} + C \Rightarrow C = 2$$

$$y = -\frac{1}{3}(4 - x^2)^{3/2} + 2$$



$$41. \int \pi \sin \pi x dx = -\cos \pi x + C$$

$$43. \int \sin 2x dx = \frac{1}{2} \int (\sin 2x)(2x) dx = -\frac{1}{2} \cos 2x + C$$

$$45. \int \frac{1}{\theta^2} \cos \frac{1}{\theta} d\theta = -\int \cos \frac{1}{\theta} \left( -\frac{1}{\theta^2} \right) d\theta = -\sin \frac{1}{\theta} + C$$



$$47. \int \sin 2x \cos 2x \, dx = \frac{1}{2} \int (\sin 2x)(2 \cos 2x) \, dx = \frac{1}{2} \frac{(\sin 2x)^2}{2} + C = \frac{1}{4} \sin^2 2x + C \quad \text{OR}$$

$$\int \sin 2x \cos 2x \, dx = -\frac{1}{2} \int (\cos 2x)(-2 \sin 2x) \, dx = -\frac{1}{2} \frac{(\cos 2x)^2}{2} + C_1 = -\frac{1}{4} \cos^2 2x + C_1 \quad \text{OR}$$

$$\int \sin 2x \cos 2x \, dx = \frac{1}{2} \int 2 \sin 2x \cos 2x \, dx = \frac{1}{2} \int \sin 4x \, dx = \frac{1}{8} \cos 4x + C_2$$

$$49. \int \tan^4 x \sec^2 x \, dx = \frac{\tan^5 x}{5} + C = \frac{1}{5} \tan^5 x + C$$

$$51. \int \frac{\csc^2 x}{\cot^3 x} \, dx = - \int (\cot x)^{-3} (-\csc^2 x) \, dx \\ = -\frac{(\cot x)^{-2}}{-2} + C = \frac{1}{2 \cot^2 x} + C = \frac{1}{2} \tan^2 x + C = \frac{1}{2} (\sec^2 x - 1) + C = \frac{1}{2} \sec^2 x + C_1$$

$$53. \int \cot^2 x \, dx = \int (\csc^2 x - 1) \, dx = -\cot x - x + C$$

$$55. f(x) = \int \cos \frac{x}{2} \, dx = 2 \sin \frac{x}{2} + C$$

Since  $f(0) = 3 = 2 \sin 0 + C$ ,  $C = 3$ . Thus,

$$f(x) = 2 \sin \frac{x}{2} + 3.$$

$$57. u = x + 2, x = u - 2, dx = du$$

$$\int x \sqrt{x+2} \, dx = \int (u-2) \sqrt{u} \, du \\ = \int (u^{3/2} - 2u^{1/2}) \, du \\ = \frac{2}{5} u^{5/2} - \frac{4}{3} u^{3/2} + C \\ = \frac{2u^{3/2}}{15} (3u - 10) + C \\ = \frac{2}{15} (x+2)^{3/2} [3(x+2) - 10] + C \\ = \frac{2}{15} (x+2)^{3/2} (3x-4) + C$$

$$59. u = 1 - x, x = 1 - u, dx = -du$$

$$\int x^2 \sqrt{1-x} \, dx = - \int (1-u)^2 \sqrt{u} \, du \\ = - \int (u^{1/2} - 2u^{3/2} + u^{5/2}) \, du \\ = - \left( \frac{2}{3} u^{3/2} - \frac{4}{5} u^{5/2} + \frac{2}{7} u^{7/2} \right) + C \\ = - \frac{2u^{3/2}}{105} (35 - 42u + 15u^2) + C \\ = - \frac{2}{105} (1-x)^{3/2} [35 - 42(1-x) + 15(1-x)^2] + C \\ = - \frac{2}{105} (1-x)^{3/2} (15x^2 + 12x + 8) + C$$

