CHAPTER 7
Integration Techniques, L'Hôpital's Rule, and Improper Integrals

Section 7.1 Basic Integration Rules **308**

Section 7.2 Integration by Parts **312**

Section 7.3 Trigonometric Integrals **321**

Section 7.4 Trigonometric Substitution **328**

Section 7.5 Partial Fractions **336**

Section 7.6 Integration by Tables and Other Integration Techniques . . **343**

Section 7.7 Indeterminate Forms and L'Hôpital's Rule **348**

Section 7.8 Improper Integrals **353**

Review Exercises **358**

Problem Solving **363**

CHAPTER 7
Integration Techniques, L'Hôpital's Rule, and Improper Integrals

Section 7.1 Basic Integration Rules

Solutions to Even-Numbered Exercises

2. (a) $\frac{d}{dx}\left[\ln\sqrt{x^2+1} + C\right] = \frac{1}{2}\left(\frac{2x}{x^2+1}\right) = \frac{x}{x^2+1}$

(b) $\frac{d}{dx}\left[\frac{2x}{(x^2+1)^2} + C\right] = \frac{(x^2+1)^2(2) - (2x)(2)(x^2+1)(2x)}{(x^2+1)^4} = \frac{2(1-3x^2)}{(x^2+1)^3}$

(c) $\frac{d}{dx}[\arctan x + C] = \frac{1}{1+x^2}$

(d) $\frac{d}{dx}[\ln(x^2+1) + C] = \frac{2x}{x^2+1}$

$\int \frac{x}{x^2+1}\,dx$ matches (a).

4. (a) $\frac{d}{dx}[2x\sin(x^2+1) + C) = 2x[\cos(x^2+1)(2x)] + 2\sin(x^2+1) = 2[2x^2\cos(x^2+1) + \sin(x^2+1)]$

(b) $\frac{d}{dx}\left[-\frac{1}{2}\sin(x^2+1) + C\right] = -\frac{1}{2}\cos(x^2+1)(2x) = -x\cos(x^2+1)$

(c) $\frac{d}{dx}\left[\frac{1}{2}\sin(x^2+1) + C\right] = \frac{1}{2}\cos(x^2+1)(2x) = x\cos(x^2+1)$

(d) $\frac{d}{dx}[-2x\sin(x^2+1) + C] = -2x[\cos(x^2+1)(2x)] - 2\sin(x^2+1) = -2[2x^2\cos(x^2+1) + \sin(x^2+1)]$

$\int x\cos(x^2+1)\,dx$ matches (c).

6. $\int \frac{2t-1}{t^2-t+2}\,dt$

$u = t^2 - t + 2,\ du = (2t-1)\,dt$

Use $\int \frac{du}{u}$.

8. $\int \frac{2}{(2t-1)^2+4}\,dt$

$u = 2t - 1,\ du = 2dt,\ a = 2$

Use $\int \frac{du}{u^2+a^2}$.

10. $\int \frac{-2x}{\sqrt{x^2-4}}\,dx$

$u = x^2 - 4,\ du = 2x\,dx,\ n = -\frac{1}{2}$

Use $\int u^n\,du$.

12. $\int \sec 3x \tan 3x\,dx$

$u = 3x,\ du = 3\,dx$

Use $\int \sec u \tan u\,du$.

14. $\int \frac{1}{x\sqrt{x^2-4}}\,dx$

$u = x,\ du = dx,\ a = 2$

Use $\int \frac{du}{u\sqrt{u^2-a^2}}$.

16. Let $u = x - 4$, $du = dx$.

$$\int 6(x-4)^5\,dx = 6\int (x-4)^5\,dx = 6\frac{(x-4)^6}{6} + C$$
$$= (x-4)^6 + C$$

18. Let $u = t - 9$, $du = dt$.

$$\int \frac{2}{(t-9)^2}\,dt = 2\int (t-9)^{-2}\,dt = \frac{-2}{t-9} + C$$

20. Let $u = 4 - 2x^2$, $du = -4x\,dx$.

$$\int x\sqrt{4-2x^2}\,dx = -\frac{1}{4}\int (4-2x^2)^{1/2}(-4x)dx$$
$$= -\frac{1}{6}(4-2x^2)^{3/2} + C$$

22. $$\int \left[x - \frac{3}{(2x+3)^2}\right]dx = \int x\,dx - \frac{3}{2}\int (2x+3)^{-2}(2)\,dx$$
$$= \frac{x^2}{2} - \frac{3}{2}\frac{(2x+3)^{-1}}{-1} + C$$
$$= \frac{x^2}{2} + \frac{3}{2(2x+3)} + C$$

24. Let $u = x^2 + 2x - 4$, $du = 2(x+1)\,dx$.

$$\int \frac{x+1}{\sqrt{x^2+2x-4}}\,dx = \frac{1}{2}\int (x^2+2x-4)^{-1/2}(2)(x+1)\,dx$$
$$= \sqrt{x^2+2x-4} + C$$

26. $$\int \frac{2x}{x-4}\,dx = \int 2\,dx + \int \frac{8}{x-4}\,dx = 2x + 8\ln|x-4| + C$$

28. $$\int \left(\frac{1}{3x-1} - \frac{1}{3x+1}\right)dx = \frac{1}{3}\int \frac{1}{3x-1}(3)\,dx - \frac{1}{3}\int \frac{1}{3x+1}(3)\,dx$$
$$= \frac{1}{3}\ln|3x-1| - \frac{1}{3}\ln|3x+1| + C = \frac{1}{3}\ln\left|\frac{3x-1}{3x+1}\right| + C$$

30. $$\int x\left(1+\frac{1}{x}\right)^3 = \int x\left(1 + \frac{3}{x} + \frac{3}{x^2} + \frac{1}{x^3}\right)dx = \int \left(x + 3 + \frac{3}{x} + \frac{1}{x^2}\right)dx = \frac{1}{2}x^2 + 3x + 3\ln|x| - \frac{1}{x} + C$$

32. $$\int \sec 4x\,dx = \frac{1}{4}\int \sec(4x)(4)\,dx$$
$$= \frac{1}{4}\ln|\sec 4x + \tan 4x| + C$$

34. Let $u = \cos x$, $du = -\sin x\,dx$.

$$\int \frac{\sin x}{\sqrt{\cos x}}\,dx = -\int (\cos x)^{-1/2}(-\sin x)\,dx$$
$$= -2\sqrt{\cos x} + C$$

36. Let $u = \cot x$, $du = -\csc^2 x\,dx$.

$$\int \csc^2 x e^{\cot x}\,dx = -\int e^{\cot x}(-\csc^2 x)\,dx = -e^{\cot x} + C$$

38. $$\int \frac{5}{3e^x - 2}\,dx = 5\int \left(\frac{1}{3e^x-2}\right)\left(\frac{e^{-x}}{e^{-x}}\right)dx$$
$$= 5\int \frac{e^{-x}}{3-2e^{-x}}\,dx$$
$$= \frac{5}{2}\int \frac{1}{3-2e^{-x}}(2e^{-x})\,dx$$
$$= \frac{5}{2}\ln|3-2e^{-x}| + C$$

40. Let $u = \ln(\cos x)$, $du = \dfrac{-\sin x}{\cos x}\,dx$

$$= -\tan x\,dx$$

$$\int (\tan x)(\ln \cos x)\,dx = -\int (\ln \cos x)(-\tan x)\,dx$$

$$= \frac{-[\ln(\cos x)]^2}{2} + C$$

42. $\displaystyle\int \frac{1 + \cos \alpha}{\sin \alpha}\,d\alpha = \int \csc \alpha\,d\alpha + \int \cot \alpha\,d\alpha$

$$= -\ln|\csc \alpha + \cot \alpha| + \ln|\sin \alpha| + C$$

44. $\displaystyle\int \frac{2}{3(\sec x - 1)}\,dx = \frac{2}{3}\int \frac{1}{\sec x - 1} \cdot \left(\frac{\sec x + 1}{\sec x + 1}\right) dx$

$$= \frac{2}{3}\int \frac{\sec x + 1}{\tan^2 x}\,dx$$

$$= \frac{2}{3}\int \frac{\sec x}{\tan^2 x}\,dx + \frac{2}{3}\int \cot^2 x\,dx$$

$$= \frac{2}{3}\int \frac{\cos x}{\sin^2 x}\,dx + \frac{2}{3}\int (\csc^2 x - 1)\,dx$$

$$= \frac{2}{3}\left(-\frac{1}{\sin x}\right) - \frac{2}{3}\cot x - \frac{2}{3}x + C$$

$$= -\frac{2}{3}[\csc x + \cot x + x] + C$$

46. $\displaystyle\int \frac{3}{t^2 + 1}\,dt = 3 \arctan t + C$

48. Let $u = \sqrt{3}x$, $du = \sqrt{3}\,dx$.

$$\int \frac{1}{4 + 3x^2}\,dx = \frac{1}{\sqrt{3}}\int \frac{\sqrt{3}}{4 + (\sqrt{3}x)^2}\,dx$$

$$= \frac{1}{2\sqrt{3}} \arctan\left(\frac{\sqrt{3}x}{2}\right) + C$$

50. Let $u = \dfrac{1}{t}$, $du = \dfrac{-1}{t^2}\,dt$.

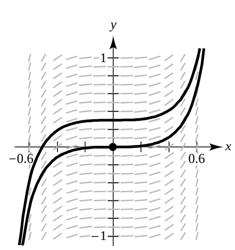
$$\int \frac{e^{1/t}}{t^2}\,dt = -\int e^{1/t}\left(\frac{-1}{t^2}\right) dt = -e^{1/t} + C$$

52. $\displaystyle\int \frac{1}{(x - 1)\sqrt{4x^2 - 8x + 3}}\,dx = \int \frac{2}{[2(x - 1)]\sqrt{[2(x - 1)]^2 - 1}}\,dx = \operatorname{arcsec}|2(x - 1)| + C$

54. $\displaystyle\int \frac{1}{\sqrt{1 - 4x - x^2}}\,dx = \int \frac{1}{\sqrt{5 - (x^2 + 4x + 4)}}\,dx = \int \frac{1}{\sqrt{5 - (x + 2)^2}}\,dx = \arcsin\left(\frac{x + 2}{\sqrt{5}}\right) + C \quad (a = \sqrt{5})$

56. $\dfrac{dy}{dx} = \tan^2(2x)$, $(0, 0)$

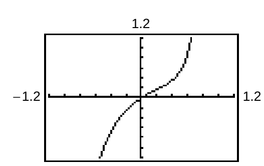
(a)



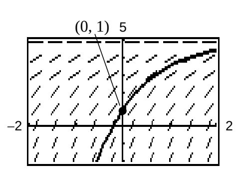
(b) $\displaystyle\int \tan^2(2x)dx = \int (\sec^2(2x) - 1)\,dx = \frac{1}{2}\tan(2x) - x + C$

$(0, 0)$: $0 = C$

$$y = \frac{1}{2}\tan(2x) - x$$



58.



60. $\displaystyle r = \int \frac{(1 + e^t)^2}{e^t}\,dt = \int \frac{1 + 2e^t + e^{2t}}{e^t}\,dt$

$$= \int (e^{-t} + 2 + e^t)\,dt = -e^{-t} + 2t + e^t + C$$

62. Let $u = 2x$, $du = 2\,dx$.

$$y = \int \frac{1}{x\sqrt{4x^2 - 1}}\,dx = \int \frac{2}{2x\sqrt{(2x)^2 - 1}}\,dx$$
$$= \operatorname{arcsec}|2x| + C$$

64. Let $u = \sin t$, $du = \cos t\,dt$.

$$\int_0^{\pi} \sin^2 t \cos t\,dt = \left[\frac{1}{3}\sin^3 t\right]_0^{\pi} = 0$$

66. Let $u = 1 - \ln x$, $du = \dfrac{-1}{x}\,dx$.

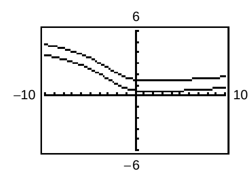
$$\int_1^e \frac{1 - \ln x}{x}\,dx = -\int_1^e (1 - \ln x)\left(\frac{-1}{x}\right)dx$$
$$= \left[-\frac{1}{2}(1 - \ln x)^2\right]_1^e = \frac{1}{2}$$

68. $$\int_1^2 \frac{x - 2}{x}\,dx = \int_1^2 \left(1 - \frac{2}{x}\right)dx$$
$$= \Big[x - 2\ln x\Big]_1^2 = 1 - \ln 4 \approx -0.386$$

70. $$\int_0^4 \frac{1}{\sqrt{25 - x^2}}\,dx = \left[\arcsin\frac{x}{5}\right]_0^4 = \arcsin\frac{4}{5} \approx 0.927$$

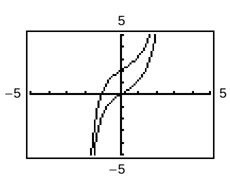
72. $$\int \frac{x - 2}{x^2 + 4x + 13}\,dx = \frac{1}{2}\ln(x^2 + 4x + 13) - \frac{4}{3}\arctan\left(\frac{x + 2}{3}\right) + C$$

The antiderivatives are vertical translations of each other.



74. $$\int \left(\frac{e^x + e^{-x}}{2}\right)^3 dx = \frac{1}{24}\left[e^{3x} + 9e^x - 9e^{-x} - e^{-3x}\right] + C$$

The antiderivatives are vertical translations of each other.



76. $$\int \sec u \tan u\,du = \sec u + \mathrm{C}$$

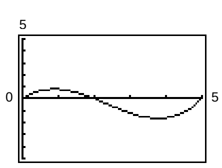
78. Arctan Rule: $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\left(\frac{4}{a}\right) + C$

80. They differ by a constant:

$$\sec^2 x + C_1 = (\tan^2 x + 1) + C_1 = \tan^2 x + C.$$

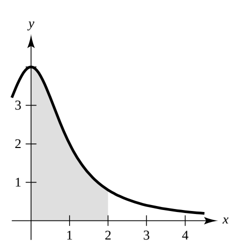
82. $f(x) = \dfrac{1}{5}(x^3 - 7x^2 + 10x)$

$\displaystyle\int_0^5 f(x)\,dx < 0$ because more area is below the x-axis than above.

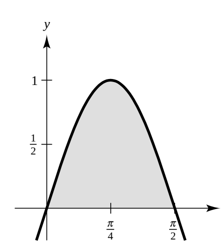


84. $$\int_0^2 \frac{4}{x^2 + 1}\,dx \approx 4$$

Matches (d).



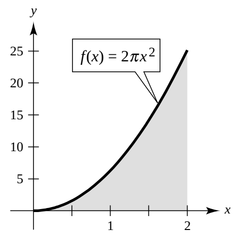
86. $$A = \int_0^{\pi/2} \sin 2x\,dx$$
$$= \left[-\frac{1}{2}\cos 2x\right]_0^{\pi/2} = 1$$



88. $\int_0^2 2\pi x^2\,dx$

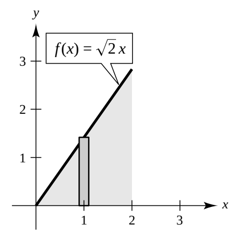
(a) Let $f(x) = 2\pi x^2$ over the interval $[0, 2]$.

$$A = \int_0^2 (2\pi x^2)\,dx$$



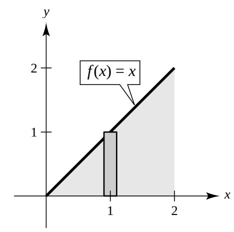
(b) Let $f(x) = \sqrt{2}\,x$ over the interval $[0, 2]$. Revolve this region about the x-axis.

$$V = \pi\int_0^2 \left(\sqrt{2}\,x\right)^2 dx$$
$$= \int_0^2 2\pi x^2\,dx$$



(c) Let $f(x) = x$ over the interval $[0, 2]$. Revolve this region about the y-axis.

$$V = 2\pi\int_0^2 x(x)\,dx$$
$$= \int_0^2 2\pi\,x^2\,dx$$

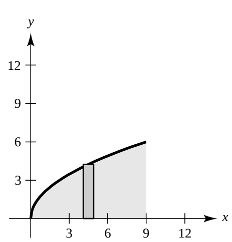


90. (a) $\frac{1}{(\pi/n) - 0}\int_0^{\pi/n} \sin(nx)\,dx = \frac{1}{\pi}\int_0^{\pi/n} \sin(nx)(n)\,dx = \left[-\frac{1}{\pi}\cos(nx)\right]_0^{\pi/n} = \frac{2}{\pi}$

(b) $\frac{1}{3 - (-3)}\int_{-3}^{3} \frac{1}{1 + x^2}\,dx = \left[\frac{1}{6}\arctan x\right]_{-3}^{3} = \frac{1}{3}\arctan 3$

92.

$$y = 2\sqrt{x}$$
$$y' = \frac{1}{\sqrt{x}}$$
$$1 + (y')^2 = 1 + \frac{1}{x} = \frac{x + 1}{x}$$
$$S = 2\pi\int_0^9 2\sqrt{x}\sqrt{\frac{x + 1}{x}}\,dx$$
$$= 2\pi\int_0^9 2\sqrt{x + 1}\,dx$$
$$= \left[4\pi\left(\frac{2}{3}\right)(x + 1)^{3/2}\right]_0^9$$
$$= \frac{8\pi}{3}\left(10\sqrt{10} - 1\right) \approx 256.545$$



94.

$$y = x^{2/3}$$
$$y' = \frac{2}{3x^{1/3}}$$
$$1 + (y')^2 = 1 + \frac{4}{9x^{2/3}}$$
$$s = \int_1^8 \sqrt{1 + \frac{4}{9x^{2/3}}}\,dx \approx 7.6337$$

Section 7.2 Integration by Parts

2. $\frac{d}{dx}[x^2 \sin x + 2x \cos x - 2 \sin x] = x^2 \cos x + 2x \sin x - 2x \sin x + 2 \cos x - 2 \cos x = x^2 \cos x$. Matches (d)

4. $\frac{d}{dx}[-x + x\ln x] = -1 + x\left(\frac{1}{x}\right) + \ln x = \ln x$. Matches (a)

6. $\int x^2 e^{2x}\,dx$

$u = x^2,\ dv = e^{2x}\,dx$

8. $\int \ln 3x\,dx$

$u = \ln 3x,\ dv = dx$

10. $\int x^2 \cos x\,dx$

$u = x^2,\ dv = \cos x\,dx$

12. $dv = e^{-x}\,dx \implies v = \int e^{-x}\,dx = -e^{-x}$

$u = x \implies du = dx$

$$2\int \frac{x}{e^x}\,dx = 2\int xe^{-x}\,dx$$

$$= 2\left[-xe^{-x} - \int -e^{-x}\,dx\right] = 2[-xe^{-x} - e^{-x}] + C$$

$$= -2xe^{-x} - 2e^{-x} + C$$

14. $\int \frac{e^{1/t}}{t^2}\,dt = -\int e^{1/t}\left(\frac{-1}{t^2}\right)dt = -e^{1/t} + C$

16. $dv = x^4\,dx \implies v = \frac{x^5}{5}$

$u = \ln x \implies du = \frac{1}{x}\,dx$

$$\int x^4 \ln x\,dx = \frac{x^5}{5}\ln x - \int \frac{x^5}{5}\left(\frac{1}{x}\right)dx = \frac{x^5}{5}\ln x - \frac{1}{5}\int x^4\,dx$$

$$= \frac{x^5}{5}\ln x - \frac{1}{25}x^5 + C \quad = \frac{x^5}{25}(5\ln x - 1) + C$$

18. Let $u = \ln x,\ du = \frac{1}{x}\,dx$.

$$\int \frac{1}{x(\ln x)^3}\,dx = \int (\ln x)^{-3}\left(\frac{1}{x}\right)dx = \frac{-1}{2(\ln x)^2} + C$$

20. $dv = \frac{1}{x^2}\,dx \implies v = \int \frac{1}{x^2}\,dx = -\frac{1}{x}$

$u = \ln x \implies du = \frac{1}{x}\,dx$

$$\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx = -\frac{\ln x}{x} - \frac{1}{x} + C$$

22. $dv = \frac{x}{(x^2+1)^2}\,dx \implies v = \int (x^2+1)^{-2}x\,dx = -\frac{1}{2(x^2+1)}$

$u = x^2e^{x^2} \implies du = (2x^3e^{x^2} + 2xe^{x^2})\,dx = 2xe^{x^2}(x^2+1)\,dx$

$$\int \frac{x^3e^{x^2}}{(x^2+1)^2}\,dx = -\frac{x^2e^{x^2}}{2(x^2+1)} + \int xe^{x^2}\,dx = -\frac{x^2e^{x^2}}{2(x^2+1)} + \frac{e^{x^2}}{2} + C = \frac{e^{x^2}}{2(x^2+1)} + C$$

24. $dv = \frac{1}{x^2}\,dx \implies v = \int \frac{1}{x^2}\,dx = -\frac{1}{x}$

$u = \ln 2x \implies du = \frac{1}{x}\,dx$

$$\int \frac{\ln(2x)}{x^2}\,dx = -\frac{\ln(2x)}{x} + \int \frac{1}{x^2}\,dx = -\frac{\ln(2x)}{x} - \frac{1}{x} + C = -\frac{\ln(2x) + 1}{x} + C$$

26. $dv = \dfrac{1}{\sqrt{2+3x}}\,dx \implies v = \displaystyle\int (2+3x)^{-1/2}\,dx = \frac{2}{3}\sqrt{2+3x}$

$u = x \implies du = dx$

$$\int \frac{x}{\sqrt{2+3x}}\,dx = \frac{2x\sqrt{2+3x}}{3} - \frac{2}{3}\int \sqrt{2+3x}\,dx$$

$$= \frac{2x\sqrt{2+3x}}{3} - \frac{4}{27}(2+3x)^{3/2} + C = \frac{2\sqrt{2+3x}}{27}[9x - 2(2+3x)] + C = \frac{2\sqrt{2+3x}}{27}(3x-4) + C$$

28. $dv = \sin x\,dx \implies v = -\cos x$

$u = x \implies du = dx$

$$\int x \sin dx = -x\cos x - \int -\cos x\,dx = -x\cos x + \sin x + C$$

30. Use integration by parts twice.

(1) $u = x^2, du = 2x\,dx, dv = \cos x\,dx, v = \sin x$

$$\int x^2 \cos x\,dx = x^2 \sin x - 2\int x \sin x\,dx$$

(2) $u = x, du = dx, dv = \sin x\,dx, v = -\cos x$

$$\int x^2 \cos x\,dx = x^2 \sin x - 2\left[-x\cos x + \int \cos x\,dx\right]$$

$$= x^2 \sin x + 2x\cos x - 2\sin x + C$$

32. $dv = \sec\theta\tan\theta\,d\theta \implies v = \displaystyle\int \sec\theta\tan\theta\,d\theta = \sec\theta$

$u = \theta \implies du = d\theta$

$$\int \theta\sec\theta\tan\theta\,d\theta = \theta\sec\theta - \int \sec\theta\,d\theta$$

$$= \theta\sec\theta - \ln|\sec\theta + \tan\theta| + C$$

34. $dv = dx \implies v = \displaystyle\int dx = x$

$u = \arccos x \implies du = -\dfrac{1}{\sqrt{1-x^2}}\,dx$

$$4\int \arccos x\,dx = 4\left[x\arccos x + \int \frac{x}{\sqrt{1-x^2}}\,dx\right]$$

$$= 4\left[x\arccos x - \sqrt{1-x^2}\right] + C$$

36. Use integration by parts twice.

(1) $dv = e^x\,dx \implies v = \displaystyle\int e^x\,dx = e^x$

$u = \cos 2x \implies du = -2\sin 2x\,dx$

(2) $dv = e^x\,dx \implies v = \displaystyle\int e^x\,dx = e^x$

$u = \sin 2x \implies du = 2\cos 2x\,dx$

$$\int e^x \cos 2x\,dx = e^x\cos 2x + 2\int e^x \sin 2x\,dx = e^x\cos 2x + 2\left(e^x\sin 2x - 2\int e^x\cos 2x\,dx\right)$$

$$5\int e^x\cos 2x\,dx = e^x\cos 2x + 2e^x\sin 2x$$

$$\int e^x\cos 2x\,dx = \frac{e^x}{5}(\cos 2x + 2\sin 2x) + C$$

38. $dv = dx \implies v = x$

$u = \ln x \implies du = \dfrac{1}{x}\,dx$

$y' = \ln x$

$$y = \int \ln x\,dx = x\ln x - \int x\left(\frac{1}{x}\right)dx = x\ln x - x + C = x(-1 + \ln x) + C$$

40. Use integration by parts twice.

(1) $dv = \sqrt{x-1}\,dx \implies v = \int (x-1)^{1/2}\,dx = \frac{2}{3}(x-1)^{3/2}$

$u = x^2 \implies du = 2x\,dx$

(2) $dv = (x-1)^{3/2}dx \implies v = \int (x-1)^{3/2}\,dx = \frac{2}{5}(x-1)^{5/2}$

$u = x \implies du = dx$

$$y = \int x^2\sqrt{x-1}\,dx$$

$$= \frac{2}{3}x^2(x-1)^{3/2} - \frac{4}{3}\int x(x-1)^{3/2}\,dx = \frac{2}{3}x^2(x-1)^{3/2} - \frac{4}{3}\left[\frac{2}{5}x(x-1)^{5/2} - \frac{2}{5}\int (x-1)^{5/2}\,dx\right]$$

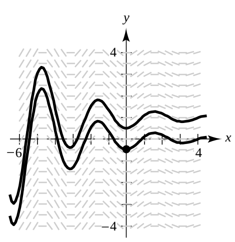
$$= \frac{2}{3}x^2(x-1)^{3/2} - \frac{8}{15}x(x-1)^{5/2} + \frac{16}{105}(x-1)^{7/2} + C = \frac{2(x-1)^{3/2}}{105}(15x^2 + 12x + 8) + C$$

42. $dv = dx \implies v = \int dx = x$

$u = \arctan\frac{x}{2} \implies du = \frac{1}{1+(x/2)^2}\left(\frac{1}{2}\right)dx = \frac{2}{4+x^2}\,dx$

$$y = \int \arctan\frac{x}{2}\,dx = x\arctan\frac{x}{2} - \int \frac{2x}{4+x^2}\,dx = x\arctan\frac{x}{2} - \ln(4+x^2) + C$$

44. (a)



(b) $\frac{dy}{dx} = e^{-x/3}\sin 2x, \left(0, -\frac{18}{37}\right)$

$y = \int e^{-x/3}\sin 2x\,dx$

Use integration by parts twice.

