Review Exercises for Chapter 4
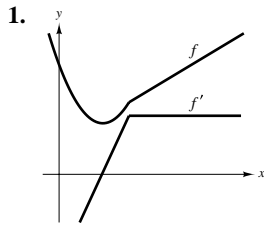
1.

3. $\int (2x^2 + x - 1)\, dx = \frac{2}{3}x^3 + \frac{1}{2}x^2 - x + C$

5. $\int \frac{x^3 + 1}{x^2}\, dx = \int \left(x + \frac{1}{x^2} \right) dx = \frac{1}{2}x^2 - \frac{1}{x} + C$

7. $\int (4x - 3 \sin x)\, dx = 2x^2 + 3 \cos x + C$

9. $f'(x) = -2x, \ (-1, 1)$

$f(x) = \int -2x\, dx = -x^2 + C$

When $x = -1$:

$y = -1 + C = 1$

$C = 2$

$y = 2 - x^2$

11. $a(t) = a$

$v(t) = \int a\, dt = at + C_1$

$v(0) = 0 + C_1 = 0$ when $C_1 = 0$.

$v(t) = at$

$s(t) = \int at\, dt = \frac{a}{2}t^2 + C_2$

$s(0) = 0 + C_2 = 0$ when $C_2 = 0$.

$s(t) = \frac{a}{2}t^2$

$s(30) = \frac{a}{2}(30)^2 = 3600$ or

$a = \frac{2(3600)}{(30)^2} = 8 \text{ ft/sec}^2$.

$v(30) = 8(30) = 240 \text{ ft/sec}$

13. $a(t) = -32$

$v(t) = -32t + 96$

$s(t) = -16t^2 + 96t$

(a) $v(t) = -32t + 96 = 0$ when $t = 3$ sec.

(b) $s(3) = -144 + 288 = 144$ ft

(c) $v(t) = -32t + 96 = \frac{96}{2}$ when $t = \frac{3}{2}$ sec.

(d) $s\left(\frac{3}{2} \right) = -16\left(\frac{9}{4} \right) + 96\left(\frac{3}{2} \right) = 108$ ft

15. (a) $\displaystyle\sum_{i=1}^{10} (2i - 1)$

(b) $\displaystyle\sum_{i=1}^{n} i^3$

(c) $\displaystyle\sum_{i=1}^{10} (4i + 2)$

17. $y = \dfrac{10}{x^2 + 1}$, $\Delta x = \dfrac{1}{2}$, $n = 4$

$$S(n) = S(4) = \frac{1}{2}\left[\frac{10}{1} + \frac{10}{(1/2)^2 + 1} + \frac{10}{(1)^2 + 1} + \frac{10}{(3/2)^2 + 1}\right]$$
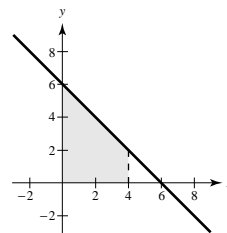
$$\approx 13.0385$$

$$s(n) = s(4) = \frac{1}{2}\left[\frac{10}{(1/2)^2 + 1} + \frac{10}{1 + 1} + \frac{10}{(3/2)^2 + 1} + \frac{10}{2^2 + 1}\right]$$

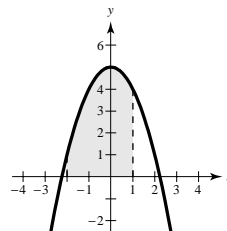$$\approx 9.0385$$

$9.0385 < \text{Area of Region} < 13.0385$

19. $y = 6 - x$, $\Delta x = \dfrac{4}{n}$, right endpoints

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(ci)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(6 - \frac{4i}{n}\right)\frac{4}{n}$$

$$= \lim_{n\to\infty} \frac{4}{n}\left[6n - \frac{4}{n}\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} \left[24 - 8\frac{n+1}{n}\right] = 24 - 8 = 16$$

