CHAPTER 5
Logarithmic, Exponential, and Other Transcendental Functions

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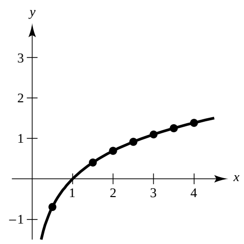
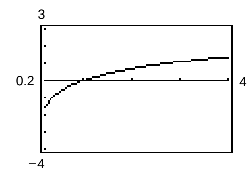
Section 5.1 The Natural Logarithmic Function: Differentiation

Solutions to Even-Numbered Exercises

2. (a)



(b)



The graphs are identical.

4. (a) $\ln 8.3 \approx 2.1163$

(b) $\int_1^{8.3} \frac{1}{t}\,dt \approx 2.1163$

6. (a) $\ln 0.6 \approx -0.5108$

(b) $\int_1^{0.6} \frac{1}{t}\,dt \approx -0.5108$

8. $f(x) = -\ln x$

Reflection in the x-axis

Matches (d)

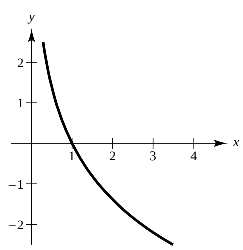
10. $f(x) = -\ln(-x)$

Reflection in the y-axis and the x-axis

Matches (c)

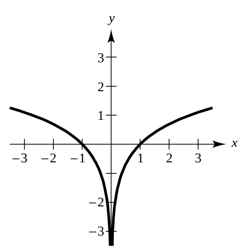
12. $f(x) = -2\ln x$

Domain: $x > 0$



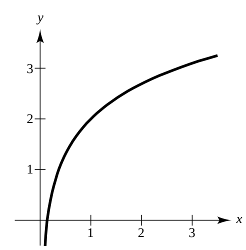
14. $f(x) = \ln|x|$

Domain: $x \neq 0$



16. $g(x) = 2 + \ln x$

Domain: $x > 0$



18. (a) $\ln 0.25 = \ln \frac{1}{4} = \ln 1 - 2\ln 2 \approx -1.3862$

(b) $\ln 24 = 3\ln 2 + \ln 3 \approx 3.1779$

(c) $\ln \sqrt[3]{12} = \frac{1}{3}(2\ln 2 + \ln 3) \approx 0.8283$

(d) $\ln \frac{1}{72} = \ln 1 - (3\ln 2 + 2\ln 3) \approx -4.2765$

20. $\ln\sqrt{2^3} = \ln 2^{3/2} = \frac{3}{2}\ln 2$

22. $\ln xyz = \ln x + \ln y + \ln z$

24. $\ln\sqrt{a-1} = \ln(a-1)^{1/2} = \left(\frac{1}{2}\right)\ln(a-1)$

26. $\ln 3e^2 = \ln 3 + 2\ln e = 2 + \ln 3$

28. $\ln \frac{1}{e} = \ln 1 - \ln e = -1$

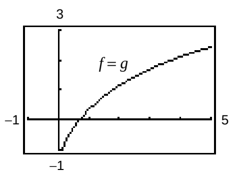
30. $3\ln x + 2\ln y - 4\ln z = \ln x^3 + \ln y^2 - \ln z^4$

$$= \ln \frac{x^3y^2}{z^4}$$

32. $2[\ln x - \ln(x+1) - \ln(x-1)] = 2\ln\frac{x}{(x+1)(x-1)} = \ln\left(\frac{x}{x^2-1}\right)^2$

34. $\frac{3}{2}[\ln(x^2+1) - \ln(x+1) - \ln(x-1)] = \frac{3}{2}\ln\frac{x^2+1}{(x+1)(x-1)} = \ln\sqrt{\left(\frac{x^2+1}{x^2-1}\right)^3}$

36.



38. $\lim_{x\to 6^-} \ln(6-x) = -\infty$

40. $\lim_{x\to 5^+} \ln\frac{x}{\sqrt{x-4}} = \ln 5 \approx 1.6094$

42. $y = \ln x^{3/2} = \frac{3}{2}\ln x$

$y' = \frac{3}{2x}$

At (1, 0), $y' = \frac{3}{2}$.

44. $y = \ln x^{1/2} = \frac{1}{2}\ln x$

$y' = \frac{1}{2x}$

At (1, 0), $y' = \frac{1}{2}$.

46. $h(x) = \ln(2x^2+1)$

$h'(x) = \frac{1}{2x^2+1}(4x) = \frac{4x}{2x^2+1}$

48. $y = x\ln x$

$\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln x = 1 + \ln x$

50. $y = \ln\sqrt{x^2-4} = \frac{1}{2}\ln(x^2-4)$

$\frac{dy}{dx} = \frac{1}{2}\left(\frac{2x}{x^2-4}\right) = \frac{x}{x^2-4}$

52. $f(x) = \ln\left(\frac{2x}{x+3}\right) = \ln 2x - \ln(x+3)$

$f'(x) = \frac{1}{x} - \frac{1}{x+3} = \frac{3}{x(x+3)}$

54. $h(t) = \frac{\ln t}{t}$

$h'(t) = \frac{t(1/t) - \ln t}{t^2} = \frac{1-\ln t}{t^2}$

56. $y = \ln(\ln x)$

$\frac{dy}{dx} = \frac{1/x}{\ln x} = \frac{1}{x\ln x}$

58. $y = \ln\sqrt[3]{\frac{x-1}{x+1}} = \frac{1}{3}[\ln(x-1) - \ln(x+1)]$

$y' = \frac{1}{3}\left[\frac{1}{x-1} - \frac{1}{x+1}\right] = \frac{1}{3}\frac{2}{x^2-1} = \frac{2}{3(x^2-1)}$

60. $f(x) = \ln\left(x + \sqrt{4+x^2}\right)$

$f'(x) = \frac{1}{x+\sqrt{4+x^2}}\left(1 + \frac{x}{\sqrt{4+x^2}}\right)$

$= \frac{1}{\sqrt{4+x^2}}$

62. $y = \frac{-\sqrt{x^2+4}}{2x^2} - \frac{1}{4}\ln\left(\frac{2+\sqrt{x^2+4}}{x}\right) = \frac{-\sqrt{x^2+4}}{2x^2} - \frac{1}{4}\ln\left(2+\sqrt{x^2+4}\right) + \frac{1}{4}\ln x$

$$\frac{dy}{dx} = \frac{-2x^2\left(x/\sqrt{x^2+4}\right) + 4x\sqrt{x^2+4}}{4x^4} - \frac{1}{4}\left(\frac{1}{2+\sqrt{x^2+4}}\right)\left(\frac{x}{\sqrt{x^2+4}}\right) + \frac{1}{4x}$$

Note that:

$$\frac{1}{2+\sqrt{x^2+4}} = \frac{1}{2+\sqrt{x^2+4}} \cdot \frac{2-\sqrt{x^2+4}}{2-\sqrt{x^2+4}} = \frac{2-\sqrt{x^2+4}}{-x^2}$$

Hence, $\frac{dy}{dx} = \frac{-1}{2x\sqrt{x^2+4}} + \frac{\sqrt{x^2+4}}{x^3} - \frac{1}{4}\frac{\left(2-\sqrt{x^2+4}\right)}{-x^2}\left(\frac{x}{\sqrt{x^2+4}}\right) + \frac{1}{4x}$

$$= \frac{-1 + (1/2)\left(2-\sqrt{x^2+4}\right)}{2x\sqrt{x^2+4}} + \frac{\sqrt{x^2+4}}{x^3} + \frac{1}{4x}$$

$$= \frac{-\sqrt{x^2+4}}{4x\sqrt{x^2+4}} + \frac{\sqrt{x^2+4}}{x^3} + \frac{1}{4x} = \frac{\sqrt{x^2+4}}{x^3}$$

64. $y = \ln|\csc x|$

$$y' = \frac{-\csc x \cdot \cot x}{\csc x} = -\cot x$$

66. $y = \ln|\sec x + \tan x|$

$$\frac{dy}{dx} = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$$

$$= \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} = \sec x$$

68. $y = \ln\sqrt{1+\sin^2 x} = \frac{1}{2}\ln(1+\sin^2 x)$

$$\frac{dy}{dx} = \left(\frac{1}{2}\right)\frac{2\sin x\cos x}{1+\sin^2 x} = \frac{\sin x \cos x}{1+\sin^2 x}$$

70. $g(x) = \int_1^{\ln x} (t^2+3)\,dt$

$$g'(x) = [(\ln x)^2 + 3]\frac{d}{dx}(\ln x) = \frac{(\ln x)^2 + 3}{x}$$

(Second Fundamental Theorem of Calculus)

72. (a) $y = 4 - x^2 - \ln\left(\frac{1}{2}x+1\right)$, $(0, 4)$

$$\frac{dy}{dx} = -2x - \frac{1}{(1/2)x+1}\left(\frac{1}{2}\right)$$

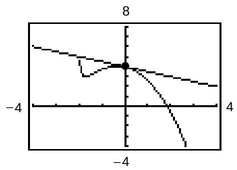
$$= -2x - \frac{1}{x+2}$$

When $x = 0$, $\frac{dy}{dx} = -\frac{1}{2}$.

Tangent line: $y - 4 = -\frac{1}{2}(x-0)$

$$y = -\frac{1}{2}x + 4$$

(b)



74. $\ln(xy) + 5x = 30$

$$\ln x + \ln y + 5x = 30$$

$$\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} + 5 = 0$$

$$\frac{1}{y}\frac{dy}{dx} = -\frac{1}{x} - 5$$

$$\frac{dy}{dx} = -\frac{y}{x} - 5y = -\left(\frac{y+5xy}{x}\right)$$

76. $y = x(\ln x) - 4x$

$$y' = x\left(\frac{1}{x}\right) + \ln x - 4 = -3 + \ln x$$

$$(x + y) - xy' = x + x\ln x - 4x - x(-3 + \ln x) = 0$$

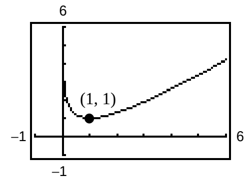
78. $y = x - \ln x$

Domain: $x > 0$

$y' = 1 - \dfrac{1}{x} = 0$ when $x = 1$.

$y'' = \dfrac{1}{x^2} > 0$

Relative minimum: $(1, 1)$



80. $y = \dfrac{\ln x}{x}$

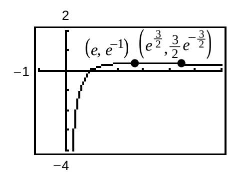
Domain: $x > 0$

$y' = \dfrac{x(1/x) - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2} = 0$ when $x = e$.

$y'' = \dfrac{x^2(-1/x) - (1 - \ln x)(2x)}{x^4} = \dfrac{2(\ln x) - 3}{x^3} = 0$ when $x = e^{3/2}$.

Relative maximum: (e, e^{-1})

Point of inflection: $\left(e^{3/2}, \frac{3}{2}e^{-3/2}\right)$



82. $y = x^2 \ln \dfrac{x}{4}$. Domain $x > 0$

$y' = x^2\left(\dfrac{1}{x}\right) + 2x\ln\dfrac{x}{4} = x\left(1 + 2\ln\dfrac{x}{4}\right) = 0$ when

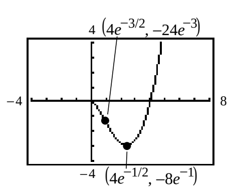
$-1 = 2\ln\dfrac{x}{4} \Rightarrow \ln\dfrac{x}{4} = -\dfrac{1}{2} \Rightarrow x = 4e^{-1/2}$

$y'' = 1 + 2\ln\dfrac{x}{4} + 2x\left(\dfrac{1}{x}\right) = 3 + 2\ln\dfrac{x}{4}$

$y'' = 0$ when $x = 4e^{-3/2}$

Relative minimum: $(4e^{-1/2}, -8e^{-1})$

Point of inflection: $(4e^{-3/2}, -24e^{-3})$



84. $f(x) = x\ln x, \quad f(1) = 0$

$f'(x) = 1 + \ln x, \quad f'(1) = 1$

$f''(x) = \dfrac{1}{x}, \quad f''(1) = 1$

$P_1(x) = f(1) + f'(1)(x - 1) = x - 1, \quad P_1(1) = 0$

$P_2(x) = f(1) + f'(1)(x - 1) + \dfrac{1}{2}f''(1)(x - 1)^2$

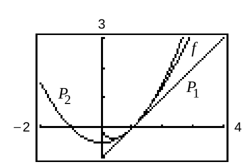
$\quad = (x - 1) + \dfrac{1}{2}(x - 1)^2, \quad P_2(1) = 0$

$P_1'(x) = 1, \quad P_1'(1) = 1$

$P_2'(x) = 1 + (x - 1) = x, \quad P_2'(1) = 1$

$P_2''(x) = x, \quad P_2''(1) = 1$

The values of f, P_1, P_2, and their first derivatives agree at $x = 1$. The values of the second derivatives of f and P_2 agree at $x = 1$.



86. Find x such that $\ln x = 3 - x$.

$$f(x) = x + (\ln x) - 3 = 0$$

$$f'(x) = 1 + \frac{1}{x}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n\left[\frac{4 - \ln x_n}{1 + x_n}\right]$$

n	1	2	3
x_n	2	2.2046	2.2079
$f(x_n)$	-0.3069	-0.0049	0.0000

Approximate root: $x = 2.208$

88.
$$y = \sqrt{(x-1)(x-2)(x-3)}$$

$$\ln y = \frac{1}{2}[\ln(x-1) + \ln(x-2) + \ln(x-3)]$$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{2}\left[\frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3}\right]$$

$$= \frac{1}{2}\left[\frac{3x^2 - 12x + 11}{(x-1)(x-2)(x-3)}\right]$$

$$\frac{dy}{dx} = \frac{3x^2 - 12x + 11}{2y}$$

$$= \frac{3x^2 - 12x + 11}{2\sqrt{(x-1)(x-2)(x-3)}}$$

90.
$$y = \sqrt{\frac{x^2 - 1}{x^2 + 1}}$$

$$\ln y = \frac{1}{2}[\ln(x^2 - 1) - \ln(x^2 + 1)]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{2x}{x^2 - 1} - \frac{2x}{x^2 + 1}\right]$$

$$\frac{dy}{dx} = \sqrt{\frac{x^2 - 1}{x^2 + 1}}\left[\frac{2x}{x^4 - 1}\right]$$

$$= \frac{(x^2 - 1)^{1/2}2x}{(x^2 + 1)^{1/2}(x^2 - 1)(x^2 + 1)}$$

$$= \frac{2x}{(x^2 + 1)^{3/2}(x^2 - 1)^{1/2}}$$

92.
$$y = \frac{(x+1)(x+2)}{(x-1)(x-2)}$$

$$\ln y = \ln(x+1) + \ln(x+2) - \ln(x-1) - \ln(x-2)$$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x+1} + \frac{1}{x+2} - \frac{1}{x-1} - \frac{1}{x-2}$$

$$\frac{dy}{dx} = y\left[\frac{-2}{x^2 - 1} + \frac{-4}{x^2 - 4}\right] = y\left[\frac{-6x^2 + 12}{(x^2 - 1)(x^2 - 4)}\right]$$

$$= \frac{(x+1)(x+2)}{(x-1)(x-2)} \cdot \frac{-6(x^2 - 2)}{(x+1)(x-1)(x+2)(x-2)}$$

$$= -\frac{6(x^2 - 2)}{(x-1)^2(x-2)^2}$$

94. The base of the natural logarithmic function is e.

96. $g(x) = \ln f(x),\ f(x) > 0$

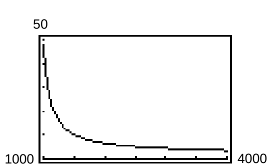
$$g'(x) = \frac{f'(x)}{f(x)}$$

(a) Yes. If the graph of g is increasing, then $g'(x) > 0$. Since $f(x) > 0$, you know that $f'(x) = g'(x)f(x)$ and thus, $f'(x) > 0$. Therefore, the graph of f is increasing.

(b) No. Let $f(x) = x^2 + 1$ (positive and concave up). $g(x) = \ln(x^2 + 1)$ is not concave up.

98. $t = \dfrac{5.315}{-6.7968 + \ln x},\ 1000 < x$

(a)



(b) $t(1167.41) \approx 20$ years

$$T = (1167.41)(20)(12) = \$280{,}178.40$$

(c) $t(1068.45) \approx 30$ years

$$T = (1068.45)(30)(12) = \$384{,}642.00$$

(d) $\dfrac{dt}{dx} = -5.315(-6.7968 + \ln x)^{-2}\left(\dfrac{1}{x}\right)$

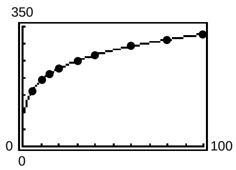
$$= -\frac{5.315}{x(-6.7968 + \ln x)^2}$$

When $x = 1167.41$, $dt/dx \approx -0.0645$. When $x = 1068.45$, $dt/dx \approx -0.1585$.

(e) There are two obvious benefits to paying a higher monthly payment:

1. The term is lower
2. The total amount paid is lower.

100. (a)

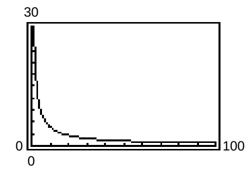


(b) $T'(p) = \dfrac{34.96}{p} + \dfrac{3.955}{\sqrt{p}}$

$T'(10) \approx 4.75 \text{ deg/lb/in}^2$

$T'(70) \approx 0.97 \text{ deg/lb/in}^2$

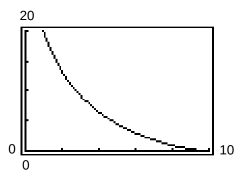
(c)



$\lim_{p\to\infty} T'(p) = 0$

102. $y = 10\ln\left(\dfrac{10 + \sqrt{100 - x^2}}{x}\right) - \sqrt{100 - x^2} = 10\left[\ln\left(10 + \sqrt{100 - x^2}\right) - \ln x\right] - \sqrt{100 - x^2}$

(a)



(c) $\displaystyle\lim_{x\to 10^-} \frac{dy}{dx} = 0$

(b) $$\frac{dy}{dx} = 10\left[\frac{-x}{\sqrt{100 - x^2}\left(10 + \sqrt{100 - x^2}\right)} - \frac{1}{x}\right] + \frac{x}{\sqrt{100 - x^2}}$$

$$= \frac{x}{\sqrt{100 - x^2}}\left[\frac{-10}{10 + \sqrt{100 - x^2}}\right] - \frac{10}{x} + \frac{x}{\sqrt{100 - x^2}}$$

$$= \frac{x}{\sqrt{100 - x^2}}\left[\frac{-10}{10 + \sqrt{100 - x^2}} + 1\right] - \frac{10}{x}$$

$$= \frac{x}{\sqrt{100 - x^2}}\left[\frac{\sqrt{100 - x^2}}{10 + \sqrt{100 - x^2}}\right] - \frac{10}{x}$$

$$= \frac{x}{10 + \sqrt{100 - x^2}} - \frac{10}{x}$$

$$= \frac{x\left(10 - \sqrt{100 - x^2}\right)}{x^2} - \frac{10}{x} = -\frac{\sqrt{100 - x^2}}{x}$$

When $x = 5$, $dy/dx = -\sqrt{3}$. When $x = 9$, $dy/dx = -\sqrt{19}/9$.

104. $y = \ln x$

$y' = \dfrac{1}{x} > 0$ for $x > 0$.

Since $\ln x$ is increasing on its entire domain $(0, \infty)$, it is a strictly monotonic function and therefore, is one-to-one.

106. False

π is a constant.