(1) $u = \sin 2x,\ du = 2\cos 2x$

$dv = e^{-x/3}\,dx, v = -3e^{-x/3}$

$$\int e^{-x/3}\sin 2x\,dx = -3e^{-x/3}\sin 2x + \int 6e^{-x/3}\cos 2x\,dx$$

(2) $u = \cos 2x,\ du = -2\sin 2x$

$dv = e^{-x/3}\,dx, v = -3e^{-x/3}$

$$\int e^{-x/3}\sin 2x\,dx = -3e^{-x/3}\sin 2x + 6\left[-3e^{-x/3}\cos 2x - \int 6e^{-x/3}\sin 2x\,dx\right] + C$$

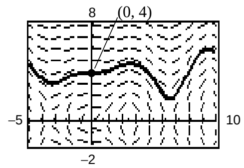
$$37\int e^{-x/3}\sin 2x\,dx = -3e^{-x/3}\sin 2x - 18e^{-x/3}\cos 2x + C$$

$$y = \int e^{-x/3}\sin 2x\,dx = \frac{1}{37}\left[-3e^{-x/3}\sin 2x - 18e^{-x/3}\cos 2x\right] + C$$

$$\left(0, \frac{-18}{37}\right): \frac{-18}{37} = \frac{1}{37}[0 - 18] + C \implies C = 0$$

$$y = \frac{-1}{37}[3e^{-x/3}\sin 2x + 18e^{-x/3}\cos 2x]$$

46. $\dfrac{dy}{dx} = \dfrac{x}{y}\sin x,\ y(0) = 4$



48. See Exercise 3.

$$\int_0^1 x^2 e^x\,dx = \Big[x^2e^x - 2xe^x + 2e^x\Big]_0^1 = e - 2 \approx 0.718$$

50. $dv = \sin 2x\,dx \implies v = \int \sin 2x\,dx = -\frac{1}{2}\cos 2x$

$u = x \implies du = dx$

$$\int x\sin 2x\,dx = \frac{-1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x\,dx$$
$$= \frac{-1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C$$
$$= \frac{1}{4}(\sin 2x - 2x\cos 2x) + C$$

Thus, $\displaystyle\int_0^{\pi} x\sin 2x\,dx = \left[\frac{1}{4}(\sin 2x - 2x\cos 2x)\right]_0^{\pi} = -\frac{\pi}{2}.$

52. $dv = x\,dx \implies v = \int x\,dx = \dfrac{x^2}{2}$

$u = \arcsin x^2 \implies du = \dfrac{2x}{\sqrt{1-x^4}}\,dx$

$$\int x\arcsin x^2\,dx = \frac{x^2}{2}\arcsin x^2 - \int \frac{x^3}{\sqrt{1-x^4}}\,dx$$
$$= \frac{x^2}{2}\arcsin x^2 + \frac{1}{4}(2)(1-x^4)^{1/2} + C$$
$$= \frac{1}{2}\Big[x^2\arcsin x^2 + \sqrt{1-x^4}\Big] + C$$

Thus, $\displaystyle\int_0^1 x\arcsin x^2 dx = \frac{1}{2}\Big[x^2\arcsin x^2 + \sqrt{1-x^4}\Big]_0^1$

$$= \frac{1}{4}(\pi - 2).$$

54. Use integration by parts twice.

(1) $dv = e^{-x},\ v = -e^{-x},\ u = \cos x,\ du = -\sin x\,dx$

$$\int e^{-x}\cos x\,dx = -e^{-x}\cos x - \int e^{-x}\sin x\,dx$$

(2) $dv = e^{-x}\,dx,\ v = -e^{-x},\ u = \sin x,\ du = \cos x\,dx$

$$\int e^{-x}\cos x\,dx = -e^{-x}\cos x - \left[-e^{-x}\sin x + \int e^{-x}\cos x\,dx\right] \implies 2\int e^{-x}\cos x\,dx = e^{-x}\sin x - e^{-x}\cos x$$

Thus,

$$\int_0^2 e^{-x}\cos x\,dx = \left[\frac{e^{-x}\sin x - e^{-x}\cos x}{2}\right]_0^2$$
$$= \frac{-e^{-2}}{2}[\sin 2 - \cos 2] + \frac{1}{2}$$

56. $dv = dx \implies v = \int dx = x$

$u = \ln(1+x^2) \implies du = \dfrac{2x}{1+x^2}\,dx$

$$\int \ln(1+x^2)dx = x\ln(1+x^2) - \int \frac{2x^2}{1+x^2}\,dx$$
$$= x\ln(1+x^2) - 2\int\left[1 - \frac{1}{1+x^2}\right]dx = x\ln(1+x^2) - 2x + 2\arctan x + C$$

Thus, $\displaystyle\int_0^1 \ln(1+x^2)\,dx = \Big[x\ln(1+x^2) - 2x + 2\arctan x\Big]_0^1 = \ln 2 - 2 + \frac{\pi}{2}.$

58. $u = x,\ du = dx,\ dv = \sec^2 x\, dx,\ v = \tan x$

$$\int x \sec^2 x\, dx = x \tan x - \int \tan x\, dx$$

Hence,

$$\int_0^{\pi/4} x \sec^2 x\, dx = \Big[x \tan x + \ln|\cos x|\Big]_0^{\pi/4}$$
$$= \left(\frac{\pi}{4} + \ln \frac{\sqrt{2}}{2}\right) - 0$$
$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

60. $$\int x^3 e^{-2x}\, dx = x^3\left(-\frac{1}{2}e^{-2x}\right) - 3x^2\left(\frac{1}{4}e^{-2x}\right) + 6x\left(-\frac{1}{8}e^{-2x}\right) - 6\left(\frac{1}{16}e^{-2x}\right) + C$$
$$= -\frac{1}{8}e^{-2x}(4x^3 + 6x^2 + 6x + 3) + C$$

Alternate signs	u and its derivatives	v' and its antiderivatives
+	x^3	e^{-2x}
−	$3x^2$	$-\frac{1}{2}e^{-2x}$
+	$6x$	$\frac{1}{4}e^{-2x}$
−	6	$-\frac{1}{8}e^{-2x}$
+	0	$\frac{1}{16}e^{-2x}$

62. $$\int x^3 \cos 2x\, dx = x^3\left(\frac{1}{2}\sin 2x\right) - 3x^2\left(-\frac{1}{4}\cos 2x\right) + 6x\left(-\frac{1}{8}\sin 2x\right) - 6\left(\frac{1}{16}\cos 2x\right) + C$$
$$= \frac{1}{2}x^3 \sin 2x + \frac{3}{4}x^2 \cos 2x - \frac{3}{4}x \sin 2x - \frac{3}{8}\cos 2x + C$$
$$= \frac{1}{8}[4x^3 \sin 2x + 6x^2 \cos 2x - 6x \sin 2x - 3\cos 2x] + C$$

Alternate signs	u and its derivatives	v' and its antiderivatives
+	x^3	$\cos 2x$
−	$3x^2$	$\frac{1}{2}\sin 2x$
+	$6x$	$-\frac{1}{4}\cos 2x$
−	6	$-\frac{1}{8}\sin 2x$
+	0	$\frac{1}{16}\cos 2x$

64. $$\int x^2(x-2)^{3/2} dx = \frac{2}{5}x^2(x-2)^{5/2} - \frac{8}{35}x(x-2)^{7/2} + \frac{16}{315}(x-2)^{9/2} + C$$
$$= \frac{2}{315}(x-2)^{5/2}(35x^2 + 40x + 32) + C$$

Alternate signs	u and its derivatives	v' and its antiderivatives
+	x^2	$(x-2)^{3/2}$
−	$2x$	$\frac{2}{5}(x-2)^{5/2}$
+	2	$\frac{4}{35}(x-2)^{7/2}$
−	0	$\frac{8}{315}(x-2)^{9/2}$

66. Answers will vary.
See pages 488, 493.

68. Yes.
$u = \ln x,\ dv = x\, dx$

70. No. Substitution.

72. No. Substitution.

74. $$\int \alpha^4 \sin \pi\alpha\, d\alpha = \frac{1}{\pi^5}[-(\alpha\pi)^4 \cos \pi\alpha + 4(\alpha\pi)^3 \sin \pi\alpha + 12(\alpha\pi)^2 \cos \pi\alpha - 24(\alpha\pi)\sin \pi\alpha - 24\cos \pi\alpha] + C$$

76. $$\int_0^5 x^4(25 - x^2)^{3/2}\, dx = \left[\frac{1{,}171{,}875 \arcsin(x/5)}{128} - \frac{x(2x^2+25)(25-x^2)^{5/2}}{16} + \frac{625x(25-x^2)^{3/2}}{64} + \frac{46{,}875x\sqrt{25-x^2}}{128}\right]_0^5$$
$$\approx 14{,}381.0699$$

78. (a) $dv = \sqrt{4+x}\,dx \implies v = \int (4+x)^{1/2}\,dx = \frac{2}{3}(4+x)^{3/2}$

$u = x \implies du = dx$

$$\int x\sqrt{4+x}\,dx = \frac{2}{3}x(4+x)^{3/2} - \frac{2}{3}\int (4+x)^{3/2}\,dx$$

$$= \frac{2}{3}x(4+x)^{3/2} - \frac{4}{15}(4+x)^{5/2} + C = \frac{2}{15}(4+x)^{3/2}(3x-8) + C$$

(b) $u = 4 + x \implies x = u - 4$ and $dx = du$

$$\int x\sqrt{4+x}\,dx = \int (u-4)u^{1/2}du = \int (u^{3/2} - 4u^{1/2})\,du$$

$$= \frac{2}{5}u^{5/2} - \frac{8}{3}u^{3/2} + C = \frac{2}{15}u^{3/2}(3u-20) + C$$

$$= \frac{2}{15}(4+x)^{3/2}[3(4+x) - 20] + C = \frac{2}{15}(4+x)^{3/2}(3x-8) + C$$

80. (a) $dv = \sqrt{4-x}\,dx \implies v = \int (4-x)^{1/2}\,dx = -\frac{2}{3}(4-x)^{3/2}$

$u = x \implies du = dx$

$$\int x\sqrt{4-x}\,dx = -\frac{2}{3}x(4-x)^{3/2} + \frac{2}{3}\int (4-x)^{3/2}\,dx$$

$$= -\frac{2}{3}x(4-x)^{3/2} - \frac{4}{15}(4-x)^{5/2} + C$$

$$= -\frac{2}{15}(4-x)^{3/2}[5x + 2(4-x)] + C$$

$$= -\frac{2}{15}(4-x)^{3/2}(3x+8) + C$$

(b) $u = 4 - x \implies x = 4 - u$ and $dx = -du$

$$\int x\sqrt{4-x}\,dx = -\int (4-u)\sqrt{u}\,du$$

$$= -\int (4u^{1/2} - u^{3/2})\,du$$

$$= -\frac{8}{3}u^{3/2} + \frac{2}{5}u^{5/2} + C$$

$$= -\frac{2}{15}u^{3/2}(20 - 3u) + C$$

$$= -\frac{2}{15}(4-x)^{3/2}[20 - 3(4-x)] + C$$

$$= -\frac{2}{15}(4-x)^{3/2}(3x+8) + C$$

82. $n = 0$: $\int e^x\,dx = e^x + C$

$n = 1$: $\int xe^x\,dx = xe^x - e^x + C = xe^x - \int e^x\,dx$

$n = 2$: $\int x^2e^x\,dx = x^2e^x - 2xe^x + 2e^x + C$

$$= x^2e^x - 2\int xe^x\,dx$$

$n = 3$: $\int x^3e^x\,dx = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$

$$= x^3e^x - 3\int x^2e^x\,dx$$

$n = 4$: $\int x^4e^x\,dx$

$$= x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24e^x + C$$

$$= x^4e^x - 4\int x^3e^x\,dx$$

In general, $\int x^ne^x\,dx = x^ne^x - n\int x^{n-1}e^x\,dx$.

(See Exercise 86)

84. $dv = \cos x\,dx \implies v = \sin x$

$u = x^n \implies du = nx^{n-1}\,dx$

$$\int x^n \cos x\,dx = x^n \sin x - n\int x^{n-1}\sin x\,dx$$

86. $dv = e^{ax}\,dx \implies v = \frac{1}{a}e^{ax}$

$u = x^n \implies du = nx^{n-1}\,dx$

$$\int x^ne^{ax}\,dx = \frac{x^ne^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$$

88. Use integration by parts twice.

(1) $dv = e^{ax}\,dx \implies v = \frac{1}{a}e^{ax}$ (2) $dv = e^{ax}\,dx \implies v = \frac{1}{a}e^{ax}$

$u = \cos bx \implies du = -b\sin bx$ $u = \sin bx \implies du = b\cos bx$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}\cos bx}{a} + \frac{b}{a}\int e^{ax}\sin bx\,dx = \frac{e^{ax}\cos bx}{a} + \frac{b}{a}\left[\frac{e^{ax}\sin bx}{a} - \frac{b}{a}\int e^{ax}\cos bx\,dx\right]$$

$$= \frac{e^{ax}\cos bx}{a} + \frac{be^{ax}\sin bx}{a^2} - \frac{b^2}{a^2}\int e^{ax}\cos bx\,dx$$

Therefore, $\left(1 + \frac{b^2}{a^2}\right)\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2}$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2 + b^2} + C.$$

90. $n = 2$ (Use formula in Exercise 84.)

$$\int x^2\cos x\,dx = x^2\sin x - 2\int x\sin x\,dx \text{ (Use formula in Exercise 83.) } (n = 1)$$

$$= x^2\sin x - 2\left[-x\cos x + \int\cos x\,dx\right] = x^2\sin x + 2x\cos x - 2\sin x + C$$

92. $n = 3, a = 2$ (Use formula in Exercise 86 three times.)

$$\int x^3e^{2x}\,dx = \frac{x^3e^{2x}}{2} - \frac{3}{2}\int x^2e^{2x}\,dx \quad (n = 3, a = 2)$$

$$= \frac{x^3e^{2x}}{2} - \frac{3}{2}\left[\frac{x^2e^{2x}}{2} - \int xe^{2x}\,dx\right] \quad (n = 2, a = 2)$$

$$= \frac{x^3e^{2x}}{2} - \frac{3x^2e^{2x}}{4} + \frac{3}{2}\left[\frac{xe^{2x}}{2} - \frac{1}{2}\int e^{2x}\,dx\right] = \frac{x^3e^{2x}}{2} - \frac{3x^2e^{2x}}{4} + \frac{3xe^{2x}}{4} - \frac{3e^{2x}}{8} + C \quad (n = 1, a = 2)$$

$$= \frac{e^{2x}}{8}(4x^3 - 6x^2 + 6x - 3) + C$$

94. $dv = e^{-x/3}\,dx \implies v = -3e^{-x/3}$

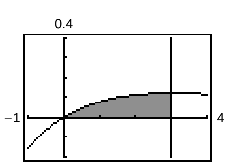
$u = x \implies du = dx$

$$A = \frac{1}{9}\int_0^3 xe^{-x/3}\,dx$$

$$= \frac{1}{9}\left(\left[-3xe^{-x/3}\right]_0^3 + 3\int_0^3 e^{-x/3}\,dx\right)$$

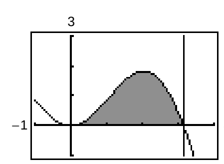
$$= \frac{1}{9}\left(\frac{-9}{e} - \left[9e^{-x/3}\right]_0^3\right)$$

$$= -\frac{1}{e} - \frac{1}{e} + 1 = 1 - \frac{2}{e} \approx 0.264$$

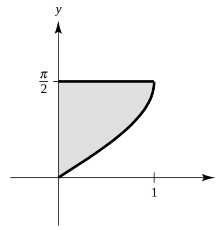


96. $A = \int_0^{\pi} x\sin x\,dx = \left[-x\cos x + \sin x\right]_0^{\pi}$

$= \pi$ (See Exercise 83.)



98. In Example 6, we showed that the centroid of an equivalent region was $(1, \pi/8)$. By symmetry, the centroid of this region is $(\pi/8, 1)$.



You can also solve this problem directly.

$$A = \int_0^1 \left(\frac{\pi}{2} - \arcsin x\right) dx = \left[\frac{\pi}{2}x - x \arcsin x - \sqrt{1 - x^2}\right]_0^1 \text{ (Example 3)}$$

$$= \left(\frac{\pi}{2} - \frac{\pi}{2} - 0\right) - (-1) = 1$$

$$\bar{x} = \frac{M_y}{A} = \int_0^1 x\left[\frac{\pi}{2} - \arcsin x\right] dx = \frac{\pi}{8}$$

$$\bar{y} = \frac{M_x}{A} = \int_0^1 \frac{(\pi/2) + \arcsin x}{2}\left[\frac{\pi}{2} - \arcsin x\right] dx = 1$$

100. (a) Average $= \int_1^2 (1.6t \ln t + 1)\,dt = \left[0.8t^2 \ln t - 0.4t^2 + t\right]_1^2 = 3.2(\ln 2) - 0.2 \approx 2.018$

(b) Average $= \int_3^4 (1.6t \ln t + 1)\,dt = \left[0.8t^2 \ln t - 0.4t^2 + t\right]_3^4 = 12.8(\ln 4) - 7.2(\ln 3) - 1.8 \approx 8.035$

102. $c(t) = 30{,}000 + 500t,\ r = 7\%,\ t_1 = 5$

$$P\int_0^5 (30{,}000 + 500t)e^{-0.07t}\,dt = 500\int_0^5 (60 + t)e^{-0.07t}\,dt$$

Let $u = 60 + t,\ dv = e^{-0.07t}\,dt,\ du = dt,\ v = -\frac{100}{7}e^{-0.07t}$.

$$P = 500\left\{\left[(60 + t)\left(-\frac{100}{7}e^{-0.07t}\right)\right]_0^5 + \frac{100}{7}\int_0^5 e^{-0.07t}\,dt\right\}$$

$$= 500\left\{\left[(60 + t)\left(-\frac{100}{7}e^{-0.07t}\right)\right]_0^5 - \left[\frac{10{,}000}{49}e^{-0.07t}\right]_0^5\right\} \approx \$131{,}528.68$$

104. $$\int_{-\pi}^{\pi} x^2 \cos nx\,dx = \left[\frac{x^2}{n}\sin nx + \frac{2x}{n^2}\cos nx - \frac{2}{n^3}\sin nx\right]_{-\pi}^{\pi}$$

$$= \frac{2\pi}{n^2}\cos n\pi + \frac{2\pi}{n^2}\cos(-n\pi)$$

$$= \frac{4\pi}{n^2}\cos n\pi$$

$$= \begin{cases} (4\pi/n^2), & \text{if } n \text{ is even} \\ -(4\pi/n^2), & \text{if } n \text{ is odd} \end{cases}$$

$$= \frac{(-1)^n 4\pi}{n^2}$$

106. For any integrable function, $\int f(x)\,dx = C + \int f(x)\,dx$, but this cannot be used to imply that $C = 0$.

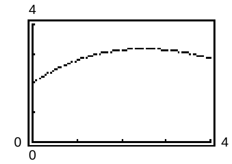
108. On $\left[0, \frac{\pi}{2}\right]$, $\sin x \le 1 \implies x \sin x \le x \implies \int_0^{\pi/2} x \sin x\,dx \le \int_0^{\pi/2} x\,dx$.

110. $f'(x) = \cos\sqrt{x}, f(0) = 2$

(a) It cannot be solved by integration.

(b) You obtain the points

n	x_n	y_n
0	0	2
1	0.05	2.05
2	0.10	2.098755
3	0.15	2.146276
⋮	⋮	⋮
80	4.0	2.8403565



Section 7.3 Trigonometric Integrals

2. (a) $y = \sec x \implies y' = \sec x \tan x = \sin x \sec^2 x.$

Matches (iii)

(b) $y = \cos x + \sec x \implies y' = -\sin x + \sec x \tan x$

$$= -\sin x + \sec^2 x \sin x$$
$$= \sin x(-1 + \sec^2 x)$$
$$= \sin x \tan^2 x \quad \text{Matches (i)}$$

(c) $y = x - \tan x + \frac{1}{3}\tan^3 x \implies y' = 1 - \sec^2 x + \tan^2 x \sec^2 x$

$$= -\tan^2 x + \tan^2 x(1 + \tan^2 x)$$
$$= \tan^4 x \quad \text{Matches (iv)}$$

(d) $y = 3x + 2\sin x\cos^3 x + 3\sin x\cos x \implies$

$$y' = 3 + 2\cos x(\cos^3 x) + 6\sin x\cos^2 x(-\sin x) + 3\cos^2 x - 3\sin^2 x$$
$$= 3 + 2\cos^4 x - 6\cos^2 x(1 - \cos^2 x) + 3\cos^2 x - 3(1 - \cos^2 x)$$
$$= 8\cos^4 x \quad \text{Matches (ii)}$$

4. $$\int \cos^3 x \sin^4 x\,dx = \int \cos x(1 - \sin^2 x)\sin^4 x\,dx$$
$$= \int (\sin^4 x - \sin^6 x)\cos x\,dx$$
$$= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C$$

6. Let $u = \cos x$, $du = -\sin x\,dx$.

$$\int \sin^3 x\,dx = \int \sin x(1 - \cos^2 x)\,dx$$
$$= \int \cos^2 x(-\sin x)\,dx + \int \sin x\,dx$$
$$= \frac{1}{3}\cos^3 x - \cos x + C$$

8. Let $u = \sin\frac{x}{3}$, $du = \frac{1}{3}\cos\frac{x}{3}\,dx$.

$$\int \cos^3\frac{x}{3}\,dx = \int\left(\cos\frac{x}{3}\right)\left(1 - \sin^2\frac{x}{3}\right)dx$$
$$= 3\int\left(1 - \sin^2\frac{x}{3}\right)\left(\frac{1}{3}\cos\frac{x}{3}\right)dx$$
$$= 3\left(\sin\frac{x}{3} - \frac{1}{3}\sin^3\frac{x}{3}\right) + C$$
$$= 3\sin\frac{x}{3} - \sin^3\frac{x}{3} + C$$

10. $\displaystyle\int \frac{\sin^5 t}{\sqrt{\cos t}}\,dt = \int \sin t(1 - \cos^2 t)^2(\cos t)^{-1/2}\,dt$

$\displaystyle= \int \sin t(1 - 2\cos^2 t + \cos^4 t)(\cos t)^{-1/2}\,dt$

$\displaystyle= \int [(\cos t)^{-1/2} - 2(\cos t)^{3/2} + (\cos t)^{7/2}]\sin t\,dt = -2(\cos t)^{1/2} + \frac{4}{5}(\cos t)^{5/2} - \frac{2}{9}(\cos t)^{9/2} + C$

12. $\displaystyle\int \sin^2 2x\,dx = \int \frac{1-\cos 4x}{2}\,dx = \frac{1}{2}\left(x - \frac{1}{4}\sin 4x\right) + C$

$\displaystyle= \frac{1}{8}(4x - \sin 4x) + C$

14. $\displaystyle\int \sin^4 2\theta\,d\theta = \int \frac{1-\cos 4\theta}{2}\cdot\frac{1-\cos 4\theta}{2}\,d\theta$

$\displaystyle= \frac{1}{4}\int (1 - 2\cos 4\theta + \cos^2 4\theta)\,d\theta$

$\displaystyle= \frac{1}{4}\int \left(1 - 2\cos 4\theta + \frac{1+\cos 8\theta}{2}\right)d\theta$

$\displaystyle= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos 4\theta + \frac{1}{2}\cos 8\theta\right)d\theta$

$\displaystyle= \frac{1}{4}\left[\frac{3}{2}\theta - \frac{1}{2}\sin 4\theta + \frac{1}{16}\sin 8\theta\right] + C$

$\displaystyle= \frac{3}{8}\theta - \frac{1}{8}\sin 4\theta + \frac{1}{64}\sin 8\theta + C$

16. Use integration by parts twice.

$\displaystyle dv = \sin^2 x\,dx = \frac{1-\cos 2x}{2} \implies v = \frac{x}{2} - \frac{\sin 2x}{4} = \frac{1}{4}(2x - \sin 2x)$

