CHAPTER 7
Integration Techniques, L'Hôpital's Rule, and Improper Integrals

C H A P T E R 7
Integration Techniques, L'Hôpital's Rule, and Improper Integrals

Section 7.1 Basic Integration Rules

Solutions to Odd-Numbered Exercises

1. (a) $\frac{d}{dx}\left[2\sqrt{x^2+1}+C\right] = 2\left(\frac{1}{2}\right)(x^2+1)^{-1/2}(2x) = \frac{2x}{\sqrt{x^2+1}}$

(b) $\frac{d}{dx}\left[\sqrt{x^2+1}+C\right] = \frac{1}{2}(x^2+1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+1}}$

(c) $\frac{d}{dx}\left[\frac{1}{2}\sqrt{x^2+1}+C\right] = \frac{1}{2}\left(\frac{1}{2}\right)(x^2+1)^{-1/2}(2x) = \frac{x}{2\sqrt{x^2+1}}$

(d) $\frac{d}{dx}\left[\ln(x^2+1)+C\right] = \frac{2x}{x^2+1}$

$\int \frac{x}{\sqrt{x^2+1}}\,dx$ matches (b).

3. (a) $\frac{d}{dx}\left[\ln\sqrt{x^2+1}+C\right] = \frac{1}{2}\left(\frac{2x}{x^2+1}\right) = \frac{x}{x^2+1}$

(b) $\frac{d}{dx}\left[\frac{2x}{(x^2+1)^2}+C\right] = \frac{(x^2+1)^2(2)-(2x)(2)(x^2+1)(2x)}{(x^2+1)^4} = \frac{2(1-3x^2)}{(x^2+1)^3}$

(c) $\frac{d}{dx}\left[\arctan x + C\right] = \frac{1}{1+x^2}$

(d) $\frac{d}{dx}\left[\ln(x^2+1)+C\right] = \frac{2x}{x^2+1}$

$\int \frac{1}{x^2+1}\,dx$ matches (c).

5. $\int (3x-2)^4\,dx$

$u = 3x-2,\ du = 3\,dx,\ n = 4$

Use $\int u^n\,du$.

7. $\int \frac{1}{\sqrt{x}(1-2\sqrt{x})}\,dx$

$u = 1-2\sqrt{x},\ du = -\frac{1}{\sqrt{x}}\,dx$

Use $\int \frac{du}{u}$.

9. $\int \frac{3}{\sqrt{1-t^2}}\,dt$

$u = t,\ du = dt,\ a = 1$

Use $\int \frac{du}{\sqrt{a^2-u^2}}$

11. $\int t\sin t^2\,dt$

$u = t^2,\ du = 2t\,dt$

Use $\int \sin u\,du$.

13. $\int \cos x e^{\sin x}\,dx$

$u = \sin x,\ du = \cos x\,dx$

Use $\int e^u\,du$.

15. Let $u = -2x + 5$, $du = -2\,dx$.

$$\int (-2x+5)^{3/2}\,dx = -\frac{1}{2}\int (-2x+5)^{3/2}(-2)\,dx$$
$$= -\frac{1}{5}(-2x+5)^{5/2} + C$$

17. Let $u = z - 4$, $du = dz$

$$\int \frac{5}{(z-4)^5}\,dz = 5\int (z-4)^{-5}\,dx = 5\,\frac{(z-4)^{-4}}{-4} + C$$
$$= \frac{-5}{4(z-4)^4} + C$$

19. Let $u = t^3 - 1$, $du = 3t^2\,dt$.

$$\int t^2\sqrt[3]{t^3-1}\,dt = \frac{1}{3}\int (t^3-1)^{1/3}(3t^2)\,dt$$
$$= \frac{1}{3}\,\frac{(t^3-1)^{4/3}}{4/3} + C$$
$$= \frac{(t^3-1)^{4/3}}{4} + C$$

21. $$\int \left[v + \frac{1}{(3v-1)^3}\right]dv = \int v\,dv + \frac{1}{3}\int (3v-1)^{-3}(3)dv$$
$$= \frac{1}{2}v^2 - \frac{1}{6(3v-1)^2} + C$$

23. Let $u = -t^3 + 9t + 1$, $du = (-3t^2 + 9)\,dt = -3(t^2 - 3)\,dt$.

$$\int \frac{t^2-3}{-t^3+9t+1}\,dt = -\frac{1}{3}\int \frac{-3(t^2-3)}{-t^3+9t+1}\,dt = -\frac{1}{3}\ln\left|-t^3+9t+1\right| + C$$

25. $$\int \frac{x^2}{x-1}\,dx = \int (x+1)\,dx + \int \frac{1}{x-1}\,dx$$
$$= \frac{1}{2}x^2 + x + \ln|x-1| + C$$

27. Let $u = 1 + e^x$, $du = e^x\,dx$.

$$\int \frac{e^x}{1+e^x}\,dx = \ln(1+e^x) + C$$

29. $$\int (1+2x^2)^2\,dx = \int (4x^4 + 4x^2 + 1)dx = \frac{4}{5}x^5 + \frac{4}{3}x^3 + x + C = \frac{x}{15}(12x^4 + 20x^2 + 15) + C$$

31. Let $u = 2\pi x^2$, $du = 4\pi x\,dx$.

$$\int x(\cos 2\pi x^2)\,dx = \frac{1}{4\pi}\int (\cos 2\pi x^2)(4\pi x)\,dx$$
$$= \frac{1}{4\pi}\sin 2\pi x^2 + C$$

33. Let $u = \pi x$, $du = \pi\,dx$.

$$\int \csc(\pi x)\cot(\pi x)\,dx = \frac{1}{\pi}\int \csc(\pi x)\cot(\pi x)\pi\,dx = -\frac{1}{\pi}\csc(\pi x) + C$$

35. Let $u = 5x$, $du = 5\,dx$.

$$\int e^{5x}\,dx = \frac{1}{5}\int e^{5x}(5)\,dx = \frac{1}{5}e^{5x} + C$$

37. Let $u = 1 + e^x$, $du = e^x\,dx$.

$$\int \frac{2}{e^{-x}+1}\,dx = 2\int \left(\frac{1}{e^{-x}+1}\right)\left(\frac{e^x}{e^x}\right)dx$$
$$= 2\int \frac{e^x}{1+e^x}\,dx = 2\ln(1+e^x) + C$$

39. $\displaystyle\int \frac{\ln x^2}{x}\,dx = 2\int (\ln x)\frac{1}{x}\,dx = 2\frac{(\ln x)^2}{2} + C = (\ln x)^2 + C$

41. $\displaystyle\int \frac{1 + \sin x}{\cos x}\,dx = \int (\sec x + \tan x)\,dx = \ln|\sec x + \tan x| + \ln|\sec x| + C = \ln|\sec x(\sec x + \tan x)| + C$

43. $$\frac{1}{\cos\theta - 1} = \frac{1}{\cos\theta - 1}\cdot\frac{\cos\theta + 1}{\cos\theta + 1} = \frac{\cos\theta + 1}{\cos^2\theta - 1} = \frac{\cos\theta + 1}{-\sin^2\theta}$$

$$= -\csc\theta\cdot\cot\theta - \csc^2\theta$$

$$\int \frac{1}{\cos\theta - 1}\,d\theta = \int (-\csc\theta\cot\theta - \csc^2\theta)\,d\theta$$

$$= \csc\theta + \cot\theta + C$$

$$= \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta} + C$$

$$= \frac{1 + \cos\theta}{\sin\theta} + C$$

45. $$\int \frac{3z + 2}{z^2 + 9}\,dz = \frac{3}{2}\int \frac{2z}{z^2 + 9}\,dz + 2\int \frac{dz}{z^2 + 9}$$

$$= \frac{3}{2}\ln(z^2 + 9) + \frac{2}{3}\arctan\left(\frac{z}{3}\right) + C$$

47. Let $u = 2t - 1$, $du = 2\,dt$.

$$\int \frac{-1}{\sqrt{1 - (2t - 1)^2}}\,dt = -\frac{1}{2}\int \frac{2}{\sqrt{1 - (2t - 1)^2}}\,dt$$

$$= -\frac{1}{2}\arcsin(2t - 1) + C$$

49. Let $u = \cos\left(\dfrac{2}{t}\right)$, $du = \dfrac{2\sin(2/t)}{t^2}\,dt$.

$$\int \frac{\tan(2/t)}{t^2}\,dt = \frac{1}{2}\int \frac{1}{\cos(2/t)}\left[\frac{2\sin(2/t)}{t^2}\right]dt$$

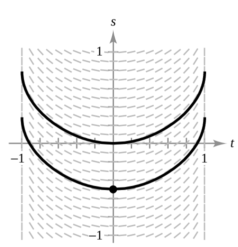
$$= \frac{1}{2}\ln\left|\cos\left(\frac{2}{t}\right)\right| + C$$

51. $\displaystyle\int \frac{3}{\sqrt{6x - x^2}}\,dx = 3\int \frac{1}{\sqrt{9 - (x - 3)^2}}\,dx = 3\arcsin\left(\frac{x - 3}{3}\right) + C$

53. $\displaystyle\int \frac{4}{4x^2 + 4x + 65}\,dx = \int \frac{1}{[x + (1/2)]^2 + 16}\,dx = \frac{1}{4}\arctan\left[\frac{x + (1/2)}{4}\right] + C = \frac{1}{4}\arctan\left(\frac{2x + 1}{8}\right) + C$

55. $\dfrac{ds}{dt} = \dfrac{t}{\sqrt{1 - t^4}}$, $\left(0, -\dfrac{1}{2}\right)$

(a)

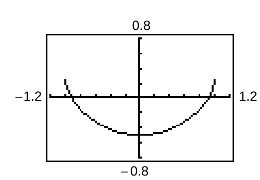


(b) $u = t^2$, $du = 2t\,dt$

$$\int \frac{t}{\sqrt{1 - t^4}}\,dt = \frac{1}{2}\int \frac{2t}{\sqrt{1 - (t^2)^2}}\,dt = \frac{1}{2}\arcsin t^2 + C$$

$$\left(0, -\frac{1}{2}\right):\ -\frac{1}{2} = \frac{1}{2}\arcsin 0 + C \Rightarrow C = -\frac{1}{2}$$

$$s = \frac{1}{2}\arcsin t^2 - \frac{1}{2}$$



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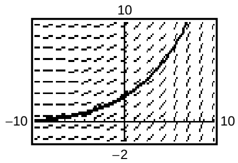


$y = 3e^{0.2x}$

59. $y = \int (1 + e^x)^2\,dx = \int (e^{2x} + 2e^x + 1)\,dx$

$= \frac{1}{2}e^{2x} + 2e^x + x + C$

61. $\frac{dy}{dx} = \frac{\sec^2 x}{4 + \tan^2 x}$

Let $u = \tan x$, $du = \sec^2 x\,dx$.

$$y = \int \frac{\sec^2 x}{4 + \tan^2 x}\,dx = \frac{1}{2}\arctan\left(\frac{\tan x}{2}\right) + C$$

63. Let $u = 2x$, $du = 2\,dx$.

$$\int_0^{\pi/4} \cos 2x\,dx = \frac{1}{2}\int_0^{\pi/4} \cos 2x(2)\,dx$$
$$= \left[\frac{1}{2}\sin 2x\right]_0^{\pi/4} = \frac{1}{2}$$

65. Let $u = -x^2$, $du = -2x\,dx$.

$$\int_0^1 xe^{-x^2}dx = -\frac{1}{2}\int_0^1 e^{-x^2}(-2x)\,dx = \left[-\frac{1}{2}e^{-x^2}\right]_0^1$$
$$= \frac{1}{2}(1 - e^{-1}) \approx 0.316$$

67. Let $u = x^2 + 9$, $du = 2x\,dx$.

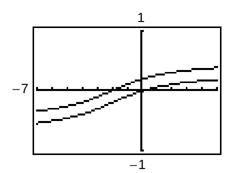
$$\int_0^4 \frac{2x}{\sqrt{x^2 + 9}}\,dx = \int_0^4 (x^2 + 9)^{-1/2}(2x)\,dx$$
$$= \left[2\sqrt{x^2 + 9}\right]_0^4 = 4$$

69. Let $u = 3x$, $du = 3\,dx$.

$$\int_0^{2/\sqrt{3}} \frac{1}{4 + 9x^2}\,dx = \frac{1}{3}\int_0^{2/\sqrt{3}} \frac{3}{4 + (3x)^2}\,dx$$
$$= \left[\frac{1}{6}\arctan\left(\frac{3x}{2}\right)\right]_0^{2/\sqrt{3}}$$
$$= \frac{\pi}{18} \approx 0.175$$

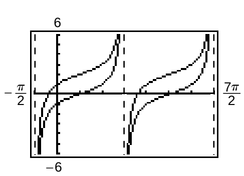
71. $\int \frac{1}{x^2 + 4x + 13}\,dx = \frac{1}{3}\arctan\left(\frac{x + 2}{3}\right) + C$

The antiderivatives are vertical translations of each other.



73. $\int \frac{1}{1 + \sin\theta}\,d\theta = \tan\theta - \sec\theta + C \left(\text{or } \frac{-2}{1 + \tan(\theta/2)}\right)$

The antiderivatives are vertical translations of each other.



75. Power Rule: $\int u^n\,du = \frac{u^{n+1}}{n + 1} + C, n \neq -1.$

$u = x^2 + 1, n = 3$

77. Log Rule: $\int \frac{du}{u} = \ln|u| + C, u = x^2 + 1.$

79. The are equivalent because

$e^{x + C_1} = e^x \cdot e^{C_1} = Ce^x, C = e^{C_1}$

81. $\sin x + \cos x = a \sin(x + b)$

$\sin x + \cos x = a \sin x \cos b + a \cos x \sin b$

$\sin x + \cos x = (a \cos b) \sin x + (a \sin b) \cos x$

Equate coefficients of like terms to obtain the following.

$1 = a \cos b \quad \text{and} \quad 1 = a \sin b$

Thus, $a = 1/\cos b$. Now, substitute for a in $1 = a \sin b$.

$$1 = \left(\frac{1}{\cos b}\right)\sin b$$

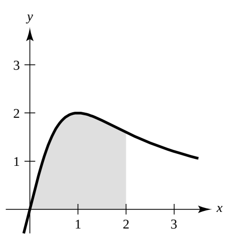
$$1 = \tan b \implies b = \frac{\pi}{4}$$

Since $b = \dfrac{\pi}{4}$, $a = \dfrac{1}{\cos(\pi/4)} = \sqrt{2}$. Thus, $\sin x + \cos x = \sqrt{2}\sin\left(x + \dfrac{\pi}{4}\right)$.

$$\int \frac{dx}{\sin x + \cos x} = \int \frac{dx}{\sqrt{2}\sin(x + (\pi/4))} = \frac{1}{\sqrt{2}}\int \csc\left(x + \frac{\pi}{4}\right) dx = -\frac{1}{\sqrt{2}} \ln\left|\csc\left(x + \frac{\pi}{4}\right) + \cot\left(x + \frac{\pi}{4}\right)\right| + C$$

83. $\displaystyle\int_0^2 \frac{4x}{x^2 + 1}\, dx \approx 3$

Matches (a).

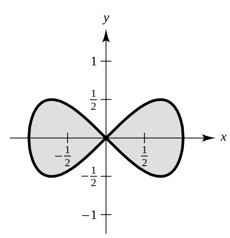


85. Let $u = 1 - x^2$, $du = -2x\, dx$.

$$A = 4\int_0^1 x\sqrt{1 - x^2}\, dx$$

$$= -2\int_0^1 (1 - x^2)^{1/2}(-2x)\, dx$$

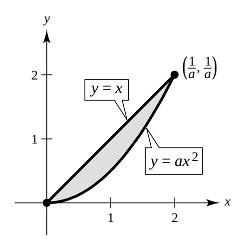
$$= \left[-\frac{4}{3}(1 - x^2)^{3/2}\right]_0^1 = \frac{4}{3}$$



87. $\displaystyle\int_0^{1/a} (x - ax^2)\, dx = \left[\frac{1}{2}x^2 - \frac{a}{3}x^3\right]_0^{1/a}$

$$= \frac{1}{6a^2}$$

Let $\dfrac{1}{6a^2} = \dfrac{2}{3}$, $12a^2 = 3$, $a = \dfrac{1}{2}$.



89. (a) **Shell Method:**

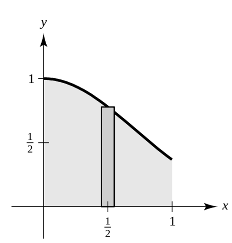
Let $u = -x^2$, $du = -2x\, dx$.

$$V = 2\pi\int_0^1 xe^{-x^2}\, dx$$

$$= -\pi\int_0^1 e^{-x^2}(-2x)\, dx$$

$$= \left[-\pi e^{-x^2}\right]_0^1$$

$$= \pi(1 - e^{-1}) \approx 1.986$$



(b) **Shell Method:**

$$V = 2\pi\int_0^b xe^{-x^2}\, dx$$

$$= \left[-\pi e^{-x^2}\right]_0^b$$

$$= \pi(1 - e^{-b^2}) = \frac{4}{3}$$

$$e^{-b^2} = \frac{3\pi - 4}{3\pi}$$

$$b = \sqrt{\ln\left(\frac{3\pi}{3\pi - 4}\right)}$$

$$\approx 0.743$$

91. $A = \int_0^4 \frac{5}{\sqrt{25 - x^2}}\,dx = \left[5 \arcsin \frac{x}{5}\right]_0^4 = 5 \arcsin \frac{4}{5}$

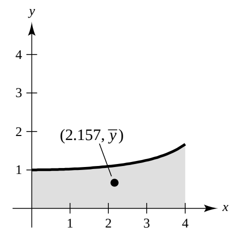
$$\bar{x} = \frac{1}{A}\int_0^4 x\left(\frac{5}{\sqrt{25 - x^2}}\right) dx$$

$$= \frac{1}{5 \arcsin(4/5)}\left(-\frac{5}{2}\right)\int_0^4 (25 - x^2)^{-1/2}(-2x)\,dx$$

$$= \frac{1}{5 \arcsin(4/5)}(-5)\left[(25 - x^2)^{1/2}\right]_0^4$$

$$= -\frac{1}{\arcsin(4/5)}[3 - 5]$$

$$= \frac{2}{\arcsin(4/5)} \approx 2.157$$



93.

$$y = \tan(\pi x)$$

$$y' = \pi \sec^2(\pi x)$$

$$1 + (y')^2 = 1 + \pi^2 \sec^4(\pi x)$$

$$s = \int_0^{1/4} \sqrt{1 + \pi^2 \sec^4(\pi x)}\,dx$$

$$\approx 1.0320$$

Section 7.2 Integration by Parts

1. $\frac{d}{dx}[\sin x - x \cos x] = \cos x - (-x \sin x + \cos x) = x \sin x.$ Matches (b)

3. $\frac{d}{dx}[x^2e^x - 2xe^x + 2e^x] = x^2e^x + 2xe^x - 2xe^x - 2e^x + 2e^x = x^2e^x.$ Matches (c)

5. $\int xe^{2x}\,dx$

$u = x,\ dv = e^{2x}\,dx$

7. $\int (\ln x)^2\,dx$

$u = (\ln x)^2,\ dv = dx$

9. $\int x \sec^2 x\,dx$

$u = x,\ dv = \sec^2 x\,dx$

11. $dv = e^{-2x}\,dx \implies v = \int e^{-2x}\,dx = -\frac{1}{2}e^{-2x}$

$u = x \implies du = dx$

$$\int xe^{-2x}\,dx = -\frac{1}{2}xe^{-2x} - \int -\frac{1}{2}e^{-2x}\,dx$$

$$= -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C = \frac{-1}{4e^{2x}}(2x + 1) + C$$

13. Use integration by parts three times.

(1) $dv = e^x\,dx \Rightarrow v = \int e^x\,dx = e^x$ (2) $dv = e^x\,dx \Rightarrow v = \int e^x\,dx = e^x$ (3) $dv = e^x\,dx \Rightarrow v = \int e^x\,dx = e^x$

$u = x^3 \Rightarrow du = 3x^2\,dx$ $u = x^2 \Rightarrow du = 2x\,dx$ $u = x \Rightarrow du = dx$

$$\int x^3e^x\,dx = x^3e^x - 3\int x^2e^x\,dx = x^3e^x - 3x^2e^x + 6\int xe^x\,dx$$

$$= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C = e^x(x^3 - 3x^2 + 6x - 6) + C$$

15. $\int x^2\,e^{x^3}\,dx = \frac{1}{3}\int e^{x^3}(3x^2)dx = \frac{1}{3}e^{x^3} + C$

17. $dv = t\,dt \Rightarrow v = \int t\,dt = \frac{t^2}{2}$

$u = \ln(t+1) \Rightarrow du = \frac{1}{t+1}\,dt$

$$\int t\ln(t+1)\,dt = \frac{t^2}{2}\ln(t+1) - \frac{1}{2}\int\frac{t^2}{t+1}\,dt$$

$$= \frac{t^2}{2}\ln(t+1) - \frac{1}{2}\int\left(t - 1 + \frac{1}{t+1}\right)dt$$

$$= \frac{t^2}{2}\ln(t+1) - \frac{1}{2}\left[\frac{t^2}{2} - t + \ln(t+1)\right] + C$$

$$= \frac{1}{4}[2(t^2-1)\ln|t+1| - t^2 + 2t] + C$$

19. Let $u = \ln x$, $du = \frac{1}{x}\,dx$.

$$\int\frac{(\ln x)^2}{x}\,dx = \int(\ln x)^2\left(\frac{1}{x}\right)dx = \frac{(\ln x)^3}{3} + C$$

21. $dv = \frac{1}{(2x+1)^2}\,dx \Rightarrow v = \int(2x+1)^{-2}\,dx = -\frac{1}{2(2x+1)}$

$u = xe^{2x} \Rightarrow du = (2xe^{2x} + e^{2x})\,dx = e^{2x}(2x+1)\,dx$

$$\int\frac{xe^{2x}}{(2x+1)^2}\,dx = -\frac{xe^{2x}}{2(2x+1)} + \int\frac{e^{2x}}{2}\,dx$$

$$= \frac{-xe^{2x}}{2(2x+1)} + \frac{e^{2x}}{4} + C$$

$$= \frac{e^{2x}}{4(2x+1)} + C$$

23. Use integration by parts twice.

(1) $dv = e^x\,dx \Rightarrow v = \int e^x\,dx = e^x$ (2) $dv = e^x\,dx \Rightarrow v = \int e^x\,dx = e^x$

$u = x^2 \Rightarrow du = 2x\,dx$ $u = x \Rightarrow du = dx$

$$\int(x^2-1)e^x\,dx = \int x^2e^x\,dx - \int e^x dx = x^2e^x - 2\int xe^x\,dx - e^x$$

$$= x^2e^x - 2\left[xe^x - \int e^x\,dx\right] - e^x = x^2e^x - 2xe^x + e^x + C = (x-1)^2e^x + C$$

25. $dv = \sqrt{x-1}\,dx \implies v = \int (x-1)^{1/2}dx = \frac{2}{3}(x-1)^{3/2}$

$u = x \implies du = dx$

$$\int x\sqrt{x-1}\,dx = \frac{2}{3}x(x-1)^{3/2} - \frac{2}{3}\int (x-1)^{3/2}\,dx$$

$$= \frac{2}{3}x(x-1)^{3/2} - \frac{4}{15}(x-1)^{5/2} + C$$

$$= \frac{2(x-1)^{3/2}}{15}(3x+2) + C$$

27. $dv = \cos x\,dx \implies v = \int \cos x\,dx = \sin x$

$u = x \implies du = dx$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$$

29. Use integration by parts three times.

(1) $u = x^3, du = 3x^2, dv = \sin x\,dx, v = -\cos x$

$$\int x^3 \sin dx = -x^3\cos x + 3\int x^2\cos x\,dx$$

(2) $u = x^2, du = 2x\,dx, dv = \cos x\,dx, v = \sin x$

$$\int x^3\sin x\,dx = -x^3\cos x + 3\left[x^2\sin x - 2\int x\sin x\,dx\right]$$

$$= -x^3\cos x + 3x^2\sin x - 6\int x\sin x\,dx$$

(3) $u = x, du = dx, dv = \sin x\,dx, v = -\cos x$

$$\int x^3\sin x\,dx = -x^3\cos x + 3x^2\sin x - 6\left[-x\cos x + \int \cos x\,dx\right]$$

$$= -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C$$

31. $u = t, du = dt, dv = \csc t\cot t\,dt, v = -\csc t$

$$\int t\csc t\cot t\,dt = -t\csc t + \int \csc t\,dt$$

$$= -t\csc t - \ln\left|\csc t + \cot t\right| + C$$

33. $dv = dx \implies v = \int dx = x$

$u = \arctan x \implies du = \frac{1}{1+x^2}\,dx$

$$\int \arctan x\,dx = x\arctan x - \int \frac{x}{1+x^2}\,dx$$

$$= x\arctan x - \frac{1}{2}\ln(1+x^2) + C$$

35. Use integration by parts twice.

(1) $dv = e^{2x}dx \implies v = \int e^{2x}\,dx = \frac{1}{2}e^{2x}$

$u = \sin x \implies du = \cos x\,dx$

(2) $dv = e^{2x}\,dx \implies v = \int e^{2x}\,dx = \frac{1}{2}e^{2x}$

$u = \cos x \implies du = -\sin x\,dx$

$$\int e^{2x}\sin x\,dx = \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\int e^{2x}\cos x\,dx = \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\left(\frac{1}{2}e^{2x}\cos x + \frac{1}{2}\int e^{2x}\sin x\,dx\right)$$

$$\frac{5}{4}\int e^{2x}\sin x\,dx = \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x$$

$$\int e^{2x}\sin x\,dx = \frac{1}{5}e^{2x}(2\sin x - \cos x) + C$$

37. $y' = xe^{x^2}$

$$y = \int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C$$

39. Use integration by parts twice.

(1) $dv = \dfrac{1}{\sqrt{2+3t}}\,dt \implies v = \displaystyle\int (2+3t)^{-1/2}\,dt = \frac{2}{3}\sqrt{2+3t}$

