

CHAPTER 5

Logarithmic, Exponential, and Other Transcendental Functions

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CHAPTER 5

Logarithmic, Exponential, and Other Transcendental Functions

Section 5.1 The Natural Logarithmic Function: Differentiation

Solutions to Odd-Numbered Exercises

1. Simpson's Rule: $n = 10$

x	0.5	1.5	2	2.5	3	3.5	4
$\int_1^x \frac{1}{t} dt$	-0.6932	0.4055	0.6932	0.9163	1.0987	1.2529	1.3865

Note: $\int_1^{0.5} \frac{1}{t} dt = -\int_{0.5}^1 \frac{1}{t} dt$

3. (a) $\ln 45 \approx 3.8067$

(b) $\int_1^{45} \frac{1}{t} dt \approx 3.8067$

5. (a) $\ln 0.8 \approx -0.2231$

(b) $\int_1^{0.8} \frac{1}{t} dt \approx -0.2231$

7. $f(x) = \ln x + 2$

Vertical shift 2 units upward

Matches (b)

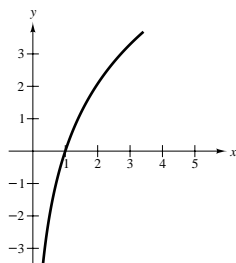
9. $f(x) = \ln(x - 1)$

Horizontal shift 1 unit to the right

Matches (a)

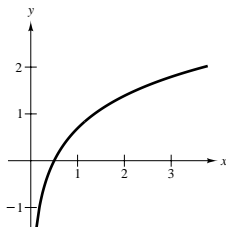
11. $f(x) = 3 \ln x$

Domain: $x > 0$



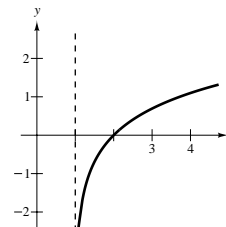
13. $f(x) = \ln 2x$

Domain: $x > 0$



15. $f(x) = \ln(x - 1)$

Domain: $x > 1$



17. (a) $\ln 6 = \ln 2 + \ln 3 \approx 1.7917$

(b) $\ln \frac{2}{3} = \ln 2 - \ln 3 \approx -0.4055$

(c) $\ln 81 = \ln 3^4 = 4 \ln 3 \approx 4.3944$

(d) $\ln \sqrt{3} = \ln 3^{1/2} = \frac{1}{2} \ln 3 \approx 0.5493$

19. $\ln \frac{2}{3} = \ln 2 - \ln 3$

21. $\ln \frac{xy}{z} = \ln x + \ln y - \ln z$

23. $\ln \sqrt[3]{a^2 + 1} = \ln(a^2 + 1)^{1/3} = \frac{1}{3} \ln(a^2 + 1)$

$$25. \ln\left(\frac{x^2 - 1}{x^3}\right)^3 = 3[\ln(x^2 - 1) - \ln x^3]$$

$$= 3[\ln(x + 1) + \ln(x - 1) - 3 \ln x]$$

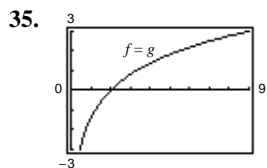
$$27. \ln z(z - 1)^2 = \ln z + \ln(z - 1)^2$$

$$= \ln z + 2 \ln(z - 1)$$

$$29. \ln(x - 2) - \ln(x + 2) = \ln \frac{x - 2}{x + 2}$$

$$31. \frac{1}{3}[2 \ln(x + 3) + \ln x - \ln(x^2 - 1)] = \frac{1}{3} \ln \frac{x(x + 3)^2}{x^2 - 1} = \ln \sqrt[3]{\frac{x(x + 3)^2}{x^2 - 1}}$$

$$33. 2 \ln 3 - \frac{1}{2} \ln(x^2 + 1) = \ln 9 - \ln \sqrt{x^2 + 1} = \ln \frac{9}{\sqrt{x^2 + 1}}$$



$$37. \lim_{x \rightarrow 3^+} \ln(x - 3) = -\infty$$

$$39. \lim_{x \rightarrow 2^-} \ln[x^2(3 - x)] = \ln 4 \approx 1.3863$$

$$41. y = \ln x^3 = 3 \ln x$$

$$y' = \frac{3}{x}$$

$$\text{At } (1, 0), y' = 3.$$

$$43. y = \ln x^2 = 2 \ln x$$

$$y' = \frac{2}{x}$$

$$\text{At } (1, 0), y' = 2.$$

$$45. g(x) = \ln x^2 = 2 \ln x$$

$$g'(x) = \frac{2}{x}$$

$$47. y = (\ln x)^4$$

$$\frac{dy}{dx} = 4(\ln x)^3 \left(\frac{1}{x}\right) = \frac{4(\ln x)^3}{x}$$

$$49. y = \ln x \sqrt{x^2 - 1} = \ln x + \frac{1}{2} \ln(x^2 - 1)$$

$$\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2} \left(\frac{2x}{x^2 - 1}\right) = \frac{2x^2 - 1}{x(x^2 - 1)}$$

$$51. f(x) = \ln \frac{x}{x^2 + 1} = \ln x - \ln(x^2 + 1)$$

$$f'(x) = \frac{1}{x} - \frac{2x}{x^2 + 1} = \frac{1 - x^2}{x(x^2 + 1)}$$

$$53. g(t) = \frac{\ln t}{t^2}$$

$$g'(t) = \frac{t^2(1/t) - 2t \ln t}{t^4} = \frac{1 - 2 \ln t}{t^3}$$

$$55. y = \ln(\ln x^2)$$

$$\frac{dy}{dx} = \frac{1}{\ln x^2} \frac{d}{dx}(\ln x^2) = \frac{(2x/x^2)}{\ln x^2} = \frac{2}{x \ln x^2} = \frac{1}{x \ln x}$$

$$57. y = \ln \sqrt{\frac{x + 1}{x - 1}} = \frac{1}{2} [\ln(x + 1) - \ln(x - 1)]$$

$$\frac{dy}{dx} = \frac{1}{2} \left[\frac{1}{x + 1} - \frac{1}{x - 1} \right] = \frac{1}{1 - x^2}$$

$$59. f(x) = \ln \frac{\sqrt{4 + x^2}}{x} = \frac{1}{2} \ln(4 + x^2) - \ln x$$

$$f'(x) = \frac{x}{4 + x^2} - \frac{1}{x} = \frac{-4}{x(x^2 + 4)}$$

$$61. y = \frac{-\sqrt{x^2+1}}{x} + \ln(x + \sqrt{x^2+1})$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{-x(x/\sqrt{x^2+1}) + \sqrt{x^2+1}}{x^2} + \left(\frac{1}{x + \sqrt{x^2+1}}\right) \left(1 + \frac{x}{\sqrt{x^2+1}}\right) \\ &= \frac{1}{x^2\sqrt{x^2+1}} + \left(\frac{1}{x + \sqrt{x^2+1}}\right) \left(\frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1}}\right) = \frac{1}{x^2\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}} = \frac{1+x^2}{x^2\sqrt{x^2+1}} = \frac{\sqrt{x^2+1}}{x^2} \end{aligned}$$

$$63. y = \ln|\sin x|$$

$$\frac{dy}{dx} = \frac{\cos x}{\sin x} = \cot x$$

$$65. y = \ln \left| \frac{\cos x}{\cos x - 1} \right|$$

$$= \ln|\cos x| - \ln|\cos x - 1|$$

$$\frac{dy}{dx} = \frac{-\sin x}{\cos x} - \frac{-\sin x}{\cos x - 1} = -\tan x + \frac{\sin x}{\cos x - 1}$$

$$67. y = \ln \left| \frac{-1 + \sin x}{2 + \sin x} \right|$$

$$= \ln|-1 + \sin x| - \ln|2 + \sin x|$$

$$\frac{dy}{dx} = \frac{\cos x}{-1 + \sin x} - \frac{\cos x}{2 + \sin x}$$

$$= \frac{3 \cos x}{(\sin x - 1)(\sin x + 2)}$$

$$69. f(x) = \sin 2x \ln x^2 = 2 \sin 2x \ln x$$

$$f'(x) = (2 \sin 2x) \left(\frac{1}{x}\right) + 4 \cos 2x \ln x$$

$$= \frac{2}{x} (\sin 2x + 2x \cos 2x \ln x)$$

$$= \frac{2}{x} (\sin 2x + x \cos 2x \ln x^2)$$

$$71. (a) y = 3x^2 - \ln x, \quad (1, 3)$$

$$\frac{dy}{dx} = 6x - \frac{1}{x}$$

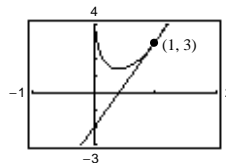
$$\text{When } x = 1, \frac{dy}{dx} = 5.$$

$$\text{Tangent line: } y - 3 = 5(x - 1)$$

$$y = 5x - 2$$

$$0 = 5x - y - 2$$

(b)



$$73. x^2 - 3 \ln y + y^2 = 10$$

$$2x - \frac{3}{y} \frac{dy}{dx} + 2y \frac{dy}{dx} = 0$$

$$2x = \frac{dy}{dx} \left(\frac{3}{y} - 2y \right)$$

$$\frac{dy}{dx} = \frac{2x}{(3/y) - 2y} = \frac{2xy}{3 - 2y^2}$$

$$75. y = 2(\ln x) + 3$$

$$y' = \frac{2}{x}$$

$$y'' = -\frac{2}{x^2}$$

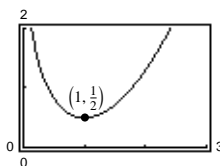
$$xy'' + y' = x \left(-\frac{2}{x^2} \right) + \frac{2}{x} = 0$$

$$77. y = \frac{x^2}{2} - \ln x$$

 Domain: $x > 0$

$$y' = x - \frac{1}{x} = \frac{(x+1)(x-1)}{x} = 0 \text{ when } x = 1.$$

$$y'' = 1 + \frac{1}{x^2} > 0$$

 Relative minimum: $(1, \frac{1}{2})$


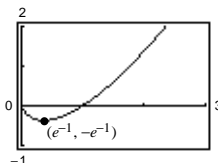
79. $y = x \ln x$

Domain: $x > 0$

$$y' = x\left(\frac{1}{x}\right) + \ln x = 1 + \ln x = 0 \text{ when } x = e^{-1}.$$

$$y'' = \frac{1}{x} > 0$$

Relative minimum: $(e^{-1}, -e^{-1})$



81. $y = \frac{x}{\ln x}$

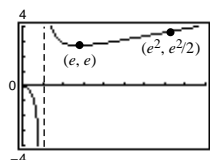
Domain: $0 < x < 1, x > 1$

$$y' = \frac{(\ln x)(1) - (x)(1/x)}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2} = 0 \text{ when } x = e.$$

$$y'' = \frac{(\ln x)^2(1/x) - (\ln x - 1)(2/x) \ln x}{(\ln x)^4} = \frac{2 - \ln x}{x(\ln x)^3} = 0 \text{ when } x = e^2.$$

Relative minimum: (e, e)

Point of inflection: $(e^2, e^2/2)$



83. $f(x) = \ln x, f(1) = 0$

$$f'(x) = \frac{1}{x}, f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2}, f''(1) = -1$$

$$P_1(x) = f(1) + f'(1)(x-1) = x-1, P_1(1) = 0$$

$$P_2(x) = f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2$$

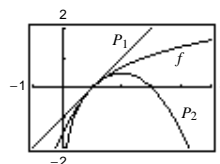
$$= (x-1) - \frac{1}{2}(x-1)^2, P_2(1) = 0$$

$$P_1'(x) = 1, P_1'(1) = 1$$

$$P_2'(x) = 1 - (x-1) = 2-x, P_2'(1) = 1$$

$$P_2''(x) = -1, P_2''(1) = -1$$

The values of f, P_1, P_2 , and their first derivatives agree at $x = 1$. The values of the second derivatives of f and P_2 agree at $x = 1$.



85. Find x such that $\ln x = -x$.

$$f(x) = (\ln x) + x = 0$$

$$f'(x) = \frac{1}{x} + 1$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n \left[\frac{1 - \ln x_n}{1 + x_n} \right]$$

n	1	2	3
x_n	0.5	0.5644	0.5671
$f(x_n)$	-0.1931	-0.0076	-0.0001

Approximate root: $x = 0.567$

87. $y = x\sqrt{x^2 - 1}$

$$\ln y = \ln x + \frac{1}{2} \ln(x^2 - 1)$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{1}{x} + \frac{x}{x^2 - 1}$$

$$\frac{dy}{dx} = y \left[\frac{2x^2 - 1}{x(x^2 - 1)} \right] = \frac{2x^2 - 1}{\sqrt{x^2 - 1}}$$

$$89. \quad y = \frac{x^2\sqrt{3x-2}}{(x-1)^2}$$

$$\ln y = 2 \ln x + \frac{1}{2} \ln(3x-2) - 2 \ln(x-1)$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{2}{x} + \frac{3}{2(3x-2)} - \frac{2}{x-1}$$

$$\frac{dy}{dx} = y \left[\frac{3x^2 - 15x + 8}{2x(3x-2)(x-1)} \right]$$

$$= \frac{3x^3 - 15x^2 + 8x}{2(x-1)^3\sqrt{3x-2}}$$

$$91. \quad y = \frac{x(x-1)^{3/2}}{\sqrt{x+1}}$$

$$\ln y = \ln x + \frac{3}{2} \ln(x-1) - \frac{1}{2} \ln(x+1)$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{1}{x} + \frac{3}{2} \left(\frac{1}{x-1} \right) - \frac{1}{2} \left(\frac{1}{x+1} \right)$$

$$\frac{dy}{dx} = \frac{y}{2} \left[\frac{2}{x} + \frac{3}{x-1} - \frac{1}{x+1} \right]$$

$$= \frac{y}{2} \left[\frac{4x^2 + 4x - 2}{x(x^2-1)} \right] = \frac{(2x^2 + 2x - 1)\sqrt{x-1}}{(x+1)^{3/2}}$$

93. Answers will vary. See Theorem 5.1 and 5.2.

95. $\ln e^x = x$ because $f(x) = \ln x$ and $g(x) = e^x$ are inverse functions.

97. (a) $f(1) \neq f(3)$

(b) $f'(x) = 1 - \frac{2}{x} = 0$ for $x = 2$.

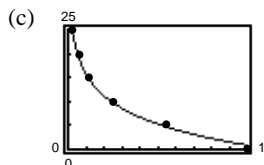
$$99. \quad \beta = 10 \log_{10} \left(\frac{I}{10^{-16}} \right) = \frac{10}{\ln 10} \ln \left(\frac{I}{10^{-16}} \right) = \frac{10}{\ln 10} [\ln I + 16 \ln 10] = 160 + 10 \log_{10} I$$

$$\beta(10^{-10}) = \frac{10}{\ln 10} [\ln 10^{-10} + 16 \ln 10] = \frac{10}{\ln 10} [-10 \ln 10 + 16 \ln 10] = \frac{10}{\ln 10} [6 \ln 10] = 60 \text{ decibels}$$

101. (a) You get an error message because $\ln h$ does not exist for $h = 0$.

(b) Reversing the data, you obtain

$$h = 0.8627 - 6.4474 \ln p.$$



(d) If $p = 0.75$, $h \approx 2.72$ km.

(e) If $h = 13$ km, $p \approx 0.15$ atmosphere.

(f) $h = 0.8627 - 6.4474 \ln p$

$$1 = -6.4474 \frac{1}{p} \frac{dp}{dh} \quad (\text{implicit differentiation})$$

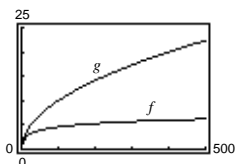
$$\frac{dp}{dh} = \frac{p}{-6.4474}$$

For $h = 5$, $p = 0.55$ and $dp/dh = -0.0853$ atmos/km.

For $h = 20$, $p = 0.06$ and $dp/dh = -0.00931$ atmos/km.

As the altitude increases, the rate of change of pressure decreases.

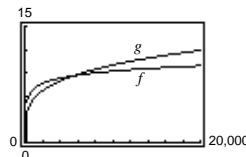
103. (a) $f(x) = \ln x$, $g(x) = \sqrt{x}$



$$f'(x) = \frac{1}{x}, \quad g'(x) = \frac{1}{2\sqrt{x}}$$

For $x > 4$, $g'(x) > f'(x)$. g is increasing at a faster rate than f for “large” values of x .

(b) $f(x) = \ln x$, $g(x) = \sqrt[4]{x}$



$$f'(x) = \frac{1}{x}, \quad g'(x) = \frac{1}{4\sqrt[3]{x^3}}$$

For $x > 256$, $g'(x) > f'(x)$. g is increasing at a faster rate than f for “large” values of x . $f(x) = \ln x$ increases very slowly for “large” values of x .

105. False

$$\ln x + \ln 25 = \ln(25x) \neq \ln(x + 25)$$

Section 5.2 The Natural Logarithmic Function: Integration

$$1. \int \frac{5}{x} dx = 5 \int \frac{1}{x} dx = 5 \ln|x| + C$$

$$3. u = x + 1, du = dx$$

$$\int \frac{1}{x+1} dx = \ln|x+1| + C$$

$$5. u = 3 - 2x, du = -2 dx$$

$$\begin{aligned} \int \frac{1}{3-2x} dx &= -\frac{1}{2} \int \frac{1}{3-2x} (-2) dx \\ &= -\frac{1}{2} \ln|3-2x| + C \end{aligned}$$

$$7. u = x^2 + 1, du = 2x dx$$

$$\begin{aligned} \int \frac{x}{x^2+1} dx &= \frac{1}{2} \int \frac{1}{x^2+1} (2x) dx \\ &= \frac{1}{2} \ln(x^2+1) + C \\ &= \ln\sqrt{x^2+1} + C \end{aligned}$$

$$\begin{aligned} 9. \int \frac{x^2-4}{x} dx &= \int \left(x - \frac{4}{x}\right) dx \\ &= \frac{x^2}{2} - 4 \ln|x| + C \end{aligned}$$

$$11. u = x^3 + 3x^2 + 9x, du = 3(x^2 + 2x + 3) dx$$

$$\begin{aligned} \int \frac{x^2+2x+3}{x^3+3x^2+9x} dx &= \frac{1}{3} \int \frac{3(x^2+2x+3)}{x^3+3x^2+9x} dx \\ &= \frac{1}{3} \ln|x^3+3x^2+9x| + C \end{aligned}$$

$$\begin{aligned} 13. \int \frac{x^2-3x+2}{x+1} dx &= \int \left(x - 4 + \frac{6}{x+1}\right) dx \\ &= \frac{x^2}{2} - 4x + 6 \ln|x+1| + C \end{aligned}$$

$$\begin{aligned} 15. \int \frac{x^3-3x^2+5}{x-3} dx &= \int \left(x^2 + \frac{5}{x-3}\right) dx \\ &= \frac{x^3}{3} + 5 \ln|x-3| + C \end{aligned}$$

$$\begin{aligned} 17. \int \frac{x^4+x-4}{x^2+2} dx &= \int \left(x^2 - 2 + \frac{x}{x^2+2}\right) dx \\ &= \frac{x^3}{3} - 2x + \frac{1}{2} \ln(x^2+2) + C \end{aligned}$$

$$19. u = \ln x, du = \frac{1}{x} dx$$

$$\int \frac{(\ln x)^2}{x} dx = \frac{1}{3} (\ln x)^3 + C$$

$$21. u = x + 1, du = dx$$

$$\begin{aligned} \int \frac{1}{\sqrt{x+1}} dx &= \int (x+1)^{-1/2} dx \\ &= 2(x+1)^{1/2} + C \\ &= 2\sqrt{x+1} + C \end{aligned}$$

$$\begin{aligned} 23. \int \frac{2x}{(x-1)^2} dx &= \int \frac{2x-2+2}{(x-1)^2} dx \\ &= \int \frac{2(x-1)}{(x-1)^2} dx + 2 \int \frac{1}{(x-1)^2} dx \\ &= 2 \int \frac{1}{x-1} dx + 2 \int \frac{1}{(x-1)^2} dx \\ &= 2 \ln|x-1| - \frac{2}{(x-1)} + C \end{aligned}$$

$$25. u = 1 + \sqrt{2x}, du = \frac{1}{\sqrt{2x}} dx \Rightarrow (u - 1) du = dx$$

$$\begin{aligned} \int \frac{1}{1 + \sqrt{2x}} dx &= \int \frac{(u - 1)}{u} du = \int \left(u - \frac{1}{u}\right) du \\ &= u - \ln|u| + C_1 \\ &= (1 + \sqrt{2x}) - \ln|1 + \sqrt{2x}| + C_1 \\ &= \sqrt{2x} - \ln(1 + \sqrt{2x}) + C \end{aligned}$$

where $C = C_1 + 1$.

$$27. u = \sqrt{x} - 3, du = \frac{1}{2\sqrt{x}} dx \Rightarrow 2(u + 3) du = dx$$

$$\begin{aligned} \int \frac{\sqrt{x}}{\sqrt{x} - 3} dx &= 2 \int \frac{(u + 3)^2}{u} du = 2 \int \frac{u^2 + 6u + 9}{u} du = 2 \int \left(u + 6 + \frac{9}{u}\right) du \\ &= 2 \left[\frac{u^2}{2} + 6u + 9 \ln|u| \right] + C_1 = u^2 + 12u + 18 \ln|u| + C_1 \\ &= (\sqrt{x} - 3)^2 + 12(\sqrt{x} - 3) + 18 \ln|\sqrt{x} - 3| + C_1 \\ &= x + 6\sqrt{x} + 18 \ln|\sqrt{x} - 3| + C \text{ where } C = C_1 - 27. \end{aligned}$$

$$29. \int \frac{\cos \theta}{\sin \theta} d\theta = \ln|\sin \theta| + C$$

$$(u = \sin \theta, du = \cos \theta d\theta)$$

$$31. \int \csc 2x dx = \frac{1}{2} \int (\csc 2x)(2) dx$$

$$= -\frac{1}{2} \ln|\csc 2x + \cot 2x| + C$$

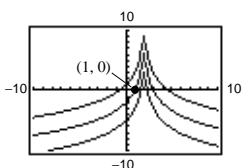
$$33. \int \frac{\cos t}{1 + \sin t} dt = \ln|1 + \sin t| + C$$

$$35. \int \frac{\sec x \tan x}{\sec x - 1} dx = \ln|\sec x - 1| + C$$

$$37. y = \int \frac{3}{2 - x} dx$$

$$= -3 \int \frac{1}{x - 2} dx$$

$$= -3 \ln|x - 2| + C$$



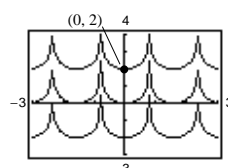
$$(1, 0): 0 = -3 \ln|1 - 2| + C \Rightarrow C = 0$$

$$y = -3 \ln|x - 2|$$

$$39. s = \int \tan(2\theta) d\theta$$

$$= \frac{1}{2} \int \tan(2\theta)(2 d\theta)$$

$$= -\frac{1}{2} \ln|\cos 2\theta| + C$$

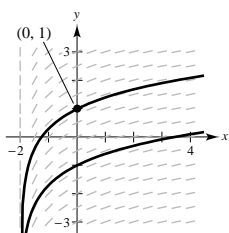


$$(0, 2): 2 = -\frac{1}{2} \ln|\cos(0)| + C \Rightarrow C = 2$$

$$s = -\frac{1}{2} \ln|\cos 2\theta| + 2$$

$$41. \frac{dy}{dx} = \frac{1}{x + 2}, (0, 1)$$

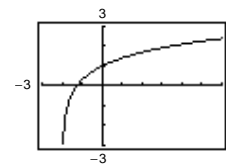
(a)



$$(b) y = \int \frac{1}{x + 2} dx = \ln|x + 2| + C$$

$$y(0) = 1 \Rightarrow 1 = \ln 2 + C \Rightarrow C = 1 - \ln 2$$

$$\text{Hence, } y = \ln|x + 2| + 1 - \ln 2 = \ln \left| \frac{x + 2}{2} \right| + 1.$$



$$43. \int_0^4 \frac{5}{3x+1} dx = \left[\frac{5}{3} \ln|3x+1| \right]_0^4$$

$$= \frac{5}{3} \ln 13 \approx 4.275$$

$$45. u = 1 + \ln x, du = \frac{1}{x} dx$$

$$\int_1^e \frac{(1 + \ln x)^2}{x} dx = \left[\frac{1}{3} (1 + \ln x)^3 \right]_1^e = \frac{7}{3}$$

$$47. \int_0^2 \frac{x^2 - 2}{x + 1} dx = \int_0^2 \left(x - 1 - \frac{1}{x+1} \right) dx$$

$$= \left[\frac{1}{2} x^2 - x - \ln|x+1| \right]_0^2 = -\ln 3$$

$$49. \int_1^2 \frac{1 - \cos \theta}{\theta - \sin \theta} d\theta = \left[\ln|\theta - \sin \theta| \right]_1^2$$

$$= \ln \left| \frac{2 - \sin 2}{1 - \sin 1} \right| \approx 1.929$$

$$51. -\ln|\cos x| + C = \ln \left| \frac{1}{\cos x} \right| + C = \ln|\sec x| + C$$

$$53. \ln|\sec x + \tan x| + C = \ln \left| \frac{(\sec x + \tan x)(\sec x - \tan x)}{(\sec x - \tan x)} \right| + C = \ln \left| \frac{\sec^2 x - \tan^2 x}{\sec x - \tan x} \right| + C$$

$$= \ln \left| \frac{1}{\sec x - \tan x} \right| + C = -\ln|\sec x - \tan x| + C$$

$$55. \int \frac{1}{1 + \sqrt{x}} dx = 2(1 + \sqrt{x}) - 2 \ln(1 + \sqrt{x}) + C_1$$

$$= 2[\sqrt{x} - \ln(1 + \sqrt{x})] + C \text{ where } C = C_1 + 2.$$

$$57. \int \cos(1-x) dx = -\sin(1-x) + C$$

$$59. \int_{\pi/4}^{\pi/2} (\csc x - \sin x) dx = \left[-\ln|\csc x + \cot x| + \cos x \right]_{\pi/4}^{\pi/2} = \ln(\sqrt{2} + 1) - \frac{\sqrt{2}}{2} \approx 0.174$$