61.  $u = 2x - 1, x = \frac{1}{2}(u + 1), dx = \frac{1}{2} du$

$$\begin{aligned} \int \frac{x^2 - 1}{\sqrt{2x - 1}} dx &= \int \frac{[(1/2)(u + 1)]^2 - 1}{\sqrt{u}} \frac{1}{2} du \\ &= \frac{1}{8} \int u^{-1/2} [(u^2 + 2u + 1) - 4] du \\ &= \frac{1}{8} \int (u^{3/2} + 2u^{1/2} - 3u^{-1/2}) du \\ &= \frac{1}{8} \left( \frac{2}{5} u^{5/2} + \frac{4}{3} u^{3/2} - 6u^{1/2} \right) + C \\ &= \frac{u^{1/2}}{60} (3u^2 + 10u - 45) + C \\ &= \frac{\sqrt{2x - 1}}{60} [3(2x - 1)^2 + 10(2x - 1) - 45] + C \\ &= \frac{1}{60} \sqrt{2x - 1} (12x^2 + 8x - 52) + C \\ &= \frac{1}{15} \sqrt{2x - 1} (3x^2 + 2x - 13) + C \end{aligned}$$

63.  $u = x + 1, x = u - 1, dx = du$

$$\begin{aligned} \int \frac{-x}{(x + 1) - \sqrt{x + 1}} dx &= \int \frac{-(u - 1)}{u - \sqrt{u}} du \\ &= - \int \frac{(\sqrt{u} + 1)(\sqrt{u} - 1)}{\sqrt{u}(\sqrt{u} - 1)} du \\ &= - \int (1 + u^{-1/2}) du \\ &= -(u + 2u^{1/2}) + C \\ &= -u - 2\sqrt{u} + C \\ &= -(x + 1) - 2\sqrt{x + 1} + C \\ &= -x - 2\sqrt{x + 1} - 1 + C \\ &= -(x + 2\sqrt{x + 1}) + C_1 \end{aligned}$$

where  $C_1 = -1 + C$ .

65. Let  $u = x^2 + 1, du = 2x dx$ .

$$\int_{-1}^1 x(x^2 + 1)^3 dx = \frac{1}{2} \int_{-1}^1 (x^2 + 1)^3 (2x) dx = \left[ \frac{1}{8} (x^2 + 1)^4 \right]_{-1}^1 = 0$$

67. Let  $u = x^3 + 1, du = 3x^2 dx$

$$\begin{aligned} \int_1^2 2x^2 \sqrt{x^3 + 1} dx &= 2 \cdot \frac{1}{3} \int_1^2 (x^3 + 1)^{1/2} (3x^2) dx \\ &= \left[ \frac{2}{3} \frac{(x^3 + 1)^{3/2}}{3/2} \right]_1^2 \\ &= \frac{4}{9} \left[ (x^3 + 1)^{3/2} \right]_1^2 \\ &= \frac{4}{9} [27 - 2\sqrt{2}] = 12 - \frac{8}{9}\sqrt{2} \end{aligned}$$

69. Let  $u = 2x + 1$ ,  $du = 2 dx$ .

$$\int_0^4 \frac{1}{\sqrt{2x+1}} dx = \frac{1}{2} \int_0^4 (2x+1)^{-1/2} (2) dx = \left[ \sqrt{2x+1} \right]_0^4 = \sqrt{9} - \sqrt{1} = 2$$

71. Let  $u = 1 + \sqrt{x}$ ,  $du = \frac{1}{2\sqrt{x}} dx$ .

$$\int_1^9 \frac{1}{\sqrt{x}(1+\sqrt{x})^2} dx = 2 \int_1^9 (1+\sqrt{x})^{-2} \left( \frac{1}{2\sqrt{x}} \right) dx = \left[ -\frac{2}{1+\sqrt{x}} \right]_1^9 = -\frac{1}{2} + 1 = \frac{1}{2}$$

73.  $u = 2 - x$ ,  $x = 2 - u$ ,  $dx = -du$

When  $x = 1$ ,  $u = 1$ . When  $x = 2$ ,  $u = 0$ .

$$\int_1^2 (x-1)\sqrt{2-x} dx = \int_1^0 -[(2-u)-1]\sqrt{u} du = \int_1^0 (u^{3/2} - u^{1/2}) du = \left[ \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} \right]_1^0 = -\left[ \frac{2}{5} - \frac{2}{3} \right] = \frac{4}{15}$$

75.  $\int_0^{\pi/2} \cos\left(\frac{2}{3}x\right) dx = \left[ \frac{3}{2} \sin\left(\frac{2}{3}x\right) \right]_0^{\pi/2} = \frac{3}{2} \left( \frac{\sqrt{3}}{2} \right) = \frac{3\sqrt{3}}{4}$