$u = t^2 \implies du = 2t\,dt$

(2) $dv = \sqrt{2+3t}\,dt \implies v = \displaystyle\int (2+3t)^{1/2}\,dt = \frac{2}{9}(2+3t)^{3/2}$

$u = t \implies du = dt$

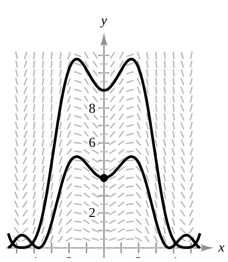
$$\begin{aligned} y = \int \frac{t^2}{\sqrt{2+3t}}\,dt &= \frac{2t^2\sqrt{2+3t}}{3} - \frac{4}{3}\int t\sqrt{2+3t}\,dt \\ &= \frac{2t^2\sqrt{2+3t}}{3} - \frac{4}{3}\left[\frac{2t}{9}(2+3t)^{3/2} - \frac{2}{9}\int (2+3t)^{3/2}\,dt\right] \\ &= \frac{2t^2\sqrt{2+3t}}{3} - \frac{8t}{27}(2+3t)^{3/2} + \frac{16}{405}(2+3t)^{5/2} + C \\ &= \frac{2\sqrt{2+3t}}{405}(27t^2 - 24t + 32) + C \end{aligned}$$

41. $(\cos y)y' = 2x$

$$\int \cos y\,dy = \int 2x\,dx$$

$$\sin y = x^2 + C$$

43. (a)



(b) $\dfrac{dy}{dx} = x\sqrt{y}\cos x,\ (0, 4)$

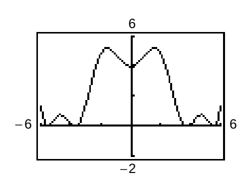
$$\int \frac{dy}{\sqrt{y}} = \int x\cos x\,dx$$

$$\int y^{-1/2}\,dy = \int x\cos x\,dx \qquad (u = x,\ du = dx,\ dv = \cos x\,dx,\ v = \sin x)$$

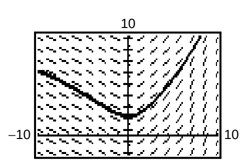
$$\begin{aligned} 2y^{1/2} &= x\sin x - \int \sin x\,dx \\ &= x\sin x + \cos x + C \end{aligned}$$

$(0, 4)$: $2(4)^{1/2} = 0 + 1 + C \implies C = 3$

$2\sqrt{y} = x\sin x + \cos x + 3$



45. $\dfrac{dy}{dx} = \dfrac{x}{y}e^{x/8},\ y(0) = 2$



47. $u = x$, $du = dx$, $dv = e^{-x/2}\,dx$, $v = -2e^{-x/2}$

$$\int xe^{-x/2}\,dx = -2xe^{-x/2} + \int 2e^{-x/2}\,dx = -2xe^{-x/2} - 4e^{-x/2} + C$$

Thus, $\displaystyle\int_0^4 xe^{-x/2}\,dx = \Big[-2xe^{-x/2} - 4e^{-x/2}\Big]_0^4$

$$= -8e^{-2} - 4e^{-2} + 4$$

$$= -12e^{-2} + 4 \approx 2.376.$$

49. See Exercise 27.

$$\int_0^{\pi/2} x\cos x\,dx = \Big[x\sin x + \cos x\Big]_0^{\pi/2} = \frac{\pi}{2} - 1$$

51. $u = \arccos x$, $du = -\dfrac{1}{\sqrt{1-x^2}}\,dx$, $dv = dx$, $v = x$

$$\int \arccos x\,dx = x\arccos x + \int \frac{x}{\sqrt{1-x^2}}\,dx = x\arccos x - \sqrt{1-x^2} + C$$

Thus, $\displaystyle\int_0^{1/2} \arccos x = \Big[x\arccos x - \sqrt{1-x^2}\Big]_0^{1/2}$

$$= \frac{1}{2}\arccos\left(\frac{1}{2}\right) - \sqrt{\frac{3}{4}} + 1$$

$$= \frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1 \approx 0.658.$$

53. Use integration by parts twice.

(1) $dv = e^x\,dx \implies v = \int e^x\,dx = e^x$

$u = \sin x \implies du = \cos x\,dx$

(2) $dv = e^x\,dx \implies v = \int e^x\,dx = e^x$

$u = \cos x \implies du = -\sin x\,dx$

$$\int e^x\sin x\,dx = e^x\sin x - \int e^x\cos x\,dx = e^x\sin x - e^x\cos x - \int e^x\sin x\,dx$$

$$2\int e^x\sin x\,dx = e^x(\sin x - \cos x)$$

$$\int e^x\sin x\,dx = \frac{e^x}{2}(\sin x - \cos x) + C$$

Thus, $\displaystyle\int_0^1 e^x\sin x\,dx = \left[\frac{e^x}{2}(\sin x - \cos x)\right]_0^1 = \frac{e}{2}(\sin 1 - \cos 1) + \frac{1}{2} = \frac{e(\sin 1 - \cos 1) + 1}{2} \approx 0.909.$

55. $dv = x^2\,dx$, $v = \dfrac{x^3}{3}$, $u = \ln x$, $du = \dfrac{1}{x}\,dx$

$$\int x^2\ln x\,dx = \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\left(\frac{1}{x}\right)dx$$

$$= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx$$

Hence, $\displaystyle\int_1^2 x^2\ln x\,dx = \left[\frac{x^3}{3}\ln x - \frac{1}{9}x^3\right]_1^2$

$$= \frac{8}{3}\ln 2 - \frac{8}{9} + \frac{1}{9} = \frac{8}{3}\ln 2 - \frac{7}{2} \approx 1.071.$$

57. $dv = x\,dx,\ v = \dfrac{x^2}{2},\ u = \operatorname{arcsec} x,\ du = \dfrac{1}{x\sqrt{x^2-1}}\,dx$

$$\int x \operatorname{arcsec} x\,dx = \frac{x^2}{2}\operatorname{arcsec} x - \int \frac{x^2/2}{x\sqrt{x^2-1}}\,dx$$
$$= \frac{x^2}{2}\operatorname{arcsec} x - \frac{1}{4}\int \frac{2x}{\sqrt{x^2-1}}\,dx$$
$$= \frac{x^2}{2}\operatorname{arcsec} x - \frac{1}{2}\sqrt{x^2-1} + C$$

Hence,

$$\int_2^4 x \operatorname{arcsec} x\,dx = \left[\frac{x^2}{2}\operatorname{arcsec} x - \frac{1}{2}\sqrt{x^2-1}\right]_2^4$$
$$= \left(8 \operatorname{arcsec} 4 - \frac{\sqrt{15}}{2}\right) - \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$$
$$= 8 \operatorname{arcsec} 4 - \frac{\sqrt{15}}{2} + \frac{\sqrt{3}}{2} - \frac{2\pi}{3}$$
$$\approx 7.380.$$

59. $$\int x^2 e^{2x}\,dx = x^2\left(\frac{1}{2}e^{2x}\right) - (2x)\left(\frac{1}{4}e^{2x}\right) + 2\left(\frac{1}{8}e^{2x}\right) + C$$
$$= \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C$$
$$= \frac{1}{4}e^{2x}(2x^2 - 2x + 1) + C$$

Alternate signs	u and its derivatives	v' and its antiderivatives
+	x^2	e^{2x}
−	$2x$	$\frac{1}{2}e^{2x}$
+	2	$\frac{1}{4}e^{2x}$
−	0	$\frac{1}{8}e^{2x}$

61. $$\int x^3 \sin x\,dx = x^3(-\cos x) - 3x^2(-\sin x) + 6x\cos x - 6\sin x + C$$
$$= -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C$$
$$= (3x^2 - 6)\sin x - (x^3 - 6x)\cos x + C$$

Alternate signs	u and its derivatives	v' and its antiderivatives
+	x^3	$\sin x$
−	$3x^2$	$-\cos x$
+	$6x$	$-\sin x$
−	6	$\cos x$
+	0	$\sin x$

63. $$\int x\sec^2 x\,dx = x\tan x + \ln\lvert\cos x\rvert + C$$

Alternate signs	u and its derivatives	v' and its antiderivatives
+	x	$\sec^2 x$
−	1	$\tan x$
+	0	$-\ln\lvert\cos x\rvert$

65. Integration by parts is based on the product rule.

67. No. Substitution.

69. Yes. $u = x^2,\ dv = e^{2x}\,dx$

71. Yes. Let $u = x$ and $du = \dfrac{1}{\sqrt{x+1}},\ dx.$

(Substitution also works. Let $u = \sqrt{x+1}$)

73. $$\int t^3e^{-4t}\,dt = -\frac{e^{-4t}}{128}(32t^3 + 24t^2 + 12t + 3) + C$$

75. $$\int_0^{\pi/2} e^{-2x}\sin 3x\,dx = \left[\frac{e^{-2x}(-2\sin 3x - 3\cos 3x)}{13}\right]_0^{\pi/2} = \frac{1}{13}(2e^{-\pi} + 3) \approx 0.2374$$

77. (a) $dv = \sqrt{2x-3}\,dx \implies v = \int (2x-3)^{1/2}\,dx = \frac{1}{3}(2x-3)^{3/2}$

$u = 2x \implies du = 2\,dx$

$$\int 2x\sqrt{2x-3}\,dx = \frac{2}{3}x(2x-3)^{3/2} - \frac{2}{3}\int (2x-3)^{3/2}\,dx$$

$$= \frac{2}{3}x(2x-3)^{3/2} - \frac{2}{15}(2x-3)^{5/2} + C$$

$$= \frac{2}{15}(2x-3)^{3/2}(3x+3) + C = \frac{2}{5}(2x-3)^{3/2}(x+1) + C$$

(b) $u = 2x - 3 \implies x = \frac{u+3}{2}$ and $dx = \frac{1}{2}\,du$

$$\int 2x\sqrt{2x-3}\,dx = \int 2\left(\frac{u+3}{2}\right)u^{1/2}\left(\frac{1}{2}\right)du = \frac{1}{2}\int (u^{3/2} + 3u^{1/2})\,du = \frac{1}{2}\left[\frac{2}{5}u^{5/2} + 2u^{3/2}\right] + C$$

$$= \frac{1}{5}u^{3/2}(u+5) + C = \frac{1}{5}(2x-3)^{3/2}[(2x-3)+5] + C = \frac{2}{5}(2x-3)^{3/2}(x+1) + C$$

79. (a) $dv = \frac{x}{\sqrt{4+x^2}}\,dx \implies v = \int (4+x^2)^{-1/2}x\,dx = \sqrt{4+x^2}$

$u = x^2 \implies du = 2x\,dx$

$$\int \frac{x^3}{\sqrt{4+x^2}}\,dx = x^2\sqrt{4+x^2} - 2\int x\sqrt{4+x^2}\,dx$$

$$= x^2\sqrt{4+x^2} - \frac{2}{3}(4+x^2)^{3/2} + C = \frac{1}{3}\sqrt{4+x^2}\,(x^2-8) + C$$

(b) $u = 4 + x^2 \implies x^2 = u - 4$ and $2x\,dx = du \implies x\,dx = \frac{1}{2}\,du$

$$\int \frac{x^3}{\sqrt{4+x^2}}\,dx = \int \frac{x^2}{\sqrt{4+x^2}}\,x\,dx = \int \frac{u-4}{\sqrt{u}}\,\frac{1}{2}\,du$$

$$= \frac{1}{2}\int (u^{1/2} - 4u^{-1/2})\,du = \frac{1}{2}\left(\frac{2}{3}u^{3/2} - 8u^{1/2}\right) + C$$

$$= \frac{1}{3}u^{1/2}(u-12) + C = \frac{1}{3}\sqrt{4+x^2}\,[(4+x^2) - 12] + C = \frac{1}{3}\sqrt{4+x^2}\,(x^2-8) + C$$

81. $n = 0$: $\int \ln x\,dx = x(\ln x - 1) + C$

$n = 1$: $\int x\ln x\,dx = \frac{x^2}{4}(2\ln x - 1) + C$

$n = 2$: $\int x^2\ln x\,dx = \frac{x^3}{9}(3\ln x - 1) + C$

$n = 3$: $\int x^3\ln x\,dx = \frac{x^4}{16}(4\ln x - 1) + C$

$n = 4$: $\int x^4\ln x\,dx = \frac{x^5}{25}(5\ln x - 1) + C$

In general, $\int x^n\ln x\,dx = \frac{x^{n+1}}{(n+1)^2}[(n+1)\ln x - 1] + C$. (See Exercise 85.)

83. $dv = \sin x\,dx \implies v = -\cos x$

$u = x^n \implies du = nx^{n-1}\,dx$

$$\int x^n \sin x\,dx = -x^n \cos x + n\int x^{n-1}\cos x\,dx$$

85. $dv = x^n\,dx \implies v = \dfrac{x^{n+1}}{n+1}$

$u = \ln x \implies du = \dfrac{1}{x}\,dx$

$$\int x^n \ln x\,dx = \frac{x^{n+1}}{n+1}\ln x - \int \frac{x^n}{n+1}\,dx$$
$$= \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + C$$
$$= \frac{x^{n+1}}{(n+1)^2}[(n+1)\ln x - 1] + C$$

87. Use integration by parts twice.

(1) $dv = e^{ax}\,dx \implies v = \dfrac{1}{a}e^{ax}$

$u = \sin bx \implies du = b\cos bx\,dx$

(2) $dv = e^{ax}\,dx \implies v = \dfrac{1}{a}e^{ax}$

$u = \cos bx \implies du = -b\sin bx\,dx$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}\sin bx}{a} - \frac{b}{a}\int e^{ax}\cos bx\,dx$$
$$= \frac{e^{ax}\sin bx}{a} - \frac{b}{a}\left[\frac{e^{ax}\cos bx}{a} + \frac{b}{a}\int e^{ax}\sin bx\,dx\right] = \frac{e^{ax}\sin bx}{a} - \frac{b^2}{a^2}\int e^{ax}\sin bx\,dx$$

Therefore, $\left(1 + \dfrac{b^2}{a^2}\right)\displaystyle\int e^{ax}\sin bx\,dx = \dfrac{e^{ax}(a\sin bx - b\cos bx)}{a^2}$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2 + b^2} + C.$$

89. $n = 3$ (Use formula in Exercise 85.)

$$\int x^3 \ln x\,dx = \frac{x^4}{16}[4\ln x - 1] + C$$

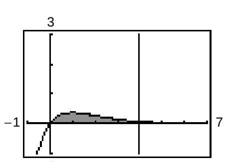
91. $a = 2$, $b = 3$ (Use formula in Exercise 88.)

$$\int e^{2x}\cos 3x\,dx = \frac{e^{2x}(2\cos 3x + 3\sin 3x)}{13} + C$$

93. $dv = e^{-x}\,dx \implies v = -e^{-x}$

$u = x \implies du = dx$

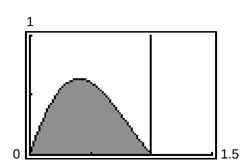
$$A = \int_0^4 xe^{-x}\,dx = \Big[-xe^{-x}\Big]_0^4 + \int_0^4 e^{-x}\,dx = \frac{-4}{e^4} - \Big[e^{-x}\Big]_0^4$$
$$= 1 - \frac{5}{e^4} \approx 0.908$$



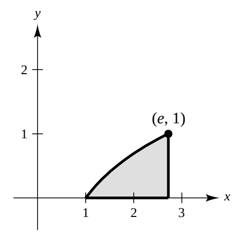
95. $A = \displaystyle\int_0^1 e^{-x}\sin(\pi x)\,dx$

$$= \left[\frac{e^{-x}(-\sin \pi x - \pi\cos \pi x)}{1 + \pi^2}\right]_0^1$$
$$= \frac{1}{1+\pi^2}\left(\frac{\pi}{e} + \pi\right) = \frac{\pi}{1+\pi^2}\left(\frac{1}{e} + 1\right)$$

≈ 0.395 (See Exercise 87.)



97. (a) $A = \int_1^e \ln x\, dx = \Big[-x + x \ln x\Big]_1^e = 1$ (See Exercise 4.)



(b) $R(x) = \ln x,\ r(x) = 0$

$$V = \pi \int_1^e (\ln x)^2\, dx$$

$$= \pi\Big[x(\ln x)^2 - 2x \ln x + 2x\Big]_1^e \text{ (Use integration by parts twice, see Exercise 7.)}$$

$$= \pi(e - 2) \approx 2.257$$

(c) $p(x) = x,\ h(x) = \ln x$

$$V = 2\pi \int_1^e x \ln x\, dx = 2\pi\left[\frac{x^2}{4}(-1 + 2 \ln x)\right]_1^e$$

$$= \frac{(e^2 + 1)\pi}{2} \approx 13.177 \text{ (See Exercise 85.)}$$

(d)

$$\bar{x} = \frac{\int_1^e x \ln x\, dx}{1} = \frac{e^2 + 1}{4} \approx 2.097$$

$$\bar{y} = \frac{\frac{1}{2}\int_1^e (\ln x)^2 dx}{1} = \frac{e - 2}{2} \approx 0.359$$

$$(\bar{x}, \bar{y}) = \left(\frac{e^2 + 1}{4}, \frac{e - 2}{2}\right) \approx (2.097, 0.359)$$

99. Average value $= \dfrac{1}{\pi}\displaystyle\int_0^\pi e^{-4t}(\cos 2t + 5 \sin 2t)\, dt$

$$= \frac{1}{\pi}\left[e^{-4t}\left(\frac{-4 \cos 2t + 2 \sin 2t}{20}\right) + 5e^{-4t}\left(\frac{-4 \sin 2t - 2 \cos 2t}{20}\right)\right]_0^\pi \text{ (From Exercises 87 and 88)}$$

$$= \frac{7}{10\pi}(1 - e^{-4\pi}) \approx 0.223$$

101. $c(t) = 100{,}000 + 4000t,\ r = 5\%,\ t_1 = 10$

$$P = \int_0^{10} (100{,}000 + 4000t)e^{-0.05t}\, dt = 4000\int_0^{10} (25 + t)e^{-0.05t}\, dt$$

Let $u = 25 + t,\ dv = e^{-0.05t}dt,\ du = dt,\ v = -\dfrac{100}{5}e^{-0.05t}$

$$P = 4000\left\{\left[(25 + t)\left(-\frac{100}{5}e^{-0.05t}\right)\right]_0^{10} + \frac{100}{5}\int_0^{10} e^{-0.05t}\, dt\right\}$$

$$= 4000\left\{\left[(25 + t)\left(-\frac{100}{5}e^{-0.05t}\right)\right]_0^{10} - \left[\frac{10{,}000}{25}e^{-0.05t}\right]_0^{10}\right\} \approx \$931{,}265$$

103. $\displaystyle\int_{-\pi}^{\pi} x \sin nx\, dx = \left[-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]_{-\pi}^{\pi}$

$$= -\frac{\pi}{n}\cos \pi n - \frac{\pi}{n}\cos(-\pi n)$$

$$= -\frac{2\pi}{n}\cos \pi n$$

$$= \begin{cases} -(2\pi/n), & \text{if } n \text{ is even} \\ (2\pi/n), & \text{if } n \text{ is odd} \end{cases}$$

105. Let $u = x$, $dv = \sin\left(\frac{n\pi}{2}x\right)dx$, $du = dx$, $v = -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)$.

$$I_1 = \int_0^1 x\sin\left(\frac{n\pi}{2}x\right)dx = \left[\frac{-2x}{n\pi}\cos\left(\frac{n\pi}{2}x\right)\right]_0^1 + \frac{2}{n\pi}\int_0^1 \cos\left(\frac{n\pi}{2}x\right)dx$$

$$= -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \left[\left(\frac{2}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}x\right)\right]_0^1$$

$$= -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}\right)$$

Let $u = (-x + 2)$, $dv = \sin\left(\frac{n\pi}{2}x\right)dx$, $du = -dx$, $v = -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)$.

$$I_2 = \int_1^2 (-x+2)\sin\left(\frac{n\pi}{2}x\right)dx = \left[\frac{-2(-x+2)}{n\pi}\cos\left(\frac{n\pi}{2}x\right)\right]_1^2 - \frac{2}{n\pi}\int_1^2 \cos\left(\frac{n\pi}{2}x\right)dx$$

$$= \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) - \left[\left(\frac{2}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}x\right)\right]_1^2$$

$$= \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}\right)$$

$$h(I_1 + I_2) = b_n = h\left[\left(\frac{2}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}\right)\right] = \frac{8h}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$$

107. Shell Method:

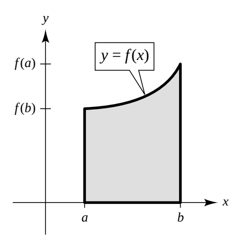
$$V = 2\pi\int_a^b x f(x)\,dx$$

$$dv = x\,dx \implies v = \frac{x^2}{2}$$

$$u = f(x) \implies du = f'(x)\,dx$$

$$V = 2\pi\left[\frac{x^2}{2}f(x) - \int\frac{x^2}{2}f'(x)dx\right]_a^b$$

$$= \pi\left[(b^2f(b) - a^2f(a)) - \int_a^b x^2 f'(x)\,dx\right]$$



Disk Method:

$$V = \pi\int_0^{f(a)}(b^2 - a^2)\,dy + \pi\int_{f(a)}^{f(b)}[b^2 - [f^{-1}(y)]^2]\,dy$$

$$= \pi(b^2 - a^2)f(a) + \pi b^2(f(b) - f(a)) - \pi\int_{f(a)}^{f(b)}[f^{-1}(y)]^2\,dy$$

$$= \pi\left[(b^2f(b) - a^2f(a)) - \int_{f(a)}^{f(b)}[f^{-1}(y)]^2\,dy\right]$$

Since $x = f^{-1}(y)$, we have $f(x) = y$ and $f'(x)dx = dy$. When $y = f(a)$, $x = a$. When $y = f(b)$, $x = b$. Thus,

$$\int_{f(a)}^{f(b)}[f^{-1}(y)]^2\,dy = \int_a^b x^2f'(x)\,dx$$

and the volumes are the same.

109. $f'(x) = xe^{-x}$

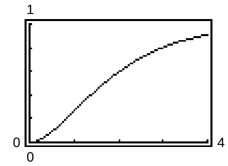
(a) $f(x) = \int xe^{-x}\,dx = -xe^{-x} - e^{-x} + C$

(Parts: $u = x,\ dv = e^{-x}\,dx$)

$f(0) = 0 = -1 + C \Rightarrow C = 1$

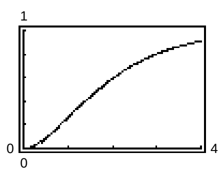
$f(x) = -xe^{-x} - e^{-x} + 1$

(b)



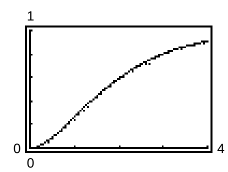
(c) You obtain the points

n	x_n	y_n
0	0	0
1	0.05	0
2	0.10	2.378×10^{-3}
3	0.15	0.0069
4	0.20	0.0134
⋮	⋮	⋮
80	4.0	0.9064



(d) You obtain the points

n	x_n	y_n
0	0	0
1	0.1	0
2	0.2	0.0090484
3	0.3	0.025423
4	0.4	0.047648
⋮	⋮	⋮
40	4.0	0.9039



(e) $f(4) = 0.9084$

The approximations are tangent line approximations. The results in (c) are better because Δx is smaller.

Section 7.3 Trigonometric Integrals

1. $f(x) = \sin^4 x + \cos^4 x$

(a) $$\sin^4 x + \cos^4 x = \left(\frac{1 - \cos 2x}{2}\right)^2 + \left(\frac{1 + \cos 2x}{2}\right)^2$$
$$= \frac{1}{4}[1 - 2\cos 2x + \cos^2 2x + 1 + 2\cos 2x + \cos^2 2x]$$
$$= \frac{1}{4}\left[2 + 2\frac{1 + \cos 4x}{2}\right]$$
$$= \frac{1}{4}[3 + \cos 4x]$$

(b) $$\sin^4 x + \cos^4 x = (\sin^2 x)^2 + \cos^4 x$$
$$= (1 - \cos^2 x)^2 + \cos^4 x$$
$$= 1 - 2\cos^2 x + 2\cos^4 x$$

(c) $$\sin^4 x + \cos^4 x = \sin^4 x + 2\sin^2 x\cos^2 x + \cos^4 x - 2\sin^2 x\cos^2 x$$
$$= (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x$$
$$= 1 - 2\sin^2 x\cos^2 x$$

—CONTINUED—

1. —CONTINUED—

(d) $$1 - 2\sin^2 x\cos^2 x = 1 - (2\sin x\cos x)(\sin x\cos x)$$
$$= 1 - (\sin 2x)\left(\frac{1}{2}\sin 2x\right)$$
$$= 1 - \frac{1}{2}\sin^2(2x)$$

(e) Four ways. There is often more than one way to rewrite a trigonometric expression.

