

# PART I

## CHAPTER 6 Applications of Integration

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# CHAPTER 6

## Applications of Integration

### Section 6.1 Area of a Region Between Two Curves

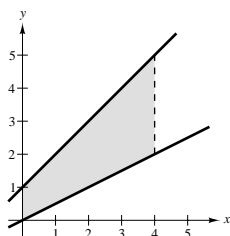
Solutions to Odd-Numbered Exercises

$$1. A = \int_0^6 [0 - (x^2 - 6x)] dx = - \int_0^6 (x^2 - 6x) dx$$

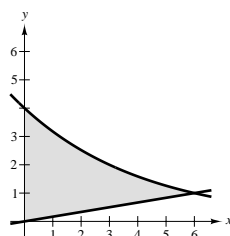
$$3. A = \int_0^3 [(-x^2 + 2x + 3) - (x^2 - 4x + 3)] dx = \int_0^3 (-2x^2 + 6x) dx$$

$$5. A = 2 \int_{-1}^0 3(x^3 - x) dx = 6 \int_{-1}^0 (x^3 - x) dx \quad \text{or} \quad -6 \int_0^1 (x^3 - x) dx$$

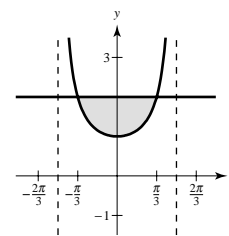
$$7. \int_0^4 \left[ (x + 1) - \frac{x}{2} \right] dx$$



$$9. \int_0^6 \left[ 4(2^{-x/3}) - \frac{x}{6} \right] dx$$



$$11. \int_{-\pi/3}^{\pi/3} [2 - \sec x] dx$$

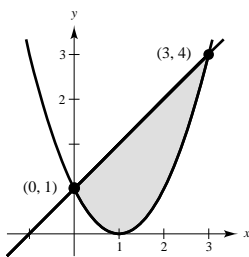


$$13. f(x) = x + 1$$

$$g(x) = (x - 1)^2$$

$$A \approx 4$$

Matches (d)

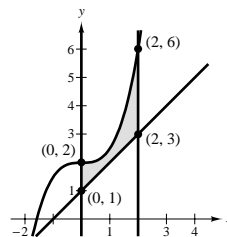


$$15. A = \int_0^2 \left[ \left( \frac{1}{2}x^3 + 2 \right) - (x + 1) \right] dx$$

$$= \int_0^2 \left( \frac{1}{2}x^3 - x + 1 \right) dx$$

$$= \left[ \frac{x^4}{8} - \frac{x^2}{2} + x \right]_0^2$$

$$= \left( \frac{16}{8} - \frac{4}{2} + 2 \right) - 0 = 2$$

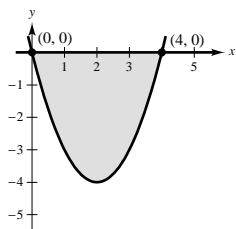


17. The points of intersection are given by:

$$x^2 - 4x = 0$$

$$x(x - 4) = 0 \text{ when } x = 0, 4$$

$$\begin{aligned} A &= \int_0^4 [g(x) - f(x)] dx \\ &= -\int_0^4 (x^2 - 4x) dx \\ &= -\left[\frac{x^3}{3} - 2x^2\right]_0^4 \\ &= \frac{32}{3} \end{aligned}$$

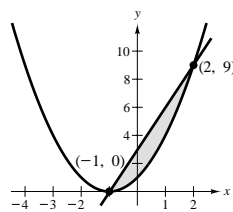


19. The points of intersection are given by:

$$x^2 + 2x + 1 = 3x + 3$$

$$(x - 2)(x + 1) = 0 \text{ when } x = -1, 2$$

$$\begin{aligned} A &= \int_{-1}^2 [g(x) - f(x)] dx \\ &= \int_{-1}^2 [(3x + 3) - (x^2 + 2x + 1)] dx \\ &= \int_{-1}^2 (2 + x - x^2) dx \\ &= \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^2 = \frac{9}{2} \end{aligned}$$



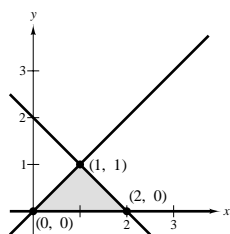
21. The points of intersection are given by:

$$x = 2 - x \text{ and } x = 0 \text{ and } 2 - x = 0$$

$$x = 1 \quad x = 0 \quad x = 2$$

$$A = \int_0^1 [(2 - y) - (y)] dy = \left[2y - y^2\right]_0^1 = 1$$

Note that if we integrate with respect to  $x$ , we need two integrals. Also, note that the region is a triangle.

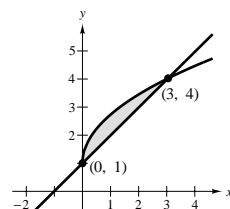


23. The points of intersection are given by:

$$\sqrt{3x} + 1 = x + 1$$

$$\sqrt{3x} = x \text{ when } x = 0, 3$$

$$\begin{aligned} A &= \int_0^3 [f(x) - g(x)] dx \\ &= \int_0^3 [(\sqrt{3x} + 1) - (x + 1)] dx \\ &= \int_0^3 [(3x)^{1/2} - x] dx \\ &= \left[\frac{2}{9}(3x)^{3/2} - \frac{x^2}{2}\right]_0^3 = \frac{3}{2} \end{aligned}$$

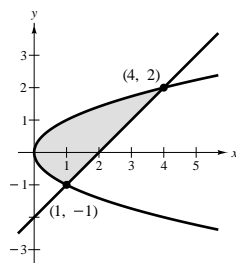


25. The points of intersection are given by:

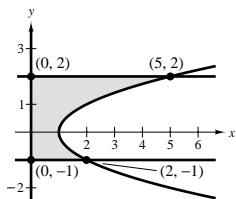
$$y^2 = y + 2$$

$$(y - 2)(y + 1) = 0 \text{ when } y = -1, 2$$

$$\begin{aligned} A &= \int_{-1}^2 [g(y) - f(y)] dy \\ &= \int_{-1}^2 [(y + 2) - y^2] dy \\ &= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_{-1}^2 = \frac{9}{2} \end{aligned}$$

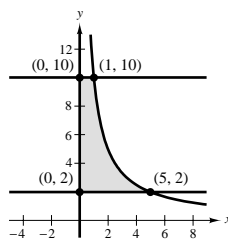


$$\begin{aligned}
 27. A &= \int_{-1}^2 [f(y) - g(y)] dy \\
 &= \int_{-1}^2 [(y^2 + 1) - 0] dy \\
 &= \left[ \frac{y^3}{3} + y \right]_{-1}^2 = 6
 \end{aligned}$$



$$29. y = \frac{10}{x} \Rightarrow x = \frac{10}{y}$$

$$\begin{aligned}
 A &= \int_2^{10} \frac{10}{y} dy \\
 &= \left[ 10 \ln y \right]_2^{10} \\
 &= 10(\ln 10 - \ln 2) \\
 &= 10 \ln 5 \approx 16.0944
 \end{aligned}$$



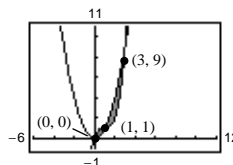
31. The points of intersection are given by:

$$x^3 - 3x^2 + 3x = x^2$$

$$x(x-1)(x-3) = 0 \quad \text{when } x = 0, 1, 3$$

$$\begin{aligned}
 A &= \int_0^1 [f(x) - g(x)] dx + \int_1^3 [g(x) - f(x)] dx \\
 &= \int_0^1 [(x^3 - 3x^2 + 3x) - x^2] dx + \int_1^3 [x^2 - (x^3 - 3x^2 + 3x)] dx \\
 &= \int_0^1 (x^3 - 4x^2 + 3x) dx + \int_1^3 (-x^3 + 4x^2 - 3x) dx \\
 &= \left[ \frac{x^4}{4} - \frac{4}{3}x^3 + \frac{3}{2}x^2 \right]_0^1 + \left[ -\frac{x^4}{4} + \frac{4}{3}x^3 - \frac{3}{2}x^2 \right]_1^3 = \frac{37}{12}
 \end{aligned}$$

Numerical Approximation:  $0.417 + 2.667 \approx 3.083$



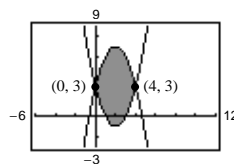
33. The points of intersection are given by:

$$x^2 - 4x + 3 = 3 + 4x - x^2$$

$$2x(x-4) = 0 \quad \text{when } x = 0, 4$$

$$\begin{aligned}
 A &= \int_0^4 [(3 + 4x - x^2) - (x^2 - 4x + 3)] dx \\
 &= \int_0^4 (-2x^2 + 8x) dx \\
 &= \left[ -\frac{2x^3}{3} + 4x^2 \right]_0^4 = \frac{64}{3}
 \end{aligned}$$

Numerical Approximation: 21.333



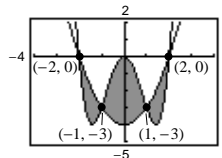
35.  $f(x) = x^4 - 4x^2$ ,  $g(x) = x^2 - 4$

The points of intersection are given by:

$$x^4 - 4x^2 = x^2 - 4$$

$$x^4 - 5x^2 + 4 = 0$$

$$(x^2 - 4)(x^2 - 1) = 0 \text{ when } x = \pm 2, \pm 1$$



By symmetry,

$$A = 2 \int_0^1 [(x^4 - 4x^2) - (x^2 - 4)] dx + 2 \int_1^2 [(x^2 - 4) - (x^4 - 4x^2)] dx$$

$$= 2 \int_0^1 (x^4 - 5x^2 + 4) dx + 2 \int_1^2 (-x^4 + 5x^2 - 4) dx$$

$$= 2 \left[ \frac{x^5}{5} - \frac{5x^3}{3} + 4x \right]_0^1 + 2 \left[ -\frac{x^5}{5} + \frac{5x^3}{3} - 4x \right]_1^2$$

$$= 2 \left[ \frac{1}{5} - \frac{5}{3} + 4 \right] + 2 \left[ \left( -\frac{32}{5} + \frac{40}{3} - 8 \right) - \left( -\frac{1}{5} + \frac{5}{3} - 4 \right) \right] = 8.$$

Numerical Approximation:  $5.067 + 2.933 = 8.0$

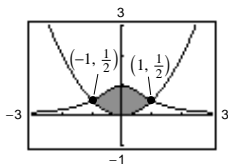
37. The points of intersection are given by:

$$\frac{1}{1+x^2} = \frac{x^2}{2}$$

$$x^4 + x^2 - 2 = 0$$

$$(x^2 + 2)(x^2 - 1) = 0$$

$$x = \pm 1$$



$$A = 2 \int_0^1 [f(x) - g(x)] dx$$

$$= 2 \int_0^1 \left[ \frac{1}{1+x^2} - \frac{x^2}{2} \right] dx$$

$$= 2 \left[ \arctan x - \frac{x^3}{6} \right]_0^1$$

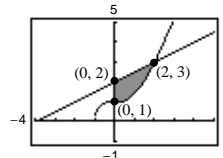
$$= 2 \left( \frac{\pi}{4} - \frac{1}{6} \right) = \frac{\pi}{2} - \frac{1}{3} \approx 1.237$$

Numerical Approximation: 1.237

39.  $\sqrt{1+x^3} \leq \frac{1}{2}x + 2$  on  $[0, 2]$

Numerical approximation: 1.759

$$A = \int_0^2 \left[ \frac{1}{2}x + 2 - \sqrt{1+x^3} \right] dx \approx 1.759$$

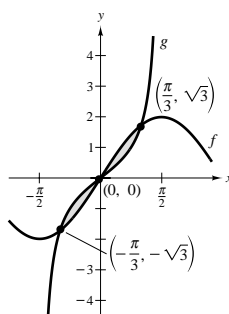


41.  $A = 2 \int_0^{\pi/3} [f(x) - g(x)] dx$

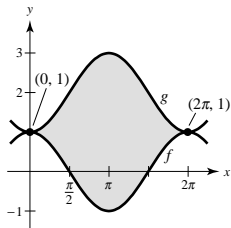
$$= 2 \int_0^{\pi/3} (2 \sin x - \tan x) dx$$

$$= 2 \left[ -2 \cos x + \ln |\cos x| \right]_0^{\pi/3}$$

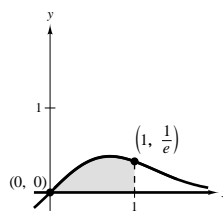
$$= 2(1 - \ln 2) \approx 0.614$$



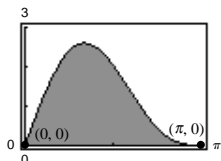
$$\begin{aligned}
 43. A &= \int_0^{2\pi} [(2 - \cos x) - \cos x] dx \\
 &= 2 \int_0^{2\pi} (1 - \cos x) dx \\
 &= 2 \left[ x - \sin x \right]_0^{2\pi} = 4\pi \approx 12.566
 \end{aligned}$$



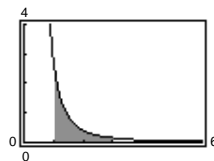
$$\begin{aligned}
 45. A &= \int_0^1 [xe^{-x^2} - 0] dx \\
 &= \left[ -\frac{1}{2}e^{-x^2} \right]_0^1 = \frac{1}{2} \left( 1 - \frac{1}{e} \right) \approx 0.316
 \end{aligned}$$



$$\begin{aligned}
 47. A &= \int_0^{\pi} [(2 \sin x + \sin 2x) - 0] dx \\
 &= \left[ -2 \cos x - \frac{1}{2} \cos 2x \right]_0^{\pi} = 4.0
 \end{aligned}$$



$$\begin{aligned}
 49. A &= \int_1^3 \left[ \frac{1}{x^2} e^{1/x} - 0 \right] dx \\
 &= \left[ -e^{1/x} \right]_1^3 = e - e^{1/3} \approx 1.323
 \end{aligned}$$

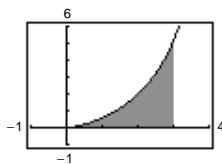


$$51. (a) y = \sqrt{\frac{x^3}{4-x}}, \quad y = 0, \quad x = 3$$

$$(b) A = \int_0^3 \sqrt{\frac{x^3}{4-x}} dx,$$

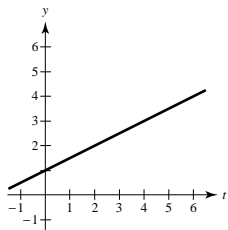
No, it cannot be evaluated by hand.

$$(c) 4.7721$$

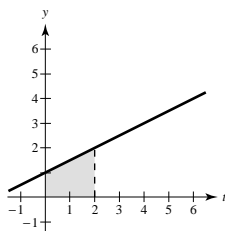


$$53. F(x) = \int_0^x \left( \frac{1}{2}t + 1 \right) dt = \left[ \frac{t^2}{4} + t \right]_0^x = \frac{x^2}{4} + x$$

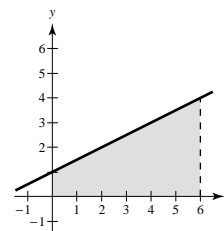
$$(a) F(0) = 0$$



$$(b) F(2) = \frac{2^2}{4} + 2 = 3$$

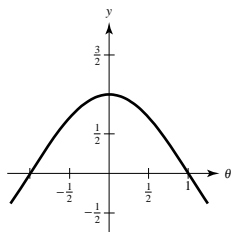


$$(c) F(6) = \frac{6^2}{4} + 6 = 15$$

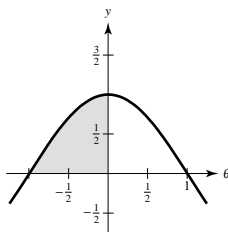


$$55. F(\alpha) = \int_{-1}^{\alpha} \cos \frac{\pi\theta}{2} d\theta = \left[ \frac{2}{\pi} \sin \frac{\pi\theta}{2} \right]_{-1}^{\alpha} = \frac{2}{\pi} \sin \frac{\pi\alpha}{2} + \frac{2}{\pi}$$

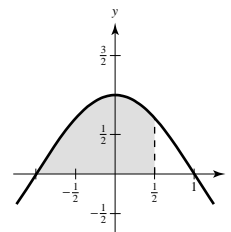
(a)  $F(-1) = 0$



(b)  $F(0) = \frac{2}{\pi} \approx 0.6366$



(c)  $F\left(\frac{1}{2}\right) = \frac{2 + \sqrt{2}}{\pi} \approx 1.0868$

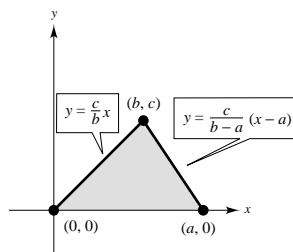


$$57. A = \int_0^c \left[ \left( \frac{b-a}{c}y + a \right) - \frac{b}{c}y \right] dy$$

$$= \int_0^c \left( -\frac{a}{c}y + a \right) dy$$

$$= \left[ -\frac{a}{2c}y^2 + ay \right]_0^c$$

$$= -\frac{ac}{2} + ac = \frac{ac}{2} \quad \left( = \frac{1}{2}(\text{base})(\text{height}) \right)$$



$$59. f(x) = x^3$$

$$f'(x) = 3x^2$$

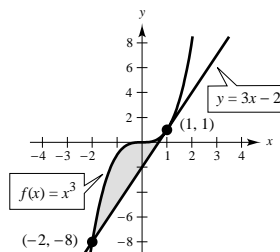
At  $(1, 1)$ ,  $f'(1) = 3$ .