$\dfrac{d}{dx}[\ln \pi] = 0$

Section 5.2 The Natural Logarithmic Function: Integration

2. $\displaystyle\int \frac{10}{x}\,dx = 10\int \frac{1}{x}\,dx = 10\ln|x| + C$

4. $u = x - 5$, $du = dx$

$$\int \frac{1}{x - 5}\,dx = \ln|x - 5| + C$$

6. $$\int \frac{1}{3x + 2}\,dx = \frac{1}{3}\int \frac{1}{3x + 2}(3)\,dx$$

$$= \frac{1}{3}\ln|3x + 2| + C$$

8. $u = 3 - x^3$, $du = -3x^2\,dx$

$$\int \frac{x^2}{3 - x^3}\,dx = -\frac{1}{3}\int \frac{1}{3 - x^3}(-3x^2)\,dx$$

$$= -\frac{1}{3}\ln|3 - x^3| + C$$

10. $u = 9 - x^2,\ du = -2x\,dx$

$$\int \frac{x}{\sqrt{9 - x^2}}\,dx = -\frac{1}{2}\int (9 - x^2)^{-1/2}(-2x)\,dx = -\sqrt{9 - x^2} + C$$

12. $$\int \frac{x(x + 2)}{x^3 + 3x^2 - 4}\,dx = \frac{1}{3}\int \frac{3x^2 + 6x}{x^3 + 3x^2 - 4}\,dx \quad (u = x^3 + 3x^2 - 4)$$

$$= \frac{1}{3}\ln\left|x^3 + 3x^2 - 4\right| + C$$

14. $$\int \frac{2x^2 + 7x - 3}{x - 2}\,dx = \int \left(2x + 11 + \frac{19}{x - 2}\right)dx$$

$$= x^2 + 11x + 19\ln|x - 2| + C$$

16. $$\int \frac{x^3 - 6x - 20}{x + 5}\,dx = \int \left(x^2 - 5x + 19 - \frac{115}{x + 5}\right)dx$$

$$= \frac{x^3}{3} - \frac{5x^2}{2} + 19x - 115\ln|x + 5| + C$$

18. $$\int \frac{x^3 - 3x^2 + 4x - 9}{x^2 + 3}\,dx = \int \left(-3 + x + \frac{x}{x^2 + 3}\right)dx$$

$$= -3x + \frac{x^2}{2} + \frac{1}{2}\ln(x^2 + 3) + C$$

20. $$\int \frac{1}{x\ln(x^3)}\,dx = \frac{1}{3}\int \frac{1}{\ln x}\cdot\frac{1}{x}\,dx$$

$$= \frac{1}{3}\ln\left|\ln|x|\right| + C$$

22. $u = 1 + x^{1/3},\ du = \dfrac{1}{3x^{2/3}}\,dx$

$$\int \frac{1}{x^{2/3}(1 + x^{1/3})}\,dx = 3\int \frac{1}{1 + x^{1/3}}\left(\frac{1}{3x^{2/3}}\right)dx$$

$$= 3\ln\left|1 + x^{1/3}\right| + C$$

24. $$\int \frac{x(x - 2)}{(x - 1)^3}\,dx = \int \frac{x^2 - 2x + 1 - 1}{(x - 1)^3}\,dx$$

$$= \int \frac{(x - 1)^2}{(x - 1)^3}\,dx - \int \frac{1}{(x - 1)^3}\,dx$$

$$= \int \frac{1}{x - 1}\,dx - \int \frac{1}{(x - 1)^3}\,dx$$

$$= \ln|x - 1| + \frac{1}{2(x - 1)^2} + C$$

26. $u = 1 + \sqrt{3x},\ du = \dfrac{3}{2\sqrt{3x}}dx \implies dx = \dfrac{2}{3}(u - 1)\,du$

$$\int \frac{1}{1 + \sqrt{3x}}\,dx = \int \frac{1}{u}\,\frac{2}{3}(u - 1)\,du$$

$$= \frac{2}{3}\int \left(1 - \frac{1}{u}\right)du$$

$$= \frac{2}{3}\left[u - \ln|u|\right] + C$$

$$= \frac{2}{3}\left[1 + \sqrt{3x} - \ln\left(1 + \sqrt{3x}\right)\right] + C$$

$$= \frac{2}{3}\sqrt{3x} - \frac{2}{3}\ln\left(1 + \sqrt{3x}\right) + C_1$$

28. $u = x^{1/3} - 1,\ du = \dfrac{1}{3x^{2/3}}\,dx \implies dx = 3(u+1)^2\,du$

$$\begin{aligned}\int \frac{\sqrt[3]{x}}{\sqrt[3]{x}-1}\,dx &= \int \frac{u+1}{u}3(u+1)^2\,du\\ &= 3\int \frac{u+1}{u}(u^2+2u+1)\,du\\ &= 3\int\left(u^2+3u+3+\frac{1}{u}\right)du\\ &= 3\left[\frac{u^3}{3}+\frac{3u^2}{2}+3u+\ln|u|\right]+C\\ &= 3\left[\frac{(x^{1/3}-1)^3}{3}+\frac{3(x^{1/3}-1)^2}{2}+3(x^{1/3}-1)+\ln|x^{1/3}-1|\right]+C\\ &= 3\ln|x^{1/3}-1|+\frac{3x^{2/3}}{2}+3x^{1/3}+x+C_1\end{aligned}$$

30. $$\int \tan 5\theta\,d\theta = \frac{1}{5}\int\frac{5\sin 5\theta}{\cos 5\theta}d\theta = -\frac{1}{5}\ln|\cos 5\theta| + C$$

32. $$\int \sec\frac{x}{2}\,dx = 2\int \sec\frac{x}{2}\left(\frac{1}{2}\right)dx = 2\ln\left|\sec\frac{x}{2}+\tan\frac{x}{2}\right| + C$$

34. $u = \cot t,\ du = -\csc^2 t\,dt$

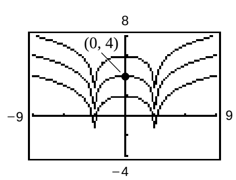
$$\int\frac{\csc^2 t}{\cot t}dt = -\ln|\cot t| + C$$

36. $$\begin{aligned}\int(\sec t+\tan t)\,dt &= \ln|\sec t+\tan t| - \ln|\cos t| + C\\ &= \ln\left|\frac{\sec t+\tan t}{\cos t}\right| + C\\ &= \ln|\sec t(\sec t+\tan t)| + C\end{aligned}$$

38. $$y = \int\frac{2x}{x^2-9}\,dx = \ln|x^2-9| + C$$

$(0, 4)$: $4 = \ln|0-9| + C \implies C = 4 - \ln 9$

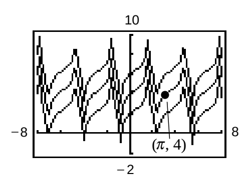
$y = \ln|x^2-9| + 4 - \ln 9$



40. $$r = \int\frac{\sec^2 t}{\tan t + 1}\,dt = \ln|\tan t + 1| + C$$

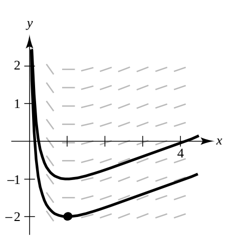
$(\pi, 4)$: $4 = \ln|0+1| + C \implies C = 4$

$r = \ln|\tan t + 1| + 4$



42. $\dfrac{dy}{dx} = \dfrac{\ln x}{x},\ (1, -2)$

(a)



(b) $$y = \int\frac{\ln x}{x}\,dx = \frac{(\ln x)^2}{2} + C$$

$y(1) = -2 \implies -2 = \dfrac{(\ln 1)^2}{2} + C \implies C = -2$

Hence, $y = \dfrac{(\ln x)^2}{2} - 2.$

44. $\displaystyle\int_{-1}^{1} \frac{1}{x+2}\,dx = \Big[\ln|x+2|\Big]_{-1}^{1}$

$= \ln 3 - \ln 1 = \ln 3$

46. $u = \ln x,\ du = \dfrac{1}{x}\,dx$

$\displaystyle\int_{e}^{e^2} \frac{1}{x\ln x}\,dx = \int_{e}^{e^2}\left(\frac{1}{\ln x}\right)\frac{1}{x}\,dx = \Big[\ln|\ln|x||\Big]_{e}^{e^2} = \ln 2$

48. $\displaystyle\int_{0}^{1} \frac{x-1}{x+1}\,dx = \int_{0}^{1} 1\,dx + \int_{0}^{1} \frac{-2}{x+1}\,dx$

$= \Big[x - 2\ln|x+1|\Big]_{0}^{1} = 1 - 2\ln 2$

50. $\displaystyle\int_{0.1}^{0.2} (\csc 2\theta - \cot 2\theta)^2\,d\theta = \int_{0.1}^{0.2} (\csc^2 2\theta - 2\csc 2\theta\cot 2\theta + \cot^2 2\theta)\,d\theta$

$\displaystyle= \int_{0.1}^{0.2} (2\csc^2 2\theta - 2\csc 2\theta\cot 2\theta - 1)\,d\theta$

$= \Big[-\cot 2\theta + \csc 2\theta - \theta\Big]_{0.1}^{0.2} \approx 0.0024$

52. $\ln|\sin x| + C = \ln\left|\dfrac{1}{\csc x}\right| + C = -\ln|\csc x| + C$

54. $-\ln|\csc x + \cot x| + C = -\ln\left|\dfrac{(\csc x + \cot x)(\csc x - \cot x)}{(\csc x - \cot x)}\right| + C = -\ln\left|\dfrac{\csc^2 x - \cot^2 x}{\csc x - \cot x}\right| + C$

$= -\ln\left|\dfrac{1}{\csc x - \cot x}\right| + C = \ln|\csc x - \cot x| + C$

56. $\displaystyle\int \frac{1-\sqrt{x}}{1+\sqrt{x}}\,dx = -(1+\sqrt{x})^2 + 6(1+\sqrt{x}) - 4\ln(1+\sqrt{x}) + C_1$

$= 4\sqrt{x} - x - 4\ln(1+\sqrt{x}) + C$ where $C = C_1 + 5$.

58. $\displaystyle\int \frac{\tan^2 2x}{\sec 2x}\,dx = \frac{1}{2}[\ln|\sec 2x + \tan 2x| - \sin 2x] + C$

60. $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{\sin^2 x - \cos^2 x}{\cos x}\,dx = \Big[\ln|\sec x + \tan x| - 2\sin x\Big]_{-\pi/4}^{\pi/4}$

$= \ln\left(\dfrac{\sqrt{2}+1}{\sqrt{2}-1}\right) - 2\sqrt{2} \approx -1.066$

Note: In Exercises 62 and 64, you can use the Second Fundamental Theorem of Calculus or integrate the function.

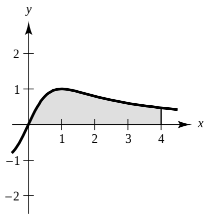
62. $\displaystyle F(x) = \int_{0}^{x} \tan t\,dt$

$F'(x) = \tan x$

64. $\displaystyle F(x) = \int_{1}^{x^2} \frac{1}{t}\,dt$

$F'(x) = \dfrac{2x}{x^2} = \dfrac{2}{x}$

66.



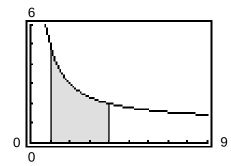
$A \approx 3$
Matches (a)

68. $A = \int_1^4 \frac{x+4}{x}\,dx = \int_1^4 \left(1 + \frac{4}{x}\right) dx$

$= \left[x + 4 \ln x\right]_1^4$

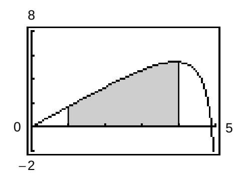
$= 4 + 4 \ln 4 - 1$

$= 3 + 4 \ln 4 \approx 8.5452$



70. $\int_1^4 (2x - \tan(0.3x))\,dx = \left[x^2 + \frac{10}{3} \ln|\cos(0.3x)|\right]_1^4$

$= \left[16 + \frac{10}{3} \ln \cos(1.2)\right] - \left[1 + \frac{10}{3} \ln \cos(0.3)\right] \approx 11.7686$



72. Substitution: $(u = x^2 + 4)$ and Power Rule

74. Substitution: $(u = \tan x)$ and Log Rule

76. Answers will vary.

78. Average value $= \frac{1}{4-2} \int_2^4 \frac{4(x+1)}{x^2}\,dx$

$= 2 \int_2^4 \left(\frac{1}{x} + \frac{1}{x^2}\right) dx$

$= 2\left[\ln x - \frac{1}{x}\right]_2^4$

$= 2\left[\ln 4 - \frac{1}{4} - \ln 2 + \frac{1}{2}\right]$

$= 2\left[\ln 2 + \frac{1}{4}\right] = \ln 4 + \frac{1}{2} \approx 1.8863$

80. Average value $= \frac{1}{2-0} \int_0^2 \sec \frac{\pi x}{6}\,dx$

$= \left[\frac{1}{2}\left(\frac{6}{\pi}\right) \ln\left|\sec \frac{\pi x}{6} + \tan \frac{\pi x}{6}\right|\right]_0^2$

$= \frac{3}{\pi}\left[\ln\left(2 + \sqrt{3}\right) - \ln(1 + 0)\right]$

$= \frac{3}{\pi} \ln\left(2 + \sqrt{3}\right)$

82. $t = \frac{10}{\ln 2} \int_{250}^{300} \frac{1}{T - 100}\,dT$

$= \frac{10}{\ln 2}\left[\ln(T - 100)\right]_{250}^{300} = \frac{10}{\ln 2}[\ln 200 - \ln 150] = \frac{10}{\ln 2}\left[\ln\left(\frac{4}{3}\right)\right] \approx 4.1504$ units of time

84. $\frac{dS}{dt} = \frac{k}{t}$

$S(t) = \int \frac{k}{t}\,dt = k \ln|t| + C = k \ln t + C$ since $t > 1$.

$S(2) = k \ln 2 + C = 200$

$S(4) = k \ln 4 + C = 300$

Solving this system yields $k = 100/\ln 2$ and $C = 100$. Thus,

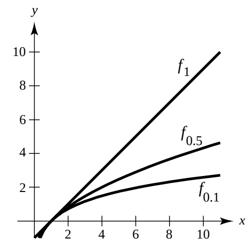
$S(t) = \frac{100 \ln t}{\ln 2} + 100 = 100\left[\frac{\ln t}{\ln 2} + 1\right].$

86. $k = 1$: $f_1(x) = x - 1$

$k = 0.5$: $f_{0.5}(x) = \dfrac{\sqrt{x} - 1}{0.5} = 2\left(\sqrt{x} - 1\right)$

$k = 0.1$: $f_{0.1}(x) = \dfrac{\sqrt[10]{x} - 1}{0.1} = 10\left(\sqrt[10]{x} - 1\right)$

$\lim\limits_{k \to 0^+} f_k(x) = \ln x$



88. False

$\dfrac{d}{dx}[\ln x] = \dfrac{1}{x}$

90. False; the integrand has a nonremovable discontinuity at $x = 0$.

Section 5.3 Inverse Functions

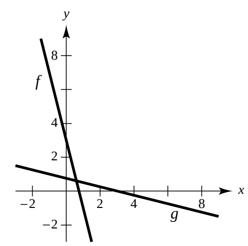
2. (a) $f(x) = 3 - 4x$

$g(x) = \dfrac{3 - x}{4}$

$f(g(x)) = f\left(\dfrac{3 - x}{4}\right) = 3 - 4\left(\dfrac{3 - x}{4}\right) = x$

$g(f(x)) = g(3 - 4x) = \dfrac{3 - (3 - 4x)}{4} = x$

(b)



4. (a) $f(x) = 1 - x^3$

$g(x) = \sqrt[3]{1 - x}$

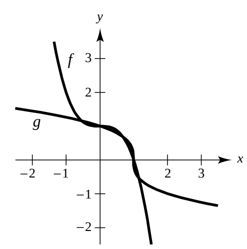
$f(g(x)) = f\left(\sqrt[3]{1 - x}\right) = 1 - \left(\sqrt[3]{1 - x}\right)^3$

$= 1 - (1 - x) = x$

$g(f(x)) = g(1 - x^3)$

$= \sqrt[3]{1 - (1 - x^3)} = \sqrt[3]{x^3} = x$

(b)



6. (a) $f(x) = 16 - x^2,\ x \geq 0$

$g(x) = \sqrt{16 - x}$

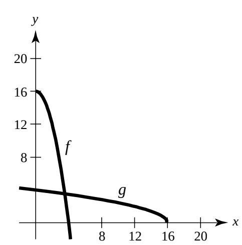
$f(g(x)) = f\left(\sqrt{16 - x}\right) = 16 - \left(\sqrt{16 - x}\right)^2$

$= 16 - (16 - x) = x$

$g(f(x)) = g(16 - x^2) = \sqrt{16 - (16 - x^2)}$

$= \sqrt{x^2} = x$

(b)



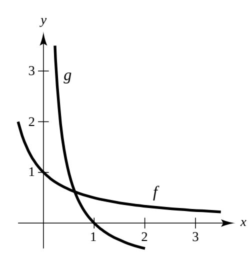
8. (a) $f(x) = \dfrac{1}{1 + x},\ x \geq 0$

$g(x) = \dfrac{1 - x}{x},\ 0 < x \leq 1$

$f(g(x)) = f\left(\dfrac{1 - x}{x}\right) = \dfrac{1}{1 + \dfrac{1 - x}{x}} = \dfrac{1}{\dfrac{1}{x}} = x$

$g(f(x)) = g\left(\dfrac{1}{1 + x}\right) = \dfrac{1 - \dfrac{1}{1 + x}}{\dfrac{1}{1 + x}} = \dfrac{x}{1 + x} \cdot \dfrac{1 + x}{1} = x$

(b)

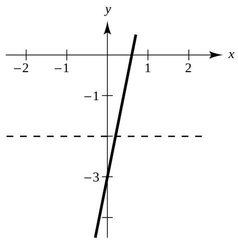


10. Matches (b)

12. Matches (d)

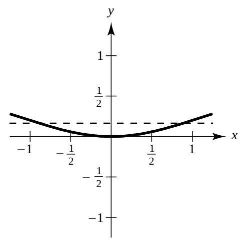
14. $f(x) = 5x - 3$

One-to-one; has an inverse



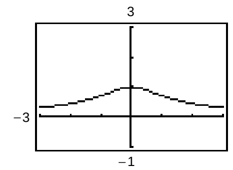
16. $F(x) = \dfrac{x^2}{x^2 + 4}$

Not one-to-one; does not have an inverse



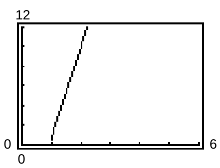
18. $g(t) = \dfrac{1}{\sqrt{t^2 + 1}}$

Not one-to-one; does not have an inverse



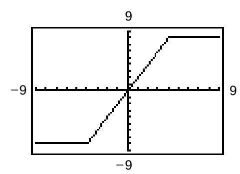
20. $f(x) = 5x\sqrt{x - 1}$

One-to-one; has an inverse



22. $h(x) = |x + 4| - |x - 4|$

Not one-to-one; does not have an inverse



24. $f(x) = \cos\dfrac{3x}{2}$

$f'(x) = -\dfrac{3}{2}\sin\dfrac{3x}{2} = 0$ when $x = 0, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \ldots$

f is not strictly monotonic on $(-\infty, \infty)$. Therefore, f does not have an inverse.

26. $f(x) = x^3 - 6x^2 + 12x$

$f'(x) = 3x^2 - 12x + 12 = 3(x - 2)^2 \geq 0$ for all x.

f is increasing on $(-\infty, \infty)$. Therefore, f is strictly monotonic and has an inverse.

28. $f(x) = \ln(x - 3),\ x > 3$

$f'(x) = \dfrac{1}{x - 3} > 0$ for $x > 3$.

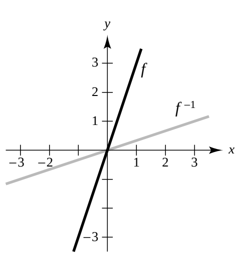
f is increasing on $(3, \infty)$. Therefore, f is strictly monotonic and has an inverse.

30. $f(x) = 3x = y$

$x = \dfrac{y}{3}$

$y = \dfrac{x}{3}$

$f^{-1}(x) = \dfrac{x}{3}$

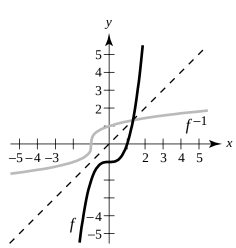


32. $f(x) = x^3 - 1 = y$

$x = \sqrt[3]{y + 1}$

$y = \sqrt[3]{x + 1}$

$f^{-1}(x) = \sqrt[3]{x + 1}$

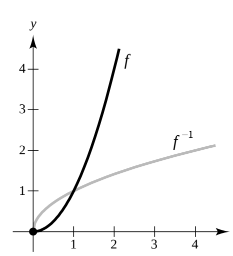


34. $f(x) = x^2 = y,\ 0 \leq x$

$x = \sqrt{y}$

$y = \sqrt{x}$

$f^{-1}(x) = \sqrt{x}$

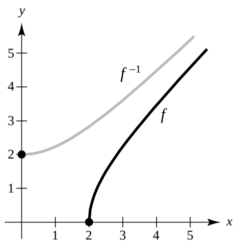


36. $f(x) = \sqrt{x^2 - 4} = y,\ x \ge 2$

$x = \sqrt{y^2 + 4}$

$y = \sqrt{x^2 + 4}$

$f^{-1}(x) = \sqrt{x^2 + 4},\ x \ge 0$

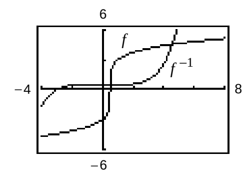


38. $f(x) = 3\sqrt[5]{2x - 1} = y$

$x = \dfrac{y^5 + 243}{486}$

$y = \dfrac{x^5 + 243}{486}$

$f^{-1}(x) = \dfrac{x^5 + 243}{486}$



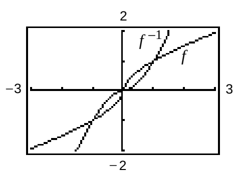
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

40. $f(x) = x^{3/5} = y$

$x = y^{5/3}$

$y = x^{5/3}$

$f^{-1}(x) = x^{5/3}$



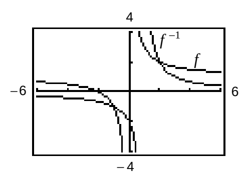
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

42. $f(x) = \dfrac{x + 2}{x} = y$

$x = \dfrac{2}{y - 1}$

$y = \dfrac{2}{x - 1}$

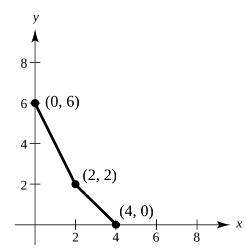
$f^{-1}(x) = \dfrac{2}{x - 1}$



The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

44.

x	0	2	4
$f^{-1}(x)$	6	2	0



46. $f(x) = k(2 - x - x^3)$ is one-to-one for all $k \ne 0$. Since $f^{-1}(3) = -2$, $f(-2) = 3 = k(2 - (-2) - (-2)^3) = 12k \implies k = \frac{1}{4}$.

48. $f(x) = |x + 2|$ on $[-2, \infty)$

$f'(x) = \dfrac{|x + 2|}{x + 2}(1) = 1 > 0$ on $(-2, \infty)$

f is increasing on $[-2, \infty)$. Therefore, f is strictly monotonic and has an inverse.

50. $f(x) = \cot x$ on $(0, \pi)$

$f'(x) = -\csc^2 x < 0$ on $(0, \pi)$

f is decreasing on $(0, \pi)$. Therefore, f is strictly monotonic and has an inverse.

52. $f(x) = \sec x$ on $\left[0, \dfrac{\pi}{2}\right)$

$f'(x) = \sec x \tan x > 0$ on $\left(0, \dfrac{\pi}{2}\right)$

f is increasing on $[0, \pi/2)$. Therefore, f is strictly monotonic and has an inverse.

54. $f(x) = 2 - \dfrac{3}{x^2} = y$ on $(0, 10)$

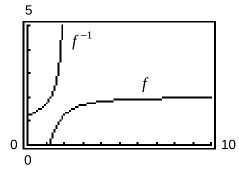
$2x^2 - 3 = x^2y$

$x^2(2 - y) = 3$

$x = \pm\sqrt{\dfrac{3}{2 - y}}$

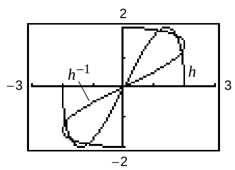
$y = \pm\sqrt{\dfrac{3}{2 - x}}$

$f^{-1}(x) = \sqrt{\dfrac{3}{2 - x}},\ x < 2$



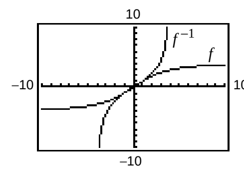
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

56. (a), (b)



(c) h is not one-to-one and does not have an inverse. The inverse relation is not an inverse function.

58. (a), (b)



(c) Yes, f is one-to-one and has an inverse. The inverse relation is an inverse function.

60. $f(x) = -3$

Not one-to-one; does not have an inverse

62. $f(x) = ax + b$

f is one-to-one; has an inverse

$ax + b = y$

$x = \dfrac{y - b}{a}$

$y = \dfrac{x - b}{a}$

$f^{-1}(x) = \dfrac{x - b}{a},\ a \neq 0$

64. $f(x) = 16 - x^4$ is one-to-one for $x \geq 0$.

$16 - x^4 = y$

$16 - y = x^4$

$\sqrt[4]{16 - y} = x$

$\sqrt[4]{16 - x} = y$

$f^{-1}(x) = \sqrt[4]{16 - x},\ x \leq 16$

66. $f(x) = |x - 3|$ is one-to-one for $x \geq 3$.

$x - 3 = y$

$x = y + 3$

$y = x + 3$

$f^{-1}(x) = x + 3,\ x \geq 0$

68. No, there could be two times $t_1 \neq t_2$ for which $h(t_1) = h(t_2)$.

70. Yes, the area function is increasing and hence one-to-one. The inverse function gives the radius r corresponding to the area A.