77.  $u = x + 1$ ,  $x = u - 1$ ,  $dx = du$

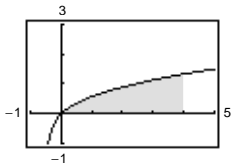
When  $x = 0$ ,  $u = 1$ . When  $x = 7$ ,  $u = 8$ .

$$\begin{aligned} \text{Area} &= \int_0^7 x \sqrt[3]{x+1} dx = \int_1^8 (u-1) \sqrt[3]{u} du \\ &= \int_1^8 (u^{4/3} - u^{1/3}) du = \left[ \frac{3}{7}u^{7/3} - \frac{3}{4}u^{4/3} \right]_1^8 = \left( \frac{384}{7} - 12 \right) - \left( \frac{3}{7} - \frac{3}{4} \right) = \frac{1209}{28} \end{aligned}$$

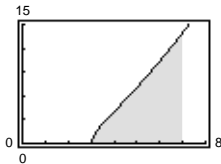
79.  $A = \int_0^\pi (2 \sin x + \sin 2x) dx = -\left[ 2 \cos x + \frac{1}{2} \cos 2x \right]_0^\pi = 4$

81.  $\text{Area} = \int_{\pi/2}^{2\pi/3} \sec^2\left(\frac{x}{2}\right) dx = 2 \int_{\pi/2}^{2\pi/3} \sec^2\left(\frac{x}{2}\right) \left(\frac{1}{2}\right) dx = \left[ 2 \tan\left(\frac{x}{2}\right) \right]_{\pi/2}^{2\pi/3} = 2(\sqrt{3} - 1)$

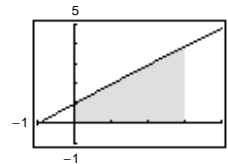
83.  $\int_0^4 \frac{x}{\sqrt{2x+1}} dx \approx 3.333 = \frac{10}{3}$



85.  $\int_3^7 x\sqrt{x-3} dx \approx 28.8 = \frac{144}{5}$



87.  $\int_0^3 \left( \theta + \cos \frac{\theta}{6} \right) d\theta \approx 7.377$



89.  $\int (2x-1)^2 dx = \frac{1}{2} \int (2x-1)^2 (2) dx = \frac{1}{6} (2x-1)^3 + C_1 = \frac{4}{3}x^3 - 2x^2 + x - \frac{1}{6} + C_1$

$$\int (2x-1)^2 dx = \int (4x^2 - 4x + 1) dx = \frac{4}{3}x^3 - 2x^2 + x + C_2$$

They differ by a constant:  $C_2 = C_1 - \frac{1}{6}$ .

91.  $f(x) = x^2(x^2 + 1)$  is even.

$$\begin{aligned}\int_{-2}^2 x^2(x^2 + 1) dx &= 2 \int_0^2 (x^4 + x^2) dx = 2 \left[ \frac{x^5}{5} + \frac{x^3}{3} \right]_0^2 \\ &= 2 \left[ \frac{32}{5} + \frac{8}{3} \right] = \frac{272}{15}\end{aligned}$$

93.  $f(x) = x(x^2 + 1)^3$  is odd.

$$\int_{-2}^2 x(x^2 + 1)^3 dx = 0$$

95.  $\int_0^2 x^2 dx = \left[ \frac{x^3}{3} \right]_0^2 = \frac{8}{3}$ ; the function  $x^2$  is an even function.

$$(a) \int_{-2}^0 x^2 dx = \int_0^2 x^2 dx = \frac{8}{3}$$

$$(b) \int_{-2}^2 x^2 dx = 2 \int_0^2 x^2 dx = \frac{16}{3}$$

$$(c) \int_0^2 (-x^2) dx = - \int_0^2 x^2 dx = -\frac{8}{3}$$

$$(d) \int_{-2}^0 3x^2 dx = 3 \int_0^2 x^2 dx = 8$$

$$97. \int_{-4}^4 (x^3 + 6x^2 - 2x - 3) dx = \int_{-4}^4 (x^3 - 2x) dx + \int_{-4}^4 (6x^2 - 3) dx = 0 + 2 \int_0^4 (6x^2 - 3) dx = 2 \left[ 2x^3 - 3x \right]_0^4 = 232$$