Tangent line:

$$y - 1 = 3(x - 1) \text{ or } y = 3x - 2$$

The tangent line intersects  $f(x) = x^3$  at  $x = -2$ .

$$A = \int_{-2}^1 [x^3 - (3x - 2)] dx = \left[ \frac{x^4}{4} - \frac{3x^2}{2} + 2x \right]_{-2}^1 = \frac{27}{4}$$



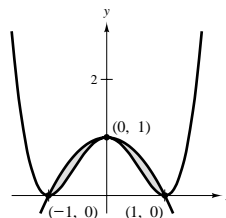
61. The variable is  $y$ .

$$63. x^4 - 2x^2 + 1 \leq 1 - x^2 \text{ on } [-1, 1]$$

$$A = \int_{-1}^1 [(1 - x^2) - (x^4 - 2x^2 + 1)] dx$$

$$= \int_{-1}^1 (x^2 - x^4) dx$$

$$= \left[ \frac{x^3}{3} - \frac{x^5}{5} \right]_{-1}^1 = \frac{4}{15}$$



You can use a single integral because  $x^4 - 2x^2 + 1 \leq 1 - x^2$  on  $[-1, 1]$ .

65. Offer 2 is better because the accumulated salary (area under the curve) is larger.

67.  $A = \int_{-3}^3 (9 - x^2) dx = 36$

$$\int_{-\sqrt{9-b}}^{\sqrt{9-b}} [(9 - x^2) - b] dx = 18$$

$$\int_0^{\sqrt{9-b}} [(9 - b) - x^2] dx = 9$$

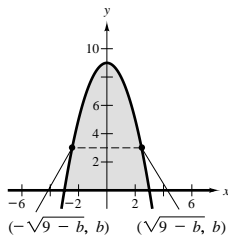
$$\left[ (9 - b)x - \frac{x^3}{3} \right]_0^{\sqrt{9-b}} = 9$$

$$\frac{2}{3} (9 - b)^{3/2} = 9$$

$$(9 - b)^{3/2} = \frac{27}{2}$$

$$9 - b = \frac{9}{\sqrt[3]{4}}$$

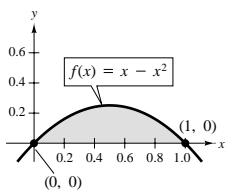
$$b = 9 - \frac{9}{\sqrt[3]{4}} \approx 3.330$$



69.  $\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n (x_i - x_i^2) \Delta x$

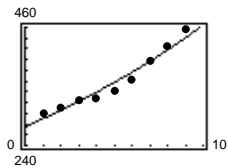
where  $x_i = \frac{i}{n}$  and  $\Delta x = \frac{1}{n}$  is the same as

$$\int_0^1 (x - x^2) dx = \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}$$

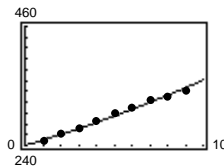


71.  $\int_0^5 [(7.21 + 0.58t) - (7.21 + 0.45t)] dt = \int_0^5 0.13t dt = \left[ \frac{0.13t^2}{2} \right]_0^5 = \$1.625 \text{ billion}$

73. (a)  $y_1 = (275.0675)(1.0537)^t = (275.0675)e^{0.0523t}$



(b)  $y_2 = (239.9407)(1.0417)^t = (239.9407)e^{0.0408t}$



(c)  $\int_{10}^{15} (y_1 - y_2) dt \approx 649.5 \text{ billion dollars}$

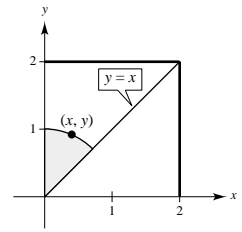
(d) No, model  $y_1 > y_2$  forever because  $1.0537 > 1.0417$ .

No, these models are not accurate. According to news reports,  $E > R$  eventually.



75. The total area is 8 times the area of the shaded region to the right. A point  $(x, y)$  is on the upper boundary of the region if

$$\begin{aligned}\sqrt{x^2 + y^2} &= 2 - y \\ x^2 + y^2 &= 4 - 4y + y^2 \\ x^2 &= 4 - 4y \\ 4y &= 4 - x^2 \\ y &= 1 - \frac{x^2}{4}.\end{aligned}$$



We now determine where this curve intersects the line  $y = x$ .

$$\begin{aligned}x &= 1 - \frac{x^2}{4} \\ x^2 + 4x - 4 &= 0 \\ x &= \frac{-4 \pm \sqrt{16 + 16}}{2} = -2 \pm 2\sqrt{2} \Rightarrow x = -2 + 2\sqrt{2}\end{aligned}$$

$$\begin{aligned}\text{Total area} &= 8 \int_0^{-2+2\sqrt{2}} \left(1 - \frac{x^2}{4} - x\right) dx \\ &= 8 \left[ x - \frac{x^3}{12} - \frac{x^2}{2} \right]_0^{-2+2\sqrt{2}} = \frac{16}{3}(4\sqrt{2} - 5) \approx 8(0.4379) = 3.503\end{aligned}$$

$$\begin{aligned}77. \text{ (a) } A &= 2 \left[ \int_0^5 \left(1 - \frac{1}{3}\sqrt{5-x}\right) dx + \int_5^{5.5} (1-0) dx \right] \\ &= 2 \left( \left[ x + \frac{2}{9}(5-x)^{3/2} \right]_0^5 + \left[ x \right]_5^{5.5} \right) = 2 \left( 5 - \frac{10\sqrt{5}}{9} + 5.5 - 5 \right) \approx 6.031 \text{ m}^2\end{aligned}$$

$$\text{(b) } V = 2A \approx 2(6.031) \approx 12.062 \text{ m}^3$$

$$\text{(c) } 5000 V \approx 5000(12.062) = 60,310 \text{ pounds}$$

79. True

81. False. Let  $f(x) = x$  and  $g(x) = 2x - x^2$ .  $f$  and  $g$  intersect at  $(1, 1)$ , the midpoint of  $[0, 2]$ . But

$$\int_a^b [f(x) - g(x)] dx = \int_0^2 [x - (2x - x^2)] dx = \frac{2}{3} \neq 0.$$

## Section 6.2 Volume: The Disk Method

$$1. V = \pi \int_0^1 (-x + 1)^2 dx = \pi \int_0^1 (x^2 - 2x + 1) dx = \pi \left[ \frac{x^3}{3} - x^2 + x \right]_0^1 = \frac{\pi}{3}$$

$$3. V = \pi \int_1^4 (\sqrt{x})^2 dx = \pi \int_1^4 x dx = \pi \left[ \frac{x^2}{2} \right]_1^4 = \frac{15\pi}{2}$$

$$5. V = \pi \int_0^1 [(x^2)^2 - (x^3)^2] dx = \pi \int_0^1 (x^4 - x^6) dx = \pi \left[ \frac{x^5}{5} - \frac{x^7}{7} \right]_0^1 = \frac{2\pi}{35}$$

$$7. y = x^2 \Rightarrow x = \sqrt{y}$$

$$\begin{aligned}V &= \pi \int_0^4 (\sqrt{y})^2 dy = \pi \int_0^4 y dy \\ &= \pi \left[ \frac{y^2}{2} \right]_0^4 = 8\pi\end{aligned}$$

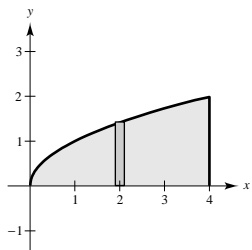
$$9. y = x^{2/3} \Rightarrow x = y^{3/2}$$

$$V = \pi \int_0^1 (y^{3/2})^2 dy = \pi \int_0^1 y^3 dy = \pi \left[ \frac{y^4}{4} \right]_0^1 = \frac{\pi}{4}$$

11.  $y = \sqrt{x}$ ,  $y = 0$ ,  $x = 4$

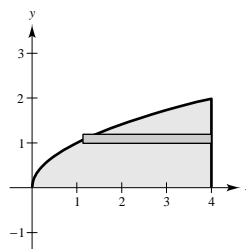
(a)  $R(x) = \sqrt{x}$ ,  $r(x) = 0$

$$\begin{aligned} V &= \pi \int_0^4 (\sqrt{x})^2 dx \\ &= \pi \int_0^4 x dx = \left[ \frac{\pi}{2} x^2 \right]_0^4 = 8\pi \end{aligned}$$



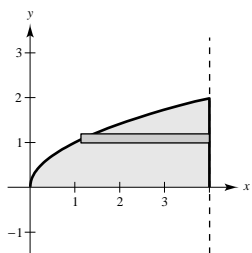
(b)  $R(y) = 4$ ,  $r(y) = y^2$

$$\begin{aligned} V &= \pi \int_0^2 (16 - y^4) dy \\ &= \pi \left[ 16y - \frac{1}{5} y^5 \right]_0^2 = \frac{128\pi}{5} \end{aligned}$$



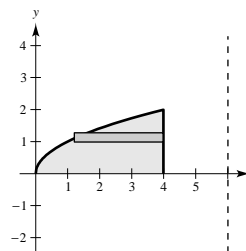
(c)  $R(y) = 4 - y^2$ ,  $r(y) = 0$

$$\begin{aligned} V &= \pi \int_0^2 (4 - y^2)^2 dy \\ &= \pi \int_0^2 (16 - 8y^2 + y^4) dy \\ &= \pi \left[ 16y - \frac{8}{3} y^3 + \frac{1}{5} y^5 \right]_0^2 = \frac{256\pi}{15} \end{aligned}$$



(d)  $R(y) = 6 - y^2$ ,  $r(y) = 2$

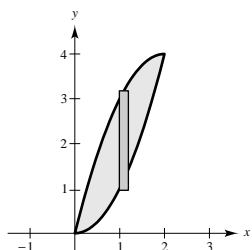
$$\begin{aligned} V &= \pi \int_0^2 [(6 - y^2)^2 - 4] dy \\ &= \pi \int_0^2 (32 - 12y^2 + y^4) dy \\ &= \pi \left[ 32y - 4y^3 + \frac{1}{5} y^5 \right]_0^2 = \frac{192\pi}{5} \end{aligned}$$



13.  $y = x^2$ ,  $y = 4x - x^2$  intersect at  $(0, 0)$  and  $(2, 4)$ .

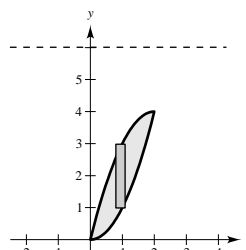
(a)  $R(x) = 4x - x^2$ ,  $r(x) = x^2$

$$\begin{aligned} V &= \pi \int_0^2 [(4x - x^2)^2 - x^4] dx \\ &= \pi \int_0^2 (16x^2 - 8x^3) dx \\ &= \pi \left[ \frac{16}{3} x^3 - 2x^4 \right]_0^2 = \frac{32\pi}{3} \end{aligned}$$



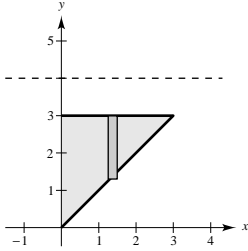
(b)  $R(x) = 6 - x^2$ ,  $r(x) = 6 - (4x - x^2)$

$$\begin{aligned} V &= \pi \int_0^2 [(6 - x^2)^2 - (6 - 4x + x^2)^2] dx \\ &= 8\pi \int_0^2 (x^3 - 5x^2 + 6x) dx \\ &= 8\pi \left[ \frac{x^4}{4} - \frac{5}{3} x^3 + 3x^2 \right]_0^2 = \frac{64\pi}{3} \end{aligned}$$



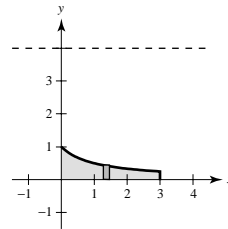
15.  $R(x) = 4 - x$ ,  $r(x) = 1$

$$\begin{aligned} V &= \pi \int_0^3 [(4-x)^2 - (1)^2] dx \\ &= \pi \int_0^3 (x^2 - 8x + 15) dx \\ &= \pi \left[ \frac{x^3}{3} - 4x^2 + 15x \right]_0^3 = 18\pi \end{aligned}$$



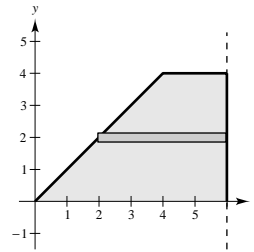
17.  $R(x) = 4$ ,  $r(x) = 4 - \frac{1}{1+x}$

$$\begin{aligned} V &= \pi \int_0^3 \left[ 4^2 - \left( 4 - \frac{1}{1+x} \right)^2 \right] dx \\ &= \pi \int_0^3 \left[ \frac{8}{1+x} - \frac{1}{(1+x)^2} \right] dx \\ &= \pi \left[ 8 \ln(1+x) + \frac{1}{1+x} \right]_0^3 \\ &= \pi \left[ 8 \ln 4 + \frac{1}{4} - 1 \right] \\ &= \left( 8 \ln 4 - \frac{3}{4} \right) \pi \approx 32.485 \end{aligned}$$



19.  $R(y) = 6 - y$ ,  $r(y) = 0$

$$\begin{aligned} V &= \pi \int_0^4 (6-y)^2 dy \\ &= \pi \int_0^4 (y^2 - 12y + 36) dy \\ &= \pi \left[ \frac{y^3}{3} - 6y^2 + 36y \right]_0^4 \\ &= \frac{208\pi}{3} \end{aligned}$$

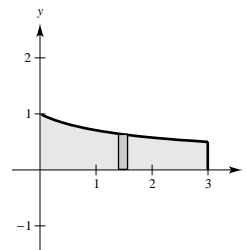


21.  $R(y) = 6 - y^2$ ,  $r(y) = 2$

$$\begin{aligned} V &= \pi \int_{-2}^2 [(6-y^2)^2 - (2)^2] dy \\ &= 2\pi \int_0^2 (y^4 - 12y^2 + 32) dy \\ &= 2\pi \left[ \frac{y^5}{5} - 4y^3 + 32y \right]_0^2 \\ &= \frac{384\pi}{5} \end{aligned}$$

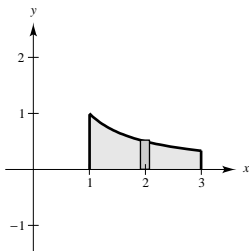
23.  $R(x) = \frac{1}{\sqrt{x+1}}$ ,  $r(x) = 0$

$$\begin{aligned} V &= \pi \int_0^3 \left( \frac{1}{\sqrt{x+1}} \right)^2 dx \\ &= \pi \int_0^3 \frac{1}{x+1} dx \\ &= \left[ \pi \ln|x+1| \right]_0^3 = \pi \ln 4 \end{aligned}$$



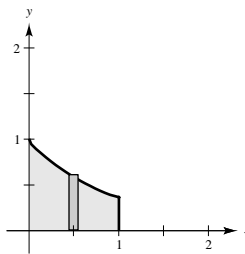
$$25. R(x) = \frac{1}{x}, r(x) = 0$$

$$\begin{aligned} V &= \pi \int_1^4 \left(\frac{1}{x}\right)^2 dx \\ &= \pi \left[ -\frac{1}{x} \right]_1^4 \\ &= \frac{3\pi}{4} \end{aligned}$$