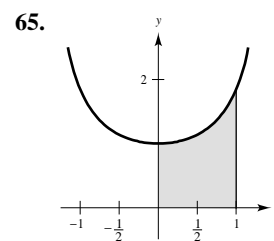
Note: In Exercises 61 and 63, you can use the Second Fundamental Theorem of Calculus or integrate the function.

$$61. F(x) = \int_1^x \frac{1}{t} dt$$

$$F'(x) = \frac{1}{x}$$

$$63. F(x) = \int_x^{3x} \frac{1}{t} dt = \int_1^{3x} \frac{1}{t} dt - \int_1^x \frac{1}{t} dt$$

$$F'(x) = \frac{3}{3x} - \frac{1}{x} = 0$$

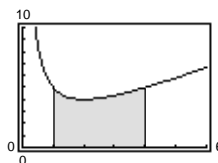


$A \approx 1.25$
Matches (d)

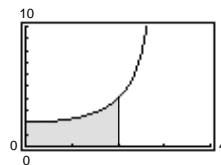
$$67. A = \int_1^4 \frac{x^2 + 4}{x} dx = \int_1^4 \left(x + \frac{4}{x} \right) dx$$

$$= \left[\frac{x^2}{2} + 4 \ln x \right]_1^4 = (8 + 4 \ln 4) - \frac{1}{2}$$

$$= \frac{15}{2} + 8 \ln 2 \approx 13.045 \text{ square units}$$



$$\begin{aligned}
 69. \int_0^2 2 \sec \frac{\pi x}{6} dx &= \frac{12}{\pi} \int_0^2 \sec \left(\frac{\pi x}{6} \right) \frac{\pi}{6} dx \\
 &= \left[\frac{12}{\pi} \ln \left| \sec \frac{\pi x}{6} + \tan \frac{\pi x}{6} \right| \right]_0^2 \\
 &= \frac{12}{\pi} \ln \left| \sec \frac{\pi}{3} + \tan \frac{\pi}{3} \right| - \frac{12}{\pi} \ln |1 + 0| \\
 &= \frac{12}{\pi} \ln(2 + \sqrt{3}) \approx 5.03041
 \end{aligned}$$



71. Power Rule

 73. Substitution: ($u = x^2 + 4$)
and Log Rule

75. Divide the polynomials:

$$\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}$$

$$\begin{aligned}
 77. \text{Average value} &= \frac{1}{4-2} \int_2^4 \frac{8}{x^2} dx = 4 \int_2^4 x^{-2} dx \\
 &= \left[-4 \frac{1}{x} \right]_2^4 \\
 &= -4 \left(\frac{1}{4} - \frac{1}{2} \right) = 1
 \end{aligned}$$

$$\begin{aligned}
 79. \text{Average value} &= \frac{1}{e-1} \int_1^e \frac{\ln x}{x} dx = \frac{1}{e-1} \left[\frac{(\ln x)^2}{2} \right]_1^e \\
 &= \frac{1}{e-1} \left(\frac{1}{2} \right) \\
 &= \frac{1}{2e-2} \approx 0.291
 \end{aligned}$$

$$81. P(t) = \int \frac{3000}{1+0.25t} dt = (3000)(4) \int \frac{0.25}{1+0.25t} dt = 12,000 \ln|1+0.25t| + C$$

$$P(0) = 12,000 \ln|1+0.25(0)| + C = 1000$$

$$C = 1000$$

$$P(t) = 12,000 \ln|1+0.25t| + 1000 = 1000[12 \ln|1+0.25t| + 1]$$

$$P(3) = 1000[12(\ln 1.75) + 1] \approx 7715$$

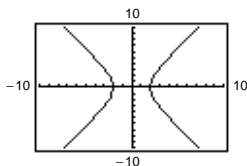
$$83. \frac{1}{50-40} \int_{40}^{50} \frac{90,000}{400+3x} dx = \left[3000 \ln|400+3x| \right]_{40}^{50} \approx \$168.27$$

$$85. (a) 2x^2 - y^2 = 8$$

$$y^2 = 2x^2 - 8$$

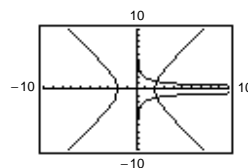
$$y_1 = \sqrt{2x^2 - 8}$$

$$y_2 = -\sqrt{2x^2 - 8}$$



$$(b) y^2 = e^{-\int(1/x)dx} = e^{-\ln x + C} = e^{\ln(1/x)}(e^C) = \frac{1}{x}k$$

$$\text{Let } k = 4 \text{ and graph } y^2 = \frac{4}{x} \quad \left(\begin{array}{l} y_1 = 2/\sqrt{x} \\ y_2 = -2/\sqrt{x} \end{array} \right)$$



$$(c) \text{ In part (a), } 2x^2 - y^2 = 8$$

$$4x - 2yy' = 0$$

$$y' = \frac{2x}{y}$$

$$\text{In part (b), } y^2 = \frac{4}{x} = 4x^{-1}$$

$$2yy' = \frac{-4}{x^2}$$

$$y' = \frac{-2}{yx^2} = \frac{-2y}{y^2 x^2} = \frac{-2y}{4x} = \frac{-y}{2x}$$

Using a graphing utility the graphs intersect at (2.214, 1.344). The slopes are 3.295 and $-0.304 = (-1)/3.295$, respectively.

87. False

$$\frac{1}{2}(\ln x) = \ln(x^{1/2}) \neq (\ln x)^{1/2}$$

89. True

$$\begin{aligned} \int \frac{1}{x} dx &= \ln|x| + C_1 \\ &= \ln|x| + \ln|C| = \ln|Cx|, C \neq 0 \end{aligned}$$

Section 5.3 Inverse Functions

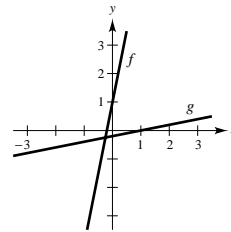
1. (a) $f(x) = 5x + 1$

$$g(x) = \frac{x-1}{5}$$

$$f(g(x)) = f\left(\frac{x-1}{5}\right) = 5\left(\frac{x-1}{5}\right) + 1 = x$$

$$g(f(x)) = g(5x+1) = \frac{(5x+1)-1}{5} = x$$

(b)



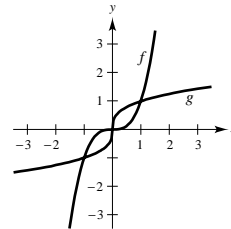
3. (a) $f(x) = x^3$

$$g(x) = \sqrt[3]{x}$$

$$f(g(x)) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x$$

$$g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x$$

(b)



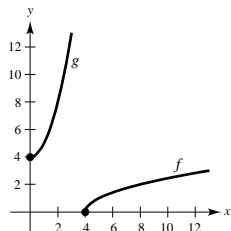
5. (a) $f(x) = \sqrt{x-4}$

$$g(x) = x^2 + 4, x \geq 0$$

$$\begin{aligned} f(g(x)) &= f(x^2 + 4) \\ &= \sqrt{(x^2 + 4) - 4} = \sqrt{x^2} = x \end{aligned}$$

$$\begin{aligned} g(f(x)) &= g(\sqrt{x-4}) \\ &= (\sqrt{x-4})^2 + 4 = x - 4 + 4 = x \end{aligned}$$

(b)



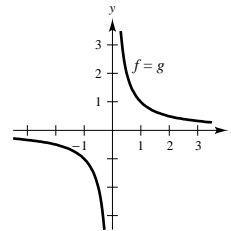
7. (a) $f(x) = \frac{1}{x}$

$$g(x) = \frac{1}{x}$$

$$f(g(x)) = \frac{1}{1/x} = x$$

$$g(f(x)) = \frac{1}{1/x} = x$$

(b)

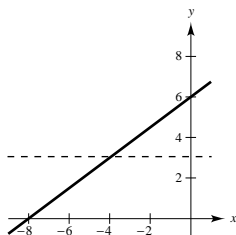


9. Matches (c)

11. Matches (a)

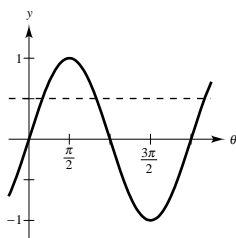
13. $f(x) = \frac{3}{4}x + 6$

One-to-one; has an inverse



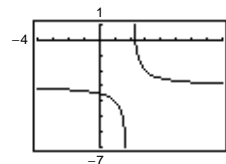
15. $f(\theta) = \sin \theta$

Not one-to-one; does not have an inverse



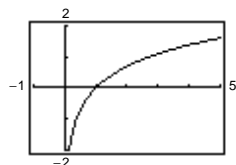
17. $h(s) = \frac{1}{s-2} - 3$

One-to-one; has an inverse



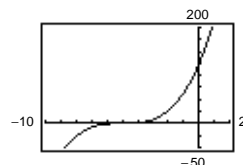
19. $f(x) = \ln x$

One-to-one; has an inverse



21. $g(x) = (x + 5)^3$

One-to-one; has an inverse



23. $f(x) = (x + a)^3 + b$

$$f'(x) = 3(x + a)^2 \geq 0 \text{ for all } x.$$

f is increasing on $(-\infty, \infty)$. Therefore, f is strictly monotonic and has an inverse.

25. $f(x) = \frac{x^4}{4} - 2x^2$

$$f'(x) = x^3 - 4x = 0 \text{ when } x = 0, 2, -2.$$

f is not strictly monotonic on $(-\infty, \infty)$. Therefore, f does not have an inverse.

27. $f(x) = 2 - x - x^3$

$$f'(x) = -1 - 3x^2 < 0 \text{ for all } x.$$

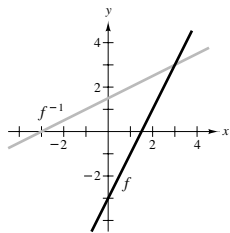
f is decreasing on $(-\infty, \infty)$. Therefore, f is strictly monotonic and has an inverse.

29. $f(x) = 2x - 3 = y$

$$x = \frac{y + 3}{2}$$

$$y = \frac{x + 3}{2}$$

$$f^{-1}(x) = \frac{x + 3}{2}$$

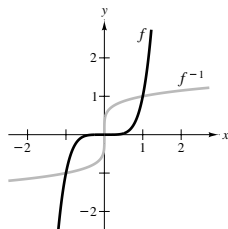


31. $f(x) = x^5 = y$

$$x = \sqrt[5]{y}$$

$$y = \sqrt[5]{x}$$

$$f^{-1}(x) = \sqrt[5]{x} = x^{1/5}$$

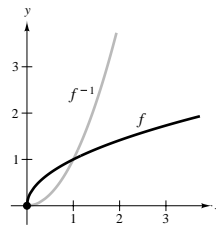


33. $f(x) = \sqrt{x} = y$

$$x = y^2$$

$$y = x^2$$

$$f^{-1}(x) = x^2, \quad x \geq 0$$

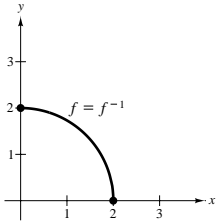


$$35. \quad f(x) = \sqrt{4-x^2} = y, \quad 0 \leq x \leq 2$$

$$x = \sqrt{4-y^2}$$

$$y = \sqrt{4-x^2}$$

$$f^{-1}(x) = \sqrt{4-x^2}, \quad 0 \leq x \leq 2$$

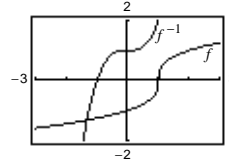


$$37. \quad f(x) = \sqrt[3]{x-1} = y$$

$$x = y^3 + 1$$

$$y = x^3 + 1$$

$$f^{-1}(x) = x^3 + 1$$



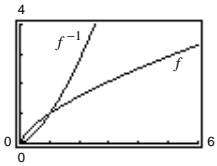
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

$$39. \quad f(x) = x^{2/3} = y, \quad x \geq 0$$

$$x = y^{3/2}$$

$$y = x^{3/2}$$

$$f^{-1}(x) = x^{3/2}, \quad x \geq 0$$



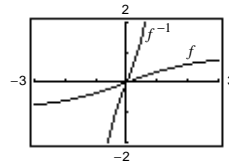
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

$$41. \quad f(x) = \frac{x}{\sqrt{x^2+7}} = y$$

$$x = \frac{\sqrt{7}y}{\sqrt{1-y^2}}$$

$$y = \frac{\sqrt{7}x}{\sqrt{1-x^2}}$$

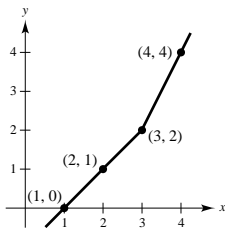
$$f^{-1}(x) = \frac{\sqrt{7}x}{\sqrt{1-x^2}}, \quad -1 < x < 1$$



The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

43.

x	1	2	3	4
$f^{-1}(x)$	0	1	2	4



45. (a) Let x be the number of pounds of the commodity costing 1.25 per pound. Since there are 50 pounds total, the amount of the second commodity is $50 - x$. The total cost is

$$y = 1.25x + 1.60(50 - x)$$

$$= -0.35x + 80 \quad 0 \leq x \leq 50.$$

- (b) We find the inverse of the original function:

$$y = -0.35x + 80$$

$$0.35x = 80 - y$$

$$x = \frac{100}{35}(80 - y)$$

$$\text{Inverse: } y = \frac{100}{35}(80 - x) = \frac{20}{7}(80 - x).$$

x represents cost and y represents pounds.

- (c) Domain of inverse is $62.5 \leq x \leq 80$.

- (d) If $x = 73$ in the inverse function,
 $y = \frac{100}{35}(80 - 73) = \frac{100}{5} = 20$ pounds.

47. $f(x) = (x - 4)^2$ on $[4, \infty)$

$$f'(x) = 2(x - 4) > 0 \text{ on } (4, \infty)$$

f is increasing on $[4, \infty)$. Therefore, f is strictly monotonic and has an inverse.

51. $f(x) = \cos x$ on $[0, \pi]$

$$f'(x) = -\sin x < 0 \text{ on } (0, \pi)$$

f is decreasing on $[0, \pi]$. Therefore, f is strictly monotonic and has an inverse.

53. $f(x) = \frac{x}{x^2 - 4} = y$ on $(-2, 2)$

$$x^2y - 4y = x$$

$$x^2y - x - 4y = 0$$

$$a = y, b = -1, c = -4y$$

$$x = \frac{1 \pm \sqrt{1 - 4(y)(-4y)}}{2y} = \frac{1 \pm \sqrt{1 + 16y^2}}{2y}$$

$$y = f^{-1}(x) = \begin{cases} (1 - \sqrt{1 + 16x^2})/2x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

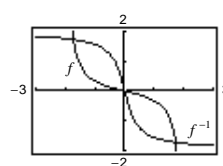
49. $f(x) = \frac{4}{x^2}$ on $(0, \infty)$

$$f'(x) = -\frac{8}{x^3} < 0 \text{ on } (0, \infty)$$

f is decreasing on $(0, \infty)$. Therefore, f is strictly monotonic and has an inverse.

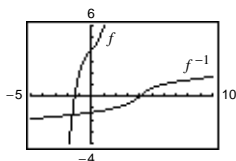
Domain: all x

Range: $-2 < y < 2$



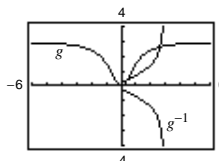
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

55. (a), (b)



(c) Yes, f is one-to-one and has an inverse. The inverse relation is an inverse function.

57. (a), (b)



(c) g is not one-to-one and does not have an inverse. The inverse relation is not an inverse function.

59. $f(x) = \sqrt{x - 2}$, Domain: $x \geq 2$

$$f'(x) = \frac{1}{2\sqrt{x - 2}} > 0 \text{ for } x > 2.$$

f is one-to-one; has an inverse

$$\sqrt{x - 2} = y$$

$$x - 2 = y^2$$

$$x = y^2 + 2$$

$$y = \sqrt{x - 2}$$

$$f^{-1}(x) = x^2 + 2, x \geq 0$$

61. $f(x) = |x - 2|, x \leq 2$

$$= -(x - 2)$$

$$= 2 - x$$

f is one-to-one; has an inverse

$$2 - x = y$$

$$2 - y = x$$

$$f^{-1}(x) = 2 - x, x \geq 0$$

63. $f(x) = (x - 3)^2$ is one-to-one for $x \geq 3$.

$$(x - 3)^2 = y$$

$$x - 3 = \sqrt{y}$$

$$x = \sqrt{y} + 3$$

$$y = \sqrt{x - 3}$$

$$f^{-1}(x) = \sqrt{x} + 3, x \geq 0$$

(Answer is not unique)

65. $f(x) = |x + 3|$ is one-to-one for $x \geq -3$.

$$x + 3 = y$$

$$x = y - 3$$

$$y = x + 3$$

$$f^{-1}(x) = x - 3, x \geq 0$$

(Answer is not unique)

67. Yes, the volume is an increasing function, and hence one-to-one. The inverse function gives the time t corresponding to the volume V .

$$71. \quad f(x) = x^3 + 2x - 1, f(1) = 2 = a$$

$$f'(x) = 3x^2 + 2$$

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = \frac{1}{3(1)^2 + 2} = \frac{1}{5}$$

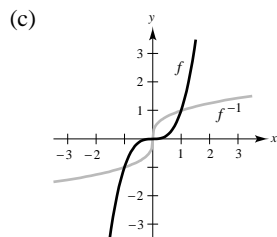
$$75. \quad f(x) = x^3 - \frac{4}{x}, f(2) = 6 = a$$

$$f'(x) = 3x^2 + \frac{4}{x^2}$$

$$(f^{-1})'(6) = \frac{1}{f'(f^{-1}(6))} = \frac{1}{f'(2)} = \frac{1}{3(2)^2 + (4/2^2)} = \frac{1}{13}$$

77. (a) Domain $f = \text{Domain } f^{-1} = (-\infty, \infty)$

(b) Range $f = \text{Range } f^{-1} = (-\infty, \infty)$



(d)

$$f(x) = x^3, \left(\frac{1}{2}, \frac{1}{8}\right)$$

$$f'(x) = 3x^2$$

$$f'\left(\frac{1}{2}\right) = \frac{3}{4}$$

$$f^{-1}(x) = \sqrt[3]{x}, \left(\frac{1}{8}, \frac{1}{2}\right)$$

$$(f^{-1})'(x) = \frac{1}{3\sqrt[3]{x^2}}$$

$$(f^{-1})'\left(\frac{1}{8}\right) = \frac{4}{3}$$

69. No, $C(t)$ is not one-to-one because long distance costs are step functions. A call lasting 2.1 minutes costs the same as one lasting 2.2 minutes.

$$73. \quad f(x) = \sin x, f\left(\frac{\pi}{6}\right) = \frac{1}{2} = a$$

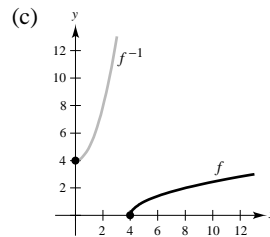
$$f'(x) = \cos x$$

$$(f^{-1})'\left(\frac{1}{2}\right) = \frac{1}{f'(f^{-1}(1/2))} = \frac{1}{f'(\pi/6)} = \frac{1}{\cos(\pi/6)}$$

$$= \frac{1}{\sqrt{3}/2} = \frac{2\sqrt{3}}{3}$$

79. (a) Domain $f = [4, \infty)$, Domain $f^{-1} = [0, \infty)$

(b) Range $f = [0, \infty)$, Range $f^{-1} = [4, \infty)$



(d)

$$f(x) = \sqrt{x-4}, (5, 1)$$

$$f'(x) = \frac{1}{2\sqrt{x-4}}$$

$$f'(5) = \frac{1}{2}$$

$$f^{-1}(x) = x^2 + 4, (1, 5)$$

$$(f^{-1})'(x) = 2x$$

$$(f^{-1})'(1) = 2$$

81. $x = y^3 - 7y^2 + 2$

$$1 = 3y^2 \frac{dy}{dx} - 14y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{3y^2 - 14y}. \text{ At } (-4, 1), \frac{dy}{dx} = \frac{1}{3 - 14} = \frac{-1}{11}.$$

Alternate solution: let $f(x) = x^3 - 7x^2 + 2$.Then $f'(x) = 3x^2 - 14x$ and $f'(1) = -11$.

Hence, $\frac{dy}{dx} = \frac{1}{-11} = \frac{-1}{11}$.

In Exercises 83 and 85, use the following.

$$f(x) = \frac{1}{8}x - 3 \text{ and } g(x) = x^3$$

$$f^{-1}(x) = 8(x + 3) \text{ and } g^{-1}(x) = \sqrt[3]{x}$$

83. $(f^{-1} \circ g^{-1})(1) = f^{-1}(g^{-1}(1)) = f^{-1}(1) = 32$

85. $(f^{-1} \circ f^{-1})(6) = f^{-1}(f^{-1}(6)) = f^{-1}(72) = 600$

In Exercises 87 and 89, use the following.

$$f(x) = x + 4 \text{ and } g(x) = 2x - 5$$

$$f^{-1}(x) = x - 4 \text{ and } g^{-1}(x) = \frac{x + 5}{2}$$

87.
$$\begin{aligned} (g^{-1} \circ f^{-1})(x) &= g^{-1}(f^{-1}(x)) \\ &= g^{-1}(x - 4) \\ &= \frac{(x - 4) + 5}{2} \\ &= \frac{x + 1}{2} \end{aligned}$$

89.
$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f(2x - 5) \\ &= (2x - 5) + 4 \\ &= 2x - 1 \end{aligned}$$

Hence, $(f \circ g)^{-1}(x) = \frac{x + 1}{2}$

(Note: $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$)

91. Answers will vary. See page 335 and Example 3.

93. $y = x^2$ on $(-\infty, \infty)$ does not have an inverse.95. f is not one-to-one because many different x -values yield the same y -value.

Example: $f(0) = f(\pi) = 0$

Not continuous at $\frac{(2n-1)\pi}{2}$, where n is an integer97. Let $(f \circ g)(x) = y$ then $x = (f \circ g)^{-1}(y)$. Also,

$$(f \circ g)(x) = y$$

$$f(g(x)) = y$$

$$g(x) = f^{-1}(y)$$

$$x = g^{-1}(f^{-1}(y))$$

$$= (g^{-1} \circ f^{-1})(y)$$

Since f and g are one-to-one functions,
 $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.99. Suppose $g(x)$ and $h(x)$ are both inverses of $f(x)$. Then the graph of $f(x)$ contains the point (a, b) if and only if the graphs of $g(x)$ and $h(x)$ contain the point (b, a) . Since the graphs of $g(x)$ and $h(x)$ are the same, $g(x) = h(x)$. Therefore, the inverse of $f(x)$ is unique.

101. False

$$\text{Let } f(x) = x^2.$$

105. Not true

$$\text{Let } f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 1 - x, & 1 < x \leq 2 \end{cases}$$

 f is one-to-one, but not strictly monotonic.