21. $y = 5 - x^2$, $\Delta x = \dfrac{3}{n}$

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(ci)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[5 - \left(-2 + \frac{3i}{n}\right)^2\right]\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[1 + \frac{12i}{n} - \frac{9i^2}{n^2}\right]$$

$$= \lim_{n\to\infty} \frac{3}{n}\left[n + \frac{12}{n}\frac{n(n+1)}{2} - \frac{9}{n^2}\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty} \left[3 + 18\frac{n+1}{n} - \frac{9}{2}\frac{(n+1)(2n+1)}{n^2}\right]$$

$$= 3 + 18 - 9 = 12$$

23. $x = 5y - y^2$, $2 \le y \le 5$, $\Delta y = \dfrac{3}{n}$

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n}\left[5\left(2 + \frac{3i}{n}\right) - \left(2 + \frac{3i}{n}\right)^2\right]\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n}\left[10 + \frac{15i}{n} - 4 - 12\frac{i}{n} - \frac{9i^2}{n^2}\right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n}\left[6 + \frac{3i}{n} - \frac{9i^2}{n^2}\right]$$

$$= \lim_{n\to\infty} \frac{3}{n}\left[6n + \frac{3}{n}\frac{n(n+1)}{2} - \frac{9}{n^2}\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \left[18 + \frac{9}{2} - 9\right] = \frac{27}{2}$$
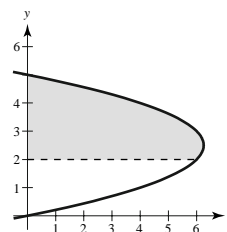
25. $\displaystyle\lim_{\|\Delta\|\to\infty} \sum_{i=1}^{n}(2c_i - 3)\,\Delta x_i = \int_{4}^{6}(2x - 3)\,dx$

27.
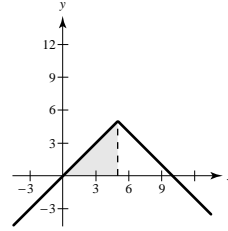
$$\int_{0}^{5}\left(5 - |x - 5|\right)dx = \int_{0}^{5}(5 - (5 - x))\,dx = \int_{0}^{5}x\,dx = \frac{25}{2}$$

(triangle)

29. (a) $\displaystyle\int_{2}^{6}[f(x) + g(x)]\,dx = \int_{2}^{6}f(x)\,dx + \int_{2}^{6}g(x)\,dx = 10 + 3 = 13$

   (b) $\displaystyle\int_{2}^{6}[f(x) - g(x)]\,dx = \int_{2}^{6}f(x)\,dx - \int_{2}^{6}g(x)\,dx = 10 - 3 = 7$

   (c) $\displaystyle\int_{2}^{6}[2f(x) - 3g(x)]\,dx = 2\int_{2}^{6}f(x)\,dx - 3\int_{2}^{6}g(x)\,dx = 2(10) - 3(3) = 11$

   (d) $\displaystyle\int_{2}^{6}5f(x)\,dx = 5\int_{2}^{6}f(x)\,dx = 5(10) = 50$

31. $\displaystyle\int_{1}^{8}\left(\sqrt[3]{x} + 1\right)dx = \left[\frac{3}{4}x^{4/3} + x\right]_{1}^{8} = \left[\frac{3}{4}(16) + 8\right] - \left[\frac{3}{4} + 1\right] = \frac{73}{4}$ (c)
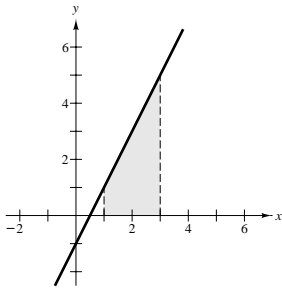
33. $\displaystyle\int_{0}^{4}(2 + x)\,dx = \left[2x + \frac{x^2}{2}\right]_{0}^{4} = 8 + \frac{16}{2} = 16$

35. $\displaystyle\int_{-1}^{1}(4t^3 - 2t)\,dt = \left[t^4 - t^2\right]_{-1}^{1} = 0$

37. $\displaystyle\int_{4}^{9}x\sqrt{x}\,dx = \int_{4}^{9}x^{3/2}\,dx = \left[\frac{2}{5}x^{5/2}\right]_{4}^{9} = \frac{2}{5}\left[(\sqrt{9})^5 - (\sqrt{4})^5\right] = \frac{2}{5}(243 - 32) = \frac{422}{5}$

39. $\displaystyle\int_{0}^{3\pi/4}\sin\theta\,d\theta = \left[-\cos\theta\right]_{0}^{3\pi/4} = -\left(-\frac{\sqrt{2}}{2}\right) + 1 = 1 + \frac{\sqrt{2}}{2} = \frac{\sqrt{2} + 2}{2}$

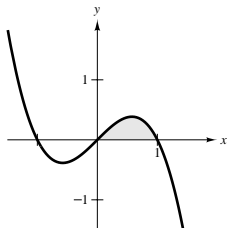**41.** $\displaystyle\int_{1}^{3}(2x - 1)\,dx = \left[x^2 - x\right]_{1}^{3} = 6$