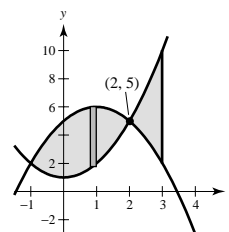


$$27. R(x) = e^{-x}, r(x) = 0$$

$$\begin{aligned} V &= \pi \int_0^1 (e^{-x})^2 dx \\ &= \pi \int_0^1 e^{-2x} dx \\ &= \left[ -\frac{\pi}{2} e^{-2x} \right]_0^1 \\ &= \frac{\pi}{2} (1 - e^{-2}) \approx 1.358 \end{aligned}$$

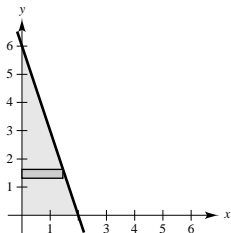


$$\begin{aligned} 29. V &= \pi \int_0^2 [(5 + 2x - x^2)^2 - (x^2 + 1)^2] dx + \pi \int_2^3 [(x^2 + 1)^2 - (5 + 2x - x^2)^2] dx \\ &= \pi \int_0^2 (-4x^3 - 8x^2 + 20x + 24) dx + \pi \int_2^3 (4x^3 + 8x^2 - 20x - 24) dx \\ &= \pi \left[ -x^4 - \frac{8}{3}x^3 + 10x^2 + 24x \right]_0^2 + \pi \left[ x^3 + \frac{8}{3}x^3 - 10x^2 - 24x \right]_2^3 \\ &= \pi \frac{152}{3} + \pi \frac{125}{3} = \frac{277\pi}{3} \end{aligned}$$

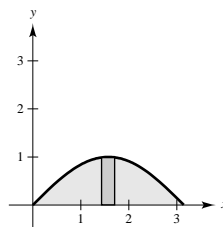


$$31. y = 6 - 3x \Rightarrow x = \frac{1}{3}(6 - y)$$

$$\begin{aligned} V &= \pi \int_0^6 \left[ \frac{1}{3}(6 - y) \right]^2 dy \\ &= \frac{\pi}{9} \int_0^6 [36 - 12y + y^2] dy \\ &= \frac{\pi}{9} \left[ 36y - 6y^2 + \frac{y^3}{3} \right]_0^6 \\ &= \frac{\pi}{9} \left[ 216 - 216 + \frac{216}{3} \right] \\ &= 8\pi \end{aligned}$$



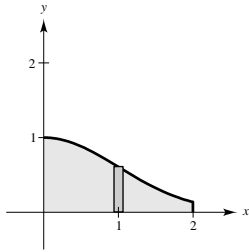
$$33. V = \pi \int_0^\pi [\sin x]^2 dx \approx 4.9348$$



$$35. V = \pi \int_0^2 [e^{-x^2}]^2 dx \approx 1.9686$$

$$39. A \approx 3$$

Matches (a)



$$37. V = \pi \int_{-1}^2 [e^{x/2} + e^{-x/2}]^2 dx \approx 49.0218$$

41. Disk Method:

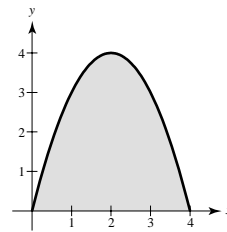
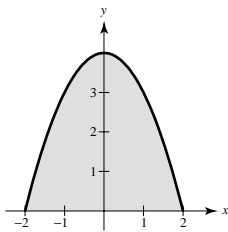
$$V = \pi \int_a^b [R(x)]^2 dx \quad \text{or} \quad V = \pi \int_c^d [R(y)]^2 dy$$

Washer Method:

$$V = \pi \int_a^b ([R(x)]^2 - [r(x)]^2) dx \quad \text{or}$$

$$V = \pi \int_c^d ([R(y)]^2 - [r(y)]^2) dy$$

43.



The volumes are the same because the solid has been translated horizontally.

$$45. R(x) = \frac{1}{2}x, \quad r(x) = 0$$

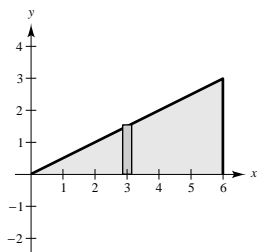
$$V = \pi \int_0^6 \frac{1}{4}x^2 dx$$

$$= \left[ \frac{\pi}{12}x^3 \right]_0^6 = 18\pi$$

**Note:**  $V = \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi(3^2)6$$

$$= 18\pi$$



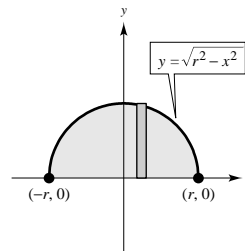
$$47. R(x) = \sqrt{r^2 - x^2}, \quad r(x) = 0$$

$$V = \pi \int_{-r}^r (r^2 - x^2) dx$$

$$= 2\pi \int_0^r (r^2 - x^2) dx$$

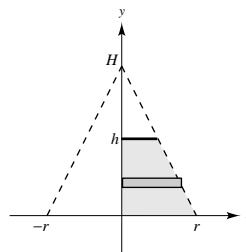
$$= 2\pi \left[ r^2x - \frac{1}{3}x^3 \right]_0^r$$

$$= 2\pi \left( r^3 - \frac{1}{3}r^3 \right) = \frac{4}{3}\pi r^3$$



$$49. x = r - \frac{r}{H}y = r\left(1 - \frac{y}{H}\right), R(y) = r\left(1 - \frac{y}{H}\right), r(y) = 0$$

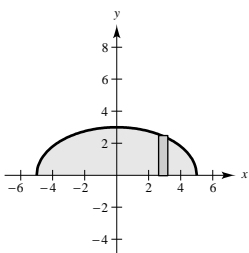
$$\begin{aligned} V &= \pi \int_0^h \left[ r\left(1 - \frac{y}{H}\right) \right]^2 dy = \pi r^2 \int_0^h \left( 1 - \frac{2}{H}y + \frac{1}{H^2}y^2 \right) dy \\ &= \pi r^2 \left[ y - \frac{1}{H}y^2 + \frac{1}{3H^2}y^3 \right]_0^h \\ &= \pi r^2 \left( h - \frac{h^2}{H} + \frac{h^3}{3H^2} \right) \\ &= \pi r^2 h \left( 1 - \frac{h}{H} + \frac{h^2}{3H^2} \right) \end{aligned}$$



$$51. V = \pi \int_0^2 \left( \frac{1}{8}x^2 \sqrt{2-x} \right)^2 dx = \frac{\pi}{64} \int_0^2 x^4(2-x) dx = \frac{\pi}{64} \left[ \frac{2x^5}{5} - \frac{x^6}{6} \right]_0^2 = \frac{\pi}{30}$$

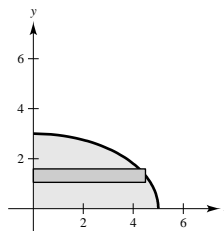
$$53. (a) R(x) = \frac{3}{5} \sqrt{25-x^2}, r(x) = 0$$

$$\begin{aligned} V &= \frac{9\pi}{25} \int_{-5}^5 (25-x^2) dx \\ &= \frac{18\pi}{25} \int_0^5 (25-x^2) dx \\ &= \frac{18\pi}{25} \left[ 25x - \frac{x^3}{3} \right]_0^5 = 60\pi \end{aligned}$$



$$(b) R(y) = \frac{5}{3} \sqrt{9-y^2}, r(y) = 0, x \geq 0$$

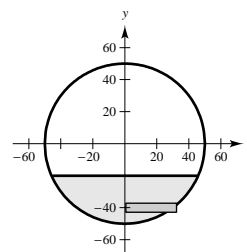
$$\begin{aligned} V &= \frac{25\pi}{9} \int_0^3 (9-y^2) dy \\ &= \frac{25\pi}{9} \left[ 9y - \frac{y^3}{3} \right]_0^3 = 50\pi \end{aligned}$$



$$55. \text{ Total volume: } V = \frac{4\pi(50)^3}{3} = \frac{500,000}{3} \pi \text{ ft}^3$$

Volume of water in the tank:

$$\begin{aligned} \pi \int_{-50}^{y_0} (\sqrt{2500-y^2})^2 dy &= \pi \int_{-50}^{y_0} (2500-y^2) dy \\ &= \pi \left[ 2500y - \frac{y^3}{3} \right]_{-50}^{y_0} \\ &= \pi \left( 2500y_0 - \frac{y_0^3}{3} + \frac{250,000}{3} \right) \end{aligned}$$



When the tank is one-fourth of its capacity:

$$\begin{aligned} \frac{1}{4} \left( \frac{500,000\pi}{3} \right) &= \pi \left( 2500y_0 - \frac{y_0^3}{3} + \frac{250,000}{3} \right) \\ 125,000 &= 7500y_0 - y_0^3 + 250,000 \end{aligned}$$

$$y_0^3 - 7500y_0 - 125,000 = 0$$

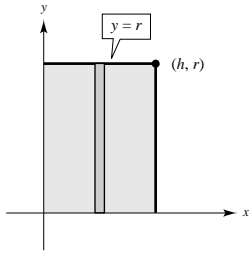
$$y_0 \approx -17.36$$

Depth:  $-17.36 - (-50) = 32.64$  feet

When the tank is three-fourths of its capacity the depth is  $100 - 32.64 = 67.36$  feet.

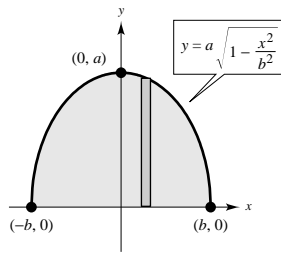
57. (a)  $\pi \int_0^h r^2 dx$  (ii)

is the volume of a right circular cylinder with radius  $r$  and height  $h$ .



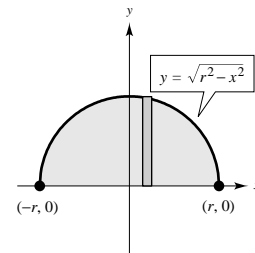
(b)  $\pi \int_{-b}^b \left( a \sqrt{1 - \frac{x^2}{b^2}} \right)^2 dx$  (iv)

is the volume of an ellipsoid with axes  $2a$  and  $2b$ .



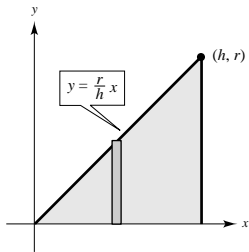
(c)  $\pi \int_{-r}^r (\sqrt{r^2 - x^2})^2 dx$  (iii)

is the volume of a sphere with radius  $r$ .



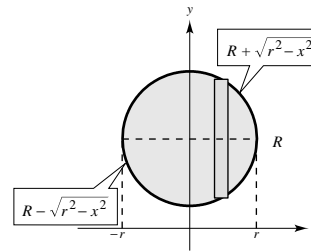
(d)  $\pi \int_0^h \left( \frac{rx}{h} \right)^2 dx$  (i)

is the volume of a right circular cone with the radius of the base as  $r$  and height  $h$ .

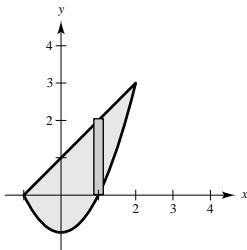


(e)  $\pi \int_{-r}^r \left[ (R + \sqrt{r^2 - x^2})^2 - (R - \sqrt{r^2 - x^2})^2 \right] dx$  (v)

is the volume of a torus with the radius of its circular cross section as  $r$  and the distance from the axis of the torus to the center of its cross section as  $R$ .



59.



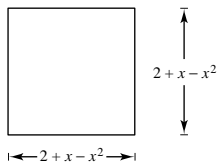
$$\text{Base of Cross Section} = (x + 1) - (x^2 - 1) = 2 + x - x^2$$

(a)  $A(x) = b^2 = (2 + x - x^2)^2$

$$= 4 + 4x - 3x^2 - 2x^3 + x^4$$

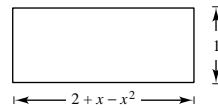
$$V = \int_{-1}^2 (4 + 4x - 3x^2 - 2x^3 + x^4) dx$$

$$= \left[ 4x + 2x^2 - x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5 \right]_{-1}^2 = \frac{81}{10}$$

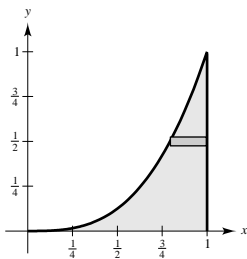


(b)  $A(x) = bh = (2 + x - x^2)1$

$$V = \int_{-1}^2 (2 + x - x^2) dx = \left[ 2x + \frac{x^2}{2} - \frac{x^3}{3} \right]_{-1}^2 = \frac{9}{2}$$



61.


 Base of Cross Section =  $1 - \sqrt[3]{y}$ 

$$(b) A(y) = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi \left(\frac{1 - \sqrt[3]{y}}{2}\right)^2 = \frac{1}{8}\pi(1 - \sqrt[3]{y})^2$$

$$V = \frac{1}{8}\pi \int_0^1 (1 - \sqrt[3]{y})^2 dy = \frac{\pi}{8} \left(\frac{1}{10}\right) = \frac{\pi}{80}$$

$$(c) A(y) = \frac{1}{2}bh = \frac{1}{2}(1 - \sqrt[3]{y})\left(\frac{\sqrt{3}}{2}\right)(1 - \sqrt[3]{y})$$

$$= \frac{\sqrt{3}}{4}(1 - \sqrt[3]{y})^2$$

$$V = \frac{\sqrt{3}}{4} \int_0^1 (1 - \sqrt[3]{y})^2 dy = \frac{\sqrt{3}}{4} \left(\frac{1}{10}\right) = \frac{\sqrt{3}}{40}$$

$$(d) A(y) = \frac{1}{2}\pi ab = \frac{\pi}{2}(2)(1 - \sqrt[3]{y})\frac{1 - \sqrt[3]{y}}{2}$$

$$= \frac{\pi}{2}(1 - \sqrt[3]{y})^2$$

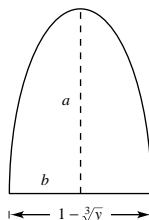
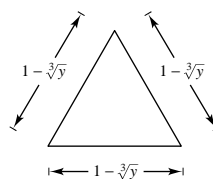
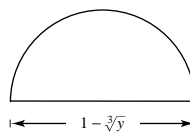
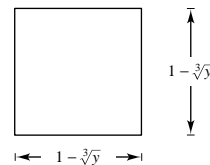
$$V = \frac{\pi}{2} \int_0^1 (1 - \sqrt[3]{y})^2 dy = \frac{\pi}{2} \left(\frac{1}{10}\right) = \frac{\pi}{20}$$

$$(a) A(y) = b^2 = (1 - \sqrt[3]{y})^2$$

$$V = \int_0^1 (1 - \sqrt[3]{y})^2 dy$$

$$= \int_0^1 (1 - 2y^{1/3} + y^{2/3}) dy$$

$$= \left[ y - \frac{3}{2}y^{4/3} + \frac{3}{5}y^{5/3} \right]_0^1 = \frac{1}{10}$$



63. Let  $A_1(x)$  and  $A_2(x)$  equal the areas of the cross sections of the two solids for  $a \leq x \leq b$ . Since  $A_1(x) = A_2(x)$ , we have

$$V_1 = \int_a^b A_1(x) dx = \int_a^b A_2(x) dx = V_2$$

Thus, the volumes are the same.

$$65. \frac{4}{3}\pi(25 - r^2)^{3/2} = \frac{1}{2}\left(\frac{4}{3}\right)\pi(125)$$

$$(25 - r^2)^{3/2} = \frac{125}{2}$$

$$25 - r^2 = \left(\frac{125}{2}\right)^{2/3}$$

$$25 - \frac{25}{(2^{2/3})} = r^2$$

$$25(1 - 2^{-2/3}) = r^2$$

$$r = 5\sqrt{1 - 2^{-2/3}} \approx 3.0415$$

67. (a) Since the cross sections are isosceles right triangles:

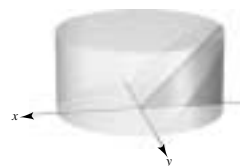
$$A(x) = \frac{1}{2}bh = \frac{1}{2}(\sqrt{r^2 - y^2})(\sqrt{r^2 - y^2}) = \frac{1}{2}(r^2 - y^2)$$

$$V = \frac{1}{2} \int_{-r}^r (r^2 - y^2) dy = \int_0^r (r^2 - y^2) dy = \left[ r^2y - \frac{y^3}{3} \right]_0^r = \frac{2}{3}r^3$$

$$(b) A(x) = \frac{1}{2}bh = \frac{1}{2}\sqrt{r^2 - y^2}(\sqrt{r^2 - y^2} \tan \theta) = \frac{\tan \theta}{2}(r^2 - y^2)$$

$$V = \frac{\tan \theta}{2} \int_{-r}^r (r^2 - y^2) dy = \tan \theta \int_0^r (r^2 - y^2) dy = \tan \theta \left[ r^2y - \frac{y^3}{3} \right]_0^r = \frac{2}{3}r^3 \tan \theta$$

As  $\theta \rightarrow 90^\circ$ ,  $V \rightarrow \infty$ .