103. True

$$107. \quad f(x) = \int_2^x \frac{dt}{\sqrt{1+t^4}}, \quad f(2) = 0$$

$$f'(x) = \frac{1}{\sqrt{1+x^4}}$$

$$(f^{-1})'(0) = \frac{1}{f'(2)} = \frac{1}{1/\sqrt{17}} = \sqrt{17}$$

Section 5.4 Exponential Functions: Differentiation and Integration

1. $e^0 = 1$

$\ln 1 = 0$

3. $\ln 2 = 0.6931$

$e^{0.6931\dots} = 2$

5. $e^{\ln x} = x$

$x = 4$

7. $e^x = 12$

$x = \ln 12 \approx 2.485$

9. $9 - 2e^x = 7$

$2e^x = 7$

$e^x = 1$

$x = 0$

11. $50e^{-x} = 30$

$e^{-x} = \frac{3}{5}$

$-x = \ln\left(\frac{3}{5}\right)$

$x = \ln\left(\frac{5}{3}\right) \approx 0.511$

13. $\ln x = 2$

$x = e^2 \approx 7.3891$

15. $\ln(x - 3) = 2$

$x - 3 = e^2$

$x = 3 + e^2 \approx 10.389$

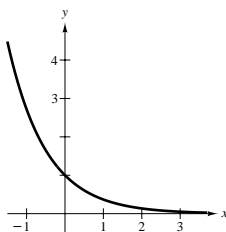
17. $\ln\sqrt{x+2} = 1$

$\sqrt{x+2} = e^1 = e$

$x + 2 = e^2$

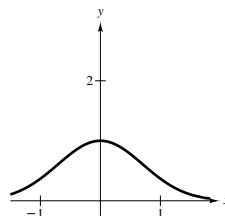
$x = e^2 - 2 \approx 5.389$

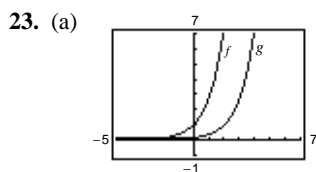
19. $y = e^{-x}$



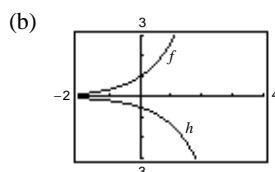
21. $y = e^{-x^2}$

Symmetric with respect to the y-axis

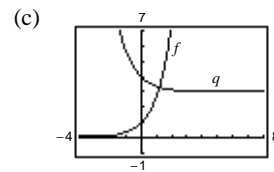
 Horizontal asymptote: $y = 0$




Horizontal shift 2 units to the right



A reflection in the x -axis and a vertical shrink



Vertical shift 3 units upward and a reflection in the y -axis

25. $y = Ce^{ax}$

Horizontal asymptote: $y = 0$

Matches (c)

27. $y = C(1 - e^{-ax})$

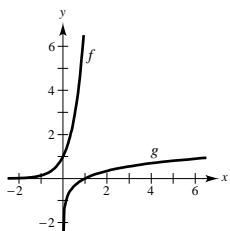
Vertical shift C units

Reflection in both the x - and y -axes

Matches (a)

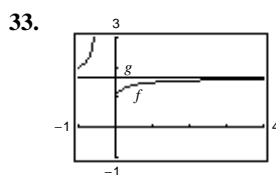
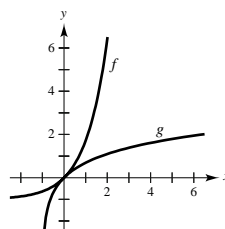
29. $f(x) = e^{2x}$

$$g(x) = \ln\sqrt{x} = \frac{1}{2}\ln x$$



31. $f(x) = e^x - 1$

$$g(x) = \ln(x + 1)$$



As $x \rightarrow \infty$, the graph of f approaches the graph of g .

$$\lim_{x \rightarrow \infty} \left(1 + \frac{0.5}{x}\right)^x = e^{0.5}$$

35. $\left(1 + \frac{1}{1,000,000}\right)^{1,000,000} \approx 2.718280469$

$$e \approx 2.718281828$$

$$e > \left(1 + \frac{1}{1,000,000}\right)^{1,000,000}$$

37. (a) $y = e^{3x}$

$$y' = 3e^{3x}$$

At $(0, 1)$, $y' = 3$.

(b) $y = e^{-3x}$

$$y' = -3e^{-3x}$$

At $(0, 1)$, $y' = -3$.

39. $f(x) = e^{2x}$

$$f'(x) = 2e^{2x}$$

41. $f(x) = e^{-2x+x^2}$

$$\frac{dy}{dx} = 2(x-1)e^{-2x+x^2}$$

43. $y = e^{\sqrt{x}}$

$$\frac{dy}{dx} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

45. $g(t) = (e^{-t} + e^t)^3$

$$g'(t) = 3(e^{-t} + e^t)^2(e^t - e^{-t})$$

47. $y = \ln e^{x^2} = x^2$

$$\frac{dy}{dx} = 2x$$

49. $y = \ln(1 + e^{2x})$

$$\frac{dy}{dx} = \frac{2e^{2x}}{1 + e^{2x}}$$

$$51. y = \frac{2}{e^x + e^{-x}} = 2(e^x + e^{-x})^{-1}$$

$$\begin{aligned} \frac{dy}{dx} &= -2(e^x + e^{-x})^{-2}(e^x - e^{-x}) \\ &= \frac{-2(e^x - e^{-x})}{(e^x + e^{-x})^2} \end{aligned}$$

$$55. f(x) = e^{-x} \ln x$$

$$f'(x) = e^{-x} \left(\frac{1}{x} \right) - e^{-x} \ln x = e^{-x} \left(\frac{1}{x} - \ln x \right)$$

$$59. \quad xe^y - 10x + 3y = 0$$

$$xe^y \frac{dy}{dx} + e^y - 10 + 3 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} (xe^y + 3) = 10 - e^y$$

$$\frac{dy}{dx} = \frac{10 - e^y}{xe^y + 3}$$

$$63. \quad y = e^x (\cos \sqrt{2}x + \sin \sqrt{2}x)$$

$$y' = e^x (-\sqrt{2} \sin \sqrt{2}x + \sqrt{2} \cos \sqrt{2}x) + e^x (\cos \sqrt{2}x + \sin \sqrt{2}x)$$

$$= e^x [(1 + \sqrt{2}) \cos \sqrt{2}x + (1 - \sqrt{2}) \sin \sqrt{2}x]$$

$$y'' = e^x [-(\sqrt{2} + 2) \sin \sqrt{2}x + (\sqrt{2} - 2) \cos \sqrt{2}x] + e^x [(1 + \sqrt{2}) \cos \sqrt{2}x + (1 - \sqrt{2}) \sin \sqrt{2}x]$$

$$= e^x [(-1 - 2\sqrt{2}) \sin \sqrt{2}x + (-1 + 2\sqrt{2}) \cos \sqrt{2}x]$$

$$-2y' + 3y = -2e^x [(1 + \sqrt{2}) \cos \sqrt{2}x + (1 - \sqrt{2}) \sin \sqrt{2}x] + 3e^x [\cos \sqrt{2}x + \sin \sqrt{2}x]$$

$$= e^x [(1 - 2\sqrt{2}) \cos \sqrt{2}x + (1 + 2\sqrt{2}) \sin \sqrt{2}x] = -y''$$

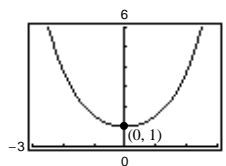
Therefore, $-2y' + 3y = -y'' \Rightarrow y'' - 2y' + 3y = 0$.

$$65. f(x) = \frac{e^x + e^{-x}}{2}$$

$$f'(x) = \frac{e^x - e^{-x}}{2} = 0 \text{ when } x = 0.$$

$$f''(x) = \frac{e^x + e^{-x}}{2} > 0$$

Relative minimum: $(0, 1)$



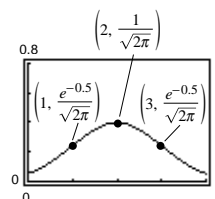
$$67. g(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-2)^2/2}$$

$$g'(x) = \frac{-1}{\sqrt{2\pi}} (x-2) e^{-(x-2)^2/2}$$

$$g''(x) = \frac{1}{\sqrt{2\pi}} (x-1)(x-3) e^{-(x-2)^2/2}$$

Relative maximum: $\left(2, \frac{1}{\sqrt{2\pi}}\right) \approx (2, 0.399)$

Points of inflection: $\left(1, \frac{1}{\sqrt{2\pi}} e^{-1/2}\right), \left(3, \frac{1}{\sqrt{2\pi}} e^{-1/2}\right) \approx (1, 0.242), (3, 0.242)$



$$53. y = x^2 e^x - 2x e^x + 2e^x = e^x (x^2 - 2x + 2)$$

$$\frac{dy}{dx} = e^x (2x - 2) + e^x (x^2 - 2x + 2) = x^2 e^x$$

$$57. y = e^x (\sin x + \cos x)$$

$$\frac{dy}{dx} = e^x (\cos x - \sin x) + (\sin x + \cos x) (e^x)$$

$$= e^x (2 \cos x) = 2e^x \cos x$$

$$61. f(x) = (3 + 2x)e^{-3x}$$

$$f'(x) = (3 + 2x)(-3e^{-3x}) + 2e^{-3x}$$

$$= (-7 - 6x)e^{-3x}$$

$$f''(x) = (-7 - 6x)(-3e^{-3x}) - 6e^{-3x}$$

$$= 3(6x + 5)e^{-3x}$$

69. $f(x) = x^2e^{-x}$

$$f'(x) = -x^2e^{-x} + 2xe^{-x} = xe^{-x}(2 - x) = 0 \text{ when } x = 0, 2.$$

$$f''(x) = -e^{-x}(2x - x^2) + e^{-x}(2 - 2x)$$

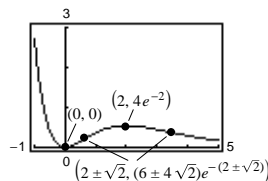
$$= e^{-x}(x^2 - 4x + 2) = 0 \text{ when } x = 2 \pm \sqrt{2}.$$

 Relative minimum: $(0, 0)$

 Relative maximum: $(2, 4e^{-2})$

$$x = 2 \pm \sqrt{2}$$

$$y = (2 \pm \sqrt{2})^2 e^{-(2 \pm \sqrt{2})}$$

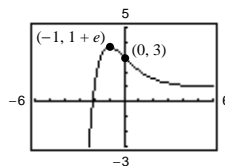
 Points of inflection: $(3.414, 0.384), (0.586, 0.191)$


71. $g(t) = 1 + (2 + t)e^{-t}$

$$g'(t) = (1 + t)e^{-t}$$

$$g''(t) = te^{-t}$$

 Relative maximum: $(-1, 1 + e) \approx (-1, 3.718)$

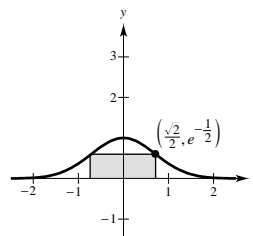
 Point of inflection: $(0, 3)$


73. $A = (\text{base})(\text{height}) = 2xe^{-x^2}$

$$\frac{dA}{dx} = -4x^2e^{-x^2} + 2e^{-x^2}$$

$$= 2e^{-x^2}(1 - 2x^2) = 0 \text{ when } x = \frac{\sqrt{2}}{2}.$$

$$A = \sqrt{2}e^{-1/2}$$



75. $y = \frac{L}{1 + ae^{-x/b}}, a > 0, b > 0, L > 0$

$$y' = \frac{-L\left(-\frac{a}{b}e^{-x/b}\right)}{(1 + ae^{-x/b})^2} = \frac{\frac{aL}{b}e^{-x/b}}{(1 + ae^{-x/b})^2}$$

$$y'' = \frac{(1 + ae^{-x/b})^2\left(\frac{-aL}{b^2}e^{-x/b}\right) - \left(\frac{aL}{b}e^{-x/b}\right)2(1 + ae^{-x/b})\left(\frac{-a}{b}e^{-x/b}\right)}{(1 + ae^{-x/b})^4}$$

$$= \frac{(1 + ae^{-x/b})\left(\frac{-aL}{b^2}e^{-x/b}\right) + 2\left(\frac{aL}{b}e^{-x/b}\right)\left(\frac{a}{b}e^{-x/b}\right)}{(1 + ae^{-x/b})^3}$$

$$= \frac{Lae^{-x/b}[ae^{-x/b} - 1]}{(1 + ae^{-x/b})^3 b^2}$$

$$y'' = 0 \text{ if } ae^{-x/b} = 1 \Rightarrow \frac{-x}{b} = \ln\left(\frac{1}{a}\right) \Rightarrow x = b \ln a$$

$$y(b \ln a) = \frac{L}{1 + ae^{-(b \ln a)/b}} = \frac{L}{1 + a(1/a)} = \frac{L}{2}$$

 Therefore, the y-coordinate of the inflection point is $L/2$.

77. $e^{-x} = x \Rightarrow f(x) = x - e^{-x}$

$$f'(x) = 1 + e^{-x}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n - e^{-x_n}}{1 + e^{-x_n}}$$

$$x_1 = 1$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 0.5379$$

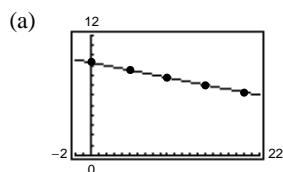
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 0.5670$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} \approx 0.5671$$

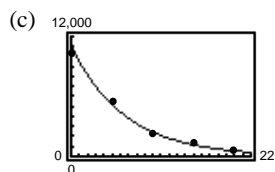
We approximate the root of f to be $x = 0.567$.

 81.

h	0	5	10	15	20
P	10,332	5,583	2,376	1,240	517
$\ln P$	9.243	8.627	7.773	7.123	6.248



$y = -0.1499h + 9.3018$ is the regression line for data $(h, \ln P)$.



83. $f(x) = e^{x/2}, f(0) = 1$

$$f'(x) = \frac{1}{2}e^{x/2}, f'(0) = \frac{1}{2}$$

$$f''(x) = \frac{1}{4}e^{x/2}, f''(0) = \frac{1}{4}$$

$$P_1(x) = 1 + \frac{1}{2}(x - 0) = \frac{x}{2} + 1, P_1(0) = 1$$

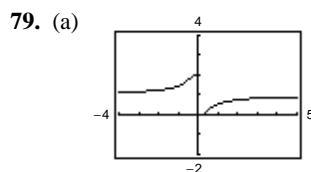
$$P_1'(x) = \frac{1}{2}, P_1'(0) = \frac{1}{2}$$

$$P_2(x) = 1 + \frac{1}{2}(x - 0) + \frac{1}{8}(x - 0)^2 = \frac{x^2}{8} + \frac{x}{2} + 1, P_2(0) = 1$$

$$P_2'(x) = \frac{1}{4}x + \frac{1}{2}, P_2'(0) = \frac{1}{2}$$

$$P_2''(x) = \frac{1}{4}, P_2''(0) = \frac{1}{4}$$

The values of f, P_1, P_2 and their first derivatives agree at $x = 0$. The values of the second derivatives of f and P_2 agree at $x = 0$.



(b) When x increases without bound, $1/x$ approaches zero, and $e^{1/x}$ approaches 1. Therefore, $f(x)$ approaches $2/(1 + 1) = 1$. Thus, $f(x)$ has a horizontal asymptote at $y = 1$. As x approaches zero from the right, $1/x$ approaches ∞ , $e^{1/x}$ approaches ∞ and $f(x)$ approaches zero. As x approaches zero from the left, $1/x$ approaches $-\infty$, $e^{1/x}$ approaches zero, and $f(x)$ approaches 2. The limit does not exist since the left limit does not equal the right limit. Therefore, $x = 0$ is a nonremovable discontinuity.

(b) $\ln P = ah + b$

$$P = e^{ah+b} = e^b e^{ah}$$

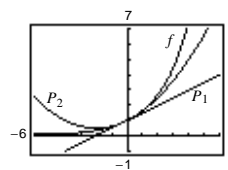
$$P = C e^{ah}, C = e^b$$

For our data, $a = -0.1499$ and $C = e^{9.3018} = 10,957.7$

$$P = 10,957.7e^{-0.1499h}$$

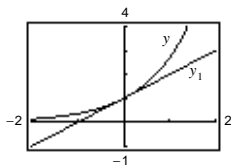
(d)
$$\begin{aligned} \frac{dP}{dh} &= (10,957.71)(-0.1499)e^{-0.1499h} \\ &= -1642.56e^{-0.1499h} \end{aligned}$$

For $h = 5$, $\frac{dP}{dh} = -776.3$. For $h = 18$, $\frac{dP}{dh} \approx -110.6$.



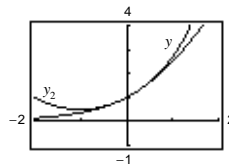
85. (a) $y = e^x$

$$y_1 = 1 + x$$



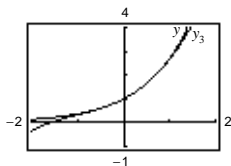
(b) $y = e^x$

$$y_2 = 1 + x + \left(\frac{x^2}{2}\right)$$



(c) $y = e^x$

$$y_3 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$



87. Let $u = 5x$, $du = 5 dx$.

$$\int e^{5x} 5 dx = e^{5x} + C$$

89. Let $u = -2x$, $du = -2 dx$.

$$\begin{aligned} \int_0^1 e^{-2x} dx &= -\frac{1}{2} \int_0^1 e^{-2x} (-2) dx = \left[-\frac{1}{2} e^{-2x} \right]_0^1 \\ &= \frac{1}{2} (1 - e^{-2}) = \frac{e^2 - 1}{2e^2} \end{aligned}$$

91. $\int x e^{-x^2} dx = -\frac{1}{2} \int e^{-x^2} (-2x) dx = -\frac{1}{2} e^{-x^2} + C$

93. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2 \int e^{\sqrt{x}} \left(\frac{1}{2\sqrt{x}} \right) dx = 2e^{\sqrt{x}} + C$

95. Let $u = 1 + e^{-x}$, $du = -e^{-x} dx$.

$$\int \frac{e^{-x}}{1 + e^{-x}} dx = -\int \frac{-e^{-x}}{1 + e^{-x}} dx = -\ln(1 + e^{-x}) + C = \ln\left(\frac{e^x}{e^x + 1}\right) + C = x - \ln(e^x + 1) + C$$

97. Let $u = \frac{3}{x}$, $du = -\frac{3}{x^2} dx$.

$$\begin{aligned} \int_1^3 \frac{e^{3/x}}{x^2} dx &= -\frac{1}{3} \int_1^3 e^{3/x} \left(-\frac{3}{x^2}\right) dx \\ &= \left[-\frac{1}{3} e^{3/x} \right]_1^3 = \frac{e}{3} (e^2 - 1) \end{aligned}$$

99. Let $u = 1 - e^x$, $du = -e^x dx$.

$$\begin{aligned} \int e^x \sqrt{1 - e^x} dx &= -\int (1 - e^x)^{1/2} (-e^x) dx \\ &= -\frac{2}{3} (1 - e^x)^{3/2} + C \end{aligned}$$

101. Let $u = e^x - e^{-x}$, $du = (e^x + e^{-x}) dx$.

$$\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx = \ln|e^x - e^{-x}| + C$$

103. $\int \frac{5 - e^x}{e^{2x}} dx = \int 5e^{-2x} dx - \int e^{-x} dx$
$$= -\frac{5}{2} e^{-2x} + e^{-x} + C$$

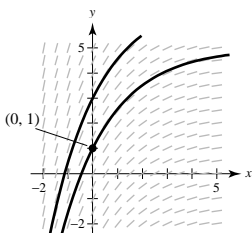
105. $\int e^{\sin \pi x} \cos \pi x dx = \frac{1}{\pi} \int e^{\sin \pi x} (\pi \cos \pi x) dx$
$$= \frac{1}{\pi} e^{\sin \pi x} + C$$

107. $\int e^{-x} \tan(e^{-x}) dx = -\int [\tan(e^{-x})] (-e^{-x}) dx$
$$= \ln|\cos(e^{-x})| + C$$

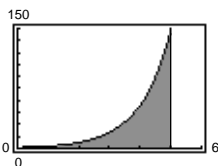
109. Let $u = ax^2$, $du = 2ax \, dx$. (Assume $a \neq 0$)

$$\begin{aligned} y &= \int x e^{ax^2} dx \\ &= \frac{1}{2a} \int e^{ax^2} (2ax) dx = \frac{1}{2a} e^{ax^2} + C \end{aligned}$$

113. (a)



115. $\int_0^5 e^x dx = [e^x]_0^5 = e^5 - 1 \approx 147.413$



119. (a) $f(u - v) = e^{u-v} = (e^u)(e^{-v}) = \frac{e^u}{e^v} = \frac{f(u)}{f(v)}$

(b) $f(kx) = e^{kx} = (e^x)^k = [f(x)]^k$.

123. $\int_0^x e^t dt \geq \int_0^x 1 dt$

$$\left[e^t \right]_0^x \geq \left[t \right]_0^x$$

$$e^x - 1 \geq x \Rightarrow e^x \geq 1 + x \text{ for } x \geq 0$$

127. Yes. $f(x) = Ce^x$, C a constant.

111. $f'(x) = \int \frac{1}{2}(e^x + e^{-x}) dx = \frac{1}{2}(e^x - e^{-x}) + C_1$

$$f'(0) = C_1 = 0$$

$$f(x) = \int \frac{1}{2}(e^x - e^{-x}) dx = \frac{1}{2}(e^x + e^{-x}) + C_2$$

$$f(0) = 1 + C_2 = 1 \Rightarrow C_2 = 0$$

$$f(x) = \frac{1}{2}(e^x + e^{-x})$$

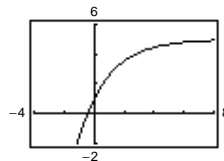
(b) $\frac{dy}{dx} = 2e^{-x/2}$, $(0, 1)$

$$y = \int 2e^{-x/2} dx = -4 \int e^{-x/2} \left(-\frac{1}{2} dx \right)$$

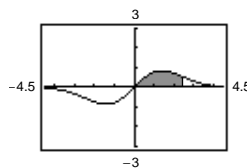
$$= -4e^{-x/2} + C$$

$$(0, 1): 1 = -4e^0 + C = -4 + C \Rightarrow C = 5$$

$$y = -4e^{-x/2} + 5$$



117. $\int_0^{\sqrt{6}} x e^{-x^2/4} dx = \left[-2e^{-x^2/4} \right]_0^{\sqrt{6}}$
 $= -2e^{-3/2} + 2 \approx 1.554$



121. $0.0665 \int_{48}^{60} e^{-0.0139(t-48)^2} dt$

Graphing Utility: $0.4772 = 47.72\%$

125. $f(x) = e^x$. Domain is $(-\infty, \infty)$ and range is $(0, \infty)$. f is continuous, increasing, one-to-one, and concave upwards on its entire domain.

$$\lim_{x \rightarrow -\infty} e^x = 0 \text{ and } \lim_{x \rightarrow \infty} e^x = \infty$$

129. $e^{-x} > 0 \Rightarrow \int_0^2 e^{-x} dx > 0$.

131. $f(x) = \frac{\ln x}{x}$

(a) $f'(x) = \frac{1 - \ln x}{x^2} = 0$ when $x = e$.

 On $(0, e)$, $f'(x) > 0 \Rightarrow f$ is increasing.

 On (e, ∞) , $f'(x) < 0 \Rightarrow f$ is decreasing.

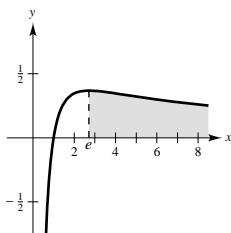
 (b) For $e \leq A < B$, we have:

$$\frac{\ln A}{A} > \frac{\ln B}{B}$$

$$B \ln A > A \ln B$$

$$\ln A^B > \ln B^A$$

$$A^B > B^A.$$

 (c) Since $e < \pi$, from part (b) we have $e^\pi > \pi^e$.