3. Let $u = \cos x$, $du = -\sin x\,dx$.

$$\int \cos^3 x\sin x\,dx = -\int \cos^3 x(-\sin x)\,dx$$
$$= -\frac{1}{4}\cos^4 x + C$$

5. Let $u = \sin 2x$, $du = 2\cos 2x\,dx$.

$$\int \sin^5 2x\cos 2x\,dx = \frac{1}{2}\int \sin^5 2x(2\cos 2x)dx$$
$$= \frac{1}{12}\sin^6 2x + C$$

7. Let $u = \cos x$, $du = -\sin x\,dx$.

$$\int \sin^5 x\cos^2 x\,dx = \int \sin x(1 - \cos^2 x)^2\cos^2 x\,dx$$
$$= -\int(\cos^2 x - 2\cos^4 x + \cos^6 x)(-\sin x)\,dx = \frac{-1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C$$

9. $$\int \cos^3\theta\sqrt{\sin\theta}\,d\theta = \int \cos\theta(1 = \sin^2\theta)(\sin\theta)^{1/2}\,d\theta$$
$$= \int[(\sin\theta)^{1/2} - (\sin\theta)^{5/2}]\cos\theta\,d\theta$$
$$= \frac{2}{3}(\sin\theta)^{3/2} - \frac{2}{7}(\sin\theta)^{7/2} + C$$

11. $$\int \cos^2 3x\,dx = \int \frac{1 + \cos 6x}{2}\,dx$$
$$= \frac{1}{2}\left(x + \frac{1}{6}\sin 6x\right) + C$$
$$= \frac{1}{12}(6x + \sin 6x) + C$$

13. $$\int \sin^2\alpha\cdot\cos^2\alpha\,d\alpha = \int \frac{1 - \cos 2\alpha}{2}\cdot\frac{1 + \cos 2\alpha}{2}\,d\alpha$$
$$= \frac{1}{4}\int(1 - \cos^2 2\alpha)\,d\alpha$$
$$= \frac{1}{4}\int\left(1 - \frac{1 + \cos 4\alpha}{2}\right)d\alpha$$
$$= \frac{1}{8}\int(1 - \cos 4\alpha)\,d\alpha$$
$$= \frac{1}{8}\left[\alpha - \frac{1}{4}\sin 4\alpha\right] + C$$
$$= \frac{1}{32}[4\alpha - \sin 4\alpha] + C$$

15. Integration by parts.

$$dv = \sin^2 x\,dx = \frac{1-\cos 2x}{2} \implies v = \frac{x}{2} - \frac{\sin 2x}{4} = \frac{1}{4}(2x - \sin 2x)$$

$$u = x \implies du = dx$$

$$\int x \sin^2 x\,dx = \frac{1}{4}x(2x - \sin 2x) - \frac{1}{4}\int (2x - \sin 2x)\,dx$$

$$= \frac{1}{4}x(2x - \sin 2x) - \frac{1}{4}\left(x^2 + \frac{1}{2}\cos 2x\right) + C = \frac{1}{8}(2x^2 - 2x\sin 2x - \cos 2x) + C$$

17. Let $u = \sin x$, $du = \cos x\,dx$.

$$\int_0^{\pi/2} \cos^3 x\,dx = \int_0^{\pi/2} (1 - \sin^2 x)\cos x\,dx$$

$$= \left[\sin x - \frac{1}{3}\sin^3 x\right]_0^{\pi/2} = \frac{2}{3}$$

19. Let $u = \sin x$, $du = \cos x\,dx$.

$$\int_0^{\pi/2} \cos^7 x\,dx = \int_0^{\pi/2} (1 - \sin^2 x)^3 \cos x\,dx = \int_0^{\pi/2} (1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x)\cos x\,dx$$

$$= \left[\sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{1}{7}\sin^7 x\right]_0^{\pi/2} = \frac{16}{35}$$

21. $\displaystyle\int \sec(3x)\,dx = \frac{1}{3}\ln\left|\sec 3x + \tan 3x\right| + C$

23. $\displaystyle\int \sec^4 5x\,dx = \int (1 + \tan^2 5x)\sec^2 5x\,dx$

$$= \frac{1}{5}\left(\tan 5x + \frac{\tan^3 5x}{3}\right) + C$$

$$= \frac{\tan 5x}{15}(3 + \tan^2 5x) + C$$

25. $dv = \sec^2 \pi x\,dx \implies v = \dfrac{1}{\pi}\tan \pi x$

$$u = \sec \pi x \implies du = \pi \sec \pi x \tan \pi x\,dx$$

$$\int \sec^3 \pi x\,dx = \frac{1}{\pi}\sec \pi x \tan \pi x - \int \sec \pi x \tan^2 \pi x\,dx = \frac{1}{\pi}\sec \pi x \tan \pi x - \int \sec \pi x(\sec^2 \pi x - 1)\,dx$$

$$2\int \sec^3 \pi x\,dx = \frac{1}{\pi}(\sec\pi x \tan \pi x + \ln|\sec\pi x + \tan \pi x|) + C_1$$

$$\int \sec^3 \pi x\,dx = \frac{1}{2\pi}(\sec \pi x \tan\pi x + \ln|\sec \pi x + \tan \pi x|) + C$$

27. $\displaystyle\int \tan^5 \frac{x}{4}\,dx = \int\left(\sec^2\frac{x}{4} - 1\right)\tan^3\frac{x}{4}\,dx$

$$= \int \tan^3\frac{x}{4}\sec^2\frac{x}{4}\,dx - \int \tan^3\frac{x}{4}\,dx$$

$$= \tan^4\frac{x}{4} - \int\left(\sec^2\frac{x}{4} - 1\right)\tan\frac{x}{4}\,dx$$

$$= \tan^4\frac{x}{4} - 2\tan^2\frac{x}{4} - 4\ln\left|\cos\frac{x}{4}\right| + C$$

29. $u = \tan x$, $du = \sec^2 x\,dx$

$$\int \sec^2 x \tan x\,dx = \frac{1}{2}\tan^2 x + C$$

31. $\int \tan^2 x \sec^2 x\,dx = \dfrac{\tan^3 x}{3} + C$

33. $\int \sec^6 4x \tan 4x\,dx = \dfrac{1}{4}\int \sec^5 4x(4\sec 4x \tan 4x)\,dx$

$= \dfrac{\sec^6 4x}{24} + C$

35. Let $u = \sec x$, $du = \sec x \tan x\,dx$.

$\int \sec^3 x \tan x\,dx = \int \sec^2 x(\sec x \tan x)\,dx$

$= \dfrac{1}{3}\sec^3 x + C$

37. $\int \dfrac{\tan^2 x}{\sec x}\,dx = \int \dfrac{(\sec^2 x - 1)}{\sec x}\,dx$

$= \int (\sec x - \cos x)\,dx$

$= \ln|\sec x + \tan x| - \sin x + C$

39. $r = \int \sin^4(\pi\theta)\,d\theta = \dfrac{1}{4}\int [1 - \cos(2\pi\theta)]^2\,d\theta$

$= \dfrac{1}{4}\int [1 - 2\cos(2\pi\theta) + \cos^2(2\pi\theta)]\,d\theta$

$= \dfrac{1}{4}\int \left[1 - 2\cos(2\pi\theta) + \dfrac{1 + \cos(4\pi\theta)}{2}\right] d\theta$

$= \dfrac{1}{4}\left[\theta - \dfrac{1}{\pi}\sin(2\pi\theta) + \dfrac{\theta}{2} + \dfrac{1}{8\pi}\sin(4\pi\theta)\right] + C$

$= \dfrac{1}{32\pi}[12\pi\theta - 8\sin(2\pi\theta) + \sin(4\pi\theta)] + C$

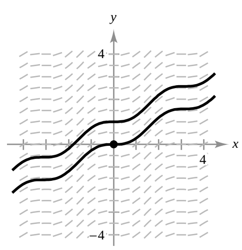
41. $y = \int \tan^3 3x \sec 3x\,dx$

$= \int (\sec^2 3x - 1)\sec 3x \tan 3x\,dx$

$= \dfrac{1}{3}\int \sec^2 3x(3\sec 3x \tan 3x)\,dx - \dfrac{1}{3}\int 3\sec 3x \tan 3x\,dx$

$= \dfrac{1}{9}\sec^3 3x - \dfrac{1}{3}\sec 3x + C$

43. (a)

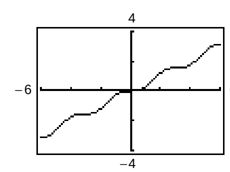


(b) $\dfrac{dy}{dx} = \sin^2 x$, $(0, 0)$

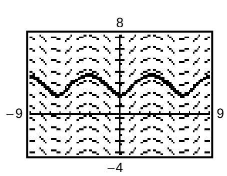
$y = \int \sin^2 x\,dx = \int \dfrac{1 - \cos 2x}{2}\,dx$

$= \dfrac{1}{2}x - \dfrac{\sin 2x}{4} + C$

$(0, 0)$: $0 = C$, $y = \dfrac{1}{2}x - \dfrac{\sin 2x}{4}$



45. $\dfrac{dy}{dx} = \dfrac{3\sin x}{y}$, $y(0) = 2$



47. $\int \sin 3x \cos 2x\,dx = \dfrac{1}{2}\int (\sin 5x + \sin x)\,dx$

$= \dfrac{-1}{2}\left(\dfrac{1}{5}\cos 5x + \cos x\right) + C$

$= \dfrac{-1}{10}(\cos 5x + 5\cos x) + C$

49. $\int \sin\theta \sin 3\theta\,d\theta = \dfrac{1}{2}\int (\cos 2\theta - \cos 4\theta)\,d\theta$

$= \dfrac{1}{2}\left(\dfrac{1}{2}\sin 2\theta - \dfrac{1}{4}\sin 4\theta\right) + C$

$= \dfrac{1}{8}(2\sin 2\theta - \sin 4\theta) + C$

51. $\int \cot^3 2x\,dx = \int (\csc^2 2x - 1)\cot 2x\,dx$

$= -\dfrac{1}{2}\int \cot 2x(-2\csc^2 2x)\,dx - \dfrac{1}{2}\int \dfrac{2\cos 2x}{\sin 2x}\,dx$

$= -\dfrac{1}{4}\cot^2 2x - \dfrac{1}{2}\ln|\sin 2x| + C$

$= \dfrac{1}{4}(\ln|\csc^2 2x| - \cot^2 2x) + C$

53. Let $u = \cot\theta$, $du = -\csc^2\theta\,d\theta$.

$$\int \csc^4\theta\,d\theta = \int \csc^2\theta(1 + \cot^2\theta)\,d\theta$$
$$= \int \csc^2\theta\,d\theta + \int \csc^2\theta\cot^2\theta\,d\theta$$
$$= -\cot\theta - \frac{1}{3}\cot^3\theta + C$$

55. $$\int \frac{\cot^2 t}{\csc t}\,dt = \int \frac{\csc^2 t - 1}{\csc t}\,dt$$
$$= \int (\csc t - \sin t)dt$$
$$= \ln|\csc t - \cot t| + \cos t + C$$

57. $$\int \frac{1}{\sec x\tan x}\,dx = \int \frac{\cos^2 x}{\sin x}\,dx = \int \frac{1 - \sin^2 x}{\sin x}\,dx$$
$$= \int (\csc x - \sin x)\,dx$$
$$= \ln|\csc x - \cot x| + \cos x + C$$

59. $$\int (\tan^4 t - \sec^4 t)\,dt = \int (\tan^2 t + \sec^2 t)(\tan^2 t - \sec^2 t)\,dt \qquad (\tan^2 t - \sec^2 t = -1)$$
$$= -\int (\tan^2 t + \sec^2 t)\,dt = -\int (2\sec^2 t - 1)\,dt = -2\tan t + t + C$$

61. $$\int_{-\pi}^{\pi} \sin^2 x\,dx = 2\int_0^{\pi} \frac{1 - \cos 2x}{2}\,dx$$
$$= \left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \pi$$

63. $$\int_0^{\pi/4} \tan^3 x\,dx = \int_0^{\pi/4} (\sec^2 x - 1)\tan x\,dx$$
$$= \int_0^{\pi/4} \sec^2 x\tan x\,dx - \int_0^{\pi/4} \frac{\sin x}{\cos x}\,dx$$
$$= \left[\frac{1}{2}\tan^2 x + \ln|\cos x|\right]_0^{\pi/4}$$
$$= \frac{1}{2}(1 - \ln 2)$$

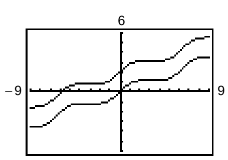
65. Let $u = 1 + \sin t$, $du = \cos t\,dt$.

$$\int_0^{\pi/2} \frac{\cos t}{1 + \sin t}\,dt = \Big[\ln|1 + \sin t|\Big]_0^{\pi/2} = \ln 2$$

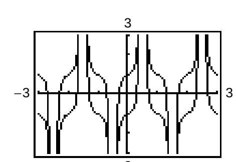
67. Let $u = \sin x$, $du = \cos x\,dx$.

$$\int_{-\pi/2}^{\pi/2} \cos^3 x\,dx = 2\int_0^{\pi/2} (1 - \sin^2 x)\cos x\,dx$$
$$= 2\left[\sin x - \frac{1}{3}\sin^3 x\right]_0^{\pi/2} = \frac{4}{3}$$

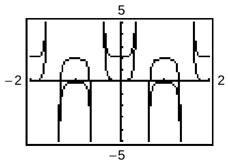
69. $$\int \cos^4\frac{x}{2}dx = \frac{1}{16}[6x + 8\sin x + \sin 2x] + C$$



71. $$\int \sec^5 \pi x\,dx = \frac{1}{4\pi}\left\{\sec^3 \pi x\tan \pi x + \frac{3}{2}[\sec \pi x\tan \pi x + \ln|\sec \pi x + \tan \pi x|]\right\} + C$$



73. $\displaystyle\int \sec^5 \pi x \tan \pi x\, dx = \frac{1}{5\pi}\sec^5 \pi x + C$



75. $\displaystyle\int_0^{\pi/4} \sin 2\theta \sin 3\theta\, d\theta = \frac{1}{2}\left[\sin\theta - \frac{1}{5}\sin 5\theta\right]_0^{\pi/4} = \frac{3\sqrt{2}}{10}$

77. $\displaystyle\int_0^{\pi/2} \sin^4 x\, dx = \frac{1}{4}\left[\frac{3x}{2} - \sin 2x + \frac{1}{8}\sin 4x\right]_0^{\pi/2}$

$$= \frac{3\pi}{16}$$

79. (a) Save one sine factor and convert the remaining sine factors to cosine. Then expand and integrate.

(b) Save one cosine factor and convert the remaining cosine factors to sine. Then expand and integrate.

(c) Make repeated use of the power reducing formula to convert the integrand to odd powers of the cosine.

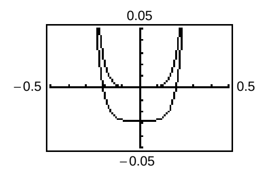
81. (a) Let $u = \tan 3x$, $du = 3\sec^2 3x\, dx$.

$$\int \sec^4 3x \tan^3 3x\, dx = \int \sec^2 3x \tan^3 3x \sec^2 3x\, dx$$
$$= \frac{1}{3}\int (\tan^2 3x + 1)\tan^3 3x(3\sec^2 3x)\, dx$$
$$= \frac{1}{3}\int (\tan^5 3x + \tan^3 3x)(3\sec^2 3x)\, dx$$
$$= \frac{\tan^6 3x}{18} + \frac{\tan^4 3x}{12} + C_1$$

Or let $u = \sec 3x$, $du = 3\sec 3x \tan 3x\, dx$.

$$\int \sec^4 3x \tan^3 3x\, dx = \int \sec^3 3x \tan^2 3x \sec 3x \tan 3x\, dx$$
$$= \frac{1}{3}\int \sec^3 3x(\sec^2 3x - 1)(3\sec 3x \tan 3x)\, dx$$
$$= \frac{\sec^6 3x}{18} - \frac{\sec^4 3x}{12} + C$$

(b)

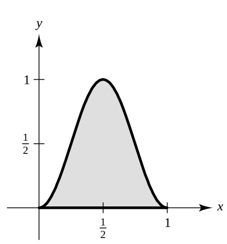


(c) $\displaystyle\frac{\sec^6 3x}{18} - \frac{\sec^4 3x}{12} + C = \frac{(1 + \tan^2 3x)^3}{18} - \frac{(1 + \tan^2 3x)^2}{12} + C$

$$= \frac{1}{18}\tan^6 3x + \frac{1}{6}\tan^4 3x + \frac{1}{6}\tan^2 3x + \frac{1}{18} - \frac{1}{12}\tan^4 3x - \frac{1}{6}\tan^2 3x - \frac{1}{12} + C$$
$$= \frac{\tan^6 3x}{18} + \frac{\tan^4 3x}{12} + \left(\frac{1}{18} - \frac{1}{12}\right) + C$$
$$= \frac{\tan^6 3x}{18} + \frac{\tan^4 3x}{12} + C_2$$

83. $\displaystyle A = \int_0^1 \sin^2(\pi x)\, dx$

$$= \int_0^1 \frac{1 - \cos(2\pi x)}{2}\, dx$$
$$= \left[\frac{x}{2} - \frac{1}{4\pi}\sin(2\pi x)\right]_0^1$$
$$= \frac{1}{2}$$



85. (a) $V = \pi\int_0^{\pi} \sin^2 x\,dx = \frac{\pi}{2}\int_0^{\pi}(1 - \cos 2x)\,dx = \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{\pi^2}{2}$

(b) $A = \int_0^{\pi} \sin x\,dx = \left[-\cos x\right]_0^{\pi} = 1 + 1 = 2$

Let $u = x$, $dv = \sin x\,dx$, $du = dx$, $v = -\cos x$.

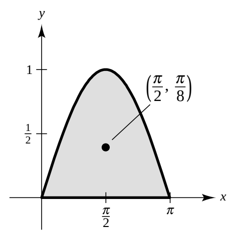
$$\bar{x} = \frac{1}{A}\int_0^{\pi} x \sin x\,dx = \frac{1}{2}\left[\left[-x\cos x\right]_0^{\pi} + \int_0^{\pi}\cos x\,dx\right] = \frac{1}{2}\left[-x\cos x + \sin x\right]_0^{\pi} = \frac{\pi}{2}$$

$$\bar{y} = \frac{1}{2A}\int_0^{\pi}\sin^2 x\,dx$$

$$= \frac{1}{8}\int_0^{\pi}(1 - \cos 2x)\,dx$$

$$= \frac{1}{8}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{\pi}{8}$$

$$(\bar{x}, \bar{y}) = \left(\frac{\pi}{2}, \frac{\pi}{8}\right)$$



87. $dv = \sin x\,dx \implies v = -\cos x$

$u = \sin^{n-1} x \implies du = (n - 1)\sin^{n-2} x \cos x\,dx$

$$\int \sin^n x\,dx = -\sin^{n-1} x \cos x + (n - 1)\int \sin^{n-2} x \cos^2 x\,dx$$

$$= -\sin^{n-1} x \cos x + (n - 1)\int \sin^{n-2} x(1 - \sin^2 x)\,dx$$

$$= -\sin^{n-1} x \cos x + (n - 1)\int \sin^{n-2} x\,dx - (n - 1)\int \sin^n x\,dx$$

Therefore, $n\int \sin^n x\,dx = -\sin^{n-1} x \cos x + (n - 1)\int \sin^{n-2} x\,dx$

$$\int \sin^n x\,dx = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n - 1}{n}\int \sin^{n-2} x\,dx.$$

89. Let $u = \sin^{n-1} x$, $du = (n - 1)\sin^{n-2} x \cos x\,dx$, $dv = \cos^m x \sin x\,dx$, $v = \frac{-\cos^{m+1} x}{m + 1}$.

$$\int \cos^m x \sin^n x\,dx = \frac{-\sin^{n-1} x \cos^{m+1} x}{m + 1} + \frac{n - 1}{m + 1}\int \sin^{n-2} x \cos^{m+2} x\,dx$$

$$= \frac{-\sin^{n-1} x \cos^{m+1} x}{m + 1} + \frac{n - 1}{m + 1}\int \sin^{n-2} x \cos^m x(1 - \sin^2 x)\,dx$$

$$= \frac{-\sin^{n-1} x \cos^{m+1} x}{m + 1} + \frac{n - 1}{m + 1}\int \sin^{n-2} x \cos^m x\,dx - \frac{n - 1}{m + 1}\int \sin^n x \cos^m x\,dx$$

$$\frac{m + n}{m + 1}\int \cos^m x \sin^n x\,dx = \frac{-\sin^{n-1} x \cos^{m+1} x}{m + 1} + \frac{n - 1}{m + 1}\int \sin^{n-2} x \cos^m x\,dx$$

$$\int \cos^m x \sin^n x\,dx = \frac{-\cos^{m+1} x \sin^{n-1}}{m + n} + \frac{n - 1}{m + n}\int \cos^m x \sin^{n-2} x\,dx$$

91. $$\int \sin^5 x\,dx = -\frac{\sin^4 x\cos x}{5} + \frac{4}{5}\int \sin^3 x\,dx$$
$$= -\frac{\sin^4 x\cos x}{5} + \frac{4}{5}\left[-\frac{\sin^2 x\cos x}{3} + \frac{2}{3}\int \sin x\,dx\right]$$
$$= -\frac{1}{5}\sin^4 x\cos x - \frac{4}{15}\sin^2 x\cos x - \frac{8}{15}\cos x + C$$
$$= -\frac{\cos x}{15}[3\sin^4 x + 4\sin^2 x + 8] + C$$

93. $$\int \sec^4\left(\frac{2\pi x}{5}\right)dx = \frac{5}{2\pi}\int \sec^4\left(\frac{2\pi x}{5}\right)\frac{2\pi}{5}\,dx$$
$$= \frac{5}{2\pi}\left[\frac{1}{3}\sec^2\left(\frac{2\pi x}{5}\right)\tan\left(\frac{2\pi x}{5}\right) + \frac{2}{3}\int \sec^2\left(\frac{2\pi x}{5}\right)\frac{2\pi}{5}\,dx\right]$$
$$= \frac{5}{6\pi}\left[\sec^2\left(\frac{2\pi x}{5}\right)\tan\left(\frac{2\pi x}{5}\right) + 2\tan\left(\frac{2\pi x}{5}\right)\right] + C$$
$$= \frac{5}{6\pi}\tan\left(\frac{2\pi x}{5}\right)\left[\sec^2\left(\frac{2\pi x}{5}\right) + 2\right] + C$$

95. (a) $f(t) = a_0 + a_1\cos\frac{\pi t}{6} + b_1\sin\frac{\pi t}{6}$ where:

$$a_0 = \frac{1}{12}\int_0^{12} f(t)\,dt$$
$$a_1 = \frac{1}{6}\int_0^{12} f(t)\cos\frac{\pi t}{6}\,dt$$
$$b_1 = \frac{1}{6}\int_0^{12} f(t)\sin\frac{\pi t}{6}\,dt$$

$$a_0 \approx \frac{12-0}{3(12)^2}[30.9 + 4(32.2) + 2(41.1) + 4(53.7) + 2(64.6) + 4(74.0) + 2(78.2) + 4(77.0) + 2(71.0) + 4(60.1) + 2(47.1) + 4(35.7) + 30.9] \approx 55.46$$

$$a_1 \approx \frac{12-0}{6(3)(12)}\left[30.9\cos 0 + 4\left(32.2\cos\frac{\pi}{6}\right) + 2\left(41.1\cos\frac{\pi}{3}\right) + 4\left(53.7\cos\frac{\pi}{2}\right) + 2\left(64.6\cos\frac{2\pi}{3}\right) + 4\left(74.0\cos\frac{5\pi}{6}\right) + 2(78.2\cos\pi) + 4\left(77.0\cos\frac{7\pi}{6}\right) + 2\left(71.0\cos\frac{4\pi}{3}\right) + 4\left(60.1\cos\frac{3\pi}{2}\right) + 2\left(47.1\cos\frac{5\pi}{3}\right) + 4\left(35.7\cos\frac{11\pi}{6}\right) + 30.9\cos 2\pi\right] \approx -23.88$$

$$b_1 \approx \frac{12-0}{6(3)(12)}\left[30.9\sin 0 + 4\left(32.2\sin\frac{\pi}{6}\right) + 2\left(41.1\sin\frac{\pi}{3}\right) + 4\left(53.7\sin\frac{\pi}{2}\right) + 2\left(64.6\sin\frac{2\pi}{3}\right) + 4\left(74.0\sin\frac{5\pi}{6}\right) + 2(78.2\sin\pi) + 4\left(77.0\sin\frac{7\pi}{6}\right) + 2\left(71.0\sin\frac{4\pi}{3}\right) + 4\left(60.1\sin\frac{3\pi}{2}\right) + 2\left(47.1\sin\frac{5\pi}{3}\right) + 4\left(35.7\sin\frac{11\pi}{6}\right) + 30.9\sin 2\pi\right] \approx -3.34$$

$$H(t) \approx 55.46 - 23.88\cos\frac{\pi t}{6} - 3.34\sin\frac{\pi t}{6}$$

—CONTINUED—

95. —CONTINUED—

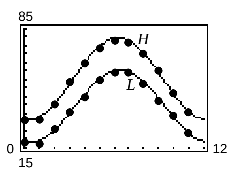
(b) $$a_0 \approx \frac{12-0}{3(12)^2}[18.0 + 4(17.7) + 2(25.8) + 4(36.1) + 2(45.4) + 4(55.2) + 2(59.9) + 4(59.4) + 2(53.1) + 4(43.2) + 2(34.3) + 4(24.2) + 18.0] \approx 39.34$$

$$a_1 \approx \frac{12-0}{6(3)(12)}\left[18.0\cos 0 + 4\left(17.7\cos\frac{\pi}{6}\right) + 2\left(25.8\cos\frac{\pi}{3}\right) + 4\left(36.1\cos\frac{\pi}{2}\right) + 2\left(45.4\cos\frac{2\pi}{3}\right) + 4\left(55.2\cos\frac{5\pi}{6}\right) + 2(59.9\cos\pi) + 4\left(59.4\cos\frac{7\pi}{6}\right) + 2\left(53.1\cos\frac{4\pi}{3}\right) + 4\left(43.2\cos\frac{3\pi}{2}\right) + 2\left(34.3\cos\frac{5\pi}{3}\right) + 4\left(24.2\cos\frac{11\pi}{6}\right) + 18\cos 2\pi\right] \approx -20.78$$

$$b_1 \approx \frac{12-0}{6(3)(12)}\left[18.0\sin 0 + 4\left(17.7\sin\frac{\pi}{6}\right) + 2\left(25.8\sin\frac{\pi}{3}\right) + 4\left(36.1\sin\frac{\pi}{2}\right) + 2\left(45.4\sin\frac{2\pi}{3}\right) + 4\left(55.2\sin\frac{5\pi}{6}\right) + 2(59.9\sin\pi) + 4\left(59.4\sin\frac{7\pi}{6}\right) + 2\left(53.1\sin\frac{4\pi}{3}\right) + 4\left(43.2\sin\frac{3\pi}{2}\right) + 2\left(34.3\sin\frac{5\pi}{3}\right) + 4\left(24.2\sin\frac{11\pi}{6}\right) + 18\sin 2\pi\right] \approx -4.33$$

$$L(t) \approx 39.34 - 20.78\cos\frac{\pi t}{6} - 4.33\sin\frac{\pi t}{6}$$

(c) The difference between the maximum and minimum temperatures is greatest in the summer.