## Section 6.3 Volume: The Shell Method

1.  $p(x) = x$

$h(x) = x$

$$V = 2\pi \int_0^2 x(x) dx = \left[ \frac{2\pi x^3}{3} \right]_0^2 = \frac{16\pi}{3}$$

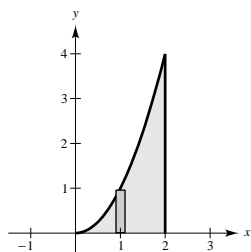
$$= \left[ \frac{2\pi x^3}{3} \right]_0^2 = \frac{16\pi}{3}$$

5.  $p(x) = x$

$h(x) = x^2$

$$V = 2\pi \int_0^2 x^3 dx$$

$$= \left[ \frac{\pi}{2} x^4 \right]_0^2 = 8\pi$$



3.  $p(x) = x$

$h(x) = \sqrt{x}$

$$V = 2\pi \int_0^4 x\sqrt{x} dx$$

$$= 2\pi \int_0^4 x^{3/2} dx$$

$$= \left[ \frac{4\pi}{5} x^{5/2} \right]_0^4 = \frac{128\pi}{5}$$

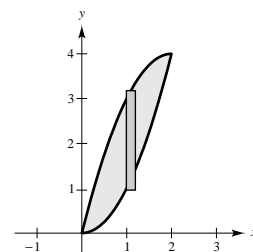
7.  $p(x) = x$

$h(x) = (4x - x^2) - x^2 = 4x - 2x^2$

$$V = 2\pi \int_0^2 x(4x - 2x^2) dx$$

$$= 4\pi \int_0^2 (2x^2 - x^3) dx$$

$$= 4\pi \left[ \frac{2}{3} x^3 - \frac{1}{4} x^4 \right]_0^2 = \frac{16\pi}{3}$$

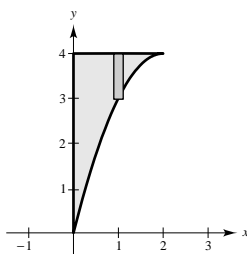


9.  $p(x) = x$

$h(x) = 4 - (4x - x^2) = x^2 - 4x + 4$

$$V = 2\pi \int_0^2 (x^3 - 4x^2 + 4x) dx$$

$$= 2\pi \left[ \frac{x^4}{4} - \frac{4}{3} x^3 + 2x^2 \right]_0^2 = \frac{8\pi}{3}$$



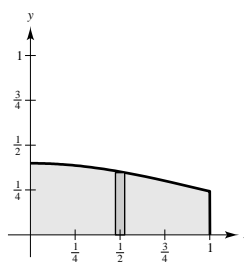
11.  $p(x) = x$

$h(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$$V = 2\pi \int_0^1 x \left( \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right) dx$$

$$= \sqrt{2\pi} \int_0^1 e^{-x^2/2} x dx$$

$$= \left[ -\sqrt{2\pi} e^{-x^2/2} \right]_0^1 = \sqrt{2\pi} \left( 1 - \frac{1}{\sqrt{e}} \right) \approx 0.986$$



13.  $p(y) = y$

$h(y) = 2 - y$

$$V = 2\pi \int_0^2 y(2 - y) dy$$

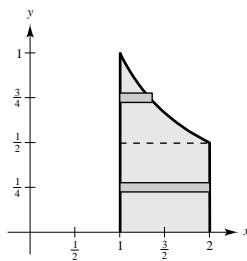
$$= 2\pi \int_0^2 (2y - y^2) dy$$

$$= 2\pi \left[ y^2 - \frac{y^3}{3} \right]_0^2 = \frac{8\pi}{3}$$

15.  $p(y) = y$  and  $h(y) = 1$  if  $0 \leq y < \frac{1}{2}$ .

$$p(y) = y \text{ and } h(y) = \frac{1}{y} - 1 \text{ if } \frac{1}{2} \leq y \leq 1.$$

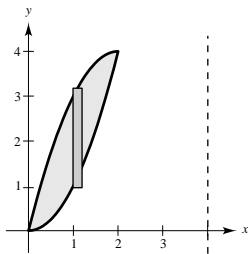
$$\begin{aligned} V &= 2\pi \int_0^{1/2} y \, dy + 2\pi \int_{1/2}^1 (1 - y) \, dy \\ &= 2\pi \left[ \frac{y^2}{2} \right]_0^{1/2} + 2\pi \left[ y - \frac{y^2}{2} \right]_{1/2}^1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2} \end{aligned}$$



17.  $p(x) = 4 - x$

$$h(x) = 4x - x^2 - x^2 = 4x - 2x^2$$

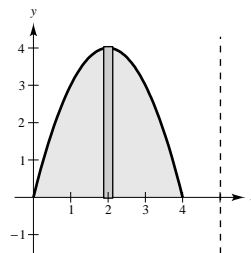
$$\begin{aligned} V &= 2\pi \int_0^2 (4 - x)(4x - 2x^2) \, dx \\ &= 2\pi(2) \int_0^2 (x^3 - 6x^2 + 8x) \, dx \\ &= 4\pi \left[ \frac{x^4}{4} - 2x^3 + 4x^2 \right]_0^2 = 16\pi \end{aligned}$$



19.  $p(x) = 5 - x$

$$h(x) = 4x - x^2$$

$$\begin{aligned} V &= 2\pi \int_0^4 (5 - x)(4x - x^2) \, dx \\ &= 2\pi \int_0^4 (x^3 - 9x^2 + 20x) \, dx \\ &= 2\pi \left[ \frac{x^4}{4} - 3x^3 + 10x^2 \right]_0^4 = 64\pi \end{aligned}$$

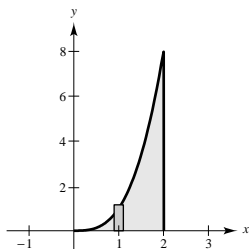


21. (a) Disk

$$R(x) = x^3$$

$$r(x) = 0$$

$$V = \pi \int_0^2 x^6 \, dx = \pi \left[ \frac{x^7}{7} \right]_0^2 = \frac{128\pi}{7}$$

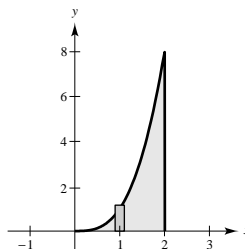


- (b) Shell

$$p(x) = x$$

$$h(x) = x^3$$

$$V = 2\pi \int_0^2 x^4 \, dx = 2\pi \left[ \frac{x^5}{5} \right]_0^2 = \frac{64\pi}{5}$$

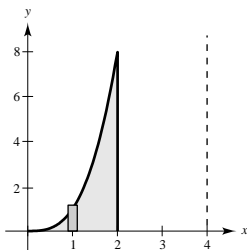


- (c) Shell

$$p(x) = 4 - x$$

$$h(x) = x^3$$

$$\begin{aligned} V &= 2\pi \int_0^2 (4 - x)x^3 \, dx \\ &= 2\pi \int_0^2 (4x^3 - x^4) \, dx \\ &= 2\pi \left[ x^4 - \frac{1}{5}x^5 \right]_0^2 = \frac{96\pi}{5} \end{aligned}$$

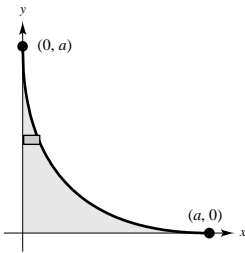


**23. (a) Shell**

$$p(y) = y$$

$$h(y) = (a^{1/2} - y^{1/2})^2$$

$$\begin{aligned} V &= 2\pi \int_0^a y(a - 2a^{1/2}y^{1/2} + y) dy \\ &= 2\pi \int_0^a (ay - 2a^{1/2}y^{3/2} + y^2) dy \\ &= 2\pi \left[ \frac{a}{2}y^2 - \frac{4a^{1/2}}{5}y^{5/2} + \frac{y^3}{3} \right]_0^a \\ &= 2\pi \left[ \frac{a^3}{2} - \frac{4a^3}{5} + \frac{a^3}{3} \right] = \frac{\pi a^3}{15} \end{aligned}$$



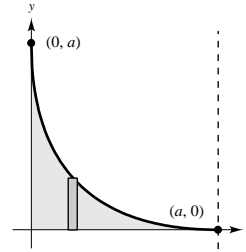
(b) Same as part (a) by symmetry

**(c) Shell**

$$p(x) = a - x$$

$$h(x) = (a^{1/2} - x^{1/2})^2$$

$$\begin{aligned} V &= 2\pi \int_0^a (a - x)(a^{1/2} - x^{1/2})^2 dx \\ &= 2\pi \int_0^a (a^2 - 2a^{3/2}x^{1/2} + 2a^{1/2}x^{3/2} - x^2) dx \\ &= 2\pi \left[ a^2x - \frac{4}{3}a^{3/2}x^{3/2} + \frac{4}{5}a^{1/2}x^{5/2} - \frac{1}{3}x^3 \right]_0^a = \frac{4\pi a^3}{15} \end{aligned}$$



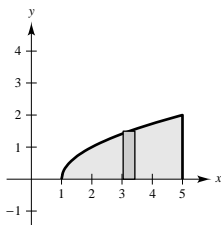
25.  $V = 2\pi \int_x^d p(y)h(y) dy$  or  $V = 2\pi \int_a^b p(x)h(x) dx$

27.  $\pi \int_1^5 (x - 1) dx = \pi \int_1^5 (\sqrt{x - 1})^2 dx$

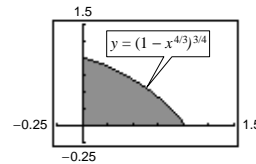
This integral represents the volume of the solid generated by revolving the region bounded by  $y = \sqrt{x - 1}$ ,  $y = 0$ , and  $x = 5$  about the  $x$ -axis by using the Disk Method.

$$2\pi \int_0^2 y[5 - (y^2 + 1)] dy$$

represents this same volume by using the Shell Method.



Disk Method

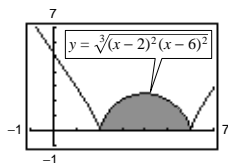
**29. (a)**


(b)  $x^{4/3} + y^{4/3} = 1$ ,  $x = 0$ ,  $y = 0$

$$y = (1 - x^{4/3})^{3/4}$$

$$V = 2\pi \int_0^1 x(1 - x^{4/3})^{3/4} dx \approx 1.5056$$

31. (a)



(b) 
$$V = 2\pi \int_2^6 x \sqrt[3]{(x-2)^2(x-6)^2} dx \approx 187.249$$

 35.  $p(x) = x$ 

$$h(x) = 2 - \frac{1}{2}x^2$$

$$V = 2\pi \int_0^2 x \left(2 - \frac{1}{2}x^2\right) dx = 2\pi \int_0^2 \left(2x - \frac{1}{2}x^3\right) dx = 2\pi \left[x^2 - \frac{1}{8}x^4\right]_0^2 = 4\pi \text{ (total volume)}$$

 Now find  $x_0$  such that

$$\pi = 2\pi \int_0^{x_0} \left(2x - \frac{1}{2}x^3\right) dx$$

$$1 = 2 \left[ x^2 - \frac{1}{8}x^4 \right]_0^{x_0}$$

$$1 = 2x_0^2 - \frac{1}{4}x_0^4$$

$$x_0^4 - 8x_0^2 + 4 = 0$$

$$x_0^2 = 4 \pm 2\sqrt{3} \quad \text{(Quadratic Formula)}$$

 Take  $x_0 = \sqrt{4 - 2\sqrt{3}}$  since the other root is too large.

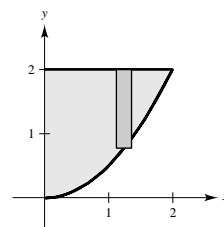
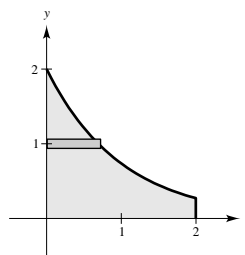
$$\text{Diameter: } 2\sqrt{4 - 2\sqrt{3}} \approx 1.464$$

$$\begin{aligned} 37. V &= 4\pi \int_{-1}^1 (2-x)\sqrt{1-x^2} dx \\ &= 8\pi \int_{-1}^1 \sqrt{1-x^2} dx - 4\pi \int_{-1}^1 x\sqrt{1-x^2} dx \\ &= 8\pi \left(\frac{\pi}{2}\right) + 2\pi \int_{-1}^1 x(1-x^2)^{1/2}(-2) dx \\ &= 4\pi^2 + \left[2\pi \left(\frac{2}{3}\right)(1-x^2)^{3/2}\right]_{-1}^1 = 4\pi^2 \end{aligned}$$

 33.  $y = 2e^{-x}, y = 0, x = 0, x = 2$ 

$$\text{Volume} \approx 7.5$$

Matches (d)

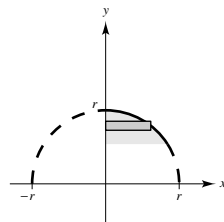


39. Disk Method

$$R(y) = \sqrt{r^2 - y^2}$$

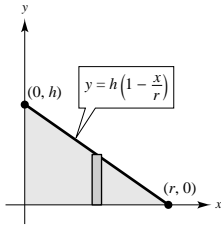
$$r(y) = 0$$

$$\begin{aligned} V &= \pi \int_{r-h}^r (r^2 - y^2) dy \\ &= \pi \left[ r^2 y - \frac{y^3}{3} \right]_{r-h}^r = \frac{1}{3} \pi h^2 (3r - h) \end{aligned}$$



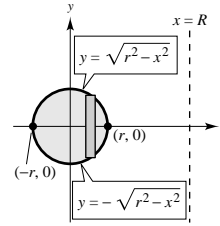
41. (a)  $2\pi \int_0^r hx \left(1 - \frac{x}{r}\right) dx$  (ii)

is the volume of a right circular cone with the radius of the base as  $r$  and height  $h$ .

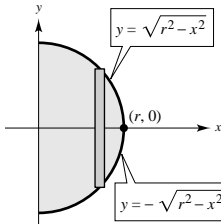


(b)  $2\pi \int_{-r}^r (R-x)(2\sqrt{r^2-x^2}) dx$  (v)

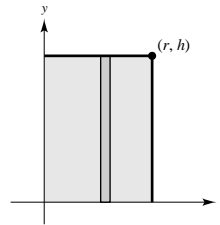
is the volume of a torus with the radius of its circular cross section as  $r$  and the distance from the axis of the torus to the center of its cross section as  $R$ .



(c)  $2\pi \int_0^r 2x\sqrt{r^2-x^2} dx$  (iii) is the volume of a sphere with radius  $r$ .

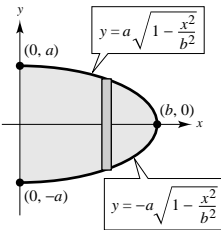


(d)  $2\pi \int_0^r hx dx$  (i) is the volume of a right circular cylinder with a radius of  $r$  and a height of  $h$ .



(e)  $2\pi \int_0^b 2ax\sqrt{1 - (x^2/b^2)} dx$  (iv)

is the volume of an ellipsoid with axes  $2a$  and  $2b$ .