43. $\displaystyle\int_{3}^{4}(x^2 - 9)\,dx = \left[\frac{x^3}{3} - 9x\right]_{3}^{4}$

$$= \left(\frac{64}{3} - 36\right) - (9 - 27)$$

$$= \frac{64}{3} - \frac{54}{3} = \frac{10}{3}$$
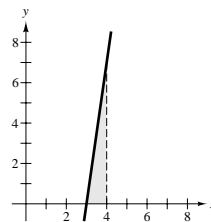
45. $\int_0^1 (x - x^3)\, dx = \left[\dfrac{x^2}{2} - \dfrac{x^4}{4}\right]_0^1 = \dfrac{1}{2} - \dfrac{1}{4} = \dfrac{1}{4}$

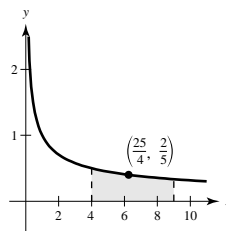**47.** Area $= \int_1^9 \dfrac{4}{\sqrt{x}}\, dx = \left[\dfrac{4x^{1/2}}{(1/2)}\right]_1^9 = 8(3 - 1) = 16$
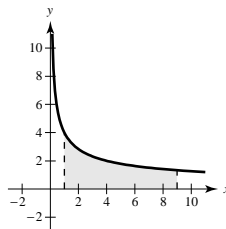
49. $\dfrac{1}{9 - 4} \int_4^9 \dfrac{1}{\sqrt{x}}\, dx = \left[\dfrac{1}{5} 2\sqrt{x}\right]_4^9 = \dfrac{2}{5}(3 - 2) = \dfrac{2}{5}$ Average value

$\dfrac{2}{5} = \dfrac{1}{\sqrt{x}}$

$\sqrt{x} = \dfrac{5}{2}$

$x = \dfrac{25}{4}$

51. $F'(x) = x^2\sqrt{1 + x^3}$

53. $F'(x) = x^2 + 3x + 2$

55. $\int (x^2 + 1)^3\, dx = \int (x^6 + 3x^4 + 3x^2 + 1)\, dx = \dfrac{x^7}{7} + \dfrac{3}{5}x^5 + x^3 + x + C$

57. $u = x^3 + 3,\, du = 3x^2\, dx$

$\int \dfrac{x^2}{\sqrt{x^3 + 3}}\, dx = \int (x^3 + 3)^{-1/2} x^2\, dx = \dfrac{1}{3} \int (x^3 + 3)^{-1/2}\, 3x^2\, dx = \dfrac{2}{3}(x^3 + 3)^{1/2} + C$

59. $u = 1 - 3x^2,\, du = -6x\, dx$

$\int x(1 - 3x^2)^4\, dx = -\dfrac{1}{6} \int (1 - 3x^2)^4(-6x\, dx) = -\dfrac{1}{30}(1 - 3x^2)^5 + C = \dfrac{1}{30}(3x^2 - 1)^5 + C$

61. $\int \sin^3 x \cos x\, dx = \dfrac{1}{4} \sin^4 x + C$

63. $\int \dfrac{\sin \theta}{\sqrt{1 - \cos \theta}}\, dx = \int (1 - \cos \theta)^{-1/2} \sin \theta\, d\theta = 2(1 - \cos \theta)^{1/2} + C = 2\sqrt{1 - \cos \theta} + C$

65. $\int \tan^n x \sec^2 x\, dx = \dfrac{\tan^{n+1} x}{n + 1} + C,\, n \neq -1$

67. $\int (1 + \sec \pi x)^2 \sec \pi x \tan \pi x\, dx = \dfrac{1}{\pi} \int (1 + \sec \pi x)^2 (\pi \sec \pi x \tan \pi x)\, dx = \dfrac{1}{3\pi}(1 + \sec \pi x)^3 + C$

69. $\int_{-1}^2 x(x^2 - 4)\, dx = \dfrac{1}{2} \int_{-1}^2 (x^2 - 4)(2x)\, dx = \dfrac{1}{2} \dfrac{(x^2 - 4)^2}{2}\Big]_{-1}^2 = \dfrac{1}{4}[0 - 9] = -\dfrac{9}{4}$

71. $\displaystyle\int_0^3 \frac{1}{\sqrt{1+x}}\,dx = \int_0^3 (1+x)^{-1/2}\,dx = \left[2(1+x)^{1/2}\right]_0^3 = 4 - 2 = 2$