97. $$\int_{-\pi}^{\pi}\cos(mx)\cos(nx)\,dx = \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_{-\pi}^{\pi} = 0,\ (m \neq n)$$

$$\int_{-\pi}^{\pi}\sin(mx)\sin(nx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi}[\cos(m-n)x - \cos(m+n)x]\,dx$$

$$= \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0,\ (m \neq n)$$

$$\int_{-\pi}^{\pi}\sin(mx)\cos(nx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi}[\sin(m+n)x + \sin(m-n)x]\,dx$$

$$= -\frac{1}{2}\left[\frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n}\right]_{-\pi}^{\pi},\ (m \neq n)$$

$$= -\frac{1}{2}\left[\left(\frac{\cos(m+n)\pi}{m+n} + \frac{\cos(m-n)\pi}{m-n}\right) - \left(\frac{\cos(m+n)(-\pi)}{m+n} + \frac{\cos(m-n)(-\pi)}{m-n}\right)\right]$$

$$= 0, \text{ since } \cos(-\theta) = \cos\theta.$$

$$\int_{-\pi}^{\pi}\sin(mx)\cos(mx)\,dx = \frac{1}{m}\left[\frac{\sin^2(mx)}{2}\right]_{-\pi}^{\pi} = 0$$

Section 7.4 Trigonometric Substitution

1. $$\frac{d}{dx}\left[4\ln\left|\frac{\sqrt{x^2+16}-4}{x}\right| + \sqrt{x^2+16}+C\right] = \frac{d}{dx}\left[4\ln\left|\sqrt{x^2+16}-4\right| - 4\ln|x| + \sqrt{x^2+16}+C\right]$$
$$= 4\left[\frac{x/\sqrt{x^2+16}}{\sqrt{x^2+16}-4}\right] - \frac{4}{x} + \frac{x}{\sqrt{x^2+16}}$$
$$= \frac{4x}{\sqrt{x^2+16}\left(\sqrt{x^2+16}-4\right)} - \frac{4}{x} + \frac{x}{\sqrt{x^2+16}}$$
$$= \frac{4x^2 - 4\sqrt{x^2+16}\left(\sqrt{x^2+16}-4\right) + x^2\left(\sqrt{x^2+16}-4\right)}{x\sqrt{x^2+16}\left(\sqrt{x^2+16}-4\right)}$$
$$= \frac{4x^2 - 4(x^2+16) + 16\sqrt{x^2+16} + x^2\sqrt{x^2+16} - 4x^2}{x\sqrt{x^2+16}\left(\sqrt{x^2+16}-4\right)}$$
$$= \frac{\sqrt{x^2+16}(x^2+16) - 4(x^2+16)}{x\sqrt{x^2+16}\left(\sqrt{x^2+16}-4\right)}$$
$$= \frac{(x^2+16)\left(\sqrt{x^2+16}-4\right)}{x\sqrt{x^2+16}\left(\sqrt{x^2+16}-4\right)} = \frac{\sqrt{x^2+16}}{x}$$

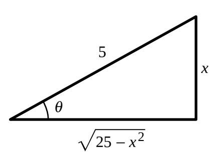
Indefinite integral: $\int \frac{\sqrt{x^2+16}}{x}\,dx$ Matches (b)

3. $$\frac{d}{dx}\left[8\arcsin\frac{x}{4} - \frac{x\sqrt{16-x^2}}{2} + C\right] = 8\frac{1/4}{\sqrt{1-(x/4)^2}} - \frac{x(1/2)(16-x^2)^{-1/2}(-2x) + \sqrt{16-x^2}}{2}$$
$$= \frac{8}{\sqrt{16-x^2}} + \frac{x^2}{2\sqrt{16-x^2}} - \frac{\sqrt{16-x^2}}{2}$$
$$= \frac{16}{2\sqrt{16-x^2}} + \frac{x^2}{2\sqrt{16-x^2}} - \frac{(16-x^2)}{2\sqrt{16-x^2}} = \frac{x^2}{\sqrt{16-x^2}}$$

Matches (a)

5. Let $x = 5\sin\theta$, $dx = 5\cos\theta\,d\theta$, $\sqrt{25-x^2} = 5\cos\theta$.

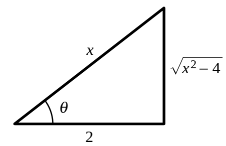
$$\int \frac{1}{(25-x^2)^{3/2}}\,dx = \int \frac{5\cos\theta}{(5\cos\theta)^3}\,d\theta$$
$$= \frac{1}{25}\int \sec^2\theta\,d\theta$$
$$= \frac{1}{25}\tan\theta + C$$
$$= \frac{x}{25\sqrt{25-x^2}} + C$$



7. Same substitution as in Exercise 5

$$\int \frac{\sqrt{25-x^2}}{x}\,dx = \int \frac{25\cos^2\theta\,d\theta}{5\sin\theta} = 5\int \frac{1-\sin^2\theta}{\sin\theta}\,d\theta = 5\int(\csc\theta - \sin\theta)\,d\theta$$
$$= 5\left[\ln|\csc\theta - \cot\theta| + \cos\theta\right] + C = 5\ln\left|\frac{5-\sqrt{25-x^2}}{x}\right| + \sqrt{25-x^2} + C$$

9. Let $x = 2\sec\theta$, $dx = 2\sec\theta\tan\theta\,d\theta$, $\sqrt{x^2 - 4} = 2\tan\theta$.

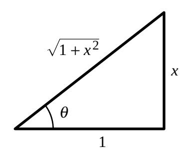


$$\int \frac{1}{\sqrt{x^2-4}}\,dx = \int \frac{2\sec\theta\tan\theta\,d\theta}{2\tan\theta} = \int \sec\theta\,d\theta = \ln\left|\sec\theta + \tan\theta\right| + C_1$$
$$= \ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + C_1$$
$$= \ln\left|x + \sqrt{x^2-4}\right| - \ln 2 + C_1 = \ln\left|x + \sqrt{x^2-4}\right| + C$$

11. Same substitution as in Exercise 9

$$\int x^3\sqrt{x^2-4}\,dx = \int (8\sec^3\theta)(2\tan\theta)(2\sec\theta\tan\theta)\,d\theta = 32\int \tan^2\theta\sec^4\theta\,d\theta$$
$$= 32\int \tan^2\theta(1 + \tan^2\theta)\sec^2\theta\,d\theta = 32\left(\frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5}\right) + C$$
$$= \frac{32}{15}\tan^3\theta[5 + 3\tan^2\theta] + C = \frac{32}{15}\frac{(x^2-4)^{3/2}}{8}\left[5 + 3\frac{(x^2-4)}{4}\right] + C$$
$$= \frac{1}{15}(x^2-4)^{3/2}[20 + 3(x^2-4)] + C = \frac{1}{15}(x^2-4)^{3/2}(3x^2+8) + C$$

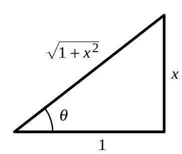
13. Let $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$, $\sqrt{1+x^2} = \sec\theta$.



$$\int x\sqrt{1+x^2}\,dx = \int \tan\theta(\sec\theta)\sec^2\theta\,d\theta = \frac{\sec^3\theta}{3} + C = \frac{1}{3}(1+x^2)^{3/2} + C$$

Note: This integral could have been evaluated with the Power Rule.

15. Same substitution as in Exercise 13



$$\int \frac{1}{(1+x^2)^2}\,dx = \int \frac{1}{\left(\sqrt{1+x^2}\right)^4}\,dx$$
$$= \int \frac{\sec^2\theta\,d\theta}{\sec^4\theta}$$
$$= \int \cos^2\theta\,d\theta = \frac{1}{2}\int (1 + \cos 2\theta)\,d\theta$$
$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]$$
$$= \frac{1}{2}[\theta + \sin\theta\cos\theta] + C$$
$$= \frac{1}{2}\left[\arctan x + \left(\frac{x}{\sqrt{1+x^2}}\right)\left(\frac{1}{\sqrt{1+x^2}}\right)\right] + C$$
$$= \frac{1}{2}\left[\arctan x + \frac{x}{1+x^2}\right] + C$$

17. Let $u = 3x$, $a = 2$, and $du = 3\,dx$.

$$\int \sqrt{4+9x^2}\,dx = \frac{1}{3}\int \sqrt{(2)^2 + (3x)^2}\,3\,dx$$
$$= \frac{1}{3}\left(\frac{1}{2}\right)\left(3x\sqrt{4+9x^2} + 4\ln\left|3x + \sqrt{4+9x^2}\right|\right) + C$$
$$= \frac{1}{2}x\sqrt{4+9x^2} + \frac{2}{3}\ln\left|3x + \sqrt{4+9x^2}\right| + C$$

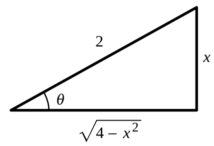
19. $$\int \frac{x}{\sqrt{x^2+9}}\,dx = \frac{1}{2}\int (x^2+9)^{-1/2}(2x)\,dx$$
$$= \sqrt{x^2+9} + C$$

(Power Rule)

21. $$\int \frac{1}{\sqrt{16-x^2}}\,dx = \arcsin\left(\frac{x}{4}\right) + C$$

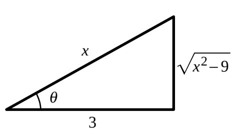
23. Let $x = 2\sin\theta$, $dx = 2\cos\theta\,d\theta$, $\sqrt{4-x^2} = 2\cos\theta$.

$$\int \sqrt{16-4x^2}\,dx = 2\int \sqrt{4-x^2}\,dx$$
$$= 2\int 2\cos\theta(2\cos\theta\,d\theta)$$
$$= 8\int \cos^2\theta\,d\theta$$
$$= 4\int (1+\cos 2\theta)\,d\theta$$
$$= 4\left[\theta + \frac{1}{2}\sin 2\theta\right] + C$$
$$= 4\theta + 4\sin\theta\cos\theta + C$$
$$= 4\arcsin\left(\frac{x}{2}\right) + x\sqrt{4-x^2} + C$$



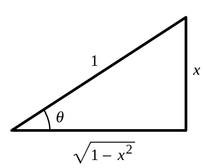
25. Let $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta\,d\theta$,
$\sqrt{x^2-9} = 3\tan\theta$.

$$\int \frac{1}{\sqrt{x^2-9}}\,dx = \int \frac{3\sec\theta\tan\theta\,d\theta}{3\tan\theta}$$
$$= \int \sec\theta\,d\theta$$
$$= \ln|\sec\theta + \tan\theta| + C_1$$
$$= \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C_1$$
$$= \ln\left|x + \sqrt{x^2-9}\right| + C$$



27. Let $x = \sin\theta$, $dx = \cos\theta\,d\theta$, $\sqrt{1-x^2} = \cos\theta$.

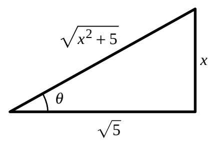
$$\int \frac{\sqrt{1-x^2}}{x^4}\,dx = \int \frac{\cos\theta(\cos\theta\,d\theta)}{\sin^4\theta}$$
$$= \int \cot^2\theta\csc^2\theta\,d\theta$$
$$= -\frac{1}{3}\cot^3\theta + C$$
$$= \frac{-(1-x^2)^{3/2}}{3x^3} + C$$



29. Same substitutions as in Exercise 28

$$\int \frac{1}{x\sqrt{4x^2+9}}\,dx = \int \frac{(3/2)\sec^2\theta\,d\theta}{(3/2)\tan\theta\, 3\sec\theta}$$
$$= \frac{1}{3}\int \csc\theta\,d\theta = -\frac{1}{3}\ln|\csc\theta + \cot\theta| + C = -\frac{1}{3}\ln\left|\frac{\sqrt{4x^2+9}+3}{2x}\right| + C$$

31. Let $x = \sqrt{5}\tan\theta$, $dx = \sqrt{5}\sec^2\theta\,d\theta$, $x^2 + 5 = 5\sec^2\theta$.

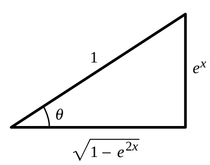


$$\int \frac{-5x}{(x^2+5)^{3/2}}\,dx = \int \frac{-5\sqrt{5}\tan\theta}{(5\sec^2\theta)^{3/2}}\sqrt{5}\sec^2\theta\,d\theta$$
$$= -\sqrt{5}\int \frac{\tan\theta}{\sec\theta}\,d\theta$$
$$= -\sqrt{5}\int \sin\theta\,d\theta$$
$$= \sqrt{5}\cos\theta + C$$
$$= \sqrt{5}\,\frac{\sqrt{5}}{\sqrt{x^2+5}} + C$$
$$= \frac{5}{\sqrt{x^2+5}} + C$$

33. Let $u = 1 + e^{2x}$, $du = 2e^{2x}\,dx$.

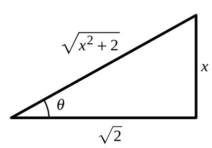
$$\int e^{2x}\sqrt{1+e^{2x}}\,dx = \frac{1}{2}\int (1+e^{2x})^{1/2}(2e^{2x})\,dx = \frac{1}{3}(1+e^{2x})^{3/2} + C$$

35. Let $e^x = \sin\theta$, $e^x\,dx = \cos\theta\,d\theta$, $\sqrt{1-e^{2x}} = \cos\theta$.



$$\int e^x\sqrt{1-e^{2x}}\,dx = \int \cos^2\theta\,d\theta$$
$$= \frac{1}{2}\int (1+\cos 2\theta)\,d\theta$$
$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]$$
$$= \frac{1}{2}(\theta + \sin\theta\cos\theta) + C = \frac{1}{2}\left(\arcsin e^x + e^x\sqrt{1-e^{2x}}\right) + C$$

37. Let $x = \sqrt{2}\tan\theta$, $dx = \sqrt{2}\sec^2\theta\,d\theta$, $x^2 + 2 = 2\sec^2\theta$.



$$\int \frac{1}{4+4x^2+x^4}\,dx = \int \frac{1}{(x^2+2)^2}\,dx$$
$$= \int \frac{\sqrt{2}\sec^2\theta\,d\theta}{4\sec^4\theta}$$
$$= \frac{\sqrt{2}}{4}\int \cos^2\theta\,d\theta$$
$$= \frac{\sqrt{2}}{4}\left(\frac{1}{2}\right)\int (1+\cos 2\theta)\,d\theta$$
$$= \frac{\sqrt{2}}{8}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$
$$= \frac{\sqrt{2}}{8}(\theta + \sin\theta\cos\theta) + C$$
$$= \frac{1}{4}\left[\frac{x}{x^2+2} + \frac{1}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}}\right] + C$$

39. Since $x > \frac{1}{2}$,

$$u = \operatorname{arcsec} 2x, \implies du = \frac{1}{x\sqrt{4x^2 - 1}}\,dx, dv = dx \implies v = x$$

$$\int \operatorname{arcsec} 2x\,dx = x \operatorname{arcsec} 2x - \int \frac{1}{\sqrt{4x^2 - 1}}\,dx$$

$$2x = \sec\theta, dx = \frac{1}{2}\sec\theta\tan\theta\,d\theta, \sqrt{4x^2 - 1} = \tan\theta$$

$$\int \operatorname{arcsec} 2x\,dx = x \operatorname{arcsec} 2x - \int \frac{(1/2)\sec\theta\tan\theta\,d\theta}{\tan\theta} = x \operatorname{arcsec} 2x - \frac{1}{2}\int \sec\theta\,d\theta$$

$$= x \operatorname{arcsec} 2x - \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + C = x \operatorname{arcsec} 2x - \frac{1}{2}\ln\left|2x + \sqrt{4x^2 - 1}\right| + C.$$

41. $$\int \frac{1}{\sqrt{4x - x^2}}\,dx = \int \frac{1}{\sqrt{4 - (x - 2)^2}}\,dx = \arcsin\left(\frac{x - 2}{2}\right) + C$$

43. Let $x + 2 = 2\tan\theta, dx = 2\sec^2\theta\,d\theta, \sqrt{(x + 2)^2 + 4} = 2\sec\theta$.

$$\int \frac{x}{\sqrt{x^2 + 4x + 8}}\,dx = \int \frac{x}{\sqrt{(x + 2)^2 + 4}}\,dx = \int \frac{(2\tan\theta - 2)(2\sec^2\theta)\,d\theta}{2\sec\theta}$$

$$= 2\int (\tan\theta - 1)(\sec\theta)\,d\theta$$

$$= 2\left[\sec\theta - \ln\left|\sec\theta + \tan\theta\right|\right] + C_1$$

$$= 2\left[\frac{\sqrt{(x + 2)^2 + 4}}{2} - \ln\left|\frac{\sqrt{(x + 2)^2 + 4}}{2} + \frac{x + 2}{2}\right|\right] + C_1$$

$$= \sqrt{x^2 + 4x + 8} - 2\left[\ln\left|\sqrt{x^2 + 4x + 8} + (x + 2)\right| - \ln 2\right] + C_1$$

$$= \sqrt{x^2 + 4x + 8} - 2\ln\left|\sqrt{x^2 + 4x + 8} + (x + 2)\right| + C$$

45. Let $t = \sin\theta, dt = \cos\theta\,d\theta, 1 - t^2 = \cos^2\theta$.

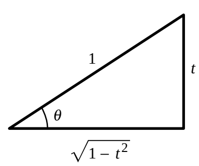
(a) $$\int \frac{t^2}{(1 - t^2)^{3/2}}\,dt = \int \frac{\sin^2\theta\cos\theta\,d\theta}{\cos^3\theta}$$

$$= \int \tan^2\theta\,d\theta$$

$$= \int (\sec^2\theta - 1)\,d\theta$$

$$= \tan\theta - \theta + C$$

$$= \frac{t}{\sqrt{1 - t^2}} - \arcsin t + C$$



Thus, $$\int_0^{\sqrt{3}/2} \frac{t^2}{(1 - t^2)^{3/2}}\,dt = \left[\frac{t}{\sqrt{1 - t^2}} - \arcsin t\right]_0^{\sqrt{3}/2} = \frac{\sqrt{3}/2}{\sqrt{1/4}} - \arcsin\frac{\sqrt{3}}{2} = \sqrt{3} - \frac{\pi}{3} \approx 0.685.$$

(b) When $t = 0, \theta = 0$. When $t = \sqrt{3}/2, \theta = \pi/3$. Thus,

$$\int_0^{\sqrt{3}/2} \frac{t^2}{(1 - t^2)^{3/2}}\,dt = \left[\tan\theta - \theta\right]_0^{\pi/3} = \sqrt{3} - \frac{\pi}{3} \approx 0.685.$$

47. (a) Let $x = 3\tan\theta$, $dx = 3\sec^2\theta\,d\theta$, $\sqrt{x^2+9} = 3\sec\theta$.

$$\int \frac{x^3}{\sqrt{x^2+9}}\,dx = \int \frac{(27\tan^3\theta)(3\sec^2\theta\,d\theta)}{3\sec\theta}$$

$$= 27\int (\sec^2\theta - 1)\sec\theta\tan\theta\,d\theta$$

$$= 27\left[\frac{1}{3}\sec^3\theta - \sec\theta\right] + C = 9\left[\sec^3\theta - 3\sec\theta\right] + C$$

$$= 9\left[\left(\frac{\sqrt{x^2+9}}{3}\right)^3 - 3\left(\frac{\sqrt{x^2+9}}{3}\right)\right] + C = \frac{1}{3}(x^2+9)^{3/2} - 9\sqrt{x^2+9} + C$$

Thus, $\displaystyle\int_0^3 \frac{x^3}{\sqrt{x^2+9}}\,dx = \left[\frac{1}{3}(x^2+9)^{3/2} - 9\sqrt{x^2+9}\right]_0^3$

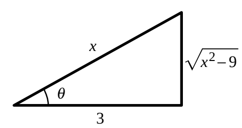
$$= \left(\frac{1}{3}\left(54\sqrt{2}\right) - 27\sqrt{2}\right) - (9 - 27)$$

$$= 18 - 9\sqrt{2} = 9\left(2 - \sqrt{2}\right) \approx 5.272.$$

(b) When $x = 0$, $\theta = 0$. When $x = 3$, $\theta = \pi/4$. Thus,

$$\int_0^3 \frac{x^3}{\sqrt{x^2+9}}\,dx = 9\left[\sec^3\theta - 3\sec\theta\right]_0^{\pi/4} = 9\left(2\sqrt{2} - 3\sqrt{2}\right) - 9(1-3) = 9\left(2 - \sqrt{2}\right) \approx 5.272.$$

49. (a) Let $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta\,d\theta$, $\sqrt{x^2-9} = 3\tan\theta$.



$$\int \frac{x^2}{\sqrt{x^2-9}}\,dx = \int \frac{9\sec^2\theta}{3\tan\theta}3\sec\theta\tan\theta\,d\theta$$

$$= 9\int \sec^3\theta\,d\theta$$

$$= 9\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int\sec\theta\,d\theta\right] \qquad \text{(7.3 Exercise 90)}$$

$$= \frac{9}{2}\left[\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right]$$

$$= \frac{9}{2}\left[\frac{x}{3}\cdot\frac{\sqrt{x^2-9}}{3} + \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right|\right]$$

Hence,

$$\int_4^6 \frac{x^2}{\sqrt{x^2-9}}\,dx = \frac{9}{2}\left[\frac{x\sqrt{x^2-9}}{9} + \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right|\right]_4^6$$

$$= \frac{9}{2}\left[\left(\frac{6\sqrt{27}}{9} + \ln\left|2 + \frac{\sqrt{27}}{3}\right|\right) - \left(\frac{4\sqrt{7}}{9} + \ln\left|\frac{4}{3} + \frac{\sqrt{7}}{3}\right|\right)\right]$$

$$= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2}\left(\ln\left(\frac{6+\sqrt{27}}{3}\right) - \ln\left(\frac{4+\sqrt{7}}{3}\right)\right)$$

$$= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2}\ln\left(\frac{6+3\sqrt{3}}{4+\sqrt{7}}\right)$$

$$= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2}\ln\left(\frac{\left(4-\sqrt{7}\right)\left(2+\sqrt{3}\right)}{3}\right) \approx 12.644.$$

—CONTINUED—

49. —CONTINUED—

(b) When $x = 4$, $\theta = \text{arcsec}\left(\frac{4}{3}\right)$.

When $x = 6$, $\theta = \text{arcsec}(2) = \frac{\pi}{3}$.

$$\int_4^6 \frac{x^2}{\sqrt{x^2-9}}\,dx = \frac{9}{2}\Big[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big]_{\text{arcsec}(4/3)}^{\pi/3}$$
$$= \frac{9}{2}\left[2\cdot\sqrt{3} + \ln\left|2+\sqrt{3}\right|\right] - \frac{9}{2}\left[\frac{4}{3}\frac{\sqrt{7}}{3} + \ln\left|\frac{4}{3}+\frac{\sqrt{7}}{3}\right|\right]$$
$$= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2}\ln\left(\frac{6+3\sqrt{3}}{4+\sqrt{7}}\right) \approx 12.644$$

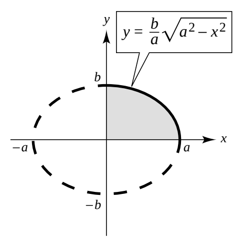
51. $\displaystyle\int \frac{x^2}{\sqrt{x^2+10x+9}}\,dx = \frac{1}{2}\sqrt{x^2+10x+9}\,(x-15) + 33\ln\left|(x+5)+\sqrt{x^2+10x+9}\right| + C$

53. $\displaystyle\int \frac{x^2}{\sqrt{x^2-1}}\,dx = \frac{1}{2}\left(x\sqrt{x^2-1} + \ln\left|x+\sqrt{x^2-1}\right|\right) + C$

55. (a) $u = a\sin\theta$ (b) $u = a\tan\theta$ (c) $u = a\sec\theta$

57. $\displaystyle A = 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$

$$= \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$
$$= \left[\frac{4b}{a}\left(\frac{1}{2}\right)\left(a^2\arcsin\frac{x}{a} + x\sqrt{a^2-x^2}\right)\right]_0^a$$
$$= \frac{2b}{a}\left(a^2\left(\frac{\pi}{2}\right)\right)$$
$$= \pi ab$$



Note: See Theorem 7.2 for $\int\sqrt{a^2-x^2}\,dx$.