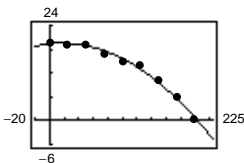


43. (a)  $V = 2\pi \int_0^{200} xf(x) dx$

$$\approx \frac{2\pi(200)}{3(8)} [0 + 4(25)(19) + 2(50)(19) + 4(75)(17) + 2(100)15 + 4(125)(14) + 2(150)(10) + 4(175)(6) + 0]$$

$$\approx 1,366,593 \text{ cubic feet}$$

(b)  $d = -0.000561x^2 + 0.0189x + 19.39$



(c)  $V \approx 2\pi \int_0^{200} xd(x) dx \approx 2\pi(213,800) = 1,343,345$  cubic feet

(d) Number gallons  $\approx V(7.48) = 10,048,221$  gallons

## Section 6.4 Arc Length and Surfaces of Revolution

1.  $(0, 0), (5, 12)$

(a)  $d = \sqrt{(5-0)^2 + (12-0)^2} = 13$

(b)  $y = \frac{12}{5}x$

$y' = \frac{12}{5}$

$s = \int_0^5 \sqrt{1 + \left(\frac{12}{5}\right)^2} dx = \left[\frac{13}{5}x\right]_0^5 = 13$

3.  $y = \frac{2}{3}x^{3/2} + 1$

$y' = x^{1/2}, [0, 1]$

$s = \int_0^1 \sqrt{1+x} dx$

$= \left[\frac{2}{3}(1+x)^{3/2}\right]_0^1$

$= \frac{2}{3}(\sqrt{8}-1) \approx 1.219$

5.  $y = \frac{3}{2}x^{2/3}$

$y' = \frac{1}{x^{1/3}}, [1, 8]$

$s = \int_1^8 \sqrt{1 + \left(\frac{1}{x^{1/3}}\right)^2} dx$

$= \int_1^8 \sqrt{\frac{x^{2/3} + 1}{x^{2/3}}} dx$

$= \frac{3}{2} \int_1^8 \sqrt{x^{2/3} + 1} \left(\frac{2}{3x^{1/3}}\right) dx$

$= \frac{3}{2} \left[\frac{2}{3}(x^{2/3} + 1)^{3/2}\right]_1^8$

$= 5\sqrt{5} - 2\sqrt{2} \approx 8.352$

7.  $y = \frac{x^4}{8} + \frac{1}{4x^2}$

$y' = \frac{1}{2}x^3 - \frac{1}{2x^3}, [1, 2]$

$1 + (y')^2 = \left(\frac{1}{2}x^3 + \frac{1}{2x^3}\right)^2, [1, 2]$

$s = \int_a^b \sqrt{1 + (y')^2} dx$

$= \int_1^2 \left(\frac{1}{2}x^3 + \frac{1}{2x^3}\right) dx$

$= \left[\frac{1}{8}x^4 - \frac{1}{4x^2}\right]_1^2 = \frac{33}{16} \approx 2.063$

9.  $y = \ln(\sin x), \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

$y' = \frac{1}{\sin x} \cos x = \cot x$

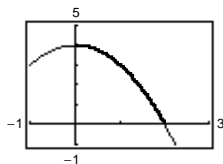
$1 + (y')^2 = 1 + \cot^2 x = \csc^2 x$

$s = \int_{\pi/4}^{3\pi/4} \csc x dx$

$= \left[\ln|\csc x - \cot x|\right]_{\pi/4}^{3\pi/4}$

$= \ln(\sqrt{2} + 1) - \ln(\sqrt{2} - 1) \approx 1.763$

11. (a)  $y = 4 - x^2, 0 \leq x \leq 2$



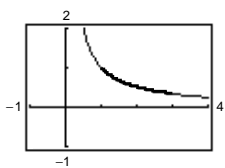
(b)  $y' = -2x$

(c)  $L \approx 4.647$

$1 + (y')^2 = 1 + 4x^2$

$L = \int_0^2 \sqrt{1 + 4x^2} dx$

13. (a)  $y = \frac{1}{x}, 1 \leq x \leq 3$



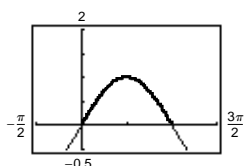
(b)  $y' = -\frac{1}{x^2}$

(c)  $L \approx 2.147$

$$1 + (y')^2 = 1 + \frac{1}{x^4}$$

$$L = \int_1^3 \sqrt{1 + \frac{1}{x^4}} dx$$

15. (a)  $y = \sin x, 0 \leq x \leq \pi$



(b)  $y' = \cos x$

(c)  $L \approx 3.820$

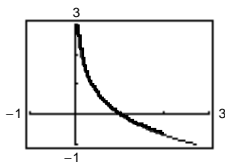
$$1 + (y')^2 = 1 + \cos^2 x$$

$$L = \int_0^\pi \sqrt{1 + \cos^2 x} dx$$

17. (a)  $x = e^{-y}, 0 \leq y \leq 2$

$y = -\ln x$

$1 \geq x \geq e^{-2} \approx 0.135$



(b)  $y' = -\frac{1}{x}$

(c)  $L \approx 2.221$

$$1 + (y')^2 = 1 + \frac{1}{x^2}$$

$$L = \int_{e^{-2}}^1 \sqrt{1 + \frac{1}{x^2}} dx$$

Alternatively, you can do all the computations with respect to  $y$ .

(a)  $x = e^{-y}, 0 \leq y \leq 2$

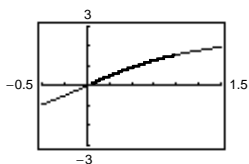
(b)  $\frac{dx}{dy} = -e^{-y}$

(c)  $L \approx 2.221$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + e^{-2y}$$

$$L = \int_0^2 \sqrt{1 + e^{-2y}} dy$$

19. (a)  $y = 2 \arctan x, 0 \leq x \leq 1$



(b)  $y' = \frac{2}{1+x^2}$

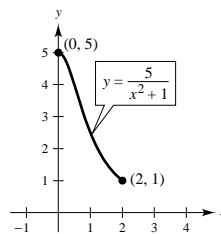
(c)  $L \approx 1.871$

$$L = \int_0^1 \sqrt{1 + \frac{4}{(1+x^2)^2}} dx$$

$$21. \int_0^2 \sqrt{1 + \left[ \frac{d}{dx} \left( \frac{5}{x^2 + 1} \right) \right]^2} dx$$

$$s \approx 5$$

Matches (b)



$$23. y = x^3, [0, 4]$$

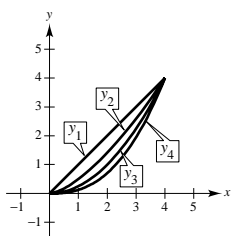
$$(a) d = \sqrt{(4 - 0)^2 + (64 - 0)^2} \approx 64.125$$

$$(b) d = \sqrt{(1 - 0)^2 + (1 - 0)^2} + \sqrt{(2 - 1)^2 + (8 - 1)^2} + \sqrt{(3 - 2)^2 + (27 - 8)^2} + \sqrt{(4 - 3)^2 + (64 - 27)^2} \approx 64.525$$

$$(c) s = \int_0^4 \sqrt{1 + (3x^2)^2} dx = \int_0^4 \sqrt{1 + 9x^4} dx \approx 64.666$$

$$(d) 64.672$$

$$25. (a)$$



$$(b) y_1, y_2, y_3, y_4$$

$$(c) y_1' = 1, L_1 = \int_0^4 \sqrt{2} dx \approx 5.657$$

$$y_2' = \frac{3}{4}x^{1/2}, L_2 = \int_0^4 \sqrt{1 + \frac{9x}{16}} dx \approx 5.759$$

$$y_3' = \frac{1}{2}x, L_3 = \int_0^4 \sqrt{1 + \frac{x^2}{4}} dx \approx 5.916$$

$$y_4' = \frac{5}{16}x^{3/2}, L_4 = \int_0^4 \sqrt{1 + \frac{25}{256}x^3} dx \approx 6.063$$

$$27. y = \frac{1}{3}[x^{3/2} - 3x^{1/2} + 2]$$

When  $x = 0$ ,  $y = \frac{2}{3}$ . Thus, the fleeing object has traveled  $\frac{2}{3}$  units when it is caught.

$$y' = \frac{1}{3} \left[ \frac{3}{2}x^{1/2} - \frac{3}{2}x^{-1/2} \right] = \left( \frac{1}{2} \right) \frac{x - 1}{x^{1/2}}$$

$$1 + (y')^2 = 1 + \frac{(x - 1)^2}{4x} = \frac{(x + 1)^2}{4x}$$

$$s = \int_0^1 \frac{x + 1}{2x^{1/2}} dx = \frac{1}{2} \int_0^1 (x^{1/2} + x^{-1/2}) dx = \frac{1}{2} \left[ \frac{2}{3}x^{3/2} + 2x^{1/2} \right]_0^1 = \frac{4}{3} = 2 \left( \frac{2}{3} \right)$$

The pursuer has traveled twice the distance that the fleeing object has traveled when it is caught.

$$29. y = 20 \cosh \frac{x}{20}, -20 \leq x \leq 20$$

$$y' = \sinh \frac{x}{20}$$

$$1 + (y')^2 = 1 + \sinh^2 \frac{x}{20} = \cosh^2 \frac{x}{20}$$

$$L = \int_{-20}^{20} \cosh \frac{x}{20} dx = 2 \int_0^{20} \cosh \frac{x}{20} dx = 2(20) \sinh \frac{x}{20} \Big|_0^{20}$$

$$= 40 \sinh(1) \approx 47.008 \text{ m.}$$



$$31. \quad y = \sqrt{9 - x^2}$$

$$y' = \frac{-x}{\sqrt{9 - x^2}}$$

$$1 + (y')^2 = \frac{9}{9 - x^2}$$

$$s = \int_0^2 \sqrt{\frac{9}{9 - x^2}} dx$$

$$= \int_0^2 \frac{3}{\sqrt{9 - x^2}} dx$$

$$= \left[ 3 \arcsin \frac{x}{3} \right]_0^2$$

$$= 3 \left( \arcsin \frac{2}{3} - \arcsin 0 \right)$$

$$= 3 \arcsin \frac{2}{3} \approx 2.1892$$

$$33. \quad y = \frac{x^3}{3}$$

$$y' = x^2, [0, 3]$$

$$S = 2\pi \int_0^3 \frac{x^3}{3} \sqrt{1 + x^4} dx$$

$$= \frac{\pi}{6} \int_0^3 (1 + x^4)^{1/2} (4x^3) dx$$

$$= \left[ \frac{\pi}{9} (1 + x^4)^{3/2} \right]_0^3$$

$$= \frac{\pi}{9} (82\sqrt{82} - 1) \approx 258.85$$

$$35. \quad y = \frac{x^3}{6} + \frac{1}{2x}$$

$$y' = \frac{x^2}{2} - \frac{1}{2x^2}$$

$$1 + (y')^2 = \left( \frac{x^2}{2} + \frac{1}{2x^2} \right)^2, [1, 2]$$

$$S = 2\pi \int_1^2 \left( \frac{x^3}{6} + \frac{1}{2x} \right) \left( \frac{x^2}{2} + \frac{1}{2x^2} \right) dx$$

$$= 2\pi \int_1^2 \left( \frac{x^5}{12} + \frac{x}{3} + \frac{1}{4x^3} \right) dx$$

$$= 2\pi \left[ \frac{x^6}{72} + \frac{x^2}{6} - \frac{1}{8x^2} \right]_1^2 = \frac{47\pi}{16}$$

$$37. \quad y = \sqrt[3]{x} + 2$$

$$y' = \frac{1}{3x^{2/3}}, [1, 8]$$

$$S = 2\pi \int_1^8 x \sqrt{1 + \frac{1}{9x^{4/3}}} dx$$

$$= \frac{2\pi}{3} \int_1^8 x^{1/3} \sqrt{9x^{4/3} + 1} dx$$

$$= \frac{\pi}{18} \int_1^8 (9x^{4/3} + 1)^{1/2} (12x^{1/3}) dx$$

$$= \left[ \frac{\pi}{27} (9x^{4/3} + 1)^{3/2} \right]_1^8$$

$$= \frac{\pi}{27} (145\sqrt{145} - 10\sqrt{10}) \approx 199.48$$

$$39. \quad y = \sin x$$

$$y' = \cos x, [0, \pi]$$

$$S = 2\pi \int_0^\pi \sin x \sqrt{1 + \cos^2 x} dx$$

$$\approx 14.4236$$

41. A rectifiable curve is one that has a finite arc length.

43. The precalculus formula is the surface area formula for the lateral surface of the frustum of a right circular cone. The representative element is

$$2\pi f(di) \sqrt{\Delta x_i^2 + \Delta y_i^2} = 2\pi f(di) \sqrt{1 + \left( \frac{\Delta y_i}{\Delta x_i} \right)^2} \Delta x_i.$$

$$45. \quad y = \frac{hx}{r}$$

$$y' = \frac{h}{r}$$

$$1 + (y')^2 = \frac{r^2 + h^2}{r^2}$$

$$S = 2\pi \int_0^r x \sqrt{\frac{r^2 + h^2}{r^2}} dx$$

$$= \left[ \frac{2\pi\sqrt{r^2 + h^2}}{r} \left( \frac{x^2}{2} \right) \right]_0^r = \pi r \sqrt{r^2 + h^2}$$

$$47. \quad y = \sqrt{9 - x^2}$$

$$y' = \frac{-x}{\sqrt{9 - x^2}}$$

$$\sqrt{1 + (y')^2} = \frac{3}{\sqrt{9 - x^2}}$$

$$S = 2\pi \int_0^2 \frac{3x}{\sqrt{9 - x^2}} dx$$

$$= -3\pi \int_0^2 \frac{-2x}{\sqrt{9 - x^2}} dx$$

$$= \left[ -6\pi\sqrt{9 - x^2} \right]_0^2$$

$$= 6\pi(3 - \sqrt{5}) \approx 14.40$$

See figure in Exercise 48.

$$49. \quad y = \frac{1}{3}x^{1/2} - x^{3/2}$$

$$y' = \frac{1}{6}x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{1}{6}(x^{-1/2} - 9x^{1/2})$$

$$1 + (y')^2 = 1 + \frac{1}{36}(x^{-1} - 18 + 81x) = \frac{1}{36}(x^{-1/2} + 9x^{1/2})^2$$

$$S = 2\pi \int_0^{1/3} \left( \frac{1}{3}x^{1/2} - x^{3/2} \right) \sqrt{\frac{1}{36}(x^{-1/2} + 9^{1/2})^2} dx = \frac{2\pi}{6} \int_0^{1/3} \left( \frac{1}{3}x^{1/2} - x^{3/2} \right) (x^{-1/2} + 9x^{1/2}) dx$$

$$= \frac{\pi}{3} \int_0^{1/3} \left( \frac{1}{3} + 2x - 9x^2 \right) dx = \frac{\pi}{3} \left[ \frac{1}{3}x + x^2 - 3x^3 \right]_0^{1/3} = \frac{\pi}{27} \text{ ft}^2 \approx 0.1164 \text{ ft}^2 \approx 16.8 \text{ in}^2$$

Amount of glass needed:  $V = \frac{\pi}{27} \left( \frac{0.015}{12} \right) \approx 0.00015 \text{ ft}^3 \approx 0.25 \text{ in}^3$

51. (a)  $y = f(x) = 0.0000001953x^4 - 0.0001804x^3 + 0.0496x^2 - 4.8323x + 536.9270$

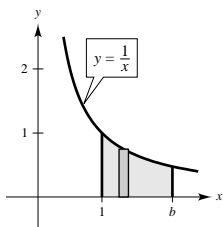
(b) Area =  $\int_0^{400} f(x) dx \approx 131,734.5$  square feet  
 $\approx 3.0$  acres

(Answers will vary.)

(c)  $L = \int_0^{400} \sqrt{1 + f'(x)^2} dx \approx 794.9$  feet

(Answers will vary.)