99. Answers will vary. See "Guidelines for Making a Change of Variables" on page 292.

101.  $f(x) = x(x^2 + 1)^2$  is odd. Hence,  $\int_{-2}^2 x(x^2 + 1)^2 dx = 0$ .

$$103. \frac{dV}{dt} = \frac{k}{(t+1)^2}$$

$$V(t) = \int \frac{k}{(t+1)^2} dt = -\frac{k}{t+1} + C$$

$$V(0) = -k + C = 500,000$$

$$V(1) = -\frac{1}{2}k + C = 400,000$$

Solving this system yields  $k = -200,000$  and  $C = 300,000$ . Thus,

$$V(t) = \frac{200,000}{t+1} + 300,000.$$

When  $t = 4$ ,  $V(4) = \$340,000$ .

$$105. \frac{1}{b-a} \int_a^b \left[ 74.50 + 43.75 \sin \frac{\pi t}{6} \right] dt = \frac{1}{b-a} \left[ 74.50t - \frac{262.5}{\pi} \cos \frac{\pi t}{6} \right]_a^b$$

$$(a) \frac{1}{3} \left[ 74.50t - \frac{262.5}{\pi} \cos \frac{\pi t}{6} \right]_0^3 = \frac{1}{3} \left( 223.5 + \frac{262.5}{\pi} \right) \approx 102.352 \text{ thousand units}$$

$$(b) \frac{1}{3} \left[ 74.50t - \frac{262.5}{\pi} \cos \frac{\pi t}{6} \right]_3^6 = \frac{1}{3} \left( 447 + \frac{262.5}{\pi} - 223.5 \right) \approx 102.352 \text{ thousand units}$$

$$(c) \frac{1}{12} \left[ 74.50t - \frac{262.5}{\pi} \cos \frac{\pi t}{6} \right]_0^{12} = \frac{1}{12} \left( 894 - \frac{262.5}{\pi} + \frac{262.5}{\pi} \right) = 74.5 \text{ thousand units}$$

$$107. \frac{1}{b-a} \int_a^b [2 \sin(60\pi t) + \cos(120\pi t)] dt = \frac{1}{b-a} \left[ -\frac{1}{30\pi} \cos(60\pi t) + \frac{1}{120\pi} \sin(120\pi t) \right]_a^b$$

$$(a) \frac{1}{(1/60) - 0} \left[ -\frac{1}{30\pi} \cos(60\pi t) + \frac{1}{120\pi} \sin(120\pi t) \right]_0^{1/60} = 60 \left[ \left( \frac{1}{30\pi} + 0 \right) - \left( -\frac{1}{30\pi} \right) \right] = \frac{4}{\pi} \approx 1.273 \text{ amps}$$

$$(b) \frac{1}{(1/240) - 0} \left[ -\frac{1}{30\pi} \cos(60\pi t) + \frac{1}{120\pi} \sin(120\pi t) \right]_0^{1/240} = 240 \left[ \left( -\frac{1}{30\sqrt{2}\pi} + \frac{1}{120\pi} \right) - \left( -\frac{1}{30\pi} \right) \right]$$

$$= \frac{2}{\pi} (5 - 2\sqrt{2}) \approx 1.382 \text{ amps}$$

$$(c) \frac{1}{(1/30) - 0} \left[ -\frac{1}{30\pi} \cos(60\pi t) + \frac{1}{120\pi} \sin(120\pi t) \right]_0^{1/30} = 30 \left[ \left( \frac{1}{30\pi} \right) - \left( -\frac{1}{30\pi} \right) \right] = 0 \text{ amps}$$

109. False

$$\int (2x + 1)^2 dx = \frac{1}{2} \int (2x + 1)^2 \cdot 2 dx = \frac{1}{6} (2x + 1)^3 + C$$

111. True

$$\int_{-10}^{10} (ax^3 + bx^2 + cx + d) dx = \int_{-10}^{10} (ax^3 + cx) dx + \int_{-10}^{10} (bx^2 + d) dx = 0 + 2 \int_0^{10} (bx^2 + d) dx$$