73. $u = 1 - y,\ y = 1 - u,\ dy = -du$

When $y = 0,\ u = 1.$ When $y = 1,\ u = 0.$

$$2\pi\int_0^1 (y+1)\sqrt{1-y}\,dy = 2\pi\int_1^0 -[(1-u)+1]\sqrt{u}\,du$$

$$= 2\pi\int_1^0 (u^{3/2} - 2u^{1/2})\,du = 2\pi\left[\frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2}\right]_1^0 = \frac{28\pi}{15}$$

75. $\displaystyle\int_0^\pi \cos\!\left(\frac{x}{2}\right)dx = 2\int_0^\pi \cos\!\left(\frac{x}{2}\right)\frac{1}{2}\,dx = \left[2\sin\!\left(\frac{x}{2}\right)\right]_0^\pi = 2$

77. $u = 1 - x,\ x = 1 - u,\ dx = -du$

When $x = a,\ u = 1 - a.$ When $x = b,\ u = 1 - b.$

$$P_{a,b} = \int_a^b \frac{15}{4}x\sqrt{1-x}\,dx = \frac{15}{4}\int_{1-a}^{1-b} -(1-u)\sqrt{u}\,du$$

$$= \frac{15}{4}\int_{1-a}^{1-b} (u^{3/2} - u^{1/2})\,du = \frac{15}{4}\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_{1-a}^{1-b} = \frac{15}{4}\left[\frac{2u^{3/2}}{15}(3u - 5)\right]_{1-a}^{1-b} = \left[-\frac{(1-x)^{3/2}}{2}(3x + 2)\right]_a^b$$

(a) $\displaystyle P_{0.50,\,0.75} = \left[-\frac{(1-x)^{3/2}}{2}(3x+2)\right]_{0.50}^{0.75} = 0.353 = 35.3\%$

(b) $\displaystyle P_{0,\,b} = \left[-\frac{(1-x)^{3/2}}{2}(3x+2)\right]_0^b = -\frac{(1-b)^{3/2}}{2}(3b+2) + 1 = 0.5$

$(1-b)^{3/2}(3b+2) = 1$

$b \approx 0.586 = 58.6\%$

79. $p = 1.20 + 0.04t$

$$C = \frac{15,000}{M}\int_t^{t+1} p\,ds$$

(a) 2000 corresponds to $t = 10.$

$$C = \frac{15,000}{M}\int_{10}^{11}[1.20 + 0.04t]\,dt$$

$$= \frac{15,000}{M}\left[1.20t + 0.02t^2\right]_{10}^{11} = \frac{24,300}{M}$$

(b) 2005 corresponds to $t = 15.$

$$C = \frac{15,000}{M}\left[1.20t + 0.02t^2\right]_{15}^{16} = \frac{27,300}{M}$$

81. Trapezoidal Rule $(n = 4)$: $\displaystyle\int_1^2 \frac{1}{1+x^3}\,dx \approx \frac{1}{8}\left[\frac{1}{1+1^3} + \frac{2}{1+(1.25)^3} + \frac{2}{1+(1.5)^3} + \frac{2}{1+(1.75)^3} + \frac{1}{1+2^3}\right] \approx 0.257$

Simpson's Rule $(n = 4)$: $\displaystyle\int_1^2 \frac{1}{1+x^3}\,dx \approx \frac{1}{12}\left[\frac{1}{1+1^3} + \frac{4}{1+(1.25)^3} + \frac{2}{1+(1.5)^3} + \frac{4}{1+(1.75)^3} + \frac{1}{1+2^3}\right] \approx 0.254$

Graphing utility: 0.254

83. Trapezoidal Rule $(n = 4)$: $\displaystyle\int_0^{\pi/2}\sqrt{x}\cos x\,dx \approx 0.637$

Simpson's Rule $(n = 4)$: 0.685

Graphing Utility: 0.704

85. (a) $R < I < T < L$

(b) $\displaystyle S(4) = \frac{4-0}{3(4)}[f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)]$

$$\approx \frac{1}{3}\left[4 + 4(2) + 2(1) + 4\!\left(\frac{1}{2}\right) + \frac{1}{4}\right] \approx 5.417$$

Problem Solving for Chapter 4

1. (a) $L(1) = \displaystyle\int_1^1 \frac{1}{t}\,dt = 0$

(b) $L'(x) = \dfrac{1}{x}$ by the Second Fundamental Theorem of Calculus.