53. (a)  $V = \pi \int_1^b \frac{1}{x^2} dx = \left[ -\frac{\pi}{x} \right]_1^b = \pi \left( 1 - \frac{1}{b} \right)$



(b)  $S = 2\pi \int_1^b \frac{1}{x} \sqrt{1 + \left( -\frac{1}{x^2} \right)^2} dx$

$$= 2\pi \int_1^b \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} dx$$

$$= 2\pi \int_1^b \frac{\sqrt{x^4 + 1}}{x^3} dx$$

—CONTINUED—

## 53. —CONTINUED—

$$(c) \lim_{b \rightarrow \infty} V = \lim_{b \rightarrow \infty} \pi \left(1 - \frac{1}{b}\right) = \pi$$

(d) Since

$$\frac{\sqrt{x^4 + 1}}{x^3} > \frac{\sqrt{x^4}}{x^3} = \frac{1}{x} > 0 \text{ on } [1, b]$$

we have

$$\int_1^b \frac{\sqrt{x^4 + 1}}{x^3} dx > \int_1^b \frac{1}{x} dx = \left[ \ln x \right]_1^b = \ln b$$

and  $\lim_{b \rightarrow \infty} \ln b \rightarrow \infty$ . Thus,

$$\lim_{b \rightarrow \infty} 2\pi \int_1^b \frac{\sqrt{x^4 + 1}}{x^3} dx = \infty.$$

55. (a) Area of circle with radius  $L$ :  $A = \pi L^2$ Area of sector with central angle  $\theta$  (in radians)

$$S = \frac{\theta}{2\pi} A = \frac{\theta}{2\pi} (\pi L^2) = \frac{1}{2} L^2 \theta$$

(b) Let  $s$  be the arc length of the sector, which is the circumference of the base of the cone. Here,  $s = L\theta = 2\pi r$ , and you have

$$S = \frac{1}{2} L^2 \theta = \frac{1}{2} L^2 \left(\frac{s}{L}\right) = \frac{1}{2} L s = \frac{1}{2} L (2\pi r) = \pi r L$$

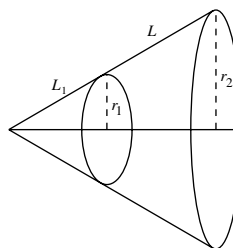
(c) The lateral surface area of the frustum is the difference of the large cone and the small one.

$$\begin{aligned} S &= \pi r_2(L + L_1) - \pi r_1 L_1 \\ &= \pi r_2 L + \pi L_1(r_2 - r_1) \end{aligned}$$

By similar triangles,  $\frac{L + L_1}{r_2} = \frac{L_1}{r_1} \Rightarrow L r_1 = L_1(r_2 - r_1)$ 

Hence,

$$\begin{aligned} S &= \pi r_2 L + \pi L_1(r_2 - r_1) = \pi r_2 L + \pi L r_1 \\ &= \pi L(r_1 + r_2). \end{aligned}$$



## Section 6.5 Work

1.  $W = Fd = (100)(10) = 1000 \text{ ft} \cdot \text{lb}$

3.  $W = Fd = (112)(4) = 448 \text{ joules (newton-meters)}$

5. Work equals force times distance,  $W = FD$ .7. Since the work equals the area under the force function, you have  $(c) < (d) < (a) < (b)$ .

9.  $F(x) = kx$

$5 = k(4)$

$k = \frac{5}{4}$

$W = \int_0^7 \frac{5}{4} x dx = \left[ \frac{5}{8} x^2 \right]_0^7$

$= \frac{245}{8} \text{ in} \cdot \text{lb}$

$= 30.625 \text{ in} \cdot \text{lb} \approx 2.55 \text{ ft} \cdot \text{lb}$

11.  $F(x) = kx$

$250 = k(30) \Rightarrow k = \frac{25}{3}$

$W = \int_{20}^{50} F(x) dx = \int_{20}^{50} \frac{25}{3} x dx = \left[ \frac{25x^2}{6} \right]_{20}^{50}$

$= 8750 \text{ n} \cdot \text{cm} = 87.5 \text{ joules or Nm}$

13.  $F(x) = kx$

$20 = k(9)$

$k = \frac{20}{9}$

$$W = \int_0^{12} \frac{20}{9}x \, dx = \left[ \frac{10}{9}x^2 \right]_0^{12} = \frac{40}{3} \text{ ft} \cdot \text{lb}$$

15.  $W = 18 = \int_0^{1/3} kx \, dx = \left[ \frac{kx^2}{2} \right]_0^{1/3} = \frac{k}{18} \Rightarrow k = 324$

$$W = \int_{1/3}^{7/12} 324x \, dx = 162x^2 \Big|_{1/3}^{7/12} = 37.125 \text{ ft} \cdot \text{lbs}$$

[Note: 4 inches =  $\frac{1}{3}$  foot]

17. Assume that Earth has a radius of 4000 miles.

$F(x) = \frac{k}{x^2}$

$s = \frac{k}{(4000)^2}$

$k = 80,000,000$

$F(x) = \frac{80,000,000}{x^2}$

(a)  $W = \int_{4000}^{4100} \frac{80,000,000}{x^2} \, dx = \left[ -\frac{80,000,000}{x} \right]_{4000}^{4100} \approx 487.8 \text{ mi} \cdot \text{tons}$

$\approx 5.15 \times 10^9 \text{ ft} \cdot \text{lb}$

(b)  $W = \int_{4000}^{4300} \frac{80,000,000}{x^2} \, dx \approx 1395.3 \text{ mi} \cdot \text{ton}$

$\approx 1.47 \times 10^{10} \text{ ft} \cdot \text{lb}$

19. Assume that the earth has a radius of 4000 miles.

$F(x) = \frac{k}{x^2}$

$10 = \frac{k}{(4000)^2}$

$k = 160,000,000$

$F(x) = \frac{160,000,000}{x^2}$

(a)  $W = \int_{4000}^{15,000} \frac{160,000,000}{x^2} \, dx = \left[ -\frac{160,000,000}{x} \right]_{4000}^{15,000} \approx -10,666.667 + 40,000$

$= 29,333.333 \text{ mi} \cdot \text{ton}$

$\approx 2.93 \times 10^4 \text{ mi} \cdot \text{ton}$

$\approx 3.10 \times 10^{11} \text{ ft} \cdot \text{lb}$

(b)  $W = \int_{4000}^{26,000} \frac{160,000,000}{x^2} \, dx = \left[ -\frac{160,000,000}{x} \right]_{4000}^{26,000} \approx -6,153.846 + 40,000$

$= 33,846.154 \text{ mi} \cdot \text{ton}$

$\approx 3.38 \times 10^4 \text{ mi} \cdot \text{ton}$

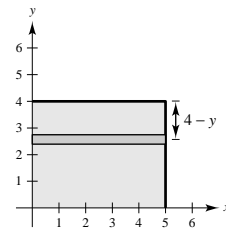
$\approx 3.57 \times 10^{11} \text{ ft} \cdot \text{lb}$

 21. Weight of each layer:  $62.4(20) \Delta y$ 

 Distance:  $4 - y$ 

(a)  $W = \int_2^4 62.4(20)(4 - y) \, dy = \left[ 4992y - 624y^2 \right]_2^4 = 2496 \text{ ft} \cdot \text{lb}$

(b)  $W = \int_0^4 62.4(20)(4 - y) \, dy = \left[ 4992y - 624y^2 \right]_0^4 = 9984 \text{ ft} \cdot \text{lb}$


 23. Volume of disk:  $\pi(2)^2 \Delta y = 4\pi \Delta y$ 

 Weight of disk of water:  $9800(4\pi) \Delta y$ 

 Distance the disk of water is moved:  $5 - y$ 

$$W = \int_0^4 (5 - y)(9800)4\pi \, dy = 39,200\pi \int_0^4 (5 - y) \, dy$$

$$= 39,200\pi \left[ 5y - \frac{y^2}{2} \right]_0^4$$

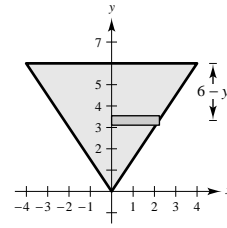
$$= 39,200\pi(12) = 470,400\pi \text{ newton-meters}$$

25. Volume of disk:  $\pi\left(\frac{2}{3}y\right)^2 \Delta y$

Weight of disk:  $62.4\pi\left(\frac{2}{3}y\right)^2 \Delta y$

Distance:  $6 - y$

$$W = \frac{4(62.4)\pi}{9} \int_0^6 (6 - y)y^2 dy = \frac{4}{9}(62.4)\pi \left[ 2y^3 - \frac{1}{4}y^4 \right]_0^6 = 2995.2\pi \text{ ft} \cdot \text{lb}$$

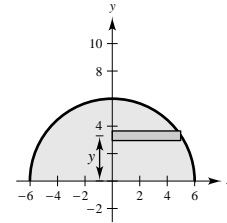


27. Volume of disk:  $\pi(\sqrt{36 - y^2})^2 \Delta y$

Weight of disk:  $62.4\pi(36 - y^2) \Delta y$

Distance:  $y$

$$\begin{aligned} W &= 62.4\pi \int_0^6 y(36 - y^2) dy \\ &= 62.4\pi \int_0^6 (36y - y^3) dy = 62.4\pi \left[ 18y^2 - \frac{1}{4}y^4 \right]_0^6 \\ &= 20,217.6\pi \text{ ft} \cdot \text{lb} \end{aligned}$$



29. Volume of layer:  $V = lwh = 4(2)\sqrt{(9/4) - y^2} \Delta y$

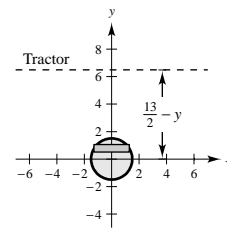
Weight of layer:  $W = 42(8)\sqrt{(9/4) - y^2} \Delta y$

Distance:  $\frac{13}{2} - y$

$$\begin{aligned} W &= \int_{-1.5}^{1.5} 42(8)\sqrt{(9/4) - y^2} \left( \frac{13}{2} - y \right) dy \\ &= 336 \left[ \frac{13}{2} \int_{-1.5}^{1.5} \sqrt{(9/4) - y^2} dy - \int_{-1.5}^{1.5} \sqrt{(9/4) - y^2} y dy \right] \end{aligned}$$

The second integral is zero since the integrand is odd and the limits of integration are symmetric to the origin. The first integral represents the area of a semicircle of radius  $\frac{3}{2}$ . Thus, the work is

$$W = 336 \left( \frac{13}{2} \right) \pi \left( \frac{3}{2} \right)^2 \left( \frac{1}{2} \right) = 2457\pi \text{ ft} \cdot \text{lb}$$



31. Weight of section of chain:  $3 \Delta y$

Distance:  $15 - y$

$$\begin{aligned} W &= 3 \int_0^{15} (15 - y) dy \\ &= \left[ -\frac{3}{2}(15 - y)^2 \right]_0^{15} \\ &= 337.5 \text{ ft} \cdot \text{lb} \end{aligned}$$

33. The lower 5 feet of chain are raised 10 feet with a constant force.

$$W_1 = 3(5)(10) = 150 \text{ ft} \cdot \text{lb}$$

The top 10 feet of chain are raised with a variable force.

Weight per section:  $3 \Delta y$

Distance:  $10 - y$

$$\begin{aligned} W_2 &= 3 \int_0^{10} (10 - y) dy = \left[ -\frac{3}{2}(10 - y)^2 \right]_0^{10} \\ &= 150 \text{ ft} \cdot \text{lb} \end{aligned}$$

$$W = W_1 + W_2 = 300 \text{ ft} \cdot \text{lb}$$

35. Weight of section of chain:  $3 \Delta y$ Distance:  $15 - 2y$ 

$$W = 3 \int_0^{7.5} (15 - 2y) dy = \left[ -\frac{3}{4}(15 - 2y)^2 \right]_0^{7.5} \\ = \frac{3}{4}(15)^2 = 168.75 \text{ ft} \cdot \text{lb}$$

39.  $p = \frac{k}{V}$

$1000 = \frac{k}{2}$

$k = 2000$

$$W = \int_2^3 \frac{2000}{V} dV = \left[ 2000 \ln |V| \right]_2^3 \\ = 2000 \ln \left( \frac{3}{2} \right) \approx 810.93 \text{ ft} \cdot \text{lb}$$

43.  $W = \int_0^5 1000[1.8 - \ln(x + 1)] dx \approx 3249.44 \text{ ft} \cdot \text{lb}$

37. Work to pull up the ball:  $W_1 = 500(15) = 7500 \text{ ft} \cdot \text{lb}$ 

Work to wind up the top 15 feet of cable: force is variable

Weight per section:  $1 \Delta y$ Distance:  $15 - x$ 

$$W_2 = \int_0^{15} (15 - x) dx = \left[ -\frac{1}{2}(15 - x)^2 \right]_0^{15} \\ = 112.5 \text{ ft} \cdot \text{lb}$$

Work to lift the lower 25 feet of cable with a constant force:

$W_3 = (1)(25)(15) = 375 \text{ ft} \cdot \text{lb}$

$W = W_1 + W_2 + W_3 = 7500 + 112.5 + 375$

$= 7987.5 \text{ ft} \cdot \text{lb}$

41.  $F(x) = \frac{k}{(2 - x)^2}$

$$W = \int_{-2}^1 \frac{k}{(2 - x)^2} dx = \left[ \frac{k}{2 - x} \right]_{-2}^1 = k \left( 1 - \frac{1}{4} \right) \\ = \frac{3k}{4} (\text{units of work})$$

45.  $W = \int_0^5 100x\sqrt{125 - x^3} dx \approx 10,330.3 \text{ ft} \cdot \text{lb}$

**Section 6.6 Moments, Centers of Mass, and Centroids**

1.  $\bar{x} = \frac{6(-5) + 3(1) + 5(3)}{6 + 3 + 5} = -\frac{6}{7}$

3.  $\bar{x} = \frac{1(7) + 1(8) + 1(12) + 1(15) + 1(18)}{1 + 1 + 1 + 1 + 1} = 12$

5. (a)  $\bar{x} = \frac{(7 + 5) + (8 + 5) + (12 + 5) + (15 + 5) + (18 + 5)}{5} = 17 = 12 + 5$

(b)  $\bar{x} = \frac{12(-6 - 3) + 1(-4 - 3) + 6(-2 - 3) + 3(0 - 3) + 11(8 - 3)}{12 + 1 + 6 + 3 + 11} = \frac{-99}{33} = -3$

7.  $50x = 75(L - x) = 75(10 - x)$

$50x = 750 - 75x$

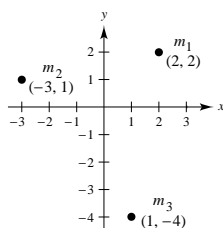
$125x = 750$

$x = 6 \text{ feet}$

9.  $\bar{x} = \frac{5(2) + 1(-3) + 3(1)}{5 + 1 + 3} = \frac{10}{9}$

$\bar{y} = \frac{5(2) + 1(1) + 3(-4)}{5 + 1 + 3} = -\frac{1}{9}$

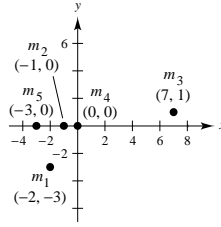
$(\bar{x}, \bar{y}) = \left( \frac{10}{9}, -\frac{1}{9} \right)$



$$11. \quad \bar{x} = \frac{3(-2) + 4(-1) + 2(7) + 1(0) + 6(-3)}{3 + 4 + 2 + 1 + 6} = -\frac{7}{8}$$

$$\bar{y} = \frac{3(-3) + 4(0) + 2(1) + 1(0) + 6(0)}{3 + 4 + 2 + 1 + 6} = -\frac{7}{16}$$

$$(\bar{x}, \bar{y}) = \left(-\frac{7}{8}, -\frac{7}{16}\right)$$



$$13. \quad m = \rho \int_0^4 \sqrt{x} \, dx = \left[ \frac{2\rho}{3} x^{3/2} \right]_0^4 = \frac{16\rho}{3}$$

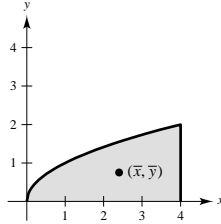
$$M_x = \rho \int_0^4 \frac{\sqrt{x}}{2} (\sqrt{x}) \, dx = \left[ \rho \frac{x^2}{4} \right]_0^4 = 4\rho$$

$$\bar{y} = \frac{M_x}{m} = 4\rho \left( \frac{3}{16\rho} \right) = \frac{3}{4}$$

$$M_y = \rho \int_0^4 x\sqrt{x} \, dx = \left[ \rho \frac{2}{5} x^{5/2} \right]_0^4 = \frac{64\rho}{5}$$

$$\bar{x} = \frac{M_y}{m} = \frac{64\rho}{5} \left( \frac{3}{16\rho} \right) = \frac{12}{5}$$

$$(\bar{x}, \bar{y}) = \left( \frac{12}{5}, \frac{3}{4} \right)$$



$$15. \quad m = \rho \int_0^1 (x^2 - x^3) \, dx = \rho \left[ \frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{\rho}{12}$$