Section 5.5 Bases Other than e and Applications

1. $y = \left(\frac{1}{2}\right)^{t/3}$

At $t_0 = 6$, $y = \left(\frac{1}{2}\right)^{6/3} = \frac{1}{4}$

3. $y = \left(\frac{1}{2}\right)^{t/7}$

At $t_0 = 10$, $y = \left(\frac{1}{2}\right)^{10/7} \approx 0.3715$

5. $\log_2 \frac{1}{8} = \log_2 2^{-3} = -3$

7. $\log_7 1 = 0$

9. (a) $2^3 = 8$

$\log_2 8 = 3$

(b) $3^{-1} = \frac{1}{3}$

$\log_3 \frac{1}{3} = -1$

11. (a) $\log_{10} 0.01 = -2$

$10^{-2} = 0.01$

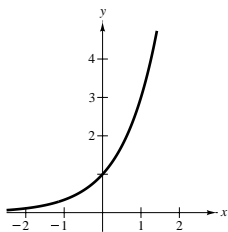
(b) $\log_{0.5} 8 = -3$

$0.5^{-3} = 8$

$\left(\frac{1}{2}\right)^{-3} = 8$

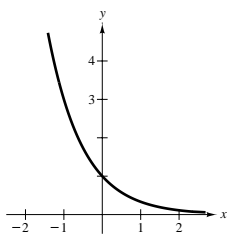
13. $y = 3^x$

x	-2	-1	0	1	2
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9



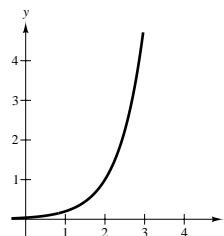
15. $y = \left(\frac{1}{3}\right)^x = 3^{-x}$

x	-2	-1	0	1	2
y	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$



17. $h(x) = 5^{x-2}$

x	-1	0	1	2	3
y	$\frac{1}{125}$	$\frac{1}{25}$	$\frac{1}{5}$	1	5



19. (a) $\log_{10} 1000 = x$

$$10^x = 1000$$

$$x = 3$$

(b) $\log_{10} 0.1 = x$

$$10^x = 0.1$$

$$x = -1$$

23. (a) $x^2 - x = \log_5 25$

$$x^2 - x = \log_5 5^2 = 2$$

$$x^2 - x - 2 = 0$$

$$(x + 1)(x - 2) = 0$$

$$x = -1 \text{ OR } x = 2$$

25. $3^{2x} = 75$

$$2x \ln 3 = \ln 75$$

$$x = \frac{1}{2} \frac{\ln 75}{\ln 3} \approx 1.965$$

29. $\left(1 + \frac{0.09}{12}\right)^{12t} = 3$

$$12t \ln\left(1 + \frac{0.09}{12}\right) = \ln 3$$

$$t = \frac{1}{12} \frac{\ln 3}{\ln\left(1 + \frac{0.09}{12}\right)} \approx 12.253$$

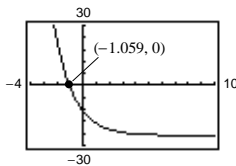
33. $\log_3 x^2 = 4.5$

$$x^2 = 3^{4.5}$$

$$x = \pm \sqrt{3^{4.5}} \approx \pm 11.845$$

35. $g(x) = 6(2^{1-x}) - 25$

Zero: $x \approx -1.059$



21. (a) $\log_3 x = -1$

$$3^{-1} = x$$

$$x = \frac{1}{3}$$

(b) $\log_2 x = -4$

$$2^{-4} = x$$

$$x = \frac{1}{16}$$

(b) $3x + 5 = \log_2 64$

$$3x + 5 = \log_2 2^6 = 6$$

$$3x = 1$$

$$x = \frac{1}{3}$$

27. $2^{3-x} = 625$

$$(3 - x) \ln 2 = \ln 625$$

$$3 - x = \frac{\ln 625}{\ln 2}$$

$$x = 3 - \frac{\ln 625}{\ln 2} \approx -6.288$$

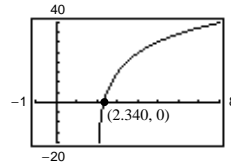
31. $\log_2(x - 1) = 5$

$$x - 1 = 2^5 = 32$$

$$x = 33$$

37. $h(s) = 32 \log_{10}(s - 2) + 15$

Zero: $s \approx 2.340$

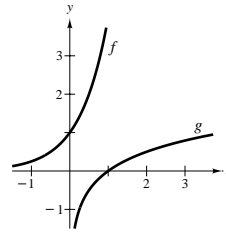


$$39. f(x) = 4^x$$

$$g(x) = \log_4 x$$

x	-2	-1	0	$\frac{1}{2}$	1
$f(x)$	$\frac{1}{16}$	$\frac{1}{4}$	1	2	4

x	$\frac{1}{16}$	$\frac{1}{4}$	1	2	4
$g(x)$	-2	-1	0	$\frac{1}{2}$	1



$$41. f(x) = 4^x$$

$$f'(x) = (\ln 4) 4^x$$

$$43. y = 5^{x-2}$$

$$\frac{dy}{dx} = (\ln 5) 5^{x-2}$$

$$45. g(t) = t^2 2^t$$

$$g'(t) = t^2 (\ln 2) 2^t + (2t) 2^t$$

$$= t 2^t (t \ln 2 + 2)$$

$$= 2^t t (2 + t \ln 2)$$

$$47. h(\theta) = 2^{-\theta} \cos \pi \theta$$

$$h'(\theta) = 2^{-\theta} (-\pi \sin \pi \theta) - (\ln 2) 2^{-\theta} \cos \pi \theta$$

$$= -2^{-\theta} [\pi \sin \pi \theta + (\ln 2) \cos \pi \theta]$$

$$49. y = \log_3 x$$

$$\frac{dy}{dx} = \frac{1}{x \ln 3}$$

$$51. f(x) = \log_2 \frac{x^2}{x-1}$$

$$= 2 \log_2 x - \log_2 (x-1)$$

$$f'(x) = \frac{2}{x \ln 2} - \frac{1}{(x-1) \ln 2}$$

$$= \frac{x-2}{(\ln 2)x(x-1)}$$

$$53. y = \log_5 \sqrt{x^2-1} = \frac{1}{2} \log_5 (x^2-1)$$

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{2x}{(x^2-1) \ln 5} = \frac{x}{(x^2-1) \ln 5}$$

$$55. g(t) = \frac{10 \log_4 t}{t} = \frac{10}{\ln 4} \left(\frac{\ln t}{t} \right)$$

$$g'(t) = \frac{10}{\ln 4} \left[\frac{t(1/t) - \ln t}{t^2} \right]$$

$$= \frac{10}{t^2 \ln 4} [1 - \ln t] = \frac{5}{t^2 \ln 2} (1 - \ln t)$$

$$57. y = x^{2/x}$$

$$\ln y = \frac{2}{x} \ln x$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{2}{x} \left(\frac{1}{x} \right) + \ln x \left(-\frac{2}{x^2} \right) = \frac{2}{x^2} (1 - \ln x)$$

$$\frac{dy}{dx} = \frac{2y}{x^2} (1 - \ln x) = 2x^{(2/x)-2} (1 - \ln x)$$

$$59. y = (x-2)^{x+1}$$

$$\ln y = (x+1) \ln(x-2)$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = (x+1) \left(\frac{1}{x-2} \right) + \ln(x-2)$$

$$\frac{dy}{dx} = y \left[\frac{x+1}{x-2} + \ln(x-2) \right]$$

$$= (x-2)^{x+1} \left[\frac{x+1}{x-2} + \ln(x-2) \right]$$

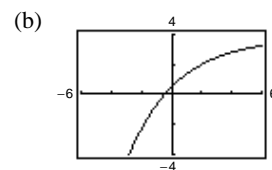
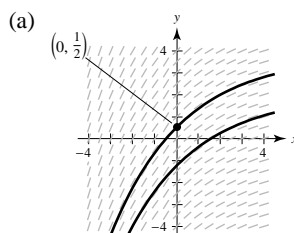
$$61. \int 3^x dx = \frac{3^x}{\ln 3} + C$$

$$\begin{aligned}
 63. \int_{-1}^2 2^x dx &= \left[\frac{2^x}{\ln 2} \right]_{-1}^2 \\
 &= \frac{1}{\ln 2} \left[4 - \frac{1}{2} \right] \\
 &= \frac{7}{2 \ln 2} = \frac{7}{\ln 4}
 \end{aligned}$$

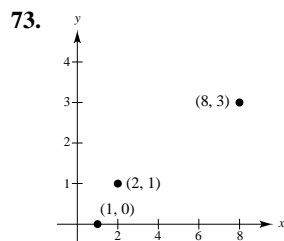
$$\begin{aligned}
 65. \int x 5^{-x^2} dx &= -\frac{1}{2} \int 5^{-x^2} (-2x) dx \\
 &= -\left(\frac{1}{2} \right) \frac{5^{-x^2}}{\ln 5} + C \\
 &= -\frac{1}{2 \ln 5} (5^{-x^2}) + C
 \end{aligned}$$

$$\begin{aligned}
 67. \int \frac{3^{2x}}{1 + 3^{2x}} dx, u = 1 + 3^{2x}, du = 2(\ln 3)3^{2x} dx \\
 \frac{1}{2 \ln 3} \int \frac{(2 \ln 3)3^{2x}}{1 + 3^{2x}} dx = \frac{1}{2 \ln 3} \ln(1 + 3^{2x}) + C
 \end{aligned}$$

$$\begin{aligned}
 69. \frac{dy}{dx} &= 0.4^{x/3}, \left(0, \frac{1}{2} \right) \\
 y &= \int 0.4^{x/3} dx = 3 \int 0.4^{x/3} \left(\frac{1}{3} dx \right) \\
 &= \frac{3}{\ln 0.4} 0.4^{x/3} + C = 3(\ln 2.5)(0.4)^{x/3} + C \\
 y &= 3 \ln 2.5 (0.4)^{x/3} + \frac{1}{2} - 3 \ln 2.5 \\
 &= \frac{3(1 - 0.4^{x/3})}{\ln 2.5} + \frac{1}{2}
 \end{aligned}$$



71. Answers will vary. Example: Growth and decay problems.



x	1	2	8
y	0	1	3

- (a) y is an exponential function of x : False
 (b) y is a logarithmic function of x : True; $y = \log_2 x$
 (c) x is an exponential function of y : True, $2^y = x$
 (d) y is a linear function of x : False

$$\begin{aligned}
 75. f(x) = \log_2 x &\Rightarrow f'(x) = \frac{1}{x \ln 2} \\
 g(x) = x^x &\Rightarrow g'(x) = x^x(1 + \ln x)
 \end{aligned}$$

[Note: Let $y = g(x)$. Then: $\ln y = \ln x^x = x \ln x$

$$\begin{aligned}
 \frac{1}{y} y' &= x \cdot \frac{1}{x} + \ln x \\
 y' &= y(1 + \ln x) \\
 y' &= x^x(1 + \ln x) = g'(x).]
 \end{aligned}$$

$$h(x) = x^2 \Rightarrow h'(x) = 2x$$

$$k(x) = 2^x \Rightarrow k'(x) = (\ln 2)2^x$$

From greatest to smallest rate of growth:

$$g(x), k(x), h(x), f(x)$$

$$77. C(t) = P(1.05)^t$$

$$\begin{aligned}
 (a) C(10) &= 24.95(1.05)^{10} \\
 &\approx \$40.64
 \end{aligned}$$

$$(b) \frac{dC}{dt} = P(\ln 1.05)(1.05)^t$$

$$\text{When } t = 1: \frac{dC}{dt} \approx 0.051P$$

$$\text{When } t = 8: \frac{dC}{dt} \approx 0.072P$$

$$\begin{aligned}
 (c) \frac{dC}{dt} &= (\ln 1.05)[P(1.05)^t] \\
 &= (\ln 1.05)C(t)
 \end{aligned}$$

The constant of proportionality is $\ln 1.05$.

79. $P = \$1000$, $r = 3\frac{1}{2}\% = 0.035$, $t = 10$

$$A = 1000\left(1 + \frac{0.035}{n}\right)^{10n}$$

$$A = 1000e^{(0.035)(10)} = 1419.07$$

n	1	2	4	12	365	Continuous
A	1410.60	1414.78	1416.91	1418.34	1419.04	1419.07

81. $P = \$1000$, $r = 5\% = 0.05$, $t = 30$

$$A = 1000\left(1 + \frac{0.05}{n}\right)^{30n}$$

$$A = 1000e^{(0.05)(30)} = 4481.69$$

n	1	2	4	12	365	Continuous
A	4321.94	4399.79	4440.21	4467.74	4481.23	4481.69

83. $100,000 = Pe^{0.05t} \Rightarrow P = 100,000e^{-0.05t}$

t	1	10	20	30	40	50
P	95,122.94	60,653.07	36,787.94	22,313.02	13,583.53	8208.50

85. $100,000 = P\left(1 + \frac{0.05}{12}\right)^{12t} \Rightarrow P = 100,000\left(1 + \frac{0.05}{12}\right)^{-12t}$

t	1	10	20	30	40	50
P	95,132.82	60,716.10	36,864.45	22,382.66	13,589.88	8251.24

87. (a) $A = 20,000\left(1 + \frac{0.06}{365}\right)^{(365)(8)} \approx \$32,320.21$

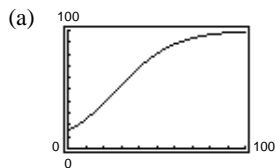
(b) $A = \$30,000$

(c) $A = 8000\left(1 + \frac{0.06}{365}\right)^{(365)(8)} + 20,000\left(1 + \frac{0.06}{365}\right)^{(365)(4)}$
 $\approx \$12,928.09 + 25,424.48 = \$38,352.57$

(d) $A = 9000\left[\left(1 + \frac{0.06}{365}\right)^{(365)(8)} + \left(1 + \frac{0.06}{365}\right)^{(365)(4)} + 1\right]$
 $\approx \$34,985.11$

Take option (c).

91. $y = \frac{300}{3 + 17e^{-0.0625x}}$



(b) If $x = 2$ (2000 egg masses), $y \approx 16.67 \approx 16.7\%$.

89. (a) $\lim_{t \rightarrow \infty} 6.7e^{(-48.1)/t} = 6.7e^0 = 6.7$ million ft^3

(b) $V' = \frac{322.27}{t^2} e^{-(48.1)/t}$

$V'(20) \approx 0.073$ million ft^3/yr

$V'(60) \approx 0.040$ million ft^3/yr

(c) If $y = 66.67\%$, then $x \approx 38.8$ or 38,800 egg masses.

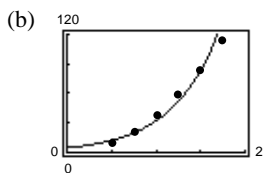
(d) $y = 300(3 + 17e^{-0.0625x})^{-1}$

$$y' = \frac{318.75e^{-0.0625x}}{(3 + 17e^{-0.0625x})^2}$$

$$y'' = \frac{19.921875e^{-0.0625x}(17e^{-0.0625x} - 3)}{(3 + 17e^{-0.0625x})^3}$$

$17e^{-0.0625x} - 3 = 0 \Rightarrow x \approx 27.8$ or 27,800 egg masses.

93. (a) $B = 4.7539(6.7744)^d = 4.7539e^{1.9132d}$



(c) $B'(d) = 9.0952e^{1.9132d}$

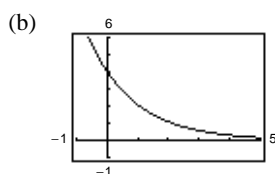
$B'(0.8) \approx 42.03$ tons/inch

$B'(1.5) \approx 160.38$ tons/inch

95. (a) $\int_0^4 f(t) dt \approx 5.67$

$\int_0^4 g(t) dt \approx 5.67$

$\int_0^4 h(t) dt \approx 5.67$



97. $P = \int_0^{10} 2000e^{-0.06t} dt$

$= \left[\frac{2000}{-0.06} e^{-0.06t} \right]_0^{10}$

$\approx \$15,039.61$

(c) The functions appear to be equal: $f(t) = g(t) = h(t)$

Analytically,

$$f(t) = 4\left(\frac{3}{8}\right)^{2t/3} = 4\left[\left(\frac{3}{8}\right)^{2/3}\right]^t = 4\left(\frac{9^{1/3}}{4}\right)^t = g(t)$$

$$h(t) = 4e^{-0.653886t} = 4[e^{-0.653886}]^t = 4(0.52002)^t$$

$$g(t) = 4\left(\frac{9^{1/3}}{4}\right)^t = 4(0.52002)^t$$

No. The definite integrals over a given interval may be equal when the functions are not equal.

99.

t	0	1	2	3	4
y	1200	720	432	259.20	155.52

$y = C(k^t)$

When $t = 0, y = 1200 \Rightarrow C = 1200$.

$y = 1200(k^t)$

$$\frac{720}{1200} = 0.6, \frac{432}{720} = 0.6, \frac{259.20}{432} = 0.6, \frac{155.52}{259.20} = 0.6$$

Let $k = 0.6$.

$y = 1200(0.6)^t$

101. False. e is an irrational number.

103. True.

$$\begin{aligned} f(g(x)) &= 2 + e^{\ln(x-2)} \\ &= 2 + x - 2 = x \end{aligned}$$

$$\begin{aligned} g(f(x)) &= \ln(2 + e^x - 2) \\ &= \ln e^x = x \end{aligned}$$

105. True.

$$\frac{d}{dx}[e^x] = e^x \text{ and } \frac{d}{dx}[e^{-x}] = -e^{-x}$$

$e^x = e^{-x}$ when $x = 0$.

$(e^0)(-e^{-0}) = -1$

$$107. \quad \frac{dy}{dt} = \frac{8}{25}y\left(\frac{5}{4} - y\right), y(0) = 1$$

$$\frac{dy}{y[(5/4) - y]} = \frac{8}{25} dt \Rightarrow \frac{4}{5} \int \left(\frac{1}{y} + \frac{1}{(5/4) - y} \right) dy = \int \frac{8}{25} dt \Rightarrow$$

$$\ln y - \ln\left(\frac{5}{4} - y\right) = \frac{2}{5}t + C$$

$$\ln\left(\frac{y}{(5/4) - y}\right) = \frac{2}{5}t + C$$

$$\frac{y}{(5/4) - y} = e^{(2/5)t + C} = C_1 e^{(2/5)t}$$

$$y(0) = 1 \Rightarrow C_1 = 4 \Rightarrow 4e^{(2/5)t} = \frac{y}{(5/4) - y}$$

$$\Rightarrow 4e^{(2/5)t}\left(\frac{5}{4} - y\right) = y \Rightarrow 5e^{(2/5)t} = 4e^{(2/5)t}y + y = (4e^{(2/5)t} + 1)y$$

$$\Rightarrow y = \frac{5e^{(2/5)t}}{4e^{(2/5)t} + 1} = \frac{5}{4 + e^{-0.4t}} = \frac{1.25}{1 + 0.25e^{-0.4t}}$$

Section 5.6 Differential Equations: Growth and Decay

$$1. \quad \frac{dy}{dx} = x + 2$$

$$y = \int (x + 2)dx = \frac{x^2}{2} + 2x + C$$

$$3. \quad \frac{dy}{dx} = y + 2$$

$$\frac{dy}{y + 2} = dx$$

$$\int \frac{1}{y + 2} dy = \int dx$$

$$\ln|y + 2| = x + C_1$$

$$y + 2 = e^{x + C_1} = Ce^x$$

$$y = Ce^x - 2$$

$$5. \quad y' = \frac{5x}{y}$$

$$yy' = 5x$$

$$\int yy' dx = \int 5x dx$$

$$\int y dy = \int 5x dx$$

$$\frac{1}{2}y^2 = \frac{5}{2}x^2 + C_1$$

$$y^2 - 5x^2 = C$$

$$7. \quad y' = \sqrt{x}y$$

$$\frac{y'}{y} = \sqrt{x}$$

$$\int \frac{y'}{y} dx = \int \sqrt{x} dx$$

$$\int \frac{dy}{y} = \int \sqrt{x} dx$$

$$\ln y = \frac{2}{3}x^{3/2} + C_1$$

$$y = e^{(2/3)x^{3/2} + C_1}$$

$$= e^{C_1} e^{(2/3)x^{3/2}}$$

$$= Ce^{(2/3)x^{3/2}}$$

9. $(1 + x^2)y' - 2xy = 0$

$$y' = \frac{2xy}{1 + x^2}$$

$$\frac{y'}{y} = \frac{2x}{1 + x^2}$$

$$\int \frac{y'}{y} dx = \int \frac{2x}{1 + x^2} dx$$

$$\int \frac{dy}{y} = \int \frac{2x}{1 + x^2} dx$$

$$\ln y = \ln(1 + x^2) + C_1$$

$$\ln y = \ln(1 + x^2) + \ln C$$

$$\ln y = \ln C(1 + x^2)$$

$$y = C(1 + x^2)$$

11. $\frac{dQ}{dt} = \frac{k}{t^2}$

$$\int \frac{dQ}{dt} dt = \int \frac{k}{t^2} dt$$

$$\int dQ = -\frac{k}{t} + C$$

$$Q = -\frac{k}{t} + C$$

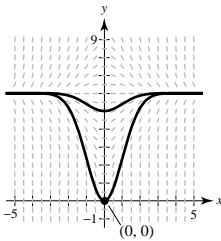
13. $\frac{dN}{ds} = k(250 - s)$

$$\int \frac{dN}{ds} ds = \int k(250 - s) ds$$

$$\int dN = -\frac{k}{2}(250 - s)^2 + C$$

$$N = -\frac{k}{2}(250 - s)^2 + C$$

15. (a)



(b) $\frac{dy}{dx} = x(6 - y), (0, 0)$

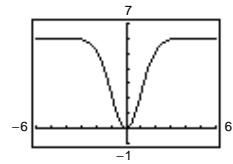
$$\frac{dy}{y - 6} = -x$$

$$\ln|y - 6| = \frac{-x^2}{2} + C$$

$$y - 6 = e^{-x^2/2 + C} = C_1 e^{-x^2/2}$$

$$y = 6 + C_1 e^{-x^2/2}$$

$$(0, 0): 0 = 6 + C_1 \Rightarrow C_1 = -6 \Rightarrow y = 6 - 6e^{-x^2/2}$$



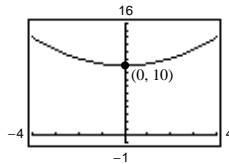
17. $\frac{dy}{dt} = \frac{1}{2}t, (0, 10)$

$$\int dy = \int \frac{1}{2}t dt$$

$$y = \frac{1}{4}t^2 + C$$

$$10 = \frac{1}{4}(0)^2 + C \Rightarrow C = 10$$

$$y = \frac{1}{4}t^2 + 10$$



19. $\frac{dy}{dt} = -\frac{1}{2}y, (0, 10)$

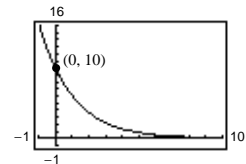
$$\int \frac{dy}{y} = \int -\frac{1}{2} dt$$

$$\ln y = -\frac{1}{2}t + C_1$$

$$y = e^{-(t/2) + C_1} = e^{C_1} e^{-t/2} = C e^{-t/2}$$

$$10 = C e^0 \Rightarrow C = 10$$

$$y = 10e^{-t/2}$$



21. $\frac{dy}{dx} = ky$

$$y = Ce^{kx} \quad (\text{Theorem 5.16})$$

$$(0, 4): 4 = Ce^0 = C$$

$$(3, 10): 10 = 4e^{3k} \Rightarrow k = \frac{1}{3} \ln\left(\frac{5}{2}\right)$$

$$\text{When } x = 6, y = 4e^{1/3 \ln(5/2)(6)} = 4e^{\ln(5/2)^2}$$

$$= 4\left(\frac{5}{2}\right)^2 = 25$$

23. $\frac{dV}{dt} = kV$

$$V = Ce^{kt} \quad (\text{Theorem 5.16})$$

$$(0, 20,000): C = 20,000$$

$$(4, 12,500): 12,500 = 20,000e^{4k} \Rightarrow k = \frac{1}{4} \ln\left(\frac{5}{8}\right)$$

$$\text{When } t = 6, V = 20,000e^{1/4 \ln(5/8)(6)} = 20,000e^{\ln(5/8)^{3/2}}$$

$$= 20,000\left(\frac{5}{8}\right)^{3/2} \approx 9882.118$$

25. $y = Ce^{kt}$, $(0, \frac{1}{2})$, $(5, 5)$

$$C = \frac{1}{2}$$

$$y = \frac{1}{2} e^{kt}$$

$$5 = \frac{1}{2} e^{5k}$$

$$k = \frac{\ln 10}{5} \approx 0.4605$$

$$y = \frac{1}{2} e^{0.4605t}$$

27. $y = Ce^{kt}$, $(1, 1)$, $(5, 5)$

$$1 = Ce^k$$

$$5 = Ce^{5k}$$

$$5Ce^k = Ce^{5k}$$

$$5e^k = e^{5k}$$

$$5 = e^{4k}$$

$$k = \frac{\ln 5}{4} \approx 0.4024$$

$$y = Ce^{0.4024t}$$

$$1 = Ce^{0.4024}$$

$$C \approx 0.6687$$

$$y = 0.6687e^{0.4024t}$$

29. A differential equation in x and y is an equation that involves x , y and derivatives of y .

31. $\frac{dy}{dx} = \frac{1}{2}xy$

$$\frac{dy}{dx} > 0 \text{ when } xy > 0. \text{ Quadrants I and III.}$$

33. Since the initial quantity is 10 grams, $y = 10e^{[\ln(1/2)/1620]t}$. When $t = 1000$, $y = 10e^{[\ln(1/2)/1620](1000)} \approx 6.52$ grams. When $t = 10,000$, $y = 10e^{[\ln(1/2)/1620](10,000)} \approx 0.14$ gram.

35. Since $y = Ce^{[\ln(1/2)/1620]t}$, we have $0.5 = Ce^{[\ln(1/2)/1620](10,000)} \Rightarrow C \approx 36.07$.

Initial quantity: 36.07 grams.