72. $f(x) = \dfrac{1}{27}(x^5 + 2x^3);\ f(-3) = \dfrac{1}{27}(-243 - 54) = -11 = a.$

$f'(x) = \dfrac{1}{27}(5x^4 + 6x^2)$

$(f^{-1})'(-11) = \dfrac{1}{f'(f^{-1}(-11))} = \dfrac{1}{f'(-3)} = \dfrac{27}{5(-3)^4 + 6(-3)^2} = \dfrac{1}{17}$

74. $f(x) = \cos 2x, f(0) = 1 = a$

$f'(x) = -2 \sin 2x$

$(f^{-1})'(1) = \dfrac{1}{f'(f^{-1}(1))} = \dfrac{1}{f'(0)} = \dfrac{1}{-2 \sin 0} = \dfrac{1}{0}$ which is undefined.

76. $f(x) = \sqrt{x-4}, f(8) = 2 = a$

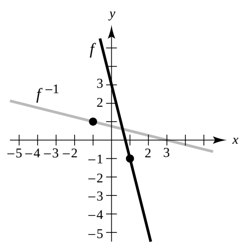
$f'(x) = \dfrac{1}{2\sqrt{x-4}}$

$(f^{-1})'(2) = \dfrac{1}{f'(f^{-1}(2))} = \dfrac{1}{f'(8)} = \dfrac{1}{1/\left(2\sqrt{8-4}\right)} = \dfrac{1}{1/4} = 4$

78. (a) Domain f = Domain $f^{-1} = (-\infty, \infty)$

(b) Range f = Range $f^{-1} = (-\infty, \infty)$

(c)



(d) $f(x) = 3 - 4x, \ (1, -1)$

$f'(x) = -4$

$f'(1) = -4$

$f^{-1}(x) = \dfrac{3-x}{4}, \ (-1, 1)$

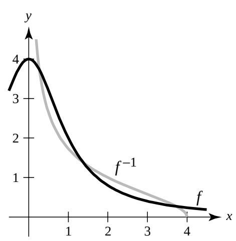
$(f^{-1})'(x) = -\dfrac{1}{4}$

$(f^{-1})'(-1) = -\dfrac{1}{4}$

80. (a) Domain $f = [0, \infty)$, Domain $f^{-1} = (0, 4]$

(b) Range $f = (0, 4]$, Range $f^{-1} = [0, \infty)$

(c)



(d) $f(x) = \dfrac{4}{1+x^2}$

$f'(x) = \dfrac{-8x}{(x^2+1)^2}, \ f'(1) = -2$

$f^{-1}(x) = \sqrt{\dfrac{4-x}{x}}$

$(f^{-1})'(x) = \dfrac{-2}{x^2\sqrt{\dfrac{4-x}{x}}}, \ (f^{-1})'(2) = -\dfrac{1}{2}$

82. $x = 2\ln(y^2 - 3)$

$1 = 2\dfrac{1}{y^2-3} 2y \dfrac{dy}{dx}$

$\dfrac{dy}{dx} = \dfrac{y^2-3}{4y}$. At $(0, 4)$, $\dfrac{dy}{dx} = \dfrac{16-3}{16} = \dfrac{13}{16}$.

In Exercises 84 and 86, use the following.

$\boldsymbol{f(x) = \frac{1}{8}x - 3}$ **and** $\boldsymbol{g(x) = x^3}$

$\boldsymbol{f^{-1}(x) = 8(x+3)}$ **and** $\boldsymbol{g^{-1}(x) = \sqrt[3]{x}}$

84. $(g^{-1} \circ f^{-1})(-3) = g^{-1}(f^{-1}(-3) = g^{-1}(0) = 0$

86. $(g^{-1} \circ g^{-1})(-4) = g^{-1}(g^{-1}(-4)) = g^{-1}\left(\sqrt[3]{-4}\right)$

$= \sqrt[3]{\sqrt[3]{-4}} = -\sqrt[9]{4}$

In Exercises 88 and 90, use the following.

$$\boldsymbol{f(x) = x + 4 \text{ and } g(x) = 2x - 5}$$

$$\boldsymbol{f^{-1}(x) = x - 4 \text{ and } g^{-1}(x) = \frac{x+5}{2}}$$

88. $(f^{-1} \circ g^{-1})(x) = f^{-1}(g^{-1}(x))$

$$= f^{-1}\left(\frac{x+5}{2}\right)$$

$$= \frac{x+5}{2} - 4$$

$$= \frac{x-3}{2}$$

90. $(g \circ f)(x) = g(f(x))$

$$= g(x + 4)$$

$$= 2(x + 4) - 5$$

$$= 2x + 3$$

Hence, $(g \circ f)^{-1}(x) = \dfrac{x-3}{2}$

(Note: $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$)

92. The graphs of f and f^{-1} are mirror images with respect to the line $y = x$.

94. Theorem 5.9: Let f be differentiable on an interval I. If f has an inverse g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))},\ f'(g(x)) \neq 0$$

96. f is not one-to-one because different x-values yield the same y-value.

Example: $f(3) = f\left(-\frac{4}{3}\right) = \frac{3}{5}$

Not continuous at ± 2.

98. If f has an inverse, then f and f^{-1} are both one-to-one. Let $(f^{-1})^{-1}(x) = y$ then $x = f^{-1}(y)$ and $f(x) = y$. Thus, $(f^{-1})^{-1} = f$.

100. If f has an inverse and $f(x_1) = f(x_2)$, then $f^{-1}(f(x_1)) = f^{-1}(f(x_2)) \implies x_1 = x_2$. Therefore, f is one-to-one. If $f(x)$ is one-to-one, then for every value b in the range, there corresponds exactly one value a in the domain. Define $g(x)$ such that the domain of g equals the range of f and $g(b) = a$. By the reflexive property of inverses, $g = f^{-1}$.

102. True; if f has a y-intercept.

104. False

Let $f(x) = x$ or $g(x) = 1/x$.

106. From Theorem 5.9, we have:

$$g'(x) = \frac{1}{f'(g(x))}$$

$$g''(x) = \frac{f'(g(x))(0) - f''(g(x))g'(x)}{[f'(g(x))]^2}$$

$$= -\frac{f''(g(x)) \cdot [1/(f'(g(x)))]}{[f'(g(x))]^2}$$

$$= -\frac{f''(g(x))}{[f'(g(x))]^3}$$

If f is increasing and concave down, then $f' > 0$ and $f'' < 0$ which implies that g is increasing and concave up.

Section 5.4 Exponential Functions: Differentiation and Integration

2. $e^{-2} = 0.1353\ldots$

$\ln 0.1353\ldots = -2$

4. $\ln 0.5 = -0.6931\ldots$

$e^{-0.6931\ldots} = \frac{1}{2}$

6. $e^{\ln 2x} = 12$

$2x = 12$

$x = 6$

8. $4e^x = 83$

$e^x = \dfrac{83}{4}$

$x = \ln\left(\dfrac{83}{4}\right) \approx 3.033$

10. $-6 + 3e^x = 8$

$3e^x = 14$

$e^x = \dfrac{14}{3}$

$x = \ln\left(\dfrac{14}{3}\right) \approx 1.540$

12. $200e^{-4x} = 15$

$e^{-4x} = \dfrac{15}{200} = \dfrac{3}{40}$

$-4x = \ln\left(\dfrac{3}{40}\right)$

$x = \dfrac{1}{4}\ln\left(\dfrac{40}{3}\right) \approx 0.648$

14. $\ln x^2 = 10$

$x^2 = e^{10}$

$x = \pm e^5 \approx \pm 148.4132$

16. $\ln 4x = 1$

$4x = e' = e$

$x = \dfrac{e}{4} \approx 0.680$

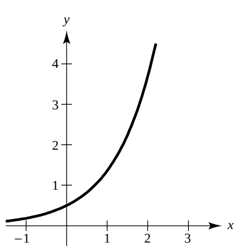
18. $\ln(x - 2)^2 = 12$

$(x - 2)^2 = e^{12}$

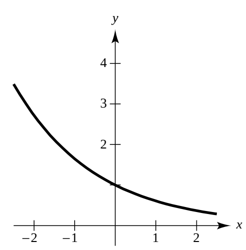
$x - 2 = e^6$

$x = 2 + e^6 \approx 405.429$

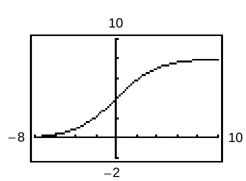
20. $y = \frac{1}{2}e^x$



22. $y = e^{-x/2}$

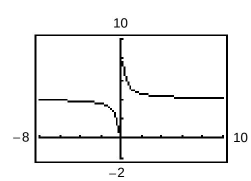


24. (a)



Horizontal asymptotes: $y = 0$ and $y = 8$

(b)



Horizontal asymptote: $y = 4$

26. $y = Ce^{-ax}$

Horizontal asymptote: $y = 0$

Reflection in the y-axis

Matches (d)

28. $y = \dfrac{C}{1 + e^{-ax}}$

$$\lim_{x \to \infty} \frac{C}{1 + e^{-ax}} = C$$

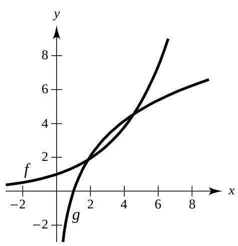
$$\lim_{x \to -\infty} \frac{C}{1 + e^{-ax}} = 0$$

Horizontal asymptotes: $y = C$ and $y = 0$

Matches (b)

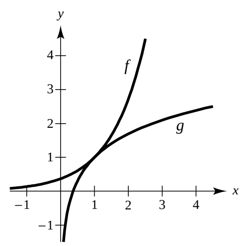
30. $f(x) = e^{x/3}$

$g(x) = \ln x^3 = 3 \ln x$



32. $f(x) = e^{x-1}$

$g(x) = 1 + \ln x$



34. In the same way,

$$\lim_{x \to \infty} \left(1 + \frac{r}{x}\right)^x = e^r \text{ for } r > 0.$$

36. $1 + 1 + \dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{24} + \dfrac{1}{120} + \dfrac{1}{720} + \dfrac{1}{5040} = 2.71\overline{825396}$

$$e \approx 2.718281828$$

$$e > 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040}$$

38. (a) $y = e^{2x}$

$y' = 2e^{2x}$

At $(0, 1)$, $y' = 2$.

(b) $y = e^{-2x}$

$y' = -2e^{-2x}$

At $(0, 1)$, $y' = -2$.

40. $f(x) = e^{1-x}$

$f'(x) = -e^{1-x}$

42. $y = e^{-x^2}$

$\dfrac{dy}{dx} = -2xe^{-x^2}$

44. $y = x^2 e^{-x}$

$\dfrac{dy}{dx} = -x^2 e^{-x} + 2xe^{-x}$

$= xe^{-x}(2 - x)$

46. $g(t) = e^{-3/t^2}$

$g'(t) = e^{-3/t^2}(6t^{-3}) = \dfrac{6}{t^3 e^{3/t^2}}$

48. $y = \ln\left(\dfrac{1 + e^x}{1 - e^x}\right)$

$= \ln(1 + e^x) - \ln(1 - e^x)$

$\dfrac{dy}{dx} = \dfrac{e^x}{1 + e^x} + \dfrac{e^x}{1 - e^x}$

$= \dfrac{2e^x}{1 - e^{2x}}$

50. $y = \ln\left(\dfrac{e^x + e^{-x}}{2}\right)$

$= \ln(e^x + e^{-x}) - \ln 2$

$\dfrac{dy}{dx} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

$= \dfrac{e^{2x} - 1}{e^{2x} + 1}$

52. $y = \dfrac{e^x - e^{-x}}{2}$

$\dfrac{dy}{dx} = \dfrac{e^x + e^{-x}}{2}$

54. $y = xe^x - e^x = e^x(x - 1)$

$\dfrac{dy}{dx} = e^x + e^x(x - 1) = xe^x$

56. $f(x) = e^3 \ln x$

$f'(x) = \dfrac{e^3}{x}$

58. $y = \ln e^x = x$

$\dfrac{dy}{dx} = 1$

60. $e^{xy} + x^2 - y^2 = 10$

$\left(x\dfrac{dy}{dx} + y\right)e^{xy} + 2x - 2y\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx}(xe^{xy} - 2y) = -ye^{xy} - 2x$

$\dfrac{dy}{dx} = -\dfrac{ye^{xy} + 2x}{xe^{xy} - 2y}$

62. $g(x) = \sqrt{x} + e^x \ln x$

$g'(x) = \dfrac{1}{2\sqrt{x}} + \dfrac{e^x}{x} + e^x \ln x$

$g''(x) = -\dfrac{1}{4x^{3/2}} + \dfrac{xe^x - e^x}{x^2} + \dfrac{e^x}{x} + e^x \ln x$

$= -\dfrac{1}{4x\sqrt{x}} + \dfrac{e^x(2x - 1)}{x^2} + e^x \ln x$

64. $y = e^x(3\cos 2x - 4\sin 2x)$

$y' = e^x(-6\sin 2x - 8\cos 2x) + e^x(3\cos 2x - 4\sin 2x)$
$= e^x(-10\sin 2x - 5\cos 2x) = -5e^x(2\sin 2x + \cos 2x)$

$y'' = -5e^x(4\cos 2x - 2\sin 2x) - 5e^x(2\sin 2x + \cos 2x) = -5e^x(5\cos 2x) = -25e^x\cos 2x$

$y'' - 2y' = -25e^x\cos 2x - 2(-5e^x)(2\sin 2x + \cos 2x) = -5e^x(3\cos 2x - 4\sin 2x) = -5y$

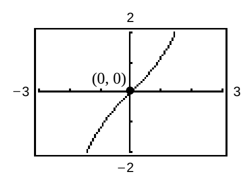
Therefore, $y'' - 2y' = -5y \Longrightarrow y'' - 2y' + 5y = 0$.

66. $f(x) = \dfrac{e^x - e^{-x}}{2}$

$f'(x) = \dfrac{e^x + e^{-x}}{2} > 0$

$f''(x) = \dfrac{e^x - e^{-x}}{2} = 0$ when $x = 0$.

Point of inflection: $(0, 0)$



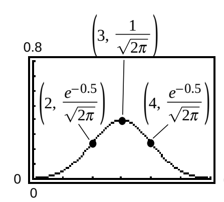
68. $g(x) = \dfrac{1}{\sqrt{2\pi}}e^{-(x-3)^2/2}$

$g'(x) = \dfrac{-1}{\sqrt{2\pi}}(x - 3)e^{-(x-3)^2/2}$

$g''(x) = \dfrac{1}{\sqrt{2\pi}}(x - 2)(x - 4)e^{-(x-3)^2/2}$

Relative maximum: $\left(3, \dfrac{1}{\sqrt{2\pi}}\right) \approx (3, 0.399)$

Points of inflection: $\left(2, \dfrac{1}{\sqrt{2\pi}}e^{-1/2}\right), \left(4, \dfrac{1}{\sqrt{2\pi}}e^{-1/2}\right) \approx (2, 0.242), (4, 0.242)$



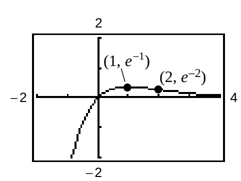
70. $f(x) = xe^{-x}$

$f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1 - x) = 0$ when $x = 1$.

$f''(x) = -e^{-x} + (-e^{-x})(1 - x) = e^{-x}(x - 2) = 0$ when $x = 2$.

Relative maximum: $(1, e^{-1})$

Point of inflection: $(2, 2e^{-2})$



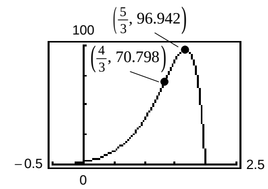
72. $f(x) = -2 + e^{3x}(4 - 2x)$

$f'(x) = e^{3x}(-2) + 3e^{3x}(4 - 2x) = e^{3x}(10 - 6x) = 0$ when $x = \frac{5}{3}$.

$f''(x) = e^{3x}(-6) + 3e^{3x}(10 - 6x) = e^{3x}(24 - 18x) = 0$ when $x = \frac{4}{3}$.

Relative maximum: $\left(\frac{5}{3}, 96.942\right)$

Point of inflection: $\left(\frac{4}{3}, 70.798\right)$



74. (a) $f(c) = f(c + x)$

$$10ce^{-c} = 10(c + x)e^{-(c+x)}$$

$$\frac{c}{e^c} = \frac{c + x}{e^{c+x}}$$

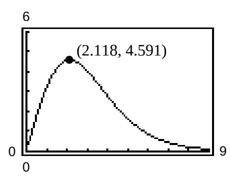
$$ce^{c+x} = (c + x)e^c$$

$$ce^x = c + x$$

$$ce^x - c = x$$

$$c = \frac{x}{e^x - 1}$$

(c) $A(x) = \dfrac{10x^2}{e^x - 1}e^{x/(1-e^x)}$

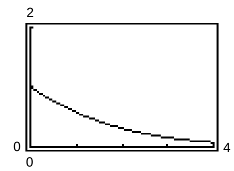


The maximum area is 4.591 for $x = 2.118$ and $f(x) = 2.547$.

(b) $A(x) = xf(c) = x\left[10\left(\dfrac{x}{e^x - 1}\right)e^{-(x/(e^x-1))}\right]$

$$= \frac{10x^2}{e^x - 1}e^{x/(1-e^x)}$$

(d) $c = \dfrac{x}{e^x - 1}$



$$\lim_{x \to 0^+} c = 1$$

$$\lim_{x \to \infty} c = 0$$

76. Let (x_0, y_0) be the desired point on $y = e^{-x}$.

$$y = e^{-x}$$

$$y' = -e^{-x} \quad \text{(Slope of tangent line)}$$

$$-\frac{1}{y'} = e^x \quad \text{(Slope of normal line)}$$

$$y - e^{-x_0} = e^{x_0}(x - x_0)$$

We want $(0, 0)$ to satisfy the equation:

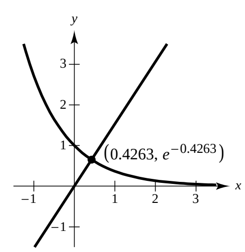
$$-e^{-x_0} = -x_0e^{x_0}$$

$$1 = x_0e^{2x_0}$$

$$x_0e^{2x_0} - 1 = 0$$

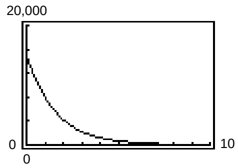
Solving by Newton's Method or using a computer, the solution is $x_0 \approx 0.4263$.

$(0.4263, e^{-0.4263})$



78. $V = 15{,}000e^{-0.6286t}$, $0 \le t \le 10$

(a)

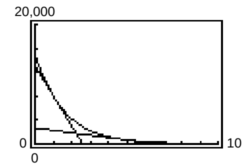


(b) $\dfrac{dV}{dt} = -9429e^{-0.6286t}$

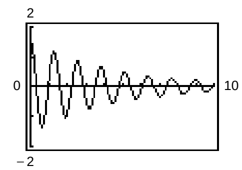
When $t = 1$, $\dfrac{dV}{dt} \approx -5028.84$.

When $t = 5$, $\dfrac{dV}{dt} \approx -406.89$.

(c)

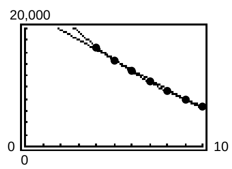


80. $1.56e^{-0.22t}\cos 4.9t \le 0.25$ (3 inches equals one-fourth foot.) Using a graphing utility or Newton's Method, we have $t \ge 7.79$ seconds.



82. (a) $V_1 = -1686.79t + 23{,}181.79$

$V_2 = 109.52t^2 - 3220.12t + 28{,}110.36$



(b) The slope represents the rate of decrease in value of the car.

(c) $V_3 = 31{,}450.77(0.8592)^t = 31{,}450.77e^{-0.1518t}$

(d) Horizontal asymptote: $\lim_{t\to\infty} V_3(t) = 0$

As $t \to \infty$, the value of the car approaches 0.

(e) $\dfrac{dV_3}{dt} = -4774.2e^{-0.1518t}$

For $t = 5$, $\dfrac{dV_3}{dt} \approx -2235$ dollars/year.

For $t = 9$, $\dfrac{dV_3}{dt} \approx -1218$ dollars/year.

84. $f(x) = e^{-x^2/2}$, $f(0) = 1$

$f'(x) = -xe^{-x^2/2}$, $f'(0) = 0$

$f''(x) = x^2e^{-x^2/2} - e^{-x^2/2} = e^{-x^2/2}(x^2 - 1)$, $f''(0) = -1$

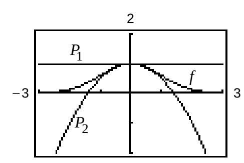
$P_1(x) = 1 + 0(x - 0) = 1$, $P_1(0) = 1$

$P_1'(x) = 0$, $P_1'(0) = 0$

$P_2(x) = 1 + 0(x - 0) - \dfrac{1}{2}(x - 0)^2 = 1 - \dfrac{x^2}{2}$, $P_2(0) = 1$

$P_2'(x) = -x$, $P_2'(0) = 0$

$P_2''(x) = -1$, $P_2''(0) = -1$



The values of f, P_1, P_2 and their first derivatives agree at $x = 0$. The values of the second derivatives of f and P_2 agree at $x = 0$.