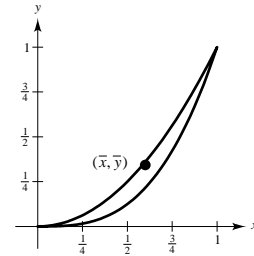
$$M_x = \rho \int_0^1 \frac{(x^2 + x^3)}{2} (x^2 - x^3) \, dx = \frac{\rho}{2} \int_0^1 (x^4 - x^6) \, dx = \frac{\rho}{2} \left[ \frac{x^5}{5} - \frac{x^7}{7} \right]_0^1 = \frac{\rho}{35}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\rho}{35} \left( \frac{12}{\rho} \right) = \frac{12}{35}$$

$$M_y = \rho \int_0^1 x(x^2 - x^3) \, dx = \rho \int_0^1 (x^3 - x^4) \, dx = \rho \left[ \frac{x^4}{4} - \frac{x^5}{5} \right]_0^1 = \frac{\rho}{20}$$

$$\bar{x} = \frac{M_y}{m} = \frac{\rho}{20} \left( \frac{12}{\rho} \right) = \frac{3}{5}$$

$$(\bar{x}, \bar{y}) = \left( \frac{3}{5}, \frac{12}{35} \right)$$



$$17. \quad m = \rho \int_0^3 [(-x^2 + 4x + 2) - (x + 2)] \, dx = -\rho \left[ \frac{x^3}{3} + \frac{3x^2}{2} \right]_0^3 = \frac{9\rho}{2}$$

$$M_x = \rho \int_0^3 \left[ \frac{(-x^2 + 4x + 2) + (x + 2)}{2} \right] [(-x^2 + 4x + 2) - (x + 2)] \, dx$$

$$= \frac{\rho}{2} \int_0^3 (-x^2 + 5x + 4)(-x^2 + 3x) \, dx = \frac{\rho}{2} \int_0^3 (x^4 - 8x^3 + 11x^2 + 12x) \, dx$$

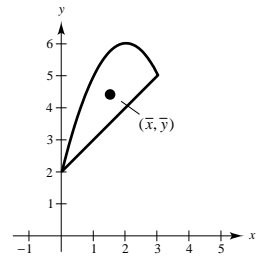
$$= \frac{\rho}{2} \left[ \frac{x^5}{5} - 2x^4 + \frac{11x^3}{3} + 6x^2 \right]_0^3 = \frac{99\rho}{5}$$

$$\bar{y} = \frac{M_x}{m} = \frac{99\rho}{5} \left( \frac{2}{9\rho} \right) = \frac{22}{5}$$

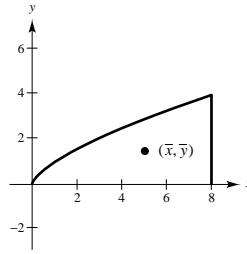
$$M_y = \rho \int_0^3 x[(-x^2 + 4x - 2) - (x + 2)] \, dx = \rho \int_0^3 (-x^3 + 3x^2) \, dx = \rho \left[ -\frac{x^4}{4} + x^3 \right]_0^3 = \frac{27\rho}{4}$$

$$\bar{x} = \frac{M_y}{m} = \frac{27\rho}{4} \left( \frac{2}{9\rho} \right) = \frac{3}{2}$$

$$(\bar{x}, \bar{y}) = \left( \frac{3}{2}, \frac{22}{5} \right)$$



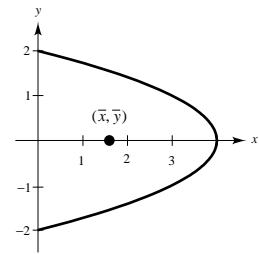
$$\begin{aligned}
 19. \quad m &= \rho \int_0^8 x^{2/3} dx = \rho \left[ \frac{3}{5} x^{5/3} \right]_0^8 = \frac{96\rho}{5} \\
 M_x &= \rho \int_0^8 \frac{x^{2/3}}{2} (x^{2/3}) dx = \frac{\rho}{2} \left[ \frac{3}{7} x^{7/3} \right]_0^8 = \frac{192\rho}{7} \\
 \bar{y} &= \frac{M_x}{m} = \frac{192\rho}{7} \left( \frac{5}{96\rho} \right) = \frac{10}{7} \\
 M_y &= \rho \int_0^8 x(x^{2/3}) dx = \rho \left[ \frac{3}{8} x^{8/3} \right]_0^8 = 96\rho \\
 \bar{x} &= \frac{M_y}{m} = 96\rho \left( \frac{5}{96\rho} \right) = 5 \\
 (\bar{x}, \bar{y}) &= \left( 5, \frac{10}{7} \right)
 \end{aligned}$$



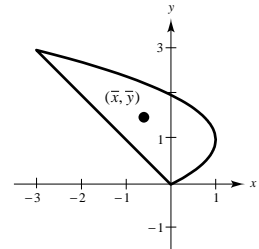
$$\begin{aligned}
 21. \quad m &= 2\rho \int_0^2 (4 - y^2) dy = 2\rho \left[ 4y - \frac{y^3}{3} \right]_0^2 = \frac{32\rho}{3} \\
 M_y &= 2\rho \int_0^2 \left( \frac{4 - y^2}{2} \right) (4 - y^2) dy = \rho \left[ 16y - \frac{8}{3}y^3 + \frac{y^5}{5} \right]_0^2 = \frac{256\rho}{15} \\
 \bar{x} &= \frac{M_y}{m} = \frac{256\rho}{15} \left( \frac{3}{32\rho} \right) = \frac{8}{5} \\
 \bar{y} &= 0 \quad (\text{by symmetry})
 \end{aligned}$$

By symmetry,  $M_x$  and  $\bar{y} = 0$ .

$$(\bar{x}, \bar{y}) = \left( \frac{8}{5}, 0 \right)$$



$$\begin{aligned}
 23. \quad m &= \rho \int_0^3 [(2y - y^2) - (-y)] dy = \rho \left[ \frac{3y^2}{2} - \frac{y^3}{3} \right]_0^3 = \frac{9\rho}{2} \\
 M_y &= \rho \int_0^3 \frac{[(2y - y^2) + (-y)]}{2} [(2y - y^2) - (-y)] dy = \frac{\rho}{2} \int_0^3 (y - y^2)(3y - y^2) dy \\
 &= \frac{\rho}{2} \int_0^3 (y^4 - 4y^3 + 3y^2) dy = \frac{\rho}{2} \left[ \frac{y^5}{5} - y^4 + y^3 \right]_0^3 = -\frac{27\rho}{10} \\
 \bar{x} &= \frac{M_y}{m} = -\frac{27\rho}{10} \left( \frac{2}{9\rho} \right) = -\frac{3}{5} \\
 M_x &= \rho \int_0^3 y[(2y - y^2) - (-y)] dy = \rho \int_0^3 (3y^2 - y^3) dy = \rho \left[ y^3 - \frac{y^4}{4} \right]_0^3 = \frac{27\rho}{4} \\
 \bar{y} &= \frac{M_x}{m} = \frac{27\rho}{4} \left( \frac{2}{9\rho} \right) = \frac{3}{2} \\
 (\bar{x}, \bar{y}) &= \left( -\frac{3}{5}, \frac{3}{2} \right)
 \end{aligned}$$



$$\begin{aligned}
 25. \quad A &= \int_0^1 (x - x^2) dx = \left[ \frac{1}{2}x^2 - \frac{x^3}{3} \right]_0^1 = \frac{1}{6} \\
 M_x &= \frac{1}{2} \int_0^1 (x^2 - x^4) dx = \frac{1}{2} \left[ \frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 = \frac{1}{2} \left( \frac{1}{3} - \frac{1}{5} \right) = \frac{1}{15} \\
 M_y &= \int_0^1 (x^2 - x^3) dx = \left[ \frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \left( \frac{1}{3} - \frac{1}{4} \right) = \frac{1}{12}
 \end{aligned}$$



$$27. A = \int_0^3 (2x + 4) dx = \left[ x^2 + 4x \right]_0^3 = 9 + 12 = 21$$

$$M_x = \frac{1}{2} \int_0^3 (2x + 4)^2 dx = \int_0^3 (2x^2 + 8x + 8) dx = \left[ \frac{2x^3}{3} + 4x^2 + 8x \right]_0^3 = 18 + 36 + 24 = 78$$

$$M_y = \int_0^3 (2x^2 + 4x) dx = \left[ \frac{2x^3}{3} + 2x^2 \right]_0^3 = 18 + 18 = 36$$

$$29. m = \rho \int_0^5 10x\sqrt{125 - x^3} dx \approx 1033.0\rho$$

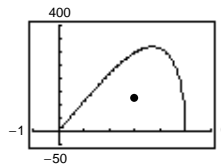
$$M_x = \rho \int_0^5 \left( \frac{10x\sqrt{125 - x^3}}{2} \right) (10x\sqrt{125 - x^3}) dx = 50\rho \int_0^5 x^2(125 - x^3) dx = \frac{3,124,375\rho}{24} \approx 130,208\rho$$

$$M_y = \rho \int_0^5 10x^2\sqrt{125 - x^3} dx = -\frac{10\rho}{3} \int_0^5 \sqrt{125 - x^3}(-3x^2) dx = \frac{12,500\sqrt{5}\rho}{9} \approx 3105.6\rho$$

$$\bar{x} = \frac{M_y}{m} \approx 3.0$$

$$\bar{y} = \frac{M_x}{m} \approx 126.0$$

Therefore, the centroid is (3.0, 126.0).

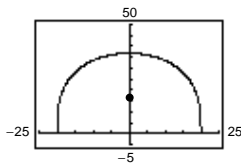


$$31. m = \rho \int_{-20}^{20} 5\sqrt[3]{400 - x^2} dx \approx 1239.76\rho$$

$$M_x = \rho \int_{-20}^{20} \frac{5\sqrt[3]{400 - x^2}}{2} (5\sqrt[3]{400 - x^2}) dx = \frac{25\rho}{2} \int_{-20}^{20} (400 - x^2)^{2/3} dx \approx 20064.27$$

$$\bar{y} = \frac{M_x}{m} \approx 16.18$$

$\bar{x} = 0$  by symmetry. Therefore, the centroid is (0, 16.2).



$$33. A = \frac{1}{2}(2a)c = ac$$

$$\frac{1}{A} = \frac{1}{ac}$$

$$\bar{x} = \left( \frac{1}{ac} \right) \frac{1}{2} \int_0^c \left[ \left( \frac{b-a}{c}y + a \right)^2 - \left( \frac{b+a}{c}y - a \right)^2 \right] dy$$

$$= \frac{1}{2ac} \int_0^c \left[ \frac{4ab}{c}y - \frac{4ab}{c^2}y^2 \right] dy$$

$$= \frac{1}{2ac} \left[ \frac{2ab}{c}y^2 - \frac{4ab}{3c^2}y^3 \right]_0^c = \frac{1}{2ac} \left( \frac{2}{3}abc \right) = \frac{b}{3}$$

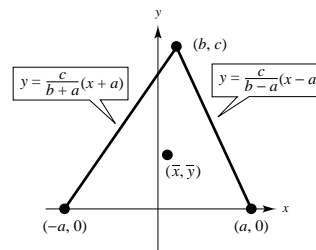
$$\bar{y} = \frac{1}{ac} \int_0^c y \left[ \left( \frac{b-a}{c}y + a \right) - \left( \frac{b+a}{c}y - a \right) \right] dy$$

$$= \frac{1}{ac} \int_0^c y \left( -\frac{2a}{c}y + 2a \right) dy = \frac{2}{c} \int_0^c \left( y - \frac{y^2}{c} \right) dy$$

$$= \frac{2}{c} \left[ \frac{y^2}{2} - \frac{y^3}{3c} \right]_0^c = \frac{c}{3}$$

$$(\bar{x}, \bar{y}) = \left( \frac{b}{3}, \frac{c}{3} \right)$$

In Exercise 566 of Section P.2, you found that  $(b/3, c/3)$  is the point of intersection of the medians.



$$35. A = \frac{c}{2}(a + b)$$

$$\frac{1}{A} = \frac{2}{c(a + b)}$$

$$\begin{aligned} \bar{x} &= \frac{2}{c(a + b)} \int_0^c x \left( \frac{b-a}{c}x + a \right) dx = \frac{2}{c(a + b)} \int_0^c \left( \frac{b-a}{c}x^2 + ax \right) dx = \frac{2}{c(a + b)} \left[ \frac{b-a}{c} \frac{x^3}{3} + \frac{ax^2}{2} \right]_0^c \\ &= \frac{2}{c(a + b)} \left[ \frac{(b-a)c^2}{3} + \frac{ac^2}{2} \right] = \frac{2}{c(a + b)} \left[ \frac{2bc^2 - 2ac^2 + 3ac^2}{6} \right] = \frac{c(2b + a)}{3(a + b)} = \frac{(a + 2b)c}{3(a + b)} \end{aligned}$$

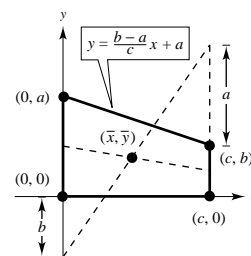
$$\begin{aligned} \bar{y} &= \frac{2}{c(a + b)} \frac{1}{2} \int_0^c \left( \frac{b-a}{c}x + a \right)^2 dx = \frac{1}{c(a + b)} \int_0^c \left[ \left( \frac{b-a}{c} \right)^2 x^2 + \frac{2a(b-a)}{c}x + a^2 \right] dx \\ &= \frac{1}{c(a + b)} \left[ \left( \frac{b-a}{c} \right)^2 \frac{x^3}{3} + \frac{2a(b-a)}{c} \frac{x^2}{2} + a^2x \right]_0^c = \frac{1}{c(a + b)} \left[ \frac{(b-a)^2c}{3} + ac(b-a) + a^2c \right] \\ &= \frac{1}{3c(a + b)} [(b^2 - 2ab + a^2)c + 3ac(b-a) + 3a^2c] \\ &= \frac{1}{3(a + b)} [b^2 - 2ab + a^2 + 3ab - 3a^2 + 3a^2] = \frac{a^2 + ab + b^2}{3(a + b)} \end{aligned}$$

$$\text{Thus, } (\bar{x}, \bar{y}) = \left( \frac{(a + 2b)c}{3(a + b)}, \frac{a^2 + ab + b^2}{3(a + b)} \right).$$

The one line passes through  $(0, a/2)$  and  $(c, b/2)$ . Its equation is  $y = \frac{b-a}{2c}x + \frac{a}{2}$ .

The other line passes through  $(0, -b)$  and  $(c, a + b)$ . Its equation is  $y = \frac{a + 2b}{c}x - b$ .

$(\bar{x}, \bar{y})$  is the point of intersection of these two lines.