59. $x^2 + y^2 = a^2$

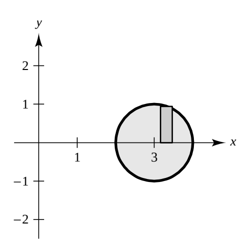
$$x = \pm\sqrt{a^2-y^2}$$

$$A = 2\int_h^a \sqrt{a^2-y^2}\,dy = \left[a^2\arcsin\left(\frac{y}{a}\right) + y\sqrt{a^2-y^2}\right]_h^a \qquad \text{(Theorem 7.2)}$$
$$= \left(a^2\frac{\pi}{2}\right) - \left(a^2\arcsin\left(\frac{h}{a}\right) + h\sqrt{a^2-h^2}\right)$$
$$= \frac{a^2\pi}{2} - a^2\arcsin\left(\frac{h}{a}\right) - h\sqrt{a^2-h^2}$$

61. Let $x - 3 = \sin\theta$, $dx = \cos\theta\,d\theta$, $\sqrt{1-(x-3)^2} = \cos\theta$.

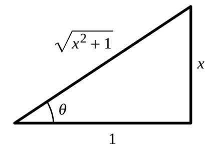
Shell Method:

$$V = 4\pi\int_2^4 x\sqrt{1-(x-3)^2}\,dx$$
$$= 4\pi\int_{-\pi/2}^{\pi/2}(3+\sin\theta)\cos^2\theta\,d\theta$$
$$= 4\pi\left[\frac{3}{2}\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta)\,d\theta + \int_{-\pi/2}^{\pi/2}\cos^2\theta\sin\theta\,d\theta\right]$$
$$= 4\pi\left[\frac{3}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) - \frac{1}{3}\cos^3\theta\right]_{-\pi/2}^{\pi/2} = 6\pi^2$$



63. $y = \ln x, y' = \dfrac{1}{x}, 1 + (y')^2 = 1 + \dfrac{1}{x^2} = \dfrac{x^2 + 1}{x^2}$

Let $x = \tan\theta, dx = \sec^2\theta\, d\theta, \sqrt{x^2+1} = \sec\theta.$



$$\begin{aligned}
s &= \int_1^5 \sqrt{\frac{x^2+1}{x^2}}\,dx = \int_1^5 \frac{\sqrt{x^2+1}}{x}\,dx\\
&= \int_a^b \frac{\sec\theta}{\tan\theta}\sec^2\theta\,d\theta = \int_a^b \frac{\sec\theta}{\tan\theta}(1+\tan^2\theta)\,d\theta\\
&= \int_a^b (\csc\theta + \sec\theta\tan\theta)\,d\theta\\
&= \Big[-\ln|\csc\theta + \cot\theta| + \sec\theta\Big]_a^b\\
&= \left[-\ln\left|\frac{\sqrt{x^2+1}}{x} + \frac{1}{x}\right| + \sqrt{x^2+1}\right]_1^5\\
&= \left[-\ln\left(\frac{\sqrt{26}+1}{5}\right) + \sqrt{26}\right] - \left[-\ln\left(\sqrt{2}+1\right) + \sqrt{2}\right]\\
&= \ln\left[\frac{5\left(\sqrt{2}+1\right)}{\sqrt{26}+1}\right] + \sqrt{26} - \sqrt{2} \approx 4.367 \text{ or } \ln\left[\frac{\sqrt{26}-1}{5\left(\sqrt{2}-1\right)}\right] + \sqrt{26} - \sqrt{2}
\end{aligned}$$

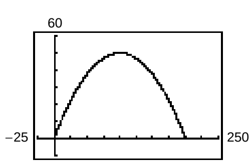
65. Length of one arch of sine curve: $y = \sin x, y' = \cos x$

$$L_1 = \int_0^\pi \sqrt{1+\cos^2 x}\,dx$$

Length of one arch of cosine curve: $y = \cos x, y' = -\sin x$

$$\begin{aligned}
L_2 &= \int_{-\pi/2}^{\pi/2} \sqrt{1+\sin^2 x}\,dx\\
&= \int_{-\pi/2}^{\pi/2} \sqrt{1+\cos^2\left(x-\frac{\pi}{2}\right)}\,dx \qquad u = x - \frac{\pi}{2}, du = dx\\
&= \int_{-\pi}^{0} \sqrt{1+\cos^2 u}\,du\\
&= \int_0^{\pi} \sqrt{1+\cos^2 u}\,du = L_1
\end{aligned}$$

67. (a)



(b) $y = 0$ for $x = 200$ (range)

(c) $y = x - 0.005x^2, y' = 1 - 0.01x, 1 + (y')^2 = 1 + (1 - 0.01x)^2$

Let $u = 1 - 0.01x, du = -0.01\,dx, a = 1.$ (See Theorem 7.2.)

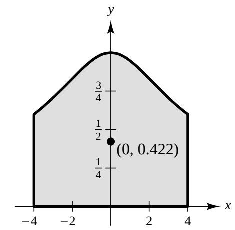
$$\begin{aligned}
s &= \int_0^{200} \sqrt{1+(1-0.01x)^2}\,dx = -100\int_0^{200} \sqrt{(1-0.01x)^2+1}\,(-0.01)\,dx\\
&= -50\left[(1-0.01x)\sqrt{(1-0.01x)^2+1} + \ln\left|(1-0.01x) + \sqrt{(1-0.01x)^2+1}\right|\right]_0^{200}\\
&= -50\left[\left(-\sqrt{2} + \ln\left|-1+\sqrt{2}\right|\right) - \left(\sqrt{2} + \ln\left|1+\sqrt{2}\right|\right)\right]\\
&= 100\sqrt{2} + 50\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) \approx 229.559
\end{aligned}$$

69. Let $x = 3\tan\theta$, $dx = 3\sec^2\theta\,d\theta$, $\sqrt{x^2+9} = 3\sec\theta$.

$$A = 2\int_0^4 \frac{3}{\sqrt{x^2+9}}\,dx = 6\int_0^4 \frac{dx}{\sqrt{x^2+9}} = 6\int_a^b \frac{3\sec^2\theta\,d\theta}{3\sec\theta}$$

$$= 6\int_a^b \sec\theta\,d\theta = \Big[6\ln\left|\sec\theta + \tan\theta\right|\Big]_a^b = \left[6\ln\left|\frac{\sqrt{x^2+9}+x}{3}\right|\right]_0^4 = 6\ln 3$$

$\bar{x} = 0$ (by symmetry)



$$\bar{y} = \frac{1}{2}\left(\frac{1}{A}\right)\int_{-4}^4 \left(\frac{3}{\sqrt{x^2+9}}\right)^2 dx$$

$$= \frac{9}{12\ln 3}\int_{-4}^4 \frac{1}{x^2+9}\,dx$$

$$= \frac{3}{4\ln 3}\left[\frac{1}{3}\arctan\frac{x}{3}\right]_{-4}^4$$

$$= \frac{2}{4\ln 3}\arctan\frac{4}{3} \approx 0.422$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{1}{2\ln 3}\arctan\frac{4}{3}\right) \approx (0, 0.422)$$

71. $y = x^2$, $\quad y' = 2x$, $\quad 1 + (y') = 1 + 4x^2$

$2x = \tan\theta$, $dx = \frac{1}{2}\sec^2\theta\,d\theta$, $\sqrt{1+4x^2} = \sec\theta$

(For $\int\sec^5\theta\,d\theta$ and $\int\sec^3\theta\,d\theta$, see Exercise 80 in Section 7.3)

$$S = 2\pi\int_0^{\sqrt{2}} x^2\sqrt{1+4x^2}\,dx = 2\pi\int_a^b \left(\frac{\tan\theta}{2}\right)^2(\sec\theta)\left(\frac{1}{2}\sec^2\theta\right)d\theta$$

$$= \frac{\pi}{4}\int_a^b \sec^3\theta\tan^2\theta\,d\theta = \frac{\pi}{4}\left[\int_a^b \sec^5\theta\,d\theta - \int_a^b \sec^3\theta\,d\theta\right]$$

$$= \frac{\pi}{4}\left\{\frac{1}{4}\left[\sec^3\theta\tan\theta + \frac{3}{2}\left(\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right)\right] - \frac{1}{2}\left(\sec\theta\tan\theta + \ln\left|\sec\theta+\tan\theta\right|\right)\right\}_a^b$$

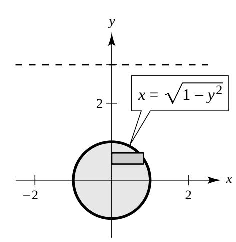
$$= \frac{\pi}{4}\left[\frac{1}{4}\left[(1+4x^2)^{3/2}(2x)\right] - \frac{1}{8}\left[(1+4x^2)^{1/2}(2x) + \ln\left|\sqrt{1+4x^2}+2x\right|\right]\right]_0^{\sqrt{2}}$$

$$= \frac{\pi}{4}\left[\frac{54\sqrt{2}}{4} - \frac{6\sqrt{2}}{6} = \frac{1}{8}\ln\left(3+2\sqrt{2}\right)\right]$$

$$= \frac{\pi}{4}\left(\frac{51\sqrt{2}}{4} - \frac{\ln\left(3+2\sqrt{2}\right)}{8}\right) = \frac{\pi}{32}\left[102\sqrt{2} - \ln\left(3+2\sqrt{2}\right)\right] \approx 13.989$$

73. (a) Area of representative rectangle: $2\sqrt{1-y^2}\,\Delta y$

Pressure: $2(62.4)(3-y)\sqrt{1-y^2}\,\Delta y$



$$F = 124.8\int_{-1}^1 (3-y)\sqrt{1-y^2}\,dy$$

$$= 124.8\left[3+\int_{-1}^1 \sqrt{1-y^2}\,dy - \int_{-1}^1 y\sqrt{1-y^2}\,dy\right]$$

$$= 124.8\left[\frac{3}{2}\left(\arcsin y + y\sqrt{1-y^2}\right) + \frac{1}{2}\left(\frac{2}{3}\right)(1-y^2)^{3/2}\right]_{-1}^1$$

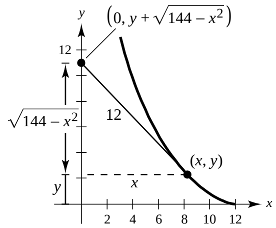
$$= (62.4)3[\arcsin 1 - \arcsin(-1)] = 187.2\pi \text{ lb}$$

(b) $F = 124.8\int_{-1}^1 (d-y)\sqrt{1-y^2}\,dy = 124.8d\int_{-1}^1 \sqrt{1-y^2}\,dy - 124.8\int_{-1}^1 y\sqrt{1-y^2}\,dy$

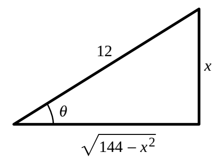
$$= 124.8\left(\frac{d}{2}\right)\left[\arcsin y + y\sqrt{1-y^2}\right]_{-1}^1 - 124.8(0) = 62.4\pi d \text{ lb}$$

75. (a) $m = \dfrac{dy}{dx} = \dfrac{y - \left(y + \sqrt{144 - x^2}\right)}{x - 0}$

$= -\dfrac{\sqrt{144 - x^2}}{x}$



(b) $y = -\displaystyle\int \frac{\sqrt{144 - x^2}}{x}\, dx$



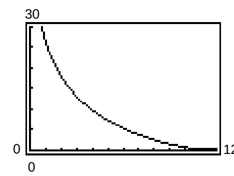
Let $x = 12 \sin \theta$, $dx = 12 \cos \theta\, d\theta$, $\sqrt{144 - x^2} = 12 \cos \theta$.

$$y = -\int \frac{12 \cos \theta}{12 \sin \theta} 12 \cos \theta\, d\theta = -12 \int \frac{1 - \sin^2 \theta}{\sin \theta}\, d\theta$$

$$= -12 \int (\csc \theta - \sin \theta)\, d\theta = -12 \ln\left|\csc \theta - \cot \theta\right| - 12 \cos \theta + C$$

$$= -12 \ln\left|\frac{12}{x} - \frac{\sqrt{144 - x^2}}{x}\right| - 12\left(\frac{\sqrt{144 - x^2}}{12}\right) + C$$

$$= -12 \ln\left|\frac{12 - \sqrt{144 - x^2}}{x}\right| - \sqrt{144 - x^2} + C$$



When $x = 12$, $y = 0 \implies C = 0$. Thus, $y = -12 \ln\left(\dfrac{12 - \sqrt{144 - x^2}}{x}\right) - \sqrt{144 - x^2}$.

Note: $\dfrac{12 - \sqrt{144 - x^2}}{x} > 0$ for $0 < x \leq 12$

(c) Vertical asymptote: $x = 0$

(d) $y + \sqrt{144 - x^2} = 12 \implies y = 12 - \sqrt{144 - x^2}$

Thus,

$$12 - \sqrt{144 - x^2} = -12 \ln\left(\frac{12 - \sqrt{144 - x^2}}{x}\right) - \sqrt{144 - x^2}$$

$$-1 = \ln\left(\frac{12 - \sqrt{144 - x^2}}{x}\right)$$

$$xe^{-1} = 12 - \sqrt{144 - x^2}$$

$$(xe^{-1} - 12)^2 = \left(-\sqrt{144 - x^2}\right)^2$$

$$x^2e^{-2} - 24xe^{-1} + 144 = 144 - x^2$$

$$x^2(e^{-2} + 1) - 24xe^{-1} = 0$$

$$x\left[x(e^{-2} + 1) - 24e^{-1}\right] = 0$$

$$x = 0 \text{ or } x = \frac{24e^{-1}}{e^{-2} + 1} \approx 7.77665.$$

Therefore,

$$s = \int_{7.77665}^{12} \sqrt{1 + \left(-\frac{\sqrt{144 - x^2}}{x}\right)^2}\, dx = \int_{7.77665}^{12} \sqrt{\frac{x^2 + (144 - x^2)}{x^2}}\, dx$$

$$= \int_{7.77665}^{12} \frac{12}{x}\, dx = \Big[12 \ln|x|\Big]_{7.77665}^{12} = 12(\ln 12 - \ln 7.77665) \approx 5.2 \text{ meters.}$$

77. True

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos\theta\, d\theta}{\cos\theta} = \int d\theta$$

79. False

$$\int_0^{\sqrt{3}} \frac{dx}{\left(\sqrt{1+x^2}\right)^3} = \int_0^{\pi/3} \frac{\sec^2\theta\, d\theta}{\sec^3\theta} = \int_0^{\pi/3} \cos\theta\, d\theta$$

81. Let $u = a\sin\theta$, $du = a\cos\theta\, d\theta$, $\sqrt{a^2-u^2} = a\cos\theta$.

$$\int \sqrt{a^2-u^2}\, du = \int a^2\cos^2\theta\, d\theta = a^2\int \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{a^2}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C = \frac{a^2}{2}(\theta + \sin\theta\cos\theta) + C$$

$$= \frac{a^2}{2}\left[\arcsin\frac{u}{a} + \left(\frac{u}{a}\right)\left(\frac{\sqrt{a^2+u^2}}{a}\right)\right] + C = \frac{1}{2}\left[a^2\arcsin\frac{u}{a} + u\sqrt{a^2-u^2}\right] + C$$

Let $u = a\sec\theta$, $du = a\sec\theta\tan\theta\, d\theta$, $\sqrt{u^2-a^2} = a\tan\theta$.

$$\int \sqrt{u^2-a^2}\, du = \int a\tan\theta(a\sec\theta\tan\theta)\, d\theta = a^2\int \tan^2\theta\sec\theta\, d\theta$$

$$= a^2\int(\sec^2\theta - 1)\sec\theta\, d\theta = a^2\int(\sec^3\theta - \sec\theta)\, d\theta$$

$$= a^2\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int\sec\theta\, d\theta\right] - a^2\int\sec\theta\, d\theta = a^2\left[\frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right|\right]$$

$$= \frac{a^2}{2}\left[\frac{u}{a}\cdot\frac{\sqrt{u^2-a^2}}{a} - \ln\left|\frac{u}{a} + \frac{\sqrt{u^2-a^2}}{a}\right|\right] + C_1$$

$$= \frac{1}{2}\left[u\sqrt{u^2-a^2} - a^2\ln\left|u + \sqrt{u^2-a^2}\right|\right] + C$$

Let $u = a\tan\theta$, $du = a\sec^2\theta\, d\theta$, $\sqrt{u^2+a^2} = a\sec\theta\, d\theta$.

$$\int \sqrt{u^2+a^2}\, du = \int (a\sec\theta)(a\sec^2\theta)\, d\theta$$

$$= a^2\int\sec^3\theta\, d\theta = a^2\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right|\right] + C_1$$

$$= \frac{a^2}{2}\left[\frac{\sqrt{u^2+a^2}}{a}\cdot\frac{u}{a} + \ln\left|\frac{\sqrt{u^2+a^2}}{a} + \frac{u}{a}\right|\right] + C_1 = \frac{1}{2}\left[u\sqrt{u^2+a^2} + a^2\ln\left|u + \sqrt{u^2+a^2}\right|\right] + C$$

Section 7.5 Partial Fractions

1. $\dfrac{5}{x^2-10x} = \dfrac{5}{x(x-10)} = \dfrac{A}{x} + \dfrac{B}{x-10}$

3. $\dfrac{2x-3}{x^3+10x} = \dfrac{2x-3}{x(x^2+10)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+10}$

5. $\dfrac{16x}{x^3-10x^2} = \dfrac{16x}{x^2(x-10)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-10}$

7. $\dfrac{1}{x^2-1} = \dfrac{1}{(x+1)(x-1)} = \dfrac{A}{x+1} + \dfrac{B}{x-1}$

$$1 = A(x-1) + B(x+1)$$

When $x = -1$, $1 = -2A$, $A = -\frac{1}{2}$.

When $x = 1$, $1 = 2B$, $B = \frac{1}{2}$.

$$\int \frac{1}{x^2-1}\, dx = -\frac{1}{2}\int\frac{1}{x+1}\, dx + \frac{1}{2}\int\frac{1}{x-1}\, dx$$

$$= -\frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C$$

$$= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$$

9. $\dfrac{3}{x^2+x-2} = \dfrac{3}{(x-1)(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{x+2}$

$3 = (x+2) + B(x-1)$

When $x = 1$, $3 = 3A$, $A = 1$.

When $x = -2$, $3 = -3B$, $B = -1$.

$$\int \frac{3}{x^2+x-2}\,dx = \int \frac{1}{x-1}\,dx - \int \frac{1}{x+2}\,dx$$
$$= \ln|x-1| - \ln|x+2| + C$$
$$= \ln\left|\frac{x-1}{x+2}\right| + C$$

11. $\dfrac{5-x}{2x^2+x-1} = \dfrac{5-x}{(2x-1)(x+1)} = \dfrac{A}{2x-1} + \dfrac{B}{x+1}$

$5 - x = A(x+1) + B(2x-1)$

When $x = \frac{1}{2}$, $\frac{9}{2} = \frac{3}{2}A$, $A = 3$.

When $x = -1$, $6 = -3B$, $B = -2$.

$$\int \frac{5-x}{2x^2+x-1}\,dx = 3\int \frac{1}{2x-1}\,dx - 2\int \frac{1}{x+1}\,dx$$
$$= \frac{3}{2}\ln|2x-1| - 2\ln|x+1| + C$$

13. $\dfrac{x^2+12x+12}{x(x+2)(x-2)} = \dfrac{A}{x} + \dfrac{B}{x+2} + \dfrac{C}{x-2}$

$x^2 + 12x + 12 = A(x+2)(x-2) + Bx(x-2) + Cx(x+2)$

When $x = 0$, $12 = -4A$, $A = -3$. When $x = -2$, $-8 = 8B$, $B = -1$. When $x = 2$, $40 = 8C$, $C = 5$.

$$\int \frac{x^2+12x+12}{x^3-4x}\,dx = 5\int \frac{1}{x-2}\,dx - \int \frac{1}{x+2}\,dx - 3\int \frac{1}{x}\,dx$$
$$= 5\ln|x-2| - \ln|x+2| - 3\ln|x| + C$$

15. $\dfrac{2x^3-4x^2-15x+5}{x^2-2x-8} = 2x + \dfrac{x+5}{(x-4)(x+2)} = 2x + \dfrac{A}{x-4} + \dfrac{B}{x+2}$

$x + 5 = A(x+2) + B(x-4)$

When $x = 4$, $9 = 6A$, $A = \frac{3}{2}$. When $x = -2$, $3 = -6B$, $B = -\frac{1}{2}$.

$$\int \frac{2x^3-4x^2-15x+5}{x^2-2x-8}\,dx = \int \left[2x + \frac{3/2}{x-4} - \frac{1/2}{x+2}\right]dx$$
$$= x^2 + \frac{3}{2}\ln|x-4| - \frac{1}{2}\ln|x+2| + C$$

17. $\dfrac{4x^2+2x-1}{x^2(x+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1}$

$4x^2 + 2x - 1 = Ax(x+1) + B(x+1) + Cx^2$

When $x = 0$, $B = -1$. When $x = -1$, $C = 1$. When $x = 1$, $A = 3$.

$$\int \frac{4x^2+2x-1}{x^3+x^2}\,dx = \int \left[\frac{3}{x} - \frac{1}{x^2} + \frac{1}{x+1}\right]dx = 3\ln|x| + \frac{1}{x} + \ln|x+1| + C$$
$$= \frac{1}{x} + \ln|x^4 + x^3| + C$$

19. $\dfrac{x^2+3x-4}{x^3-4x^2+4x} = \dfrac{x^2+3x-4}{x(x-2)^2} = \dfrac{A}{x} + \dfrac{B}{(x-2)} + \dfrac{C}{(x-2)^2}$

$x^2 + 3x - 4 = A(x-2)^2 + Bx(x-2) + Cx$

When $x = 0$, $-4 = -4A \implies A = -1$. When $x = 2$, $6 = 2C \implies C = 3$. When $x = 1$, $0 = -1 - B + 3 \implies B = 2$.

$$\int \frac{x^2+3x-4}{x^3-4x^2+4x}\,dx = \int \frac{-1}{x}\,dx + \int \frac{2}{(x-2)}\,dx + \int \frac{3}{(x-2)^2}\,dx$$
$$= -\ln|x| + 2\ln|x-2| - \frac{3}{(x-2)} + C$$

21. $\dfrac{x^2 - 1}{x(x^2 + 1)} = \dfrac{A}{x} + \dfrac{Bx + C}{x^2 + 1}$

$$x^2 - 1 = A(x^2 + 1) + (Bx + C)x$$

When $x = 0$, $A = -1$. When $x = 1$, $0 = -2 + B + C$. When $x = -1$, $0 = -2 + B + C$. Solving these equations we have $A = -1$, $B = 2$, $C = 0$.

$$\int \frac{x^2 - 1}{x^3 + x}\,dx = -\int \frac{1}{x}\,dx + \int \frac{2x}{x^2 + 1}\,dx$$

$$= \ln|x^2 + 1| - \ln|x| + C$$

$$= \ln\left|\frac{x^2 + 1}{x}\right| + C$$

23. $\dfrac{x^2}{x^4 - 2x^2 - 8} = \dfrac{A}{x - 2} + \dfrac{B}{x + 2} + \dfrac{Cx + D}{x^2 + 2}$

$$x^2 = A(x + 2)(x^2 + 2) + B(x - 2)(x^2 + 2) + (Cx + D)(x + 2)(x - 2)$$

When $x = 2$, $4 = 24A$. When $x = -2$, $4 = -24B$. When $x = 0$, $0 = 4A - 4B - 4D$, and when $x = 1$, $1 = 9A - 3B - 3C - 3D$. Solving these equations we have $A = \frac{1}{6}$, $B = -\frac{1}{6}$, $C = 0$, $D = \frac{1}{3}$.

$$\int \frac{x^2}{x^4 - 2x^2 - 8}\,dx = \frac{1}{6}\left[\int \frac{1}{x - 2}\,dx - \int \frac{1}{x + 2}\,dx + 2\int \frac{1}{x^2 + 2}\,dx\right]$$

$$= \frac{1}{6}\left[\ln\left|\frac{x - 2}{x + 2}\right| + \sqrt{2}\arctan\frac{x}{\sqrt{2}}\right] + C$$

25. $\dfrac{x}{(2x - 1)(2x + 1)(4x^2 + 1)} = \dfrac{A}{2x - 1} + \dfrac{B}{2x + 1} + \dfrac{Cx + D}{4x^2 + 1}$

$$x = A(2x + 1)(4x^2 + 1) + B(2x - 1)(4x^2 + 1) + (Cx + D)(2x - 1)(2x + 1)$$

When $x = \frac{1}{2}$, $\frac{1}{2} = 4A$. When $x = -\frac{1}{2}$, $-\frac{1}{2} = -4B$. When $x = 0$, $0 = A - B - D$, and when $x = 1$, $1 = 15A + 5B + 3C + 3D$. Solving these equations we have $A = \frac{1}{8}$, $B = \frac{1}{8}$, $C = -\frac{1}{2}$, $D = 0$.

$$\int \frac{x}{16x^4 - 1}\,dx = \frac{1}{8}\left[\int \frac{1}{2x - 1}\,dx + \int \frac{1}{2x + 1}\,dx - 4\int \frac{x}{4x^2 + 1}\,dx\right]$$

$$= \frac{1}{16}\ln\left|\frac{4x^2 - 1}{4x^2 + 1}\right| + C$$

27. $\dfrac{x^2 + 5}{(x + 1)(x^2 - 2x + 3)} = \dfrac{A}{x + 1} + \dfrac{Bx + C}{x^2 - 2x + 3}$

$$x^2 + 5 = A(x^2 - 2x + 3) + (Bx + C)(x + 1)$$

$$= (A + B)x^2 + (-2A + B + C)x + (3A + C)$$

When $x = -1$, $A = 1$. By equating coefficients of like terms, we have $A + B = 1$, $-2A + B + C = 0$, $3A + C = 5$. Solving these equations we have $A = 1$, $B = 0$, $C = 2$.

$$\int \frac{x^2 + 5}{x^3 - x^2 + x + 3}\,dx = \int \frac{1}{x + 1}\,dx + 2\int \frac{1}{(x - 1)^2 + 2}\,dx$$

$$= \ln|x + 1| + \sqrt{2}\arctan\left(\frac{x - 1}{\sqrt{2}}\right) + C$$

29. $\dfrac{3}{(2x+1)(x+2)} = \dfrac{A}{2x+1} + \dfrac{B}{x+2}$

$$3 = A(x+2) + B(2x+1)$$

When $x = -\frac{1}{2}$, $A = 2$. When $x = -2$, $B = -1$.

$$\int_0^1 \frac{3}{2x^2+5x+2}\,dx = \int_0^1 \frac{2}{2x+1}\,dx - \int_0^1 \frac{1}{x+2}\,dx$$
$$= \Big[\ln|2x-1| - \ln|x+2|\Big]_0^1$$
$$= \ln 2$$

31. $\dfrac{x+1}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$

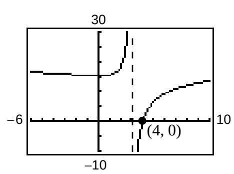
$$x + 1 = A(x^2+1) + (Bx+C)x$$

When $x = 0$, $A = 1$. When $x = 1$, $2 = 2A + B + C$. When $x = -1$, $0 = 2A + B - C$. Solving these equations we have $A = 1$, $B = -1$, $C = 1$.