86. n^{th} term is $x^n/n!$ in polynomial:

$$y_4 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

Conjecture: $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$

88. Let $u = -x^4$, $du = -4x^3\,dx$.

$$\int e^{-x^4}(-4x^3)\,dx = e^{-x^4} + C$$

90. $\displaystyle\int_3^4 e^{3-x}\,dx = \Big[-e^{3-x}\Big]_3^4 = -e^{-1} + 1 = 1 - \frac{1}{e}$

92. $\displaystyle\int x^2 e^{x^3/2}\,dx = \frac{2}{3}\int e^{x^3/2}\left(\frac{3x^2}{2}\right)dx = \frac{2}{3}e^{x^3/2} + C$

94. $\displaystyle\int \frac{e^{1/x^2}}{x^3}\,dx = -\frac{1}{2}\int e^{1/x^2}\left(\frac{-2}{x^3}\right)dx = -\frac{1}{2}e^{1/x^2} + C$

96. Let $u = 1 + e^{2x}$, $du = 2e^{2x}\,dx$.

$$\int \frac{e^{2x}}{1+e^{2x}}\,dx = \frac{1}{2}\int \frac{2e^{2x}}{1+e^{2x}}\,dx = \frac{1}{2}\ln(1+e^{2x}) + C$$

98. Let $u = \dfrac{-x^2}{2}$, $du = -x\,dx$.

$$\int_0^{\sqrt{2}} xe^{-x^2/2}\,dx = -\int_0^{\sqrt{2}} e^{-x^2/2}(-x)\,dx = \Big[-e^{-x^2/2}\Big]_0^{\sqrt{2}} = 1 - e^{-1} = \frac{e-1}{e}$$

100. Let $u = e^x + e^{-x}$, $du = (e^x - e^{-x})dx$.

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \ln(e^x + e^{-x}) + C$$

102. Let $u = e^x + e^{-x}$, $du = (e^x - e^{-x})\,dx$.

$$\int \frac{2e^x - 2e^{-x}}{(e^x + e^{-x})^2}\,dx = 2\int (e^x + e^{-x})^{-2}(e^x - e^{-x})\,dx$$

$$= \frac{-2}{e^x + e^{-x}} + C$$

104. $\displaystyle\int \frac{e^{2x} + 2e^x + 1}{e^x}\,dx = \int (e^x + 2 + e^{-x})\,dx$

$$= e^x + 2x - e^{-x} + C$$

106. $\displaystyle\int e^{\sec 2x}\sec 2x \tan 2x\,dx = \frac{1}{2}e^{\sec 2x} + C$

$(u = \sec 2x,\ du = 2\sec 2x \tan 2x)$

108. $\displaystyle\int \ln(e^{2x-1})\,dx = \int (2x-1)\,dx$

$$= x^2 - x + C$$

110. $\displaystyle y = \int (e^x - e^{-x})^2\,dx$

$$= \int (e^{2x} - 2 + e^{-2x})\,dx$$

$$= \frac{1}{2}e^{2x} - 2x - \frac{1}{2}e^{-2x} + C$$

112. $\displaystyle f'(x) = \int (\sin x + e^{2x})\,dx = -\cos x + \frac{1}{2}e^{2x} + C_1$

$$f'(0) = -1 + \frac{1}{2} + C_1 = \frac{1}{2} \implies C_1 = 1$$

$$f'(x) = -\cos x + \frac{1}{2}e^{2x} + 1$$

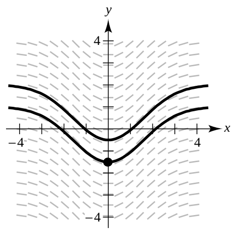
$$f(x) = \int \left(-\cos x + \frac{1}{2}e^{2x} + 1\right)dx$$

$$= -\sin x + \frac{1}{4}e^{2x} + x + C_2$$

$$f(0) = \frac{1}{4} + C_2 = \frac{1}{4} \implies C_2 = 0$$

$$f(x) = x - \sin x + \frac{1}{4}e^{2x}$$

114. (a)



(b) $\dfrac{dy}{dx} = xe^{-0.2x^2}, \quad \left(0, -\dfrac{3}{2}\right)$

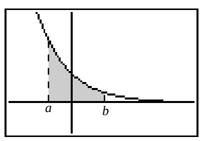
$$y = \int xe^{-0.2x^2}\,dx = \frac{1}{-0.4}\int e^{-0.2x^2}(-0.4x)\,dx$$

$$= -\frac{1}{0.4}e^{-0.2x^2} + C = -2.5e^{-0.2x^2} + C$$

$$\left(0, -\frac{3}{2}\right):\ -\frac{3}{2} = -2.5e^0 + C = -2.5 + C \implies C = 1$$

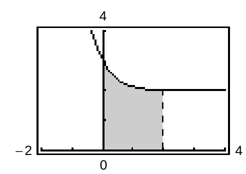
$$y = -2.5e^{-0.2x^2} + 1$$

116. $\displaystyle\int_a^b e^{-x}\,dx = \Big[-e^{-x}\Big]_a^b = e^{-a} - e^{-b}$



118. $\displaystyle\int_0^2 (e^{-2x} + 2)\,dx = \left[-\frac{1}{2}e^{-2x} + 2x\right]_0^2$

$$= -\frac{1}{2}e^{-4} + 4 + \frac{1}{2} \approx 4.491$$



120. (a) $\displaystyle\int_0^4 \sqrt{x}\,e^x\,dx,\ n = 12$

Midpoint Rule: 92.1898

Trapezoidal Rule: 93.8371

Simpson's Rule: 92.7385

Graphing Utility: 92.7437

(b) $\displaystyle\int_0^2 2xe^{-x}\,dx,\ n = 12$

Midpoint Rule: 1.1906

Trapezoidal Rule: 1.1827

Simpson's Rule: 1.1880

Graphing Utility: 1.18799

122. $\displaystyle\int_0^x 0.3^{-0.3t}\,dt = \frac{1}{2}$

$$\Big[-e^{-0.3t}\Big]_0^x = \frac{1}{2}$$

$$-e^{-0.3x} + 1 = \frac{1}{2}$$

$$e^{-0.3x} = \frac{1}{2}$$

$$-0.3x = \ln\frac{1}{2} = -\ln 2$$

$$x = \frac{\ln 2}{0.3} \approx 2.31 \text{ minutes}$$

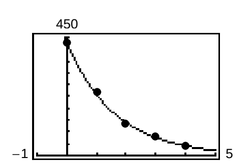
124.

t	0	1	2	3	4
R	425	240	118	71	36
$\ln R$	6.052	5.481	4.771	4.263	3.584

(a) $\ln R = -0.6155t + 6.0609$

$R = e^{-0.6155t + 6.0609} = 428.78e^{-0.6155t}$

(b)



(c) $\displaystyle\int_0^4 R(t)\,dt = \int_0^4 428.78e^{-0.6155t}\,dt \approx 637.2 \text{ liters}$

126. The graphs of $f(x) = \ln x$ and $g(x) = e^x$ are mirror images across the line $y = x$.

128. (a) Log Rule: $(u = e^x + 1)$

(b) Substitution: $(u = x^2)$

130. $\ln \dfrac{e^a}{e^b} = \ln e^a - \ln e^b = a - b$

$\ln e^{a-b} = a - b$

Therefore, $\ln \dfrac{e^a}{e^b} = \ln e^{a-b}$ and since $y = \ln x$ is one-to-one, we have $\dfrac{e^a}{e^b} = e^{a-b}$.

Section 5.5 Bases Other than *e* and Applications

2. $y = \left(\dfrac{1}{2}\right)^{t/8}$

At $t_0 = 16, y = \left(\dfrac{1}{2}\right)^{16/8} = \dfrac{1}{4}$

4. $y = \left(\dfrac{1}{2}\right)^{t/5}$

At $t_0 = 2, y = \left(\dfrac{1}{2}\right)^{2/5} \approx 0.7579$

6. $\log_{27} 9 = \log_{27} 27^{2/3} = \frac{2}{3}$

8. $\log_a \dfrac{1}{a} = \log_a 1 - \log_a a = -1$

10. (a) $27^{2/3} = 9$

$\log_{27} 9 = \dfrac{2}{3}$

(b) $16^{3/4} = 8$

$\log_{16} 8 = \dfrac{3}{4}$

12. (a) $\log_3 \frac{1}{9} = -2$

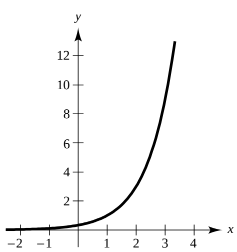
$3^{-2} = \frac{1}{9}$

(b) $49^{1/2} = 7$

$\log_{49} 7 = \frac{1}{2}$

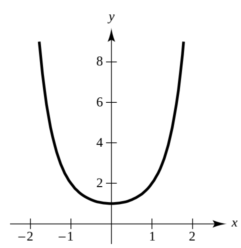
14. $y = 3^{x-1}$

x	-1	0	1	2	3
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9



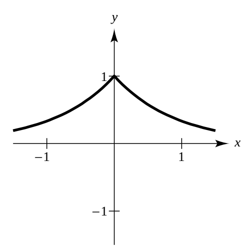
16. $y = 2^{x^2}$

x	-2	-1	0	1	2
y	16	2	1	2	16



18. $y = 3^{-|x|}$

x	0	± 1	± 2
y	1	$\frac{1}{3}$	$\frac{1}{9}$



20. (a) $\log_3 \dfrac{1}{81} = x$

$3^x = \dfrac{1}{81}$

$x = -4$

(b) $\log_6 36 = x$

$6^x = 36$

$x = 2$

22. (a) $\log_b 27 = 3$

$b^3 = 27$

$b = 3$

(b) $\log_b 125 = 3$

$b^3 = 125$

$b = 5$

24. (a) $\log_3 x + \log_3(x-2) = 1$

$$\log_3[x(x-2)] = 1$$
$$x(x-2) = 3^1$$
$$x^2 - 2x - 3 = 0$$
$$(x+1)(x-3) = 0$$
$$x = -1 \text{ OR } x = 3$$

$x = 3$ is the only solution since the domain of the logarithmic function is the set of all *positive* real numbers.

(b) $\log_{10}(x+3) - \log_{10} x = 1$

$$\log_{10}\frac{x+3}{x} = 1$$
$$\frac{x+3}{x} = 10^1$$
$$x + 3 = 10x$$
$$3 = 9x$$
$$x = \frac{1}{3}$$

26. $5^{6x} = 8320$

$$6x \ln 5 = \ln 8320$$
$$x = \frac{\ln 8320}{6 \ln 5} \approx 0.935$$

28. $3(5^{x-1}) = 86$

$$5^{x-1} = \frac{86}{3}$$
$$(x-1)\ln 5 = \ln\left(\frac{86}{3}\right)$$
$$x - 1 = \frac{\ln\left(\frac{86}{3}\right)}{\ln 5}$$
$$x = 1 + \frac{\ln\left(\frac{86}{3}\right)}{\ln 5} \approx 3.085$$

30. $\left(1 + \frac{0.10}{365}\right)^{365t} = 2$

$$365t \ln\left(1 + \frac{0.10}{365}\right) = \ln 2$$
$$t = \frac{1}{365}\frac{\ln 2}{\ln\left(1 + \frac{0.10}{365}\right)} \approx 6.932$$

32. $\log_{10}(t-3) = 2.6$

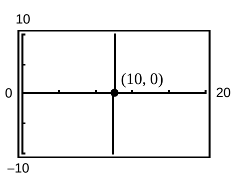
$$t - 3 = 10^{2.6}$$
$$t = 3 + 10^{2.6} \approx 401.107$$

34. $\log_5 \sqrt{x-4} = 3.2$

$$\sqrt{x-4} = 5^{3.2}$$
$$x - 4 = (5^{3.2})^2 = 5^{6.4}$$
$$x = 4 + 5^{6.4} \approx 29{,}748.593$$

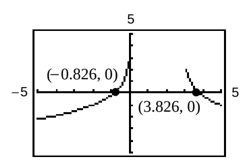
36. $f(t) = 300(1.0075^{12t}) - 735.41$

Zero: $t \approx 10$



38. $g(x) = 1 - 2\log_{10}[x(x-3)]$

Zeros: $x \approx -0.826, 3.826$

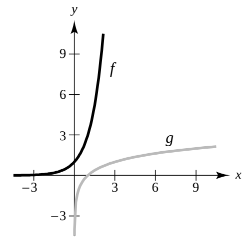


40. $f(x) = 3^x$

$g(x) = \log_3 x$

x	-2	-1	0	1	2
$f(x)$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$g(x)$	-2	-1	0	1	2



42. $g(x) = 2^{-x}$

$g'(x) = -(\ln 2)\, 2^{-x}$

44. $y = x(6^{-2x})$

$$\frac{dy}{dx} = x[-2(\ln 6)6^{-2x}) + 6^{-2x}$$
$$= 6^{-2x}[-2x(\ln 6) + 1]$$
$$= 6^{-2x}(1 - 2x \ln 6)$$

46. $f(t) = \dfrac{3^{2t}}{t}$

$$f'(t) = \frac{t(2 \ln 3)\, 3^{2t} - 3^{2t}}{t^2}$$
$$= \frac{3^{2t}\,(2t \ln 3 - 1)}{t^2}$$

48. $g(\alpha) = 5^{-\alpha/2} \sin 2\alpha$

$g'(\alpha) = 5^{-\alpha/2}\, 2 \cos 2\alpha - \frac{1}{2}(\ln 5)\, 5^{-\alpha/2} \sin 2\alpha$

50. $y = \log_{10} (2x) = \log_{10} 2 + \log_{10} x$

$$\frac{dy}{dx} = 0 + \frac{1}{x \ln 10} = \frac{1}{x \ln 10}$$

52. $h(x) = \log_3 \dfrac{x\sqrt{x - 1}}{2}$

$$= \log_3 x + \frac{1}{2} \log_3 (x - 1) - \log_3 2$$
$$h'(x) = \frac{1}{x \ln 3} + \frac{1}{2} \cdot \frac{1}{(x - 1) \ln 3} - 0$$
$$= \frac{1}{\ln 3}\left[\frac{1}{x} + \frac{1}{2(x - 1)}\right]$$
$$= \frac{1}{\ln 3}\left[\frac{3x - 2}{2x(x - 1)}\right]$$

54. $y = \log_{10} \dfrac{x^2 - 1}{x}$

$$= \log_{10}(x^2 - 1) - \log_{10} x$$
$$\frac{dy}{dx} = \frac{2x}{(x^2 - 1) \ln 10} - \frac{1}{x \ln 10}$$
$$= \frac{1}{\ln 10}\left[\frac{2x}{x^2 - 1} - \frac{1}{x}\right]$$
$$= \frac{1}{\ln 10}\left[\frac{x^2 + 1}{x(x^2 - 1)}\right]$$

56. $f(t) = t^{3/2} \log_2 \sqrt{t + 1} = t^{3/2}\, \dfrac{1}{2}\, \dfrac{\ln(t + 1)}{\ln 2}$

$$f'(t) = \frac{1}{2 \ln 2}\left[t^{3/2}\, \frac{1}{t + 1} + \frac{3}{2} t^{1/2} \ln(t + 1)\right]$$

58. $y = x^{x-1}$

$$\ln y = (x - 1)(\ln x)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = (x - 1)\left(\frac{1}{x}\right) + \ln x$$
$$\frac{dy}{dx} = y\left[\frac{x - 1}{x} + \ln x\right]$$
$$= x^{x-2}\,(x - 1 + x \ln x)$$

60. $y = (1 + x)^{1/x}$

$$\ln y = \frac{1}{x} \ln(1 + x)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x}\left(\frac{1}{1 + x}\right) + \ln(1 + x)\left(-\frac{1}{x^2}\right)$$
$$\frac{dy}{dx} = \frac{y}{x}\left[\frac{1}{x + 1} - \frac{\ln(x + 1)}{x}\right]$$
$$= \frac{(1 + x)^{1/x}}{x}\left[\frac{1}{x + 1} - \frac{\ln(x + 1)}{x}\right]$$

62. $\displaystyle\int 5^{-x}\, dx = \frac{-5^{-x}}{\ln 5} + C$

64. $\displaystyle\int_{-2}^{0} (3^3 - 5^2)\, dx = \int_{-2}^{0} (27 - 25)\, dx$

$$= \int_{-2}^{0} 2\, dx$$
$$= \Big[2x\Big]_{-2}^{0} = 4$$

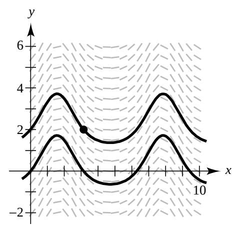
66. $\displaystyle\int (3-x)\,7^{(3-x)^2}\,dx = -\frac{1}{2}\int -2(3-x)\,7^{(3-x)^2}\,dx$

$$= -\frac{1}{2\ln 7}\left[7^{(3-x)^2}\right] + C$$

68. $\displaystyle\int 2^{\sin x}\cos x\,dx,\ u = \sin x,\ du = \cos x\,dx$

$$\frac{1}{\ln 2}2^{\sin x} + C$$

70. (a)



(b) $\dfrac{dy}{dx} = e^{\sin x}\cos x \qquad (\pi, 2)$

$$y = \int e^{\sin x}\cos x\,dx = e^{\sin x} + C$$

$(\pi, 2)$: $2 = e^{\sin \pi} + C = 1 + C \Rightarrow C = 1$

$$y = e^{\sin x} + 1$$

72. $\log_b x = \dfrac{\ln x}{\ln b} = \dfrac{\log_{10} x}{\log_{10} b}$

74. $f(x) = \log_{10} x$

(a) Domain: $x > 0$

(b) $y = \log_{10} x$

$10^y = x$

$f^{-1}(x) = 10^x$

(c) $\log_{10} 1000 = \log_{10} 10^3 = 3$

$\log_{10} 10{,}000 = \log_{10} 10^4 = 4$

If $1000 \le x \le 10{,}000$, then $3 \le f(x) \le 4$.

(d) If $f(x) < 0$, then $0 < x < 1$.

(e) $f(x) + 1 = \log_{10} x + \log_{10} 10$

$= \log_{10}(10x)$

x must have been increased by a factor of 10.

(f) $\log_{10}\left(\dfrac{x_1}{x_2}\right) = \log_{10} x_1 - \log_{10} x_2$

$= 3n - n = 2n$

Thus, $x_1/x_2 = 10^{2n} = 100^n$.

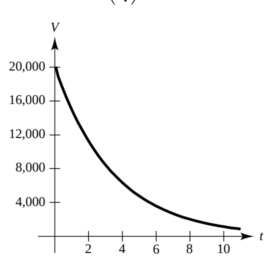
76. $f(x) = a^x$

(a) $f(u+v) = a^{u+v} = a^u a^v = f(u)\,f(v)$

(b) $f(2x) = a^{2x} = (a^x)^2 = [f(x)]^2$

78. $V(t) = 20{,}000\left(\dfrac{3}{4}\right)^t$

(a)



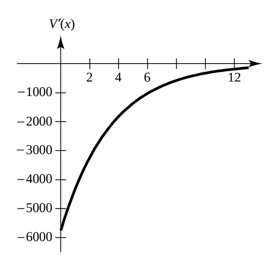
$V(2) = 20{,}000\left(\dfrac{3}{4}\right)^2 = \$11{,}250$

(b) $\dfrac{dV}{dt} = 20{,}000\left(\ln\dfrac{3}{4}\right)\left(\dfrac{3}{4}\right)^t$

When $t = 1$: $\dfrac{dV}{dt} \approx -4315.23$

When $t = 4$: $\dfrac{dV}{dt} \approx -1820.49$

(c)



Horizontal asymptote: $v' = 0$

As the car ages, it is worth less each year and depreciates less each year, but the value of the car will never reach \$0.

80. $P = \$2500,\ r = 6\% = 0.06,\ t = 20$

$$A = 2500\left(1 + \frac{0.06}{n}\right)^{20n}$$

$$A = 2500e^{(0.06)(20)} = 8300.29$$

n	1	2	4	12	365	Continuous
A	8017.84	8155.09	8226.66	8275.51	8299.47	8300.29

82. $P = \$5000, r = 7\% = 0.07, t = 25$

$$A = 5000\left(1 + \frac{0.07}{n}\right)^{25n}$$

$$A = 5000e^{0.07(25)}$$

n	1	2	4	12	365	Continuous
A	27,137.16	27,924.63	28,340.78	28,627.09	28,768.19	28,773.01

84. $100{,}000 = Pe^{0.06t} \implies P = 100{,}000e^{-0.06t}$

t	1	10	20	30	40	50
P	94,176.45	54,881.16	30,119.42	16,529.89	9071.80	4978.71

86. $100{,}000 = P\left(1 + \frac{0.07}{365}\right)^{365t} \implies P = 100{,}000\left(1 + \frac{0.07}{365}\right)^{-365t}$

t	1	10	20	30	40	50
P	93,240.01	49,661.86	24,663.01	12,248.11	6082.64	3020.75

88. Let $P = \$100,\ 0 \le t \le 20.$

(a) $A = 100e^{0.03t}$

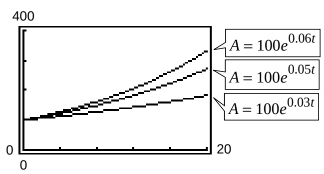
$A(20) \approx 182.21$

(b) $A = 100e^{0.05t}$

$A(20) \approx 271.83$

(c) $A = 100e^{0.06t}$

$A(20) \approx 332.01$



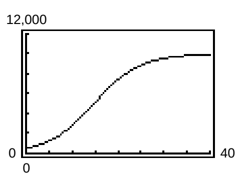
90. (a) $\lim_{n\to\infty} \frac{0.86}{1 + e^{-0.25n}} = 0.86$ or 86%

(b) $$P' = \frac{-0.86(-0.25)(e^{-0.25n})}{(1 + e^{-0.25n})} = \frac{0.215e^{-0.25n}}{(1 + e^{-0.25n})}$$

$P'(3) \approx 0.069$

$P'(10) \approx 0.016$

92. (a)



(b) Limiting size: 10,000 fish

(c) $$p(t) = \frac{10{,}000}{1 + 19e^{-t/5}}$$

$$p'(t) = \frac{e^{-t/5}}{(1 + 19e^{t/5})^2}\left(\frac{19}{5}\right)(10{,}000)$$

$$= \frac{38{,}000e^{-t/5}}{(1 + 19e^{-t/5})^2}$$

$p'(1) \approx 113.5$ fish/month

$p'(10) \approx 403.2$ fish/month

(d) $$p''(t) = -\frac{38{,}000}{5}(e^{-t/5})\left[\frac{1 - 19e^{-t/5}}{(1 + 19e^{-t/5})^3}\right] = 0$$

$$19e^{-t/5} = 1$$

$$\frac{t}{5} = \ln 19$$

$$t = 5 \ln 19 \approx 14.72$$

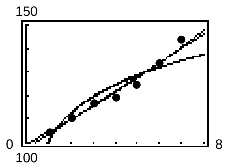
94. (a) $y_1 = 6.0536x + 97.5571$

$y_2 = 100.0751 + 17.8148 \ln x$

$y_3 = 99.4557(1.0506)^x$

$y_4 = 101.2875x^{0.1471}$

(b)



y_3 seems best.

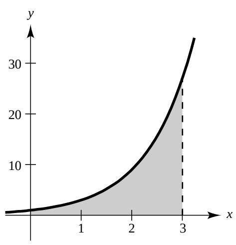
(c) The slope of 6.0536 is the annual rate of change in the amount given to philanthropy.

(d) For 1996, $x = 6$ and $y_1' = 6.0536$, $y_2' \approx 2.9691$,

$y_3' \approx 6.6015$, $y_4' \approx 3.2321$.

y_3 is increasing at the greatest rate in 1996.

96. $A = \int_0^3 3^x\,dx = \left[\frac{3^x}{\ln 3}\right]_0^3 = \frac{26}{\ln 3} \approx 23.666$



98.

x	1	10^{-1}	10^{-2}	10^{-4}	10^{-6}
$(1 + x)^{1/x}$	2	2.594	2.705	2.718	2.718

100.

t	0	1	2	3	4
y	600	630	661.50	694.58	729.30

$y = C(k^t)$

When $t = 0$, $y = 600 \implies C = 600$.

$y = 600(k^t)$

$\frac{630}{600} = 1.05$, $\frac{661.50}{630} = 1.05$, $\frac{694.58}{661.50} \approx 1.05$, $\frac{729.30}{694.58} \approx 1.05$

Let $k = 1.05$.