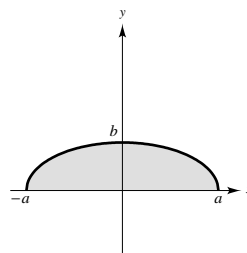
37.  $\bar{x} = 0$  by symmetry

$$A = \frac{1}{2} \pi ab$$

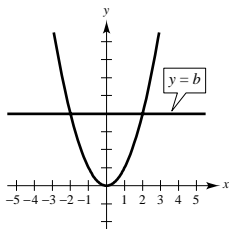
$$\frac{1}{A} = \frac{2}{\pi ab}$$

$$\begin{aligned} \bar{y} &= \frac{2}{\pi ab} \frac{1}{2} \int_{-a}^a \left( \frac{b}{a} \sqrt{a^2 - x^2} \right)^2 dx \\ &= \frac{1}{\pi ab} \left( \frac{b^2}{a^2} \right) \left[ a^2x - \frac{x^3}{3} \right]_{-a}^a = \frac{b}{\pi a^3} \left[ \frac{4a^3}{3} \right] = \frac{4b}{3\pi} \end{aligned}$$

$$(\bar{x}, \bar{y}) = \left( 0, \frac{4b}{3\pi} \right)$$



39. (a)



(b)  $\bar{x} = 0$  by symmetry

$$(c) M_y = \int_{-\sqrt{b}}^{\sqrt{b}} x(b - x^2) dx = 0 \text{ because } bx - x^3 \text{ is odd}$$

(d)  $\bar{y} > \frac{b}{2}$  since there is more area above  $y = \frac{b}{2}$  than below

$$\begin{aligned} (e) M_x &= \int_{-\sqrt{b}}^{\sqrt{b}} \frac{(b + x^2)(b - x^2)}{2} dx \\ &= \int_{-\sqrt{b}}^{\sqrt{b}} \frac{b^2 - x^4}{2} dx = \frac{1}{2} \left[ b^2x - \frac{x^5}{5} \right]_{-\sqrt{b}}^{\sqrt{b}} \\ &= b^2\sqrt{b} - \frac{b^2\sqrt{b}}{5} = \frac{4b^2\sqrt{b}}{5} \end{aligned}$$

$$\begin{aligned} A &= \int_{-\sqrt{b}}^{\sqrt{b}} (b - x^2) dx = \left[ bx - \frac{x^3}{3} \right]_{-\sqrt{b}}^{\sqrt{b}} \\ &= \left( b\sqrt{b} - \frac{b\sqrt{b}}{3} \right) 2 = 4 \frac{b\sqrt{b}}{3} \end{aligned}$$

$$\bar{y} = \frac{M_x}{A} = \frac{4b^2\sqrt{b}/5}{4b\sqrt{b}/3} = \frac{3}{5}b.$$

41. (a)
- $\bar{x} = 0$
- by symmetry

$$A = 2 \int_0^{40} f(x) dx = \frac{2(40)}{3(4)} [30 + 4(29) + 2(26) + 4(20) + 0] = \frac{20}{3}(278) = \frac{5560}{3}$$

$$M_x = \int_{-40}^{40} \frac{f(x)^2}{2} dx = \frac{40}{3(4)} [30^2 + 4(29)^2 + 2(26)^2 + 4(20)^2 + 0] = \frac{10}{3}(7216) = \frac{72160}{3}$$

$$\bar{y} = \frac{M_x}{A} = \frac{72160/3}{5560/3} = \frac{72160}{5560} \approx 12.98$$

$$(\bar{x}, \bar{y}) = (0, 12.98)$$

$$(b) y = (-1.02 \times 10^{-5})x^4 - 0.0019x^2 + 29.28$$

$$(c) \bar{y} = \frac{M_x}{A} \approx \frac{23697.68}{1843.54} \approx 12.85$$

$$(\bar{x}, \bar{y}) = (0, 12.85)$$

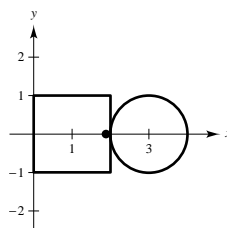
43. Centroids of the given regions: (1, 0) and (3, 0)

$$\text{Area: } A = 4 + \pi$$

$$\bar{x} = \frac{4(1) + \pi(3)}{4 + \pi} = \frac{4 + 3\pi}{4 + \pi}$$

$$\bar{y} = \frac{4(0) + \pi(0)}{4 + \pi} = 0$$

$$(\bar{x}, \bar{y}) = \left( \frac{4 + 3\pi}{4 + \pi}, 0 \right) \approx (1.88, 0)$$



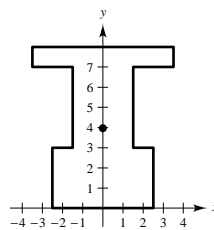
45. Centroids of the given regions:
- $\left(0, \frac{3}{2}\right)$
- , (0, 5), and
- $\left(0, \frac{15}{2}\right)$

$$\text{Area: } A = 15 + 12 + 7 = 34$$

$$\bar{x} = \frac{15(0) + 12(0) + 7(0)}{34} = 0$$

$$\bar{y} = \frac{15(3/2) + 12(5) + 7(15/2)}{34} = \frac{135}{34}$$

$$(\bar{x}, \bar{y}) = \left( 0, \frac{135}{34} \right)$$



47. Centroids of the given regions: (1, 0) and (3, 0)

$$\text{Mass: } 4 + 2\pi$$

$$\bar{x} = \frac{4(1) + 2\pi(3)}{4 + 2\pi} = \frac{2 + 3\pi}{2 + \pi}$$

$$\bar{y} = 0$$

$$(\bar{x}, \bar{y}) = \left( \frac{2 + 3\pi}{2 + \pi}, 0 \right) \approx (2.22, 0)$$

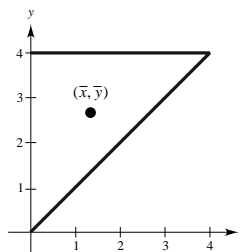
49.  $V = 2\pi rA = 2\pi(5)(16\pi) = 160\pi^2 \approx 1579.14$

$$51. A = \frac{1}{2}(4)(4) = 8$$

$$\bar{y} = \left(\frac{1}{8}\right)\frac{1}{2}\int_0^4 (4+x)(4-x) dx = \frac{1}{16}\left[16x - \frac{x^3}{3}\right]_0^4 = \frac{8}{3}$$

$$r = \bar{y} = \frac{8}{3}$$

$$V = 2\pi rA = 2\pi\left(\frac{8}{3}\right)(8) = \frac{128\pi}{3} \approx 134.04$$



$$53. m = m_1 + \cdots + m_n$$

$$M_y = m_1x_1 + \cdots + m_nx_n$$

$$M_x = m_1y_1 + \cdots + m_ny_n$$

$$\bar{x} = \frac{M_y}{m}, \bar{y} = \frac{M_x}{m}$$

$$55. (a) \text{ Yes. } (\bar{x}, \bar{y}) = \left(\frac{5}{6}, \frac{5}{18} + 2\right) = \left(\frac{5}{6}, \frac{41}{18}\right)$$

$$(b) \text{ Yes. } (\bar{x}, \bar{y}) = \left(\frac{5}{6} + 2, \frac{5}{18}\right) = \left(\frac{17}{6}, \frac{5}{18}\right)$$

$$(c) \text{ Yes. } (\bar{x}, \bar{y}) = \left(\frac{5}{6}, -\frac{5}{18}\right)$$

(d) No.

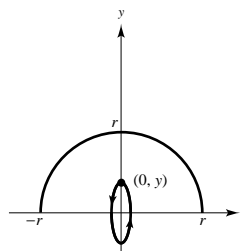
57. The surface area of the sphere is  $S = 4\pi r^2$ . The arc length of  $C$  is  $s = \pi r$ . The distance traveled by the centroid is

$$d = \frac{S}{s} = \frac{4\pi r^2}{\pi r} = 4r.$$

This distance is also the circumference of the circle of radius  $y$ .

$$d = 2\pi y$$

Thus,  $2\pi y = 4r$  and we have  $y = 2r/\pi$ . Therefore, the centroid of the semicircle  $y = \sqrt{r^2 - x^2}$  is  $(0, 2r/\pi)$ .



$$59. A = \int_0^1 x^n dx = \left[\frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{n+1}$$

$$m = \rho A = \frac{\rho}{n+1}$$

$$M_x = \frac{\rho}{2} \int_0^1 (x^n)^2 dx = \left[\frac{\rho}{2} \cdot \frac{x^{2n+1}}{2n+1}\right]_0^1 = \frac{\rho}{2(2n+1)}$$

$$M_y = \rho \int_0^1 x(x^n) dx = \left[\rho \cdot \frac{x^{n+2}}{n+2}\right]_0^1 = \frac{\rho}{n+2}$$

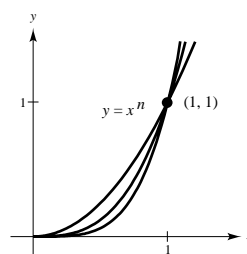
$$\bar{x} = \frac{M_y}{m} = \frac{n+1}{n+2}$$

$$\bar{y} = \frac{M_x}{m} = \frac{n+1}{2(2n+1)} = \frac{n+1}{4n+2}$$

$$\text{Centroid: } \left(\frac{n+1}{n+2}, \frac{n+1}{4n+2}\right)$$

$$\text{As } n \rightarrow \infty, (\bar{x}, \bar{y}) \rightarrow \left(1, \frac{1}{4}\right).$$

The graph approaches the  $x$ -axis and the line  $x = 1$  as  $n \rightarrow \infty$ .



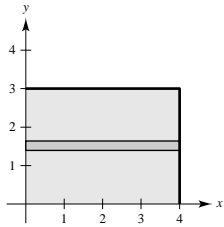
## Section 6.7 Fluid Pressure and Fluid Force

$$1. F = PA = [62.4(5)](3) = 936 \text{ lb}$$

$$5. h(y) = 3 - y$$

$$L(y) = 4$$

$$\begin{aligned} F &= 62.4 \int_0^3 (3 - y)(4) dy \\ &= 249.6 \int_0^3 (3 - y) dy \\ &= 249.6 \left[ 3y - \frac{y^2}{2} \right]_0^3 = 1123.2 \text{ lb} \end{aligned}$$

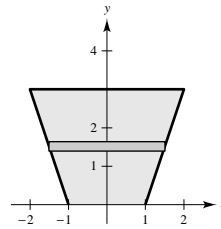


$$\begin{aligned} 3. F &= 62.4(h + 2)(6) - (62.4)(h)(6) \\ &= 62.4(2)(6) = 748.8 \text{ lb} \end{aligned}$$

$$7. h(y) = 3 - y$$

$$L(y) = 2\left(\frac{y}{3} + 1\right)$$

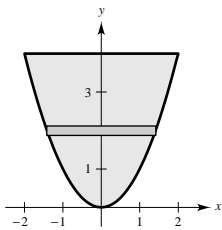
$$\begin{aligned} F &= 2(62.4) \int_0^3 (3 - y)\left(\frac{y}{3} + 1\right) dy \\ &= 124.8 \int_0^3 \left(3 - \frac{y^2}{3}\right) dy \\ &= 124.8 \left[ 3y - \frac{y^3}{9} \right]_0^3 = 748.8 \text{ lb} \end{aligned}$$



$$9. h(y) = 4 - y$$

$$L(y) = 2\sqrt{y}$$

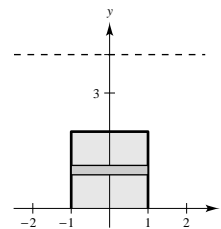
$$\begin{aligned} F &= 2(62.4) \int_0^4 (4 - y)\sqrt{y} dy \\ &= 124.8 \int_0^4 (4y^{1/2} - y^{3/2}) dy \\ &= 124.8 \left[ \frac{8y^{3/2}}{3} - \frac{2y^{5/2}}{5} \right]_0^4 = 1064.96 \text{ lb} \end{aligned}$$



$$11. h(y) = 4 - y$$

$$L(y) = 2$$

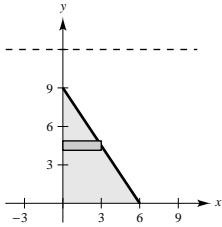
$$\begin{aligned} F &= 9800 \int_0^2 2(4 - y) dy \\ &= 9800 \left[ 8y - y^2 \right]_0^2 = 117,600 \text{ Newtons} \end{aligned}$$



13.  $h(y) = 12 - y$

$$L(y) = 6 - \frac{2y}{3}$$

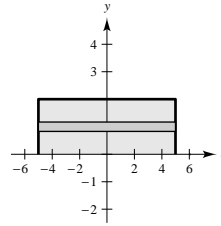
$$\begin{aligned} F &= 9800 \int_0^9 (12 - y) \left(6 - \frac{2y}{3}\right) dy \\ &= 9800 \left[ 72y - 7y^2 + \frac{2y^3}{9} \right]_0^9 = 2,381,400 \text{ Newtons} \end{aligned}$$



15.  $h(y) = 2 - y$

$$L(y) = 10$$

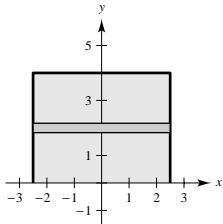
$$\begin{aligned} F &= 140.7 \int_0^2 (2 - y)(10) dy \\ &= 1407 \int_0^2 (2 - y) dy \\ &= 1407 \left[ 2y - \frac{y^2}{2} \right]_0^2 = 2814 \text{ lb} \end{aligned}$$



17.  $h(y) = 4 - y$

$$L(y) = 6$$

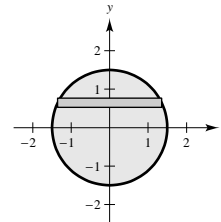
$$\begin{aligned} F &= 140.7 \int_0^4 (4 - y)(6) dy \\ &= 844.2 \int_0^4 (4 - y) dy \\ &= 844.2 \left[ 4y - \frac{y^2}{2} \right]_0^4 = 6753.6 \text{ lb} \end{aligned}$$



19.  $h(y) = -y$

$$L(y) = 2 \left( \frac{1}{2} \right) \sqrt{9 - 4y^2}$$

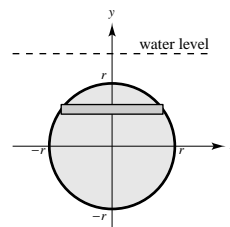
$$\begin{aligned} F &= 42 \int_{-3/2}^0 (-y) \sqrt{9 - 4y^2} dy \\ &= \frac{42}{8} \int_{-3/2}^0 (9 - 4y^2)^{1/2} (-8y) dy \\ &= \left[ \left( \frac{21}{4} \right) \left( \frac{2}{3} \right) (9 - 4y^2)^{3/2} \right]_{-3/2}^0 = 94.5 \text{ lb} \end{aligned}$$



21.  $h(y) = k - y$

$$L(y) = 2\sqrt{r^2 - y^2}$$

$$\begin{aligned} F &= w \int_{-r}^r (k - y) \sqrt{r^2 - y^2} (2) dy \\ &= w \left[ 2k \int_{-r}^r \sqrt{r^2 - y^2} dy + \int_{-r}^r \sqrt{r^2 - y^2} (-2y) dy \right] \end{aligned}$$



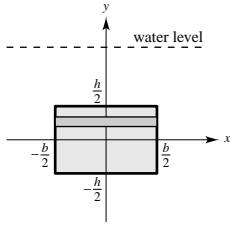
The second integral is zero since its integrand is odd and the limits of integration are symmetric to the origin. The first integral is the area of a semicircle with radius  $r$ .

$$F = w \left[ (2k) \frac{\pi r^2}{2} + 0 \right] = wk\pi r^2$$

23.  $h(y) = k - y$

$$L(y) = b$$

$$\begin{aligned} F &= w \int_{-h/2}^{h/2} (k - y)b \, dy \\ &= wb \left[ ky - \frac{y^2}{2} \right]_{-h/2}^{h/2} = wb(hk) = wkhb \end{aligned}$$



25. From Exercise 22:

$$F = 64(15)(1)(1) = 960 \text{ lb}$$

27.  $h(y) = 4 - y$

$$F = 62.4 \int_0^4 (4 - y)L(y) \, dy$$

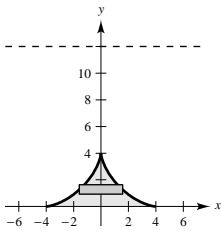
Using Simpson's Rule with  $n = 8$  we have:

$$\begin{aligned} F &\approx 62.4 \left( \frac{4 - 0}{3(8)} \right) [0 + 4(3.5)(3) + 2(3)(5) + 4(2.5)(8) + 2(2)(9) + 4(1.5)(10) + 2(1)(10.25) + 4(0.5)(10.5) + 0] \\ &= 3010.8 \text{ lb} \end{aligned}$$

29.  $h(y) = 12 - y$

$$L(y) = 2(4^{2/3} - y^{2/3})^{3/2}$$

$$\begin{aligned} F &= 62.4 \int_0^4 2(12 - y)(4^{2/3} - y^{2/3})^{3/2} \, dy \\ &\approx 6448.73 \text{ lb} \end{aligned}$$



31. (a) If the fluid force is one half of 1123.2 lb, and the height of the water is  $b$ , then

$$h(y) = b - y$$

$$L(y) = 4$$

$$F = 62.4 \int_0^b (b - y)(4) \, dy = \frac{1}{2}(1123.2)$$

$$\int_0^b (b - y) \, dy = 2.25$$

$$\left[ by - \frac{y^2}{2} \right]_0^b = 2.25$$

$$b^2 - \frac{b^2}{2} = 2.25$$

$$b^2 = 4.5 \Rightarrow b \approx 2.12 \text{ ft.}$$

(b) The pressure increases with increasing depth.

33.  $F = F_w = w \int_c^d h(y)L(y) \, dy$ , see page 471.