$u = x^2 \implies du = 2x\,dx$

$\displaystyle dv = \sin 2x\,dx \implies v = -\frac{1}{2}\cos 2x$

$u = x \implies du = dx$

$\displaystyle\int x^2\sin^2 x\,dx = \frac{1}{4}x^2(2x - \sin 2x) - \frac{1}{2}\int (2x^2 - x\sin 2x)\,dx$

$\displaystyle= \frac{1}{2}x^3 - \frac{1}{4}x^2\sin 2x - \frac{1}{3}x^3 + \frac{1}{2}\int x\sin 2x\,dx$

$\displaystyle= \frac{1}{6}x^3 - \frac{1}{4}x^2\sin 2x + \frac{1}{2}\left[-\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x\,dx\right]$

$\displaystyle= \frac{1}{6}x^3 - \frac{1}{4}x^2\sin 2x - \frac{1}{4}x\cos 2x + \frac{1}{8}\sin 2x + C$

$\displaystyle= \frac{1}{24}(4x^3 - 6x^2\sin 2x - 6x\cos 2x + 3\sin 2x) + C$

18. Let $u = \sin x$, $du = \cos x\,dx$.

$\displaystyle\int_0^{\pi/2} \cos^5 x\,dx = \int_0^{\pi/2} (1 - \sin^2 x)^2\cos x\,dx$

$\displaystyle= \int_0^{\pi/2} (1 - 2\sin^2 x + \sin^4 x)\cos x\,dx$

$\displaystyle= \left[\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x\right]_0^{\pi/2}$

$\displaystyle= \frac{8}{15}$

20. $\displaystyle\int_0^{\pi/2} \sin^2 x\,dx = \frac{1}{2}\int_0^{\pi/2} (1 - \cos 2x)\,dx$

$\displaystyle= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi/2} = \frac{\pi}{4}$

22. $\int \sec^2(2x-1)\,dx = \frac{1}{2}\tan(2x-1) + C$

24. $\int \sec^6 3x\,dx = \int (1+\tan^2 3x)^2 \sec^2 3x\,dx$

$= \int (1 + 2\tan^2 3x + \tan^4 3x)\sec^2 3x\,dx$

$= \frac{1}{3}\tan 3x + \frac{2}{9}\tan^3 3x + \frac{1}{15}\tan^5 3x + C$

26. $\int \tan^2 x\,dx = \int (\sec^2 x - 1)dx = \tan x - x + C$

28. $\int \tan^3 \frac{\pi x}{2} \sec^2 \frac{\pi x}{2}\,dx = \frac{1}{2\pi}\tan^4 \frac{\pi x}{2} + C$

30. Let $u = \sec 2t$, $du = 2\sec 2t \tan 2t$

$\int \tan^3 2t \cdot \sec^3 2t\,dt = \int (\sec^2 2t - 1)\sec^3 2t \cdot \tan 2t\,dt$

$= \int (\sec^4 2t - \sec^2 2\text{t})(\sec 2t \tan 2t)\,dt$

$= \frac{\sec^5 2t}{10} - \frac{\sec^3 2t}{6} + C$

32. $\int \tan^5 2x \sec^2 2x\,dx = \frac{1}{12}\tan^6 2x + C$

34. $\int \sec^2 \frac{x}{2} \tan \frac{x}{2}\,dx = 2\int \sec \frac{x}{2}\left(\frac{1}{2}\sec \frac{x}{2} \tan \frac{x}{2}\right)dx$

$= \sec^2 \frac{x}{2} + C$

or $\int \sec^2 \frac{x}{2} \tan \frac{x}{2}\,dx = 2\int \tan \frac{x}{2}\left(\frac{1}{2}\sec^2 \frac{x}{2}\right)dx$

$= \tan^2 \frac{x}{2} + C$

36. $\int \tan^3 3x\,dx = \int (\sec^2 3x - 1)\tan 3x\,dx$

$= \frac{1}{3}\int \tan 3x(3\sec^2 3x)\,dx + \frac{1}{3}\int \frac{-3\sin 3x}{\cos 3x}\,dx$

$= \frac{1}{6}\tan^2 3x + \frac{1}{3}\ln|\cos 3x| + C$

38. $\int \frac{\tan^2 x}{\sec^5 x} = \int \frac{\sin^2 x}{\cos^2 x} \cdot \cos^5 x\,dx$

$= \int \sin^2 x \cdot \cos^3 x\,dx$

$= \int \sin^2 x(1 - \sin^2 x)\cos x\,dx$

$= \int (\sin^2 x - \sin^4 x)\cos x\,dx$

$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$

40. $s = \int \sin^2 \frac{\alpha}{2} \cos^2 \frac{\alpha}{2}\,d\alpha$

$= \int \left(\frac{1-\cos\alpha}{2}\right)\left(\frac{1+\cos\alpha}{2}\right)d\alpha = \int \frac{1-\cos^2\alpha}{4}\,d\alpha$

$= \frac{1}{4}\int \sin^2\alpha\,d\alpha = \frac{1}{8}\int (1 - \cos 2\alpha)\,d\alpha$

$= \frac{1}{8}\left[\theta - \frac{\sin 2\alpha}{2}\right] + C$

$= \frac{1}{16}(2\alpha - \sin 2\alpha) + C$

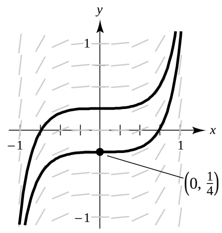
42. $y = \int \sqrt{\tan x}\,\sec^4 x\,dx$

$= \int \tan^{1/2} x(\tan^2 x + 1)\sec^2 x\,dx$

$= \int (\tan^{5/2} x + \tan^{1/2} x)\sec^2 x\,dx$

$= \frac{2}{7}\tan^{7/2} x + \frac{2}{3}\tan^{3/2} x + C$

44. (a)



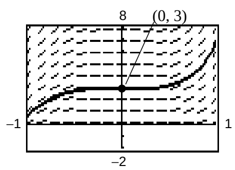
(b) $\dfrac{dy}{dx} = \sec^2 x \tan^2 x, \left(0, -\dfrac{1}{4}\right)$

$$y = \int \sec^2 x \tan^2 x\,dx \quad u = \tan x, du = \sec^2 x\,dx$$

$$y = \frac{\tan^3 x}{3} + C$$

$$\left(0, -\frac{1}{4}\right): \ -\frac{1}{4} = C \implies y = \frac{1}{3}\tan^3 x - \frac{1}{4}$$

46. $\dfrac{dy}{dx} = 3\sqrt{y}\tan^2 x, y(0) = 3$



48. $$\int \cos 4\theta \cos(-3\theta)\,d\theta = \int \cos 4\theta \cos 3\theta\,d\theta$$
$$= \frac{1}{2}\int (\cos 7\theta + \cos\theta)\,d\theta$$
$$= \frac{\sin 7\theta}{14} + \frac{\sin\theta}{2} + C$$

50. $$\int \sin(-4x)\cos 3x\,dx = -\int \sin 4x \cos 3x\,dx$$
$$= -\frac{1}{2}\int(\sin x + \sin 7x)dx$$
$$= -\frac{1}{2}\left[-\cos x - \frac{1}{7}\cos 7x\right] + C$$
$$= \frac{1}{14}[7\cos x + \cos 7x] + C$$

52. Let $u = \tan\dfrac{x}{2}, du = \dfrac{1}{2}\sec^2\dfrac{x}{2}\,dx$.

$$\int \tan^4\frac{x}{2}\sec^4\frac{x}{2}\,dx = \int \tan^4\frac{x}{2}\left(\tan^2\frac{x}{2} + 1\right)\sec^2\frac{x}{2}\,dx$$
$$= 2\int\left(\tan^6\frac{x}{2} + \tan^4\frac{x}{2}\right)\left(\frac{1}{2}\sec^2\frac{x}{2}\right)dx$$
$$= \frac{2}{7}\tan^7\frac{x}{2} + \frac{2}{5}\tan^5\frac{x}{2} + C$$

54. $u = \cot 3x, du = -3\csc^2 3x\,dx$

$$\int \csc^2 3x \cot 3x\,dx = -\frac{1}{3}\int \cot 3x(-3\csc^2 3x)\,dx$$
$$= -\frac{1}{6}\cot^2 3x + C$$

56. $$\int \frac{\cot^3 t}{\csc t}\,dt = \int \frac{\cos^3 t}{\sin^2 t}\,dt = \int \frac{(1 - \sin^2 t)\cos t}{\sin^2 t}\,dt$$
$$= \int \frac{\cos t}{\sin^2 t}\,dt - \int \cos t\,dt$$
$$= \frac{-1}{\sin t} - \sin t + C$$
$$= -\csc t - \sin t + C$$

58. $$\int \frac{\sin^2 x - \cos^2 x}{\cos x}\,dx = \int \frac{1 - 2\cos^2 x}{\cos x}\,dx = \int (\sec x - 2\cos x)\,dx = \ln|\sec x + \tan x| - 2\sin x + C$$

60. $$\int \frac{1 - \sec t}{\cos t - 1}\,dt = \int \frac{\cos t - 1}{(\cos t - 1)\cos t}\,dt$$
$$= \int \sec t\,dt = \ln|\sec t + \tan t| + C$$

62. $$\int_0^{\pi/3} \tan^2 x\,dx = \int_0^{\pi/3} (\sec^2 x - 1)\,dx$$
$$= \Big[\tan x - x\Big]_0^{\pi/3} = \sqrt{3} - \frac{\pi}{3}$$

64. Let $u = \tan t, du = \sec^2 t\,dt$.

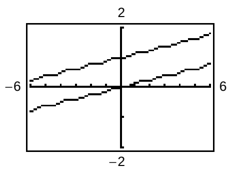
$$\int_0^{\pi/4} \sec^2 t\sqrt{\tan t}\,dt = \left[\frac{2}{3}\tan^{3/2} t\right]_0^{\pi/4} = \frac{2}{3}$$

66. $$\int_{-\pi}^{\pi} \sin 3\theta \cos\theta\,d\theta = \frac{1}{2}\int_{-\pi}^{\pi}(\sin 4\theta + \sin 2\theta)\,d\theta$$
$$= -\frac{1}{2}\left[\frac{1}{4}\cos 4\theta + \frac{1}{2}\cos 2\theta\right]_{-\pi}^{\pi} = 0$$

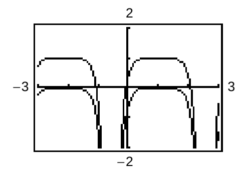
68. $\displaystyle\int_{-\pi/2}^{\pi/2}(\sin^2 x + 1)\,dx = \int_{-\pi/2}^{\pi/2}\left(\frac{1-\cos 2x}{2} + 1\right)dx$

$\displaystyle = \int_{-\pi/2}^{\pi/2}\left(\frac{3}{2} - \frac{1}{2}\cos 2x\right)dx = \left[\frac{3}{2}x - \frac{1}{4}\sin 2x\right]_{-\pi/2}^{\pi/2} = \frac{3\pi}{2}$

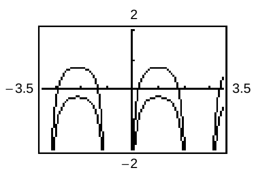
70. $\displaystyle\int \sin^2 x\cos^2 x\,dx = \frac{1}{32}[4x - \sin 4x] + C$



72. $\displaystyle\int \tan^3(1-x)\,dx = -\frac{\tan^2(1-x)}{2} - \ln|\cos(1-x)| + C$



74. $\displaystyle\int \sec^4(1-x)\tan(1-x)\,dx = -\frac{\sec^4(1-x)}{4} + C$



76. $\displaystyle\int_0^{\pi/2}(1-\cos\theta)^2\,d\theta = \left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/2}$

$\displaystyle = \frac{3\pi}{4} - 2$

78. $\displaystyle\int_0^{\pi/2}\sin^6 x\,dx = \frac{1}{8}\left[\frac{5x}{2} - 2\sin 2x + \frac{3}{8}\sin 4x + \frac{1}{6}\sin^3 2x\right]_0^{\pi/2} = \frac{5\pi}{32}$

80. See guidelines on page 500.

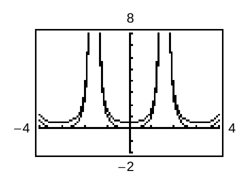
82. (a) Let $u = \tan x$, $du = \sec^2 x\,dx$.

$\displaystyle\int \sec^2 x\tan x\,dx = \frac{1}{2}\tan^2 x + C_1$

Or let $u = \sec x$, $du = \sec x\tan x\,dx$.

$\displaystyle\int \sec x(\sec x\tan x)\,dx = \frac{1}{2}\sec^2 x + C$

(b)



(c) $\displaystyle\frac{1}{2}\sec^2 x + C = \frac{1}{2}(\tan^2 x + 1) + C = \frac{1}{2}\tan^2 x + \left(\frac{1}{2} + C\right) = \frac{1}{2}\tan^2 x + C_2$

84. Disks

$R(x) = \tan x$

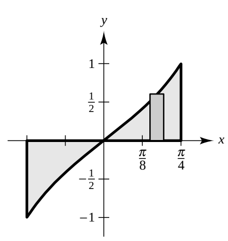
$r(x) = 0$

$\displaystyle V = 2\pi\int_0^{\pi/4}\tan^2 x\,dx$

$\displaystyle = 2\pi\int_0^{\pi/4}(\sec^2 x - 1)\,dx$

$\displaystyle = 2\pi\Big[\tan x - x\Big]_0^{\pi/4}$

$\displaystyle = 2\pi\left(1 - \frac{\pi}{4}\right) \approx 1.348$



86. (a) $V = \pi\int_0^{\pi/2} \cos^2 x\,dx = \frac{\pi}{2}\int_0^{\pi/2}(1 + \cos 2x)\,dx = \frac{\pi}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{\pi/2} = \frac{\pi^2}{4}$

(b) $A = \int_0^{\pi/2} \cos x\,dx = \left[\sin x\right]_0^{\pi/2} = 1$

Let $u = x$, $dv = \cos x\,dx$, $du = dx$, $v = \sin x$.

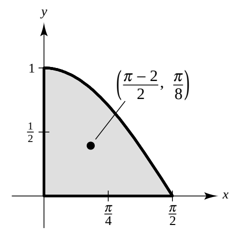
$$\bar{x} = \int_0^{\pi/2} x\cos x\,dx = \left[x\sin x\right]_0^{\pi/2} - \int_0^{\pi/2}\sin x\,dx = \left[x\sin x + \cos x\right]_0^{\pi/2} = \frac{\pi}{2} - 1 = \frac{\pi - 2}{2}$$

$$\bar{y} = \frac{1}{2}\int_0^{\pi/2}\cos^2 x\,dx$$

$$= \frac{1}{4}\int_0^{\pi/2}(1 + \cos 2x)\,dx$$

$$= \frac{1}{4}\left[x + \frac{1}{2}\sin 2x\right]_0^{\pi/2} = \frac{\pi}{8}$$

$$(\bar{x}, \bar{y}) = \left(\frac{\pi - 2}{2}, \frac{\pi}{8}\right)$$



88. $dv = \cos x\,dx \implies v = \sin x$

$u = \cos^{n-1} x \implies du = -(n-1)\cos^{n-2} x\sin x\,dx$

$$\int \cos^n x\,dx = \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\sin^2 x\,dx$$

$$= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x(1 - \cos^2 x)\,dx$$

$$= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx$$

Therefore, $n\int \cos^n x\,dx = \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\,dx$

$$\int \cos^n x\,dx = \frac{\cos^{n-1} x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$$

90. Let $u = \sec^{n-2} x$, $du = (n-2)\sec^{n-2} x\tan x\,dx$, $dv = \sec^2 x\,dx$, $v = \tan x$.

$$\int \sec^n x\,dx = \sec^{n-2} x\tan x - \int (n-2)\sec^{n-2} x\tan^2 x\,dx$$

$$= \sec^{n-2} x\tan x - (n-2)\int \sec^{n-2} x(\sec^2 x - 1)\,dx$$

$$= \sec^{n-2} x\tan x - (n-2)\left[\int \sec^n x\,dx - \int \sec^{n-2} x\,dx\right]$$

$$(n-1)\int \sec^n x\,dx = \sec^{n-2} x\tan x + (n-2)\int \sec^{n-2} x\,dx$$

$$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2} x\tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$$

92. $\int \cos^4 x\,dx = \frac{\cos^3 x\sin x}{4} + \frac{3}{4}\int \cos^2 x\,dx = \frac{\cos^3 x\sin x}{4} + \frac{3}{4}\left[\frac{\cos x\sin x}{2} + \frac{1}{2}\int dx\right]$

$$= \frac{1}{4}\cos^3 x\sin x + \frac{3}{8}\cos x\sin x + \frac{3}{8}x + C = \frac{1}{8}\left[2\cos^3 x\sin x + 3\cos x\sin x + 3x\right] + C$$

94. $\int \sin^4 x \cos^2 x\,dx = -\dfrac{\cos^3 x \sin^3 x}{6} + \dfrac{1}{2}\int \cos^2 x \sin^2 x\,dx$

$$= -\frac{\cos^3 x \sin^3 x}{6} + \frac{1}{2}\left[-\frac{\cos^3 x \sin x}{4} + \frac{1}{4}\int \cos^2 x\,dx\right]$$

$$= -\frac{1}{6}\cos^3 x \sin^3 x - \frac{1}{8}\cos^3 x \sin x + \frac{1}{8}\left[\frac{\cos x \sin x}{2} + \frac{x}{2}\right] + C$$

$$= -\frac{1}{48}[8\cos^3 x \sin^3 x + 6\cos^3 x \sin x - 3\cos x \sin x - 3x] + C$$

96. (a) n is odd and $n \geq 3$.

$$\int_0^{\pi/2} \cos^n x\,dx = \left[\frac{\cos^{n-1} x \sin x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x\,dx$$

$$= \frac{n-1}{n}\left[\left[\frac{\cos^{n-3} x \sin x}{n-2}\right]_0^{\pi/2} + \frac{n-3}{n-2}\int_0^{\pi/2} \cos^{n-4} x\,dx\right]$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2}\left[\left[\frac{\cos^{n-5} x \sin x}{n-4}\right]_0^{\pi/2} + \frac{n-5}{n-4}\int_0^{\pi/2} \cos^{n-6} x\,dx\right]$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4}\int_0^{\pi/2} \cos^{n-6} x\,dx$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \int_0^{\pi/2} \cos x\,dx$$

$$= \left[\frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots (\sin x)\right]_0^{\pi/2}$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots 1 \qquad \text{(Reverse the order)}$$

$$= (1)\left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{n-1}{n}\right)$$

$$= \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{n-1}{n}\right)$$

(b) n is even and $n \geq 2$.

$$\int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \int_0^{\pi/2} \cos^2 x\,dx \qquad \text{(From part (a).)}$$

$$= \left[\frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \left(\frac{x}{2} + \frac{1}{4}\sin 2x\right)\right]_0^{\pi/2}$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{\pi}{4} \qquad \text{(Reverse the order)}$$

$$= \left(\frac{\pi}{2} \cdot \frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right)$$

$$= \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)$$

Section 7.4 Trigonometric Substitution

2. $$\frac{d}{dx}\left[8\ln\left|\sqrt{x^2-16}+x\right|+\frac{1}{2}x\sqrt{x^2-16}+C\right]=8\left[\frac{\left(x/\sqrt{x^2-16}\right)+1}{\sqrt{x^2-16}+x}\right]+\frac{1}{2}x\left(\frac{x}{\sqrt{x^2-16}}\right)+\frac{1}{2}\sqrt{x^2-16}$$

$$=\frac{8\left(x+\sqrt{x^2-16}\right)}{\sqrt{x^2-16}\left(\sqrt{x^2-16}+x\right)}+\frac{x^2}{2\sqrt{x^2+16}}+\frac{\sqrt{x^2-16}}{2}$$

$$=\frac{16+x^2+x^2-16}{2\sqrt{x^2-16}}$$

$$=\frac{x^2}{\sqrt{x^2-16}}$$

Indefinite integral: $\displaystyle\int\frac{x^2}{\sqrt{x^2-16}}$ Matches (d)

4. $$\frac{d}{dx}\left[8\arcsin\frac{x-3}{4}+\frac{(x-3)\sqrt{7+6x-x^2}}{2}+C\right]=8\left[\frac{1}{\sqrt{1-[(x-3)/4]^2}}\cdot\frac{1}{4}\right]+\frac{1}{2}(x-3)\frac{3-x}{\sqrt{7+6x-x^2}}+\frac{1}{2}\sqrt{7+6x-x^2}$$

$$=\frac{8}{\sqrt{16-(x-3)^2}}-\frac{(x-3)^2}{2\sqrt{16-(x-3)^2}}+\frac{\sqrt{16-(x-3)^2}}{2}$$

$$=\frac{16-(x^2-6x+9)+16-(x^2-6x+9)}{2\sqrt{16-(x-3)^2}}$$

$$=\frac{2[16-(x-3)^2]}{2\sqrt{16-(x-3)^2}}$$

$$=\sqrt{16-(x-3)^2}$$

$$=\sqrt{7+6x-x^2}$$

Indefinite integral: $\displaystyle\int\sqrt{7+6x-x^2}\,dx$ Matches (c)

6. Same substitution as in Exercise 5.

$$\int\frac{10}{x^2\sqrt{25-x^2}}\,dx=10\int\frac{5\cos\theta\,d\theta}{(25\sin^2\theta)(5\cos\theta)}=\frac{2}{5}\int\csc^2\theta\,d\theta=-\frac{2}{5}\cot\theta+C=\frac{-2\sqrt{25-x^2}}{5x}+C$$

8. Same substitution as in Exercise 5

$$\int\frac{x^2}{\sqrt{25-x^2}}\,dx=\int\frac{25\sin^2\theta}{5\cos\theta}(5\cos\theta)\,d\theta=\frac{25}{2}\int(1-\cos2\theta)\,d\theta$$

$$=\frac{25}{2}\left(\theta-\frac{1}{2}\sin2\theta\right)+C=\frac{25}{2}(\theta-\sin\theta\cos\theta)+C$$

$$=\frac{25}{2}\left[\arcsin\left(\frac{x}{5}\right)-\left(\frac{x}{5}\right)\left(\frac{\sqrt{25-x^2}}{5}\right)\right]+C=\frac{1}{2}\left[25\arcsin\left(\frac{x}{5}\right)-x\sqrt{25-x^2}\right]+C$$

10. Same substitution as in Exercise 9

$$\int\frac{\sqrt{x^2-4}}{x}\,dx=\int\frac{2\tan\theta}{2\sec\theta}(2\sec\theta\tan\theta)\,d\theta=2\int\tan^2\theta\,d\theta=2\int(\sec^2\theta-1)\,d\theta$$

$$=2(\tan\theta-\theta)+C=2\left[\frac{\sqrt{x^2-4}}{2}-\operatorname{arcsec}\left(\frac{x}{2}\right)\right]+C=\sqrt{x^2-4}-2\operatorname{arcsec}\left(\frac{x}{2}\right)+C$$

12. Same substitution as in Exercise 9

$$\int \frac{x^3}{\sqrt{x^2-4}}\,dx = \int \frac{8\sec^3\theta}{2\tan\theta}(2\sec\theta\tan\theta)\,d\theta = 8\int \sec^4\theta\,d\theta$$

$$= 8\int (1+\tan^2\theta)\sec^2\theta\,d\theta = 8\left(\tan\theta + \frac{\tan^3\theta}{3}\right) + C = \frac{8}{3}\tan\theta(3+\tan^2\theta) + C$$

$$= \frac{8}{3}\left(\frac{\sqrt{x^2-4}}{2}\right)\left(3 + \frac{x^2-4}{4}\right) + C = \frac{1}{3}\sqrt{x^2-4}\,(12 + x^2 - 4) + C = \frac{1}{3}\sqrt{x^2-4}\,(x^2+8) + C$$

14. Same substitution as in Exercise 13.

$$\int \frac{9x^3}{\sqrt{1+x^2}}\,dx = 9\int \frac{\tan^3\theta}{\sec\theta}\sec^2\theta\,d\theta = 9\int (\sec^2\theta - 1)\sec\theta\tan\theta\,d\theta = 9\left[\frac{\sec^3\theta}{3} - \sec\theta\right] + C$$

$$= 3\sec\theta(\sec^2\theta - 3) + C = 3\sqrt{1+x^2}[(1+x^2) - 3] + C = 3\sqrt{1+x^2}(x^2-2) + C$$

16. Same substitution as in Exercise 13

$$\int \frac{x^2}{(1+x^2)^2}dx = \int \frac{x^2}{\left(\sqrt{1+x^2}\right)^4}\,dx = \int \frac{\tan^2\theta\sec^2\theta\,d\theta}{\sec^4\theta} = \int \sin^2\theta\,d\theta$$

$$= \frac{1}{2}\int (1-\cos 2\theta)\,d\theta = \frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right] = \frac{1}{2}[\theta - \sin\theta\cos\theta] + C$$

$$= \frac{1}{2}\left[\arctan x - \left(\frac{x}{\sqrt{1+x^2}}\right)\left(\frac{1}{\sqrt{1+x^2}}\right)\right] + C = \frac{1}{2}\left[\arctan x - \frac{x}{1+x^2}\right] + C$$

18. Let $u = x$, $a = 1$, and $du = dx$.

$$\int \sqrt{1+x^2}\,dx = \frac{1}{2}\left(x\sqrt{1+x^2} + \ln\left|x + \sqrt{1+x^2}\right|\right) + C$$