$$\int_1^2 \frac{x+1}{x(x^2+1)}\,dx = \int_1^2 \frac{1}{x}\,dx - \int_1^2 \frac{x}{x^2+1}\,dx + \int_1^2 \frac{1}{x^2+1}\,dx$$
$$= \left[\ln|x| - \frac{1}{2}\ln(x^2+1) + \arctan x\right]_1^2$$
$$= \frac{1}{2}\ln\frac{8}{5} - \frac{\pi}{4} + \arctan 2$$
$$\approx 0.557$$

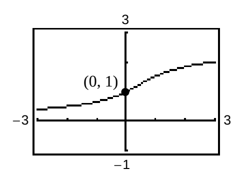
33. $\displaystyle\int \frac{3x\,dx}{x^2-6x+9} = 3\ln|x-3| - \frac{9}{x-3} + C$

$(4, 0)$: $\quad 3\ln|4-3| - \dfrac{9}{4-3} + C = 0 \Longrightarrow C = 9$



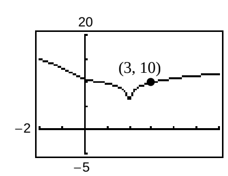
35. $\displaystyle\int \frac{x^2+x+2}{(x^2+2)^2}\,dx = \frac{\sqrt{2}}{2}\arctan\frac{x}{\sqrt{2}} - \frac{1}{2(x^2+2)} + C$

$(0, 1)$: $\; 0 - \dfrac{1}{4} + C = 1 \Longrightarrow C = \dfrac{5}{4}$



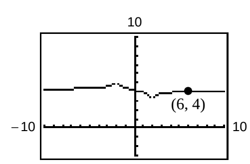
37. $\displaystyle\int \frac{2x^2-2x+3}{x^3-x^2-x-2}\,dx = \ln|x-2| + \frac{1}{2}\ln|x^2+x+1| - \sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$

$(3, 10)$: $\; 0 + \dfrac{1}{2}\ln 13 - \sqrt{3}\arctan\dfrac{7}{\sqrt{3}} + C = 10 \Longrightarrow C = 10 - \dfrac{1}{2}\ln 13 + \sqrt{3}\arctan\dfrac{7}{\sqrt{3}}$



39. $\displaystyle\int \frac{1}{x^2-4}\,dx = \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$

$(6, 4)$: $\; \dfrac{1}{4}\ln\left|\dfrac{4}{8}\right| + C = 4 \Longrightarrow C = 4 - \dfrac{1}{4}\ln\dfrac{1}{2} = 4 + \dfrac{1}{4}\ln 2$



41. Let $u = \cos x \;\; du = \sin x\,dx$.

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$$

$$1 = A(u-1) + Bu$$

When $u = 0$, $A = -1$. When $u = 1$, $B = 1$, $u = \cos x$, $du = -\sin x\,dx$.

$$\int \frac{\sin x}{\cos x(\cos x - 1)}\,dx = -\int \frac{1}{u(u-1)}\,du$$

$$= \int \frac{1}{u}\,du - \int \frac{1}{u-1}\,du$$

$$= \ln|u| - \ln|u-1| + C$$

$$= \ln\left|\frac{u}{u-1}\right| + C$$

$$= \ln\left|\frac{\cos x}{\cos x - 1}\right| + C$$

43. $$\int \frac{3\cos x}{\sin^2 x + \sin x - 2}\,dx = 3\int \frac{1}{u^2 + u - 2}\,du$$

$$= \ln\left|\frac{u-1}{u+2}\right| + C$$

$$= \ln\left|\frac{-1 + \sin x}{2 + \sin x}\right| + C$$

(From Exercise 9 with $u = \sin x$, $du = \cos x\,dx$)

45. Let $u = e^x$, $du = e^x\,dx$.

$$\frac{1}{(u-1)(u+4)} = \frac{A}{u-1} + \frac{B}{u+4}$$

$$1 = A(u+4) + B(u-1)$$

When $u = 1$, $A = \frac{1}{5}$. When $u = -4$, $B = -\frac{1}{5}$, $u = e^x$, $du = e^x\,dx$.

$$\int \frac{e^x}{(e^x - 1)(e^x + 4)}\,dx = \int \frac{1}{(u-1)(u+4)}\,du$$

$$= \frac{1}{5}\left(\int \frac{1}{u-1}\,du - \int \frac{1}{u+4}\,du\right)$$

$$= \frac{1}{5}\ln\left|\frac{u-1}{u+4}\right| + C$$

$$= \frac{1}{5}\ln\left|\frac{e^x - 1}{e^x + 4}\right| + C$$

47. $$\frac{1}{x(a+bx)} = \frac{A}{x} + \frac{B}{a+bx}$$

$$1 = A(a+bx) + Bx$$

When $x = 0$, $1 = aA \Longrightarrow A = 1/a$.
When $x = -a/b$, $1 = -(a/b)B \Longrightarrow B = -b/a$.

$$\int \frac{1}{x(a+bx)}\,dx = \frac{1}{a}\int\left(\frac{1}{x} - \frac{b}{a+bx}\right)dx$$

$$= \frac{1}{a}\big(\ln|x| - \ln|a+bx|\big) + C$$

$$= \frac{1}{a}\ln\left|\frac{x}{a+bx}\right| + C$$

49. $$\frac{x}{(a+bx)^2} = \frac{A}{a+bx} + \frac{B}{(a+bx)^2}$$

$$x = A(a+bx) + B$$

When $x = -a/b$, $B = -a/b$.
When $x = 0$, $0 = aA + B \implies A = 1/b$.

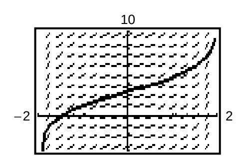
$$\int \frac{x}{(a+bx)^2}\,dx = \int\left(\frac{1/b}{a+bx} + \frac{-a/b}{(a+bx)^2}\right)dx$$

$$= \frac{1}{b}\int \frac{1}{a+bx}\,dx - \frac{a}{b}\int \frac{1}{(a+bx)^2}\,dx$$

$$= \frac{1}{b^2}\ln|a+bx| + \frac{a}{b^2}\left(\frac{1}{a+bx}\right) + C$$

$$= \frac{1}{b^2}\left(\frac{a}{a+bx} + \ln|a+bx|\right) + C$$

51. $\dfrac{dy}{dx} = \dfrac{6}{4 - x^2}$, $y(0) = 3$



53. Dividing x^3 by $x - 5$.

55. (a) Substitution: $u = x^2 + 2x - 8$

(b) Partial fractions

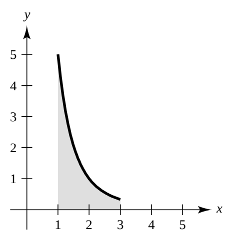
(c) Trigonometric substitution (tan) or inverse tangent rule

57. $$\text{Average Cost} = \frac{1}{80-75}\int_{75}^{80}\frac{124p}{(10+p)(100-p)}\,dp$$
$$= \frac{1}{5}\int_{75}^{80}\left(\frac{-124}{(10+p)11} + \frac{1240}{(100-p)11}\right)dp$$
$$= \frac{1}{5}\left[\frac{-124}{11}\ln(10+p) - \frac{1240}{11}\ln(100-p)\right]_{75}^{80}$$
$$\approx \frac{1}{5}(24.51) = 4.9$$

Approximately $490,000.

59. $$A = \int_1^3 \frac{10}{x(x^2+1)}\,dx \approx 3$$

Matches (c)



61. $$\frac{1}{(x+1)(n-x)} = \frac{A}{x+1} + \frac{B}{n-x},\ A = B = \frac{1}{n+1}$$
$$\frac{1}{n+1}\int\left(\frac{1}{x+1} + \frac{1}{n-x}\right)dx = kt + C$$
$$\frac{1}{n+1}\ln\left|\frac{x+1}{n-x}\right| = kt + C$$

When $t = 0$, $x = 0$, $C = \frac{1}{n+1}\ln\frac{1}{n}$.

$$\frac{1}{n+1}\ln\left|\frac{x+1}{n-x}\right| = kt + \frac{1}{n+1}\ln\frac{1}{n}$$
$$\frac{1}{n+1}\left[\ln\left|\frac{x+1}{n-x}\right| - \ln\frac{1}{n}\right] = kt$$
$$\ln\frac{nx+n}{n-x} = (n+1)kt$$
$$\frac{nx+n}{n-x} = e^{(n+1)kt}$$
$$x = \frac{n[e^{(n+1)kt} - 1]}{n + e^{(n+1)kt}}$$

Note: $\lim_{t\to\infty} x = n$

63. $\dfrac{x}{1+x^4} = \dfrac{Ax+B}{x^2+\sqrt{2}\,x+1} + \dfrac{Cx+D}{x^2-\sqrt{2}\,x+1}$

$$x = (Ax+B)\left(x^2-\sqrt{2}\,x+1\right) + (Cx+D)\left(x^2+\sqrt{2}\,x+1\right)$$
$$= (A+C)x^3 + \left(B+D-\sqrt{2}\,A+\sqrt{2}\,C\right)x^2 + \left(A+C-\sqrt{2}\,B+\sqrt{2}\,D\right)x + (B+D)$$

$0 = A + C \implies C = -A$

$0 = B + D - \sqrt{2}\,A + \sqrt{2}\,C$ } $-2\sqrt{2}\,A = 0 \implies A = 0$ and $C = 0$

$1 = A + C - \sqrt{2}\,B + \sqrt{2}\,D$ } $-2\sqrt{2}\,B = 1 \implies B = -\dfrac{\sqrt{2}}{4}$ and $D = \dfrac{\sqrt{2}}{4}$

$0 = B + D \implies D = -B$

Thus,

$$\int_0^1 \frac{x}{1+x^4}\,dx = \int_0^1 \left[\frac{-\sqrt{2}/4}{x^2+\sqrt{2}\,x+1} + \frac{\sqrt{2}/4}{x^2-\sqrt{2}\,x+1}\right]dx$$
$$= \frac{\sqrt{2}}{4}\int_0^1 \left[\frac{-1}{\left[x+\left(\sqrt{2}/2\right)\right]^2+(1/2)} + \frac{1}{\left[x-\left(\sqrt{2}/2\right)\right]^2+(1/2)}\right]dx$$
$$= \frac{\sqrt{2}}{4}\cdot\frac{1}{1/\sqrt{2}}\left[-\arctan\left(\frac{x+\left(\sqrt{2}/2\right)}{1/\sqrt{2}}\right) + \arctan\left(\frac{x-\left(\sqrt{2}/2\right)}{1/\sqrt{2}}\right)\right]_0^1$$
$$= \frac{1}{2}\left[-\arctan\left(\sqrt{2}\,x+1\right) + \arctan\left(\sqrt{2}\,x-1\right)\right]_0^1$$
$$= \frac{1}{2}\left[\left(-\arctan\left(\sqrt{2}+1\right) + \arctan\left(\sqrt{2}-1\right)\right) - \left(-\arctan 1 + \arctan(-1)\right)\right]$$
$$= \frac{1}{2}\left[\arctan\left(\sqrt{2}-1\right) - \arctan\left(\sqrt{2}+1\right) + \frac{\pi}{4} + \frac{\pi}{4}\right].$$

Since $\arctan x - \arctan y = \arctan[(x-y)/(1+xy)]$, we have:

$$\int_0^1 \frac{x}{1+x^4}\,dx = \frac{1}{2}\left[\arctan\left(\frac{\left(\sqrt{2}-1\right)-\left(\sqrt{2}+1\right)}{1+\left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)}\right) + \frac{\pi}{2}\right] = \frac{1}{2}\left[\arctan\left(\frac{-2}{2}\right) + \frac{\pi}{2}\right] = \frac{1}{2}\left[-\frac{\pi}{4} + \frac{\pi}{2}\right] = \frac{\pi}{8}$$

Section 7.6 Integration by Tables and Other Integration Techniques

1. By Formula 6: $\displaystyle\int \frac{x^2}{1+x}\,dx = -\frac{x}{2}(2-x) + \ln|1+x| + C$

3. By Formula 26: $\displaystyle\int e^x\sqrt{1+e^{2x}}\,dx = \frac{1}{2}\left[e^x\sqrt{e^{2x}+1} + \ln\left|e^x+\sqrt{e^{2x}+1}\right|\right] + C$

$u = e^x,\ du = e^x\,dx$

5. By Formula 44: $\displaystyle\int \frac{1}{x^2\sqrt{1-x^2}}\,dx = -\frac{\sqrt{1-x^2}}{x} + C$

7. By Formulas 50 and 48: $\int \sin^4(2x)\,dx = \frac{1}{2}\int \sin^4(2x)(2)\,dx$

$$= \frac{1}{2}\left[\frac{-\sin^3(2x)\cos(2x)}{4} + \frac{3}{4}\int \sin^2(2x)(2)\,dx\right]$$

$$= \frac{1}{2}\left[\frac{-\sin^3(2x)\cos(2x)}{4} + \frac{3}{8}(2x - \sin 2x \cos 2x)\right] + C$$

$$= \frac{1}{16}(6x - 3\sin 2x \cos 2x - 2\sin^3 2x \cos 2x) + C$$

9. By Formula 57: $\int \frac{1}{\sqrt{x}\left(1 - \cos\sqrt{x}\right)}\,dx = 2\int \frac{1}{1 - \cos\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right)dx$

$$= -2\left(\cot\sqrt{x} + \csc\sqrt{x}\right) + C$$

$u = \sqrt{x},\ du = \frac{1}{2\sqrt{x}}\,dx$

11. By Formula 84:

$$\int \frac{1}{1 + e^{2x}}\,dx = x - \frac{1}{2}\ln(1 + e^{2x}) + C$$

13. By Formula 89:

$$\int x^3 \ln x\,dx = \frac{x^4}{16}\left(4\ln|x| - 1\right) + C$$

15. (a) By Formulas 83 and 82: $\int x^2e^x\,dx = x^2e^x - 2\int xe^x\,dx$

$$= x^2e^x - 2[(x - 1)e^x + C_1]$$

$$= x^2e^x - 2xe^x + 2e^x + C$$

(b) Integration by parts: $u = x^2,\ du = 2x\,dx,\ dv = e^x\,dx,\ v = e^x$

$$\int x^2e^x\,dx = x^2e^x - \int 2xe^x\,dx$$

Parts again: $u = 2x,\ du = 2\,dx,\ dv = e^x\,dx,\ v = e^x$

$$\int x^2e^x\,dx = x^2e^x - \left[2xe^x - \int 2e^x\,dx\right] = x^2e^x - 2xe^x + 2e^x + C$$

17. (a) By Formula: 12, $a = b = 1$, $u = x$, and

$$\int \frac{1}{x^2(x + 1)}\,dx = \frac{-1}{1}\left(\frac{1}{x} + \frac{1}{1}\ln\left|\frac{x}{1 + x}\right|\right) + C$$

$$= \frac{-1}{x} - \ln\left|\frac{x}{1 + x}\right| + C$$

$$= \frac{-1}{x} + \ln\left|\frac{x + 1}{x}\right| + C$$

(b) Partial fractions:

$$\frac{1}{x^2(x + 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x + 1}$$

$$1 = Ax(x + 1) + B(x + 1) + Cx^2$$

$x = 0$: $1 = B$

$x = -1$: $1 = C$

$x = 1$: $1 = 2A + 2 + 1 \Longrightarrow A = -1$

$$\int \frac{1}{x^2(x + 1)}\,dx = \int\left[\frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x + 1}\right]dx$$

$$= -\ln|x| - \frac{1}{x} + \ln|x + 1| + C$$

$$= -\frac{1}{x} - \ln\left|\frac{x}{x + 1}\right| + C$$

19. By Formula 81: $\int xe^{x^2} = \frac{1}{2}e^{x^2} + C$

21. By Formula 79: $\int x \operatorname{arcsec}(x^2 + 1)\, dx = \frac{1}{2}\int \operatorname{arcsec}(x^2 + 1)(2x)\, dx$

$$= \frac{1}{2}\left[(x^2 + 1) \operatorname{arcsec}(x^2 + 1) - \ln\left((x^2 + 1) + \sqrt{x^4 + 2x^2}\right)\right] + C$$

$u = x^2 + 1$, $du = 2x\, dx$

23. By Formula 89: $\int x^2 \ln x\, dx = \frac{x^3}{9}\left(-1 + 3\ln|x|\right) + C$

25. By Formula 35: $\int \frac{1}{x^2\sqrt{x^2 - 4}}\, dx = \frac{\sqrt{x^2 - 4}}{4x} + C$

27. By Formula 4: $\int \frac{2x}{(1 - 3x)^2}\, dx = 2\int \frac{x}{(1 - 3x)^2}\, dx = \frac{2}{9}\left(\ln|1 - 3x| + \frac{1}{1 - 3x}\right) + C$

29. By Formula 76:

$$\int e^x \arccos e^x\, dx = e^x \arccos e^x - \sqrt{1 - e^{2x}} + C$$

$u = e^x$, $du = e^x\, dx$

31. By Formula 73:

$$\int \frac{x}{1 - \sec x^2}\, dx = \frac{1}{2}\int \frac{2x}{1 - \sec x^2}\, dx$$

$$= \frac{1}{2}(x^2 + \cot x^2 + \csc x^2) + C$$

33. By Formula 23: $\int \frac{\cos x}{1 + \sin^2 x}\, dx = \arctan(\sin x) + C$

$u = \sin x$, $du = \cos x\, dx$

35. By Formula 14: $\int \frac{\cos\theta}{3 + 2\sin\theta + \sin^2\theta}\, d\theta = \frac{\sqrt{2}}{2}\arctan\left(\frac{1 + \sin\theta}{\sqrt{2}}\right) + C$

$u = \sin\theta$, $du = \cos\theta\, d\theta$

37. By Formula 35: $\int \frac{1}{x^2\sqrt{2 + 9x^2}}\, dx = 3\int \frac{3}{(3x)^2\sqrt{\left(\sqrt{2}\right)^2 + (3x)^2}}\, dx$

$$= -\frac{3\sqrt{2 + 9x^2}}{6x} + C$$

$$= -\frac{\sqrt{2 + 9x^2}}{2x} + C$$

39. By Formulas 54 and 55:

$$\int t^3 \cos t\, dt = t^3 \sin t - 3\int t^2 \sin t\, dt$$

$$= t^3 \sin t - 3\left[-t^2 \cos t + 2\int t \cos t\, dt\right]$$

$$= t^3 \sin t + 3t^2 \cos t - 6\left[t \sin t - \int \sin t\, dt\right]$$

$$= t^3 \sin t + 3t^2 \cos t - 6t \sin t - 6\cos t + C$$

41. By Formula 3: $\displaystyle\int \frac{\ln x}{x(3 + 2\ln x)}\,dx = \frac{1}{4}\left(2\ln|x| - 3\ln|3 + 2\ln|x||\right) + C$

$u = \ln x,\ du = \dfrac{1}{x}\,dx$

43. By Formulas 1, 25, and 33: $\displaystyle\int \frac{x}{(x^2 - 6x + 10)^2}\,dx = \frac{1}{2}\int \frac{2x - 6 + 6}{(x^2 - 6x + 10)^2}\,dx$

$$= \frac{1}{2}\int (x^2 - 6x + 10)^{-2}(2x - 6)\,dx + 3\int \frac{1}{[(x-3)^2 + 1]^2}\,dx$$

$$= -\frac{1}{2(x^2 - 6x + 10)} + \frac{3}{2}\left[\frac{x - 3}{x^2 - 6x + 10} + \arctan(x - 3)\right] + C$$

$$= \frac{3x - 10}{2(x^2 - 6x + 10)} + \frac{3}{2}\arctan(x - 3) + C$$

45. By Formula 31: $\displaystyle\int \frac{x}{\sqrt{x^4 - 6x^2 + 5}}\,dx = \frac{1}{2}\int \frac{2x}{\sqrt{(x^2 - 3)^2 - 4}}\,dx$

$$= \frac{1}{2}\ln\left|x^2 - 3 + \sqrt{x^4 - 6x^2 + 5}\right| + C$$

$u = x^2 - 3,\ du = 2x\,dx$

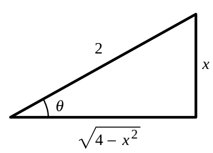
47. $\displaystyle\int \frac{x^3}{\sqrt{4 - x^2}}\,dx = \int \frac{8\sin^3\theta(2\cos\theta\,d\theta)}{2\cos\theta}$

$$= 8\int (1 - \cos^2\theta)\sin\theta\,d\theta$$

$$= 8\int [\sin\theta - \cos^2\theta(\sin\theta)]\,d\theta$$

$$= -8\cos\theta + \frac{8\cos^3\theta}{3} + C$$

$$= \frac{-\sqrt{4 - x^2}}{3}(x^2 + 8) + C$$



$x = 2\sin\theta,\ dx = 2\cos\theta\,d\theta,\ \sqrt{4 - x^2} = 2\cos\theta$

49. By Formula 8: $\displaystyle\int \frac{e^{3x}}{(1 + e^x)^3}\,dx = \int \frac{(e^x)^2}{(1 + e^x)^3}(e^x)\,dx$

$$= \frac{2}{1 + e^x} - \frac{1}{2(1 + e^x)^2} + \ln|1 + e^x| + C$$

$u = e^x,\ du = e^x\,dx$

51. $\displaystyle\frac{u^2}{(a + bu)^2} = \frac{1}{b^2} - \frac{(2a/b)u + (a^2/b^2)}{(a + bu)^2} = \frac{1}{b^2} + \frac{A}{a + bu} + \frac{B}{(a + bu)^2}$

$-\dfrac{2a}{b}u - \dfrac{a^2}{b^2} = A(a + bu) + B = (aA + B) + bAu$

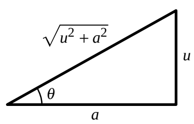
Equating the coefficients of like terms we have $aA + B = -a^2/b^2$ and $bA = -2a/b$. Solving these equations we have $A = -2a/b^2$ and $B = a^2/b^2$.

$$\int \frac{u^2}{(a + bu)^2}\,du = \frac{1}{b^2}\int du - \frac{2a}{b^2}\left(\frac{1}{b}\right)\int \frac{1}{a + bu}b\,du + \frac{a^2}{b^2}\left(\frac{1}{b}\right)\int \frac{1}{(a + bu)^2}b\,du = \frac{1}{b^2}u - \frac{2a}{b^3}\ln|a + bu| - \frac{a^2}{b^3}\left(\frac{1}{a + bu}\right) + C$$

$$= \frac{1}{b^3}\left(bu - \frac{a^2}{a + bu} - 2a\ln|a + bu|\right) + C$$

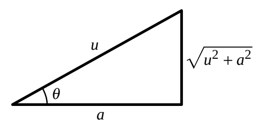
53. When we have $u^2 + a^2$:

$$u = a\tan\theta$$
$$du = a\sec^2\theta\, d\theta$$
$$u^2 + a^2 = a^2\sec^2\theta$$
$$\int \frac{1}{(u^2+a^2)^{3/2}}\,du = \int \frac{a\sec^2\theta\, d\theta}{a^3\sec^3\theta}$$
$$= \frac{1}{a^2}\int \cos\theta\, d\theta$$
$$= \frac{1}{a^2}\sin\theta + C$$
$$= \frac{u}{a^2\sqrt{u^2+a^2}} + C$$



When we have $u^2 - a^2$:

$$u = a\sec\theta$$
$$du = a\sec\theta\tan\theta\, d\theta$$
$$u^2 - a^2 = a^2\tan^2\theta$$
$$\int \frac{1}{(u^2-a^2)^{3/2}}\,du = \int \frac{a\sec\theta\tan\theta\, d\theta}{a^3\tan^3\theta}$$
$$= \frac{1}{a^2}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$
$$= -\frac{1}{a^2}\csc\theta + C$$
$$= \frac{-u}{a^2\sqrt{u^2-a^2}} + C$$



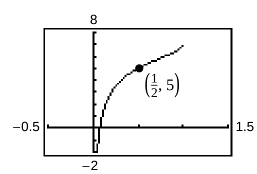
55. $$\int (\arctan u)\,du = u\arctan u - \frac{1}{2}\int \frac{2u}{1+u^2}\,du$$
$$= u\arctan u - \frac{1}{2}\ln(1+u^2) + C$$
$$= u\arctan u - \ln\sqrt{1+u^2} + C$$

$w = \arctan u,\ dv = du,\ dw = \dfrac{du}{1+u^2},\ v = u$

57. $$\int \frac{1}{x^{3/2}\sqrt{1-x}}\,dx = \frac{-2\sqrt{1-x}}{\sqrt{x}} + C$$

$\left(\dfrac{1}{2}, 5\right)$: $\dfrac{-2\sqrt{1/2}}{\sqrt{1/2}} + C = 5 \Rightarrow C = 7$

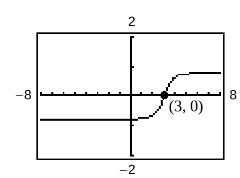
$$y = \frac{-2\sqrt{1-x}}{\sqrt{x}} + 7$$



59. $$\int \frac{1}{(x^2-6x+10)^2}\,dx = \frac{1}{2}\left[\tan^{-1}(x-3) + \frac{x-3}{x^2-6x+10}\right] + C$$

$(3, 0)$: $\dfrac{1}{2}\left[0 + \dfrac{0}{10}\right] + C = 0 \Rightarrow C = 0$

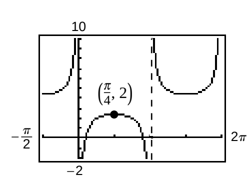
$$y = \frac{1}{2}\left[\tan^{-1}(x-3) + \frac{x-3}{x^2-6x+10}\right]$$



61. $$\int \frac{1}{\sin\theta\tan\theta}\,d\theta = -\csc\theta + C$$

$\left(\dfrac{\pi}{4}, 2\right)$: $-\dfrac{2}{\sqrt{2}} + C = 2 \Rightarrow C = 2 + \sqrt{2}$

$$y = -\csc\theta + 2 + \sqrt{2}$$



63. $\displaystyle\int \frac{1}{2 - 3\sin\theta}\,d\theta = \int\left[\frac{\dfrac{2\,du}{1+u^2}}{2 - 3\left(\dfrac{2u}{1+u^2}\right)}\right]$

$\displaystyle = \int 2\frac{2}{(1+u^2) - 6u}\,du$

$\displaystyle = \int \frac{1}{u^2 - 3u + 1}\,du$

$\displaystyle = \int \frac{1}{\left(u - \frac{3}{2}\right)^2 - \frac{5}{4}}\,du$

$\displaystyle = \frac{1}{\sqrt{5}}\ln\left|\frac{\left(u - \frac{3}{2}\right) - \frac{\sqrt{5}}{2}}{\left(u - \frac{3}{2}\right) + \frac{\sqrt{5}}{2}}\right| + C$

$\displaystyle = \frac{1}{\sqrt{5}}\ln\left|\frac{2u - 3 - \sqrt{5}}{2u - 3 + \sqrt{5}}\right| + C$

$\displaystyle = \frac{1}{\sqrt{5}}\ln\left|\frac{2\tan\left(\frac{\theta}{2}\right) - 3 - \sqrt{5}}{2\tan\left(\frac{\theta}{2}\right) - 3 + \sqrt{5}}\right| + C$

$u = \tan\dfrac{\theta}{2}$

65. $\displaystyle\int_0^{\pi/2} \frac{1}{1 + \sin\theta + \cos\theta}\,d\theta = \int_0^1\left[\frac{\dfrac{2\,du}{1+u^2}}{1 + \dfrac{2u}{1+u^2} + \dfrac{1-u^2}{1+u^2}}\right]$

$\displaystyle = \int_0^1 \frac{1}{1+u}\,du$

$\displaystyle = \Big[\ln|1+u|\Big]_0^1$

$= \ln 2$

$u = \tan\dfrac{\theta}{2}$

67. $\displaystyle\int \frac{\sin\theta}{3 - 2\cos\theta}\,d\theta = \frac{1}{2}\int\frac{2\sin\theta}{3 - 2\cos\theta}\,d\theta$

$\displaystyle = \frac{1}{2}\ln|u| + C$

$\displaystyle = \frac{1}{2}\ln(3 - 2\cos\theta) + C$

$u = 3 - 2\cos\theta,\ du = 2\sin\theta\,d\theta$

69. $\displaystyle\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}}\,d\theta = 2\int \cos\sqrt{\theta}\left(\frac{1}{2\sqrt{\theta}}\right)d\theta$

$= 2\sin\sqrt{\theta} + C$

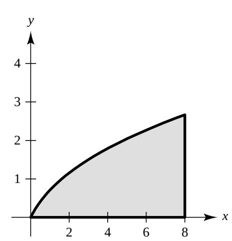
$u = \sqrt{\theta},\ du = \dfrac{1}{2\sqrt{\theta}}\,d\theta$

71. $\displaystyle A = \int_0^8 \frac{x}{\sqrt{x+1}}\,dx$

$\displaystyle = \left[\frac{-2(2-x)}{3}\sqrt{x+1}\right]_0^8$

$\displaystyle = 12 - \left(-\frac{4}{3}\right)$

$\displaystyle = \frac{40}{3} \approx 13.333$ square units



73. Arctangent Formula, Formula 23,

$\displaystyle\int \frac{1}{u^2+1}\,du,\ u = e^x$

75. Substitution: $u = x^2,\ du = 2x\,dx$

Then Formula 81.