$y = 600(1.05)^t$

102. True.

$$f(e^{n+1}) - f(e^n) = \ln e^{n+1} - \ln e^n = n + 1 - n = 1$$

104. True.

$$\frac{d^n y}{dx^n} = Ce^x = y \text{ for } n = 1, 2, 3, \ldots$$

106. True.

$f(x) = g(x)e^x = 0 \implies$

$g(x) = 0$ since $e^x > 0$ for all x.

108. $y = x^{\sin x}$

$$\ln y = \ln x^{\sin x} = \sin x \cdot \ln x$$

$$\frac{y'}{y} = \sin x\left(\frac{1}{x}\right) + \cos x \cdot \ln x$$

$$y' = x^{\sin x}\left[\frac{\sin x}{x} + \cos x \cdot \ln x\right]$$

At $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$,

$$y' = \left(\frac{\pi}{2}\right)^{\sin(\pi/2)}\left[\frac{\sin(\pi/2)}{\pi/2} + \cos\left(\frac{\pi}{2}\right)\ln\left(\frac{\pi}{2}\right)\right]$$

$$= \frac{\pi}{2}\left[\frac{2}{\pi} + 0\right] = 1$$

Tangent line: $y - \frac{\pi}{2} = 1\left(x - \frac{\pi}{2}\right)$

$$y = x$$

Section 5.6 Differential Equations: Growth and Decay

2. $\frac{dy}{dx} = 4 - x$

$$y = \int (4 - x)\,dx = 4x - \frac{x^2}{2} + C$$

4. $\frac{dy}{dx} = 4 - y$

$$\frac{dy}{4 - y} = dx$$

$$\int \frac{-1}{4 - y}\,dy = \int -dx$$

$$\ln|4 - y|\,dy = -x + C_1$$

$$4 - y = e^{-x + C_1} = Ce^{-x}$$

$$y = 4 - Ce^{-x}$$

6. $y' = \frac{\sqrt{x}}{3y}$

$$3yy' = \sqrt{x}$$

$$\int 3yy'\,dx = \int \sqrt{x}\,dx$$

$$\frac{3y^2}{2} = \frac{2}{3}x^{3/2} + C_1$$

$$9y^2 - 4x^{3/2} = C$$

8. $y' = x(1 + y)$

$$\frac{y'}{1 + y} = x$$

$$\int \frac{y'}{1 + y}\,dx = \int x\,dx$$

$$\int \frac{dy}{1 + y} = \int x\,dx$$

$$\ln(1 + y) = \frac{x^2}{2} + C_1$$

$$1 + y = e^{(x^2/2) + C_1}$$

$$y = e^{C_1}e^{x^2/2} - 1$$

$$= Ce^{x^2/2} - 1$$

10. $xy + y' = 100x$

$$y' = 100x + xy = x(100 - y)$$

$$\frac{y'}{100 - y} = x$$

$$\int \frac{y'}{100 - y}\,dx = \int x\,dx$$

$$\int \frac{1}{100 - y}\,dy = \int x\,dx$$

$$-\ln(100 - y) = \frac{x^2}{2} + C_1$$

$$\ln(100 - y) = -\frac{x^2}{2} - C_1$$

$$100 - y = e^{-(x^2/2) - C_1}$$

$$-y = e^{-C_1}e^{-x^2/2} - 100$$

$$y = 100 - Ce^{-x^2/2}$$

12. $\frac{dP}{dt} = k(10 - t)$

$$\int \frac{dP}{dt}\,dt = \int k(10 - t)\,dt$$

$$\int dP = -\frac{k}{2}(10 - t)^2 + C$$

$$P = -\frac{k}{2}(10 - t)^2 + C$$

14. $\frac{dy}{dx} = kx(L - y)$

$$\frac{1}{L - y}\frac{dy}{dx} = kx$$

$$\int \frac{1}{L - y}\frac{dy}{dx}\,dx = \int kx\,dx$$

$$\int \frac{1}{L - y}\,dy = \frac{kx^2}{2} + C_1$$

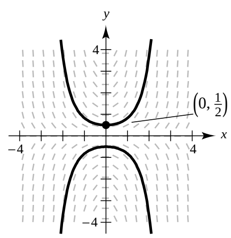
$$-\ln(L - y) = \frac{kx^2}{2} + C_1$$

$$L - y = e^{-(kx^2/2) - C_1}$$

$$-y = -L + e^{-C_1}e^{-kx^2/2}$$

$$y = L - Ce^{-kx^2/2}$$

16. (a)



(b) $\frac{dy}{dx} = xy,\ \left(0, \frac{1}{2}\right)$

$$\frac{dy}{y} = x\,dx$$

$$\ln|y| = \frac{x^2}{2} + C$$

$$y = e^{x^2/2 + C} = C_1e^{x^2/2}$$

$$\left(0, \frac{1}{2}\right):\ \frac{1}{2} = C_1e^0 \implies C_1 = \frac{1}{2} \implies y = \frac{1}{2}e^{x^2/2}$$

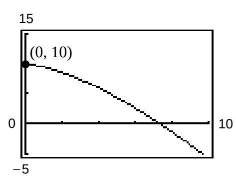
18. $\frac{dy}{dt} = -\frac{3}{4}\sqrt{t},\quad (0, 10)$

$$\int dy = \int -\frac{3}{4}\sqrt{t}\,dt$$

$$y = -\frac{1}{2}t^{3/2} + C$$

$$10 = -\frac{1}{2}(0)^{3/2} + C \implies C = 10$$

$$y = -\frac{1}{2}t^{3/2} + 10$$



20. $\frac{dy}{dt} = \frac{3}{4}y,\ (0, 10)$

$$\int \frac{dy}{y} = \int \frac{3}{4}\,dt$$

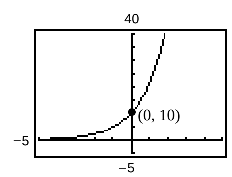
$$\ln y = \frac{3}{4}t + C_1$$

$$y = e^{(3/4)t + C_1}$$

$$= e^{C_1}e^{(3/4)t} = Ce^{3t/4}$$

$$10 = Ce^0 \implies C = 10$$

$$y = 10e^{3t/4}$$



22. $\frac{dN}{dt} = kN$

$N = Ce^{kt}$ (Theorem 5.16)

$(0, 250)$: $C = 250$

$(1, 400)$: $400 = 250e^k \implies k = \ln\frac{400}{250} = \ln\frac{8}{5}$

When $t = 4$, $N = 250e^{4\ln(8/5)} = 250e^{\ln(8/5)^4}$

$$= 250\left(\frac{8}{5}\right)^4 = \frac{8192}{5}$$

24. $\frac{dP}{dt} = kP$

$P = Ce^{kt}$ (Theorem 5.16)

$(0, 5000)$: $C = 5000$

$(1, 4750)$: $4750 = 5000e^k \implies k = \ln\left(\frac{19}{20}\right)$

When $t = 5$, $P = 5000e^{\ln(19/20)(5)}$

$$= 5000\left(\frac{19}{20}\right)^5 \approx 3868.905$$

26. $y = Ce^{kt},\ (0, 4),\ \left(5, \frac{1}{2}\right)$

$$C = 4$$
$$y = 4e^{kt}$$
$$\frac{1}{2} = 4e^{5k}$$
$$k = \frac{\ln(1/8)}{5} \approx -0.4159$$
$$y = 4e^{-0.4159t}$$

28. $y = Ce^{kt},\ \left(3, \frac{1}{2}\right),\ (4, 5)$

$$\frac{1}{2} = Ce^{3k}$$
$$5 = Ce^{4k}$$
$$2Ce^{3k} = \frac{1}{5}Ce^{4k}$$
$$10e^{3k} = e^{4k}$$
$$10 = e^{k}$$
$$k = \ln 10 \approx 2.3026$$
$$y = Ce^{2.3026t}$$
$$5 = Ce^{2.3026(4)}$$
$$C \approx 0.0005$$
$$y = 0.0005e^{2.3026t}$$

30. $y' = \frac{dy}{dt} = ky$

32. $\frac{dy}{dx} = \frac{1}{2}x^2y$

$\frac{dy}{dx} > 0$ when $y > 0$. Quadrants I and II.

34. Since $y = Ce^{[\ln(1/2)/1620]t}$, we have $1.5 = Ce^{[\ln(1/2)/1620](1000)} \implies C \approx 2.30$ which implies that the initial quantity is 2.30 grams. When $t = 10{,}000$, we have $y = 2.30e^{[\ln(1/2)/1620](10{,}000)} \approx 0.03$ gram.

36. Since $y = Ce^{[\ln(1/2)/5730]t}$, we have $2.0 = Ce^{[\ln(1/2)/5730](10{,}000)} \implies C \approx 6.70$ which implies that the initial quantity is 6.70 grams. When $t = 1000$, we have $y = 6.70e^{[\ln(1/2)/5730](1000)} \approx 5.94$ grams.

38. Since $y = Ce^{[\ln(1/2)/5730]t}$, we have $3.2 = Ce^{[\ln(1/2)/5730]1000} \implies C \approx 3.61$.

Initial quantity: 3.61 grams.

When $t = 10{,}000$, we have $y \approx 1.08$ grams.

40. Since $y = Ce^{[\ln(1/2)/24{,}360]t}$, we have $0.4 = Ce^{[\ln(1/2)/24{,}360](10{,}000)} \implies C \approx 0.53$ which implies that the initial quantity is 0.53 gram. When $t = 1000$, we have $y = 0.53e^{[\ln(1/2)/24{,}360](1000)} \approx 0.52$ gram.

42. Since $\frac{dy}{dx} = ky,\ y = Ce^{kt}$ or $y = y_0e^{kt}$.

$$\frac{1}{2}y_0 = y_0e^{5730k}$$
$$k = -\frac{\ln 2}{5730}$$
$$0.15y_0 = y_0e^{(-\ln 2/5730)t}$$
$$\ln 0.15 = -\frac{(\ln 2)t}{5730}$$
$$t = -\frac{5730 \ln 0.15}{\ln 2} \approx 15{,}682.8 \text{ years.}$$

44. Since $A = 20{,}000e^{0.055t}$, the time to double is given by $40{,}000 = 20{,}000e^{0.055t}$ and we have

$$2 = e^{0.055t}$$
$$\ln 2 = 0.055t$$
$$t = \frac{\ln 2}{0.055} \approx 12.6 \text{ years.}$$

Amount after 10 years:
$A = 20{,}000e^{(0.055)(10)} \approx \$34{,}665.06$

46. Since $A = 10{,}000e^{rt}$ and $A = 20{,}000$ when $t = 5$, we have the following.

$$20{,}000 = 10{,}000e^{5r}$$
$$r = \frac{\ln 2}{5} \approx 0.1386 = 13.86\%$$

Amount after 10 years: $A = 10{,}000e^{[(\ln 2)/5](10)} = \$40{,}000$

48. Since $A = 2000e^{rt}$ and $A = 5436.56$ when $t = 10$, we have the following.

$$5436.56 = 2000e^{10r}$$
$$r = \frac{\ln(5436.56/2000)}{10} \approx 0.10 = 10\%$$

The time to double is given by

$$4000 = 2000e^{0.10t}$$
$$t = \frac{\ln 2}{0.10} \approx 6.93 \text{ years.}$$

50. $500{,}000 = P\left(1 + \frac{0.06}{12}\right)^{(12)(40)}$

$$P = 500{,}000(1.005)^{-480} \approx \$45{,}631.04$$

52. $500{,}000 = P\left(1 + \frac{0.09}{12}\right)^{(12)(25)}$

$$P = 500{,}000\left(1 + \frac{0.09}{12}\right)^{-300}$$
$$\approx \$53{,}143.92$$

54. (a) $2000 = 1000(1 + 0.6)^t$

$$2 = 1.06^t$$
$$\ln 2 = t \ln 1.06$$
$$t = \frac{\ln 2}{\ln 1.06} \approx 11.90 \text{ years}$$

(b) $2000 = 1000\left(1 + \frac{0.06}{12}\right)^{12t}$

$$2 = \left(1 + \frac{0.06}{12}\right)^{12t}$$
$$\ln 2 = 12t \ln\left(1 + \frac{0.06}{12}\right)$$
$$t = \frac{1}{12}\frac{\ln 2}{\ln\left(1 + \frac{0.06}{12}\right)} \approx 11.58 \text{ years}$$

(c) $2000 = 1000\left(1 + \frac{0.06}{365}\right)^{365t}$

$$2 = \left(1 + \frac{0.06}{365}\right)^{365t}$$
$$\ln 2 = 365t \ln\left(1 + \frac{0.06}{365}\right)$$
$$t = \frac{1}{365}\frac{\ln 2}{\ln\left(1 + \frac{0.06}{365}\right)} \approx 11.55 \text{ years}$$

(d) $2000 = 1000e^{0.06t}$

$$2 = e^{0.06t}$$
$$\ln 2 = 0.06t$$
$$t = \frac{\ln 2}{0.06} \approx 11.55 \text{ years}$$

56. (a) $2000 = 1000(1 + 0.055)^t$

$$2 = 1.055^t$$
$$\ln 2 = t \ln 1.055$$
$$t = \frac{\ln 2}{\ln 1.055} \approx 12.95 \text{ years}$$

(b) $2000 = 1000\left(1 + \frac{0.055}{12}\right)^{12t}$

$$2 = \left(1 + \frac{0.055}{12}\right)^{12t}$$
$$\ln 2 = 12t \ln\left(1 + \frac{0.055}{12}\right)$$
$$t = \frac{1}{12}\frac{\ln 2}{\ln\left(1 + \frac{0.055}{12}\right)} \approx 12.63 \text{ years}$$

(c) $2000 = 1000\left(1 + \frac{0.055}{365}\right)^{365t}$

$$2 = \left(1 + \frac{0.055}{365}\right)^{365t}$$
$$\ln 2 = 365t \ln\left(1 + \frac{0.055}{365}\right)$$
$$t = \frac{1}{365}\frac{\ln 2}{\ln\left(1 + \frac{0.055}{365}\right)} \approx 12.60 \text{ years}$$

(d) $2000 = 1000e^{0.055t}$

$$2 = e^{0.055t}$$
$$\ln 2 = 0.055t$$
$$t = \frac{\ln 2}{0.055} \approx 12.60 \text{ years}$$

58. $P = Ce^{kt} = Ce^{0.031t}$

$P(-1) = 11.6 = Ce^{0.031(-1)} \implies C = 11.9652$

$P = 11.9652e^{0.031t}$

$P(10) \approx 16.31$ or 16,310,000 people in 2010

60. $P = Ce^{kt} = Ce^{-0.004t}$

$P(-1) = 3.6 = Ce^{-0.004(-1)} \implies C = 3.5856$

$P = 3.5856e^{-0.004t}$

$P(10) \approx 3.45$ or 3,450,000 people in 2010

62. (a) $N = 100.1596(1.2455)^t$

(b) $N = 400$ when $t = 6.3$ hours (graphing utility)

Analytically,

$$400 = 100.1596(1.2455)^t$$

$$1.2455^t = \frac{400}{100.1596} = 3.9936$$

$$t \ln 1.2455 = \ln 3.9936$$

$$t = \frac{\ln 3.9936}{\ln 1.2455} \approx 6.3 \text{ hours.}$$

64. $y = Ce^{kt}$, $(0, 742{,}000)$, $(2, 632{,}000)$

$$C = 742{,}000$$

$$632{,}000 = 742{,}000e^{2k}$$

$$k = \frac{\ln(632/742)}{2} \approx -0.0802$$

$$y \approx 742{,}000e^{-0.0802t}$$

When $t = 4$, $y \approx \$538{,}372$.

66. (a) $20 = 30(1 - e^{30k})$

$$30e^{30k} = 10$$

$$k = \frac{\ln(1/3)}{30} = \frac{-\ln 3}{30} \approx -0.0366$$

$$N \approx 30(1 - e^{-0.0366t})$$

(b) $25 = 30(1 - e^{-0.0366t})$

$$e^{-0.0366t} = \frac{1}{6}$$

$$t = \frac{-\ln 6}{-0.0366} \approx 49 \text{ days}$$

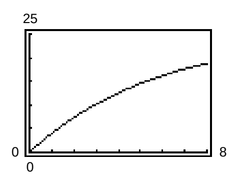
68. $S = 25(1 - e^{kt})$

(a) $4 = 25(1 - e^{k(1)}) \implies 1 - e^k = \dfrac{4}{25} \implies e^k = \dfrac{21}{25} \implies k = \ln\left(\dfrac{21}{25}\right) \approx -0.1744$

(b) 25,000 units $\left(\lim_{t\to\infty} S = 25\right)$

(c) When $t = 5$, $S \approx 14.545$ which is 14,545 units.

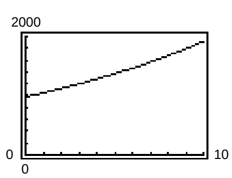
(d)



70. (a) $R = 979.3993(1.0694)^t = 979.3993e^{0.0671t}$

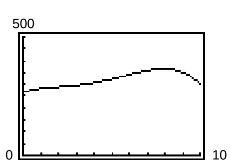
$I = -0.1385t^4 + 2.1770t^3 - 9.9755t^2 + 23.8513t + 266.4923$

(b)

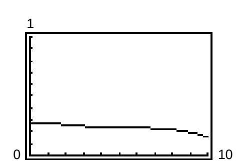


Rate of growth $= R'(t) = 65.7e^{0.0671t}$

(c)



(d) $P(t) = \dfrac{I}{R}$



72. $93 = 10 \log_{10} \dfrac{I}{10^{-16}} = 10(\log_{10} I + 16)$

$-6.7 = \log_{10} I \implies I = 10^{-6.7}$

$80 = 10 \log_{10} \dfrac{I}{10^{-16}} = 10(\log_{10} I + 16)$

$-8 = \log_{10} I \implies I = 10^{-8}$

Percentage decrease: $\left(\dfrac{10^{-6.7} - 10^{-8}}{10^{-6.7}}\right)(100) \approx 95\%$

74. Since $\dfrac{dy}{dt} = k(y - 80)$

$$\int \frac{1}{y - 80}\, dy = \int k\, dt$$

$\ln(y - 80) = kt + C.$

When $t = 0$, $y = 1500$. Thus, $C = \ln 1420$.

When $t = 1$, $y = 1120$. Thus,

$k(1) + \ln 1420 = \ln(1120 - 80)$

$$k = \ln 1040 - \ln 1420 = \ln \frac{104}{142}.$$

Thus, $y = 1420e^{[\ln(104/142)]t} + 80$.

When $t = 5$, $y \approx 379.2°$.

76. True

78. True

Section 5.7 Differential Equations: Separation of Variables

2. Differential equation: $y' = \dfrac{2xy}{x^2 - y^2}$

Solution: $x^2 + y^2 = Cy$

Check: $2x + 2yy' = Cy'$

$$y' = \frac{-2x}{(2y - C)}$$

$$y' = \frac{-2xy}{2y^2 - Cy} = \frac{-2xy}{2y^2 - (x^2 + y^2)} = \frac{-2xy}{y^2 - x^2} = \frac{2xy}{x^2 - y^2}$$

4. Differential equation: $y'' + 2y' + 2y = 0$

Solution: $y = C_1e^{-x}\cos x + C_2e^{-x}\sin x$

Check:

$$y' = -(C_1 + C_2)e^{-x}\sin x + (-C_1 + C_2)e^{-x}\cos x$$

$$y'' = 2\,C_1e^{-x}\sin x - 2C_2e^{-x}\cos x$$

$$y'' + 2y' + 2y = 2C_1e^{-x}\sin x - 2C_2e^{-x}\cos x +$$

$$2(-(C_1 + C_2)e^{-x}\sin x + (-C_1 + C_2)e^{-x}\cos x) + 2(C_1e^{-x}\cos x + C_2e^{-x}\sin x)$$

$$= (2C_1 - 2C_1 - 2C_2 + 2C_2)e^{-x}\sin x + (-2C_2 - 2C_1 + 2C_2 + 2C_1)e^{-x}\cos x = 0$$

6. $y = \frac{2}{3}(e^{-2x} + e^x)$

$y' = \frac{2}{3}(-2e^{-2x} + e^x)$

$y'' = \frac{2}{3}(4e^{-2x} + e^x)$

Substituting, $y'' + 2y' = \frac{2}{3}(4e^{-2x} + e^x) + 2\left(\frac{2}{3}\right)(-2e^{-2x} + e^x) = 2e^x$.

In Exercises 8–12, the differential equation is $y^{(4)} - 16y = 0$.

8. $y = 3\cos 2x$

$y^{(4)} = 48\cos 2x$

$y^{(4)} - 16y = 48\cos 2x - 48\cos 2x = 0,$ Yes.

10. $y = 5\ln x$

$y^{(4)} = -\dfrac{30}{x^4}$

$y^{(4)} - 16y = -\dfrac{30}{x^4} - 80\ln x \neq 0,$ No.

12. $y = 3e^{2x} - 4\sin 2x$

$y^{(4)} = 48e^{2x} - 64\sin 2x$

$y^{(4)} - 16y = (24e^{2x} - 64\sin 2x) - 16(3e^{2x} - 4\sin 2x) = 0,$ Yes

In 14–18, the differential equation is $xy' - 2y = x^3e^x$.

14. $y = x^2e^x,\ y' = x^2e^x + 2xe^x = e^x(x^2 + 2x)$

$xy' - 2y = x(e^x(x^2 + 2x)) - 2(x^2e^x) = x^3e^x,$ Yes.

16. $y = \sin x, y' = \cos x$

$xy' - 2y = x(\cos x) - 2(\sin x) \neq x^3e^x,$ No.

18. $y = x^2e^x - 5x^2, y' = x^2e^x + 2xe^x - 10x$

$xy' - 2y = x[x^2e^x + 2xe^x - 10x] - 2[x^2e^x - 5x^2] = x^3e^x,$ Yes.

20. $y = A\sin \omega t$

$\dfrac{d^2y}{dt^2} = -A\omega^2 \sin \omega t$

Since $(d^2y/dt^2) + 16y = 0$, we have

$-A\omega^2 \sin \omega t + 16A\sin \omega t = 0.$

Thus, $\omega^2 = 16$ and $\omega = \pm 4$.