$L'(1) = 1$

(c) $L(x) = 1 = \displaystyle\int_1^x \frac{1}{t}\,dt$ for $x \approx 2.718$

$\displaystyle\int_1^{2.718} \frac{1}{t}\,dt = 0.999896$ (**Note:** The exact value of x is e, the base of the natural logarithm function.)
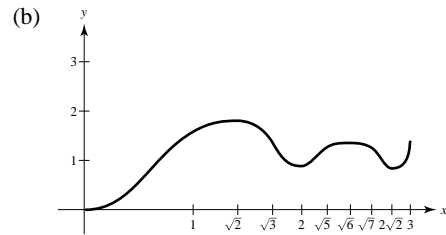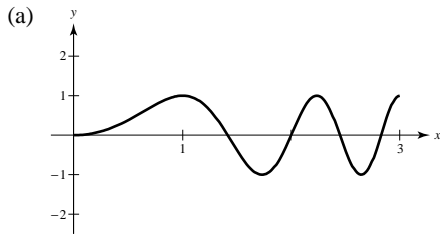
(d) We first show that $\displaystyle\int_1^{x_1} \frac{1}{t}\,dt = \int_{1/x_1}^1 \frac{1}{t}\,dt$.

To see this, let $u = \dfrac{t}{x_1}$ and $du = \dfrac{1}{x_1}\,dt$.

Then $\displaystyle\int_1^{x_1} \frac{1}{t}\,dt = \int_{1/x_1}^1 \frac{1}{ux_1}(x_1\,du) = \int_{1/x_1}^1 \frac{1}{u}\,du = \int_{1/x_1}^1 \frac{1}{t}\,dt$.

Now, $L(x_1 x_2) = \displaystyle\int_1^{x_1 x_2} \frac{1}{t}\,dt = \int_{1/x_1}^{x_2} \frac{1}{u}\,du\left(\text{using } u = \frac{t}{x_1}\right)$

$= \displaystyle\int_{1/x_1}^1 \frac{1}{u}\,du + \int_1^{x_2} \frac{1}{u}\,du$

$= \displaystyle\int_1^{x_1} \frac{1}{u}\,du + \int_1^{x_2} \frac{1}{u}\,du$

$= L(x_1) + L(x_2).$

3. $S(x) = \displaystyle\int_0^x \sin\left(\frac{\pi t^2}{2}\right)dt$

(a)

(b)

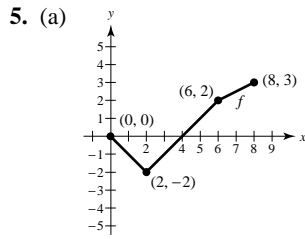The zeros of $y = \sin\dfrac{\pi x^2}{2}$ correspond to the relative extrema of $S(x)$.

(c) $S'(x) = \sin\dfrac{\pi x^2}{2} = 0 \implies \dfrac{\pi x^2}{2} = n\pi \implies x^2 = 2n \implies x = \sqrt{2n}$, n integer.

Relative maximum at $x = \sqrt{2} \approx 1.4142$ and $x = \sqrt{6} \approx 2.4495$

Relative minimum at $x = 2$ and $x = \sqrt{8} \approx 2.8284$

(d) $S''(x) = \cos\left(\dfrac{\pi x^2}{2}\right)(\pi x) = 0 \implies \dfrac{\pi x^2}{2} = \dfrac{\pi}{2} + n\pi \implies x^2 = 1 + 2n \implies x = \sqrt{1 + 2n}$, n integer

Points of inflection at $x = 1$, $\sqrt{3}$, $\sqrt{5}$, and $\sqrt{7}$.

5. (a)

(b)

x	0	1	2	3	4	5	6	7	8
$F(x)$	0	$-\dfrac{1}{2}$	-2	$-\dfrac{7}{2}$	-4	$-\dfrac{7}{2}$	-2	$\dfrac{1}{4}$	3

(c) $f(x) = \begin{cases} -x, & 0 \le x < 2 \\ x - 4, & 2 \le x < 6 \\ \dfrac{1}{2}x - 1, & 6 \le x \le 8 \end{cases}$

$F(x) = \displaystyle\int_0^x f(t)\, dt = \begin{cases} (-x^2/2), & 0 \le x < 2 \\ (x^2/2) - 4x + 4, & 2 \le x < 6 \\ (1/4)x^2 - x - 5, & 6 \le x \le 8 \end{cases}$

$F'(x) = f(x)$. F is decreasing on $(0, 4)$ and increasing on $(4, 8)$. Therefore, the minimum is -4 at $x = 4$, and the maximum is 3 at $x = 8$.