77. Cannot be integrated.

79. Answers will vary. For example,

$$\int (2x)e^{2x}\,dx$$

can be integrated by first letting $u = 2x$ and then using Formula 82.

81. $W = \int_0^5 2000xe^{-x}\,dx$

$= -2000\int_0^5 -xe^{-x}\,dx$

$= 2000\int_0^5 (-x)e^{-x}(-1)\,dx$

$= 2000\Big[(-x)e^{-x} - e^{-x}\Big]_0^5$

$= 2000\left(-\dfrac{6}{e^5} + 1\right)$

≈ 1919.145 ft · lbs

83. (a) $V = 20(2)\int_0^3 \dfrac{2}{\sqrt{1+y^2}}\,dy$

$= \Big[80 \ln\left|y + \sqrt{1+y^2}\right|\Big]_0^3$

$= 80 \ln\left(3 + \sqrt{10}\right)$

≈ 145.5 cubic feet

$W = 148\left(80 \ln\left(3 + \sqrt{10}\right)\right)$

$= 11{,}840 \ln\left(3 + \sqrt{10}\right)$

$\approx 21{,}530.4$ lb

(b) By symmetry, $\bar{x} = 0$.

$$M = \rho(2)\int_0^3 \frac{2}{\sqrt{1+y^2}}\,dy = \Big[4\rho \ln\left|y + \sqrt{1+y^2}\right|\Big]_0^3 = 4\rho \ln\left(3 + \sqrt{10}\right)$$

$$M_x = 2\rho\int_0^3 \frac{2y}{\sqrt{1+y^2}}\,dy = \Big[4\rho\sqrt{1+y^2}\Big]_0^3 = 4\rho\left(\sqrt{10} - 1\right)$$

$$\bar{y} = \frac{M_x}{M} = \frac{4\rho\left(\sqrt{10} - 1\right)}{4\rho \ln\left(3 + \sqrt{10}\right)} \approx 1.19$$

Centroid: $(\bar{x}, \bar{y}) \approx (0, 1.19)$

85. (a) $\int_0^4 \dfrac{k}{2+3x}\,dx = 10$

$$k = \frac{10}{\int_0^4 \frac{1}{2+3x}\,dx} \approx \frac{10}{0.6486}$$

$= 15.417 \quad \left(= \dfrac{30}{\ln 7}\right)$

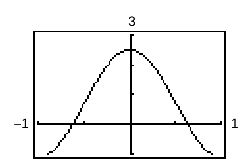
(b) $\int_0^4 \dfrac{15.417}{2+3x}\,dx$

87. False. You might need to convert your integral using substitution or algebra.

Section 7.7 Indeterminate Forms and L'Hôpital's Rule

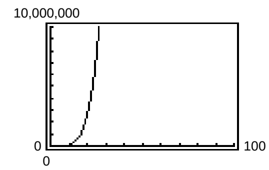
1. $\lim_{x\to 0} \dfrac{\sin 5x}{\sin 2x} \approx 2.5\left(\text{exact: } \dfrac{5}{2}\right)$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	2.4132	2.4991	2.500	2.500	2.4991	2.4132



3. $\lim_{x\to\infty} x^5e^{-x/100} \approx 0$

x	1	10	10^2	10^3	10^4	10^5
$f(x)$	0.9901	90,484	3.7×10^9	4.5×10^{10}	0	0



5. (a) $\lim_{x\to3} \frac{2(x-3)}{x^2-9} = \lim_{x\to3} \frac{2(x-3)}{(x+3)(x-3)} = \lim_{x\to3} \frac{2}{x+3} = \frac{1}{3}$

(b) $\lim_{x\to3} \frac{2(x-3)}{x^2-9} = \lim_{x\to3} \frac{(d/dx)[2(x-3)]}{(d/dx)[x^2-9]} = \lim_{x\to3} \frac{2}{2x} = \frac{2}{6} = \frac{1}{3}$

7. (a) $\lim_{x\to3} \frac{\sqrt{x+1}-2}{x-3} = \lim_{x\to3} \frac{\sqrt{x+1}-2}{x-3} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} = \lim_{x\to3} \frac{(x+1)-4}{(x-3)\left[\sqrt{x+1}+2\right]} = \lim_{x\to3} \frac{1}{\sqrt{x+1}+2} = \frac{1}{4}$

(b) $\lim_{x\to3} \frac{\sqrt{x+1}-2}{x-3} = \lim_{x\to3} \frac{(d/dx)\left[\sqrt{x+1}-2\right]}{(d/dx)[x-3]} = \lim_{x\to3} \frac{1/\left(2\sqrt{x+1}\right)}{1} = \frac{1}{4}$

9. (a) $\lim_{x\to\infty} \frac{5x^2-3x+1}{3x^2-5} = \lim_{x\to\infty} \frac{5-(3/x)+(1/x^2)}{3-(5/x^2)} = \frac{5}{3}$

(b) $\lim_{x\to\infty} \frac{5x^2-3x+1}{3x^2-5} = \lim_{x\to\infty} \frac{(d/dx)[5x^2-3x+1]}{(d/dx)[3x^2-5]} = \lim_{x\to\infty} \frac{10x-3}{6x} = \lim_{x\to\infty} \frac{(d/dx)[10x-3]}{(d/dx)[6x]} = \lim_{x\to\infty} \frac{10}{6} = \frac{5}{3}$

11. $\lim_{x\to2} \frac{x^2-x-2}{x-2} = \lim_{x\to2} \frac{2x-1}{1} = 3$

13. $\lim_{x\to0} \frac{\sqrt{4-x^2}-2}{x} = \lim_{x\to0} \frac{-x/\sqrt{4-x^2}}{1} = 0$

15. $\lim_{x\to0} \frac{e^x-(1-x)}{x} = \lim_{x\to0} \frac{e^x+1}{1} = 2$

17. Case 1: $n = 1$

$$\lim_{x\to0^+} \frac{e^x-(1+x)}{x} = \lim_{x\to0^+} \frac{e^x-1}{1} = 0$$

Case 2: $n = 2$

$$\lim_{x\to0^+} \frac{e^x-(1+x)}{x^2} = \lim_{x\to0^+} \frac{e^x-1}{2x} = \lim_{x\to0^+} \frac{e^x}{2} = \frac{1}{2}$$

Case 3: $n \geq 3$

$$\lim_{x\to0^+} \frac{e^x-(1+x)}{x^n} = \lim_{x\to0^+} \frac{e^x-1}{nx^{n-1}} = \lim_{x\to0^+} \frac{e^x}{n(n-1)x^{n-2}} = \infty$$

19. $\lim_{x\to0} \frac{\sin 2x}{\sin 3x} = \lim_{x\to0} \frac{2\cos 2x}{3\cos 3x} = \frac{2}{3}$

21. $\lim_{x\to0} \frac{\arcsin x}{x} = \lim_{x\to0} \frac{1/\sqrt{1-x^2}}{1} = 1$

23. $\lim_{x\to\infty} \frac{3x^2-2x+1}{2x^2+3} = \lim_{x\to\infty} \frac{6x-2}{4x}$

$= \lim_{x\to\infty} \frac{6}{4} = \frac{3}{2}$

25. $\lim_{x\to\infty} \frac{x^2+2x+3}{x-1} = \lim_{x\to\infty} \frac{2x+2}{1} = \infty$

27. $\lim_{x\to\infty} \frac{x^3}{e^{x/2}} = \lim_{x\to\infty} \frac{3x^2}{(1/2)e^{x/2}}$

$= \lim_{x\to\infty} \frac{6x}{(1/4)e^{x/2}} = \lim_{x\to\infty} \frac{6}{(1/8)e^{x/2}} = 0$

29. $\displaystyle\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+(1/x^2)}} = 1$

Note: L'Hôpital's Rule does not work on this limit. See Exercise 79.

31. $\displaystyle\lim_{x\to\infty} \frac{\cos x}{x} = 0$ by Squeeze Theorem

$$\left(\frac{\cos x}{x} \le \frac{1}{x}\right)$$

33. $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x^2} = \lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0$

35. $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$

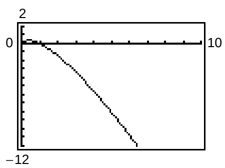
37. (a) $\displaystyle\lim_{x\to 0^+} (-x\ln x) = (-0)(-\infty) = (0)(\infty)$

(b) $\displaystyle\lim_{x\to 0^+} (-x\ln x) = \lim_{x\to 0^+} \frac{\ln x}{-1/x}$

$$= \lim_{x\to 0^+} \frac{1/x}{1/x^2}$$

$$= \lim_{x\to 0^+} x = 0$$

(c)



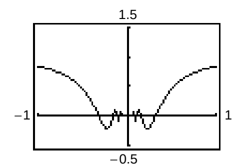
39. (a) $\displaystyle\lim_{x\to\infty} \left(x \sin\frac{1}{x}\right) = (\infty)(0)$

(b) $\displaystyle\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{x\to\infty} \frac{\sin(1/x)}{1/x}$

$$= \lim_{x\to\infty} \frac{(-1/x^2)\cos(1/x)}{-1/x^2}$$

$$= \lim_{x\to\infty} \cos\left(\frac{1}{x}\right) = 1$$

(c)



41. (a) $\displaystyle\lim_{x\to 0^+} x^{1/x} = 0^\infty = 0$, not indeterminant
(See Exercise 95)

(b) Let $y = x^{1/x}$

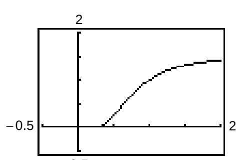
$$\ln y = \ln x^{1/x} = \frac{1}{x}\ln x.$$

Since $x \to 0^+$, $\dfrac{1}{x}\ln x \to (\infty)(-\infty) = -\infty$. Hence,

$$\ln y \to -\infty \implies y \to 0^+.$$

Therefore, $\displaystyle\lim_{x\to 0^+} x^{1/x} = 0$.

(c)



43. (a) $\displaystyle\lim_{x\to\infty} x^{1/x} = \infty^0$

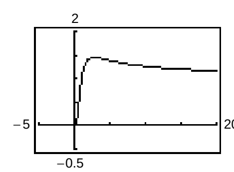
(b) Let $y = \displaystyle\lim_{x\to\infty} x^{1/x}$.

$$\ln y = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \left(\frac{1/x}{1}\right) = 0$$

Thus, $\ln y = 0 \implies y = e^0 = 1$. Therefore,

$$\lim_{x\to\infty} x^{1/x} = 1.$$

(c)



45. (a) $\displaystyle\lim_{x\to 0^+} (1+x)^{1/x} = 1^\infty$

(b) Let $y = \displaystyle\lim_{x\to 0^+} (1+x)^{1/x}$.

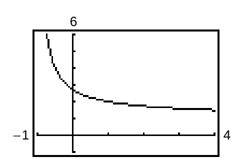
$$\ln y = \lim_{x\to 0^+} \frac{\ln(1+x)}{x}$$

$$= \lim_{x\to 0^+} \left(\frac{1/(1+x)}{1}\right) = 1$$

Thus, $\ln y = 1 \implies y = e^1 = e$.

Therefore, $\displaystyle\lim_{x\to 0^+} (1+x)^{1/x} = e$.

(c)



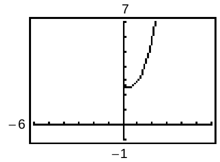
47. (a) $\lim_{x\to 0^+} [3(x)^{x/2}] = 0^0$

(b) Let $y = \lim_{x\to 0^+} 3(x)^{x/2}$.

$$\ln y = \lim_{x\to 0^+}\left[\ln 3 + \frac{x}{2}\ln x\right]$$
$$= \lim_{x\to 0^+}\left[\ln 3 + \frac{\ln x}{2/x}\right]$$
$$= \lim_{x\to 0^+} \ln 3 + \lim_{x\to 0^+} \frac{1/x}{-2/x^2}$$
$$= \lim_{x\to 0^+} \ln 3 - \lim_{x\to 0^+} \frac{x}{2}$$
$$= \ln 3$$

Hence, $\lim_{x\to 0^+} 3(x)^{x/2} = 3$.

(c)



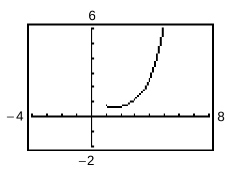
49. (a) $\lim_{x\to 1^+} (\ln x)^{x-1} = 0^0$

(b) Let $y = \lim_{x 1^+} (\ln x)^{x-1}$

$$= \lim_{x\to 1^+} (x-1)\ln x = 0$$

Hence, $\lim_{x\to 1^+} (\ln x)^{x-1} = 1$

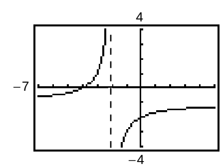
(c)



51. (a) $\lim_{x\to 2^+}\left(\frac{8}{x^2-4} - \frac{x}{x-2}\right) = \infty - \infty$

(b) $$\lim_{x\to 2^+}\left(\frac{8}{x^2-4} - \frac{x}{x-2}\right) = \lim_{x\to 2^+}\frac{8 - x(x+2)}{x^2-4}$$
$$= \lim_{x\to 2^+}\frac{(2-x)(4+x)}{(x+2)(x-2)}$$
$$= \lim_{x\to 2^+}\frac{-(x+4)}{x+2} = \frac{-3}{2}$$

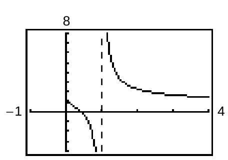
(c)



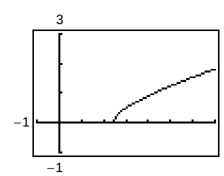
53. (a) $\lim_{x\to 1^+}\left(\frac{3}{\ln x} - \frac{2}{x-1}\right) = \infty - \infty$

(b) $$\lim_{x\to 1^+}\left(\frac{3}{\ln x} - \frac{2}{x-1}\right) = \lim_{x\to 1^+}\frac{3x - 3 - 2\ln x}{(x-1)\ln x}$$
$$= \lim_{x\to 1^+}\frac{3 - (2/x)}{[(x-1)/x] + \ln x} = \infty$$

(c)

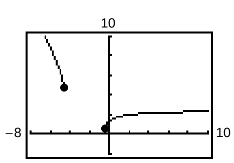


55. (a)



(b) $$\lim_{x\to 3}\frac{x-3}{\ln(2x-5)} = \lim_{x\to 3}\frac{1}{2/(2x-5)}$$
$$= \lim_{x\to 3}\frac{2x-5}{2} = \frac{1}{2}$$

57. (a)



(b) $$\lim_{x\to\infty}\left(\sqrt{x^2+5x+2} - x\right) = \lim_{x\to\infty}\left(\sqrt{x^2+5x+2} - x\right)\frac{\left(\sqrt{x^2+5x+2} + x\right)}{\left(\sqrt{x^2+5x+2} + x\right)}$$
$$= \lim_{x\to\infty}\frac{(x^2+5x+2) - x^2}{\sqrt{x^2+5x+2} + x}$$
$$= \lim_{x\to\infty}\frac{5x+2}{\sqrt{x^2+5x+2} + x}$$
$$= \lim_{x\to\infty}\frac{5 + (2/x)}{\sqrt{1 + (5/x) + (2/x^2)} + 1} = \frac{5}{2}$$

59. $\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, 1^\infty, 0^0, \infty - \infty$

61. (a) Let $f(x) = x^2 - 25$ and $g(x) = x - 5$.

(b) Let $f(x) = (x - 5)^2$ and $g(x) = x^2 - 25$.

(c) Let $f(x) = x^2 - 25$ and $g(x) = (x - 5)^3$.

63. $\lim_{x \to \infty} \frac{x^2}{e^{5x}} = \lim_{x \to \infty} \frac{2x}{5e^{5x}} = \lim_{x \to \infty} \frac{2}{25e^{5x}} = 0$

65.
$$\lim_{x\to\infty} \frac{(\ln x)^3}{x} = \lim_{x\to\infty} \frac{3(\ln x)^2(1/x)}{1} = \lim_{x\to\infty} \frac{3(\ln x)^2}{x} = \lim_{x\to\infty} \frac{6(\ln x)(1/x)}{1} = \lim_{x\to\infty} \frac{6(\ln x)}{x} = \lim_{x\to\infty} \frac{6}{x} = 0$$

67.
$$\lim_{x\to\infty} \frac{(\ln x)^n}{x^m} = \lim_{x\to\infty} \frac{n(\ln x)^{n-1}/x}{mx^{m-1}} = \lim_{x\to\infty} \frac{n(\ln x)^{n-1}}{mx^m} = \lim_{x\to\infty} \frac{n(n-1)(\ln x)^{n-2}}{m^2x^m} = \cdots = \lim_{x\to\infty} \frac{n!}{m^n x^m} = 0$$

69.

x	10	10^2	10^4	10^6	10^8	10^{10}
$\frac{(\ln x)^4}{x}$	2.811	4.498	0.720	0.036	0.001	0.000

71. $y = x^{1/x}, x > 0$

Horizontal asymptote: $y = 1$ (See Exercise 37)

$$\ln y = \frac{1}{x} \ln x$$

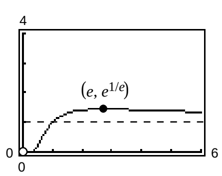
$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\left(\frac{1}{x}\right) + (\ln x)\left(-\frac{1}{x^2}\right)$$

$$\frac{dy}{dx} = x^{1/x}\left(\frac{1}{x^2}\right)(1 - \ln x) = x^{(1/x)-2}(1 - \ln x) = 0$$

Critical number: $x = e$

Intervals:	$(0, e)$	(e, ∞)
Sign of dy/dx:	$+$	$-$
$y = f(x)$:	Increasing	Decreasing

Relative maximum: $(e, e^{1/e})$



73. $y = 2xe^{-x}$

$$\lim_{x \to \infty} \frac{2x}{e^x} = \lim_{x \to \infty} \frac{2}{e^x} = 0$$

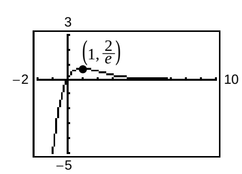
Horizontal asymptote: $y = 0$

$$\frac{dy}{dx} = 2x(-e^{-x}) + 2e^{-x} = 2e^{-x}(1 - x) = 0$$

Critical number: $x = 1$

Intervals:	$(-\infty, 1)$	$(1, \infty)$
Sign of dy/dx:	$+$	$-$
$y = f(x)$:	Increasing	Decreasing

Relative maximum: $\left(1, \frac{2}{e}\right)$



75. $\lim_{x \to 0} \frac{e^{2x} - 1}{e^x} = \frac{0}{1} = 0$

Limit is not of the form $0/0$ or ∞/∞.
L'Hôpital's Rule does not apply.

77. $\lim_{x \to \infty} x \cos \frac{1}{x} = \infty(1) = \infty$

Limit is not of the form $0/0$ or ∞/∞.
L'Hôpital's Rule does not apply.

79. (a) $\displaystyle\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{x/x}{\sqrt{x^2+1}/x}$

$\displaystyle = \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}/\sqrt{x^2}}$

$\displaystyle = \lim_{x\to\infty} \frac{1}{\sqrt{1+(1/x^2)}}$

$\displaystyle = \frac{1}{\sqrt{1+0}} = 1$

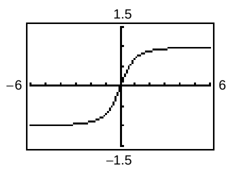
(b) $\displaystyle\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{1}{x/\sqrt{x^2+1}}$

$\displaystyle = \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x} = \lim_{x\to\infty} \frac{x/\sqrt{x^2+1}}{1}$

$\displaystyle = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$

Applying L'Hôpital's rule twice results in the original limit, so L'Hôpital's rule fails.