20. $$\int \frac{x}{\sqrt{9-x^2}}\,dx = -\frac{1}{2}\int (9-x^2)^{-1/2}(-2x)\,dx$$

$$= -(9-x^2)^{1/2} + C \quad \text{(Power Rule)}$$

22. $$\int \frac{1}{\sqrt{25-x^2}}\,dx = \arcsin\frac{x}{5} + C$$

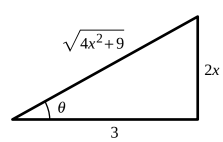
24. Let $u = 16 - 4x^2$, $du = -8x\,dx$.

$$\int x\sqrt{16-4x^2}\,dx = -\frac{1}{8}\int (16-4x^2)^{1/2}(-8x)\,dx = \left[-\frac{1}{12}(16-4x^2)^{3/2}\right] + C = -\frac{2}{3}(4-x^2)^{3/2} + C$$

26. Let $u = 1 - t^2$, $du = -2t\,dt$.

$$\int \frac{t}{(1-t^2)^{3/2}}\,dt = -\frac{1}{2}\int (1-t^2)^{-3/2}(-2t)\,dt = \frac{1}{\sqrt{1-t^2}} + C$$

28. Let $2x = 3\tan\theta$, $dx = \frac{3}{2}\sec^2\theta\,d\theta$, $\sqrt{4x^2+9} = 3\sec\theta$.



$$\int \frac{\sqrt{4x^2+9}}{x^4}\,dx = \int \frac{3\sec\theta[(3/2)\sec^2\theta\,d\theta]}{(3/2)^4\tan^4\theta}$$

$$= \frac{8}{9}\int \frac{\cos\theta}{\sin^4\theta}\,d\theta$$

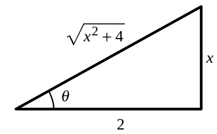
$$= \frac{-8}{27\sin^3\theta} + C$$

$$= -\frac{8}{27}\csc^3\theta + C$$

$$= \frac{-(4x^2+9)^{3/2}}{27x^3} + C$$

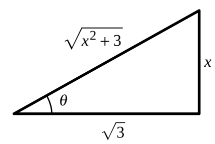
30. Let $2x = 4\tan\theta$, $dx = 2\sec^2\theta\, d\theta$, $\sqrt{4x^2 + 16} = 4\sec\theta$.

$$\int \frac{1}{x\sqrt{4x^2+16}}\,dx = \int \frac{2\sec^2\theta\,d\theta}{2\tan\theta(4\sec\theta)}$$
$$= \frac{1}{4}\int \frac{\sec\theta}{\tan\theta}\,d\theta = \frac{1}{4}\int \csc\theta\,d\theta$$
$$= -\frac{1}{4}\ln\left|\csc\theta + \cot\theta\right| + C = -\frac{1}{4}\ln\left|\frac{\sqrt{x^2+4}+2}{x}\right| + C$$



32. Let $x = \sqrt{3}\tan\theta$, $dx = \sqrt{3}\sec^2\theta\,d\theta$, $x^2 + 3 = 3\sec^2\theta$.

$$\int \frac{1}{(x^2+3)^{3/2}}\,dx = \int \frac{\sqrt{3}\sec^2\theta\,d\theta}{3\sqrt{3}\sec^3\theta}$$
$$= \frac{1}{3}\int \cos\theta\,d\theta$$
$$= \frac{1}{3}\sin\theta + C$$
$$= \frac{x}{3\sqrt{x^2+3}} + C$$

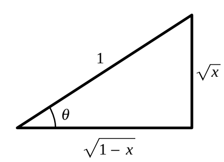


34. Let $u = x^2 + 2x + 2$, $du = (2x + 2)\,dx$.

$$\int (x+1)\sqrt{x^2+2x+2}\,dx = \frac{1}{2}\int (x^2+2x+2)^{1/2}(2x+2)\,dx = \frac{1}{3}(x^2+2x+2)^{3/2} + C$$

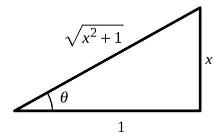
36. Let $\sqrt{x} = \sin\theta$, $x = \sin^2\theta$, $dx = 2\sin\theta\cos\theta\,d\theta$, $\sqrt{1-x} = \cos\theta$.

$$\int \frac{\sqrt{1-x}}{\sqrt{x}}\,dx = \int \frac{\cos\theta(2\sin\theta\cos\theta\,d\theta)}{\sin\theta}$$
$$= 2\int \cos^2\theta\,d\theta$$
$$= \int (1+\cos 2\theta)\,d\theta$$
$$= (\theta + \sin\theta\cos\theta) + C$$
$$= \arcsin\sqrt{x} + \sqrt{x}\sqrt{1-x} + C$$



38. Let $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$, $x^2 + 1 = \sec^2\theta$.

$$\int \frac{x^3+x+1}{x^4+2x^2+1}\,dx = \frac{1}{4}\int \frac{4x^3+4x}{x^4+2x^2+1}\,dx + \int \frac{1}{(x^2+1)^2}\,dx$$
$$= \frac{1}{4}\ln(x^4+2x^2+1) + \int \frac{\sec^2\theta\,d\theta}{\sec^4\theta}$$
$$= \frac{1}{2}\ln(x^2+1) + \frac{1}{2}\int (1+\cos 2\theta)\,d\theta$$
$$= \frac{1}{2}\ln(x^2+1) + \frac{1}{2}(\theta + \sin\theta\cos\theta) + C$$
$$= \frac{1}{2}\left[\ln(x^2+1) + \arctan x + \frac{x}{x^2+1}\right] + C$$



40. $u = \arcsin x, \implies du = \dfrac{1}{\sqrt{1-x^2}}\,dx, dv = x\,dx \implies v = \dfrac{x^2}{2}$

$$\int x \arcsin x\,dx = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,dx$$

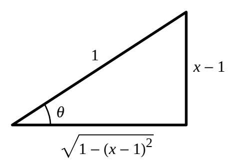
$x = \sin\theta, dx = \cos\theta\,d\theta, \sqrt{1-x^2} = \cos\theta$

$$\int x \arcsin x\,dx = \frac{x^2}{2}\arcsin x = \frac{1}{2}\int \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta = \frac{x^2}{2}\arcsin x - \frac{1}{4}\int (1-\cos 2\theta)\,d\theta$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\left[\theta - \frac{1}{2}\sin 2\theta\right] + C = \frac{x^2}{2}\arcsin x - \frac{1}{4}[\theta - \sin\theta\cos\theta] + C$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\left[\arcsin x - x\sqrt{1-x^2}\right] + C = \frac{1}{4}\left[(2x^2-1)\arcsin x + x\sqrt{1-x^2}\right] + C$$

42. Let $x - 1 = \sin\theta, dx = \cos\theta\,d\theta, \sqrt{1-(x-1)^2} = \sqrt{2x-x^2} = \cos\theta.$



$$\int \frac{x^2}{\sqrt{2x-x^2}}\,dx = \int \frac{x^2}{\sqrt{1-(x-1)^2}}\,dx$$

$$= \int \frac{(1+\sin\theta)^2(\cos\theta\,d\theta)}{\cos\theta}$$

$$= \int (1 + 2\sin\theta + \sin^2\theta)\,d\theta$$

$$= \int \left(\frac{3}{2} + 2\sin\theta - \frac{1}{2}\cos 2\theta\right)d\theta$$

$$= \frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin 2\theta + C$$

$$= \frac{3}{2}\theta - 2\cos\theta - \frac{1}{2}\sin\theta\cos\theta + C$$

$$= \frac{3}{2}\arcsin(x-1) - 2\sqrt{2x-x^2} - \frac{1}{2}(x-1)\sqrt{2x-x^2} + C$$

$$= \frac{3}{2}\arcsin(x-1) - \frac{1}{2}\sqrt{2x-x^2}(x+3) + C$$

44. Let $x - 3 = 2\sec\theta, dx = 2\sec\theta\tan\theta\,d\theta, \sqrt{(x-3)^2-4} = 2\tan\theta.$

$$\int \frac{x}{\sqrt{x^2-6x+5}}\,dx = \int \frac{x}{\sqrt{(x-3)^2-4}}\,dx = \int \frac{(2\sec\theta+3)}{2\tan\theta}(2\sec\theta\tan\theta)\,d\theta$$

$$= \int (2\sec^2\theta + 3\sec\theta)\,d\theta$$

$$= 2\tan\theta + 3\ln\left|\sec\theta + \tan\theta\right| + C_1$$

$$= 2\left(\frac{\sqrt{(x-3)^3-4}}{2}\right) + 3\ln\left|\frac{x-3}{2} + \frac{\sqrt{(x-3)^2-4}}{2}\right| + C_1$$

$$= \sqrt{x^2-6x+5} + 3\ln\left|(x-3) + \sqrt{x^2-6x+5}\right| + C$$

46. Same substitution as in Exercise 45

(a) $$\int \frac{1}{(1-t^2)^{5/2}}\,dt = \int \frac{\cos\theta\,d\theta}{\cos^5\theta} = \int \sec^4\theta\,d\theta = \int (\tan^2\theta + 1)\sec^2\theta\,d\theta$$

$$= \frac{1}{3}\tan^3\theta + \tan\theta + C = \frac{1}{3}\left(\frac{t}{\sqrt{1-t^2}}\right)^3 + \frac{t}{\sqrt{1-t^2}} + C$$

Thus, $$\int_0^{\sqrt{3}/2} \frac{1}{(1-t^2)^{5/2}}\,dt = \left[\frac{t^3}{3(1-t^2)^{3/2}} + \frac{t}{\sqrt{1-t^2}}\right]_0^{\sqrt{3}/2}$$

$$= \frac{3\sqrt{3}/8}{3(1/4)^{3/2}} + \frac{\sqrt{3}/2}{\sqrt{1/4}} = \sqrt{3} + \sqrt{3} = 2\sqrt{3} \approx 3.464.$$

(b) When $t = 0$, $\theta = 0$. When $t = \sqrt{3}/2$, $\theta = \pi/3$. Thus,

$$\int_0^{\sqrt{3}/2} \frac{1}{(1-t^2)^{5/2}}\,dt = \left[\frac{1}{3}\tan^3\theta + \tan\theta\right]_0^{\pi/3} = \frac{1}{3}\left(\sqrt{3}\right)^3 + \sqrt{3} = 2\sqrt{3} \approx 3.464.$$

48. (a) Let $5x = 3\sin\theta$, $dx = \frac{3}{5}\cos\theta\,d\theta$, $\sqrt{9-25x^2} = 3\cos\theta$.

$$\int \sqrt{9-25x^2}\,dx = \int (3\cos\theta)\frac{3}{5}\cos\theta\,d\theta$$

$$= \frac{9}{5}\int \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \frac{9}{10}\left[\theta + \frac{1}{2}\sin 2\theta\right] + C$$

$$= \frac{9}{10}[\theta + \sin\theta\cos\theta] + C$$

$$= \frac{9}{10}\left[\arcsin\frac{5x}{3} + \frac{5x}{3}\cdot\frac{\sqrt{9-25x^2}}{3}\right] + C$$

Thus, $$\int_0^{3/5} \sqrt{9-25x^2}\,dx = \frac{9}{10}\left[\arcsin\frac{5x}{3} + \frac{5x\sqrt{9-25x^2}}{9}\right]_0^{3/5} = \frac{9}{10}\left[\frac{\pi}{2}\right] = \frac{9\pi}{20}.$$

(b) When $x = 0$, $\theta = 0$. When $x = \frac{3}{5}$, $\theta = \frac{\pi}{2}$.

Thus, $$\int_0^{3/5} \sqrt{9-25x^2}\,dx = \left[\frac{9}{10}(\theta + \sin\theta\cos\theta)\right]_0^{\pi/2} = \frac{9}{10}\left(\frac{\pi}{2}\right) = \frac{9\pi}{20}.$$

50. (a) Let $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta\,d\theta$, $\sqrt{x^2-9} = 3\tan\theta$.

$$\int \frac{\sqrt{x^2-9}}{x^2}\,dx = \int \frac{3\tan\theta}{9\sec^2\theta}3\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{\tan^2\theta}{\sec\theta}\,d\theta$$

$$= \int \frac{\sin^2\theta}{\cos\theta}\,d\theta$$

$$= \int \frac{1-\cos^2\theta}{\cos\theta}\,d\theta$$

$$= \int (\sec\theta - \cos\theta)\,d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| - \sin\theta + C$$

$$= \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| - \frac{\sqrt{x^2-9}}{x} + C$$

—CONTINUED—

50. —CONTINUED—

Hence, $\displaystyle\int_3^6 \frac{\sqrt{x^2-9}}{x^2}\,dx = \left[\ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| - \frac{\sqrt{x^2-9}}{x}\right]_3^6 = \ln\left|2+\sqrt{3}\right| - \frac{\sqrt{3}}{2}.$

(b) When $x = 3$, $\theta = 0$; when $x = 6$, $\theta = \dfrac{\pi}{3}$.

Hence, $\displaystyle\int_3^6 \frac{\sqrt{x^2-9}}{x^2}\,dx = \Big[\ln|\sec\theta + \tan\theta| - \sin\theta\Big]_0^{\pi/3} = \ln\left|2+\sqrt{3}\right| - \frac{\sqrt{3}}{2}.$

52. $\displaystyle\int (x^2+2x+11)^{3/2}\,dx = \frac{1}{4}(x+1)(x^2+2x+26)\sqrt{x^2+2x+11} + \frac{75}{2}\ln\left|\sqrt{x^2+2x+11} + (x+1)\right| + C$

54. $\displaystyle\int x^2\sqrt{x^2-4}\,dx = \frac{1}{4}x^3\sqrt{x^2-4} - \frac{1}{2}x\sqrt{x^2-4} - 2\ln\left|x+\sqrt{x^2-4}\right| + C$

56. (a) Substitution: $u = x^2+1$, $du = 2x\,dx$

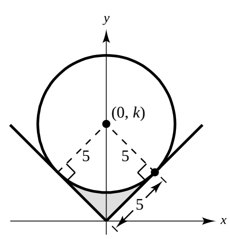
(b) Trigonometric substitution: $x = \sec\theta$

58. (a) $x^2 + (y-k)^2 = 25$

Radius of circle $= 5$

$k^2 = 5^2 + 5^2 = 50$

$k = 5\sqrt{2}$



(b) Area $=$ square $- \dfrac{1}{4}$(circle)

$= 25 - \dfrac{1}{4}\pi(5)^2 = 25\left(1 - \dfrac{\pi}{4}\right)$

(c) Area $= r^2 - \dfrac{1}{4}\pi r^2 = r^2\left(1 - \dfrac{\pi}{4}\right)$

60. (a) Place the center of the circle at $(0, 1)$; $x^2 + (y-1)^2 = 1$. The depth d satisfies $0 \le d \le 2$. The volume is

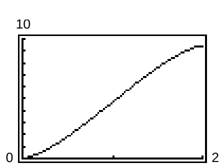
$$V = 3\cdot 2\int_0^d \sqrt{1-(y-1)^2}\,dy$$

$$= 6\cdot\frac{1}{2}\Big[\arcsin(y-1) + (y-1)\sqrt{1-(y-1)^2}\Big]_0^d \text{ (Theorem 7.2 (1))}$$

$$= 3\Big[\arcsin(d-1) + (d-1)\sqrt{1-(d-1)^2} - \arcsin(-1)\Big]$$

$$= \frac{3\pi}{2} + 3\arcsin(d-1) + 3(d-1)\sqrt{2d-d^2}.$$

(b)



(c) The full tank holds $3\pi \approx 9.4248$ cubic meters. The horizontal lines

$$y = \frac{3\pi}{4},\ y = \frac{3\pi}{2},\ y = \frac{9\pi}{4}$$

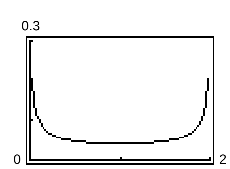
intersect the curve at $d = 0.596$, 1. 0, 1.404. The dipstick would have these markings on it.

(d) $\displaystyle V = 6\int_0^d \sqrt{1-(y-1)^2}\,dy$

$$\frac{dV}{dt} = \frac{dV}{dd}\cdot\frac{dd}{dt} = 6\sqrt{1-(d-1)^2}\cdot d'(t) = \frac{1}{4}$$

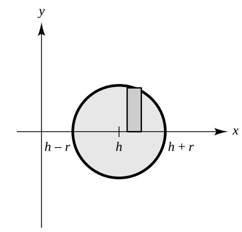
$$\Rightarrow d'(t) = \frac{1}{24\sqrt{1-(d-1)^2}}$$

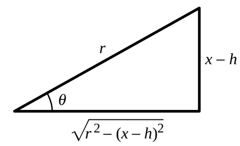
(e)



The minimum occurs at $d = 1$, which is the widest part of the tank.

62. Let $x - h = r\sin\theta$, $dx = r\cos\theta\, d\theta$, $\sqrt{r^2 - (x-h)^2} = r\cos\theta$.





Shell Method:

$$V = 4\pi\int_{h-r}^{h+r} x\sqrt{r^2-(x-h)^2}\,dx$$

$$= 4\pi\int_{-\pi/2}^{\pi/2}(h + r\sin\theta)r\cos\theta(r\cos\theta)\,d\theta = 4\pi r^2\int_{-\pi/2}^{\pi/2}(h + r\sin\theta)\cos^2\theta\,d\theta$$

$$= 4\pi r^2\left[\frac{h}{2}\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta)\,d\theta + r\int_{-\pi/2}^{\pi/2}\sin\theta\cos^2\theta\,d\theta\right]$$

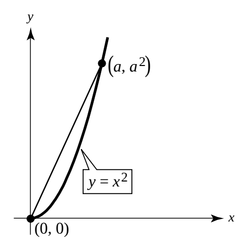
$$= 2\pi r^2 h\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\pi/2}^{\pi/2} - \left[4\pi r^3\left(\frac{\cos^3\theta}{3}\right)\right]_{-\pi/2}^{\pi/2} = 2\pi^2 r^2 h$$

64. $y = \frac{1}{2}x^2$, $y' = x$, $1 + (y')^2 = 1 + x^2$

$$s = \int_0^4 \sqrt{1+x^2}\,dx = \left[\frac{1}{2}\left(x\sqrt{x^2+1} + \ln\left|x + \sqrt{x^2+1}\right|\right)\right]_0^4 \quad \text{(Theorem 7.2)}$$

$$= \frac{1}{2}\left[4\sqrt{17} + \ln\left(4+\sqrt{17}\right)\right] \approx 9.2936$$

66. (a) Along line: $d_1 = \sqrt{a^2+a^4} = a\sqrt{1+a^2}$



Along parabola: $y = x^2$, $y' = 2x$

$$d_2 = \int_0^a \sqrt{1+4x^2}\,dx$$

$$= \frac{1}{4}\left[2x\sqrt{4x^2+1} + \ln\left|2x + \sqrt{4x^2+1}\right|\right]_0^a \quad \text{(Theorem 7.2)}$$

$$= \frac{1}{4}\left[2a\sqrt{4a^2+1} + \ln\left(2a + \sqrt{4a^2+1}\right)\right]$$

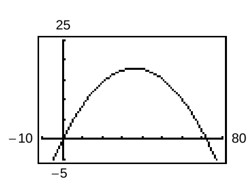
(b) For $a = 1$, $d_1 = \sqrt{2}$ and $d_2 = \frac{\sqrt{5}}{2} + \frac{1}{4}\ln\left(2+\sqrt{5}\right) \approx 1.4789$.

For $a = 10$, $d_1 = 10\sqrt{101} \approx 100.4988$

$d_2 \approx 101.0473$.

(c) As a increases, $d_2 - d_1 \to 0$.

68. (a)



(b) $y = 0$ for $x = 72$

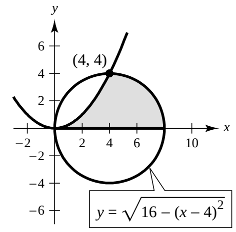
(c) $y = x - \frac{x^2}{72}$, $y' = 1 - \frac{x}{36}$, $1 + (y')^2 = 1 + \left(1 - \frac{x}{36}\right)^2$

$$s = \int_0^{72}\sqrt{1 + \left(1-\frac{x}{36}\right)^2}\,dx = -36\int_0^{72}\sqrt{1+\left(1-\frac{x}{36}\right)^2}\left(-\frac{1}{36}\right)dx$$

$$= -\frac{36}{2}\left[\left(1-\frac{x}{36}\right)\sqrt{1+\left(1-\frac{x}{36}\right)^2} + \ln\left|\left(1-\frac{x}{36}\right) + \sqrt{1+\left(1-\frac{x}{36}\right)^2}\right|\right]_0^{72}$$

$$= -18\left[\left(-\sqrt{2} + \ln\left|-1+\sqrt{2}\right|\right) - \left(\sqrt{2} + \ln\left|1+\sqrt{2}\right|\right)\right] = 36\sqrt{2} + 18\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) \approx 82.641$$

70. First find where the curves intersect.



$$y^2 = 16 - (x-4)^2 = \frac{1}{16}x^4$$

$$16^2 - 16(x-4)^2 = x^4$$

$$16^2 - 16x^2 + 128x - 16^2 = x^4$$

$$x^4 + 16x^2 - 128x = 0$$

$$x(x-4)(x^2 + 4x + 32) \implies x = 0, 4$$

$$A = \int_0^4 \frac{1}{4}x^2 dx + \frac{1}{4}\pi(4)^2 = \frac{1}{12}x^3\Big]_0^4 + 4\pi = \frac{16}{3} + 4\pi$$

$$M_y = \int_0^4 x\left[\frac{1}{4}x^2\right]dx + \int_4^8 x\sqrt{16-(x-4)^2}\,dx$$

$$= \frac{x^4}{16}\Big]_0^4 + \int_4^8 (x-4)\sqrt{16-(x-4)^2}\,dx + \int_4^8 4\sqrt{16-(x-4)^2}\,dx$$

$$= 16 + \left[\frac{-1}{3}(16-(x-4)^2)^{3/2}\right]_4^8 + 2\left[16\arcsin\frac{x-4}{4} + (x-4)\sqrt{16-(x-4)^2}\right]_4^8$$

$$= 16 + \frac{1}{3}16^{3/2} + 2\left[16\left(\frac{\pi}{2}\right)\right] = 16 + \frac{64}{3} + 16\pi = \frac{112}{3} + 16\pi$$

$$M_x = \int_0^4 \frac{1}{2}\left(\frac{1}{4}x^2\right)^2 dx + \int_4^8 \frac{1}{2}(16-(x-4)^2)\,dx$$

$$= \left[\frac{1}{32}\cdot\frac{x^5}{5}\right]_0^4 + \left[8x - \frac{(x-4)^3}{6}\right]_4^8$$

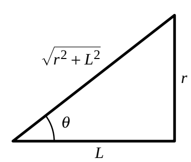
$$= \frac{32}{5} + \left(64 - \frac{64}{6}\right) - 32 = \frac{416}{15}$$

$$\bar{x} = \frac{M_y}{A} = \frac{112/3 + 16\pi}{16/3 + 4\pi} = \frac{112 + 48\pi}{16 + 12\pi} = \frac{28 + 12\pi}{4 + 3\pi} \approx 4.89$$

$$\bar{y} = \frac{M_x}{A} = \frac{416/15}{(16/3) + 4\pi} = \frac{104}{5(4 + 3\pi)} \approx 1.55$$

$$(\bar{x}, \bar{y}) \approx (4.89, 1.55)$$

72. Let $r = L\tan\theta$, $dr = L\sec^2\theta\,d\theta$, $r^2 + L^2 = L^2\sec^2\theta$.



$$\frac{1}{R}\int_0^R \frac{2mL}{(r^2 + L^2)^{3/2}}\,dr = \frac{2mL}{R}\int_a^b \frac{L\sec^2\theta\,d\theta}{L^3\sec^3\theta}$$

$$= \frac{2m}{RL}\int_a^b \cos\theta\,d\theta$$

$$= \left[\frac{2m}{RL}\sin\theta\right]_a^b$$

$$= \left[\frac{2m}{RL}\frac{r}{\sqrt{r^2 + L^2}}\right]_0^R$$

$$= \frac{2m}{L\sqrt{R^2 + L^2}}$$

74. (a) $F_{\text{inside}} = 48\int_{-1}^{0.8} (0.8 - y)(2)\sqrt{1-y^2}\,dy$

$= 96\left[0.8\int_{-1}^{0.8} \sqrt{1-y^2}\,dy - \int_{-1}^{0.8} y\sqrt{1-y^2}\,dy\right]$

$= 96\left[\frac{0.8}{2}\left(\arcsin y + y\sqrt{1-y^2}\right) + \frac{1}{3}(1-y^2)^{3/2}\right]_{-1}^{0.8} \approx 96(1.263) \approx 121.3 \text{ lbs}$

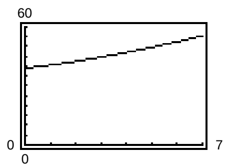
(b) $F_{\text{outside}} = 64\int_{-1}^{0.4} (0.4 - y)(2)\sqrt{1-y^2}\,dy$

$= 128\left[0.4\int_{-1}^{0.4} \sqrt{1-y^2}\,dy - \int_{-1}^{0.4} y\sqrt{1-y^2}\,dy\right]$

$= 128\left[\frac{0.4}{2}\left(\arcsin y + y\sqrt{1-y^2}\right) + \frac{1}{3}(1-y^2)^{3/2}\right]_{-1}^{0.4} \approx 92.98$

76. $S = \sqrt{1520.4 + 111.2t + 15.8t^2}$

(a)



(b) $S'(t) = \frac{1}{2}(1520.4 + 111.2t + 15.8t^2)^{-1/2}(111.2 + 31.6t)$

$S'(5) \approx 2.71$

(c) Average value $= \frac{1}{2}\int_{10}^{12} S(t)\,dt \approx 68.24$

78. False

$\int \frac{\sqrt{x^2-1}}{x}\,dx = \int \frac{\tan\theta}{\sec\theta}(\sec\theta\tan\theta\,d\theta)$

$= \int \tan^2\theta\,d\theta$

80. True

$\int_{-1}^{1} x^2\sqrt{1-x^2}\,dx = 2\int_{0}^{1} x^2\sqrt{1-x^2}\,dx = 2\int_{0}^{\pi/2} (\sin^2\theta)(\cos\theta)(\cos\theta\,d\theta) = 2\int_{0}^{\pi/2} \sin^2\theta\cos^2\theta\,d\theta$

Section 7.5 Partial Fractions

2. $\frac{4x^2+3}{(x-5)^3} = \frac{A}{x-5} + \frac{B}{(x-5)^2} + \frac{C}{(x-5)^3}$

4. $\frac{x-2}{x^2+4x+3} = \frac{x-2}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3}$

6. $\frac{2x-1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$

8. $\frac{1}{4x^2-9} = \frac{1}{(2x-3)(2x+3)} = \frac{A}{2x-3} + \frac{B}{2x+3}$

$1 = A(2x+3) + B(2x-3)$

When $x = \frac{3}{2}$, $1 = 6A$, $A = \frac{1}{6}$.