When $t = 1000$, we have $y = Ce^{[\ln(1/2)/1620](1000)} \approx 23.51$ grams.

37. Since the initial quantity is 5 grams, we have $y = 5.0e^{[\ln(1/2)/5730]t}$.

When $t = 1000$, $y \approx 4.43$ g.

When $t = 10,000$, $y \approx 1.49$ g.

39. Since $y = Ce^{[\ln(1/2)/24,360]t}$, we have $2.1 = Ce^{[\ln(1/2)/24,360](1000)} \Rightarrow C \approx 2.16$. Thus, the initial quantity is 2.16 grams. When $t = 10,000$, $y = 2.16e^{[\ln(1/2)/24,360](10,000)} \approx 1.63$ grams.

41. Since $\frac{dy}{dx} = ky$, $y = Ce^{kt}$ or $y = y_0e^{kt}$.

$$\frac{1}{2}y_0 = y_0e^{1620k}$$

$$k = \frac{-\ln 2}{1620}$$

$$y = y_0e^{-(\ln 2)t/1620}$$

When $t = 100$, $y = y_0e^{-(\ln 2)/16.2} \approx y_0(0.9581)$.

Therefore, 95.81% of the present amount still exists.

43. Since $A = 1000e^{0.06t}$, the time to double is given by $2000 = 1000e^{0.06t}$ and we have

$$2 = e^{0.06t}$$

$$\ln 2 = 0.06t$$

$$t = \frac{\ln 2}{0.06} \approx 11.55 \text{ years.}$$

Amount after 10 years: $A = 1000e^{(0.06)(10)} \approx \1822.12

45. Since $A = 750e^{rt}$ and $A = 1500$ when $t = 7.75$, we have the following.

$$1500 = 750e^{7.75r}$$

$$r = \frac{\ln 2}{7.75} \approx 0.0894 = 8.94\%$$

Amount after 10 years: $A = 750e^{0.0894(10)} \approx \1833.67

49. $500,000 = P\left(1 + \frac{0.075}{12}\right)^{(12)(20)}$

$$P = 500,000\left(1 + \frac{0.075}{12}\right)^{-240}$$

$$\approx \$112,087.09$$

53. (a) $2000 = 1000(1 + 0.07)^t$

$$2 = 1.07^t$$

$$\ln 2 = t \ln 1.07$$

$$t = \frac{\ln 2}{\ln 1.07} \approx 10.24 \text{ years}$$

- (b) $2000 = 1000\left(1 + \frac{0.07}{12}\right)^{12t}$

$$2 = \left(1 + \frac{0.007}{12}\right)^{12t}$$

$$\ln 2 = 12t \ln\left(1 + \frac{0.07}{12}\right)$$

$$t = \frac{\ln 2}{12 \ln\left(1 + \frac{0.07}{12}\right)} \approx 9.93 \text{ years}$$

55. (a) $2000 = 1000(1 + 0.085)^t$

$$2 = 1.085^t$$

$$\ln 2 = t \ln 1.085$$

$$t = \frac{\ln 2}{\ln 1.085} \approx 8.50 \text{ years}$$

- (b) $2000 = 1000\left(1 + \frac{0.085}{12}\right)^{12t}$

$$2 = \left(1 + \frac{0.085}{12}\right)^{12t}$$

$$\ln 2 = 12t \ln\left(1 + \frac{0.085}{12}\right)$$

$$t = \frac{1}{12} \frac{\ln 2}{\ln\left(1 + \frac{0.085}{12}\right)} \approx 8.18 \text{ years}$$

47. Since $A = 500e^{rt}$ and $A = 1292.85$ when $t = 10$, we have the following.

$$1292.85 = 500e^{10r}$$

$$r = \frac{\ln(1292.85/500)}{10} \approx 0.0950 = 9.50\%$$

The time to double is given by

$$1000 = 500e^{0.0950r}$$

$$t = \frac{\ln 2}{0.095} \approx 7.30 \text{ years.}$$

51. $500,000 = P\left(1 + \frac{0.08}{12}\right)^{(12)(35)}$

$$P = 500,000\left(1 + \frac{0.08}{12}\right)^{-420}$$

$$= \$30,688.87$$

- (c) $2000 = 1000\left(1 + \frac{0.07}{365}\right)^{365t}$

$$2 = \left(1 + \frac{0.07}{365}\right)^{365t}$$

$$\ln 2 = 365t \ln\left(1 + \frac{0.07}{365}\right)$$

$$t = \frac{\ln 2}{365 \ln\left(1 + \frac{0.07}{365}\right)} \approx 9.90 \text{ years}$$

- (d) $2000 = 1000e^{(0.07)t}$

$$2 = e^{0.07t}$$

$$\ln 2 = 0.07t$$

$$t = \frac{\ln 2}{0.07} \approx 9.90 \text{ years}$$

- (c) $2000 = 1000\left(1 + \frac{0.085}{365}\right)^{365t}$

$$2 = \left(1 + \frac{0.085}{365}\right)^{365t}$$

$$\ln 2 = 365t \ln\left(1 + \frac{0.085}{365}\right)$$

$$t = \frac{1}{365} \frac{\ln 2}{\ln\left(1 + \frac{0.085}{365}\right)} \approx 8.16 \text{ years}$$

- (d) $2000 = 1000e^{0.085t}$

$$2 = e^{0.085t}$$

$$\ln 2 = 0.085t$$

$$t = \frac{\ln 2}{0.085} \approx 8.15 \text{ years}$$

57. $P = Ce^{kt} = Ce^{-0.009t}$

$$P(-1) = 8.2 = Ce^{-0.009(-1)} \Rightarrow C = 8.1265$$

$$P = 8.1265e^{-0.009t}$$

$$P(10) \approx 7.43 \quad \text{or} \quad 7,430,000 \text{ people in 2010}$$

61. If $k < 0$, the population decreases.If $k > 0$, the population increases.

65. (a) $19 = 30(1 - e^{20k})$

$$30e^{20k} = 11$$

$$k = \frac{\ln(11/30)}{20} \approx -0.0502$$

$$N \approx 30(1 - e^{-0.0502t})$$

67. $S = Ce^{k/t}$

(a) $S = 5$ when $t = 1$

$$5 = Ce^k$$

$$\lim_{t \rightarrow \infty} Ce^{k/t} = C = 30$$

$$5 = 30e^k$$

$$k = \ln \frac{1}{6} \approx -1.7918$$

$$S \approx 30e^{-1.7918/t}$$

59. $P = Ce^{kt} = Ce^{0.036t}$

$$P(-1) = 4.6 = Ce^{0.036(-1)} \Rightarrow C = 4.7686$$

$$P = 4.7686e^{0.036t}$$

$$P(10) \approx 6.83 \quad \text{or} \quad 6,830,000 \text{ people in 2010}$$

63. $P = Ce^{kx}$, $(0, 760)$, $(1000, 672.71)$

$$C = 760$$

$$672.71 = 760e^{1000k}$$

$$x = \frac{\ln(672.71/760)}{1000} \approx -0.000122$$

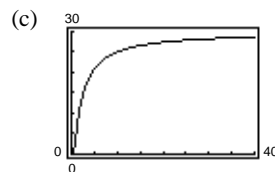
$$P \approx 760e^{-0.000122x}$$

When $x = 3000$, $P \approx 527.06$ mm Hg.

(b) $25 = 30(1 - e^{-0.0502t})$

$$e^{-0.0502t} = \frac{1}{6}$$

$$t = \frac{-\ln 6}{-0.0502} \approx 36 \text{ days}$$

(b) When $t = 5$, $S \approx 20.9646$ which is 20,965 units.

69. $A(t) = V(t)e^{-0.10t} = 100,000e^{0.8\sqrt{t}}e^{-0.10t} = 100,000e^{0.8\sqrt{t}-0.10t}$

$$\frac{dA}{dt} = 100,000 \left(\frac{0.4}{\sqrt{t}} - 0.10 \right) e^{0.8\sqrt{t}-0.10t} = 0 \text{ when } 16.$$

The timber should be harvested in the year 2014, $(1998 + 16)$. **Note:** You could also use a graphing utility to graph $A(t)$ and find the maximum of $A(t)$. Use the viewing rectangle $0 \leq x \leq 30$ and $0 \leq y \leq 600,000$.

71. $\beta(I) = 10 \log_{10} \frac{I}{I_0}$, $I_0 = 10^{-16}$

(a) $\beta(10^{-14}) = 10 \log_{10} \frac{10^{-14}}{10^{-16}} = 20$ decibels

(b) $\beta(10^{-9}) = 10 \log_{10} \frac{10^{-9}}{10^{-16}} = 70$ decibels

(c) $\beta(10^{-6.5}) = 10 \log_{10} \frac{10^{-6.5}}{10^{-16}} = 95$ decibels

(d) $\beta(10^{-4}) = 10 \log_{10} \frac{10^{-4}}{10^{-16}} = 120$ decibels

73. $R = \frac{\ln I - 0}{\ln 10}$, $I = e^{R \ln 10} = 10^R$

(a) $8.3 = \frac{\ln I - 0}{\ln 10}$

$$I = 10^{8.3} \approx 199,526,231.5$$

(b) $2R = \frac{\ln I - 0}{\ln 10}$

$$I = e^{2R \ln 10} = e^{2R \ln 10} = (e^{R \ln 10})^2 = (10^R)^2$$

Increases by a factor of $e^{2R \ln 10}$ or 10^R .

(c) $\frac{dR}{dI} = \frac{1}{I \ln 10}$

75. False. If $y = Ce^{kt}$, $y' = Cke^{kt} \neq \text{constant}$.

77. True

Section 5.7 Differential Equations: Separation of Variables1. Differential equation: $y' = 4y$

Solution: $y = Ce^{4x}$

Check: $y' = 4Ce^{4x} = 4y$

3. Differential equation: $y'' + y = 0$

Solution: $y = C_1 \cos x + C_2 \sin x$

Check: $y' = -C_1 \sin x + C_2 \cos x$

$y'' = -C_1 \cos x - C_2 \sin x$

$y'' + y = -C_1 \cos x - C_2 \sin x + C_1 \cos x + C_2 \sin x = 0$

5. $y = -\cos x \ln|\sec x + \tan x|$

$$y' = (-\cos x) \frac{1}{\sec x + \tan x} (\sec x \cdot \tan x + \sec^2 x) + \sin x \ln|\sec x + \tan x|$$

$$= \frac{(-\cos x)}{\sec x + \tan x} (\sec x)(\tan x + \sec x) + \sin x \ln|\sec x + \tan x|$$

$$= -1 + \sin x \ln|\sec x + \tan x|$$

$$y'' = (\sin x) \frac{1}{\sec x + \tan x} (\sec x \cdot \tan x + \sec^2 x) + \cos x \ln|\sec x + \tan x|$$

$$= (\sin x)(\sec x) + \cos x \ln|\sec x + \tan x|$$

Substituting,

$$y'' + y = (\sin x)(\sec x) + \cos x \ln|\sec x + \tan x| - \cos x \ln|\sec x + \tan x|$$

$$= \tan x.$$

In Exercises 7–11, the differential equation is $y^{(4)} - 16y = 0$.7. $y = 3 \cos x$

$y^{(4)} = 3 \cos x$

$y^{(4)} - 16y = -45 \cos x \neq 0,$

No.

9. $y = e^{-2x}$

$y^{(4)} = 16e^{-2x}$

$y^{(4)} - 16y = 16e^{-2x} - 16e^{-2x} = 0,$

Yes.

11. $y = C_1 e^{2x} + C_2 e^{-2x} + C_3 \sin 2x + C_4 \cos 2x$

$y^{(4)} = 16C_1 e^{2x} + 16C_2 e^{-2x} + 16C_3 \sin 2x + 16C_4 \cos 2x$

$y^{(4)} - 16y = 0,$

Yes.

In 13–17, the differential equation is $xy' - 2y = x^3e^x$.

13. $y = x^2, y' = 2x$

$$xy' - 2y = x(2x) - 2(x^2) = 0 \neq x^3e^x$$

No.

17. $y = \ln x, y' = \frac{1}{x}$

$$xy' - 2y = x\left(\frac{1}{x}\right) - 2 \ln x \neq x^3e^x, \quad \text{No.}$$

19. $y = Ce^{kx}$

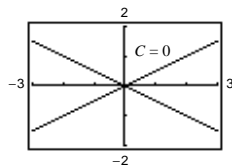
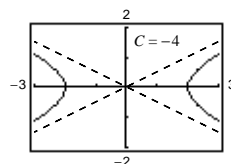
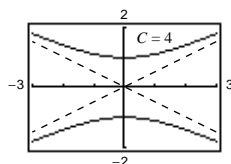
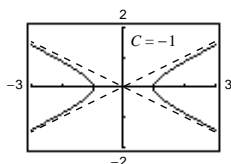
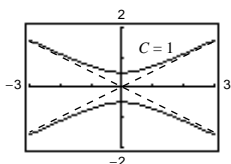
$$\frac{dy}{dx} = Cke^{kx}$$

Since $dy/dx = 0.07y$, we have $Cke^{kx} = 0.07Ce^{kx}$.
Thus, $k = 0.07$.

23. Differential equation: $4yy' - x = 0$

General solution: $4y^2 - x^2 = C$

Particular solutions: $C = 0$, Two intersecting lines
 $C = \pm 1, C = \pm 4$, Hyperbolas



25. Differential equation: $y' + 2y = 0$

General Solution: $y = Ce^{-2x}$

$$y' + 2y = C(-2)e^{-2x} + 2(Ce^{-2x}) = 0$$

Initial condition: $y(0) = 3, 3 = Ce^0 = C$

Particular solution: $y = 3e^{-2x}$

27. Differential equation: $y'' + 9y = 0$

General solution: $y = C_1 \sin 3x + C_2 \cos 3x$

$$y' = 3C_1 \cos 3x - 3C_2 \sin 3x,$$

$$y'' = -9C_1 \sin 3x - 9C_2 \cos 3x$$

$$y'' + 9y = (-9C_1 \sin 3x - 9C_2 \cos 3x) + 9(C_1 \sin 3x + C_2 \cos 3x) = 0$$

Initial conditions: $y\left(\frac{\pi}{6}\right) = 2, y'\left(\frac{\pi}{6}\right) = 1$

$$2 = C_1 \sin\left(\frac{\pi}{2}\right) + C_2 \cos\left(\frac{\pi}{2}\right) \Rightarrow C_1 = 2$$

$$y' = 3C_1 \cos 3x - 3C_2 \sin 3x$$

$$\begin{aligned} 1 &= 3C_1 \cos\left(\frac{\pi}{2}\right) - 3C_2 \sin\left(\frac{\pi}{2}\right) \\ &= -3C_2 \Rightarrow C_2 = -\frac{1}{3} \end{aligned}$$

Particular solution: $y = 2 \sin 3x - \frac{1}{3} \cos 3x$

29. Differential equation: $x^2y'' - 3xy' + 3y = 0$

General solution: $y = C_1x + C_2x^3$

$y' = C_1 + 3C_2x^2, y'' = 6C_2x$

$x^2y'' - 3xy' + 3y = x^2(6C_2x) - 3x(C_1 + 3C_2x^2) +$

$3(C_1x + C_2x^3) = 0$

Initial conditions: $y(2) = 0, y'(2) = 4$

$0 = 2C_1 + 8C_2$

$y' = C_1 + 3C_2x^2$

$4 = C_1 + 12C_2$

$$\left. \begin{array}{l} C_1 + 4C_2 = 0 \\ C_1 + 12C_2 = 4 \end{array} \right\} C_2 = \frac{1}{2}, C_1 = -2$$

Particular solution: $y = -2x + \frac{1}{2}x^3$

31. $\frac{dy}{dx} = 3x^2$

$y = \int 3x^2 dx = x^3 + C$

33. $\frac{dy}{dx} = \frac{x}{1+x^2}$

$y = \int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C$

$(u = 1+x^2, du = 2x dx)$

35. $\frac{dy}{dx} = \frac{x-2}{x} = 1 - \frac{2}{x}$

$y = \int \left[1 - \frac{2}{x} \right] dx$

$= x - 2 \ln|x| + C = x - \ln x^2 + C$

37. $\frac{dy}{dx} = \sin 2x$

$y = \int \sin 2x dx = -\frac{1}{2} \cos 2x + C$

$(u = 2x, du = 2dx)$

39. $\frac{dy}{dx} = x\sqrt{x-3}$ Let $u = \sqrt{x-3}$, then $x = u^2 + 3$ and $dx = 2u du$.

$y = \int x\sqrt{x-3} dx = \int (u^2 + 3)(u)(2u) du$

$= 2 \int (u^4 + 3u^2) du = 2 \left(\frac{u^5}{5} + u^3 \right) + C = \frac{2}{5} (x-3)^{5/2} + 2(x-3)^{3/2} + C$

41. $\frac{dy}{dx} = xe^{x^2}$

$y = \int xe^{x^2} dx = \frac{1}{2} e^{x^2} + C$

$(u = x^2, du = 2x dx)$

43. $\frac{dy}{dx} = \frac{x}{y}$

$\int y dy = \int x dx$

$\frac{y^2}{2} = \frac{x^2}{2} + C_1$

$y^2 - x^2 = C$

45. $\frac{dr}{ds} = 0.05r$

$\int \frac{dr}{r} = \int 0.05 ds$

$\ln|r| = 0.05s + C_1$

$r = e^{0.05s + C_1} = Ce^{0.05s}$

47. $(2+x)y' = 3y$

$\int \frac{dy}{y} = \int \frac{3}{2+x} dx$

$\ln y = 3 \ln(2+x) + \ln C = \ln C(2+x)^3$

$y = C(x+2)^3$

49. $yy' = \sin x$

$$\int y dy = \int \sin x dx$$

$$\frac{y^2}{2} = -\cos x + C_1$$

$$y^2 = -2 \cos x + C$$

53. $y \ln x - xy' = 0$

$$\int \frac{dy}{y} = \int \frac{\ln x}{x} dx \quad \left(u = \ln x, du = \frac{dx}{x} \right)$$

$$\ln y = \frac{1}{2}(\ln x)^2 + C_1$$

$$y = e^{(1/2)(\ln x)^2 + C_1} = Ce^{(\ln x)^2/2}$$

57. $y(x+1) + y' = 0$

$$\int \frac{dy}{y} = -\int (x+1) dx$$

$$\ln y = -\frac{(x+1)^2}{2} + C_1$$

$$y = Ce^{-(x+1)^2/2}$$

Initial condition: $y(-2) = 1$, $1 = Ce^{-1/2}$, $C = e^{1/2}$

Particular solution: $y = e^{[1-(x+1)^2]/2} = e^{-(x^2+2x)/2}$

61. $\frac{du}{dv} = uv \sin v^2$

$$\int \frac{du}{u} = \int v \sin v^2 dv$$

$$\ln u = -\frac{1}{2} \cos v^2 + C_1$$

$$u = Ce^{-(\cos v^2)/2}$$

Initial condition: $u(0) = 1$, $C = \frac{1}{e^{-1/2}} = e^{1/2}$

Particular solution: $u = e^{(1-\cos v^2)/2}$

51. $\sqrt{1-4x^2} \frac{dy}{dx} = x$

$$dy = \frac{x}{\sqrt{1-4x^2}} dx$$

$$\int dy = \int \frac{x}{\sqrt{1-4x^2}} dx$$

$$= -\frac{1}{8} \int (1-4x^2)^{-1/2} (-8x dx)$$

$$y = -\frac{1}{4}(1-4x^2)^{1/2} + C$$

55. $yy' - e^x = 0$

$$\int y dy = \int e^x dx$$

$$\frac{y^2}{2} = e^x + C_1$$

$$y^2 = 2e^x + C$$

Initial condition: $y(0) = 4$, $16 = 2 + C$, $C = 14$

Particular solution: $y^2 = 2e^x + 14$

59. $y(1+x^2) \frac{dy}{dx} = x(1+y^2)$

$$\frac{y}{1+y^2} dy = \frac{x}{1+x^2} dx$$

$$\frac{1}{2} \ln(1+y^2) = \frac{1}{2} \ln(1+x^2) + C_1$$

$$\ln(1+y^2) = \ln(1+x^2) + \ln C = \ln[C(1+x^2)]$$

$$1+y^2 = C(1+x^2)$$

$$y(0) = \sqrt{3}: 1+3 = C \Rightarrow C = 4$$

$$1+y^2 = 4(1+x^2)$$

$$y^2 = 3+4x^2$$

63. $dP - kP dt = 0$

$$\int \frac{dP}{P} = k \int dt$$

$$\ln P = kt + C_1$$

$$P = Ce^{kt}$$

Initial condition: $P(0) = P_0$, $P_0 = Ce^0 = C$

Particular solution: $P = P_0 e^{kt}$

$$65. \quad \frac{dy}{dx} = \frac{-9x}{16y}$$

$$\int 16y \, dy = -\int 9x \, dx$$

$$8y^2 = \frac{-9}{2}x^2 + C$$

$$\text{Initial condition: } y(1) = 1, \quad 8 = -\frac{9}{2} + C, \quad C = \frac{25}{2}$$

$$\text{Particular solution: } 8y^2 = \frac{-9}{2}x^2 + \frac{25}{2},$$

$$16y^2 + 9x^2 = 25$$

$$69. \quad f(x, y) = x^3 - 4xy^2 + y^3$$

$$f(tx, ty) = t^3 x^3 - 4t x t^2 y^2 + t^3 y^3$$

$$= t^3(x^3 - 4xy^2 + y^3)$$

Homogeneous of degree 3

$$73. \quad f(x, y) = 2 \ln xy$$

$$f(tx, ty) = 2 \ln tx ty$$

$$= 2 \ln t^2 xy = 2(\ln t^2 + \ln xy)$$

Not homogeneous

$$77. \quad y' = \frac{x+y}{2x}, \quad y = vx$$

$$v + x \frac{dv}{dx} = \frac{x+vx}{2x}$$

$$x \frac{dv}{dx} = \frac{1+v}{2} - v$$

$$2 \int \frac{dv}{1-v} = \int \frac{dx}{x}$$

$$-\ln(1-v)^2 = \ln|x| + \ln C = \ln|Cx|$$

$$\frac{1}{(1-v)^2} = |Cx|$$

$$\frac{1}{[1-(y/x)]^2} = |Cx|$$

$$\frac{x^2}{(x-y)^2} = |Cx|$$

$$|x| = C(x-y)^2$$

$$67. \quad m = \frac{dy}{dx} = \frac{0-y}{(x+2)-x} = -\frac{y}{2}$$

$$\int \frac{dy}{y} = \int -\frac{1}{2} dx$$

$$\ln y = -\frac{1}{2}x + C_1$$

$$y = Ce^{-x/2}$$

$$71. \quad f(x, y) = \frac{x^2 y^2}{\sqrt{x^2 + y^2}}$$

$$f(tx, ty) = \frac{t^4 x^2 y^2}{\sqrt{t^2 x^2 + t^2 y^2}} = t^3 \frac{x^2 y^2}{\sqrt{x^2 + y^2}}$$

Homogeneous of degree 3

$$75. \quad f(x, y) = 2 \ln \frac{x}{y}$$

$$f(tx, ty) = 2 \ln \frac{tx}{ty} = 2 \ln \frac{x}{y}$$

Homogeneous degree 0

$$79. \quad y' = \frac{x-y}{x+y}, \quad y = vx$$

$$v + x \frac{dv}{dx} = \frac{x-xv}{x+xv}$$

$$v \, dx + x \, dv = \frac{1-v}{1+v} dx$$

$$\int \frac{v+1}{v^2+2v-1} dv = -\int \frac{dx}{x}$$

$$\frac{1}{2} \ln|v^2+2v-1| = -\ln|x| + \ln C_1 = \ln \left| \frac{C_1}{x} \right|$$

$$|v^2+2v-1| = \frac{C}{x^2}$$

$$\left| \frac{y^2}{x^2} + 2\frac{y}{x} - 1 \right| = \frac{C}{x^2}$$

$$|y^2 + 2xy - x^2| = C$$

$$81. \quad y' = \frac{xy}{x^2 - y^2}, y = vx$$

$$v + x \frac{dv}{dx} = \frac{x^2 v}{x^2 - x^2 v^2}$$

$$v dx + x dv = \frac{v}{1 - v^2} dx$$

$$\int \frac{1 - v^2}{v^3} dv = \int \frac{dx}{x}$$

$$-\frac{1}{2v^2} - \ln|v| = \ln|x| + \ln C_1 = \ln|C_1 x|$$

$$\frac{-1}{2v^2} = \ln|C_1 x v|$$

$$\frac{-x^2}{2y^2} = \ln|C_1 y|$$

$$y = C e^{-x^2/2y^2}$$

$$85. \quad \left(x \sec \frac{y}{x} + y \right) dx - x dy = 0, y = vx$$

$$(x \sec v + xv) dx - x(v dx + x dv) = 0$$

$$(\sec v + v) dx = v dx + x dv$$

$$\int \cos v dv = \int \frac{dx}{x}$$

$$\sin v = \ln x + \ln C_1$$

$$x = C e^{\sin v}$$

$$= C e^{\sin(y/x)}$$

$$\text{Initial condition: } y(1) = 0, 1 = C e^0 = C$$

$$\text{Particular solution: } x = e^{\sin(y/x)}$$

$$83. \quad x dy - (2xe^{-y/x} + y) dx = 0, y = vx$$

$$x(v dx + x dv) - (2xe^{-v} + vx) dx = 0$$

$$\int e^v dv = \int \frac{2}{x} dx$$

$$e^v = \ln C_1 x^2$$

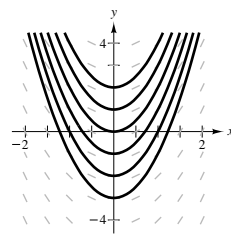
$$e^{y/x} = \ln C_1 + \ln x^2$$

$$e^{y/x} = C + \ln x^2$$

$$\text{Initial condition: } y(1) = 0, 1 = C$$

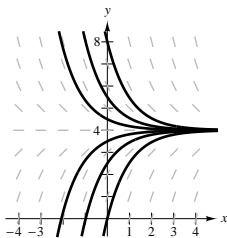
$$\text{Particular solution: } e^{y/x} = 1 + \ln x^2$$

$$87. \quad \frac{dy}{dx} = x$$



$$y = \int x dx = \frac{1}{2}x^2 + C$$

$$89. \quad \frac{dy}{dx} = 4 - y$$



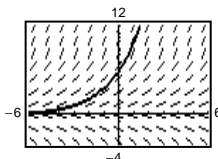
$$\int \frac{dy}{4 - y} = \int dx$$

$$\ln|4 - y| = -x + C_1$$

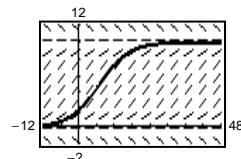
$$4 - y = e^{-x + C_1}$$

$$y = 4 + C e^{-x}$$

$$91. \quad \frac{dy}{dx} = 0.5y, y(0) = 6$$



$$93. \quad \frac{dy}{dx} = 0.02y(10 - y), y(0) = 2$$



95. $\frac{dy}{dt} = ky, y = Ce^{kt}$

Initial conditions: $y(0) = y_0$

$$y(1620) = \frac{y_0}{2}$$

$$C = y_0$$

$$\frac{y_0}{2} = y_0 e^{1620k}$$

$$k = \frac{\ln(1/2)}{1620}$$

Particular solution: $y = y_0 e^{-t(\ln 2)/1620}$

When $t = 25$, $y \approx 0.989y_0$, $y = 98.9\%$ of y_0 .