Odd Even

113. True

$$4 \int \sin x \cos x dx = 2 \int \sin 2x dx = -\cos 2x + C$$

115. Let  $u = x + h$ , then  $du = dx$ . When  $x = a$ ,  $u = a + h$ . When  $x = b$ ,  $u = b + h$ . Thus,

$$\int_a^b f(x + h) dx = \int_{a+h}^{b+h} f(u) du = \int_{a+h}^{b+h} f(x) dx.$$

## Section 4.6 Numerical Integration

1. Exact:  $\int_0^2 x^2 dx = \left[ \frac{1}{3} x^3 \right]_0^2 = \frac{8}{3} \approx 2.6667$

Trapezoidal:  $\int_0^2 x^2 dx \approx \frac{1}{4} \left[ 0 + 2\left(\frac{1}{2}\right)^2 + 2(1)^2 + 2\left(\frac{3}{2}\right)^2 + (2)^2 \right] = \frac{11}{4} = 2.7500$

Simpson's:  $\int_0^2 x^2 dx \approx \frac{1}{6} \left[ 0 + 4\left(\frac{1}{2}\right)^2 + 2(1)^2 + 4\left(\frac{3}{2}\right)^2 + (2)^2 \right] = \frac{8}{3} \approx 2.6667$

3. Exact:  $\int_0^2 x^3 dx = \left[ \frac{x^4}{4} \right]_0^2 = 4.000$

Trapezoidal:  $\int_0^2 x^3 dx \approx \frac{1}{4} \left[ 0 + 2\left(\frac{1}{2}\right)^3 + 2(1)^3 + 2\left(\frac{3}{2}\right)^3 + (2)^3 \right] = \frac{17}{4} = 4.2500$

Simpson's:  $\int_0^2 x^3 dx \approx \frac{1}{6} \left[ 0 + 4\left(\frac{1}{2}\right)^3 + 2(1)^3 + 4\left(\frac{3}{2}\right)^3 + (2)^3 \right] = \frac{24}{6} = 4.0000$

5. Exact:  $\int_0^2 x^3 dx = \left[ \frac{1}{4}x^4 \right]_0^2 = 4.0000$
- Trapezoidal:  $\int_0^2 x^3 dx \approx \frac{1}{8} \left[ 0 + 2\left(\frac{1}{4}\right)^3 + 2\left(\frac{2}{4}\right)^3 + 2\left(\frac{3}{4}\right)^3 + 2(1)^3 + 2\left(\frac{5}{4}\right)^3 + 2\left(\frac{6}{4}\right)^3 + 2\left(\frac{7}{4}\right)^3 + 8 \right] = 4.0625$
- Simpson's:  $\int_0^2 x^3 dx \approx \frac{1}{12} \left[ 0 + 4\left(\frac{1}{4}\right)^3 + 2\left(\frac{2}{4}\right)^3 + 4\left(\frac{3}{4}\right)^3 + 2(1)^3 + 4\left(\frac{5}{4}\right)^3 + 2\left(\frac{6}{4}\right)^3 + 4\left(\frac{7}{4}\right)^3 + 8 \right] = 4.0000$
7. Exact:  $\int_4^9 \sqrt{x} dx = \left[ \frac{2}{3}x^{3/2} \right]_4^9 = 18 - \frac{16}{3} = \frac{38}{3} \approx 12.6667$
- Trapezoidal:  $\int_4^9 \sqrt{x} dx \approx \frac{5}{16} \left[ 2 + 2\sqrt{\frac{37}{8}} + 2\sqrt{\frac{21}{4}} + 2\sqrt{\frac{47}{8}} + 2\sqrt{\frac{26}{4}} + 2\sqrt{\frac{57}{8}} + 2\sqrt{\frac{31}{4}} + 2\sqrt{\frac{67}{8}} + 3 \right]$   
 $\approx 12.6640$
- Simpson's:  $\int_4^9 \sqrt{x} dx \approx \frac{5}{24} \left[ 2 + 4\sqrt{\frac{37}{8}} + \sqrt{21} + 4\sqrt{\frac{47}{8}} + \sqrt{26} + 4\sqrt{\frac{57}{8}} + \sqrt{31} + 4\sqrt{\frac{67}{8}} + 3 \right] \approx 12.6667$
9. Exact:  $\int_1^2 \frac{1}{(x+1)^2} dx = \left[ -\frac{1}{x+1} \right]_1^2 = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6} \approx 0.1667$
- Trapezoidal:  $\int_1^2 \frac{1}{(x+1)^2} dx \approx \frac{1}{8} \left[ \frac{1}{4} + 2\left(\frac{1}{((5/4)+1)^2}\right) + 2\left(\frac{1}{((3/2)+1)^2}\right) + 2\left(\frac{1}{((7/4)+1)^2}\right) + \frac{1}{9} \right]$   
 $= \frac{1}{8} \left( \frac{1}{4} + \frac{32}{81} + \frac{8}{25} + \frac{32}{121} + \frac{1}{9} \right) \approx 0.1676$
- Simpson's:  $\int_1^2 \frac{1}{(x+1)^2} dx \approx \frac{1}{12} \left[ \frac{1}{4} + 4\left(\frac{1}{((5/4)+1)^2}\right) + 2\left(\frac{1}{((3/2)+1)^2}\right) + 4\left(\frac{1}{((7/4)+1)^2}\right) + \frac{1}{9} \right]$   
 $= \frac{1}{12} \left( \frac{1}{4} + \frac{64}{81} + \frac{8}{25} + \frac{64}{121} + \frac{1}{9} \right) \approx 0.1667$
11. Trapezoidal:  $\int_0^2 \sqrt{1+x^3} dx \approx \frac{1}{4} [1 + 2\sqrt{1+(1/8)} + 2\sqrt{2} + 2\sqrt{1+(27/8)} + 3] \approx 3.283$
- Simpson's:  $\int_0^2 \sqrt{1+x^3} dx \approx \frac{1}{6} [1 + 4\sqrt{1+(1/8)} + 2\sqrt{2} + 4\sqrt{1+(27/8)} + 3] \approx 3.240$
- Graphing utility: 3.241
13.  $\int_0^1 \sqrt{x}\sqrt{1-x} dx = \int_0^1 \sqrt{x(1-x)} dx$
- Trapezoidal:  $\int_0^1 \sqrt{x(1-x)} dx \approx \frac{1}{8} \left[ 0 + 2\sqrt{\frac{1}{4}\left(1-\frac{1}{4}\right)} + 2\sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)} + 2\sqrt{\frac{3}{4}\left(1-\frac{3}{4}\right)} \right] \approx 0.342$
- Simpson's:  $\int_0^1 \sqrt{x(1-x)} dx \approx \frac{1}{12} \left[ 0 + 4\sqrt{\frac{1}{4}\left(1-\frac{1}{4}\right)} + 2\sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)} + 4\sqrt{\frac{3}{4}\left(1-\frac{3}{4}\right)} \right] \approx 0.372$
- Graphing utility: 0.393