When $x = -\frac{3}{2}$, $1 = -6B$, $B = -\frac{1}{6}$.

$\int \frac{1}{4x^2-9}\,dx = \frac{1}{6}\left[\int \frac{1}{2x-3}\,dx - \int \frac{1}{2x+3}\,dx\right]$

$= \frac{1}{12}[\ln|2x-3| - \ln|2x+3|] + C$

$= \frac{1}{12}\ln\left|\frac{2x-3}{2x+3}\right| + C$

10. $\int \frac{x+1}{x^2+4x+3}\,dx = \int \frac{(x+1)}{(x+1)(x+3)}\,dx$

$= \int \frac{1}{x+3}\,dx = \ln|x+3| + C$

12. $\dfrac{5x^2 - 12x - 12}{x(x-2)(x+2)} = \dfrac{A}{x} + \dfrac{B}{x-2} + \dfrac{C}{x+2}$

$5x^2 - 12x - 12 = A(x^2 - 4) + Bx(x+2) + Cx(x-2)$

When $x = 0$, $-12 = -4A \implies A = 3$. When $x = 2$, $-16 = 8B \implies B = -2$. When $x = -2$, $32 = 8C \implies C = 4$.

$$\int \frac{5x^2 - 12x - 12}{x^3 - 4x}\,dx = \int \frac{3}{x}\,dx + \int \frac{-2}{x-2}\,dx + \int \frac{4}{x+2}\,dx = 3\ln|x| - 2\ln|x-2| + 4\ln|x+2| + C$$

14. $\dfrac{x^3 - x + 3}{x^2 + x - 2} = x - 1 + \dfrac{2x+1}{(x+2)(x-1)} = x - 1 + \dfrac{A}{x+2} + \dfrac{B}{x-1}$

$2x + 1 = A(x-1) + B(x+2)$

When $x = -2$, $-3 = -3A$, $A = 1$. When $x = 1$, $3 = 3B$, $B = 1$.

$$\int \frac{x^3 - x + 3}{x^2 + x - 2}\,dx = \int \left[x - 1 + \frac{1}{x+2} + \frac{1}{x-1}\right]dx$$

$$= \frac{x^2}{2} - x + \ln|x+2| + \ln|x-1| + C = \frac{x^2}{2} - x + \ln|x^2 + x - 2| + C$$

16. $\dfrac{x+2}{x(x-4)} = \dfrac{A}{x-4} + \dfrac{B}{x}$

$x + 2 = Ax + B(x-4)$

When $x = 4$, $6 = 4A$, $A = \frac{3}{2}$.
When $x = 0$, $2 = -4B$, $B = -\frac{1}{2}$.

$$\int \frac{x+2}{x^2 - 4x}\,dx = \int \left[\frac{3/2}{x-4} - \frac{1/2}{x}\right]dx$$

$$= \frac{3}{2}\ln|x-4| - \frac{1}{2}\ln|x| + C$$

18. $\dfrac{2x-3}{(x-1)^2} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2}$

$2x - 3 = A(x-1) + B$

When $x = 1$, $B = -1$. When $x = 0$, $A = 2$.

$$\int \frac{2x-3}{(x-1)^2}\,dx = \int \left[\frac{2}{x-1} - \frac{1}{(x-1)^2}\right]dx$$

$$= 2\ln|x-1| + \frac{1}{x-1} + C$$

20. $\dfrac{4x^2}{x^3 + x^2 - x - 1} = \dfrac{4x^2}{x^2(x+1) - (x+1)} = \dfrac{4x^2}{(x^2-1)(x+1)} = \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$

$4x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$

When $x = -1$, $4 = -2C \implies C = -2$. When $x = 1$, $4 = 4A \implies A = 1$. When $x = 0$, $0 = 1 - B + 2 \implies B = 3$.

$$\int \frac{4x^2}{x^3 + x^2 - x - 1}\,dx = \int \frac{1}{x-1}\,dx + \int \frac{3}{x+1}\,dx - \int \frac{2}{(x+1)^2}\,dx$$

$$= \ln|x-1| + 3\ln|x+1| + \frac{2}{(x+1)} + C$$

22. $\dfrac{6x}{x^3 - 8} = \dfrac{6x}{(x-2)(x^2 + 2x + 4)} = \dfrac{A}{x-2} + \dfrac{Bx + C}{x^2 + 2x + 4}$

$6x = A(x^2 + 2x + 4) + (Bx + C)(x-2)$

When $x = 2$, $12 = 12A \implies A = 1$. When $x = 0$, $0 = 4 - 2C \implies C = 2$. When $x = 1$, $6 = 7 + (B+2)(-1) \implies B = -1$.

$$\int \frac{6x}{x^3 - 8}\,dx = \int \frac{1}{x-2}\,dx + \int \frac{-x+2}{x^2 + 2x + 4}\,dx = \int \frac{1}{x-2}\,dx + \int \frac{-x-1}{x^2 + 2x + 4}\,dx + \int \frac{3}{(x^2 + 2x + 1) + 3}\,dx$$

$$= \ln|x-2| - \frac{1}{2}\ln|x^2 + 2x + 4| + \frac{3}{\sqrt{3}}\arctan\left(\frac{x+1}{\sqrt{3}}\right) + C$$

$$= \ln|x-2| - \frac{1}{2}\ln|x^2 + 2x + 4| + \sqrt{3}\arctan\left(\frac{\sqrt{3}(x+1)}{3}\right) + C$$

24. $\dfrac{x^2 - x + 9}{(x^2+9)^2} = \dfrac{Ax+B}{x^2+9} + \dfrac{Cx+D}{(x^2+9)^2}$

$$x^2 - x + 9 = (Ax+B)(x^2+9) + Cx + D$$
$$= Ax^3 + Bx^2 + (9A + C)x + (9B + D)$$

By equating coefficients of like terms, we have $A = 0$, $B = 1$, $D = 0$, and $C = -1$.

$$\int \frac{x^2 - x - 9}{(x^2+9)^2} = \int \frac{1}{x^2+9}\,dx - \int \frac{x}{(x^2+9)^2}\,dx$$
$$= \frac{1}{3}\arctan\left(\frac{x}{3}\right) + \frac{1}{2(x^2+9)} + C$$

26. $\dfrac{x^2 - 4x + 7}{(x+1)(x^2-2x+3)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2-2x+3}$

$$x^2 - 4x + 7 = A(x^2 - 2x + 3) + (Bx + C)(x + 1)$$

When $x = -1$, $12 = 6A$. When $x = 0$, $7 = 3A + C$. When $x = 1$, $4 = 2A + 2B + 2C$. Solving these equations we have $A = 2$, $B = -1$, $C = 1$.

$$\int \frac{x^2 - 4x + 7}{x^3 - x^2 + x + 3}\,dx = 2\int \frac{1}{x+1}\,dx + \int \frac{-x+1}{x^2-2x+3}\,dx$$
$$= 2\ln|x+1| - \frac{1}{2}\ln|x^2 - 2x + 3| + C$$

28. $\dfrac{x^2 + x + 3}{(x^2+3)^2} = \dfrac{Ax+B}{x^2+3} + \dfrac{Cx+D}{(x^2+3)^2}$

$$x^2 + x + 3 = (Ax+B)(x^2+3) + Cx + D$$
$$= Ax^3 + Bx^2 + (3A + C)x + (3B + D)$$

By equating coefficients of like terms, we have $A = 0$, $B = 1$, $3A + C = 1$, $3B + D = 3$. Solving these equations we have $A = 0$, $B = 1$, $C = 1$, $D = 0$.

$$\int \frac{x^2 + x + 3}{x^4 + 6x^2 + 9}\,dx = \int\left[\frac{1}{x^2+3} + \frac{x}{(x^2+3)^2}\right]dx$$
$$= \frac{1}{\sqrt{3}}\arctan\frac{x}{\sqrt{3}} - \frac{1}{2(x^2+3)} + C$$

30. $\dfrac{x-1}{x^2(x+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1}$

$$x - 1 = Ax(x+1) + B(x+1) + Cx^2$$

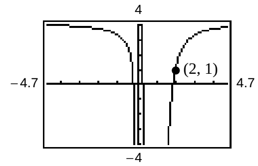
When $x = 0$, $B = -1$. When $x = -1$, $C = -2$. When $x = 1$, $0 = 2A + 2B + C$. Solving these equations we have $A = 2$, $B = -1$, $C = -2$.

$$\int_1^5 \frac{x-1}{x^2(x+1)}\,dx = 2\int_1^5 \frac{1}{x}\,dx - \int_1^5 \frac{1}{x^2}\,dx - 2\int_1^5 \frac{1}{x+1}\,dx$$
$$= \left[2\ln|x| + \frac{1}{x} - 2\ln|x+1|\right]_1^5$$
$$= \left[2\ln\left|\frac{x}{x+1}\right| + \frac{1}{x}\right]_1^5$$
$$= 2\ln\frac{5}{3} - \frac{4}{5}$$

32. $\displaystyle\int_0^1 \frac{x^2 - x}{x^2 + x + 1}\,dx = \int_0^1 dx - \int_0^1 \frac{2x + 1}{x^2 + x + 1}\,dx = \Big[x - \ln|x^2 + x + 1|\Big]_0^1 = 1 - \ln 3$

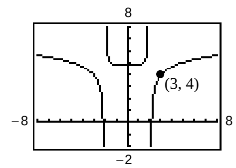
34. $\displaystyle\int \frac{6x^2 + 1}{x^2(x - 1)^3}\,dx = 3\ln\left|\frac{x - 1}{x}\right| + \frac{1}{x} + \frac{2}{x - 1} - \frac{7}{2(x - 1)^2} + C$

$(2, 1)$: $\displaystyle 3\ln\left|\frac{1}{2}\right| + \frac{1}{2} + \frac{2}{1} - \frac{7}{2} + C = 1 \Rightarrow C = 2 - 3\ln\frac{1}{2}$



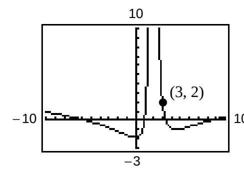
36. $\displaystyle\int \frac{x^3}{(x^2 - 4)^2}\,dx = \frac{1}{2}\ln|x^2 - 4| - \frac{2}{x^2 - 4} + C$

$(3, 4)$: $\displaystyle \frac{1}{2}\ln 5 - \frac{2}{5} + C = 4 \Rightarrow C = \frac{22}{5} - \frac{1}{2}\ln 5$



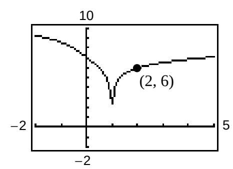
38. $\displaystyle\int \frac{x(2x - 9)}{x^3 - 6x^2 + 12x - 8}\,dx = 2\ln|x - 2| + \frac{1}{x - 2} + \frac{5}{(x - 2)^2} + C$

$(3, 2)$: $0 + 1 + 5 + C = 2 \Rightarrow C = -4$



40. $\displaystyle\int \frac{x^2 - x + 2}{x^3 - x^2 + x - 1}\,dx = -\arctan x + \ln|x - 1| + C$

$(2, 6)$: $-\arctan 2 + 0 + C = 6 \Rightarrow C = 6 + \arctan 2$



42. Let $u = \cos x$, $du = \sin x\,dx$.

$$\frac{1}{u(u + 1)} = \frac{A}{u} + \frac{B}{u + 1}$$

$$1 = A(u + 1) + Bu$$

When $u = 0, A = 1$. When $u = -1, B = -1, u = \cos x$.
$du = -\sin dx$.

$$\begin{aligned}\int \frac{\sin x}{\cos x + \cos^2 x}\,dx &= -\int \frac{1}{u(u + 1)}\,du \\ &= \int \frac{1}{u + 1}\,du - \int \frac{1}{u}\,du \\ &= \ln|u + 1| - \ln|u| + C \\ &= \ln\left|\frac{u + 1}{u}\right| + C \\ &= \ln\left|\frac{\cos x + 1}{\cos x}\right| + C \\ &= \ln|1 + \sec x| + C\end{aligned}$$

44. $\displaystyle\frac{1}{u(u + 1)} = \frac{A}{u} + \frac{B}{u + 1}$, $u = \tan x$, $du = \sec^2 x\,dx$

$$1 = A(u + 1) + Bu$$

When $u = 0, A = 1$.
When $u = -1, 1 = -B \Rightarrow B = -1$.

$$\begin{aligned}\int \frac{\sec^2 x\,dx}{\tan x(\tan x + 1)} &= \int \frac{1}{u(u + 1)}\,du \\ &= \int\left(\frac{1}{u} - \frac{1}{u + 1}\right)du \\ &= \ln|u| - \ln|u + 1| + C \\ &= \ln\left|\frac{u}{u + 1}\right| + C \\ &= \ln\left|\frac{\tan x}{\tan x + 1}\right| + C\end{aligned}$$

46. Let $u = e^x$, $du = e^x\,dx$.

$$\frac{1}{(u^2+1)(u-1)} = \frac{A}{u-1} + \frac{Bu+C}{u^2+1}$$

$$1 = A(u^2+1) + (Bu+C)(u-1)$$

When $u = 1$, $A = \frac{1}{2}$.

When $u = 0$, $1 = A - C$.

When $u = -1$, $1 = 2A + 2B - 2C$. Solving these equations we have $A = \frac{1}{2}$, $B = -\frac{1}{2}$, $C = -\frac{1}{2}$, $u = e^x$, $du = e^x\,dx$.

$$\int \frac{e^x}{(e^{2x}+1)(e^x-1)}\,dx = \int \frac{1}{(u^2+1)(u-1)}\,du$$

$$= \frac{1}{2}\left(\int \frac{1}{u-1}\,du - \int \frac{u+1}{u^2+1}\,du\right)$$

$$= \frac{1}{2}\left(\ln|u-1| - \frac{1}{2}\ln|u^2+1| - \arctan u\right) + C$$

$$= \frac{1}{4}\left(2\ln|e^x-1| - \ln|e^{2x}+1| - 2\arctan e^x\right) + C$$

48. $\dfrac{1}{a^2-x^2} = \dfrac{A}{a-x} + \dfrac{B}{a+x}$

$$1 = A(a+x) + B(a-x)$$

When $x = a$, $A = 1/2a$.
When $x = -a$, $B = 1/2a$.

$$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\int\left(\frac{1}{a-x} + \frac{1}{a+x}\right)dx$$

$$= \frac{1}{2a}\left(-\ln|a-x| + \ln|a+x|\right) + C$$

$$= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$$

50. $\dfrac{1}{x^2(a+bx)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{a+bx}$

$$1 = Ax(a+bx) + B(a+bx) + Cx^2$$

When $x = 0$, $1 = Ba \implies B = 1/a$.
When $x = -a/b$, $1 = C(a^2/b^2) \implies C = b^2/a^2$.
When $x = 1$, $1 = (a+b)A + (a+b)B + C \implies$
$A = -b/a^2$.

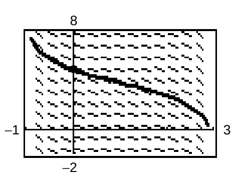
$$\int \frac{1}{x^2(a+bx)}\,dx = \int\left(\frac{-b/a^2}{x} + \frac{1/a}{x^2} + \frac{b^2/a^2}{a+bx}\right)dx$$

$$= -\frac{b}{a^2}\ln|x| - \frac{1}{ax} + \frac{b}{a^2}\ln|a+bx| + C$$

$$= -\frac{1}{ax} + \frac{b}{a^2}\ln\left|\frac{a+bx}{x}\right| + C$$

$$= -\frac{1}{ax} - \frac{b}{a^2}\ln\left|\frac{x}{a+bx}\right| + C$$

52. $\dfrac{dy}{dx} = \dfrac{4}{(x^2-2x-3)}$, $y(0) = 5$



54. (a) $\dfrac{N(x)}{D(x)} = \dfrac{A_1}{px+q} + \dfrac{A_2}{(px+q)^2} + \cdots + \dfrac{A_m}{(px+q)^m}$

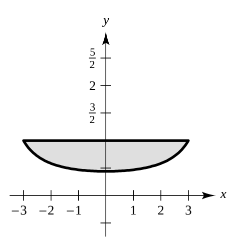
(b) $\dfrac{N(x)}{D(x)} = \dfrac{A_1+B_1x}{(ax^2+bx+c)} + \cdots + \dfrac{A_n+B_nx}{(ax^2+bx+c)^n}$

56. $A = 2\displaystyle\int_0^3\left(1 - \frac{7}{16-x^2}\right)dx$

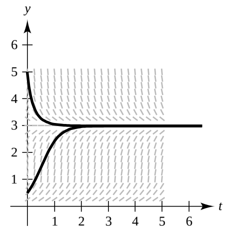
$$= 2\int_0^3 dx - 14\int_0^3 \frac{1}{16-x^2}\,dx$$

$$= \left[2x - \frac{14}{8}\ln\left|\frac{4+x}{4-x}\right|\right]_0^3 \quad \text{(From Exercise 46)}$$

$$= 6 - \frac{7}{4}\ln 7 \approx 2.595$$



58. (a)



(b) The slope is negative because the function is decreasing.

(c) For $y > 0$, $\lim_{t \to \infty} y(t) = 3$.

(d) $$\frac{dy}{y(L-y)} = \frac{A}{y} + \frac{B}{L-y}$$

$$1 = A(L-y) + By \implies A = \frac{1}{L}, B = \frac{1}{L}$$

$$\int \frac{dy}{y(L-y)} = \int k\,dt$$

$$\frac{1}{L}\left[\int \frac{1}{y}dy + \int \frac{1}{L-y}dy\right] = \int k\,dt$$

$$\frac{1}{L}\left[\ln|y| - \ln|L-y|\right] = kt + C_1$$

$$\ln\left|\frac{y}{L-y}\right| = kLt + LC_1$$

$$C_2 e^{kLt} = \frac{y}{L-y}$$

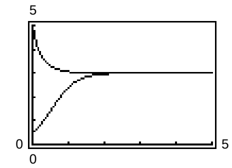
When $t = 0$, $\frac{y_0}{L-y_0} = C_2 \implies \frac{y}{L-y} = \frac{y_0}{L-y_0}e^{kLt}$.

Solving for y, you obtain $y = \frac{y_0 L}{y_0 + (L-y_0)e^{-kLt}}$.

(e) $k = 1, L = 3$

(i) $y(0) = 5$: $\quad y = \frac{15}{5 - 2e^{-3t}}$

(ii) $y(0) = \frac{1}{2}$: $\quad y = \frac{3/2}{(1/2) + (5/2)e^{-3t}} = \frac{3}{1 + 5e^{-3t}}$



(f) $$\frac{dy}{dt} = ky(L-y)$$

$$\frac{d^2y}{dt^2} = k\left[y\left(\frac{-dy}{dt}\right) + (L-y)\frac{dy}{dt}\right] = 0$$

$$\implies y\frac{dy}{dt} = (L-y)\frac{dy}{dt}$$

$$\implies \quad y = \frac{L}{2}$$

From the first derivative test, this is a maximum.

60. (a) $$V = \pi\int_0^3 \left(\frac{2x}{x^2+1}\right)^2 dx = 4\pi\int_0^3 \frac{x^2}{(x^2+1)^2}\,dx$$

$$= 4\pi\int_0^3 \left(\frac{1}{x^2+1} - \frac{1}{(x^2+1)^2}\right) dx \qquad \text{(partial fractions)}$$

$$= 4\pi\left[\arctan x - \frac{1}{2}\left(\arctan x + \frac{x}{x^2+1}\right)\right]_0^3 \qquad \text{(trigonometric substitution)}$$

$$= 2\pi\left[\arctan x - \frac{x}{x^2+1}\right]_0^3 = 2\pi\left[\arctan 3 - \frac{3}{10}\right] \approx 5.963$$

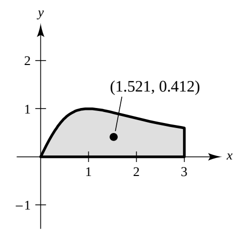
—CONTINUED—

60. —CONTINUED—

(b) $A = \int_0^3 \frac{2x}{x^2+1}\,dx = \Big[\ln(x^2+1)\Big]_0^3 = \ln 10$

$\bar{x} = \frac{1}{A}\int_0^3 \frac{2x^2}{x^2+1}\,dx = \frac{1}{\ln 10}\int_0^3\left(2 - \frac{2}{x^2+1}\right)dx$

$= \frac{1}{\ln 10}\Big[2x - 2\arctan x\Big]_0^3 = \frac{2}{\ln 10}[3 - \arctan 3] \approx 1.521$



$\bar{y} = \frac{1}{A}\left(\frac{1}{2}\right)\int_0^3\left(\frac{2x}{x^2+1}\right)^2 dx = \frac{2}{\ln 10}\int_0^3 \frac{x^2}{(x^2+1)^2}\,dx$

$= \frac{2}{\ln 10}\int_0^3\left(\frac{1}{x^2+1} - \frac{1}{(x^2+1)^2}\right)dx$ (partial fractions)

$= \frac{2}{\ln 10}\left[\arctan x - \frac{1}{2}\left(\arctan x + \frac{x}{x^2+1}\right)\right]_0^3$ (trigonometric substitution)

$= \frac{2}{\ln 10}\left[\frac{1}{2}\arctan x - \frac{x}{2(x^2+1)}\right]_0^3 = \frac{1}{\ln 10}\left[\arctan x - \frac{x}{x^2+1}\right]_0^3 = \frac{1}{\ln 10}\left[\arctan 3 - \frac{3}{10}\right] \approx 0.412$

$(\bar{x}, \bar{y}) \approx (1.521, 0.412)$

62. (a) $\frac{1}{(y_0 - x)(z_0 - x)} = \frac{A}{y_0 - x} + \frac{B}{z_0 - x}, A = \frac{1}{z_0 - y_0}, B = -\frac{1}{z_0 - y_0}$ (Assume $y_0 \neq z_0$)

$\frac{1}{z_0 - y_0}\int\left(\frac{1}{y_0 - x} - \frac{1}{z_0 - x}\right)dx = kt + C$

$\frac{1}{z_0 - y_0}\ln\left|\frac{z_0 - x}{y_0 - x}\right| = kt + C$, when $t = 0, x = 0$

$C = \frac{1}{z_0 - y_0}\ln\frac{z_0}{y_0}$

$\frac{1}{z_0 - y_0}\left[\ln\left|\frac{z_0 - x}{y_0 - x}\right| - \ln\left(\frac{z_0}{y_0}\right)\right] = kt$

$\ln\left[\frac{y_0(z_0 - x)}{z_0(y_0 - x)}\right] = (z_0 - y_0)kt$

$\frac{y_0(z_0 - x)}{z_0(y_0 - x)} = e^{(z_0 - y_0)kt}$

$x = \frac{y_0 z_0[e^{(z_0 - y_0)kt} - 1]}{z_0 e^{(z_0 - y_0)kt} - y_0}$

(b) (1) If $y_0 < z_0$, $\lim_{t\to\infty} x = y_0$.

(2) If $y_0 > z_0$, $\lim_{t\to\infty} x = z_0$.

(c) If $y_0 = z_0$, then the original equation is

$\int \frac{1}{(y_0 - x)^2}\,dx = \int k\,dt$

$(y_0 - x)^{-1} = kt + C_1$

$x = 0$ when $t = 0 \implies \frac{1}{y_0} = C_1$

$\frac{1}{y_0 - x} = kt + \frac{1}{y_0} = \frac{kt\,y_0 + 1}{y_0}$

$y_0 - x = \frac{y_0}{kt\,y_0 + 1}$

$x = y_0 - \frac{y_0}{kt\,y_0 + 1}$

As $t \to \infty$, $x \to y_0 = x_0$.