99. $\frac{dy}{dx} = ky(y - 4)$

The direction field satisfies $(dy/dx) = 0$ along $y = 0$ and $y = 4$. Matches (c).

101. $\frac{dw}{dt} = k(1200 - w)$

$$\int \frac{dw}{1200 - w} = \int k dt$$

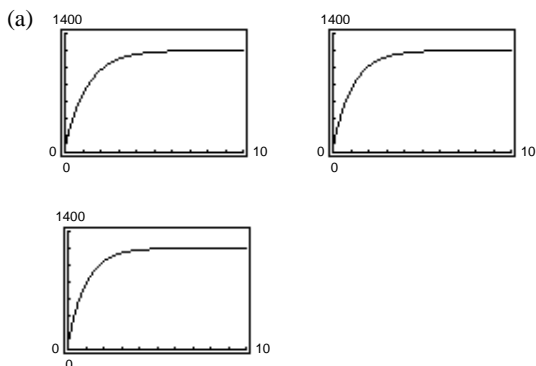
$$\ln(1200 - w) = -kt + C_1$$

$$1200 - w = e^{-kt + C_1} = Ce^{-kt}$$

$$w = 1200 - Ce^{-kt}$$

$$w(0) = 60 = 1200 - C \Rightarrow C = 1200 - 60 = 1140$$

$$w = 1200 - 1140e^{-kt}$$



(b) $k = 0.8$: $t = 1.31$ years

$k = 0.9$: $t = 1.16$ years

$k = 1.0$: $t = 1.05$ years

(c) Maximum weight: 1200 pounds

$$\lim_{t \rightarrow \infty} w = 1200$$

97. $\frac{dy}{dx} = k(y - 4)$

The direction field satisfies $(dy/dx) = 0$ along $y = 4$; but not along $y = 0$. Matches (a).

103. (a) $\frac{dv}{dt} = k(W - v)$

$$\int \frac{dv}{W - v} = \int k dt$$

$$-\ln(W - v) = kt + C_1$$

$$v = W - Ce^{-kt}$$

Initial conditions:

$$W = 20, v = 0 \text{ when } t = 0, \text{ and}$$

$$v = 5 \text{ when } t = 1.$$

$$C = 20, k = -\ln(3/4)$$

Particular solution:

$$v = 20(1 - e^{\ln(3/4)t}) \approx 20(1 - e^{-0.2877t})$$

$$\begin{aligned} \text{(b) } s &= \int 20(1 - e^{-0.2877t}) dt \\ &\approx 20[t + 3.4761e^{-0.2877t}] + C \end{aligned}$$

Since $S(0) = 0$, $C \approx -69.5$ and we have

$$s \approx 20t + 69.5(e^{-0.2877t} - 1).$$

105. Given family (circles): $x^2 + y^2 = C$

$$2x + 2yy' = 0$$

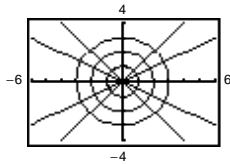
$$y' = -\frac{x}{y}$$

Orthogonal trajectory (lines): $y' = \frac{y}{x}$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln y = \ln x + \ln K$$

$$y = Kx$$



107. Given family (parabolas): $x^2 = Cy$

$$2x = Cy'$$

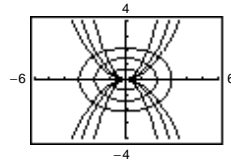
$$y' = \frac{2x}{C} = \frac{2x}{x^2/y} = \frac{2y}{x}$$

Orthogonal trajectory (ellipses): $y' = -\frac{x}{2y}$

$$2 \int y \, dy = - \int x \, dx$$

$$y^2 = -\frac{x^2}{2} + K_1$$

$$x^2 + 2y^2 = K$$



109. Given family: $y^2 = Cx^3$

$$2yy' = 3Cx^2$$

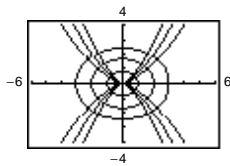
$$y' = \frac{3Cx^2}{2y} = \frac{3x^2}{2y} \left(\frac{y^2}{x^3} \right) = \frac{3y}{2x}$$

Orthogonal trajectory (ellipses): $y' = -\frac{2x}{3y}$

$$3 \int y \, dy = -2 \int x \, dx$$

$$\frac{3y^2}{2} = -x^2 + K_1$$

$$3y^2 + 2x^2 = K$$



111. A general solution of order n has n arbitrary constants while in a particular solution initial conditions are given in order to solve for all these constants.

113. $M(x, y)dx + N(x, y)dy = 0$, where M and N are homogeneous functions of the same degree.

115. False. Consider Example 2. $y = x^3$ is a solution to $xy' - 3y = 0$, but $y = x^3 + 1$ is not a solution.

117. False

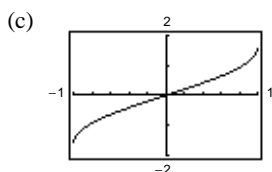
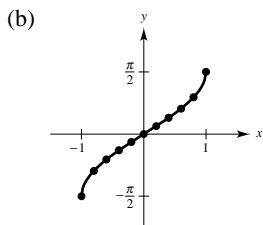
$$f(tx, ty) = t^2x^2 + t^2xy + 2 \\ \neq t^2f(x, y)$$

Section 5.8 Inverse Trigonometric Functions: Differentiation

1. $y = \arcsin x$

(a)

x	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
y	-1.571	-0.927	-0.644	-0.412	-0.201	0	0.201	0.412	0.644	0.927	1.571



(d) Symmetric about origin:
 $\arcsin(-x) = -\arcsin x$
 Intercept: $(0, 0)$

3. False.

$$\arccos \frac{1}{2} = \frac{\pi}{3}$$

since the range is $[0, \pi]$.

5. $\arcsin \frac{1}{2} = \frac{\pi}{6}$

7. $\arccos \frac{1}{2} = \frac{\pi}{3}$

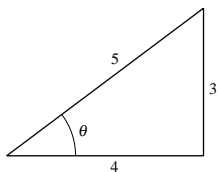
9. $\arctan \frac{\sqrt{3}}{3} = \frac{\pi}{6}$

11. $\operatorname{arccsc}(-\sqrt{2}) = -\frac{\pi}{4}$

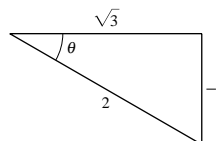
13. $\arccos(-0.8) \approx 2.50$

15. $\operatorname{arcsec}(1.269) = \arccos\left(\frac{1}{1.269}\right) \approx 0.66$

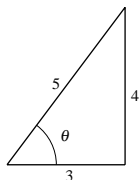
17. (a) $\sin(\arctan \frac{3}{4}) = \frac{3}{5}$



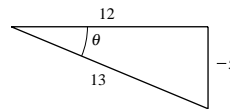
19. (a) $\cot\left[\arcsin\left(-\frac{1}{2}\right)\right] = \cot\left(-\frac{\pi}{6}\right) = -\sqrt{3}$



(b) $\sec(\arcsin \frac{4}{5}) = \frac{5}{3}$



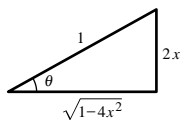
(b) $\csc\left[\arctan\left(-\frac{5}{12}\right)\right] = -\frac{13}{5}$



21. $y = \cos(\arcsin 2x)$

$$\theta = \arcsin 2x$$

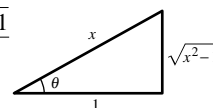
$$y = \cos \theta = \sqrt{1 - 4x^2}$$



23. $y = \sin(\operatorname{arcsec} x)$

$$\theta = \operatorname{arcsec} x, 0 \leq \theta \leq \pi, \theta \neq \frac{\pi}{2}$$

$$y = \sin \theta = \frac{\sqrt{x^2 - 1}}{|x|}$$

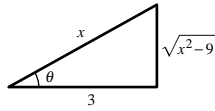


The absolute value bars on x are necessary because of the restriction $0 \leq \theta \leq \pi, \theta \neq \pi/2$, and $\sin \theta$ for this domain must always be nonnegative.

25. $y = \tan\left(\operatorname{arcsec} \frac{x}{3}\right)$

$$\theta = \operatorname{arcsec} \frac{x}{3}$$

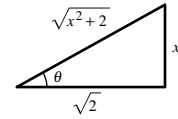
$$y = \tan \theta = \frac{\sqrt{x^2 - 9}}{3}$$



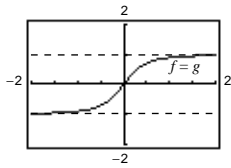
27. $y = \csc\left(\arctan \frac{x}{\sqrt{2}}\right)$

$$\theta = \arctan \frac{x}{\sqrt{2}}$$

$$y = \csc \theta = \frac{\sqrt{x^2 + 2}}{x}$$



29. $\sin(\arctan 2x) = \frac{2x}{\sqrt{1 + 4x^2}}$

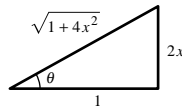


Asymptotes: $y = \pm 1$

$$\arctan 2x = \theta$$

$$\tan \theta = 2x$$

$$\sin \theta = \frac{2x}{\sqrt{1 + 4x^2}}$$



33. $\arcsin \sqrt{2x} = \arccos \sqrt{x}$

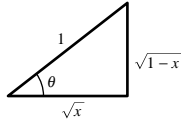
$$\sqrt{2x} = \sin(\arccos \sqrt{x})$$

$$\sqrt{2x} = \sqrt{1 - x}, 0 \leq x \leq 1$$

$$2x = 1 - x$$

$$3x = 1$$

$$x = \frac{1}{3}$$



35. (a) $\operatorname{arccsc} x = \arcsin \frac{1}{x}, |x| \geq 1$

Let $y = \operatorname{arccsc} x$. Then for

$$-\frac{\pi}{2} \leq y < 0 \text{ and } 0 < y \leq \frac{\pi}{2},$$

$$\csc y = x \Rightarrow \sin y = 1/x. \text{ Thus, } y = \arcsin(1/x).$$

Therefore, $\operatorname{arccsc} x = \arcsin(1/x)$.

(b) $\arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}, x > 0$

Let $y = \arctan x + \arctan(1/x)$. Then,

$$\tan y = \frac{\tan(\arctan x) + \tan[\arctan(1/x)]}{1 - \tan(\arctan x) \tan[\arctan(1/x)]}$$

$$= \frac{x + (1/x)}{1 - x(1/x)}$$

$$= \frac{x + (1/x)}{0} \text{ (which is undefined).}$$

Thus, $y = \pi/2$. Therefore, $\arctan x + \arctan(1/x) = \pi/2$.

37. $f(x) = \arcsin(x - 1)$

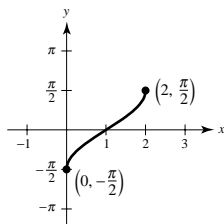
$$x - 1 = \sin y$$

$$x = 1 + \sin y$$

$$\text{Domain: } [0, 2]$$

$$\text{Range: } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$f(x)$ is the graph of $\arcsin x$ shifted 1 unit to the right.



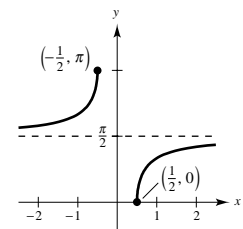
39. $f(x) = \operatorname{arcsec} 2x$

$$2x = \sec y$$

$$x = \frac{1}{2} \sec y$$

$$\text{Domain: } \left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$$

$$\text{Range: } \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$$



$$41. f(x) = 2 \arcsin(x - 1)$$

$$f'(x) = \frac{2}{\sqrt{1 - (x - 1)^2}} = \frac{2}{\sqrt{2x - x^2}}$$

$$45. f(x) = \arctan \frac{x}{a}$$

$$f'(x) = \frac{1/a}{1 + (x^2/a^2)} = \frac{a}{a^2 + x^2}$$

$$49. h(t) = \sin(\arccos t) = \sqrt{1 - t^2}$$

$$h'(t) = \frac{1}{2}(1 - t^2)^{-1/2}(-2t) = \frac{-t}{\sqrt{1 - t^2}}$$

$$53. y = \frac{1}{2} \left(\ln \frac{x+1}{x-1} + \arctan x \right)$$

$$= \frac{1}{4} [\ln(x+1) - \ln(x-1)] + \frac{1}{2} \arctan x$$

$$\frac{dy}{dx} = \frac{1}{4} \left(\frac{1}{x+1} - \frac{1}{x-1} \right) + \frac{1/2}{1+x^2} = \frac{1}{1-x^4}$$

$$57. y = 8 \arcsin \frac{x}{4} - \frac{x\sqrt{16-x^2}}{2}$$

$$\begin{aligned} y' &= 2 \frac{1}{\sqrt{1 - (x/4)^2}} - \frac{\sqrt{16-x^2}}{2} - \frac{x}{4}(16-x^2)^{-1/2}(-2x) \\ &= \frac{8}{\sqrt{16-x^2}} - \frac{\sqrt{16-x^2}}{2} + \frac{x^2}{2\sqrt{16-x^2}} \\ &= \frac{16 - (16-x^2) + x^2}{2\sqrt{16-x^2}} = \frac{x^2}{\sqrt{16-x^2}} \end{aligned}$$

$$61. f(x) = \arcsin x, a = \frac{1}{2}$$

$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$f''(x) = \frac{x}{(1-x^2)^{3/2}}$$

$$P_1(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) = \frac{\pi}{6} + \frac{2\sqrt{3}}{3}\left(x - \frac{1}{2}\right)$$

$$P_2(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) + \frac{1}{2}f''\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right)^2 = \frac{\pi}{6} + \frac{2\sqrt{3}}{3}\left(x - \frac{1}{2}\right) + \frac{2\sqrt{3}}{9}\left(x - \frac{1}{2}\right)^2$$

$$43. g(x) = 3 \arccos \frac{x}{2}$$

$$g'(x) = \frac{-3(1/2)}{\sqrt{1 - (x^2/4)}} = \frac{-3}{\sqrt{4-x^2}}$$

$$47. g(x) = \frac{\arcsin 3x}{x}$$

$$\begin{aligned} g'(x) &= \frac{x(3/\sqrt{1-9x^2}) - \arcsin 3x}{x^2} \\ &= \frac{3x - \sqrt{1-9x^2} \arcsin 3x}{x^2 \sqrt{1-9x^2}} \end{aligned}$$

$$51. y = x \arccos x - \sqrt{1-x^2}$$

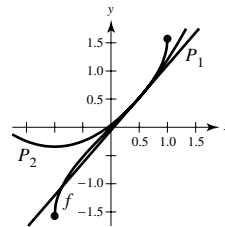
$$\begin{aligned} y' &= \arccos x - \frac{x}{\sqrt{1-x^2}} - \frac{1}{2}(1-x^2)^{-1/2}(-2x) \\ &= \arccos x \end{aligned}$$

$$55. y = x \arcsin x + \sqrt{1-x^2}$$

$$\frac{dy}{dx} = x \left(\frac{1}{\sqrt{1-x^2}} \right) + \arcsin x - \frac{x}{\sqrt{1-x^2}} = \arcsin x$$

$$59. y = \arctan x + \frac{x}{1+x^2}$$

$$\begin{aligned} y' &= \frac{1}{1+x^2} + \frac{(1+x^2) - x(2x)}{(1+x^2)^2} \\ &= \frac{(1+x^2) + (1-x^2)}{(1+x^2)^2} \\ &= \frac{2}{(1+x^2)^2} \end{aligned}$$



$$\begin{aligned}
 63. \quad f(x) &= \operatorname{arcsec} x - x \\
 f'(x) &= \frac{1}{|x|\sqrt{x^2 - 1}} - 1 \\
 &= 0 \text{ when } |x|\sqrt{x^2 - 1} = 1. \\
 x^2(x^2 - 1) &= 1 \\
 x^4 - x^2 - 1 &= 0 \text{ when } x^2 = \frac{1 + \sqrt{5}}{2} \text{ or} \\
 x &= \pm \sqrt{\frac{1 + \sqrt{5}}{2}} = \pm 1.272
 \end{aligned}$$

Relative maximum: (1.272, -0.606)

Relative minimum: (-1.272, 3.747)

67. The trigonometric functions are not one-to-one on $(-\infty, \infty)$, so their domains must be restricted to intervals on which they are one-to-one.

$$\begin{aligned}
 71. \quad (a) \quad \cot \theta &= \frac{x}{5} \\
 \theta &= \operatorname{arccot}\left(\frac{x}{5}\right)
 \end{aligned}$$

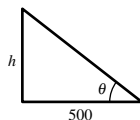
$$\begin{aligned}
 73. \quad (a) \quad h(t) &= -16t^2 + 256 \\
 -16t^2 + 256 &= 0 \text{ when } t = 4 \text{ sec.}
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad \tan \theta &= \frac{h}{500} = \frac{-16t^2 + 256}{500} \\
 \theta &= \arctan\left[\frac{16}{500}(-t^2 + 16)\right]
 \end{aligned}$$

$$\frac{d\theta}{dt} = \frac{-8t/125}{1 + [(4/125)(-t^2 + 16)]^2} = \frac{-1000t}{15,625 + 16(16 - t^2)^2}$$

When $t = 1$, $d\theta/dt \approx -0.0520$ rad/sec.

When $t = 2$, $d\theta/dt \approx -0.1116$ rad/sec.



$$\begin{aligned}
 65. \quad f(x) &= \arctan x - \arctan(x - 4) \\
 f'(x) &= \frac{1}{1 + x^2} - \frac{1}{1 + (x - 4)^2} = 0 \\
 1 + x^2 &= 1 + (x - 4)^2 \\
 0 &= -8x + 16 \\
 x &= 2
 \end{aligned}$$

By the First Derivative Test, (2, 2.214) is a relative maximum.

$$\begin{aligned}
 69. \quad y &= \operatorname{arccot} x, 0 < y < \pi \\
 x &= \cot y \\
 \tan y &= \frac{1}{x}
 \end{aligned}$$

So, graph the function

$$y = \arctan\left(\frac{1}{x}\right) \text{ for } x > 0 \text{ and } y = \arctan\left(\frac{1}{x}\right) + \pi \text{ for } x < 0.$$

$$(b) \quad \frac{d\theta}{dt} = \frac{-\frac{1}{5} \frac{dx}{dt}}{1 + \left(\frac{x}{5}\right)^2} = \frac{-5}{x^2 + 25} \frac{dx}{dt}$$

$$\text{If } \frac{dx}{dt} = -400 \text{ and } x = 10, \frac{d\theta}{dt} = 16 \text{ rad/hr.}$$

$$\text{If } \frac{dx}{dt} = -400 \text{ and } x = 3, \frac{d\theta}{dt} \approx 58.824 \text{ rad/hr.}$$

$$75. \tan(\arctan x + \arctan y) = \frac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x)\tan(\arctan y)} = \frac{x + y}{1 - xy}, xy \neq 1$$

Therefore,

$$\arctan x + \arctan y = \arctan\left(\frac{x + y}{1 - xy}\right), xy \neq 1.$$

Let $x = \frac{1}{2}$ and $y = \frac{1}{3}$.

$$\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \arctan \frac{(1/2) + (1/3)}{1 - [(1/2) \cdot (1/3)]} = \arctan \frac{5/6}{1 - (1/6)} = \arctan \frac{5/6}{5/6} = \arctan 1 = \frac{\pi}{4}$$

$$77. f(x) = kx + \sin x$$

$$f'(x) = k + \cos x \geq 0 \text{ for } k \geq 1$$

$$f'(x) = k + \cos x \leq 0 \text{ for } k \leq -1$$

Therefore, $f(x) = kx + \sin x$ is strictly monotonic and has an inverse for $k \leq -1$ or $k \geq 1$.