(d) $F''(x) = f'(x) = \begin{cases} -1, & 0 < x < 2 \\ 1, & 2 < x < 6 \\ \dfrac{1}{2}, & 6 < x < 8 \end{cases}$

$x = 2$ is a point of inflection, whereas $x = 6$ is not. (f is not continuous at $x = 6$.)

7. (a) $\displaystyle\int_{-1}^{1} \cos x \, dx \approx \cos\left(-\frac{1}{\sqrt{3}}\right) + \cos\left(\frac{1}{\sqrt{3}}\right) = 2\cos\left(\frac{1}{\sqrt{3}}\right) \approx 1.6758$

$\displaystyle\int_{-1}^{1} \cos x \, dx = \sin x \Big]_{-1}^{1} = 2\sin(1) \approx 1.6829$

Error. $|1.6829 - 1.6758| = 0.0071$

(b) $\displaystyle\int_{-1}^{1} \frac{1}{1 + x^2} \, dx \approx \frac{1}{1 + (1/3)} + \frac{1}{1 + (1/3)} = \frac{3}{2}$

(Note: exact answer is $\pi/2 \approx 1.5708$)

(c) Let $p(x) = ax^3 + bx^2 + cx + d$.

$\displaystyle\int_{-1}^{1} p(x) \, dx = \left[\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx \right]_{-1}^{1} = \frac{2b}{3} + 2d$

$p\left(-\frac{1}{\sqrt{3}}\right) + p\left(\frac{1}{\sqrt{3}}\right) = \left(\frac{b}{3} + d\right) + \left(\frac{b}{3} + d\right) = \frac{2b}{3} + 2d$

9. Consider $F(x) = [f(x)]^2 \implies F'(x) = 2f(x)f'(x)$. Thus,

$\displaystyle\int_a^b f(x) f'(x) \, dx = \int_a^b \frac{1}{2} F'(x) \, dx$

$= \left[\frac{1}{2} F(x) \right]_a^b$

$= \frac{1}{2} [F(b) - F(a)]$

$= \frac{1}{2} [f(b)^2 - f(a)^2]$

11. Consider $\displaystyle\int_0^1 x^5 \, dx = \frac{x^6}{6} \Big]_0^1 = \frac{1}{6}$.

The corresponding Riemann Sum using right endpoints is

$S(n) = \frac{1}{n}\left[\left(\frac{1}{n}\right)^5 + \left(\frac{2}{n}\right)^5 + \cdots + \left(\frac{n}{n}\right)^5 \right]$

$= \frac{1}{n^6}[1^5 + 2^5 + \cdots + n^5]$

Thus, $\displaystyle\lim_{n\to\infty} S(n) = \lim_{n\to\infty} \frac{1^5 + 2^5 + \cdots + n^5}{n^6} = \frac{1}{6}$.

13. By Theorem 4.8, $0 < f(x) \le M \implies \int_a^b f(x)\, dx \le \int_a^b M\, dx = M(b - a)$.

Similarly, $m \le f(x) \implies m(b - a) = \int_a^b m\, dx \le \int_a^b f(x)\, dx$.

Thus, $m(b - a) \le \int_a^b f(x)\, dx \le M(b - a)$. On the interval $[0, 1]$, $1 \le \sqrt{1 + x^4} \le \sqrt{2}$ and $b - a = 1$.

Thus, $1 \le \int_0^1 \sqrt{1 + x^4}\, dx \le \sqrt{2}$. $\left(\textbf{Note: } \int_0^1 \sqrt{1 + x^4}\, dx \approx 1.0894\right)$

15. Since $-|f(x)| \le f(x) \le |f(x)|$,

$$-\int_a^b |f(x)|\, dx \le \int_a^b f(x)\, dx \le \int_a^b |f(x)|\, dx \implies \left| \int_a^b f(x)\, dx \right| \le \int_a^b |f(x)|\, dx.$$

17. $\dfrac{1}{365} \displaystyle\int_0^{365} 100{,}000 \left[1 + \sin \dfrac{2\pi(t - 60)}{365} \right] dt = \dfrac{100{,}000}{365} \left[t - \dfrac{365}{2\pi} \cos \dfrac{2\pi(t - 60)}{365} \right]_0^{365} = 100{,}000 \text{ lbs.}$