Section 7.6 Integration by Tables and Other Integration Techniques

2. By Formula 13: $(b = 2, a = -5)$

$$\frac{2}{3}\int \frac{1}{x^2(2x - 5)^2}\,dx = \frac{2}{3}\left(\frac{-1}{25}\right)\left[\frac{-5 + 4x}{x(-5 + 2x)} + \frac{4}{-5}\ln\left|\frac{x}{2x - 5}\right|\right] + C$$
$$= \frac{8}{375}\ln\left|\frac{x}{2x - 5}\right| - \frac{2}{75}\frac{(4x - 5)}{x(2x - 5)} + C$$

4. By Formula 29: $(a = 3)$

$$\frac{1}{3}\int \frac{\sqrt{x^2 - 9}}{x}\,dx = \frac{1}{3}\sqrt{x^2 - 9} - \operatorname{arcsec}\frac{|x|}{3} + C$$

6. By Formula 41: $\displaystyle\int \frac{x}{\sqrt{9 - x^4}}\,dx = \frac{1}{2}\int \frac{2x}{\sqrt{3^2 - (x^2)^2}}\,dx$

$$= \frac{1}{2}\arcsin\frac{x^2}{3} + C$$

8. By Formulas 51 and 47: $\displaystyle\int \frac{\cos^3\sqrt{x}}{\sqrt{x}}\,dx = 2\int \cos^3\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right)dx$

$$= 2\left[\frac{\cos^2\sqrt{x}\sin\sqrt{x}}{3} + \frac{2}{3}\int \cos\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right)dx\right] = \frac{2}{3}\sin\sqrt{x}\left(\cos^2\sqrt{x} + 2\right) + C$$

$u = \sqrt{x},\ du = \dfrac{1}{2\sqrt{x}}\,dx$

10. By Formula 71:

$$\int \frac{1}{1 - \tan 5x}\,dx = \frac{1}{5}\int \frac{1}{1 - \tan 5x}(5)\,dx$$
$$= \frac{1}{5}\left(\frac{1}{2}\right)\left(u - \ln|\cos u - \sin u|\right) + C$$
$$= \frac{1}{10}\left(5x - \ln|\cos 5x - \sin 5x|\right) + C$$

$u = 5x,\ du = 5\,dx$

12. By Formula 85: $\left(a = -\frac{1}{2}, b = 2\right)$

$$\int e^{-x/2}\sin 2x\,dx = \frac{e^{-x/2}}{(1/4) + 4}\left(-\frac{1}{2}\sin 2x - 2\cos 2x\right) + C$$
$$= \frac{4}{17}e^{-x/2}\left(-\frac{1}{2}\sin 2x - 2\cos 2x\right) + C$$

14. By Formulas 90 and 91: $\displaystyle\int (\ln x)^3\,dx = x(\ln x)^3 - 3\int (\ln x)^2\,dx$

$$= x(\ln x)^3 - 3x[2 - 2\ln x + (\ln x)^2] + C$$
$$= x[(\ln x)^3 - 3(\ln x)^2 + 6\ln x - 6] + C$$

16. (a) By Formula 89: $\displaystyle\int x^4\ln x\,dx = \frac{x^5}{5^2}[-1 + (4 + 1)\ln x] + C = \frac{-x^5}{25} + \frac{1}{5}x^5\ln x + C$

(b) Integration by parts: $u = \ln x,\ du = \dfrac{1}{x}\,dx,\ dv = x^4\,dx,\ v = \dfrac{x^5}{5}$

$$\int x^4\ln x\,dx = \frac{x^5}{5}\ln x - \int \frac{x^5}{5}\frac{1}{x}\,dx = \frac{x^5}{5}\ln x - \frac{x^5}{25} + C$$

18. (a) By Formula 24: $a = \sqrt{75}$, $x = u$, and

$$\int \frac{1}{x^2 - 75}\,dx = \frac{1}{2\sqrt{75}} \ln\left|\frac{x - \sqrt{75}}{x + \sqrt{75}}\right| + C$$

$$= \frac{\sqrt{3}}{30} \ln\left|\frac{x - \sqrt{75}}{x + \sqrt{75}}\right| + C$$

(b) Partial fractions:

$$\frac{1}{x^2 - 75} = \frac{A}{x - \sqrt{75}} + \frac{B}{x + \sqrt{75}}$$

$$1 = A\left(x + \sqrt{75}\right) + B\left(x - \sqrt{75}\right)$$

$$x = \sqrt{75}:\ 1 = 2A\sqrt{75} \Rightarrow A = \frac{1}{2\sqrt{75}} = \frac{1}{10\sqrt{3}} = \frac{\sqrt{3}}{30}$$

$$x = -\sqrt{75}:\ 1 = -2B\sqrt{75} \Rightarrow B = -\frac{\sqrt{3}}{30}$$

$$\int \frac{1}{x^2 - 75}\,dx = \int \left[\frac{\sqrt{3}/30}{x - \sqrt{75}} - \frac{\sqrt{3}/30}{x + \sqrt{75}}\right] dx$$

$$= \frac{\sqrt{3}}{30} \ln\left|\frac{x - \sqrt{75}}{x + \sqrt{75}}\right| + C$$

20. By Formula 21: $\displaystyle\int \frac{x}{\sqrt{1 + x}}\,dx = -\frac{2}{3}(2 - x)\sqrt{1 + x} + C$

22. By Formula 79: $\displaystyle\int \operatorname{arcsec} 2x\,dx = \frac{1}{2}\left[2x \operatorname{arcsec} 2x - \ln\left|2x + \sqrt{4x^2 - 1}\right|\right] + C$

$u = 2x$, $du = 2\,dx$

24. By Formula 52: $\displaystyle\int x \sin x\,dx = \sin x - x \cos x + C$

26. By Formula 7: $\displaystyle\int \frac{x^2}{(3x - 5)^2}\,dx = \frac{1}{27}\left(3x - \frac{25}{3x - 5} + 10 \ln|3x - 5|\right) + C$

28. By Formula 14: $\displaystyle\int \frac{1}{x^2 + 2x + 2}\,dx = \frac{2}{\sqrt{4}} \arctan\left(\frac{2x + 2}{2}\right) + C = \arctan(x + 1) + C$

30. By Formula 56:

$$\int \frac{\theta^2}{1 - \sin \theta^3}\,d\theta = \frac{1}{3}\int \frac{1}{1 - \sin \theta^3} 3\theta^2\,d\theta$$

$$= \frac{1}{3}(\tan \theta^3 + \sec \theta^3) + C$$

32. By Formula 71:

$$\int \frac{e^x}{1 - \tan e^x}\,dx = \frac{1}{2}\left(e^x - \ln\left|\cos e^x - \sin e^x\right|\right) + C$$

$u = e^x$, $du = e^x\,dx$

34. By Formula 23: $\displaystyle\int \frac{1}{t[1 + (\ln t)^2]}\,dt = \int \frac{1}{1 + (\ln t)^2}\left(\frac{1}{t}\right) dt = \arctan(\ln t) + C$

$u = \ln t$, $du = \dfrac{1}{t}\,dt$

36. By Formula 26: $\displaystyle\int \sqrt{3 + x^2}\,dx = \frac{1}{2}\left(x\sqrt{x^2 + 3} + 3 \ln\left|x + \sqrt{x^2 + 3}\right|\right) + C$

38. By Formula 27: $\displaystyle\int x^2\sqrt{2 + (3x)^2}\,dx = \frac{1}{27}\int (3x)^2\sqrt{\left(\sqrt{2}\right)^2 + (3x)^2}\,3\,dx$

$$= \frac{1}{8(27)}\left[3x(18x^2 + 2)\sqrt{2 + 9x^2} - 4 \ln\left|3x + \sqrt{2 + 9x^2}\right|\right] + C$$

40. By Formula 77: $\int \sqrt{x}\arctan(x^{3/2})\,dx = \frac{2}{3}\int \arctan(x^{3/2})\left(\frac{3}{2}\sqrt{x}\right)dx$

$$= \frac{2}{3}\left[x^{3/2}\arctan(x^{3/2}) - \ln\sqrt{1+x^3}\right] + C$$

42. By Formula 45: $\int \frac{e^x}{(1-e^{2x})^{3/2}}\,dx = \frac{e^x}{\sqrt{1-e^{2x}}} + C$

$u = e^x,\ du = e^x\,dx$

44. By Formula 27:

$$\int (2x-3)^2\sqrt{(2x-3)^2+4}\,dx = \frac{1}{2}\int (2x-3)^2\sqrt{(2x-3)^2+4}(2)\,dx$$

$$= \frac{1}{8}(2x-3)[(2x-3)^2+2]\sqrt{(2x-3)^2+4} - \ln\left|2x-3+\sqrt{(2x-3)^2+4}\right| + C$$

$u = 2x-3,\ du = 2\,dx$

46. By Formula 31: $\int \frac{\cos x}{\sqrt{\sin^2 x + 1}}\,dx = \ln\left|\sin x + \sqrt{\sin^2 x + 1}\right| + C$

$u = \sin x,\ du = \cos x\,dx$

48. $\int \sqrt{\frac{3-x}{3+x}}\,dx = \int \frac{3-x}{\sqrt{9-x^2}}\,dx$

$$= 3\int \frac{1}{\sqrt{9-x^2}}\,dx + \int \frac{-x}{\sqrt{9-x^2}}\,dx$$

$$= 3\arcsin\frac{x}{3} + \sqrt{9-x^2} + C$$

50. By Formula 67:

$$\int \tan^3\theta\,d\theta = \frac{\tan^2\theta}{2} - \int \tan\theta\,d\theta$$

$$= \frac{\tan^2\theta}{2} + \ln|\cos x| + C$$

52. Integration by parts: $w = u^n,\ dw = nu^{n-1}\,du,\ dv = \frac{du}{\sqrt{a+bu}},\ v = \frac{2}{b}\sqrt{a+bu}$

$$\int \frac{u^n}{\sqrt{a+bu}}\,du = \frac{2u^n}{b}\sqrt{a+bu} - \frac{2n}{b}\int u^{n-1}\sqrt{a+bu}\,du$$

$$= \frac{2u^n}{b}\sqrt{a+bu} - \frac{2n}{b}\int u^{n-1}\sqrt{a+bu}\cdot\frac{\sqrt{a+bu}}{\sqrt{a+bu}}\,du$$

$$= \frac{2u^n}{b}\sqrt{a+bu} - \frac{2n}{b}\int \frac{au^{n-1}+bu^n}{\sqrt{a+bu}}\,du$$

$$= \frac{2u^n}{b}\sqrt{a+bu} - \frac{2na}{b}\int \frac{u^{n-1}}{\sqrt{a+bu}}\,du - 2n\int \frac{u^n}{\sqrt{a+bu}}\,du$$

Therefore, $(2n+1)\int \frac{u^n}{\sqrt{a+bu}}\,du = \frac{2}{b}\left[u^n\sqrt{a+bu} - na\int \frac{u^{n-1}}{\sqrt{a+bu}}\,du\right]$ and

$$\int \frac{u^n}{\sqrt{a+bu}} = \frac{2}{(2n+1)b}\left[u^n\sqrt{a+bu} - na\int \frac{u^{n-1}}{\sqrt{a+bu}}\,du\right].$$

54. $\int u^n(\cos u)\,du = u^n\sin u - n\int u^{n-1}(\sin u)\,du$

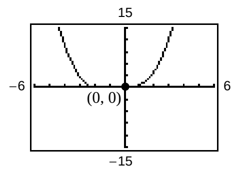
$w = u^n,\ dv = \cos u\,du,\ dw = nu^{n-1}\,du,\ v = \sin u$

56. $\displaystyle\int (\ln u)^n\,du = u(\ln u)^n - \int n(\ln u)^{n-1}\left(\frac{1}{u}\right)u\,du = u(\ln u)^n - n\int (\ln u)^{n-1}\,du$

$w = (\ln u)^n,\ dv = du,\ dw = n(\ln u)^{n-1}\left(\dfrac{1}{u}\right)du,\ v = u$

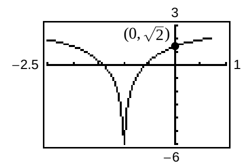
58. $\displaystyle\int x\sqrt{x^2+2x}\,dx = \frac{1}{6}\left[2(x^2+2x)^{3/2} - 3(x+1)\sqrt{x^2+2x} + 3\ln\left|x+1+\sqrt{x^2+2x}\right|\right] + C$

$(0, 0)$: $\dfrac{1}{6}[3\ln|1|] + C = 0 \Rightarrow C = 0$



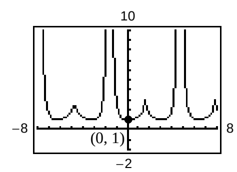
60. $\displaystyle\int \frac{\sqrt{2-2x-x^2}}{x+1}\,dx = \sqrt{2-2x-x^2} - \sqrt{3}\ln\left|\frac{\sqrt{3}+\sqrt{2-2x-x^2}}{x+1}\right| + C$

$\left(0, \sqrt{2}\right)$: $\sqrt{2} - \sqrt{3}\ln\left(\sqrt{3}+\sqrt{2}\right) + C = \sqrt{2} \Rightarrow C = \sqrt{3}\ln\left(\sqrt{3}+\sqrt{2}\right)$



62. $\displaystyle\int \frac{\sin\theta}{(\cos\theta)(1+\sin\theta)}\,d\theta = \frac{1}{2}\left[\frac{-\sin\theta}{1+\sin\theta} + \ln\left|\frac{1+\sin\theta}{\cos\theta}\right|\right] + C$

$(0, 1)$: $C = 1 \Rightarrow y = \dfrac{1}{2}\left[\dfrac{-\sin\theta}{1+\sin\theta} + \ln\left|\dfrac{1+\sin\theta}{\cos\theta}\right|\right] + 1$



64. $\displaystyle\int \frac{\sin\theta}{1+\cos^2\theta}\,d\theta = -\int \frac{-\sin\theta}{1+(\cos\theta)^2}\,d\theta$

$\displaystyle = -\arctan(\cos\theta) + C$

66. $\displaystyle\int_0^{\pi/2} \frac{1}{3-2\cos\theta}\,d\theta = \int_0^1 \left[\frac{\dfrac{2u}{1+u^2}}{3 - \dfrac{2(1-u^2)}{1+u^2}}\right]$

$\displaystyle = 2\int_0^1 \frac{1}{5u^2+1}\,du$

$\displaystyle = \left[\frac{2}{\sqrt{5}}\arctan\left(\sqrt{5}\,u\right)\right]_0^1$

$\displaystyle = \frac{2}{\sqrt{5}}\arctan\sqrt{5}$

$u = \tan\dfrac{\theta}{2}$

68. $\displaystyle\int \frac{\cos\theta}{1+\cos\theta}\,d\theta = \int \frac{\cos\theta(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}\,d\theta$

$\displaystyle = \int \frac{\cos\theta - \cos^2\theta}{\sin^2\theta}\,d\theta$

$\displaystyle = \int (\csc\theta\cot\theta - \cot^2\theta)\,d\theta$

$\displaystyle = \int (\csc\theta\cot\theta - (\csc^2\theta - 1))\,d\theta$

$= -\csc\theta + \cot\theta + \theta + C$

70. $\displaystyle\int \frac{1}{\sec\theta - \tan\theta}\,d\theta = \int \frac{1}{(1/\cos\theta) - (\sin\theta/\cos\theta)}\,d\theta$

$\displaystyle = -\int \frac{-\cos\theta}{1-\sin\theta}\,d\theta$

$= -\ln|1-\sin\theta| + C$

$u = 1 - \sin\theta,\ du = -\cos\theta\,d\theta$

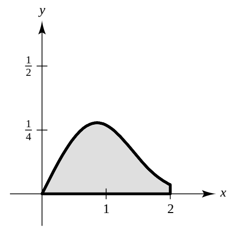
72. $A = \int_0^2 \frac{x}{1 + e^{x^2}}\, dx$

$= \frac{1}{2}\int_0^2 \frac{2x\, dx}{1 + e^{x^2}}$

$= \frac{1}{2}\left[x^2 - \ln(1 + e^{x^2})\right]_0^2$

$= \frac{1}{2}\left[4 - \ln(1 + e^4)\right] + \frac{1}{2}\ln 2$

≈ 0.337 square units



74. Log Rule: $\int \frac{1}{u}\, du, u = e^x + 1$

76. Integration by parts

78. Formula 16 with $u = e^{2x}$

80. A reduction formula reduces an integral to the sum of a function and a simpler integral. For example, see Formula 50, 54.

82. $W = \int_0^5 \frac{500x}{\sqrt{26 - x^2}}\, dx$

$= -250\int_0^5 (26 - x^2)^{-1/2}(-2x)\, dx$

$= \left[-500\sqrt{26 - x^2}\right]_0^5$

$= 500\left(\sqrt{26} - 1\right)$

≈ 2049.51 ft $\cdot$ lbs

84. $\frac{1}{2 - 0}\int_0^2 \frac{5000}{1 + e^{4.8 - 1.9t}}\, dt = \frac{2500}{-1.9}\int_0^2 \frac{-1.9\, dt}{1 + e^{4.8 - 1.9t}}$

$= -\frac{2500}{1.9}\left[(4.8 - 1.9t) - \ln(1 + e^{4.8 - 1.9t})\right]_0^2$

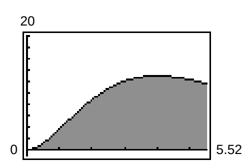
$= -\frac{2500}{1.9}[(1 - \ln(1 + e)) - (4.8 - \ln(1 + e^{4.8}))]$

$= \frac{2500}{1.9}\left[3.8 + \ln\left(\frac{1 + e}{1 + e^{4.8}}\right)\right] \approx 401.4$

86. (a) $\int_0^k 6x^2e^{-x/2}\, dx = 50$

By trial and error, $k = 5.51897$.

(b) $\int_0^{5.51897} 6x^2e^{-x/2}\, dx$

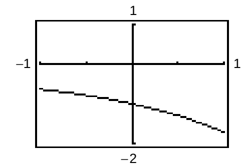


88. True

Section 7.7 Indeterminate Forms and L'Hôpital's Rule

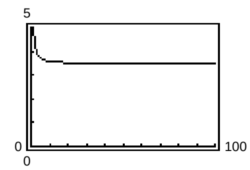
2. $\lim_{x \to 0} \frac{1 - e^x}{x} \approx -1$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	-0.9516	-0.9950	-0.9995	-1.00005	-1.005	-1.0517



4. $\lim_{x \to \infty} \frac{6x}{\sqrt{3x^2 - 2x}} \approx 3.4641 \left(\text{exact: } \frac{6}{\sqrt{3}}\right)$

x	1	10	10^2	10^3	10^4	10^5
$f(x)$	6	3.5857	3.4757	3.4653	3.4642	3.4641



6. (a) $\lim_{x \to -1} \frac{2x^2 - x - 3}{x + 1} = \lim_{x \to -1} \frac{(2x - 3)(x + 1)}{x + 1} = \lim_{x \to -1} (2x - 3) = -5$

(b) $\lim_{x \to -1} \frac{2x^2 - x - 3}{x + 1} = \lim_{x \to -1} \frac{(d/dx)[2x^2 - x - 3]}{(d/dx)[x + 1]} = \lim_{x \to -1} \frac{4x - 1}{1} = -5$

8. (a) $\lim_{x \to 0} \frac{\sin 4x}{2x} = \lim_{x \to 0} 2\left(\frac{\sin 4x}{4x}\right) = 2(1) = 2$

(b) $\lim_{x \to 0} \frac{\sin 4x}{2x} = \lim_{x \to 0} \frac{(d/dx)[\sin 4x]}{(d/dx)[2x]} = \lim_{x \to 0} \frac{4 \cos 4x}{2} = 2$

10. (a) $\lim_{x \to \infty} \frac{2x + 1}{4x^2 + x} = \lim_{x \to \infty} \frac{(2/x) + (1/x^2)}{4 + (1/x)} = \frac{0}{4} = 0$

(b) $\lim_{x \to \infty} \frac{2x + 1}{4x^2 + x} = \lim_{x \to \infty} \frac{(d/dx)[2x + 1]}{(d/dx)[4x^2 + x]} = \lim_{x \to \infty} \frac{2}{8x + 1} = 0$

12. $\lim_{x \to -1} \frac{x^2 - x - 2}{x + 1} = \lim_{x \to -1} \frac{2x - 1}{1} = -3$

14. $\lim_{x \to 2^-} \frac{\sqrt{4 - x^2}}{x - 2} = \lim_{x \to 2^-} \frac{-x/\sqrt{4 - x^2}}{1}$

$= \lim_{x \to 2^-} \frac{-x}{\sqrt{4 - x^2}} = -\infty$

16. $\lim_{x \to 0^+} \frac{e^x - (1 + x)}{x^3} = \lim_{x \to 0^+} \frac{e^x - 1}{3x^2}$

$= \lim_{x \to 0^+} \frac{e^x}{6x} = \infty$

18. $\lim_{x \to 1} \frac{\ln x^2}{x^2 - 1} = \lim_{x \to 1} \frac{2 \ln x}{x^2 - 1}$

$= \lim_{x \to 1} \frac{2/x}{2x}$

$= \lim_{x \to 1} \frac{1}{x^2} = 1$

20. $\lim_{x \to 0} \frac{\sin ax}{\sin bx} = \lim_{x \to 0} \frac{a \cos ax}{b \cos bx} = \frac{a}{b}$

22. $\lim_{x \to 1} \frac{\arctan x - (\pi/4)}{x - 1} = \lim_{x \to 1} \frac{1/(1 + x^2)}{1} = \frac{1}{2}$

24. $\lim_{x \to \infty} \frac{x - 1}{x^2 + 2x + 3} = \lim_{x \to \infty} \frac{1}{2x + 2} = 0$

26. $\lim_{x \to \infty} \frac{x^3}{x + 1} = \lim_{x \to \infty} \frac{3x^2}{1} = \infty$

28. $\lim_{x \to \infty} \frac{x^2}{e^x} = \lim_{x \to \infty} \frac{2x}{e^x} = \lim_{x \to \infty} \frac{2}{e^x} = 0$

30. $\lim\limits_{x\to\infty} \dfrac{x^2}{\sqrt{x^2+1}} = \lim\limits_{x\to\infty} \dfrac{x}{\sqrt{1+(1/x)^2}} = \infty$

32. $\lim\limits_{x\to\infty} \dfrac{\sin x}{x-\pi} = 0$

Note: Use the Squeeze Theorem for $x > \pi$.

$$-\frac{1}{x-\pi} \le \frac{\sin x}{x-\pi} \le \frac{1}{x-\pi}$$

34. $\lim\limits_{x\to\infty} \dfrac{\ln x^4}{x^3} = \lim\limits_{x\to\infty} \dfrac{4\ln x}{x^3} = \lim\limits_{x\to\infty} \dfrac{4/x}{3x^2}$

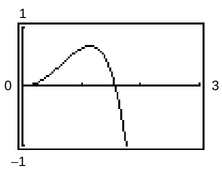
$= \lim\limits_{x\to\infty} \dfrac{4}{3x^3} = 0$

36. $\lim\limits_{x\to\infty} \dfrac{e^{x/2}}{x} = \lim\limits_{x\to\infty} \dfrac{(1/2)e^{x/2}}{1} = \infty$

38. (a) $\lim\limits_{x\to 0^+} x^3 \cot x = (0)(\infty)$

(b) $\lim\limits_{x\to 0^+} x^3 \cot x = \lim\limits_{x\to 0^+} \dfrac{x^3}{\tan x} = \lim\limits_{x\to 0^+} \dfrac{3x^2}{\sec^2 x} = 0$

(c)



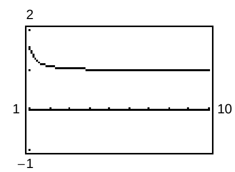
40. (a) $\lim\limits_{x\to\infty} \left(x \tan \dfrac{1}{x}\right) = (\infty)(0)$

(b) $\lim\limits_{x\to\infty} x \tan \dfrac{1}{x} = \lim\limits_{x\to\infty} \dfrac{\tan(1/x)}{1/x}$

$= \lim\limits_{x\to\infty} \dfrac{-(1/x^2)\sec^2(1/x)}{-(1/x^2)}$

$= \lim\limits_{x\to\infty} \sec^2\left(\dfrac{1}{x}\right) = 1$

(c)



42. (a) $\lim\limits_{x\to 0^+} (e^x + x)^{2/x} = 1^\infty$

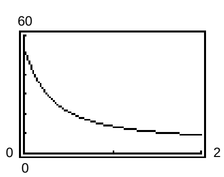
(b) Let $y = \lim\limits_{x\to 0^+} (e^x + x)^{2/x}$.

$\ln y = \lim\limits_{x\to 0^+} \dfrac{2\ln(e^x + x)}{x}$

$= \lim\limits_{x\to 0^+} \dfrac{2(e^x+1)/(e^x+x)}{1} = 4$

Thus, $\ln y = 4 \implies y = e^4 \approx 54.598$.

(c)



44. (a) $\lim\limits_{x\to\infty} \left(1 + \dfrac{1}{x}\right)^x = 1^\infty$

(b) Let $y = \lim\limits_{x\to\infty} \left(1 + \dfrac{1}{x}\right)^x$.