22. $2x^2 - y^2 = C$ passes through $(3, 4)$

$2(9) - 16 = C \implies C = 2$

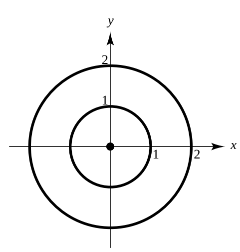
Particular solution: $2x^2 - y^2 = 2$

24. Differential equation: $yy' + x = 0$

General solution: $x^2 + y^2 = C$

Particular solutions: $C = 0$, Point

$C = 1, C = 4$, Circles



26. Differential equation: $3x + 2yy' = 0$

General solution: $3x^2 + 2y^2 = C$

$6x + 4yy' = 0$

$2(3x + 2yy') = 0$

$3x + 2yy' = 0$

Initial condition:
$y(1) = 3$: $3(1)^2 + 2(3)^2 = 3 + 18 = 21 = C$

Particular solution: $3x^2 + 2y^2 = 21$

28. Differential equation: $xy'' + y' = 0$

General solution: $y = C_1 + C_2 \ln x$

$y' = C_2\left(\frac{1}{x}\right), y'' = -C_2\left(\frac{1}{x^2}\right)$

$xy'' + y' = x\left(-C_2\frac{1}{x^2}\right) + C_2\frac{1}{x} = 0$

Initial conditions: $y(2) = 0,\ y'(2) = \frac{1}{2}$

$0 = C_1 + C_2 \ln 2$

$y' = \frac{C_2}{x}$

$\frac{1}{2} = \frac{C_2}{2} \implies C_2 = 1,\ C_1 = -\ln 2$

Particular solution: $y = -\ln 2 + \ln x = \ln\frac{x}{2}$

30. Differential equation: $9y'' - 12y' + 4y = 0$

General solution: $y = e^{2x/3}(C_1 + C_2 x)$

$y' = \frac{2}{3}e^{2x/3}(C_1 + C_2x) + C_2e^{2x/3} = e^{2x/3}\left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2x\right)$

$y'' = \frac{2}{3}e^{2x/3}\left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2x\right) + e^{2x/3}\frac{2}{3}C_2 = \frac{2}{3}e^{2x/3}\left(\frac{2}{3}C_1 + 2C_2 + \frac{2}{3}C_2x\right)$

$9y'' - 12y' + 4y = 9\left(\frac{2}{3}e^{2x/3}\right)\left(\frac{2}{3}C_1 + 2C_2 + \frac{2}{3}C_2x\right) - 12(e^{2x/3})\left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2x\right) + 4(e^{2x/3})(C_1 + C_2x) = 0$

Initial conditions: $y(0) = 4,\ y(3) = 0$

$0 = e^2(C_1 + 3C_2)$

$4 = (1)(C_1 + 0) \implies C_1 = 4$

$0 = e^2(4 + 3C_2) \implies C_2 = -\frac{4}{3}$

Particular solution: $y = e^{2x/3}\left(4 - \frac{4}{3}x\right)$

32. $\frac{dy}{dx} = x^3 - 4x$

$y = \int (x^3 - 4x)\,dx = \frac{x^4}{4} - 2x^2 + C$

34. $\frac{dy}{dx} = \frac{e^x}{1 + e^x}$

$y = \int \frac{e^x}{1 + e^x}\,dx = \ln(1 + e^x) + C$

36. $\frac{dy}{dx} = x \cos x^2$

$y = \int x\cos(x^2)\,dx = \frac{1}{2}\sin(x^2) + C$

$(u = x^2,\ du = 2x\,dx)$

38. $\frac{dy}{dx} = \tan^2 x = \sec^2 x - 1$

$y = \int (\sec^2 x - 1)\,dx = \tan x - x + C$

40. $\frac{dy}{dx} = x\sqrt{5 - x}$. Let $u = \sqrt{5 - x}$, $u^2 = 5 - x$, $dx = -2u\,du$

$$\begin{aligned} y = \int x\sqrt{5 - x}\,dx &= \int (5 - u^2)u(-2u)\,du \\ &= \int (-10u^2 + 2u^4)\,du \\ &= \frac{-10u^3}{3} + \frac{2u^5}{5} + C \\ &= -\frac{10}{3}(5 - x)^{3/2} + \frac{2}{5}(5 - x)^{5/2} + C \end{aligned}$$

42. $\dfrac{dy}{dx} = 5e^{-x/2}$

$$y = \int 5e^{-x/2}\,dx = 5(-2)\int e^{-x/2}\left(-\frac{1}{2}\right)dx$$
$$= -10e^{-x/2} + C$$

44. $\dfrac{dy}{dx} = \dfrac{x^2 + 2}{3y^2}$

$$\int 3y^2\,dy = \int (x^2 + 2)dx$$
$$y^3 = \frac{x^3}{3} + 2x + C$$

46. $\dfrac{dr}{ds} = 0.05s$

$$\int dr = \int 0.05s\,ds$$
$$r = 0.025s^2 + C$$

48. $xy' = y$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$
$$\ln y = \ln x + \ln C = \ln Cx$$
$$y = Cx$$

50. $y\dfrac{dy}{dx} = 6\cos \pi x$

$$\int y\,dy = \int 6\cos \pi x\,dx$$
$$\frac{y^2}{2} = \frac{6}{\pi}\sin \pi x + C_1$$
$$y^2 = \frac{12}{\pi}\sin \pi x + C$$

52. $\sqrt{x^2 - 9}\dfrac{dy}{dx} = 5x$

$$\int dy = \int \frac{5x}{\sqrt{x^2 - 9}}\,dx$$
$$y = 5(x^2 - 9)^{1/2} + C$$

54. $4y\dfrac{dy}{dx} = 3e^x$

$$\int 4y\,dy = \int 3e^x\,dx$$
$$2y^2 = 3e^x + C$$

56. $\sqrt{x} + \sqrt{y}\,y' = 0$

$$\int y^{1/2}\,dy = -\int x^{1/2}\,dx$$
$$\frac{2}{3}y^{3/2} = -\frac{2}{3}x^{3/2} + C_1$$
$$y^{3/2} + x^{3/2} = C$$

Initial condition: $y(1) = 4$,
$(4)^{3/2} + (1)^{3/2} = 8 + 1 = 9 = C$

Particular solution: $y^{3/2} + x^{3/2} = 9$

58. $2xy' - \ln x^2 = 0$

$$2x\frac{dy}{dx} = 2\ln x$$
$$\int dy = \int \frac{\ln x}{x}\,dx$$
$$y = \frac{(\ln x)^2}{2} + C$$

$y(1) = 2$: $2 = C$

$$y = \frac{1}{2}(\ln x)^2 + 2$$

60. $y\sqrt{1 - x^2}\,\dfrac{dy}{dx} = x\sqrt{1 - y^2}$

$$\int (1 - y^2)^{-1/2}y\,dy = \int (1 - x^2)^{-1/2}\,x\,dx$$
$$-(1 - y^2)^{1/2} = -(1 - x^2)^{1/2} + C$$

$y(0) = 1$: $\quad 0 = -1 + C \implies C = 1$

$$\sqrt{1 - y^2} = \sqrt{1 - x^2} - 1$$

62. $\dfrac{dr}{ds} = e^{r-2s}$

$$\int e^{-r}dr = \int e^{-2s}\,ds$$
$$-e^{-r} = -\frac{1}{2}e^{-2s} + C$$

$r(0) = 0$: $-1 = -\dfrac{1}{2} + C \implies C = -\dfrac{1}{2}$

$$-e^{-r} = -\frac{1}{2}e^{-2s} - \frac{1}{2}$$
$$e^{-r} = \frac{1}{2}e^{-2s} + \frac{1}{2}$$
$$-r = \ln\left(\frac{1}{2}e^{-2s} + \frac{1}{2}\right) = \ln\left(\frac{1 + e^{-2s}}{2}\right)$$
$$r = \ln\left(\frac{2}{1 + e^{-2s}}\right)$$

64. $dT + k(T - 70)\,dt = 0$

$$\int \frac{dT}{T - 70} = -k\int dt$$

$$\ln(T - 70) = -kt + C_1$$

$$T - 70 = Ce^{-kt}$$

Initial condition: $T(0) = 140$;
$140 - 70 = 70 = Ce^0 = C$

Particular solution: $T - 70 = 70e^{-kt}$, $T = 70(1 + e^{-kt})$

66. $\dfrac{dy}{dx} = \dfrac{2y}{3x}$

$$\int \frac{3}{y}\,dy = \int \frac{2}{x}\,dx$$

$$\ln y^3 = \ln x^2 + \ln C$$

$$y^3 = Cx^2$$

Initial condition: $y(8) = 2$, $2^3 = C(8^2)$, $C = \dfrac{1}{8}$

Particular solution: $8y^3 = x^2$, $y = \dfrac{1}{2}x^{2/3}$

68. $m = \dfrac{dy}{dx} = \dfrac{y - 0}{x - 0} = \dfrac{y}{x}$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln y = \ln x + C_1 = \ln x + \ln C = \ln Cx$$

$$y = Cx$$

70. $f(x, y) = x^3 + 3x^2y^2 - 2y^2$

$f(tx, ty) = t^3x^3 + 3t^4x^2y^2 - 2t^2y^2$

Not homogeneous

72. $f(x, y) = \dfrac{xy}{\sqrt{x^2 + y^2}}$

$$f(tx, ty) = \frac{tx\,ty}{\sqrt{t^2x^2 + t^2y^2}}$$

$$= \frac{t^2xy}{t\sqrt{x^2 + y^2}} = t\frac{xy}{\sqrt{x^2 + y^2}}$$

Homogeneous of degree 1

74. $f(x, y) = \tan(x + y)$

$f(tx, ty) = \tan(tx + ty) = \tan[t(x + y)]$

Not homogeneous

76. $f(x, y) = \tan\dfrac{y}{x}$

$$f(tx, ty) = \tan\frac{ty}{tx} = \tan\frac{y}{x}$$

Homogeneous of degree 0

78. $y' = \dfrac{(x^3 + y^3)}{xy^2}$

$$xy^2\,dy = (x^3 + y^3)\,dx$$

$$y = vx, \quad dy = x\,dv + v\,dx$$

$$x(vx)^2(x\,dv + v\,dx) = (x^3 + (vx)^3)\,dx$$

$$x^4v^2\,dv + x^3v^3\,dx = x^3\,dx + v^3x^3\,dx$$

$$xv^2\,dv = dx$$

$$\int v^2\,dv = \int \frac{1}{x}\,dx$$

$$\frac{v^3}{3} = \ln|x| + C$$

$$\left(\frac{y}{x}\right)^3 = 3\ln|x| + C$$

$$y^3 = 3x^3\ln|x| + Cx^3$$

80. $y' = \dfrac{x^2 + y^2}{2xy}$, $y = vx$

$$v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{2x^2v}$$

$$2v\,dx + 2x\,dv = \frac{1 + v^2}{v}\,dx$$

$$\int \frac{2v}{v^2 - 1}\,dv = -\int \frac{dx}{x}$$

$$\ln(v^2 - 1) = -\ln x + \ln C = \ln\frac{C}{x}$$

$$v^2 - 1 = \frac{C}{x}$$

$$\frac{y^2}{x^2} - 1 = \frac{C}{x}$$

$$y^2 - x^2 = Cx$$

82. $y' = \dfrac{2x + 3y}{x}, y = vx$

$$v + x\frac{dv}{dx} = \frac{2x + 3vx}{x} = 2 + 3v$$

$$x\frac{dv}{dx} = 2 + 2v \implies \int \frac{dv}{1 + v} = 2\int \frac{dx}{x}$$

$$\ln|1 + v| = \ln x^2 + \ln C = \ln x^2C$$

$$1 + v = x^2C$$

$$1 + \frac{y}{x} = x^2C$$

$$\frac{y}{x} = x^2C - 1$$

$$y = Cx^3 - x$$

84. $-y^2\,dx + x(x + y)\,dy = 0, y = vx$

$$-x^2v^2\,dx + (x^2 + x^2v)(v\,dx + x\,dv) = 0$$

$$\int \frac{1 + v}{v}\,dv = -\int \frac{dx}{x}$$

$$v + \ln v = -\ln x + \ln C_1 = \ln \frac{C_1}{x}$$

$$v = \ln \frac{C_1}{xv}$$

$$\frac{C_1}{vx} = e^v$$

$$\frac{C_1}{y} = e^{y/x}$$

$$y = Ce^{-y/x}$$

Initial condition: $y(1) = 1, 1 = Ce^{-1} \implies C = e$

Particular solution: $y = e^{1-y/x}$

86. $(2x^2 + y^2)\,dx + xy\,dy = 0$

Let $y = vx, dy = x\,dv + v\,dx$.

$$(2x^2 + v^2x^2)\,dx + x(vx)(x\,dv + v\,dx) = 0$$

$$(2x^2 + 2x^2v^2)\,dx + x^3v\,dv = 0$$

$$(2 + 2v^2)\,dx = -xv\,dv$$

$$\frac{-2}{x}\,dx = \frac{v}{1 + v^2}\,dv$$

$$-2\ln x = \frac{1}{2}\ln(1 + v^2) + C_1$$

$$\ln x^{-2} = \ln(1 + v^2)^{1/2} + \ln C$$

$$x^{-2} = C(1 + v^2)^{1/2}$$

$$\frac{1}{x^2} = C\left(1 + \frac{y^2}{x^2}\right)^{1/2} = \frac{C}{x}(x^2 + y^2)^{1/2}$$

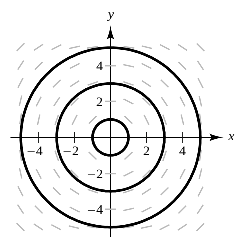
$$\frac{1}{x} = C(x^2 + y^2)^{1/2}$$

$y(1) = 0$: $1 = C(1 + 0) \implies C = 1$

$$\frac{1}{x} = \sqrt{x^2 + y^2}$$

$$1 = x\sqrt{x^2 + y^2}$$

88. $\dfrac{dy}{dx} = -\dfrac{x}{y}$



$$y\,dy = -x\,dx$$

$$\frac{y^2}{2} = \frac{-x^2}{2} + C_1$$

$$y^2 + x^2 = C$$

90. $$\frac{dy}{dx} = 0.25x(4 - y)$$

$$\frac{dy}{4 - y} = 0.25x\,dx$$

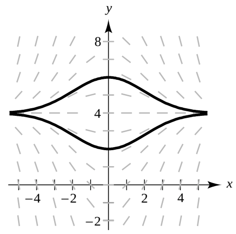
$$\int \frac{dy}{y - 4} = \int -0.25x\,dx$$

$$= \;= -\frac{1}{4}\int x\,dx$$

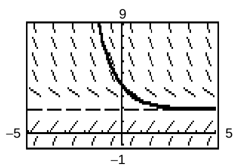
$$\ln|y - 4| = -\frac{1}{8}x^2 + C_1$$

$$y - 4 = e^{C_1 - (1/8)x^2} = Ce^{-(1/8)x^2}$$

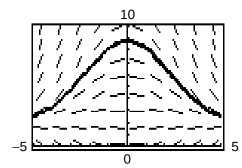
$$y = 4 + Ce^{-(1/8)x^2}$$



92. $\frac{dy}{dx} = 2 - y,\; y(0) = 4$



94. $\frac{dy}{dx} = 0.2x(2 - y),\; y(0) = 9$



96. $\frac{dy}{dt} = ky,\; y = Ce^{kt}$

Initial conditions: $y(0) = 20,\; y(1) = 16$

$$20 = Ce^0 = C$$

$$16 = 20e^k$$

$$k = \ln\frac{4}{5}$$

Particular solution: $y = 20e^{t\ln(4/5)}$

When 75% has been changed:

$$5 = 20e^{t\ln(4/5)}$$

$$\frac{1}{4} = e^{t\ln(4/5)}$$

$$t = \frac{\ln(1/4)}{\ln(4/5)} \approx 6.2 \text{ hr}$$

98. $\frac{dy}{dx} = k(x - 4)$

The direction field satisfies $(dy/dx) = 0$ along $x = 4$: Matches (b).

100. $\frac{dy}{dx} = ky^2$

The direction field satisfies $(dy/dx) = 0$ along $y = 0$, and grows more positive as y increases. Matches (d).

102. From Exercise 101,

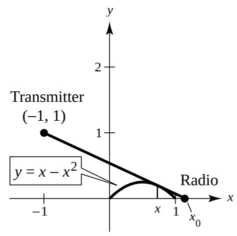
$$w = 1200 - Ce^{-kt},\; k = 1$$

$$w = 1200 - Ce^{-t}$$

$$w(0) = w_0 = 1200 - C \implies C = 1200 - w_0$$

$$w = 1200 - (1200 - w_0)e^{-t}$$

104. Let the radio receiver be located at $(x_0, 0)$. The tangent line to $y = x - x^2$ joins $(-1, 1)$ and $(x_0, 0)$.



(a) If (x, y) is the point of tangency on the $y = x - x^2$, then

$$1 - 2x = \frac{y - 1}{x + 1} = \frac{x - x^2 - 1}{x + 1}$$

$$x - 2x^2 + 1 - 2x = x - x^2 - 1$$

$$x^2 + 2x - 2 = 0$$

$$x = \left(\frac{-2 \pm \sqrt{4 + 8}}{2}\right) = -1 + \sqrt{3}$$

$$y = x - x^2 = 3\sqrt{3} - 5$$

Then $\dfrac{1 - 0}{-1 - x_0} = \dfrac{1 - 3\sqrt{3} + 5}{-1 + 1 - \sqrt{3}} = \dfrac{6 - 3\sqrt{3}}{-\sqrt{3}}$

$$\sqrt{3} = (1 + x_0)(6 - 3\sqrt{3})$$

$$= 6 - 3\sqrt{3} + x_0(6 - 3\sqrt{3})$$

$$x_0 = \frac{4\sqrt{3} - 6}{6 - 3\sqrt{3}} \approx 1.155$$

(b) Now let the transmitter be located at $(-1, h)$.

$$1 - 2x = \frac{y - h}{x + 1} = \frac{x - x^2 - h}{x + 1}$$

$$x - 2x^2 + 1 - 2x = x - x^2 - h$$

$$x^2 + 2x - h - 1 = 0$$

$$x = \frac{\left(-2 \pm \sqrt{4 + 4(h + 1)}\right)}{2}$$

$$= -1 + \sqrt{2 + h}$$

$$y = x - x^2$$

$$= 3\sqrt{2 + h} - h - 4$$

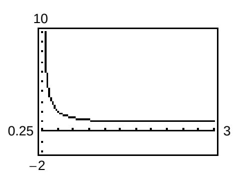
Then, $\dfrac{h - 0}{-1 - x_0} = \dfrac{h - \left(3\sqrt{2 + h} - h - 4\right)}{-1 - \left(-1 + \sqrt{2 + h}\right)}$

$$= \frac{2h + 4 - 3\sqrt{2 + h}}{-\sqrt{2 + h}}$$

$$\frac{x_0 + 1}{h} = \frac{\sqrt{2 + h}}{2h + 4 - 3\sqrt{2 + h}}$$

$$x_0 = \frac{h\sqrt{2 + h}}{2h + 4 - 3\sqrt{2 + h}} - 1$$

(c)



There is a vertical asymptote at $h = \frac{1}{4}$, which is the height of the mountain.

106. Given family (hyperbolas): $x^2 - 2y^2 = C$

$$2x - 4yy' = 0$$

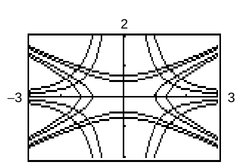
$$y' = \frac{x}{2y}$$

Orthogonal trajectory: $y' = \dfrac{-2y}{x}$

$$\int \frac{dy}{y} = -\int \frac{2}{x}\,dx$$

$$\ln y = -2 \ln x + \ln k$$

$$y = kx^{-2} = \frac{k}{x^2}$$



108. Given family (parabolas): $y^2 = 2Cx$

$$2yy' = 2C$$

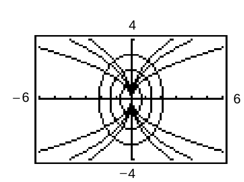
$$y' = \frac{C}{y} = \frac{y^2}{2x}\left(\frac{1}{y}\right) = \frac{y}{2x}$$

Orthogonal trajectory (ellipse): $y' = -\dfrac{2x}{y}$

$$\int y\,dy = -\int 2x\,dx$$

$$\frac{y^2}{2} = -x^2 + K_1$$

$$2x^2 + y^2 = K$$



110. Given family (exponential functions): $y = Ce^x$

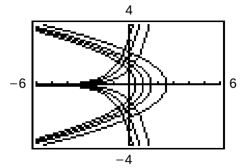
$$y' = Ce^x = y$$

Orthogonal trajectory (parabolas): $y' = -\frac{1}{y}$

$$\int y\,dy = -\int dx$$

$$\frac{y^2}{2} = -x + K_1$$

$$y^2 = -2x + K$$



112. The number of initial conditions matches the number of constants in the general solution.

114. Two families of curves are mutually orthogonal if each curve in the first family intersects each curve in the second family at right angles.

116. True

$$\frac{dy}{dx} = (x - 2)(y + 1)$$

118. True

$$x^2 + y^2 = 2Cy \qquad x^2 + y^2 = 2Kx$$

$$\frac{dy}{dx} = \frac{x}{C - y} \qquad \frac{dy}{dx} = \frac{K - x}{y}$$

$$\frac{x}{C-y} \cdot \frac{K-x}{y} = \frac{Kx - x^2}{Cy - y^2} = \frac{2Kx - 2x^2}{2Cy - 2y^2} = \frac{x^2 + y^2 - 2x^2}{x^2 + y^2 - 2y^2} = \frac{y^2 - x^2}{x^2 - y^2} = -1$$

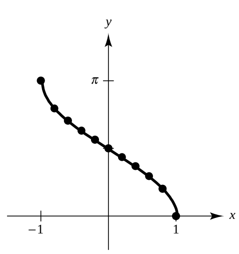
Section 5.8 Inverse Trigonometric Functions: Differentiation

2. $y = \arccos x$

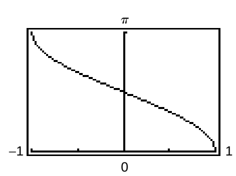
(a)

x	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
y	3.142	2.499	2.214	1.982	1.772	1.571	1.369	1.159	0.927	0.634	0

(b)



(c)



(d) Intercepts: $\left(0, \frac{\pi}{2}\right)$ and $(1, 0)$

No symmetry

4. $\left(\_\_, \frac{\pi}{4}\right) = \left(1, \frac{\pi}{4}\right)$

$\left(\_\_, -\frac{\pi}{6}\right) = \left(-\frac{\sqrt{3}}{3}, -\frac{\pi}{6}\right)$

$\left(-\sqrt{3}, \_\_\right) = \left(-\sqrt{3}, -\frac{\pi}{3}\right)$

6. $\arcsin 0 = 0$

8. $\arccos 0 = \dfrac{\pi}{2}$

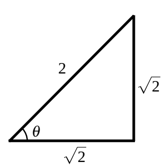
10. $\text{arc cot}\left(-\sqrt{3}\right) = \dfrac{5\pi}{6}$

12. $\arccos\left(-\dfrac{\sqrt{3}}{2}\right) = \dfrac{5\pi}{6}$

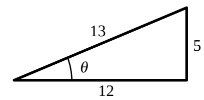
14. $\arcsin(-0.39) \approx -0.40$

16. $\arctan(-3) \approx -1.25$

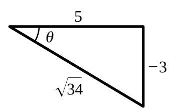
18. (a) $\tan\left(\arccos \dfrac{\sqrt{2}}{2}\right) = \tan\left(\dfrac{\pi}{4}\right) = 1$



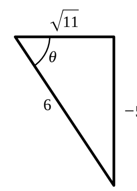
(b) $\cos\left(\arcsin \dfrac{5}{13}\right) = \dfrac{12}{13}$



20. (a) $\sec\left[\arctan\left(-\dfrac{3}{5}\right)\right] = \dfrac{\sqrt{34}}{5}$



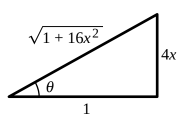
(b) $\tan\left[\arcsin\left(-\dfrac{5}{6}\right)\right] = -\dfrac{5\sqrt{11}}{11}$



22. $y = \sec(\text{arc tan } 4x)$

$\theta = \arctan 4x$

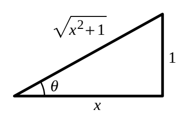
$y = \sec \theta = \sqrt{1 + 16x^2}$



24. $y = \cos(\text{arccot } x)$

$\theta = \text{arccot } x$

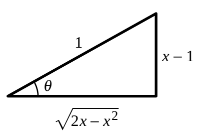
$y = \cos \theta = \dfrac{x}{\sqrt{x^2 + 1}}$



26. $y = \sec[\arcsin(x - 1)]$

$\theta = \arcsin(x - 1)$

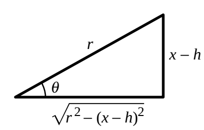
$y = \sec \theta = \dfrac{1}{\sqrt{2x - x^2}}$



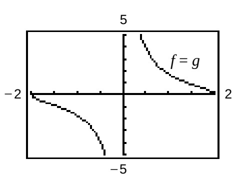
28. $y = \cos\left(\arcsin \dfrac{x - h}{r}\right)$

$\theta = \arcsin \dfrac{x - h}{r}$

$y = \cos \theta = \dfrac{\sqrt{r^2 - (x - h)^2}}{r}$



30.