15. Trapezoidal:  $\int_0^{\sqrt{\pi/2}} \cos(x^2) dx \approx \frac{\sqrt{\pi/2}}{8} \left[ \cos 0 + 2 \cos\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + 2 \cos\left(\frac{\sqrt{\pi/2}}{2}\right)^2 + 2 \cos\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + \cos\left(\sqrt{\frac{\pi}{2}}\right)^2 \right]$   
 $\approx 0.957$

Simpson's:  $\int_0^{\sqrt{\pi/2}} \cos(x^2) dx \approx \frac{\sqrt{\pi/2}}{12} \left[ \cos 0 + 4 \cos\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + 2 \cos\left(\frac{\sqrt{\pi/2}}{2}\right)^2 + 4 \cos\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + \cos\left(\sqrt{\frac{\pi}{2}}\right)^2 \right]$   
 $\approx 0.978$

Graphing utility: 0.977

17. Trapezoidal:  $\int_1^{1.1} \sin x^2 dx \approx \frac{1}{80} [\sin(1) + 2 \sin(1.025)^2 + 2 \sin(1.05)^2 + 2 \sin(1.075)^2 + \sin(1.1)^2] \approx 0.089$

Simpson's:  $\int_1^{1.1} \sin x^2 dx \approx \frac{1}{120} [\sin(1) + 4 \sin(1.025)^2 + 2 \sin(1.05)^2 + 4 \sin(1.075)^2 + \sin(1.1)^2] \approx 0.089$