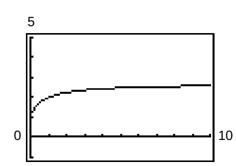
$\ln y = \lim\limits_{x\to\infty} \left[x \ln\left(1 + \dfrac{1}{x}\right)\right] = \lim\limits_{x\to\infty} \dfrac{\ln[1 + (1/x)]}{1/x}$

$= \lim\limits_{x\to\infty} \dfrac{\left[\dfrac{(-1/x^2)}{1 + (1/x)}\right]}{(-1/x^2)} = \lim\limits_{x\to\infty} \dfrac{1}{1 + (1/x)} = 1$

Thus, $\ln y = 1 \implies y = e^1 = e$. Therefore,

$\lim\limits_{x\to\infty} \left(1 + \dfrac{1}{x}\right)^x = e.$

(c)



46. (a) $\lim_{x \to \infty} (1 + x)^{1/x} = \infty^0$

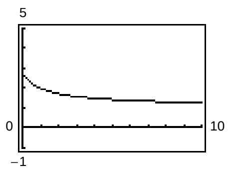
(b) Let $y = \lim_{x \to \infty} (1 + x)^{1/x}$.

$$\ln y = \lim_{x \to \infty} \frac{\ln(1 + x)}{x}$$

$$= \lim_{x \to \infty} \left(\frac{1/(1 + x)}{1}\right) = 0$$

Thus, $\ln y = 0 \implies y = e^0 = 1$.
Therefore, $\lim_{x \to \infty} (1 + x)^{1/x} = 1$.

(c)



48. (a) $\lim_{x \to 4^+} [3(x - 4)]^{x-4} = 0^0$

(b) Let $y = \lim_{x \to 4^+} [3(x - 4)]^{x-4}$.

$$\ln y = \lim_{x \to 4^+} (x - 4)\ln[3(x - 4)]$$

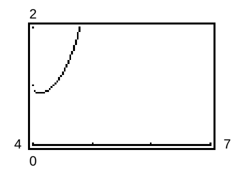
$$= \lim_{x \to 4^+} \frac{\ln[3(x - 4)]}{1/(x - 4)}$$

$$= \lim_{x \to 4^+} \frac{1/(x - 4)}{-1/(x - 4)^2}$$

$$= \lim_{x \to 4^+} [-(x - 4)] = 0$$

Hence, $\lim_{x \to 4^+} [3(x - 4)]^{x-4} = 1$.

(c)



50. (a) $\lim_{x \to 0^+} \left[\cos\left(\frac{\pi}{2} - x\right)\right]^x = 0^0$

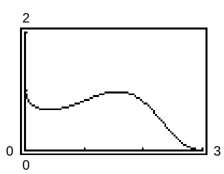
(b) Let $y = \lim_{x \to 0^+} \left[\cos\left(\frac{\pi}{2} - x\right)\right]^x$.

$$\ln y = \lim_{x \to 0^+} x \ln\left[\cos\left(\frac{\pi}{2} - x\right)\right]$$

$$= 0 \cdot 0 = 0$$

Hence, $\lim_{x \to 0^+} \left[\cos\left(\frac{\pi}{2} - x\right)\right]^x = 1$.

(c)



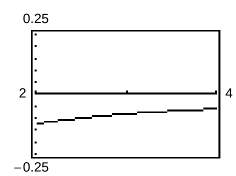
52. (a) $\lim_{x \to 2^+} \left(\frac{1}{x^2 - 4} - \frac{\sqrt{x - 1}}{x^2 - 4}\right) = \infty - \infty$

(b) $$\lim_{x \to 2^+} \left(\frac{1}{x^2 - 4} - \frac{\sqrt{x - 1}}{x^2 - 4}\right) = \lim_{x \to 2^+} \frac{1 - \sqrt{x - 1}}{x^2 - 4}$$

$$= \lim_{x \to 2^+} \frac{-1/\left(2\sqrt{x - 1}\right)}{2x}$$

$$= \lim_{x \to 2^+} \frac{-1}{4x\sqrt{x - 1}} = \frac{-1}{8}$$

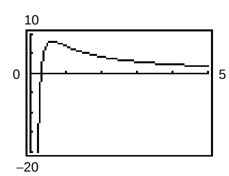
(c)



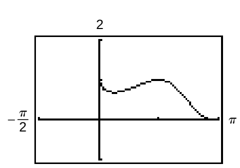
54. (a) $\lim_{x \to 0^+} \left(\frac{10}{x} - \frac{3}{x^2}\right) = \infty - \infty$

(b) $\lim_{x \to 0^+} \left(\frac{10}{x} - \frac{3}{x^2}\right) = \lim_{x \to 0^+} \left(\frac{10x - 3}{x^2}\right) = -\infty$

(c)



56. (a)

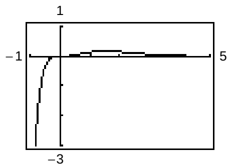


(b) Let $y = (\sin x)^x$, then $\ln y = x \ln(\sin x)$.

$$\lim_{x \to 0^+} \frac{\ln(\sin x)}{1/x} = \lim_{x \to 0^+} \frac{\cos x/\sin x}{-1/x^2} = \lim_{x \to 0^+} \frac{-x^2}{\tan x} = \lim_{x \to 0^+} \frac{-2x}{\sec^2 x} = 0$$

Therefore, since $\ln y = 0$, $y = 1$ and $\lim_{x \to 0^+} (\sin x)^x = 1$.

58. (a)



(b) $\lim_{x\to\infty} \frac{x^3}{e^{2x}} = \lim_{x\to\infty} \frac{3x^2}{2e^{2x}} = \lim_{x\to\infty} \frac{6x}{4e^{2x}} = \lim_{x\to\infty} \frac{6}{8e^{2x}} = 0$

60. See Theorem 7.4.

62. Let $f(x) = x + 25$ and $g(x) = x$.

64. $\lim_{x\to\infty} \frac{x^3}{e^{2x}} = \lim_{x\to\infty} \frac{3x^2}{2e^{2x}} = \lim_{x\to\infty} \frac{6x}{4e^{2x}} = \lim_{x\to\infty} \frac{6}{8e^{2x}} = 0$

66.
$$\lim_{x\to\infty} \frac{(\ln x)^2}{x^3} = \lim_{x\to\infty} \frac{(2\ln x)/x}{3x^2}$$
$$= \lim_{x\to\infty} \frac{2\ln x}{3x^3}$$
$$= \lim_{x\to\infty} \frac{2/x}{9x^2} = \lim_{x\to\infty} \frac{2}{9x^3} = 0$$

68.
$$\lim_{x\to\infty} \frac{x^m}{e^{nx}} = \lim_{x\to\infty} \frac{mx^{m-1}}{ne^{nx}}$$
$$= \lim_{x\to\infty} \frac{m(m-1)x^{m-2}}{n^2e^{nx}}$$
$$= \cdots = \lim_{x\to\infty} \frac{m!}{n^m e^{nx}} = 0$$

70.

x	1	5	10	20	30	40	50	100
$\frac{e^x}{x^5}$	2.718	0.047	0.220	151.614	4.40×10^5	2.30×10^9	1.66×10^{13}	2.69×10^{33}

72. $y = x^x, x > 0$

$\lim_{x\to\infty} x^x = \infty$ and $\lim_{x\to 0^+} x^x = 1$

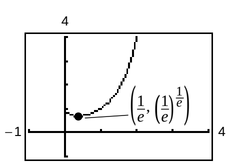
No horizontal asymptotes

$$\ln y = x \ln x$$
$$\frac{1}{y}\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln x$$
$$\frac{dy}{dx} = x^x(1 + \ln x) = 0$$

Critical number: $x = e^{-1}$

Intervals:	$(0, e^{-1})$	$(e^{-1}, 0)$
Sign of dy/dx:	$-$	$+$
$y = f(x)$:	Decreasing	Increasing

Relative minimum: $(e^{-1}, (e^{-1})^{e^{-1}}) = \left(\frac{1}{e}, \left(\frac{1}{e}\right)^{1/e}\right)$



74. $y = \frac{\ln x}{x}$

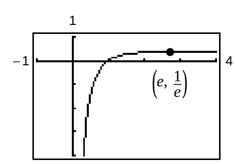
Horizontal asymptote: $y = 0$ (See Exercise 29)

$$\frac{dy}{dx} = \frac{x(1/x) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2} = 0$$

Critical number: $x = e$

Intervals:	$(0, e)$	(e, ∞)
Sign of dy/dx:	$+$	$-$
$y = f(x)$:	Increasing	Decreasing

Relative maximum: $\left(e, \frac{1}{e}\right)$



76. $\lim_{x\to\infty} \frac{\sin \pi x - 1}{x} = 0$ (Numerator is bounded)

Limit is not of the form $0/0$ or ∞/∞.
L'Hôpital's Rule does not apply.

78. $\lim_{x\to\infty} \frac{e^{-x}}{1 + e^{-x}} = \frac{0}{1 + 0} = 0$

Limit is not of the form $0/0$ or ∞/∞.
L'Hôpital's Rule does not apply.

80. $A = P\left(1 + \frac{r}{n}\right)^{nt}$

$$\ln A = \ln P + nt \ln\left(1 + \frac{r}{n}\right) = \ln P + \frac{\ln\left(1 + \frac{r}{n}\right)}{\frac{1}{nt}}$$

$$\lim_{n\to\infty}\left[\frac{\ln\left(1 + \frac{r}{n}\right)}{\frac{1}{nt}}\right] = \lim_{n\to\infty}\left[\frac{-\frac{r}{n^2}\left(\frac{1}{1 + (r/n)}\right)}{-\left(\frac{1}{n^2 t}\right)}\right] = \lim_{n\to\infty}\left[rt\left(\frac{1}{1 + \frac{r}{n}}\right)\right] = rt$$

Since $\lim_{n\to\infty} \ln A = \ln P + rt$, we have $\lim_{n\to\infty} A = e^{(\ln P + rt)} = e^{\ln P}e^{rt} = Pe^{rt}$. Alternatively,

$$\lim_{n\to\infty} A = \lim_{n\to\infty} P\left(1 + \frac{r}{n}\right)^{nt} = \lim_{n\to\infty} P\left[\left(1 + \frac{r}{n}\right)^{n/r}\right]^{rt} = Pe^{rt}.$$

82. Let N be a fixed value for n. Then

$$\lim_{x\to\infty} \frac{x^{N-1}}{e^x} = \lim_{x\to\infty} \frac{(N-1)x^{N-2}}{e^x} = \lim_{x\to\infty} \frac{(N-1)(N-2)x^{N-3}}{e^x} = \ldots = \lim_{x\to\infty}\left[\frac{(N-1)!}{e^x}\right] = 0.$$ (See Exercise 68)

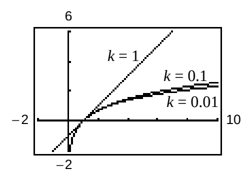
84. $f(x) = \dfrac{x^k - 1}{k}$

$k = 1, \quad f(x) = x - 1$

$k = 0.1, \quad f(x) = \dfrac{x^{0.1} - 1}{0.1} = 10(x^{0.1} - 1)$

$k = 0.01, \quad f(x) = \dfrac{x^{0.01} - 1}{0.01} = 100(x^{0.01} - 1)$

$$\lim_{k\to 0^+} \frac{x^k - 1}{k} = \lim_{k\to 0^+} \frac{x^k(\ln x)}{1} = \ln x$$



86. $f(x) = \frac{1}{x}$, $g(x) = x^2 - 4$, $[1, 2]$

$$\frac{f(2) - f(1)}{g(2) - g(1)} = \frac{f'(c)}{g'(c)}$$

$$\frac{-1/2}{3} = \frac{-1/c^2}{2c}$$

$$-\frac{1}{6} = -\frac{1}{2c^3}$$

$$2c^3 = 6$$

$$c = \sqrt[3]{3}$$

88. $f(x) = \ln x$, $g(x) = x^3$, $[1, 4]$

$$\frac{f(4) - f(1)}{g(4) - g(1)} = \frac{f'(c)}{g'(c)}$$

$$\frac{\ln 4}{63} = \frac{1/c}{3c^2} = \frac{1}{3c^3}$$

$$3c^3 \ln 4 = 63$$

$$c^3 = \frac{21}{\ln 4}$$

$$c = \sqrt[3]{\frac{21}{\ln 4}} \approx 2.474$$

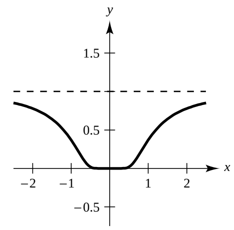
90. False. If $y = e^x/x^2$, then

$$y' = \frac{x^2e^x - 2xe^x}{x^4} = \frac{xe^x(x - 2)}{x^4} = \frac{e^x(x - 2)}{x^3}.$$

92. False. Let $f(x) = x$ and $g(x) = x + 1$. Then

$$\lim_{x\to\infty} \frac{x}{x + 1} = 1, \text{ but } \lim_{x\to\infty} [x - (x + 1)] = -1.$$

94. $g(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$



$$g'(0) = \lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \frac{e^{-1/x^2}}{x}$$

Let $y = \dfrac{e^{-1/x^2}}{x}$, then $\ln y = \ln\left(\dfrac{e^{-1/x^2}}{x}\right) = -\dfrac{1}{x^2} - \ln x = \dfrac{-1 - x^2 \ln x}{x^2}$. Since

$$\lim_{x \to 0} x^2 \ln x = \lim_{x \to 0} \frac{\ln x}{1/x^2} = \lim_{x \to 0} \frac{1/x}{-2/x^3} = \lim_{x \to 0} \left(-\frac{x^2}{2}\right) = 0$$

we have $\displaystyle\lim_{x \to 0} \left(\frac{-1 - x^2 \ln x}{x^2}\right) = -\infty$. Thus, $\displaystyle\lim_{x \to 0} y = e^{-\infty} = 0 \implies g'(0) = 0$.

Note: The graph appears to support this conclusion—the tangent line is horizontal at $(0, 0)$.

96. $\displaystyle\lim_{x \to a} f(x)^{g(x)}$

$$y = f(x)^{g(x)}$$

$$\ln y = g(x) \ln f(x)$$

$$\lim_{x \to a} g(x) \ln f(x) = (-\infty)(-\infty) = \infty$$

As $x \to a$, $\ln y \implies \infty$, and hence $y = \infty$. Thus,

$$\lim_{x \to a} f(x)^{g(x)} = \infty.$$

98. $\displaystyle\lim_{x \to 0^+} x^{\ln 2/(1 + \ln x)}$

Let $y = x^{\ln 2/(1 + \ln x)}$, then:

$$\ln y = \frac{\ln 2}{1 + \ln x} \cdot \ln x = \frac{(\ln 2)(\ln x)}{1 + \ln x}$$

$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} \frac{(\ln 2)(\ln x)}{1 + \ln x} = \lim_{x \to 0^+} \frac{(\ln 2)/x}{1/x}$$

$$= \lim_{x \to 0^+} (\ln 2) = \ln 2$$

Thus, $\displaystyle\lim_{x \to \infty} y = e^{\ln 2} = 2$.

Section 7.8 Improper Integrals

2. Infinite discontinuity at $x = 3$.

$$\int_3^4 \frac{1}{(x - 3)^{3/2}}\, dx = \lim_{b \to 3^+} \int_b^4 (x - 3)^{-3/2}\, dx$$

$$= \lim_{b \to 3^+} \Big[-2(x - 3)^{-1/2}\Big]_b^4$$

$$= \lim_{b \to 3^+} \left[-2 + \frac{2}{\sqrt{b - 3}}\right] = \infty$$

Diverges

4. Infinite discontinuity at $x = 1$.

$$\int_0^2 \frac{1}{(x - 1)^{2/3}}\, dx = \int_0^1 \frac{1}{(x - 1)^{2/3}}\, dx + \int_1^2 \frac{1}{(x - 1)^{2/3}}\, dx$$

$$= \lim_{b \to 1^-} \int_0^b \frac{1}{(x - 1)^{2/3}}\, dx + \lim_{c \to 1^+} \int_c^2 \frac{1}{(x - 1)^{2/3}}\, dx$$

$$= \lim_{b \to 1^-} \Big[3\sqrt[3]{x - 1}\Big]_0^b + \lim_{c \to 1^+} \Big[3\sqrt[3]{x - 1}\Big]_c^2 = (0 + 3) + (3 - 0) = 6$$

Converges

6. Infinite limit of integration.

$$\int_{-\infty}^{0} e^{2x}\,dx = \lim_{b\to-\infty}\int_{b}^{0} e^{2x}\,dx$$
$$= \lim_{b\to-\infty}\left[\frac{1}{2}e^{2x}\right]_b^0 = \frac{1}{2} - 0 = \frac{1}{2}$$

Converges

8. $\int_0^{\infty} e^{-x}\,dx \neq 0$. You need to evaluate the limit.

$$\lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty}\left[-e^{-x}\right]_0^b$$
$$= \lim_{b\to\infty}\left[-e^{-b} + 1\right] = 1$$

10. $\int_1^{\infty}\frac{5}{x^3}\,dx = \lim_{b\to\infty}\int_1^b \frac{5}{x^3}\,dx$

$$= \lim_{b\to\infty}\left[-\frac{5}{2}x^{-2}\right]_1^b = \frac{5}{2}$$

12. $\int_1^{\infty}\frac{4}{\sqrt[4]{x}}\,dx = \lim_{b\to\infty}\int_1^b 4x^{-1/4}\,dx$

$$= \lim_{b\to\infty}\left[\frac{16}{3}x^{3/4}\right]_1^b = \infty \quad \text{Diverges}$$

14. $\int_0^{\infty} xe^{-x/2}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x/2}\,dx = \lim_{b\to\infty}\left[e^{-x/2}(-2x-4)\right]_0^b = \lim_{b\to\infty} e^{-b/2}(-2b-4) + 4 = 4$

16. $\int_0^{\infty}(x-1)e^{-x}\,dx = \lim_{b\to\infty}\int_0^b (x-1)e^{-x}\,dx = \lim_{b\to\infty}\left[-xe^{-x}\right]_0^b = \lim_{b\to\infty}\left(\frac{-b}{e^b} + 0\right) = 0$ by L'Hôpital's Rule.

18. $\int_0^{\infty} e^{-ax}\sin bx\,dx = \lim_{c\to\infty}\left[\frac{e^{-ax}(-a\sin bx - b\cos bx)}{a^2+b^2}\right]_0^c$

$$= 0 - \frac{-b}{a^2+b^2} = \frac{b}{a^2+b^2}$$

20. $\int_1^{\infty}\frac{\ln x}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x}\,dx$

$$= \lim_{b\to\infty}\left[\frac{(\ln x)^2}{2}\right]_1^b = \infty \quad \text{Diverges}$$

22. $\int_0^{\infty}\frac{x^3}{(x^2+1)^2}\,dx = \lim_{b\to\infty}\int_0^b \frac{x}{x^2+1}\,dx - \lim_{b\to\infty}\int_0^b \frac{x}{(x^2+1)^2}\,dx$

$$= \lim_{b\to\infty}\left[\frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)}\right]_0^b$$
$$= \infty - \frac{1}{2}$$

Diverges

24. $\int_0^{\infty}\frac{e^x}{1+e^x}\,dx = \lim_{b\to\infty}\left[\ln(1+e^x)\right]_0^b = \infty - \ln 2$

Diverges

26. $\int_0^{\infty}\sin\frac{x}{2}\,dx = \lim_{b\to\infty}\left[-2\cos\frac{x}{2}\right]_0^b$

Diverges since $\cos\frac{x}{2}$ does not approach a limit as $x\to\infty$.

28. $\int_0^4 \frac{8}{x}\,dx = \lim_{b\to 0^+}\int_b^4 \frac{8}{x}\,dx = \lim_{b\to 0^+}\left[8\ln x\right]_b^4 = \infty$

Diverges

30. $\int_0^6 \frac{4}{\sqrt{6-x}}\,dx = \lim_{b\to 6^-}\int_0^b 4(6-x)^{-1/2}\,dx$

$$= \lim_{b\to 6^-}\left[-8(6-x)^{1/2}\right]_0^b$$
$$= -8(0) + 8\sqrt{6}$$
$$= 8\sqrt{6}$$

32. $\int_0^e \ln x^2\,dx = \lim_{b\to 0^+}\int_0^e 2\ln x\,dx$

$$= \lim_{b\to 0^+}\left[2x\ln x - 2x\right]_b^e$$
$$= \lim_{b\to 0^+}\left[(2e - 2e) - (2b\ln b - 2b)\right]$$
$$= 0$$

34. $\int_0^{\pi/2}\sec\theta\,d\theta = \lim_{b\to(\pi/2)}\left[\ln|\sec\theta + \tan\theta|\right]_0^b = \infty,$

Diverges

36. $\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx = \lim_{b\to 2^-}\left[\arcsin\left(\frac{x}{2}\right)\right]_0^b = \frac{\pi}{2}$

38. $\displaystyle\int_0^2 \frac{1}{4-x^2}\,dx = \lim_{b\to 2^-}\int_0^b \frac{1}{4}\left(\frac{1}{2+x}+\frac{1}{2-x}\right)dx = \lim_{b\to 2^-}\left[\frac{1}{4}\ln\left|\frac{2+x}{2-x}\right|\right]_0^b = \infty - 0$

Diverges

40. $\displaystyle\int_1^3 \frac{2}{(x-2)^{8/3}}\,dx = \int_1^2 2(x-2)^{-8/3}\,dx + \int_2^3 2(x-2)^{-8/3}\,dx$

$\displaystyle= \lim_{b\to 2^-}\int_1^b 2(x-2)^{-8/3}\,dx + \lim_{c\to 2^+}\int_c^3 2(x-2)^{-8/3}\,dx$

$\displaystyle= \lim_{b\to 2^-}\left[-\frac{6}{5}(x-2)^{-5/3}\right]_1^b + \lim_{c\to 2^+}\left[-\frac{6}{5}(x-2)^{-5/3}\right]_c^3 = \infty$

Diverges

42. $\displaystyle\int \frac{1}{x\ln x}\,dx = \ln|\ln|x|| + C$

Thus,

$\displaystyle\int_1^\infty \frac{1}{x\ln x}\,dx = \int_1^e \frac{1}{x\ln x}\,dx + \int_e^\infty \frac{1}{x\ln x}\,dx$

$\displaystyle= \lim_{b\to 1^+}\Big[\ln(\ln x)\Big]_b^e + \lim_{c\to\infty}\Big[\ln(\ln x)\Big]_e^c.$

Diverges

44. If $p = 1$, $\displaystyle\int_0^1 \frac{1}{x}\,dx = \lim_{a\to 0^+}\Big[\ln x\Big]_a^1 = \lim_{a\to 0^+} -\ln a = \infty.$

Diverges. If $p \neq 1$,

$\displaystyle\int_0^1 \frac{1}{x^p}\,dx = \lim_{a\to 0^+}\left[\frac{x^{1-p}}{1-p}\right]_a^1 = \lim_{a\to 0^+}\left[\frac{1}{1-p} - \frac{a^{1-p}}{1-p}\right].$

This converges to $\dfrac{1}{1-p}$ if $1 - p > 0$ or $p < 1$.

46. (a) Assume $\displaystyle\int_a^\infty g(x)\,dx = L$ (converges).

Since $0 \le f(x) \le g(x)$ on $[a, \infty)$, $\displaystyle 0 \le \int_a^\infty f(x)\,dx \le \int_a^\infty g(x)\,dx = L$ and $\displaystyle\int_a^\infty f(x)\,dx$ converges.

(b) $\displaystyle\int_a^\infty g(x)\,dx$ diverges, because otherwise, by part (a), if $\displaystyle\int_a^\infty g(x)\,dx$ converges, then so does $\displaystyle\int_a^\infty f(x)\,dx$.

48. $\displaystyle\int_0^1 \frac{1}{\sqrt[3]{x}}\,dx = \frac{1}{1-(1/3)} = \frac{3}{2}$ converges.

(See Exercise 44, $p = \frac{1}{3}$.)

50. $\displaystyle\int_0^\infty x^4 e^{-x}\,dx$ converges.

(See Exercise 45.)

52. Since $\dfrac{1}{\sqrt{x-1}} \ge \dfrac{1}{x}$ on $[2, \infty)$ and $\displaystyle\int_2^\infty \frac{1}{x}\,dx$ diverges by Exercise 43, $\displaystyle\int_2^\infty \frac{1}{\sqrt{x-1}}\,dx$ diverges.

54. Since $\dfrac{1}{\sqrt{x}(1+x)} \le \dfrac{1}{x^{3/2}}$ on $[1, \infty)$ and $\displaystyle\int_1^\infty \frac{1}{x^{3/2}}\,dx$ converges by Exercise 43, $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}(1+x)}\,dx$ converges.

56. $\dfrac{1}{\sqrt{x}\ln x} \ge \dfrac{1}{x}$ since $\sqrt{x}\ln x < x$ on $[2, \infty)$. Since $\displaystyle\int_2^\infty \frac{1}{x}\,dx$ diverges by Exercise 43, $\displaystyle\int_2^\infty \frac{1}{\sqrt{x}\ln x}\,dx$ diverges.