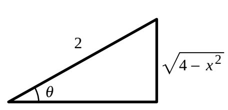


Asymptote: $x = 0$

$\arccos \dfrac{x}{2} = \theta$

$\cos \theta = \dfrac{x}{2}$

$\tan \theta = \dfrac{\sqrt{4 - x^2}}{x}$



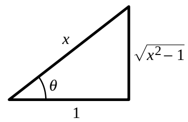
32. $\arctan(2x - 5) = -1$

$2x - 5 = \tan(-1)$

$x = \dfrac{1}{2}(\tan(-1) + 5) \approx 1.721$

34. $\arccos x = \operatorname{arcsec} x$

$$x = \cos(\operatorname{arcsec} x)$$
$$x = \frac{1}{x}$$
$$x^2 = 1$$
$$x = \pm 1$$



36. (a) $\arcsin(-x) = -\arcsin x,\ |x| \le 1.$

Let $y = \arcsin(-x)$. Then,

$$-x = \sin y \implies x = -\sin y \implies x = \sin(-y).$$

Thus, $-y = \arcsin x \implies y = -\arcsin x$. Therefore, $\arcsin(-x) = -\arcsin x$.

(b) $\arccos(-x) = \pi - \arccos x,\ |x| \le 1.$

Let $y = \arccos(-x)$. Then,

$$-x = \cos y \implies x = -\cos y \implies x = \cos(\pi - y).$$

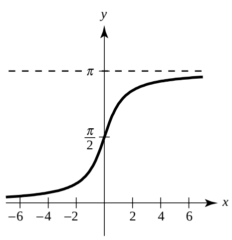
Thus, $\pi - y = \arccos x \implies y = \pi - \arccos x$. Therefore, $\arccos(-x) = \pi - \arccos x$.

38. $f(x) = \arctan x + \dfrac{\pi}{2}$

$$x = \tan\left(y - \frac{\pi}{2}\right)$$

Domain: $(-\infty, \infty)$

Range: $(0, \pi)$



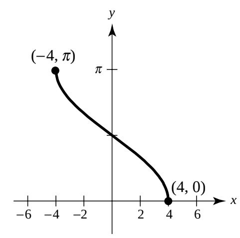
$f(x)$ is the graph of $\arctan x$ shifted $\pi/2$ units upward.

40. $f(x) = \arccos\left(\dfrac{x}{4}\right)$

$$\frac{x}{4} = \cos y$$
$$x = 4\cos y$$

Domain: $[-4, 4]$

Range: $[0, \pi]$



42. $f(t) = \arcsin t^2$

$$f'(t) = \frac{2t}{\sqrt{1 - t^4}}$$

44. $f(x) = \operatorname{arcsec} 2x$

$$f'(x) = \frac{2}{|2x|\sqrt{4x^2 - 1}} = \frac{1}{|x|\sqrt{4x^2 - 1}}$$

46. $f(x) = \arctan \sqrt{x}$

$$f'(x) = \left(\frac{1}{1 + x}\right)\left(\frac{1}{2\sqrt{x}}\right) = \frac{1}{2\sqrt{x}(1 + x)}$$

48. $h(x) = x^2 \arctan x$

$$h'(x) = 2x \arctan x + \frac{x^2}{1 + x^2}$$

50. $f(x) = \arcsin x + \arccos x = \dfrac{\pi}{2}$

$$f'(x) = 0$$

52. $y = \ln(t^2 + 4) - \dfrac{1}{2}\arctan\dfrac{t}{2}$

$$y' = \frac{2t}{t^2 + 4} - \frac{1}{2} \cdot \frac{1}{1 + \left(\frac{t}{2}\right)^2}\left(\frac{1}{2}\right)$$
$$= \frac{2t}{t^2 + 4} - \frac{1}{t^2 + 4} = \frac{2t - 1}{t^2 + 4}$$

54. $y = \dfrac{1}{2}\left[x\sqrt{4 - x^2} + 4\arcsin\left(\dfrac{x}{2}\right)\right]$

$$y' = \frac{1}{2}\left[x\frac{1}{2}(4 - x^2)^{-1/2}(-2x) + \sqrt{4 - x^2} + 2\frac{1}{\sqrt{1 - (x/2)^2}}\right]$$
$$= \frac{1}{2}\left[\frac{-x^2}{\sqrt{4 - x^2}} + \sqrt{4 - x^2} + \frac{4}{\sqrt{4 - x^2}}\right]$$
$$= \sqrt{4 - x^2}$$

56. $y = x \arctan 2x - \dfrac{1}{4}\ln(1 + 4x^2)$

$$\frac{dy}{dx} = \frac{2x}{1 + 4x^2} + \arctan(2x) - \frac{1}{4}\left(\frac{8x}{1 + 4x^2}\right) = \arctan(2x)$$

58. $y = 25 \arcsin \frac{x}{5} - x\sqrt{25 - x^2}$

$$y' = 5\frac{1}{\sqrt{1 - (x/2)^2}} - \sqrt{25 - x^2} - x\frac{1}{2}(25 - x^2)^{-1/2}(-2x)$$
$$= \frac{25}{\sqrt{25 - x^2}} - \frac{(25 - x^2)}{\sqrt{25 - x^2}} + \frac{x^2}{\sqrt{25 - x^2}}$$
$$= \frac{2x^2}{\sqrt{25 - x^2}}$$

60. $y = \arctan \frac{x}{2} - \frac{1}{2(x^2 + 4)}$

$$y' = \frac{1}{2}\frac{1}{1 + (x/2)^2} + \frac{1}{2}(x^2 + 4)^{-2}(2x)$$
$$= \frac{2}{x^2 + 4} + \frac{x}{(x^2 + 4)^2}$$
$$= \frac{2x^2 + 8 + x}{(x^2 + 4)^2}$$

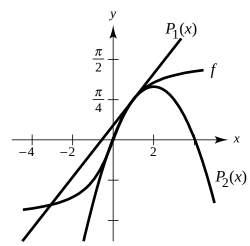
62. $f(x) = \arctan x,\ a = 1$

$$f'(x) = \frac{1}{1 + x^2}$$

$$f''(x) = \frac{-2x}{(1 + x^2)^2}$$

$$P_1(x) = f(1) + f'(1)(x - 1) = \frac{\pi}{4} + \frac{1}{2}(x - 1)$$

$$P_2(x) = f(1) + f'(1)(x - 1) + \frac{1}{2}f''(1)(x - 1)^2 = \frac{\pi}{4} + \frac{1}{2}(x - 1) - \frac{1}{4}(x - 1)^2$$



64. $f(x) = \arcsin x - 2x$

$$f'(x) = \frac{1}{\sqrt{1 - x^2}} - 2 = 0 \text{ when } \sqrt{1 - x^2} = \frac{1}{2} \text{ or}$$
$$x = \pm\frac{\sqrt{3}}{2}.$$

$$f''(x) = \frac{x}{(1 - x^2)^{3/2}}$$

$$f''\left(\frac{\sqrt{3}}{2}\right) > 0$$

Relative minimum: $\left(\frac{\sqrt{3}}{2}, -0.68\right)$

$$f''\left(-\frac{\sqrt{3}}{2}\right) < 0$$

Relative maximum: $\left(-\frac{\sqrt{3}}{2}, 0.68\right)$

66. $f(x) = \arcsin x - 2 \arctan x$

$$f'(x) = \frac{1}{\sqrt{1 - x^2}} - \frac{2}{1 + x^2} = 0$$
$$1 + x^2 = 2\sqrt{1 - x^2}$$
$$1 + 2x^2 + x^4 = 4(1 - x^2)$$
$$x^4 + 6x^2 - 3 = 0$$
$$x = \pm 0.681$$

By the First Derivative Test, $(-0.681, 0.447)$ is a relative maximum and $(0.681, -0.447)$ is a relative minimum.

68. $\arctan 0 = 0$. π is not in the range of $y = \arctan x$.

70. The derivatives are algebraic. See Theorem 5.18.

72. (a) $\cot \theta = \frac{x}{3}$

$$\theta = \operatorname{arccot}\left(\frac{x}{3}\right)$$

(b) $\frac{d\theta}{dt} = \frac{-3}{x^2 + 9}\frac{dx}{dt}$

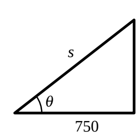
If $x = 10$, $\frac{d\theta}{dt} \approx 11.001$ rad/hr.

If $x = 3$, $\frac{d\theta}{dt} \approx 66.667$ rad/hr.

A lower altitude results in a greater rate of change of θ.

74. $\cos \theta = \frac{750}{s}$

$$\theta = \arccos\left(\frac{750}{s}\right)$$



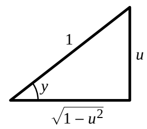
$$\frac{d\theta}{dt} = \frac{d\theta}{ds} \cdot \frac{ds}{dt} = \frac{-1}{\sqrt{1 - (750/s)^2}}\left(\frac{-750}{s^2}\right)\frac{ds}{dt}$$
$$= \frac{750}{s\sqrt{s^2 - 750^2}}\frac{ds}{dt}$$

76. (a) Let $y = \arcsin u$. Then

$$\sin y = u$$

$$\cos y \cdot y' = u'$$

$$\frac{dy}{dx} = \frac{u'}{\cos y} = \frac{u'}{\sqrt{1 - u^2}}.$$

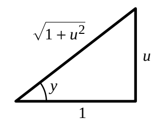


(b) Let $y = \arctan u$. Then

$$\tan y = u$$

$$\sec^2 y \frac{dy}{dx} = u'$$

$$\frac{dy}{dx} = \frac{u'}{\sec^2 y} = \frac{u'}{1 + u^2}.$$

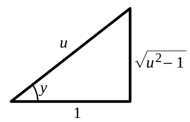


(c) Let $y = \operatorname{arcsec} u$. Then

$$\sec y = u$$

$$\sec y \tan y \frac{dy}{dx} = u'$$

$$\frac{dy}{dx} = \frac{u'}{\sec y \tan y} = \frac{u'}{|u|\sqrt{u^2 - 1}}.$$



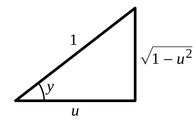
Note: The absolute value sign in the formula for the derivative of arcsec u is necessary because the inverse secant function has a positive slope at every value in its domain.

(d) Let $y = \arccos u$. Then

$$\cos y = u$$

$$-\sin y \frac{dy}{dx} = u'$$

$$\frac{dy}{dx} = -\frac{u'}{\sin y} = -\frac{u'}{\sqrt{1 - u^2}}.$$

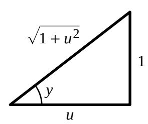


(e) Let $y = \operatorname{arccot} u$. Then

$$\cot y = u$$

$$-\csc^2 y \frac{dy}{dx} = u'$$

$$\frac{dy}{dx} = \frac{u'}{-\csc^2 y} = -\frac{u'}{1 + u^2}.$$

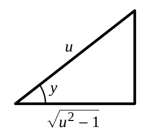


(f) Let $y = \operatorname{arccsc} u$. Then

$$\csc y = u$$

$$-\csc y \cot y \frac{dy}{dx} = u'$$

$$\frac{dy}{dx} = \frac{u'}{-\csc y \cot y} = -\frac{u'}{|u|\sqrt{u^2 - 1}}$$



Note: The absolute value sign in the formula for the derivative of arccsc u is necessary because the inverse cosecant function has a negative slope at every value in its domain.

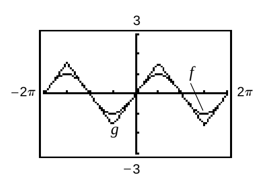
78. $f(x) = \sin x$

$g(x) = \arcsin(\sin x)$

(a) The range of $y = \arcsin x$ is $-\pi/2 \le y \le \pi/2$.

(b) Maximum: $\pi/2$

Minimum: $-\pi/2$



80. False

The range of $y = \arcsin x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

82. False

$$\arcsin^2 0 + \arccos^2 0 = 0 + \left(\frac{\pi}{2}\right)^2 \neq 1$$

Section 5.9 Inverse Trigonometric Functions: Integration

2. $\displaystyle\int \frac{3}{\sqrt{1 - 4x^2}}\,dx = \frac{3}{2}\int \frac{2}{\sqrt{1 - 4x^2}}\,dx = \frac{3}{2}\arcsin(2x) + C$

4. $\displaystyle\int_0^1 \frac{1}{\sqrt{4 - x^2}}\,dx = \left[\arcsin \frac{x}{2}\right]_0^1 = \frac{\pi}{6}$

6. $\displaystyle\int \frac{4}{1 + 9x^2}\,dx = \frac{4}{3}\int \frac{3}{1 + 9x^2}\,dx = \frac{4}{3}\arctan(3x) + C$

8. $\displaystyle\int_{\sqrt{3}}^3 \frac{1}{9 + x^2}\,dx = \left[\frac{1}{3}\arctan \frac{x}{3}\right]_{\sqrt{3}}^3 = \frac{\pi}{36}$

10. $\displaystyle\int \frac{1}{4 + (x - 1)^2}\,dx = \frac{1}{2}\arctan\left(\frac{x - 1}{2}\right) + C$

12. $\displaystyle\int \frac{x^4 - 1}{x^2 + 1}\,dx = \int (x^2 - 1)\,dx = \frac{1}{3}x^3 - x + C$

14. Let $u = t^2$, $du = 2t\,dt$.

$$\int \frac{t}{t^4 + 16}\,dt = \frac{1}{2}\int \frac{1}{(4)^2 + (t^2)^2}(2t)\,dt = \frac{1}{8}\arctan\frac{t^2}{4} + C$$

16. Let $u = x^2$, $du = 2x\,dx$.

$$\int \frac{1}{x\sqrt{x^4 - 4}}\,dx = \frac{1}{2}\int \frac{1}{x^2\sqrt{(x^2)^2 - 2^2}}(2x)\,dx$$

$$= \frac{1}{4}\operatorname{arcsec}\frac{x^2}{2} + C$$

18. Let $u = \arccos x$, $du = -\frac{1}{\sqrt{1 - x^2}}\,dx$.

$$\int_0^{1/\sqrt{2}} \frac{\arccos x}{\sqrt{1 - x^2}}\,dx = -\int_0^{1/\sqrt{2}} \frac{-\arccos x}{\sqrt{1 - x^2}}\,dx$$

$$= \left[-\frac{1}{2}\arccos^2 x\right]_0^{1/\sqrt{2}} = \frac{3\pi^2}{32} \approx 0.925$$

20. Let $u = 1 + x^2$, $du = 2x\,dx$.

$$\int_{-\sqrt{3}}^{0} \frac{x}{1 + x^2}\,dx = \frac{1}{2}\int_{-\sqrt{3}}^{0} \frac{1}{1 + x^2}(2x)\,dx$$

$$= \left[\frac{1}{2}\ln(1 + x^2)\right]_{-\sqrt{3}}^{0} = -\ln 2$$

22. $$\int_1^2 \frac{1}{3 + (x - 2)^2}\,dx = \int_1^2 \frac{1}{\left(\sqrt{3}\right)^2 + (x - 2)^2}\,dx = \left[\frac{1}{\sqrt{3}}\arctan\left(\frac{x - 2}{\sqrt{3}}\right)\right]_1^2 = \frac{\sqrt{3}\pi}{18}$$

24. $$\int_0^{\pi/2} \frac{\cos x}{1 + \sin^2 x}\,dx = \arctan(\sin x)\Big]_0^{\pi/2} = \frac{\pi}{4}$$

26. $\int \frac{3}{2\sqrt{x}(1 + x)}\,dx$. $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}\,dx$, $dx = 2u\,du$

$$\frac{3}{2}\int \frac{2u\,du}{u(1 + u^2)} = 3\int \frac{du}{1 + u^2} = 3\arctan u + C$$

$$= 3\arctan\sqrt{x} + C$$

28. $$\int \frac{4x + 3}{\sqrt{1 - x^2}}\,dx = (-2)\int \frac{-2x}{\sqrt{1 - x^2}}\,dx + 3\int \frac{1}{\sqrt{1 - x^2}}\,dx = -4\sqrt{1 - x^2} + 3\arcsin x + C$$

30. $$\int \frac{x - 2}{(x + 1)^2 + 4}\,dx = \frac{1}{2}\int \frac{2x + 2}{(x + 1)^2 + 4}\,dx - \int \frac{3}{(x + 1)^2 + 4}\,dx$$

$$= \frac{1}{2}\ln(x^2 + 2x + 5) - \frac{3}{2}\arctan\left(\frac{x + 1}{2}\right) + C$$

32. $$\int_{-2}^{2} \frac{dx}{x^2 + 4x + 13} = \int_{-2}^{2} \frac{dx}{(x + 2)^2 + 9} = \left[\frac{1}{3}\arctan\left(\frac{x + 2}{3}\right)\right]_{-2}^{2} = \frac{1}{3}\arctan\left(\frac{4}{3}\right)$$

34. $$\int \frac{2x - 5}{x^2 + 2x + 2}\,dx = \int \frac{2x + 2}{x^2 + 2x + 2}\,dx - 7\int \frac{1}{1 + (x + 1)^2}\,dx = \ln|x^2 + 2x + 2| - 7\arctan(x + 1) + C$$

36. $$\int \frac{2}{\sqrt{-x^2 + 4x}}\,dx = \int \frac{2}{\sqrt{4 - (x^2 - 4x + 4)}}\,dx$$

$$= \int \frac{2}{\sqrt{4 - (x - 2)^2}}\,dx$$

$$= 2\arcsin\left(\frac{x - 2}{2}\right) + C$$

38. Let $u = x^2 - 2x$, $du = (2x - 2)\,dx$.

$$\int \frac{x - 1}{\sqrt{x^2 - 2x}}\,dx = \frac{1}{2}\int (x^2 - 2x)^{-1/2}(2x - 2)\,dx$$

$$= \sqrt{x^2 - 2x} + C$$

40. $$\int \frac{1}{(x - 1)\sqrt{x^2 - 2x}}\,dx = \int \frac{1}{(x - 1)\sqrt{(x - 1)^2 - 1}}\,dx = \operatorname{arcsec}|x - 1| + C$$

42. Let $u = x^2 - 4$, $du = 2x\,dx$.

$$\int \frac{x}{\sqrt{9 + 8x^2 - x^4}}\,dx = \frac{1}{2}\int \frac{2x}{\sqrt{25 - (x^2 - 4)^2}}\,dx = \frac{1}{2}\arcsin\left(\frac{x^2 - 4}{5}\right) + C$$

44. Let $u = \sqrt{x - 2}$, $u^2 + 2 = x$, $2u\,du = dx$

$$\int \frac{\sqrt{x - 2}}{x + 1}\,dx = \int \frac{2u^2}{u^2 + 3}\,du = \int \frac{2u^2 + 6 - 6}{u^2 + 3}\,du = 2\int du - 6\int \frac{1}{u^2 + 3}\,du$$

$$= 2u - \frac{6}{\sqrt{3}}\arctan\frac{u}{\sqrt{3}} + C = 2\sqrt{x - 2} - 2\sqrt{3}\arctan\sqrt{\frac{x - 2}{3}} + C$$

46. The term is $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$: $x^2 + 3x = x^2 + 3x + \frac{9}{4} - \frac{9}{4} = \left(x + \frac{3}{2}\right)^2 - \frac{9}{4}$

48. (a) $\int e^{x^2}\,dx$ cannot be evaluated using the basic integration rules.

(b) $\int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C$, $u = x^2$

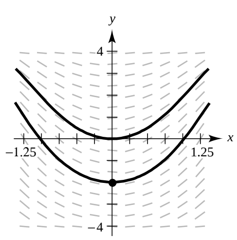
(c) $\int \frac{1}{x^2}e^{1/x}\,dx = -e^{1/x} + C$, $u = \frac{1}{x}$

50. (a) $\int \frac{1}{1 + x^4}\,dx$ cannot be evaluated using the basic integration rules.

(b) $\int \frac{x}{1 + x^4}\,dx = \frac{1}{2}\int \frac{2x}{1 + (x^2)^2}\,dx = \frac{1}{2}\arctan(x^2) + C$, $u = x^2$

(c) $\int \frac{x^3}{1 + x^4}\,dx = \frac{1}{4}\int \frac{4x^3}{1 + x^4}\,dx = \frac{1}{4}\ln(1 + x^4) + C$, $u = 1 + x^4$

52. (a)



(b) $\frac{dy}{dx} = x\sqrt{16 - y^2}$, $(0, -2)$

$$\frac{dy}{\sqrt{16 - y^2}} = x\,dx$$

$$\arcsin\left(\frac{y}{4}\right) = \frac{x^2}{2} + C$$

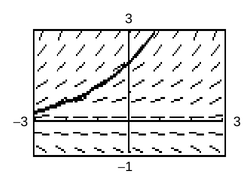
$$(0, -2): \arcsin\left(-\frac{2}{4}\right) = C \implies C = -\frac{\pi}{6}$$

$$\arcsin\left(\frac{y}{4}\right) = \frac{x^2}{2} - \frac{\pi}{6}$$

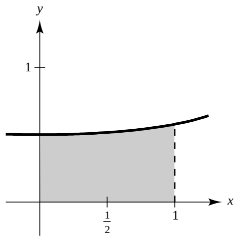
$$\frac{y}{4} = \sin\left(\frac{x^2}{2} - \frac{\pi}{6}\right)$$

$$y = 4\sin\left(\frac{x^2}{2} - \frac{\pi}{6}\right)$$

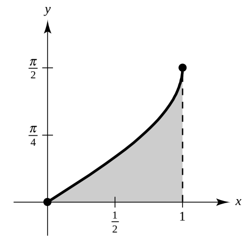
54. $\frac{dy}{dx} = \frac{2y}{\sqrt{16 - x^2}}$, $y(0) = 2$



56. $A = \int_0^1 \frac{1}{\sqrt{4 - x^2}}\,dx = \left[\arcsin \frac{x}{2}\right]_0^1 = \frac{\pi}{6}$



58. $\int_0^1 \arcsin x\,dx \approx 0.571$



60. $F(x) = \frac{1}{2}\int_x^{x+2} \frac{2}{t^2 + 1}\,dt$

(a) $F(x)$ represents the average value of $f(x)$ over the interval $[x, x + 2]$. Maximum at $x = -1$, since the graph is greatest on $[-1, 1]$.