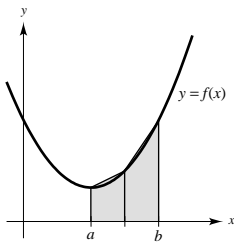
Graphing utility: 0.089

19. Trapezoidal:  $\int_0^{\pi/4} x \tan x dx \approx \frac{\pi}{32} \left[ 0 + 2\left(\frac{\pi}{16}\right) \tan\left(\frac{\pi}{16}\right) + 2\left(\frac{2\pi}{16}\right) \tan\left(\frac{2\pi}{16}\right) + 2\left(\frac{3\pi}{16}\right) \tan\left(\frac{3\pi}{16}\right) + \frac{\pi}{4} \right] \approx 0.194$

Simpson's:  $\int_0^{\pi/4} x \tan x dx \approx \frac{\pi}{48} \left[ 0 + 4\left(\frac{\pi}{16}\right) \tan\left(\frac{\pi}{16}\right) + 2\left(\frac{2\pi}{16}\right) \tan\left(\frac{2\pi}{16}\right) + 4\left(\frac{3\pi}{16}\right) \tan\left(\frac{3\pi}{16}\right) + \frac{\pi}{4} \right] \approx 0.186$

Graphing utility: 0.186

21. (a)



The Trapezoidal Rule overestimates the area if the graph of the integrand is concave up.

23.  $f(x) = x^3$

$$f'(x) = 3x^2$$

$$f''(x) = 6x$$

$$f'''(x) = 6$$

$$f^{(4)}(x) = 0$$

(a) Trapezoidal: Error  $\leq \frac{(2-0)^3}{12(4^2)}(12) = 0.5$  since

$$f''(x) \text{ is maximum in } [0, 2] \text{ when } x = 2.$$

(b) Simpson's: Error  $\leq \frac{(2-0)^5}{180(4^4)}(0) = 0$  since

$$f^{(4)}(x) = 0.$$

25.  $f''(x) = \frac{2}{x^3}$  in  $[1, 3]$ .

(a)  $|f''(x)|$  is maximum when  $x = 1$  and  $|f''(1)| = 2$ .

Trapezoidal: Error  $\leq \frac{2^3}{12n^2}(2) < 0.00001$ ,  $n^2 > 133,333.33$ ,  $n > 365.15$ ; let  $n = 366$ .

$$f^{(4)}(x) = \frac{24}{x^5} \text{ in } [1, 3]$$

(b)  $|f^{(4)}(x)|$  is maximum when  $x = 1$  and when  $|f^{(4)}(1)| = 24$ .

Simpson's: Error  $\leq \frac{2^5}{180n^4}(24) < 0.00001$ ,  $n^4 > 426,666.67$ ,  $n > 25.56$ ; let  $n = 26$ .

27.  $f(x) = \sqrt{1+x}$

(a)  $f''(x) = -\frac{1}{4(1+x)^{3/2}}$  in  $[0, 2]$ .

$|f''(x)|$  is maximum when  $x = 0$  and  $|f''(0)| = \frac{1}{4}$ .

Trapezoidal: Error  $\leq \frac{8}{12n^2} \left(\frac{1}{4}\right) < 0.00001$ ,  $n^2 > 16,666.67$ ,  $n > 129.10$ ; let  $n = 130$ .

(b)  $f^{(4)}(x) = \frac{-15}{16(1+x)^{7/2}}$  in  $[0, 2]$

$|f^{(4)}(x)|$  is maximum when  $x = 0$  and  $|f^{(4)}(0)| = \frac{15}{16}$ .

Simpson's: Error  $\leq \frac{32}{180n^4} \left(\frac{15}{16}\right) < 0.00001$ ,  $n^4 > 16,666.67$ ,  $n > 11.36$ ; let  $n = 12$ .

29.  $f(x) = \tan(x^2)$

(a)  $f''(x) = 2 \sec^2(x^2)[1 + 4x^2 \tan(x^2)]$  in  $[0, 1]$ .

$|f''(x)|$  is maximum when  $x = 1$  and  $|f''(1)| \approx 49.5305$ .

Trapezoidal: Error  $\leq \frac{(1-0)^3}{12n^2} (49.5305) < 0.00001$ ,  $n^2 > 412,754.17$ ,  $n > 642.46$ ; let  $n = 643$ .

(b)  $f^{(4)}(x) = 8 \sec^2(x^2)[12x^2 + (3 + 32x^4) \tan(x^2) + 36x^2 \tan^2(x^2) + 48x^4 \tan^3(x^2)]$  in  $[0, 1]$

$|f^{(4)}(x)|$  is maximum when  $x = 1$  and  $|f^{(4)}(1)| \approx 9184.4734$ .