58. See the definitions, pages 540, 543.

60. Answers will vary.

(a) $\displaystyle\int_{-\infty}^\infty \frac{e^x}{1+e^{2x}}\,dx$ Converges (Example 4)

(b) $\displaystyle\int_{-\infty}^\infty x\,dx$ Diverges

62. $f(t) = t$

$\displaystyle F(s) = \int_0^\infty te^{-st}\,dt = \lim_{b\to\infty}\left[\frac{1}{s^2}(-st-1)e^{-st}\right]_0^b$

$\displaystyle= \frac{1}{s^2},\ s > 0$

64. $f(t) = e^{at}$

$$F(s) = \int_0^\infty e^{at}e^{-st}\,dt = \int_0^\infty e^{t(a-s)}\,dt$$
$$= \lim_{b\to\infty}\left[\frac{1}{a-s}e^{t(a-s)}\right]_0^b$$
$$= 0 - \frac{1}{a-s} = \frac{1}{s-a},\ s > a$$

66. $f(t) = \sin at$

$$F(s) = \int_0^\infty e^{-st}\sin at\,dt$$
$$= \lim_{b\to\infty}\left[\frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right]_0^b$$
$$= 0 + \frac{a}{s^2+a^2} = \frac{a}{s^2+a^2},\ s > 0$$

68. $f(t) = \sinh at$

$$F(s) = \int_0^\infty e^{-st}\sinh at\,dt = \int_0^\infty e^{-st}\left(\frac{e^{at}-e^{-at}}{2}\right)dt = \frac{1}{2}\int_0^\infty\left[e^{t(-s+a)} - e^{t(-s-a)}\right]dt$$
$$= \lim_{b\to\infty}\frac{1}{2}\left[\frac{1}{(-s+a)}e^{t(-s+a)} - \frac{1}{(-s-a)}e^{t(-s-a)}\right]_0^b = 0 - \frac{1}{2}\left[\frac{1}{(-s+a)} - \frac{1}{(-s-a)}\right]$$
$$= \frac{-1}{2}\left[\frac{1}{(-s+a)} - \frac{1}{(-s-a)}\right] = \frac{a}{s^2-a^2},\ s > |a|$$

70. (a) $A = \int_1^\infty \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^\infty = 1$

(b) **Disk:**

$$V = \pi\int_1^\infty \frac{1}{x^4}\,dx = \lim_{b\to\infty}\left[-\frac{\pi}{3x^3}\right]_1^b = \frac{\pi}{3}$$

(c) **Shell:**

$$V = 2\pi\int_1^\infty x\left(\frac{1}{x^2}\right)dx = \lim_{b\to\infty}\left[2\pi(\ln x)\right]_1^b = \infty$$

Diverges

72. $(x-2)^2 + y^2 = 1$

$$2(x-2) + 2yy' = 0$$
$$y' = \frac{-(x-2)}{y}$$
$$\sqrt{1+(y')^2} = \sqrt{1+[(x-2)^2/y^2]} = \frac{1}{y}\ \text{(Assume } y > 0.)$$
$$S = 4\pi\int_1^3 \frac{x}{y}\,dx = 4\pi\int_1^3 \frac{x}{\sqrt{1-(x-2)^2}}\,dx = 4\pi\int_1^3\left[\frac{x-2}{\sqrt{1-(x-2)^2}} + \frac{2}{\sqrt{1-(x-2)^2}}\right]dx$$
$$= \lim_{\substack{a\to1^+\\ b\to3^-}}\left\{4\pi\left[-\sqrt{1-(x-2)^2} + 2\arcsin(x-2)\right]_a^b\right\} = 4\pi[0 + 2\arcsin(1) - 2\arcsin(-1)] = 8\pi^2$$

74. (a) $F(x) = \frac{K}{x^2},\ 5 = \frac{K}{(4000)^2},\ K = 80{,}000{,}000$

$$W = \int_{4000}^\infty \frac{80{,}000{,}000}{x^2}\,dx = \lim_{b\to\infty}\left[\frac{-80{,}000{,}000}{x}\right]_{4000}^b = 20{,}000 \text{ mi-ton}$$

(b) $$\frac{W}{2} = 10{,}000 = \left[\frac{-80{,}000{,}000}{x}\right]_{4000}^b = \frac{-80{,}000{,}000}{b} + 20{,}000$$

$$\frac{80{,}000{,}000}{b} = 10{,}000$$
$$b = 8000$$

Therefore, 4000 miles *above* the earth's surface.

76. (a) $\displaystyle\int_{-\infty}^{\infty} \frac{2}{5}e^{-2t/5}\,dt = \int_0^{\infty} \frac{2}{5}e^{-2t/5}\,dt = \lim_{b\to\infty}\Big[-e^{-2t5}\Big]_0^b = 1$

(b) $\displaystyle\int_0^4 \frac{2}{5}e^{-2t/5}\,dt = \Big[-e^{-2t/5}\Big]_0^4 = -e^{-8/5} + 1$

$\approx 0.7981 = 79.81\%$

(c) $\displaystyle\int_0^{\infty} t\left[\frac{2}{5}e^{-2t/5}\right]dt = \lim_{b\to\infty}\left[-te^{2t/5} - \frac{5}{2}e^{-2t/5}\right]_0^b = \frac{5}{2}$

78. (a) $\displaystyle C = 650{,}000 + \int_0^5 25{,}000(1 + 0.08t)e^{-0.06t}\,dt$

$\displaystyle = 650{,}000 + 25{,}000\left[-\frac{1}{0.06}e^{-0.06t} - 0.08\left(\frac{t}{0.06}e^{-0.06t} + \frac{1}{(0.06)^2}e^{-0.06t}\right)\right]_0^5 \approx \$778{,}512.58$

(b) $\displaystyle C = 650{,}000 + \int_0^{10} 25{,}000(1 + 0.08t)e^{-0.06t}\,dt$

$\displaystyle = 650{,}000 + 25{,}000\left[-\frac{1}{0.06}e^{-0.06t} - 0.08\left(\frac{t}{0.06}e^{-0.06t} + \frac{1}{(0.06)^2}e^{-0.06t}\right)\right]_0^{10} \approx \$905{,}718.14$

(c) $\displaystyle C = 650{,}000 + \int_0^{\infty} 25{,}000(1 + 0.08t)e^{-0.06t}\,dt$

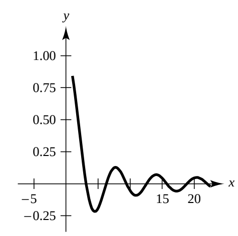
$\displaystyle = 650{,}000 + 25{,}000\lim_{b\to\infty}\left[-\frac{t}{0.06}e^{-0.06t} - 0.08\left(\frac{t}{0.06}e^{-0.06t+}\frac{1}{(0.06)^2}e^{-0.06t}\right)\right]_0^b \approx \$1{,}622{,}222.22$

80. (a) $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty}\Big[\ln|x|\Big]_1^b = \infty$

$\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = 1$

$\displaystyle\int_1^{\infty} \frac{1}{x^n}\,dx$ will converge if $n > 1$ and will diverge if $n \le 1$.

(b) It would appear to converge.



(c) Let $dv = \sin x\,dx \implies v = -\cos x$

$\displaystyle u = \frac{1}{x} \implies du = -\frac{1}{x^2}\,dx$

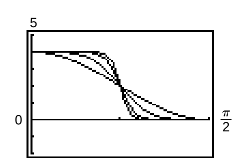
$\displaystyle\int_1^{\infty} \frac{\sin x}{x}\,dx = \lim_{b\to 0}\left[-\frac{\cos x}{x}\right]_1^b - \int_1^{\infty} \frac{\cos x}{x^2}\,dx$

$\displaystyle = \cos 1 - \int_1^{\infty} \frac{\cos x}{x^2}\,dx$

Converges

82. (a) Yes, the integral is not defined at $x = \pi/2$.

(b)



(c) As $n \to \infty$, the integral approaches $4(\pi/4) = \pi$.

(d) $\displaystyle I_n = \int_0^{\pi/2} \frac{4}{1 + (\tan x)^n}\,dx$

$I_2 \approx 3.14159$

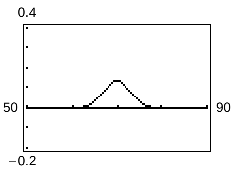
$I_4 \approx 3.14159$

$I_8 \approx 3.14159$

$I_{12} \approx 3.14159$

84. (a) $f(x) = \dfrac{1}{3\sqrt{2\pi}} e^{-(x-70)^2/18}$

$\displaystyle\int_{50}^{90} f(x)\,dx \approx 1.0$



(b) $P(72 \le x < \infty) \approx 0.2525$

(c) $0.5 - P(70 \le x \le 72) \approx 0.5 - 0.2475 = 0.2525$

These are the same answers because by symmetry,

$P(70 \le x < \infty) = 0.5$

and

$$0.5 = P(70 \le x < \infty)$$
$$= P(70 \le x \le 72) + P(72 \le x < \infty).$$

86. False. This is equivalent to Exercise 85.

88. True

Review Exercises for Chapter 7

2. $\displaystyle\int xe^{x^2-1}\,dx = \frac{1}{2}\int e^{x^2-1}(2x)\,dx$

$$= \frac{1}{2}e^{x^2-1} + C$$

4. $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int (1-x^2)^{-1/2}(-2x)\,dx$

$$= -\frac{1}{2}\frac{(1-x^2)^{1/2}}{1/2} + C$$

$$= -\sqrt{1-x^2} + C$$

6. $\displaystyle\int 2x\sqrt{2x-3}\,dx = \int (u^4 + 3u^2)\,du = \frac{u^5}{5} + u^3 + C$

$$= \frac{2(2x-3)^{3/2}}{5}(x+1) + C$$

$u = \sqrt{2x-3},\ x = \dfrac{u^2+3}{2},\ dx = u\,du$

8. $\dfrac{x^4 + 2x^2 + x + 1}{x^4 + 2x^2 + 1} = 1 + \dfrac{x}{(x^2+1)^2}$

$$\int \frac{x^4 + 2x^2 + x + 1}{(x^2+1)^2}\,dx = \int dx + \frac{1}{2}\int \frac{2x}{(x^2+1)^2}\,dx$$

$$= x - \frac{1}{2(x^2+1)} + C$$

10. $\displaystyle\int (x^2-1)e^x\,dx = (x^2-1)e^x - 2\int xe^x\,dx = (x^2-1)e^x - 2xe^x + 2\int e^x\,dx = e^x(x^2 - 2x + 1) + 1$

(1) $dv = e^x\,dx \implies v = e^x$

$u = x^2 - 1 \implies du = 2x\,dx$

(2) $dv = e^x\,dx \implies v = e^x$

$u = x \implies du = dx$

12. $u = \arctan 2x,\ du = \dfrac{2}{1+4x^2}\,dx,\ dv = dx, v = x$

$$\int \arctan 2x\,dx = x\arctan 2x - \int \frac{2x}{1+4x^2}\,dx$$

$$= x\arctan 2x - \frac{1}{4}\ln(1+4x^2) + C$$

14. $\displaystyle\int \ln\sqrt{x^2-1}\,dx = \frac{1}{2}\int \ln(x^2-1)\,dx$

$$= \frac{1}{2}x\ln|x^2-1| - \int \frac{x^2}{x^2-1}\,dx$$

$$= \frac{1}{2}x\ln|x^2-1| - \int dx - \int \frac{1}{x^2-1}\,dx$$

$$= \frac{1}{2}x\ln|x^2-1| - x - \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$$

$dv = dx \implies v = x$

$u = \ln(x^2-1) \implies du = \dfrac{2x}{x^2-1}\,dx$

16. $\int e^x \arctan(e^x)\,dx = e^x \arctan(e^x) - \int \frac{e^{2x}}{1+e^{2x}}\,dx$

$$= e^x \arctan(e^x) - \frac{1}{2}\ln(1+e^{2x}) + C$$

$dv = e^x\,dx \implies v = e^x$

$u = \arctan e^x \implies du = \frac{e^x}{1+e^{2x}}\,dx$

18. $\int \sin^2 \frac{\pi x}{2}\,dx = \int \frac{1}{2}(1 - \cos \pi x)\,dx = \frac{1}{2}\left[x - \frac{1}{\pi}\sin \pi x\right] + C = \frac{1}{2\pi}[\pi x - \sin \pi x] + C$

20. $\int \tan\theta \sec^4\theta\,d\theta = \int (\tan^3\theta + \tan\theta)\sec^2\theta\,d\theta = \frac{1}{4}\tan^4\theta + \frac{1}{2}\tan^2\theta + C_1$

or

$\int \tan\theta \sec^4\theta\,d\theta = \int \sec^3\theta(\sec\theta\tan\theta)\,d\theta = \frac{1}{4}\sec^4\theta + C_2$

22. $\int \cos 2\theta(\sin\theta + \cos\theta)^2\,d\theta = \int (\cos^2\theta - \sin^2\theta)(\sin\theta + \cos\theta)^2\,d\theta$

$$= \int (\sin\theta + \cos\theta)^3(\cos\theta - \sin\theta)\,d\theta = \frac{1}{4}(\sin\theta + \cos\theta)^4 + C$$

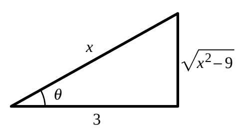
24. $\int \frac{\sqrt{x^2-9}}{x}\,dx = \int \frac{3\tan\theta}{3\sec\theta}(3\sec\theta\tan\theta\,d\theta)$

$$= 3\int \tan^2\theta\,d\theta$$

$$= 3\int (\sec^2\theta - 1)\,d\theta$$

$$= 3(\tan\theta - \theta) + C$$

$$= \sqrt{x^2-9} - 3\,\text{arcsec}\left(\frac{x}{3}\right) + C$$



$x = 3\sec\theta,\ dx = 3\sec\theta\tan\theta\,d\theta,\ \sqrt{x^2-9} = 3\tan\theta$

26. $\int \sqrt{9-4x^2}\,dx = \frac{1}{2}\int \sqrt{9-(2x)^2}\,(2)\,dx$

$$= \frac{1}{2}\cdot\frac{1}{2}\left[9\arcsin\frac{2x}{3} + 2x\sqrt{9-4x^2}\right] + C$$

$$= \frac{9}{4}\arcsin\frac{2x}{3} + \frac{x}{2}\sqrt{9-4x^2} + C$$

28. $\int \frac{\sin\theta}{1+2\cos^2\theta}\,d\theta = \frac{-1}{\sqrt{2}}\int \frac{1}{1+2\cos^2\theta}\left(-\sqrt{2}\sin\theta\right)d\theta$

$$= \frac{-1}{\sqrt{2}}\arctan\left(\sqrt{2}\cos\theta\right) + C$$

$u = \sqrt{2}\cos\theta,\ du = -\sqrt{2}\sin\theta\,d\theta$

30. (a) $$\int x\sqrt{4+x}\,dx = 64\int \tan^3\theta\sec^3\theta\,d\theta$$
$$= 64\int(\sec^4\theta - \sec^2\theta)\sec\theta\tan\theta\,d\theta$$
$$= \frac{64\sec^3\theta}{15}(3\sec^3\theta - 5) + C$$
$$= \frac{2(4+x)^{3/2}}{15}(3x-8) + C$$

$x = 4\tan^2\theta,\ dx = 8\tan\theta\sec^2\theta\,d\theta,$

$\sqrt{4+x} = 2\sec\theta$

(b) $$\int x\sqrt{4+x}\,dx = 2\int(u^4 - 4u^2)\,du$$
$$= \frac{2u^3}{15}(3u^2 - 20) + C$$
$$= \frac{2(4+x)^{3/2}}{15}(3x-8) + C$$

$u^2 = 4+x,\ dx = 2u\,du$

(c) $$\int x\sqrt{4+x}\,dx = \int(u^{3/2} - 4u^{1/2})\,du$$
$$= \frac{2u^{3/2}}{15}(3u - 20) + C$$
$$= \frac{2(4+x)^{3/2}}{15}(3x-8) + C$$

$u = 4+x,\ du = dx$

(d) $$\int x\sqrt{4+x}\,dx = \frac{2x}{3}(4+x)^{3/2} - \frac{2}{3}\int(4+x)^{3/2}\,dx$$
$$= \frac{2x}{3}(4+x)^{3/2} - \frac{4}{15}(4+x)^{5/2} + C$$
$$= \frac{2(4+x)^{3/2}}{15}(3x-8) + C$$

$dv = \sqrt{4+x}\,dx \implies v = \frac{2}{3}(4+x)^{3/2}$

$u = x \implies du = dx$

32. $$\frac{2x^3 - 5x^2 + 4x - 4}{x^2 - x} = 2x - 3 + \frac{4}{x} - \frac{3}{x-1}$$
$$\int\frac{2x^3 - 5x^2 + 4x - 4}{x^2 - x}\,dx = \int\left(2x - 3 + \frac{4}{x} - \frac{3}{x-1}\right)dx = x^2 - 3x + 4\ln|x| - 3\ln|x-1| + C$$

34. $$\frac{4x-2}{3(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

$4x - 2 = 3A(x-1) + 3B$

Let $x = 1$: $2 = 3B \implies B = \frac{2}{3}$

Let $x = 2$: $6 = 3A + 3B \implies A = \frac{4}{3}$

$$\int\frac{4x-2}{3(x-1)^2}\,dx = \frac{4}{3}\int\frac{1}{x-1}\,dx + \frac{2}{3}\int\frac{1}{(x-1)^2}\,dx = \frac{4}{3}\ln|x-1| - \frac{2}{3(x-1)} + C = \frac{2}{3}\left(2\ln|x-1| - \frac{1}{x-1}\right) + C$$

36. $$\int\frac{\sec^2\theta}{\tan\theta(\tan\theta - 1)}\,d\theta = \int\frac{1}{u(u-1)}\,du = \int\frac{1}{u-1}\,du - \int\frac{1}{u}\,du$$
$$= \ln|u-1| - \ln|u| + C = \ln\left|\frac{\tan\theta - 1}{\tan\theta}\right| + C = \ln|1 - \cot\theta| + C$$

$u = \tan\theta,\ du = \sec^2\theta\,d\theta$

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$$

$1 = A(u-1) + Bu$

Let $u = 0$: $1 = -A \implies A = -1$

Let $u = 1$: $1 = B$

38. $\displaystyle\int \frac{x}{\sqrt{2+3x}}\,dx = \frac{-2(4-3x)}{27}\sqrt{2+3x} + C$ (Formula 21)

$$= \frac{6x-8}{27}\sqrt{2+3x} + C$$

40. $\displaystyle\int \frac{x}{1+e^{x^2}}\,dx = \frac{1}{2}\int \frac{1}{1+e^u}\,du$ $(u = x^2)$

$$= \frac{1}{2}\left[u - \ln(1+e^u)\right] + C \quad \text{(Formula 84)}$$

$$= \frac{1}{2}\left[x^2 - \ln(1+e^{x^2})\right] + C$$

42. $\displaystyle\int \frac{3}{2x\sqrt{9x^2-1}}\,dx = \frac{3}{2}\int \frac{1}{3x\sqrt{(3x)^2-1}}\,3\,dx$ $(u = 3x)$

$$= \frac{3}{2}\operatorname{arcsec}|3x| + C \quad \text{(Formula 33)}$$

44. $\displaystyle\int \frac{1}{1+\tan \pi x}\,dx = \frac{1}{\pi}\int \frac{1}{1+\tan \pi x}(\pi)dx$ $(u = \pi x)$

$$= \frac{1}{\pi}\frac{1}{2}\left[\pi x + \ln|\cos \pi x + \sin \pi x|\right] + C \quad \text{(Formula 71)}$$

46. $\displaystyle\int \tan^n x\,dx = \int \tan^{n-2} x(\sec^2 x - 1)\,dx$

$$= \int \tan^{n-2} x \sec^2 x\,dx - \int \tan^{n-2} x\,dx$$

$$= \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx$$

48. $\displaystyle\int \frac{\csc\sqrt{2x}}{\sqrt{x}}\,dx = \sqrt{2}\int \csc\sqrt{2x}\left(\frac{1}{\sqrt{2x}}\right)dx$

$$= -\sqrt{2}\ln\left|\csc\sqrt{2x} + \cot\sqrt{2x}\right| + C$$

$u = \sqrt{2x},\ du = \dfrac{1}{\sqrt{2x}}\,dx$

50. $\displaystyle\int \sqrt{1+\sqrt{x}}\,dx = \int u(4u^3 - 4u)\,du = \int (4u^4 - 4u^2)\,du = \frac{4u^5}{5} - \frac{4u^3}{3} + C = \frac{4}{15}\left(1+\sqrt{x}\right)^{3/2}\left(3\sqrt{x} - 2\right) + C$

$u = \sqrt{1+\sqrt{x}},\ x = u^4 - 2u^2 + 1,\ dx = (4u^3 - 4u)\,du$

52. $\dfrac{3x^3+4x}{(x^2+1)^2} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{(x^2+1)^2}$

$3x^3 + 4x = (Ax+B)(x^2+1) + Cx + D$

$\quad = Ax^3 + Bx^2 + (A+C)x + (B+D)$

$A = 3, B = 0, A + C = 4 \implies C = 1,$

$B + D = 0 \implies D = 0$

$$\int \frac{3x^3+4x}{(x^2+1)^2}\,dx = 3\int \frac{x}{x^2+1}\,dx + \int \frac{x}{(x^2+1)^2}\,dx$$

$$= \frac{3}{2}\ln(x^2+1) - \frac{1}{2(x^2+1)} + C$$

54. $\displaystyle\int (\sin\theta + \cos\theta)^2\,d\theta = \int (\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta)\,d\theta$

$$= \int (1 + \sin 2\theta)\,d\theta = \theta - \frac{1}{2}\cos 2\theta + C = \frac{1}{2}(2\theta - \cos 2\theta) + C$$

56. $$y = \int \frac{\sqrt{4-x^2}}{2x}\,dx = \int \frac{2\cos\theta(2\cos\theta)\,d\theta}{4\sin\theta}$$
$$= \int (\csc\theta - \sin\theta)\,d\theta$$
$$= \left[-\ln\left|\csc\theta + \cos\theta\right| + \cos\theta\right] + C$$
$$= -\ln\left|\frac{2+\sqrt{4-x^2}}{x}\right| + \frac{\sqrt{4-x^2}}{2} + C$$

$x = 2\sin\theta,\ dx = 2\cos\theta\,d\theta,\ \sqrt{4-x^2} = 2\cos\theta$

58. $$y = \int \sqrt{1-\cos\theta}\,d\theta = \int \frac{\sin\theta}{\sqrt{1+\cos\theta}}\,d\theta = -\int (1+\cos\theta)^{-1/2}(-\sin\theta)d\theta = -2\sqrt{1+\cos\theta} + C$$

$u = 1 + \cos\theta,\ du = -\sin\theta\,d\theta$

60. $$\int_0^1 \frac{x}{(x-2)(x-4)}\,dx = \Big[2\ln|x-4| - \ln|x-2|\Big]_0^1$$
$$= 2\ln 3 - 2\ln 4 + \ln 2$$
$$= \ln\frac{9}{8} \approx 0.118$$

62. $$\int_0^2 xe^{3x}\,dx = \left[\frac{e^{3x}}{9}(3x-1)\right]_0^2 = \frac{1}{9}(5e^6+1) \approx 224.238$$

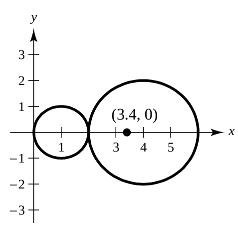
64. $$\int_0^3 \frac{x}{\sqrt{1+x}}\,dx = \left[\frac{-2(2-x)}{3}\sqrt{1+x}\right]_0^3 = \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$$

66. $$A = \int_0^4 \frac{1}{25-x^2}\,dx$$
$$= \left[-\frac{1}{10}\ln\left|\frac{x-5}{x+5}\right|\right]_0^4 = -\frac{1}{10}\ln\frac{1}{9} = \frac{1}{10}\ln 9 \approx 0.220$$

68. By symmetry, $\bar{y} = 0$.

$$A = \pi + 4\pi = 5\pi$$
$$\bar{x} = \frac{1(\pi) + 4(4\pi)}{\pi + 4\pi}$$
$$= \frac{17\pi}{5\pi} = 3.4$$

$(\bar{x}, \bar{y}) = (3.4, 0)$



70. $$s = \int_0^{\pi} \sqrt{1+\sin^2 2x}\,dx \approx 3.82$$

72. $$\lim_{x\to 0}\frac{\sin\pi x}{\sin 2\pi x} = \lim_{x\to 0}\frac{\pi\cos\pi x}{2\pi\cos 2\pi x} = \frac{\pi}{2\pi} = \frac{1}{2}$$

74. $$\lim_{x\to\infty} xe^{-x^2} = \lim_{x\to\infty}\frac{x}{e^{x^2}} = \lim_{x\to\infty}\frac{1}{2xe^{x^2}} = 0$$

76. $$y = \lim_{x\to 1^+}(x-1)^{\ln x}$$

$$\ln y = \lim_{x\to 1^+}\left[(\ln x)\ln(x-1)\right]$$
$$= \lim_{x\to 1^+}\left[\frac{\ln(x-1)}{\frac{1}{\ln x}}\right] = \lim_{x\to 1^+}\left[\frac{\frac{1}{x-1}}{\left(\frac{1}{x}\right)\frac{-1}{\ln^2 x}}\right] = \lim_{x\to 1^+}\left[\frac{-\ln^2 x}{\frac{x-1}{x}}\right] = \lim_{x\to 1^+}\left[\frac{-2\left(\frac{1}{x}\right)(\ln x)}{\frac{1}{x^2}}\right]$$
$$= \lim_{x\to 1^+} 2x(\ln x) = 0$$

Since $\ln y = 0$, $y = 1$.