79. True

$$\frac{d}{dx}[\arctan x] = \frac{1}{1 + x^2} > 0 \text{ for all } x.$$

81. True

$$\frac{d}{dx}[\arctan(\tan x)] = \frac{\sec^2 x}{1 + \tan^2 x} = \frac{\sec^2 x}{\sec^2 x} = 1$$

Section 5.9 Inverse Trigonometric Functions: Integration

$$1. \int \frac{5}{\sqrt{9 - x^2}} dx = 5 \arcsin\left(\frac{x}{3}\right) + C$$

3. Let $u = 3x$, $du = 3 dx$.

$$\int_0^{1/6} \frac{1}{\sqrt{1 - 9x^2}} dx = \frac{1}{3} \int_0^{1/6} \frac{1}{\sqrt{1 - (3x)^2}} (3) dx = \left[\frac{1}{3} \arcsin(3x) \right]_0^{1/6} = \frac{\pi}{18}$$

$$5. \int \frac{7}{16 + x^2} dx = \frac{7}{4} \arctan\left(\frac{x}{4}\right) + C$$

7. Let $u = 2x$, $du = 2 dx$.

$$\int_0^{\sqrt{3}/2} \frac{1}{1 + 4x^2} dx = \frac{1}{2} \int_0^{\sqrt{3}/2} \frac{2}{1 + (2x)^2} dx = \left[\frac{1}{2} \arctan(2x) \right]_0^{\sqrt{3}/2} = \frac{\pi}{6}$$

$$9. \int \frac{1}{x\sqrt{4x^2 - 1}} dx = \int \frac{2}{2x\sqrt{(2x)^2 - 1}} dx = \operatorname{arcsec}|2x| + C$$

$$11. \int \frac{x^3}{x^2 + 1} dx = \int \left[x - \frac{x}{x^2 + 1} \right] dx = \int x dx - \frac{1}{2} \int \frac{2x}{x^2 + 1} dx = \frac{1}{2}x^2 - \frac{1}{2} \ln(x^2 + 1) + C \quad (\text{Use long division.})$$

$$13. \int \frac{1}{\sqrt{1 - (x + 1)^2}} dx = \arcsin(x + 1) + C$$

15. Let $u = t^2$, $du = 2t dt$.

$$\int \frac{t}{\sqrt{1 - t^4}} dt = \frac{1}{2} \int \frac{1}{\sqrt{1 - (t^2)^2}} (2t) dt = \frac{1}{2} \arcsin(t^2) + C$$

17. Let $u = \arcsin x$, $du = \frac{1}{\sqrt{1-x^2}} dx$.

$$\int_0^{1/\sqrt{2}} \frac{\arcsin x}{\sqrt{1-x^2}} dx = \left[\frac{1}{2} \arcsin^2 x \right]_0^{1/\sqrt{2}} = \frac{\pi^2}{32} \approx 0.308$$

19. Let $u = 1 - x^2$, $du = -2x dx$.

$$\begin{aligned} \int_{-1/2}^0 \frac{x}{\sqrt{1-x^2}} dx &= -\frac{1}{2} \int_{-1/2}^0 (1-x^2)^{-1/2} (-2x) dx \\ &= \left[-\sqrt{1-x^2} \right]_{-1/2}^0 = \frac{\sqrt{3}-2}{2} \\ &\approx -0.134 \end{aligned}$$

21. Let $u = e^{2x}$, $du = 2e^{2x} dx$.

$$\int \frac{e^{2x}}{4 + e^{4x}} dx = \frac{1}{2} \int \frac{2e^{2x}}{4 + (e^{2x})^2} dx = \frac{1}{4} \arctan \frac{e^{2x}}{2} + C$$

23. Let $u = \cos x$, $du = -\sin x dx$.

$$\begin{aligned} \int_{\pi/2}^{\pi} \frac{\sin x}{1 + \cos^2 x} dx &= -\int_{\pi/2}^{\pi} \frac{-\sin x}{1 + \cos^2 x} dx \\ &= \left[-\arctan(\cos x) \right]_{\pi/2}^{\pi} = \frac{\pi}{4} \end{aligned}$$

25. $\int \frac{1}{\sqrt{x}\sqrt{1-x}} dx$. $u = \sqrt{x}$, $x = u^2$, $dx = 2u du$

$$\begin{aligned} \int \frac{1}{u\sqrt{1-u^2}} (2u du) &= 2 \int \frac{du}{\sqrt{1-u^2}} = 2 \arcsin u + C \\ &= 2 \arcsin \sqrt{x} + C \end{aligned}$$

27. $\int \frac{x-3}{x^2+1} dx = \frac{1}{2} \int \frac{2x}{x^2+1} dx - 3 \int \frac{1}{x^2+1} dx$

$$= \frac{1}{2} \ln(x^2+1) - 3 \arctan x + C$$

29. $\int \frac{x+5}{\sqrt{9-(x-3)^2}} dx = \int \frac{(x-3)}{\sqrt{9-(x-3)^2}} dx + \int \frac{8}{\sqrt{9-(x-3)^2}} dx$

$$= -\sqrt{9-(x-3)^2} - 8 \arcsin\left(\frac{x-3}{3}\right) + C$$

$$= -\sqrt{6x-x^2} + 8 \arcsin\left(\frac{x}{3} - 1\right) + C$$

31. $\int_0^2 \frac{1}{x^2-2x+2} dx = \int_0^2 \frac{1}{1+(x-1)^2} dx = \left[\arctan(x-1) \right]_0^2 = \frac{\pi}{2}$

33. $\int \frac{2x}{x^2+6x+13} dx = \int \frac{2x+6}{x^2+6x+13} dx - 6 \int \frac{1}{x^2+6x+13} dx = \int \frac{2x+6}{x^2+6x+13} dx - 6 \int \frac{1}{4+(x+3)^2} dx$

$$= \ln|x^2+6x+13| - 3 \arctan\left(\frac{x+3}{2}\right) + C$$

35. $\int \frac{1}{\sqrt{-x^2-4x}} dx = \int \frac{1}{\sqrt{4-(x+2)^2}} dx = \arcsin\left(\frac{x+2}{2}\right) + C$

37. Let $u = -x^2 - 4x$, $du = (-2x - 4) dx$.

$$\int \frac{x+2}{\sqrt{-x^2-4x}} dx = -\frac{1}{2} \int (-x^2-4x)^{-1/2} (-2x-4) dx = -\sqrt{-x^2-4x} + C$$

39. $\int_2^3 \frac{2x-3}{\sqrt{4x-x^2}} dx = \int_2^3 \frac{2x-4}{\sqrt{4x-x^2}} dx + \int_2^3 \frac{1}{\sqrt{4x-x^2}} dx = -\int_2^3 (4x-x^2)^{-1/2} (4-2x) dx + \int_2^3 \frac{1}{\sqrt{4-(x-2)^2}} dx$

$$= \left[-2\sqrt{4x-x^2} + \arcsin\left(\frac{x-2}{2}\right) \right]_2^3 = 4 - 2\sqrt{3} + \frac{\pi}{6} \approx 1.059$$

41. Let $u = x^2 + 1$, $du = 2x dx$.

$$\int \frac{x}{x^4 + 2x^2 + 2} dx = \frac{1}{2} \int \frac{2x}{(x^2 + 1)^2 + 1} dx = \frac{1}{2} \arctan(x^2 + 1) + C$$

43. Let $u = \sqrt{e^t - 3}$. Then $u^2 + 3 = e^t$, $2u du = e^t dt$, and $\frac{2u du}{u^2 + 3} = dt$.

$$\begin{aligned} \int \sqrt{e^t - 3} dt &= \int \frac{2u^2}{u^2 + 3} du = \int 2 du - \int 6 \frac{1}{u^2 + 3} du \\ &= 2u - 2\sqrt{3} \arctan \frac{u}{\sqrt{3}} + C = 2\sqrt{e^t - 3} - 2\sqrt{3} \arctan \sqrt{\frac{e^t - 3}{3}} + C \end{aligned}$$

45. A perfect square trinomial is an expression in x with three terms that factor as a perfect square.

Example: $x^2 + 6x + 9 = (x + 3)^2$

47. (a) $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$, $u = x$ (b) $\int \frac{x}{\sqrt{1-x^2}} dx = -\sqrt{1-x^2} + C$, $u = 1-x^2$

- (c) $\int \frac{1}{x\sqrt{1-x^2}} dx$ cannot be evaluated using the basic integration rules.

49. (a) $\int \sqrt{x-1} dx = \frac{2}{3}(x-1)^{3/2} + C$, $u = x-1$

- (b) Let $u = \sqrt{x-1}$. Then $x = u^2 + 1$ and $dx = 2u du$.

$$\int x\sqrt{x-1} dx = \int (u^2 + 1)(u)(2u) du = 2 \int (u^4 + u^2) du = 2 \left(\frac{u^5}{5} + \frac{u^3}{3} \right) + C$$

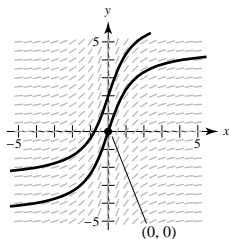
$$= \frac{2}{15} u^3(3u^2 + 5) + C = \frac{2}{15} (x-1)^{3/2} [3(x-1) + 5] + C = \frac{2}{15} (x-1)^{3/2} (3x+2) + C$$

- (c) Let $u = \sqrt{x-1}$. Then $x = u^2 + 1$ and $dx = 2u du$.

$$\int \frac{x}{\sqrt{x-1}} dx = \int \frac{u^2 + 1}{u} (2u) du = 2 \int (u^2 + 1) du = 2 \left(\frac{u^3}{3} + u \right) + C = \frac{2}{3} u(u^2 + 3) + C = \frac{2}{3} \sqrt{x-1} (x+2) + C$$

Note: In (b) and (c), substitution was necessary *before* the basic integration rules could be used.

51. (a)

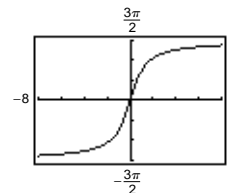


(b) $\frac{dy}{dx} = \frac{3}{1+x^2}$, $(0, 0)$

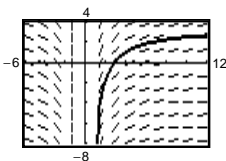
$$y = 3 \int \frac{dx}{1+x^2} = 3 \arctan x + C$$

$$(0, 0): 0 = 3 \arctan(0) + C \Rightarrow C = 0$$

$$y = 3 \arctan x$$



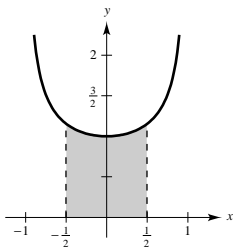
53. $\frac{dy}{dx} = \frac{10}{x\sqrt{x^2-1}}$, $y(3) = 0$



$$\begin{aligned} 55. A &= \int_1^3 \frac{1}{x^2 - 2x + 1 + 4} dx = \int_1^3 \frac{1}{(x-1)^2 + 2^2} dx \\ &= \left[\frac{1}{2} \arctan \left(\frac{x-1}{2} \right) \right]_1^3 = \frac{1}{2} \arctan(1) = \frac{\pi}{8} \approx 0.3927 \end{aligned}$$

57. Area $\approx (1)(1) = 1$

Matches (c)



59. (a) $\int_0^1 \frac{4}{1+x^2} dx = \left[4 \arctan x \right]_0^1 = 4 \arctan 1 - 4 \arctan 0 = 4\left(\frac{\pi}{4}\right) - 4(0) = \pi$

 (b) Let $n = 6$.

$$4 \int_0^1 \frac{4}{1+x^2} dx \approx 4\left(\frac{1}{36}\right) \left[1 + \frac{4}{1+(1/36)} + \frac{2}{1+(1/9)} + \frac{4}{1+(1/4)} + \frac{2}{1+(4/9)} + \frac{4}{1+(25/36)} + \frac{1}{2} \right] \approx 3.1415918$$

(c) 3.1415927

61. (a) $\frac{d}{dx} \left[\arcsin\left(\frac{u}{a}\right) + C \right] = \frac{1}{\sqrt{1-(u^2/a^2)}} \left(\frac{u'}{a}\right) = \frac{u'}{\sqrt{a^2-u^2}}$

Thus, $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C$.

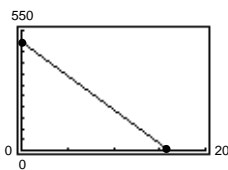
(b) $\frac{d}{dx} \left[\frac{1}{a} \arctan \frac{u}{a} + C \right] = \frac{1}{a} \left[\frac{u'/a}{1+(u/a)^2} \right] = \frac{1}{a^2} \left[\frac{u'}{(a^2+u^2)/a^2} \right] = \frac{u'}{a^2+u^2}$

Thus, $\int \frac{du}{a^2+u^2} = \int \frac{u'}{a^2+u^2} dx = \frac{1}{a} \arctan \frac{u}{a} + C$.

 (c) Assume $u > 0$.

$$\frac{d}{dx} \left[\frac{1}{a} \operatorname{arccsc} \frac{u}{a} + C \right] = \frac{1}{a} \left[\frac{u'/a}{(u/a)\sqrt{(u/a)^2-1}} \right] = \frac{1}{a} \left[\frac{u'}{u\sqrt{u^2-a^2}/a^2} \right] = \frac{u'}{u\sqrt{u^2-a^2}}. \text{ The case } u < 0 \text{ is handled in a similar manner.}$$

Thus, $\int \frac{du}{u\sqrt{u^2-a^2}} = \int \frac{u'}{u\sqrt{u^2-a^2}} dx = \frac{1}{a} \operatorname{arccsc} \frac{|u|}{a} + C$.

 63. (a) $v(t) = -32t + 500$


(b) $s(t) = \int v(t) dt = \int (-32t + 500) dt$

$$= -16t^2 + 500t + C$$

$$s(0) = -16(0) + 500(0) + C = 0 \implies C = 0$$

$$s(t) = -16t^2 + 500t$$

When the object reaches its maximum height,

$$v(t) = 0.$$

$$v(t) = -32t + 500 = 0$$

$$-32t = -500$$

$$t = 15.625$$

$$s(15.625) = -16(15.625)^2 + 500(15.625)$$

$$= 3906.25 \text{ ft (Maximum height)}$$

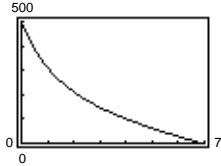
63. —CONTINUED—

$$\begin{aligned}
 \text{(c)} \quad \int \frac{1}{32 + kv^2} dv &= - \int dt \\
 \frac{1}{\sqrt{32k}} \arctan\left(\sqrt{\frac{k}{32}} v\right) &= -t + C_1 \\
 \arctan\left(\sqrt{\frac{k}{32}} v\right) &= -\sqrt{32k}t + C \\
 \sqrt{\frac{k}{32}} v &= \tan(C - \sqrt{32k}t) \\
 v &= \sqrt{\frac{32}{k}} \tan(C - \sqrt{32k}t)
 \end{aligned}$$

When $t = 0$, $v = 500$, $C = \arctan(500\sqrt{k/32})$, and we have

$$v(t) = \sqrt{\frac{32}{k}} \tan\left[\arctan\left(500\sqrt{\frac{k}{32}}\right) - \sqrt{32k}t\right].$$

$$\text{(d) When } k = 0.001, v(t) = \sqrt{32,000} \tan\left[\arctan(500\sqrt{0.00003125}) - \sqrt{0.032}t\right].$$



$v(t) = 0$ when $t_0 \approx 6.86$ sec.

$$\text{(e) } h = \int_0^{6.86} \sqrt{32,000} \tan\left[\arctan(500\sqrt{0.00003125}) - \sqrt{0.032}t\right] dt$$

Simpson's Rule: $n = 10$; $h \approx 1088$ feet

(f) Air resistance lowers the maximum height.

Section 5.10 Hyperbolic Functions

$$1. \text{ (a) } \sinh 3 = \frac{e^3 - e^{-3}}{2} \approx 10.018$$

$$\text{(b) } \tanh(-2) = \frac{\sinh(-2)}{\cosh(-2)} = \frac{e^{-2} - e^2}{e^{-2} + e^2} \approx -0.964$$

$$3. \text{ (a) } \operatorname{csch}(\ln 2) = \frac{2}{e^{\ln 2} - e^{-\ln 2}} = \frac{2}{2 - (1/2)} = \frac{4}{3}$$

$$\begin{aligned}
 \text{(b) } \operatorname{coth}(\ln 5) &= \frac{\cosh(\ln 5)}{\sinh(\ln 5)} = \frac{e^{\ln 5} + e^{-\ln 5}}{e^{\ln 5} - e^{-\ln 5}} \\
 &= \frac{5 + (1/5)}{5 - (1/5)} = \frac{13}{12}
 \end{aligned}$$

$$5. \text{ (a) } \cosh^{-1}(2) = \ln(2 + \sqrt{3}) \approx 1.317$$

$$\text{(b) } \operatorname{sech}^{-1}\left(\frac{2}{3}\right) = \ln\left(\frac{1 + \sqrt{1 - (4/9)}}{2/3}\right) \approx 0.962$$

$$7. \tanh^2 x + \operatorname{sech}^2 x = \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 + \left(\frac{2}{e^x + e^{-x}}\right)^2 = \frac{e^{2x} - 2 + e^{-2x} + 4}{(e^x + e^{-x})^2} = \frac{e^{2x} + 2 + e^{-2x}}{e^{2x} + 2 + e^{-2x}} = 1$$

$$\begin{aligned}
 9. \sinh x \cosh y + \cosh x \sinh y &= \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right) \\
 &= \frac{1}{4}[e^{x+y} - e^{-x+y} + e^{x-y} - e^{-(x+y)} + e^{x+y} + e^{-x+y} - e^{x-y} - e^{-(x+y)}] \\
 &= \frac{1}{4}[2(e^{x+y} - e^{-(x+y)})] = \frac{e^{(x+y)} - e^{-(x+y)}}{2} = \sinh(x+y)
 \end{aligned}$$

$$\begin{aligned}
 11. 3 \sinh x + 4 \sinh^3 x &= \sinh x(3 + 4 \sinh^2 x) = \left(\frac{e^x - e^{-x}}{2}\right)\left[3 + 4\left(\frac{e^x - e^{-x}}{2}\right)^2\right] \\
 &= \left(\frac{e^x - e^{-x}}{2}\right)[3 + e^{2x} - 2 + e^{-2x}] = \frac{1}{2}(e^x - e^{-x})(e^{2x} + e^{-2x} + 1) \\
 &= \frac{1}{2}[e^{3x} + e^{-x} + e^x - e^x - e^{-3x} - e^{-x}] = \frac{e^{3x} - e^{-3x}}{2} = \sinh(3x)
 \end{aligned}$$

$$13. \quad \sinh x = \frac{3}{2}$$

$$\cosh^2 x - \left(\frac{3}{2}\right)^2 = 1 \Rightarrow \cosh^2 x = \frac{13}{4} \Rightarrow \cosh x = \frac{\sqrt{13}}{2}$$

$$\tanh x = \frac{3/2}{\sqrt{13}/2} = \frac{3\sqrt{13}}{13}$$

$$\operatorname{csch} x = \frac{1}{3/2} = \frac{2}{3}$$

$$\operatorname{sech} x = \frac{1}{\sqrt{13}/2} = \frac{2\sqrt{13}}{13}$$

$$\operatorname{coth} x = \frac{1}{3/\sqrt{13}} = \frac{\sqrt{13}}{3}$$

$$15. \quad y = \sinh(1 - x^2)$$

$$y' = -2x \cosh(1 - x^2)$$

$$17. \quad f(x) = \ln(\sinh x)$$

$$f'(x) = \frac{1}{\sinh}(\cosh x) = \operatorname{coth} x$$

$$19. \quad y = \ln\left(\tanh \frac{x}{2}\right)$$

$$y' = \frac{1/2}{\tanh(x/2)} \operatorname{sech}^2\left(\frac{x}{2}\right) = \frac{1}{2 \sinh(x/2) \cosh(x/2)}$$

$$= \frac{1}{\sinh x} = \operatorname{csch} x$$

$$21. \quad h(x) = \frac{1}{4} \sinh(2x) - \frac{x}{2}$$

$$h'(x) = \frac{1}{2} \cosh(2x) - \frac{1}{2} = \frac{\cosh(2x) - 1}{2} = \sinh^2 x$$

$$23. \quad f(t) = \arctan(\sinh t)$$

$$f'(t) = \frac{1}{1 + \sinh^2 t}(\cosh t)$$

$$= \frac{\cosh t}{\cosh^2 t} = \operatorname{sech} t$$

$$25. \quad \text{Let } y = g(x).$$

$$y = x^{\cosh x}$$

$$\ln y = \cosh x \ln x$$

$$\frac{1}{y} \left(\frac{dy}{dx}\right) = \frac{\cosh x}{x} + \sinh x \ln x$$

$$\frac{dy}{dx} = \frac{y}{x} [\cosh x + x(\sinh x) \ln x]$$

$$= \frac{x^{\cosh x}}{x} [\cosh x + x(\sinh x) \ln x]$$

$$27. y = (\cosh x - \sinh x)^2$$

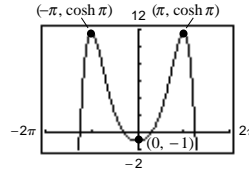
$$\begin{aligned} y' &= 2(\cosh x - \sinh x)(\sinh x - \cosh x) \\ &= -2(\cosh x - \sinh x)^2 = -2e^{-2x} \end{aligned}$$

$$29. f(x) = \sin x \sinh x - \cos x \cosh x, -4 \leq x \leq 4$$

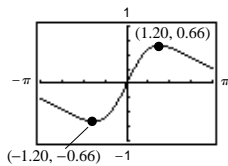
$$\begin{aligned} f'(x) &= \sin x \cosh x + \cos x \sinh x - \cos x \sinh x + \sin x \cosh x \\ &= 2 \sin x \cosh x = 0 \text{ when } x = 0, \pm\pi. \end{aligned}$$

Relative maxima: $(\pm\pi, \cosh \pi)$

Relative minimum: $(0, -1)$



$$31. g(x) = x \operatorname{sech} x = \frac{x}{\cosh x}$$



Relative maximum: $(1.20, 0.66)$

Relative minimum: $(-1.20, -0.66)$

$$33. y = a \sinh x$$

$$y' = a \cosh x$$

$$y'' = a \sinh x$$

$$y''' = a \cosh x$$

Therefore, $y''' - y' = 0$.

$$35. f(x) = \tanh x$$

$$f(1) = \tanh(1) \approx 0.7616$$

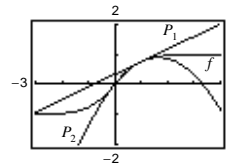
$$f'(x) = \operatorname{sech}^2 x$$

$$f'(1) = \frac{1}{\cosh^2(1)} \approx 0.4200$$

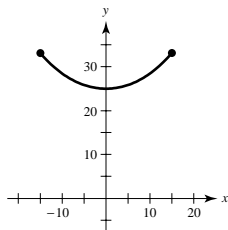
$$f''(x) = -2 \operatorname{sech}^2 x \cdot \tanh x \quad f''(1) \approx -0.6397$$

$$P_1(x) = f(1) + f'(1)(x - 1) = 0.7616 + 0.42(x - 1)$$

$$P_2(x) = 0.7616 + 0.42(x - 1) - \frac{0.6397}{2}(x - 1)^2$$



$$37. (a) y = 10 + 15 \cosh \frac{x}{15}, -15 \leq x \leq 15$$



$$(b) \text{ At } x = \pm 15, y = 10 + 15 \cosh(1) \approx 33.146.$$

$$\text{ At } x = 0, y = 10 + 15 \cosh(0) = 25.$$

$$(c) y' = \sinh \frac{x}{15}. \text{ At } x = 15, y' = \sinh(1) \approx 1.175$$

$$39. \text{ Let } u = 1 - 2x, du = -2 dx.$$

$$\begin{aligned} \int \sinh(1 - 2x) dx &= -\frac{1}{2} \int \sinh(u) (-2) du \\ &= -\frac{1}{2} \cosh(1 - 2x) + C \end{aligned}$$

$$41. \text{ Let } u = \cosh(x - 1), du = \sinh(x - 1) dx.$$

$$\int \cosh^2(x - 1) \sinh(x - 1) dx = \frac{1}{3} \cosh^3(x - 1) + C$$

43. Let $u = \sinh x$, $du = \cosh x dx$.

$$\int \frac{\cosh x}{\sinh x} dx = \ln|\sinh x| + C$$

45. Let $u = \frac{x^2}{2}$, $du = x dx$.

$$\int x \operatorname{csch}^2 \frac{x^2}{2} dx = \int \left(\operatorname{csch}^2 \frac{x^2}{2} \right) x dx = -\operatorname{coth} \frac{x^2}{2} + C$$

47. Let $u = \frac{1}{x}$, $du = -\frac{1}{x^2} dx$.

$$\int \frac{\operatorname{csch}(1/x) \operatorname{coth}(1/x)}{x^2} dx = -\int \operatorname{csch} \frac{1}{x} \operatorname{coth} \frac{1}{x} \left(-\frac{1}{x^2} \right) dx = \operatorname{csch} \frac{1}{x} + C$$

49. $\int_0^4 \frac{1}{25-x^2} dx = \left[\frac{1}{10} \ln \left| \frac{5+x}{5-x} \right| \right]_0^4 = \frac{1}{10} \ln 9 = \frac{1}{5} \ln 3$ 51. Let $u = 2x$, $du = 2 dx$.

$$\int_0^{\sqrt{2}/4} \frac{1}{\sqrt{1-(2x)^2}} (2) dx = \left[\arcsin(2x) \right]_0^{\sqrt{2}/4} = \frac{\pi}{4}$$

53. Let $u = x^2$, $du = 2x dx$.

$$\int \frac{x}{x^4+1} dx = \frac{1}{2} \int \frac{2x}{(x^2)^2+1} dx = \frac{1}{2} \arctan(x^2) + C$$

55. $y = \cosh^{-1}(3x)$

$$y' = \frac{3}{\sqrt{9x^2-1}}$$

57. $y = \sinh^{-1}(\tan x)$

$$y' = \frac{1}{\sqrt{\tan^2 x + 1}} (\sec^2 x) = |\sec x|$$

59. $y = \operatorname{coth}^{-1}(\sin 2x)$

$$y' = \frac{1}{1-\sin^2 2x} (2 \cos 2x) = 2 \sec 2x$$

61. $y = 2x \sinh^{-1}(2x) - \sqrt{1+4x^2}$

$$y' = 2x \left(\frac{2}{\sqrt{1+4x^2}} \right) + 2 \sinh^{-1}(2x) - \frac{4x}{\sqrt{1+4x^2}} = 2 \sinh^{-1}(2x)$$

63. See page 395.

65. $y = a \operatorname{sech}^{-1} \left(\frac{x}{a} \right) - \sqrt{a^2 - x^2}$

$$\frac{dy}{dx} = \frac{-1}{(x/a)\sqrt{1-(x^2/a^2)}} + \frac{x}{\sqrt{a^2-x^2}} = \frac{-a^2}{x\sqrt{a^2-x^2}} + \frac{x}{\sqrt{a^2-x^2}} = \frac{x^2-a^2}{x\sqrt{a^2-x^2}} = \frac{-\sqrt{a^2-x^2}}{x}$$

67. $\int \frac{1}{\sqrt{1+e^{2x}}} dx = \int \frac{e^x}{e^x \sqrt{1+(e^x)^2}} dx = -\operatorname{csch}^{-1}(e^x) + C = -\ln \left(\frac{1+\sqrt{1+e^{2x}}}{e^x} \right) + C$ 69. Let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}} dx$.