(b) $F(x) = \Big[\arctan t\Big]_x^{x+2} = \arctan(x + 2) - \arctan x$

$$F'(x) = \frac{1}{1 + (x + 2)^2} - \frac{1}{1 + x^2} = \frac{(1 + x^2) - (x^2 + 4x + 5)}{(x^2 + 1)(x^2 + 4x + 5)} = \frac{-4(x + 1)}{(x^2 + 1)(x^2 + 4x + 5)} = 0 \text{ when } x = -1.$$

62. $\int \frac{1}{\sqrt{6x - x^2}}\,dx$

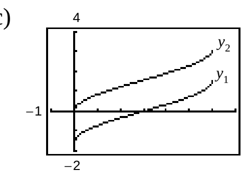
(a) $6x - x^2 = 9 - (x^2 - 6x + 9) = 9 - (x - 3)^2$

$$\int \frac{1}{\sqrt{6x - x^2}}\,dx = \int \frac{dx}{\sqrt{9 - (x - 3)^2}} = \arcsin\left(\frac{x - 3}{3}\right) + C$$

(b) $u = \sqrt{x}$, $u^2 = x$, $2u\,du = dx$

$$\int \frac{1}{\sqrt{6u^2 - u^4}}(2u\,du) = \int \frac{2}{\sqrt{6 - u^2}}\,du = 2\arcsin\left(\frac{u}{\sqrt{6}}\right) + C = 2\arcsin\left(\frac{\sqrt{x}}{\sqrt{6}}\right) + C$$

(c)



The antiderivatives differ by a constant, $\pi/2$.

Domain: $[0, 6]$

64. Let $f(x) = \arctan x - \frac{x}{1 + x^2}$

$$f'(x) = \frac{1}{1 + x^2} - \frac{1 - x^2}{(1 + x^2)^2} = \frac{2x^2}{(1 + x^2)} > 0 \text{ for } x > 0.$$

Since $f(0) = 0$ and f is increasing for $x > 0$, $\arctan x - \frac{x}{1 + x^2} > 0$ for $x > 0$. Thus,

$$\arctan x > \frac{x}{1 + x^2}.$$

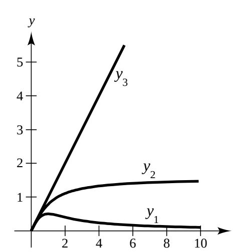
Let $g(x) = x - \arctan x$

$$g'(x) = 1 - \frac{1}{1 + x^2} = \frac{x^2}{1 + x^2} > 0 \text{ for } x > 0.$$

Since $g(0) = 0$ and g is increasing for $x > 0$, $x - \arctan x > 0$ for $x > 0$. Thus, $x > \arctan x$.

Therefore,

$$\frac{x}{1 + x^2} < \arctan x < x.$$



Section 5.10 Hyperbolic Functions

2. (a) $\cosh(0) = \dfrac{e^0 + e^0}{2} = 1$

(b) $\operatorname{sech}(1) = \dfrac{2}{e + e^{-1}} \approx 0.648$

4. (a) $\sinh^{-1}(0) = 0$

(b) $\tanh^{-1}(0) = 0$

6. (a) $\operatorname{csch}^{-1}(2) = \ln\left(\dfrac{1 + \sqrt{5}}{2}\right) \approx 0.481$

(b) $\coth^{-1}(3) = \dfrac{1}{2}\ln\left(\dfrac{4}{2}\right) \approx 0.347$

8. $\dfrac{1 + \cosh 2x}{2} = \dfrac{1 + (e^{2x} + e^{-2x})/2}{2} = \dfrac{e^{2x} + 2 + e^{-2x}}{4} = \left(\dfrac{e^x + e^{-x}}{2}\right)^2 = \cosh^2 x$

10. $2 \sinh x \cosh x = 2\left(\dfrac{e^x - e^{-x}}{2}\right)\left(\dfrac{e^x + e^{-x}}{2}\right) = \dfrac{e^{2x} - e^{-2x}}{2} = \sinh 2x$

12. $2 \cosh\left(\dfrac{x + y}{2}\right) \cosh\left(\dfrac{x - y}{2}\right) = 2\left[\dfrac{e^{(x+y)/2} + e^{-(x+y)/2}}{2}\right]\left[\dfrac{e^{(x-y)/2} + e^{-(x-y)/2}}{2}\right]$

$$= 2\left[\frac{e^x + e^y + e^{-y} + e^{-x}}{4}\right] = \frac{e^x + e^{-x}}{2} + \frac{e^y + e^{-y}}{2}$$

$$= \cosh x + \cosh y$$

14. $\tanh x = \dfrac{1}{2}$

$\left(\dfrac{1}{2}\right)^2 + \operatorname{sech}^2 x = 1 \implies \operatorname{sech}^2 x = \dfrac{3}{4} \implies \operatorname{sech} x = \dfrac{\sqrt{3}}{2}$

$\cosh x = \dfrac{1}{\sqrt{3}/2} = \dfrac{2\sqrt{3}}{3}$

$\coth x = \dfrac{1}{1/2} = 2$

$\sinh x = \tanh x \cosh x = \left(\dfrac{1}{2}\right)\left(\dfrac{2\sqrt{3}}{3}\right) = \dfrac{\sqrt{3}}{3}$

$\operatorname{csch} x = \dfrac{1}{\sqrt{3}/3} = \sqrt{3}$

Putting these in order:

$\sinh x = \dfrac{\sqrt{3}}{3}$ $\quad\operatorname{csch} x = \sqrt{3}$

$\cosh x = \dfrac{2\sqrt{3}}{3}$ $\quad\operatorname{sech} x = \dfrac{\sqrt{3}}{2}$

$\tanh x = \dfrac{1}{2}$ $\quad\coth x = 2$

16. $y = \coth(3x)$

$y' = -3 \operatorname{csch}^2(3x)$

18. $g(x) = \ln(\cosh x)$

$g'(x) = \dfrac{1}{\cosh x}(\sinh x) = \tanh x$

20. $y = x \cosh x - \sinh x$

$y' = x \sinh x + \cosh x - \cosh x = x \sinh x$

22. $h(t) = t - \coth t$

$h'(t) = 1 + \operatorname{csch}^2 t = \coth^2 t$

24. $g(x) = \text{sech}^2\, 3x$

$g'(x) = -2\,\text{sech}(3x)\,\text{sech}(3x)\tanh(3x)(3)$

$= -6\,\text{sech}^2\, 3x \tanh 3x$

26. $f(x) = e^{\sinh x}$

$f'(x) = (\cosh x)(e^{\sinh x})$

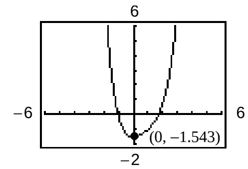
28. $y = \text{sech}(x + 1)$

$y' = -\text{sech}(x + 1)\tanh(x + 1)$

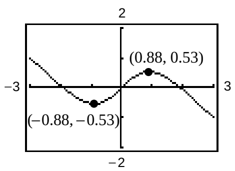
30. $f(x) = x\sinh(x - 1) - \cosh(x - 1)$

$f'(x) = x\cosh(x - 1) + \sinh(x - 1) - \sinh(x - 1) = x\cosh(x - 1)$

$f'(x) = 0$ for $x = 0$. By the First Derivative Test, $(0, -\cosh(-1)) \approx (0, -1.543)$ is a relative minimum.



32. $h(x) = 2\tanh x - x$



Relative maximum: $(0.88, 0.53)$

Relative minimum: $(-0.88, -0.53)$

34. $y = a\cosh x$

$y' = a\sinh x$

$y'' = a\cosh x$

Therefore, $y'' - y = 0$.

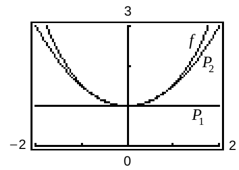
36. $f(x) = \cosh x$ $\quad f(1) = \cosh(0) \approx 1$

$f'(x) = \sinh x$ $\quad f'(1) = \sinh(0) \approx 0$

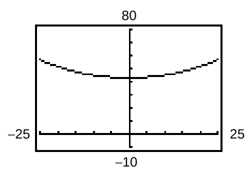
$f''(x) = \cosh x$ $\quad f''(1) = \cosh(0) \approx 1$

$P_1(x) = f(0) + f'(0)(x - 0) = 1$

$P_2(x) = 1 + \frac{1}{2}x^2$



38. (a) $y = 18 + 25\cosh\dfrac{x}{25}, -25 \le x \le 25$



(b) At $x = \pm 25$, $y = 18 + 25\cosh(1) \approx 56.577$.

At $x = 0$, $y = 18 + 25 = 43$.

(c) $y' = \sinh\dfrac{x}{25}$. At $x = 25$, $y' = \sinh(1) \approx 1.175$

40. Let $u = \sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}}\,dx$.

$$\int \frac{\cosh\sqrt{x}}{\sqrt{x}}\,dx = 2\int \cosh\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right)dx = 2\sinh\sqrt{x} + C$$

42. Let $u = \cosh x$, $du = \sinh x\,dx$.

$$\int \frac{\sinh}{1 + \sinh^2 x}\,dx = \int \frac{\sinh x}{\cosh^2 x}\,dx = \frac{-1}{\cosh x} + C$$

$$= -\text{sech}\, x + C$$

44. Let $u = 2x - 1$, $du = 2\,dx$.

$$\int \text{sech}^2(2x - 1)\,dx = \frac{1}{2}\int \text{sech}^2(2x - 1)(2)\,dx$$

$$= \frac{1}{2}\tanh(2x - 1) + C$$

46. Let $u = \text{sech}\, x$, $du = -\text{sech}\, x\tanh x\,dx$.

$$\int \text{sech}^3 x\tanh x\,dx = -\int \text{sech}^2 x(-\text{sech}\, x\tanh x)\,dx$$

$$= -\frac{1}{3}\text{sech}^3 x + C$$

48. $\displaystyle\int \cosh^2 x\,dx = \int \frac{1+\cosh 2x}{2}\,dx$

$\displaystyle = \frac{1}{2}\left[x + \frac{\sinh 2x}{2}\right] + C$

$\displaystyle = \frac{1}{2}x + \frac{1}{4}\sinh 2x + C$

50. $\displaystyle\int_0^4 \frac{1}{\sqrt{25-x^2}}\,dx = \left[\arcsin\frac{x}{5}\right]_0^4 = \arcsin\frac{4}{5}$

52. $\displaystyle\int \frac{2}{x\sqrt{1+4x^2}}\,dx = 2\int \frac{1}{(2x)\sqrt{1+(2x)^2}}(2)\,dx = -2\ln\left(\frac{1+\sqrt{1+4x^2}}{|2x|}\right) + C$

54. Let $u = \sinh x$, $du = \cosh x\,dx$.

$\displaystyle\int \frac{\cosh x}{\sqrt{9-\sinh^2 x}}\,dx = \arcsin\left(\frac{\sinh x}{3}\right) + C$

$\displaystyle = \arcsin\left(\frac{e^x - e^{-x}}{6}\right) + C$

56. $\displaystyle y = \tanh^{-1}\left(\frac{x}{2}\right)$

$\displaystyle y' = \frac{1}{1-(x/2)^2}\left(\frac{1}{2}\right) = \frac{2}{4-x^2}$

58. $\displaystyle y = \operatorname{sech}^{-1}(\cos 2x),\ 0 < x < \frac{\pi}{4}$

$\displaystyle y' = \frac{-1}{\cos 2x\sqrt{1-\cos^2 2x}}(-2\sin 2x) = \frac{2\sin 2x}{\cos 2x|\sin 2x|} = \frac{2}{\cos 2x} = 2\sec 2x,$

since $\sin 2x \geq 0$ for $0 < x < \pi/4$.

60. $y = (\operatorname{csch}^{-1} x)^2$

$\displaystyle y' = 2\operatorname{csch}^{-1} x\left(\frac{-1}{|x|\sqrt{1+x^2}}\right) = \frac{-2\operatorname{csch}^{-1} x}{|x|\sqrt{1+x^2}}$

62. $\displaystyle y = x\tanh^{-1} x + \ln\sqrt{1-x^2} = x\tanh^{-1} x + \frac{1}{2}\ln(1-x^2)$

$\displaystyle y' = x\left(\frac{1}{1-x^2}\right) + \tanh^{-1} x + \frac{-x}{1-x^2} = \tanh^{-1} x$

64. See page 401, Theorem 5.22.

66. Equation of tangent line through $P = (x_0, y_0)$:

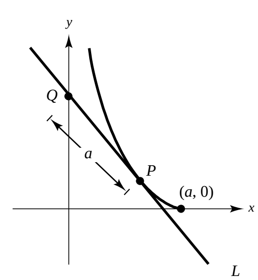
$\displaystyle y - a\operatorname{sech}^{-1}\frac{x_0}{a} + \sqrt{a^2 - x_0^2} = -\frac{\sqrt{a^2-x_0^2}}{x_0}(x - x_0)$

When $x = 0$,

$\displaystyle y = a\operatorname{sech}^{-1}\frac{x_0}{a} - \sqrt{a^2-x_0^2} + \sqrt{a^2-x_0^2} = a\operatorname{sech}^{-1}\frac{x_0}{a}.$

Hence, Q is the point $[0, a\operatorname{sech}^{-1}(x_0/a)]$.

Distance from P to Q: $d = \sqrt{x_0^2 + \left(-\sqrt{a^2-x_0^2}\right)^2} = a$



68. $\displaystyle\int \frac{x}{9-x^4}\,dx = -\frac{1}{2}\int \frac{-2x}{9-(x^2)^2}\,dx = -\frac{1}{2}\left(\frac{1}{6}\right)\ln\left|\frac{3-x^2}{3+x^2}\right| + C$

$\displaystyle = -\frac{1}{12}\ln\left|\frac{3-x^2}{3+x^2}\right| + C$

70. Let $u = x^{3/2}$, $du = \frac{3}{2}\sqrt{x}\,dx$.

$$\int \frac{\sqrt{x}}{\sqrt{1+x^3}}\,dx = \frac{2}{3}\int \frac{1}{\sqrt{1+(x^{3/2})^2}}\left(\frac{3}{2}\sqrt{x}\right)dx = \frac{2}{3}\sinh^{-1}(x^{3/2}) + C = \frac{2}{3}\ln\left(x^{3/2} + \sqrt{1+x^3}\right) + C$$

72. $$\int \frac{dx}{(x+2)\sqrt{x^2+4x+8}} = \int \frac{dx}{(x+2)\sqrt{(x+2)^2+4}}$$

$$= -\frac{1}{2}\ln\left(\frac{2+\sqrt{(x+2)^2+4}}{|x+2|}\right) + C$$

74. $$\int \frac{1}{(x+1)\sqrt{2x^2+4x+8}}\,dx = \int \frac{1}{(x+1)\sqrt{2(x+1)^2+6}}\,dx$$

$$= \frac{1}{\sqrt{2}}\int \frac{1}{(x+1)\sqrt{(x+1)^2+\left(\sqrt{3}\right)^2}}\,dx = -\frac{1}{\sqrt{6}}\ln\left(\frac{\sqrt{3}+\sqrt{(x+1)^2+3}}{x+1}\right) + C$$

76. Let $u = 2(x-1)$, $du = 2\,dx$.

$$y = \int \frac{1}{(x-1)\sqrt{-4x^2+8x-1}}\,dx = \int \frac{2}{2(x-1)\sqrt{\left(\sqrt{3}\right)^2 - [2(x-1)]^2}}\,dx = -\frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}+\sqrt{-4x^2+8x-1}}{2(x-1)}\right| + C$$

78. $$y = \int \frac{1-2x}{4x-x^2}\,dx = \int \frac{4-2x}{4x-x^2}\,dx + 3\int \frac{1}{(x-2)^2-4}\,dx$$

$$= \ln|4x-x^2| + \frac{3}{4}\ln\left|\frac{(x-2)-2}{(x-2)+2}\right| + C = \ln|4x-x^2| + \frac{3}{4}\ln\left|\frac{x-4}{x}\right| + C$$

80. $$A = \int_0^2 \tanh 2x\,dx$$

$$= \int_0^2 \frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}\,dx$$

$$= \frac{1}{2}\int_0^2 \frac{1}{e^{2x}+e^{-2x}}(2)(e^{2x}-e^{-2x})\,dx$$

$$= \left[\frac{1}{2}\ln(e^{2x}+e^{-2x})\right]_0^2$$

$$= \frac{1}{2}\ln(e^4+e^{-4}) - \frac{1}{2}\ln 2$$

$$= \ln\sqrt{\frac{e^4+e^{-4}}{2}} \approx 1.654$$

82. $$A = \int_3^5 \frac{6}{\sqrt{x^2-4}}\,dx$$

$$= \left[6\ln\left(x+\sqrt{x^2-4}\right)\right]_3^5$$

$$= 6\ln\left(5+\sqrt{21}\right) - 6\ln\left(3+\sqrt{5}\right)$$

$$= 6\ln\left(\frac{5+\sqrt{21}}{3+\sqrt{5}}\right) \approx 3.626$$

84. (a) $v(t) = -32t$

(b) $s(t) = \int v(t)\,dt = \int (-32t)\,dt = -16t^2 + C$

$s(0) = -16(0)^2 + C = 400 \implies C = 400$

$s(t) = -16t^2 + 400$

—CONTINUED—

84. —CONTINUED—

(c)
$$\frac{dv}{dt} = -32 + kv^2$$
$$\int \frac{dv}{kv^2 - 32} = \int dt$$
$$\int \frac{dv}{32 - kv^2} = -\int dt$$

Let $u = \sqrt{k}\,v$, then $du = \sqrt{k}\,dv$.

$$\frac{1}{\sqrt{k}} \cdot \frac{1}{2\sqrt{32}} \ln\left|\frac{\sqrt{32} + \sqrt{k}\,v}{\sqrt{32} - \sqrt{k}\,v}\right| = -t + C$$

Since $v(0) = 0$, $C = 0$.

$$\ln\left|\frac{\sqrt{32} + \sqrt{k}\,v}{\sqrt{32} - \sqrt{k}\,v}\right| = -2\sqrt{32k}\,t$$
$$\frac{\sqrt{32} + \sqrt{k}\,v}{\sqrt{32} - \sqrt{k}\,v} = e^{-2\sqrt{32k}\,t}$$
$$\sqrt{32} + \sqrt{k}\,v = e^{-2\sqrt{32k}\,t}\left(\sqrt{32} - \sqrt{k}\,v\right)$$
$$v\left(\sqrt{k} + \sqrt{k}\,e^{-2\sqrt{32k}\,t}\right) = \sqrt{32}\left(e^{-2\sqrt{32k}\,t} - 1\right)$$
$$v = \frac{\sqrt{32}\left(e^{-2\sqrt{32k}\,t} - 1\right)}{\sqrt{k}\left(e^{-2\sqrt{32k}\,t} + 1\right)} \cdot \frac{e^{\sqrt{32k}\,t}}{e^{\sqrt{32k}\,t}}$$
$$= \frac{\sqrt{32}}{\sqrt{k}}\left[\frac{-\left(e^{\sqrt{32k}\,t} - e^{-\sqrt{32k}\,t}\right)}{e^{\sqrt{32k}\,t} + e^{-\sqrt{32k}\,t}}\right]$$
$$= -\frac{\sqrt{32}}{\sqrt{k}}\tanh\left(\sqrt{32k}\,t\right)$$

(d) $$\lim_{t\to\infty}\left[-\frac{\sqrt{32}}{\sqrt{k}}\tanh\left(\sqrt{32k}\,t\right)\right] = -\frac{\sqrt{32}}{\sqrt{k}}.$$

The velocity is bounded by $-\sqrt{32}/\sqrt{k}$.

(e) Since $\int \tanh(ct)\,dt = (1/c)\ln\cosh(ct)$ (which can be verified by differentiation), then

$$s(t) = \int -\frac{\sqrt{32}}{\sqrt{k}}\tanh\left(\sqrt{32k}\,t\right)dt$$
$$= -\frac{\sqrt{32}}{\sqrt{k}}\frac{1}{\sqrt{32k}}\ln\left[\cosh\left(\sqrt{32k}\,t\right)\right] + C$$
$$= -\frac{1}{k}\ln\left[\cosh\left(\sqrt{32k}\,t\right)\right] + C.$$

When $t = 0$,

$$s(0) = C$$
$$= 400 \implies 400 - (1/k)\ln\left[\cosh\left(\sqrt{32k}\,t\right)\right].$$

When $k = 0.01$,

$$s_2(t) = 400 - 100\ln\left(\cosh\sqrt{0.32}\,t\right)$$
$$s_1(t) = -16t^2 + 400.$$

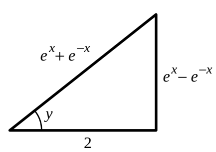
$s_1(t) = 0$ when $t = 5$ seconds.

$s_2(t) = 0$ when $t \approx 8.3$ seconds

When air resistance is not neglected, it takes approximately 3.3 more seconds to reach the ground.

86. Let $y = \arcsin(\tanh x)$. Then,

$$\sin y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \text{ and}$$
$$\tan y = \frac{e^x - e^{-x}}{2} = \sinh x.$$



Thus, $y = \arctan(\sinh x)$. Therefore,

$$\arctan(\sinh x) = \arcsin(\tanh x).$$

88.
$$y = \operatorname{sech}^{-1} x$$
$$\operatorname{sech} y = x$$
$$-(\operatorname{sech} y)(\tanh y)y' = 1$$
$$y' = \frac{-1}{(\operatorname{sech} y)(\tanh y)} = \frac{-1}{(\operatorname{sech} y)\sqrt{1 - \operatorname{sech}^2 y}} = \frac{-1}{x\sqrt{1 - x^2}}$$

90.
$$y = \sinh^{-1} x$$
$$\sinh y = x$$
$$(\cosh y)y' = 1$$
$$y' = \frac{1}{\cosh y} = \frac{1}{\sqrt{\sinh^2 y + 1}} = \frac{1}{\sqrt{x^2 + 1}}$$