Simpson's: Error  $\leq \frac{(1-0)^5}{180n^4} (9184.4734) < 0.00001$ ,  $n^4 > 5,102,485.22$ ,  $n > 47.53$ ; let  $n = 48$ .

31. Let  $f(x) = Ax^3 + Bx^2 + Cx + D$ . Then  $f^{(4)}(x) = 0$ .

Simpson's: Error  $\leq \frac{(b-a)^5}{180n^4} (0) = 0$

Therefore, Simpson's Rule is exact when approximating the integral of a cubic polynomial.

Example:  $\int_0^1 x^3 dx = \frac{1}{6} \left[ 0 + 4 \left(\frac{1}{2}\right)^3 + 1 \right] = \frac{1}{4}$

This is the exact value of the integral.

33.  $f(x) = \sqrt{2 + 3x^2}$  on  $[0, 4]$ .

$n$	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	12.7771	15.3965	18.4340	15.6055	15.4845
8	14.0868	15.4480	16.9152	15.5010	15.4662
10	14.3569	15.4544	16.6197	15.4883	15.4658
12	14.5386	15.4578	16.4242	15.4814	15.4657
16	14.7674	15.4613	16.1816	15.4745	15.4657
20	14.9056	15.4628	16.0370	15.4713	15.4657



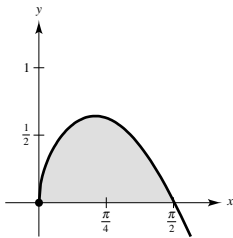
35.  $f(x) = \sin\sqrt{x}$  on  $[0, 4]$ .

$n$	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	2.8163	3.5456	3.7256	3.2709	3.3996
8	3.1809	3.5053	3.6356	3.4083	3.4541
10	3.2478	3.4990	3.6115	3.4296	3.4624
12	3.2909	3.4952	3.5940	3.4425	3.4674
16	3.3431	3.4910	3.5704	3.4568	3.4730
20	3.3734	3.4888	3.5552	3.4643	3.4759

37.  $A = \int_0^{\pi/2} \sqrt{x} \cos x \, dx$

Simpson's Rule:  $n = 14$

$$\int_0^{\pi/2} \sqrt{x} \cos x \, dx \approx \frac{\pi}{84} \left[ \sqrt{0} \cos 0 + 4\sqrt{\frac{\pi}{28}} \cos \frac{\pi}{28} + 2\sqrt{\frac{\pi}{14}} \cos \frac{\pi}{14} + 4\sqrt{\frac{3\pi}{28}} \cos \frac{3\pi}{28} + \cdots + \sqrt{\frac{\pi}{2}} \cos \frac{\pi}{2} \right] \approx 0.701$$



39.  $W = \int_0^5 100x\sqrt{125 - x^3} \, dx$

Simpson's Rule:  $n = 12$

$$\int_0^5 100x\sqrt{125 - x^3} \, dx \approx \frac{5}{3(12)} \left[ 0 + 400\left(\frac{5}{12}\right)\sqrt{125 - \left(\frac{5}{12}\right)^3} + 200\left(\frac{10}{12}\right)\sqrt{125 - \left(\frac{10}{12}\right)^3} + 400\left(\frac{15}{12}\right)\sqrt{125 - \left(\frac{15}{12}\right)^3} + \cdots + 0 \right] \approx 10,233.58 \text{ ft} \cdot \text{lb}$$

41.  $\int_0^{1/2} \frac{6}{\sqrt{1-x^2}} \, dx$  Simpson's Rule,  $n = 6$

$$\pi \approx \frac{\left(\frac{1}{2} - 0\right)}{3(6)} [6 + 4(6.0209) + 2(6.0851) + 4(6.1968) + 2(6.3640) + 4(6.6002) + 6.9282] \approx \frac{1}{36} [113.098] \approx 3.1416$$

43. Area  $\approx \frac{1000}{2(10)} [125 + 2(125) + 2(120) + 2(112) + 2(90) + 2(90) + 2(95) + 2(88) + 2(75) + 2(35)] = 89,250 \text{ sq m}$

45.  $\int_0^t \sin\sqrt{x} \, dx = 2, n = 10$

By trial and error, we obtain  $t \approx 2.477$ .