$$\int \frac{1}{\sqrt{x}\sqrt{1+x}} dx = 2 \int \frac{1}{\sqrt{1+(\sqrt{x})^2}} \left(\frac{1}{2\sqrt{x}} \right) dx = 2 \sinh^{-1} \sqrt{x} + C = 2 \ln(\sqrt{x} + \sqrt{1+x}) + C$$

71. $\int \frac{-1}{4x-x^2} dx = \int \frac{1}{(x-2)^2-4} dx = \frac{1}{4} \ln \left| \frac{(x-2)-2}{(x-2)+2} \right| = \frac{1}{4} \ln \left| \frac{x-4}{x} \right| + C$

$$\begin{aligned}
 73. \int \frac{1}{1-4x-2x^2} dx &= \int \frac{1}{3-2(x+1)^2} dx = \frac{-1}{\sqrt{2}} \int \frac{\sqrt{2}}{[\sqrt{2}(x+1)]^2 - (\sqrt{3})^2} dx \\
 &= \frac{-1}{2\sqrt{6}} \ln \left| \frac{\sqrt{2}(x+1) - \sqrt{3}}{\sqrt{2}(x+1) + \sqrt{3}} \right| + C = \frac{1}{2\sqrt{6}} \ln \left| \frac{\sqrt{2}(x+1) + \sqrt{3}}{\sqrt{2}(x+1) - \sqrt{3}} \right| + C
 \end{aligned}$$

75. Let $u = 4x - 1$, $du = 4 dx$.

$$y = \int \frac{1}{\sqrt{80+8x-16x^2}} dx = \frac{1}{4} \int \frac{4}{\sqrt{81-(4x-1)^2}} dx = \frac{1}{4} \arcsin\left(\frac{4x-1}{9}\right) + C$$

$$\begin{aligned}
 77. y &= \int \frac{x^3 - 21x}{5 + 4x - x^2} dx = \int \left(-x - 4 + \frac{20}{5 + 4x - x^2}\right) dx = \int (-x - 4) dx + 20 \int \frac{1}{3^2 - (x-2)^2} dx \\
 &= -\frac{x^2}{2} - 4x + \frac{20}{6} \ln \left| \frac{(x-2) + 3}{(x-2) - 3} \right| + C = -\frac{x^2}{2} - 4x + \frac{10}{3} \ln \left| \frac{x+1}{x-5} \right| + C = \frac{-x^2}{2} - 4x - \frac{10}{3} \ln \left| \frac{x-5}{x+1} \right| + C
 \end{aligned}$$

$$\begin{aligned}
 79. A &= 2 \int_0^4 \operatorname{sech} \frac{x}{2} dx \\
 &= 2 \int_0^4 \frac{2}{e^{x/2} + e^{-x/2}} dx \\
 &= 4 \int_0^4 \frac{e^{x/2}}{(e^{x/2})^2 + 1} dx \\
 &= \left[8 \arctan(e^{x/2}) \right]_0^4 \\
 &= 8 \arctan(e^2) - 2\pi \approx 5.207
 \end{aligned}$$

$$\begin{aligned}
 81. A &= \int_0^2 \frac{5x}{\sqrt{x^4+1}} dx \\
 &= \frac{5}{2} \int_0^2 \frac{2x}{\sqrt{(x^2)^2+1}} dx \\
 &= \left[\frac{5}{2} \ln(x^2 + \sqrt{x^4+1}) \right]_0^2 \\
 &= \frac{5}{2} \ln(4 + \sqrt{17}) \approx 5.237
 \end{aligned}$$

$$\begin{aligned}
 83. \int \frac{3k}{16} dt &= \int \frac{1}{x^2 - 12x + 32} dx \\
 \frac{3kt}{16} &= \int \frac{1}{(x-6)^2 - 4} dx = \frac{1}{2(2)} \ln \left| \frac{(x-6) - 2}{(x-6) + 2} \right| + C = \frac{1}{4} \ln \left| \frac{x-8}{x-4} \right| + C
 \end{aligned}$$

When $x = 0$: $t = 0$

$$C = -\frac{1}{4} \ln(2)$$

When $x = 1$: $t = 10$

$$\frac{30k}{16} = \frac{1}{4} \ln \left| \frac{-7}{-3} \right| - \frac{1}{4} \ln(2) = \frac{1}{4} \ln\left(\frac{7}{6}\right)$$

$$k = \frac{2}{15} \ln\left(\frac{7}{6}\right)$$

When $t = 20$: $\left(\frac{3}{16}\right)\left(\frac{2}{15}\right) \ln\left(\frac{7}{6}\right)(20) = \frac{1}{4} \ln \frac{x-8}{2x-8}$

$$\ln\left(\frac{7}{6}\right)^2 = \ln \frac{x-8}{2x-8}$$

$$\frac{49}{36} = \frac{x-8}{2x-8}$$

$$62x = 104$$

$$x = \frac{104}{62} = \frac{52}{31} \approx 1.677 \text{ kg}$$

85. As k increases, the time required for the object to reach the ground increases.

$$87. y = \cosh x = \frac{e^x + e^{-x}}{2}$$

$$y' = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$89. y = \cosh^{-1} x$$

$$\cosh y = x$$

$$(\sinh y)(y') = 1$$

$$y' = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2 - 1}}$$

$$91. y = \operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$

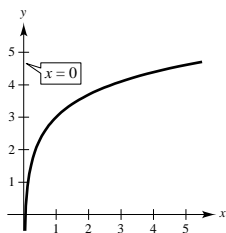
$$y' = -2(e^x + e^{-x})^{-2}(e^x - e^{-x}) = \left(\frac{-2}{e^x + e^{-x}}\right)\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = -\operatorname{sech} x \tanh x$$

Review Exercises for Chapter 5

1. $f(x) = \ln x + 3$

Vertical shift 3 units upward

Vertical asymptote: $x = 0$



$$3. \ln \sqrt[5]{\frac{4x^2 - 1}{4x^2 + 1}} = \frac{1}{5} \ln \frac{(2x - 1)(2x + 1)}{4x^2 + 1} = \frac{1}{5} [\ln(2x - 1) + \ln(2x + 1) - \ln(4x^2 + 1)]$$

$$5. \ln 3 + \frac{1}{3} \ln(4 - x^2) - \ln x = \ln 3 + \ln \sqrt[3]{4 - x^2} - \ln x = \ln \left(\frac{3\sqrt[3]{4 - x^2}}{x} \right)$$

$$7. \ln \sqrt{x + 1} = 2$$

$$\sqrt{x + 1} = e^2$$

$$x + 1 = e^4$$

$$x = e^4 - 1 \approx 53.598$$

$$9. g(x) = \ln \sqrt{x} = \frac{1}{2} \ln x$$

$$g'(x) = \frac{1}{2x}$$

$$11. f(x) = x\sqrt{\ln x}$$

$$f'(x) = \left(\frac{x}{2}\right)(\ln x)^{-1/2}\left(\frac{1}{x}\right) + \sqrt{\ln x}$$

$$= \frac{1}{2\sqrt{\ln x}} + \sqrt{\ln x} = \frac{1 + 2\ln x}{2\sqrt{\ln x}}$$

$$13. y = \frac{1}{b^2} \left[\ln(a + bx) + \frac{a}{a + bx} \right]$$

$$\frac{dy}{dx} = \frac{1}{b^2} \left[\frac{b}{a + bx} - \frac{ab}{(a + bx)^2} \right] = \frac{x}{(a + bx)^2}$$

$$15. y = -\frac{1}{a} \ln \left(\frac{a + bx}{x} \right) = -\frac{1}{a} [\ln(a + bx) - \ln x]$$

$$\frac{dy}{dx} = -\frac{1}{a} \left(\frac{b}{a + bx} - \frac{1}{x} \right) = \frac{1}{x(a + bx)}$$

$$17. u = 7x - 2, du = 7dx$$

$$\int \frac{1}{7x - 2} dx = \frac{1}{7} \int \frac{1}{7x - 2} (7) dx = \frac{1}{7} \ln|7x - 2| + C$$

$$\begin{aligned}
 19. \int \frac{\sin x}{1 + \cos x} dx &= -\int \frac{-\sin x}{1 + \cos x} dx \\
 &= -\ln|1 + \cos x| + C
 \end{aligned}$$

$$23. \int_0^{\pi/3} \sec \theta d\theta = \left[\ln|\sec \theta + \tan \theta| \right]_0^{\pi/3} = \ln(2 + \sqrt{3})$$

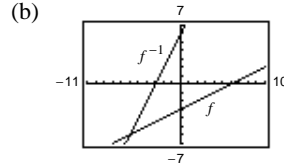
$$\begin{aligned}
 25. (a) \quad f(x) &= \frac{1}{2}x - 3 \\
 y &= \frac{1}{2}x - 3 \\
 2(y + 3) &= x \\
 2(x + 3) &= y \\
 f^{-1}(x) &= 2x + 6
 \end{aligned}$$

$$\begin{aligned}
 27. (a) \quad f(x) &= \sqrt{x + 1} \\
 y &= \sqrt{x + 1} \\
 y^2 - 1 &= x \\
 x^2 - 1 &= y \\
 f^{-1}(x) &= x^2 - 1, x \geq 0
 \end{aligned}$$

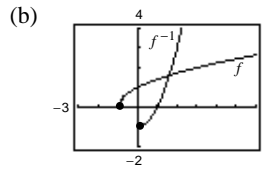
$$\begin{aligned}
 29. (a) \quad f(x) &= \sqrt[3]{x + 1} \\
 y &= \sqrt[3]{x + 1} \\
 y^3 - 1 &= x \\
 x^3 - 1 &= y \\
 f^{-1}(x) &= x^3 - 1
 \end{aligned}$$

$$\begin{aligned}
 31. \quad f(x) &= x^3 + 2 \\
 f^{-1}(x) &= (x - 2)^{1/3} \\
 (f^{-1})'(x) &= \frac{1}{3}(x - 2)^{-2/3} \\
 (f^{-1})'(-1) &= \frac{1}{3}(-1 - 2)^{-2/3} = \frac{1}{3(-3)^{2/3}} \\
 &= \frac{1}{3^{5/3}} \approx 0.160
 \end{aligned}$$

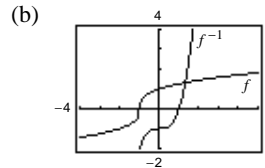
$$21. \int_1^4 \frac{x + 1}{x} dx = \int_1^4 \left(1 + \frac{1}{x}\right) dx = \left[x + \ln|x| \right]_1^4 = 3 + \ln 4$$



$$\begin{aligned}
 (c) \quad f^{-1}(f(x)) &= f^{-1}\left(\frac{1}{2}x - 3\right) = 2\left(\frac{1}{2}x - 3\right) + 6 = x \\
 f(f^{-1}(x)) &= f(2x + 6) = \frac{1}{2}(2x + 6) - 3 = x
 \end{aligned}$$



$$\begin{aligned}
 (c) \quad f^{-1}(f(x)) &= f^{-1}(\sqrt{x + 1}) = \sqrt{(x^2 - 1)^2} - 1 = x \\
 f(f^{-1}(x)) &= f(x^2 - 1) = \sqrt{(x^2 - 1) + 1} \\
 &= \sqrt{x^2} = x \text{ for } x \geq 0.
 \end{aligned}$$



$$\begin{aligned}
 (c) \quad f^{-1}(f(x)) &= f^{-1}(\sqrt[3]{x + 1}) = (\sqrt[3]{x + 1})^3 - 1 = x \\
 f(f^{-1}(x)) &= f(x^3 - 1) = \sqrt[3]{(x^3 - 1) + 1} = x
 \end{aligned}$$

$$\begin{aligned}
 33. \quad f(x) &= \tan x \\
 f\left(\frac{\pi}{6}\right) &= \frac{\sqrt{3}}{3} \\
 f'(x) &= \sec^2 x \\
 f'\left(\frac{\pi}{6}\right) &= \frac{4}{3} \\
 (f^{-1})'\left(\frac{\sqrt{3}}{3}\right) &= \frac{1}{f'(\pi/6)} = \frac{3}{4}
 \end{aligned}$$

35. (a) $f(x) = \ln \sqrt{x}$

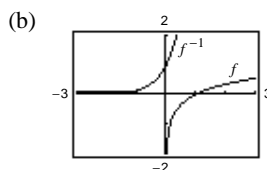
$$y = \ln \sqrt{x}$$

$$e^y = \sqrt{x}$$

$$e^{2y} = x$$

$$e^{2x} = y$$

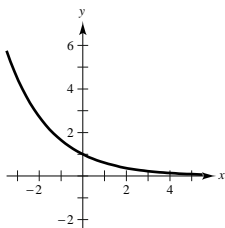
$$f^{-1}(x) = e^{2x}$$



(c) $f^{-1}(f(x)) = f^{-1}(\ln \sqrt{x}) = e^{2 \ln \sqrt{x}} = e^{\ln x} = x$

$$f(f^{-1}(x)) = f(e^{2x}) = \ln \sqrt{e^{2x}} = \ln e^x = x$$

37. $y = e^{-x/2}$



41. $g(t) = t^2 e^t$

$$g'(t) = t^2 e^t + 2te^t = te^t(t + 2)$$

45. $g(x) = \frac{x^2}{e^x}$

$$g'(x) = \frac{e^x(2x) - x^2 e^x}{e^{2x}} = \frac{x(2-x)}{e^x}$$

49. Let $u = -3x^2$, $du = -6x dx$.

$$\int x e^{-3x^2} dx = -\frac{1}{6} \int e^{-3x^2} (-6x) dx = -\frac{1}{6} e^{-3x^2} + C$$

53.
$$\int x e^{1-x^2} dx = -\frac{1}{2} \int e^{1-x^2} (-2x) dx$$
$$= -\frac{1}{2} e^{1-x^2} + C$$

57. $y = e^x(a \cos 3x + b \sin 3x)$

$$y' = e^x(-3a \sin 3x + 3b \cos 3x) + e^x(a \cos 3x + b \sin 3x)$$

$$= e^x[(-3a + b) \sin 3x + (a + 3b) \cos 3x]$$

$$y'' = e^x[3(-3a + b) \cos 3x - 3(a + 3b) \sin 3x] + e^x[(-3a + b) \sin 3x + (a + 3b) \cos 3x]$$

$$= e^x[(-6a - 8b) \sin 3x + (-8a + 6b) \cos 3x]$$

$$y'' - 2y' + 10y = e^x[(-6a - 8b) - 2(-3a + b) + 10b] \sin 3x + [(-8a + 6b) - 2(a + 3b) + 10a] \cos 3x = 0$$

43. $y = \sqrt{e^{2x} + e^{-2x}}$

$$y' = \frac{1}{2}(e^{2x} + e^{-2x})^{-1/2}(2e^{2x} - 2e^{-2x}) = \frac{e^{2x} - e^{-2x}}{\sqrt{e^{2x} + e^{-2x}}}$$

47. $y(\ln x) + y^2 = 0$

$$y\left(\frac{1}{x}\right) + (\ln x)\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) = 0$$

$$(2y + \ln x)\frac{dy}{dx} = \frac{-y}{x}$$

$$\frac{dy}{dx} = \frac{-y}{x(2y + \ln x)}$$

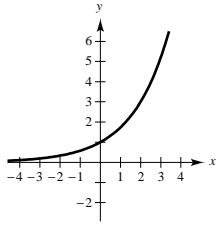
51.
$$\int \frac{e^{4x} - e^{2x} + 1}{e^x} dx = \int (e^{3x} - e^x + e^{-x}) dx$$
$$= \frac{1}{3} e^{3x} - e^x - e^{-x} + C$$
$$= \frac{e^{4x} - 3e^{2x} - 3}{3e^x} + C$$

55. Let $u = e^x - 1$, $du = e^x dx$.

$$\int \frac{e^x}{e^x - 1} dx = \ln|e^x - 1| + C$$

$$59. \text{Area} = \int_0^4 x e^{-x^2} dx = \left[-\frac{1}{2} e^{-x^2} \right]_0^4 = -\frac{1}{2}(e^{-16} - 1) \approx 0.500$$

$$61. y = 3^{3/2}$$



$$65. f(x) = 3^{x-1}$$

$$f'(x) = 3^{x-1} \ln 3$$

$$69. g(x) = \log_3 \sqrt{1-x} = \frac{1}{2} \log_3(1-x)$$

$$g'(x) = \frac{1}{2} \frac{-1}{(1-x) \ln 3} = \frac{1}{2(x-1) \ln 3}$$

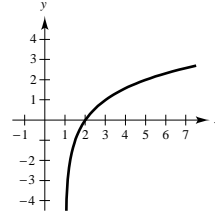
$$73. (a) y = x^a$$

$$y' = ax^{a-1}$$

$$(b) y = a^x$$

$$y' = (\ln a)a^x$$

$$63. y = \log_2(x-1)$$



$$67. y = x^{2x+1}$$

$$\ln y = (2x+1) \ln x$$

$$\frac{y'}{y} = \frac{2x+1}{x} + 2 \ln x$$

$$y' = y \left(\frac{2x+1}{x} + 2 \ln x \right) = x^{2x+1} \left(\frac{2x+1}{x} + 2 \ln x \right)$$

$$71. \int (x+1)5^{(x+1)^2} dx = \frac{1}{2} \frac{1}{\ln 5} 5^{(x+1)^2} + C$$

$$(c) y = x^x$$

$$\ln y = x \ln x$$

$$\frac{1}{y} y' = x \cdot \frac{1}{x} + (1) \ln x$$

$$y' = y(1 + \ln x)$$

$$y' = x^x(1 + \ln x)$$

$$(d) y = a^a$$

$$y' = 0$$

$$75. 10,000 = P e^{(0.07)(15)}$$

$$P = \frac{10,000}{e^{1.05}} \approx \$3499.38$$

$$77. P(h) = 30e^{kh}$$

$$P(18,000) = 30e^{18,000k} = 15$$

$$k = \frac{\ln(1/2)}{18,000} = \frac{-\ln 2}{18,000}$$

$$P(h) = 30e^{-(h \ln 2)/18,000}$$

$$P(35,000) = 30e^{-(35,000 \ln 2)/18,000} \approx 7.79 \text{ inches}$$

$$79. P = C e^{0.015t}$$

$$2C = C e^{0.015t}$$

$$2 = e^{0.015t}$$

$$\ln 2 = 0.015t$$

$$t = \frac{\ln 2}{0.015} \approx 46.21 \text{ years}$$

$$81. \frac{dy}{dx} = \frac{x^2 + 3}{x}$$

$$\int dy = \int \left(x + \frac{3}{x} \right) dx$$

$$y = \frac{x^2}{2} + 3 \ln|x| + C$$

83. $y' - 2xy = 0$

$$\frac{dy}{dx} = 2xy$$

$$\int \frac{1}{y} dy = \int 2x dx$$

$$\ln|y| = x^2 + C_1$$

$$e^{x^2+C_1} = y$$

$$y = Ce^{x^2}$$

85. $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$ (homogeneous differential equation)

$$(x^2 + y^2) dx - 2xy dy = 0$$

Let $y = vx$, $dy = x dv + v dx$.

$$(x^2 + v^2x^2) dx - 2x(vx)(x dv + v dx) = 0$$

$$(x^2 + v^2x^2 - 2x^2v^2) dx - 2x^3v dv = 0$$

$$(x^2 - x^2v^2) dx = 2x^3v dv$$

$$(1 - v^2) dx = 2x dv$$

$$\int \frac{dx}{x} = \int \frac{2v}{1 - v^2} dv$$

$$\ln|x| = -\ln|1 - v^2| + C_1 = -\ln|1 - v^2| + \ln C$$

$$x = \frac{C}{1 - v^2} = \frac{C}{1 - (y/x)^2} = \frac{Cx^2}{x^2 - y^2}$$

$$1 = \frac{Cx}{x^2 - y^2} \quad \text{or} \quad C_1 = \frac{x}{x^2 - y^2}$$

87. $y = C_1x + C_2x^3$

$$y' = C_1 + 3C_2x^2$$

$$y'' = 6C_2x$$

$$\begin{aligned} x^2y'' - 3xy' + 3y &= x^2(6C_2x) - 3x(C_1 + 3C_2x^2) + (C_1x + C_2x^3) \\ &= 6C_2x^3 - 3C_1x - 9C_2x^3 + 3C_1x + 3C_2x^3 = 0 \end{aligned}$$

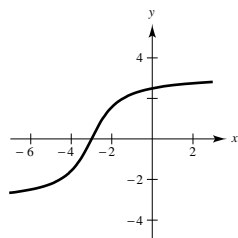
$$x = 2, y = 0: 0 = 2C_1 + 8C_2 \Rightarrow C_1 = -4C_2$$

$$x = 2, y' = 4: 4 = C_1 + 12C_2$$

$$4 = (-4C_2) + 12C_2 = 8C_2 \Rightarrow C_2 = \frac{1}{2}, C_1 = -2$$

$$y = -2x + \frac{1}{2}x^3$$

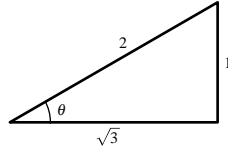
89. $f(x) = 2 \arctan(x + 3)$



91. (a) Let $\theta = \arcsin \frac{1}{2}$

$$\sin \theta = \frac{1}{2}$$

$$\sin\left(\arcsin \frac{1}{2}\right) = \sin \theta = \frac{1}{2}.$$



(b) Let $\theta = \arcsin \frac{1}{2}$

$$\sin \theta = \frac{1}{2}$$

$$\cos\left(\arcsin \frac{1}{2}\right) = \cos \theta = \frac{\sqrt{3}}{2}.$$

93. $y = \tan(\arcsin x) = \frac{x}{\sqrt{1-x^2}}$
 $y' = \frac{(1-x^2)^{1/2} + x^2(1-x^2)^{-1/2}}{1-x^2} = (1-x^2)^{-3/2}$

95. $y = x \operatorname{arcsec} x$

$$y' = \frac{x}{|x|\sqrt{x^2-1}} + \operatorname{arcsec} x$$

97. $y = x(\arcsin x)^2 - 2x + 2\sqrt{1-x^2} \arcsin x$
 $y' = \frac{2x \arcsin x}{\sqrt{1-x^2}} + (\arcsin x)^2 - 2 + \frac{2\sqrt{1-x^2}}{\sqrt{1-x^2}} - \frac{2x}{\sqrt{1-x^2}} \arcsin x = (\arcsin x)^2$

99. Let $u = e^{2x}$, $du = 2e^{2x} dx$.

$$\int \frac{1}{e^{2x} + e^{-2x}} dx = \int \frac{e^{2x}}{1 + e^{4x}} dx = \frac{1}{2} \int \frac{1}{1 + (e^{2x})^2} (2e^{2x}) dx = \frac{1}{2} \arctan(e^{2x}) + C$$

101. Let $u = x^2$, $du = 2x dx$.

$$\int \frac{x}{\sqrt{1-x^4}} dx = \frac{1}{2} \int \frac{1}{\sqrt{1-(x^2)^2}} (2x) dx = \frac{1}{2} \arcsin x^2 + C$$

103. Let $u = 16 + x^2$, $du = 2x dx$.

$$\int \frac{x}{16+x^2} dx = \frac{1}{2} \int \frac{1}{16+x^2} (2x) dx = \frac{1}{2} \ln(16+x^2) + C$$

105. Let $u = \arctan\left(\frac{x}{2}\right)$, $du = \frac{2}{4+x^2} dx$.

$$\int \frac{\arctan(x/2)}{4+x^2} dx = \frac{1}{2} \int \left(\arctan \frac{x}{2}\right) \left(\frac{2}{4+x^2}\right) dx = \frac{1}{4} \left(\arctan \frac{x}{2}\right)^2 + C$$

107. $\int \frac{dy}{\sqrt{A^2-y^2}} = \int \sqrt{\frac{k}{m}} dt$
 $\arcsin\left(\frac{y}{A}\right) = \sqrt{\frac{k}{m}} t + C$

109. $y = 2x - \cosh \sqrt{x}$

$$y' = 2 - \frac{1}{2\sqrt{x}} (\sinh \sqrt{x}) = 2 - \frac{\sinh \sqrt{x}}{2\sqrt{x}}$$

Since $y = 0$ when $t = 0$, you have $C = 0$. Thus,

$$\sin\left(\sqrt{\frac{k}{m}} t\right) = \frac{y}{A}$$

$$y = A \sin\left(\sqrt{\frac{k}{m}} t\right)$$

111. Let $u = x^2$, $du = 2x dx$.

$$\int \frac{x}{\sqrt{x^4-1}} dx = \frac{1}{2} \int \frac{1}{\sqrt{(x^2)^2-1}} (2x) dx = \frac{1}{2} \ln(x^2 + \sqrt{x^4-1}) + C$$