(c)



81. $\displaystyle\lim_{k\to 0} \frac{32\left(1 - e^{-kt} + \dfrac{v_0ke^{-kt}}{32}\right)}{k} = \lim_{k\to 0} \frac{32(1-e^{-kt})}{k} + \lim_{k\to 0}(v_0e^{-kt})$

$\displaystyle = \lim_{k\to 0} \frac{32(0+te^{-kt})}{1} + \lim_{k\to 0}\left(\frac{v_0}{e^{kt}}\right) = 32t + v_0$

83. Area of triangle: $\frac{1}{2}(2x)(1-\cos x) = x - x\cos x$

Shaded area: Area of rectangle − Area under curve

$$2x(1-\cos x) - 2\int_0^x (1-\cos t)\,dt = 2x(1-\cos x) - 2\Big[t - \sin t\Big]_0^x$$

$$= 2x(1-\cos x) - 2(x - \sin x) = 2\sin x - 2x\cos x$$

Ratio: $\displaystyle\lim_{x\to 0} \frac{x - x\cos x}{2\sin x - 2x\cos x} = \lim_{x\to 0} \frac{1 + x\sin x - \cos x}{2\cos x + 2x\sin x - 2\cos x}$

$\displaystyle = \lim_{x\to 0} \frac{1 + x\sin x - \cos x}{2x\sin x}$

$\displaystyle = \lim_{x\to 0} \frac{x\cos x + \sin x + \sin x}{2x\cos x + 2\sin x}$

$\displaystyle = \lim_{x\to 0} \frac{x\cos x + 2\sin x}{2x\cos x + 2\sin x} \cdot \frac{1/\cos x}{1/\cos x}$

$\displaystyle = \lim_{x\to 0} \frac{x + 2\tan x}{2x + 2\tan x}$

$\displaystyle = \lim_{x\to 0} \frac{1 + 2\sec^2 x}{2 + 2\sec^2 x} = \frac{3}{4}$

85. $f(x) = x^3, g(x) = x^2 + 1, [0, 1]$

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{f(1) - f(0)}{g(1) - g(0)} = \frac{3c^2}{2c}$$

$$\frac{1}{1} = \frac{3c}{2}$$

$$c = \frac{2}{3}$$

87. $f(x) = \sin x, g(x) = \cos x, \left[0, \dfrac{\pi}{2}\right]$

$$\frac{f(\pi/2) - f(0)}{g(\pi/2) - g(0)} = \frac{f'(c)}{g'(c)}$$

$$\frac{1}{-1} = \frac{\cos c}{-\sin c}$$

$$-1 = -\cot c$$

$$c = \frac{\pi}{4}$$

89. False. L'Hôpital's Rule does not apply since

$$\lim_{x \to 0} (x^2 + x + 1) \neq 0.$$

$$\lim_{x \to 0} \frac{x^2 + x + 1}{x} = \lim_{x \to 0} \left(x + 1 + \frac{1}{x}\right) = 1 + \infty = \infty$$

91. True

93. (a) $\sin \theta = BD$

$\cos \theta = DO \implies AD = 1 - \cos \theta$

$$\text{Area } \triangle ABD = \frac{1}{2}bh = \frac{1}{2}(1 - \cos \theta) \sin \theta = \frac{1}{2} \sin \theta - \frac{1}{2} \sin \theta \cos \theta$$

(b) Area of sector: $\frac{1}{2}\theta$

$$\text{Shaded area: } \frac{1}{2}\theta - \text{Area } \triangle OBD = \frac{1}{2}\theta - \frac{1}{2}(\cos \theta)(\sin \theta) = \frac{1}{2}\theta - \frac{1}{2} \sin \theta \cos \theta$$

(c) $$R = \frac{(1/2) \sin \theta - (1/2) \sin \theta \cos \theta}{(1/2)\theta - (1/2) \sin \theta \cos \theta} = \frac{\sin \theta - \sin \theta \cos \theta}{\theta - \sin \theta \cos \theta}$$

(d) $$\lim_{\theta \to 0} R = \lim_{\theta \to 0} \frac{\sin \theta - (1/2) \sin 2\theta}{\theta - (1/2) \sin 2\theta}$$

$$= \lim_{\theta \to 0} \frac{\cos \theta - \cos 2\theta}{1 - \cos 2\theta} = \lim_{\theta \to 0} \frac{-\sin \theta + 2 \sin 2\theta}{2 \sin 2\theta} = \lim_{\theta \to 0} \frac{-\cos \theta + 4 \cos 2\theta}{4 \cos 2\theta} = \frac{3}{4}$$

95. $\lim_{x \to a} f(x)^{g(x)}$

$$y = f(x)^{g(x)}$$

$$\ln y = g(x) \ln f(x)$$

$$\lim_{x \to a} g(x) \ln f(x) = (\infty)(-\infty) = -\infty$$

As $x \to a$, $\ln y \implies -\infty$, and hence $y = 0$. Thus,

$$\lim_{x \to a} f(x)^{g(x)} = 0.$$

97. $$f'(a)(b - a) - \int_a^b f''(t)(t - b)\, dt = f'(a)(b - a) - \left\{\Big[f'(t)(t - b)\Big]_a^b - \int_a^b f'(t)\, dt\right\}$$

$$= f'(a)(b - a) + f'(a)(a - b) + \Big[f(t)\Big]_a^b = f(b) - f(a)$$

$$dv = f''(t)dt \implies v = f'(t)$$

$$u = t - b \implies du = dt$$

Section 7.8 Improper Integrals

1. Infinite discontinuity at $x = 0$.

$$\int_0^4 \frac{1}{\sqrt{x}}\, dx = \lim_{b \to 0^+} \int_b^4 \frac{1}{\sqrt{x}}\, dx$$

$$= \lim_{b \to 0^+} \Big[2\sqrt{x}\Big]_b^4$$

$$= \lim_{b \to 0^+} \left(4 - 2\sqrt{b}\right) = 4$$

Converges

3. Infinite discontinuity at $x = 1$.

$$\int_0^2 \frac{1}{(x-1)^2}\,dx = \int_0^1 \frac{1}{(x-1)^2}\,dx + \int_1^2 \frac{1}{(x-1)^2}\,dx$$

$$= \lim_{b\to 1^-}\int_0^b \frac{1}{(x-1)^2}\,dx + \lim_{c\to 1^+}\int_c^2 \frac{1}{(x-1)^2}\,dx$$

$$= \lim_{b\to 1^-}\left[-\frac{1}{x-1}\right]_0^b + \lim_{c\to 1^+}\left[-\frac{1}{x-1}\right]_c^2 = (\infty - 1) + (-1 + \infty)$$

Diverges

5. Infinite limit of integration.

$$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx$$

$$= \lim_{b\to\infty}\left[-e^{-x}\right]_0^b = 0 + 1 = 1$$

Converges

7. $\displaystyle\int_{-1}^1 \frac{1}{x^2}\,dx \neq -2$

because the integrand is not defined at $x = 0$.
Diverges

9. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx$

$$= \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = 1$$

11. $\displaystyle\int_1^\infty \frac{3}{\sqrt[3]{x}}\,dx = \lim_{b\to\infty}\int_1^b 3x^{-1/3}\,dx$

$$= \lim_{b\to\infty}\left[\frac{9}{2}x^{2/3}\right]_1^b = \infty$$

Diverges

13. $\displaystyle\int_{-\infty}^0 xe^{-2x}\,dx = \lim_{b\to-\infty}\int_b^0 xe^{-2x}\,dx = \lim_{b\to-\infty}\frac{1}{4}\Big[(-2x-1)e^{-2x}\Big]_b^0 = \lim_{b\to-\infty}\frac{1}{4}\left[-1 + (2b+1)e^{-2b}\right] = -\infty$ (Integration by parts)

Diverges

15. $\displaystyle\int_0^\infty x^2e^{-x}\,dx = \lim_{b\to\infty}\int_0^b x^2e^{-x}\,dx = \lim_{b\to\infty}\left[-e^{-x}(x^2+2x+2)\right]_0^b = \lim_{b\to\infty}\left(-\frac{b^2+2b+2}{e^b} + 2\right) = 2$

Since $\displaystyle\lim_{b\to\infty}\left(-\frac{b^2+2b+2}{e^b}\right) = 0$ by L'Hôpital's Rule.

17. $\displaystyle\int_0^\infty e^{-x}\cos x\,dx = \lim_{b\to\infty}\frac{1}{2}\Big[e^{-x}(-\cos x + \sin x)\Big]_0^b$

$$= \frac{1}{2}[0 - (-1)] = \frac{1}{2}$$

19. $\displaystyle\int_4^\infty \frac{1}{x(\ln x)^3}\,dx = \lim_{b\to\infty}\int_4^b (\ln x)^{-3}\frac{1}{x}\,dx$

$$= \lim_{b\to\infty}\left[-\frac{1}{2}(\ln x)^{-2}\right]_4^b$$

$$= -\frac{1}{2}(\ln b)^{-2} + \frac{1}{2}(\ln 4)^{-2}$$

$$= \frac{1}{2}\,\frac{1}{(2\ln 2)^2} = \frac{1}{8(\ln 2)^2}$$

21. $\displaystyle\int_{-\infty}^\infty \frac{2}{4+x^2}\,dx = \int_{-\infty}^0 \frac{2}{4+x^2}\,dx + \int_0^\infty \frac{2}{4+x^2}\,dx$

$$= \lim_{b\to-\infty}\int_b^0 \frac{2}{4+x^2}\,dx + \lim_{c\to\infty}\int_0^c \frac{2}{4+x^2}\,dx$$

$$= \lim_{b\to-\infty}\left[\arctan\!\left(\frac{x}{2}\right)\right]_b^0 + \lim_{c\to\infty}\left[\arctan\!\left(\frac{x}{2}\right)\right]_0^c$$

$$= \left(0 - \left(-\frac{\pi}{2}\right)\right) + \left(\frac{\pi}{2} - 0\right) = \pi$$

23. $\displaystyle\int_0^{\infty} \frac{1}{e^x + e^{-x}}\,dx = \lim_{b\to\infty} \int_0^b \frac{e^x}{1 + e^{2x}}\,dx$

$$= \lim_{b\to\infty} \Big[\arctan(e^x)\Big]_0^b$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

25. $\displaystyle\int_0^{\infty} \cos \pi x\,dx = \lim_{b\to\infty} \left[\frac{1}{\pi}\sin \pi x\right]_0^b$

Diverges since $\sin \pi x$ does not approach a limit as $x \to \infty$.

27. $\displaystyle\int_0^1 \frac{1}{x^2}\,dx = \lim_{b\to 0^+} \left[\frac{-1}{x}\right]_b^1 = -1 + \infty$

Diverges

29. $\displaystyle\int_0^8 \frac{1}{\sqrt[3]{8-x}}\,dx = \lim_{b\to 8^-} \int_0^b \frac{1}{\sqrt[3]{8-x}}\,dx = \lim_{b\to 8^-} \left[\frac{-3}{2}(8-x)^{2/3}\right]_0^b = 6$

31. $\displaystyle\int_0^1 x \ln x\,dx = \lim_{b\to 0^+} \left[\frac{x^2}{2}\ln|x| - \frac{x^2}{4}\right]_b^1 = \lim_{b\to 0^+} \left[\frac{-1}{4} - \frac{b^2 \ln b}{2} + \frac{b^2}{4}\right] = \frac{-1}{4}$ since $\displaystyle\lim_{b\to 0^+} (b^2 \ln b) = 0$ by L'Hôpital's Rule.

33. $\displaystyle\int_0^{\pi/2} \tan\theta\,d\theta = \lim_{b\to(\pi/2)^-} \Big[\ln|\sec\theta|\Big]_0^b = \infty$

Diverges

35. $\displaystyle\int_2^4 \frac{2}{x\sqrt{x^2-4}}\,dx = \lim_{b\to 2^+} \int_b^4 \frac{2}{x\sqrt{x^2-4}}\,dx$

$$= \lim_{b\to 2^+} \left[\operatorname{arcsec}\left|\frac{x}{2}\right|\right]_b^4$$

$$= \lim_{b\to 2^+} \left(\operatorname{arcsec} 2 - \operatorname{arcsec}\left(\frac{b}{2}\right)\right)$$

$$= \frac{\pi}{3} - 0 = \frac{\pi}{3}$$

37. $\displaystyle\int_2^4 \frac{1}{\sqrt{x^2-4}} = \lim_{b\to 2^+} \Big[\ln\left|x + \sqrt{x^2-4}\right|\Big]_b^4$

$$= \ln\left(4 + 2\sqrt{3}\right) - \ln 2$$

$$= \ln\left(2 + \sqrt{3}\right) \approx 1.317$$

39. $\displaystyle\int_0^2 \frac{1}{\sqrt[3]{x-1}}\,dx = \int_0^1 \frac{1}{\sqrt[3]{x-1}}\,dx + \int_1^2 \frac{1}{\sqrt[3]{x-1}}\,dx$

$$= \lim_{b\to 1^-} \left[\frac{3}{2}(x-1)^{2/3}\right]_0^b + \lim_{c\to 1^+} \left[\frac{3}{2}(x-1)^{2/3}\right]_c^2 = \frac{-3}{2} + \frac{3}{2} = 0$$

41. $\displaystyle\int_0^{\infty} \frac{4}{\sqrt{x}(x+6)}\,dx = \int_0^1 \frac{4}{\sqrt{x}(x+6)}\,dx + \int_1^{\infty} \frac{4}{\sqrt{x}(x+6)}\,dx$

Let $u = \sqrt{x}$, $u^2 = x$, $2u\,du = dx$.

$$\int \frac{4}{\sqrt{x}(x+6)}\,dx = \int \frac{4(2u\,du)}{u(u^2+6)} = 8\int \frac{du}{u^2+6} = \frac{8}{\sqrt{6}}\arctan\left(\frac{4}{\sqrt{6}}\right) + C = \frac{8}{\sqrt{6}}\arctan\left(\frac{\sqrt{x}}{\sqrt{6}}\right) + C$$

Thus, $\displaystyle\int_0^{\infty} \frac{4}{\sqrt{x}(x+6)}\,dx = \lim_{b\to 0^+} \left[\frac{8}{\sqrt{6}}\arctan\left(\frac{\sqrt{x}}{\sqrt{6}}\right)\right]_b^1 + \lim_{c\to\infty} \left[\frac{8}{\sqrt{6}}\arctan\left(\frac{\sqrt{x}}{\sqrt{6}}\right)\right]_1^c$

$$= \left(\frac{8}{\sqrt{6}}\arctan\left(\frac{1}{\sqrt{6}}\right) - \frac{8}{\sqrt{6}}0\right) + \left(\frac{8}{\sqrt{6}}\frac{\pi}{2} - \frac{8}{\sqrt{6}}\arctan\left(\frac{1}{\sqrt{6}}\right)\right)$$

$$= \frac{8\pi}{2\sqrt{6}} = \frac{2\pi\sqrt{6}}{3}.$$

43. If $p = 1$, $\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln x\Big]_1^b$.

Diverges. For $p \neq 1$,

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\left[\frac{x^{1-p}}{1-p}\right]_1^b = \lim_{b\to\infty}\left[\frac{b^{1-p}}{1-p} - \frac{1}{1-p}\right].$$

This converges to $\frac{1}{p-1}$ if $1 - p < 0$ or $p > 1$.

45. For $n = 1$ we have

$$\begin{aligned}\int_0^\infty xe^{-x}\,dx &= \lim_{b\to\infty}\int_0^b xe^{-x}\,dx\\ &= \lim_{b\to\infty}\Big[-e^{-x}x - e^{-x}\Big]_0^b \qquad \text{(Parts: } u = x,\ dv = e^{-x}\,dx)\\ &= \lim_{b\to\infty}\left[-e^{-b}b - e^{-b} + 1\right]\\ &= \lim_{b\to\infty}\left[\frac{-b}{e^b} - \frac{1}{e^b} + 1\right] = 1 \quad \text{(L'Hôpital's Rule)}\end{aligned}$$

Assume that $\int_0^\infty x^n e^{-x}\,dx$ converges. Then for $n + 1$ we have

$$\int x^{n+1}e^{-x}\,dx = -x^{n+1}e^{-x} + (n+1)\int x^n e^{-x}\,dx$$

by parts ($u = x^{n+1}$, $du = (n+1)x^n\,dx$, $dv = e^{-x}\,dx$, $v = -e^{-x}$).

Thus,

$$\int_0^\infty x^{n+1}e^{-x}\,dx = \lim_{b\to\infty}\Big[-x^{n+1}e^{-x}\Big]_0^b + (n+1)\int_0^\infty x^n e^{-x}\,dx = 0 + (n+1)\int_0^\infty x^n e^{-x}\,dx,$$ which converges.

47. $\int_0^1 \frac{1}{x^3}\,dx$ diverges.

(See Exercise 44, $p = 3 \not< 1$.)

49. $\int_1^\infty \frac{1}{x^3}\,dx = \frac{1}{3-1} = \frac{1}{2}$ converges.

(See Exercise 43, $p = 3$.)

51. Since $\frac{1}{x^2+5} \le \frac{1}{x^2}$ on $[1, \infty)$ and $\int_1^\infty \frac{1}{x^2}\,dx$ converges by Exercise 43, $\int_1^\infty \frac{1}{x^2+5}\,dx$ converges.

53. Since $\frac{1}{\sqrt[3]{x(x-1)}} \ge \frac{1}{\sqrt[3]{x^2}}$ on $[2, \infty)$ and $\int_2^\infty \frac{1}{\sqrt[3]{x^2}}\,dx$ diverges by Exercise 43, $\int_2^\infty \frac{1}{\sqrt[3]{x(x-1)}}\,dx$ diverges.

55. Since $e^{-x^2} \le e^{-x}$ on $[1, \infty)$ and $\int_0^\infty e^{-x}\,dx$ converges (see Exercise 5), $\int_0^\infty e^{-x^2}\,dx$ converges.

57. Answers will vary. See pages 540, 543.

59. $\int_{-1}^1 \frac{1}{x^3}\,dx = \int_{-1}^0 \frac{1}{x^3}\,dx + \int_0^1 \frac{1}{x^3}\,dx$

These two integrals diverge by Exercise 44.

61. $f(t) = 1$

$$F(s) = \int_0^\infty e^{-st}\,dx = \lim_{b\to\infty}\left[-\frac{1}{s}e^{-st}\right]_0^b = \frac{1}{s},\ s > 0$$

63. $f(t) = t^2$

$$F(s) = \int_0^\infty t^2e^{-st}\,dx = \lim_{b\to\infty}\left[\frac{1}{s^3}(-s^2t^2 - 2st - 2)e^{-st}\right]_0^b = \frac{2}{s^3},\ s > 0$$

65. $f(t) = \cos at$

$$F(s) = \int_0^\infty e^{-st} \cos at\, dt$$
$$= \lim_{b\to\infty} \left[\frac{e^{-st}}{s^2 + a^2}(-s\cos at + a \sin at)\right]_0^b$$
$$= 0 + \frac{s}{s^2+a^2} = \frac{s}{s^2+a^2},\ s > 0$$

67. $f(t) = \cosh at$

$$F(s) = \int_0^\infty e^{-st}\cosh at\, dt = \int_0^\infty e^{-st}\left(\frac{e^{at}+e^{-at}}{2}\right) dt = \frac{1}{2}\int_0^\infty \left[e^{t(-s+a)} + e^{t(-s-a)}\right] dt$$
$$= \lim_{b\to\infty} \frac{1}{2}\left[\frac{1}{(-s+a)}e^{t(-s+a)} + \frac{1}{(-s-a)}e^{t(-s-a)}\right]_0^b = 0 - \frac{1}{2}\left[\frac{1}{(-s+a)} + \frac{1}{(-s-a)}\right]$$
$$= \frac{-1}{2}\left[\frac{1}{(-s+a)} + \frac{1}{(-s-a)}\right] = \frac{s}{s^2-a^2},\ s > |a|$$

69. (a) $A = \int_0^\infty e^{-x}\, dx$

$$= \lim_{b\to\infty}\left[-e^{-x}\right]_0^b = 0 - (-1) = 1$$

(b) **Disk:**

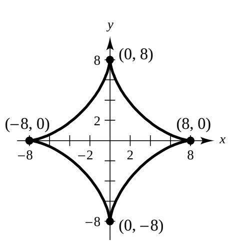
$$V = \pi\int_0^\infty (e^{-x})^2\, dx$$
$$= \lim_{b\to\infty} \pi\left[-\frac{1}{2}e^{-2x}\right]_0^b = \frac{\pi}{2}$$

(c) **Shell:**

$$V = 2\pi\int_0^\infty xe^{-x}\, dx$$
$$= \lim_{b\to\infty}\left\{2\pi\left[-e^{-x}(x+1)\right]_0^b\right\} = 2\pi$$

71. $x^{2/3} + y^{2/3} = 4$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$
$$y' = \frac{-y^{1/3}}{x^{1/3}}$$
$$\sqrt{1+(y')^2} = \sqrt{1+\frac{y^{2/3}}{x^{2/3}}} = \sqrt{\frac{x^{2/3}+y^{2/3}}{x^{2/3}}} = \sqrt{\frac{4}{x^{2/3}}} = \frac{2}{x^{1/3}}$$
$$s = 4\int_0^8 \frac{2}{x^{1/3}}\, dx = \lim_{b\to 0^+}\left[8\cdot\frac{3}{2}x^{2/3}\right]_b^8 = 48$$



73. $\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}\,dx$

(a) $\Gamma(1) = \int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\Big[-e^{-x}\Big]_0^b = 1$

$\Gamma(2) = \int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty}\Big[-e^{-x}(x+1)\Big]_0^b = 1$

$\Gamma(3) = \int_0^\infty x^2e^{-x}\,dx = \lim_{b\to\infty}\Big[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\Big]_0^b = 2$

(b) $\Gamma(n+1) = \int_0^\infty x^ne^{-x}\,dx = \lim_{b\to\infty}\Big[-x^ne^{-x}\Big]_0^b + \lim_{b\to\infty} n\int_0^b x^{n-1}e^{-x}\,dx = 0 + n\Gamma(n)$ $\quad(u = x^n,\ dv = e^{-x}\,dx)$

(c) $\Gamma(n) = (n-1)!$

75. (a) $\int_{-\infty}^\infty \frac{1}{7}e^{-t/7}\,dt = \int_0^\infty \frac{1}{7}e^{-t/7}\,dt = \lim_{b\to\infty}\Big[-e^{-t/7}\Big]_0^b = 1$

(b) $\int_0^4 \frac{1}{7}e^{-t/7}\,dt = \Big[-e^{-t/7}\Big]_0^4 = -e^{-t/7} + 1$

$\approx 0.4353 = 43.53\%$

(c) $\int_0^\infty t\Big[\frac{1}{7}e^{-t/7}\Big]\,dt = \lim_{b\to\infty}\Big[-te^{-t/7} - 7e^{-t/7}\Big]_0^b$

$= 0 + 7 = 7$

77. (a) $C = 650{,}000 + \int_0^5 25{,}000\,e^{-0.06t}\,dt = 650{,}000 - \Big[\frac{25{,}000}{0.06}e^{-0.06t}\Big]_0^5 \approx \$757{,}992.41$

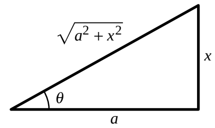
(b) $C = 650{,}000 + \int_0^{10} 25{,}000e^{-0.06t}\,dt \approx \$837{,}995.15$

(c) $C = 650{,}000 + \int_0^\infty 25{,}000e^{-0.06t}\,dt = 650{,}000 - \lim_{b\to\infty}\Big[\frac{25{,}000}{0.06}e^{-0.06t}\Big]_0^b \approx \$1{,}066{,}666.67$

79. Let $x = a\tan\theta,\ dx = a\sec^2\theta\,d\theta,\ \sqrt{a^2+x^2} = a\sec\theta.$

$$\int \frac{1}{(a^2+x^2)^{3/2}}\,dx = \int \frac{a\sec^2\theta\,d\theta}{a^3\sec^3\theta} = \frac{1}{a^2}\int\cos\theta\,d\theta$$

$$= \frac{1}{a^2}\sin\theta = \frac{1}{a^2}\frac{x}{\sqrt{a^2+x^2}}$$



Hence,

$$P = k\int_1^\infty \frac{1}{(a^2+x^2)^{3/2}}\,dx = \frac{k}{a^2}\lim_{b\to\infty}\Big[\frac{x}{\sqrt{a^2+x^2}}\Big]_1^b$$

$$= \frac{k}{a^2}\Big[1 - \frac{1}{\sqrt{a^2+1}}\Big] = \frac{k\big(\sqrt{a^2+1}-1\big)}{a^2\sqrt{a^2+1}}.$$

81. $\frac{10}{x^2-2x} = \frac{10}{x(x-2)} \implies x = 0, 2.$

You must analyze three improper integrals, and each must converge in order for the original integral to converge.

$$\int_0^3 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx$$

83. For $n = 1$,

$$I_1 = \int_0^\infty \frac{x}{(x^2+1)^4}\,dx = \lim_{b\to\infty} \frac{1}{2}\int_0^b (x^2+1)^{-4}(2x\,dx) = \lim_{b\to\infty}\left[-\frac{1}{6}\frac{1}{(x^2+1)^3}\right]_0^b = \frac{1}{6}.$$

For $n > 1$,

$$I_n = \int_0^\infty \frac{x^{2n-1}}{(x^2+1)^{n+3}}\,dx = \lim_{b\to\infty}\left[\frac{-x^{2n-2}}{2(n+2)(x^2+1)^n} + 2\right]_0^b + \frac{n-1}{n+2}\int_0^\infty \frac{x^{2n-3}}{(x^2+1)^{n+2}}\,dx = 0 + \frac{n-1}{n+2}(I_{n-1})$$

$$u = x^{2n-2},\ du = (2n-2)x^{2n-3}\,dx,\ dv = \frac{x}{(x^2+1)^{n+3}}\,dx,\ v = \frac{-1}{2(n+2)(x^2+1)^{n+2}}$$

(a) $\displaystyle\int_0^\infty \frac{x}{(x^2+1)^4}\,dx = \lim_{b\to\infty}\left[-\frac{1}{6(x^2+1)^3}\right]_0^b = \frac{1}{6}$

(b) $\displaystyle\int_0^\infty \frac{x^3}{(x^2+1)^5}\,dx = \frac{1}{4}\int_0^\infty \frac{x}{(x^2+1)^4}\,dx = \frac{1}{4}\left(\frac{1}{6}\right) = \frac{1}{24}$

(c) $\displaystyle\int_0^\infty \frac{x^5}{(x^2+1)^6} = \frac{2}{5}\int_0^\infty \frac{x^3}{(x^2+1)^5}\,dx = \frac{2}{5}\left(\frac{1}{24}\right) = \frac{1}{60}$

85. False. $f(x) = 1/(x+1)$ is continuous on $[0, \infty)$, $\lim_{x\to\infty} 1/(x+1) = 0$, but $\displaystyle\int_0^\infty \frac{1}{x+1}\,dx = \lim_{b\to\infty}\Big[\ln|x+1|\Big]_0^b = \infty.$

Diverges

87. True

Review Exercises for Chapter 7

1.
$$\int x\sqrt{x^2-1}\,dx = \frac{1}{2}\int (x^2-1)^{1/2}(2x)\,dx$$
$$= \frac{1}{2}\frac{(x^2-1)^{3/2}}{3/2} + C$$
$$= \frac{1}{3}(x^2-1)^{3/2} + C$$

3.
$$\int \frac{x}{x^2-1}\,dx = \frac{1}{2}\int \frac{2x}{x^2-1}\,dx$$
$$= \frac{1}{2}\ln|x^2-1| + C$$

5. $\displaystyle\int \frac{\ln(2x)}{x}\,dx = \frac{(\ln 2x)^2}{2} + C$

7. $\displaystyle\int \frac{16}{\sqrt{16-x^2}}\,dx = 16\arcsin\left(\frac{x}{4}\right) + C$

9.
$$\int e^{2x}\sin 3x\,dx = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\int e^{2x}\cos 3x\,dx$$
$$= -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\left(\frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\int e^{2x}\sin 3x\,dx\right)$$
$$\frac{13}{9}\int e^{2x}\sin 3x\,dx = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{9}e^{2x}\sin 3x$$
$$\int e^{2x}\sin 3x\,dx = \frac{e^{2x}}{13}(2\sin 3x - 3\cos 3x) + C$$

(1) $dv = \sin 3x\,dx \implies v = -\frac{1}{3}\cos 3x$

$u = e^{2x} \implies du = 2e^{2x}\,dx$

(2) $dv = \cos 3x\,dx \implies v = \frac{1}{3}\sin 3x$

$u = e^{2x} \implies du = 2e^{2x}\,dx$

11. $u = x,\ du = dx,\ dv = (x-5)^{1/2}\,dx,\ v = \frac{2}{3}(x-5)^{3/2}$

$$\int x\sqrt{x-5}\,dx = \frac{2}{3}x(x-5)^{3/2} - \int \frac{2}{3}(x-5)^{3/2}\,dx$$
$$= \frac{2}{3}x(x-5)^{3/2} - \frac{4}{15}(x-5)^{5/2} + C$$
$$= (x-5)^{3/2}\left[\frac{2}{3}x - \frac{4}{15}(x-5)\right] + C$$
$$= (x-5)^{3/2}\left[\frac{6}{15}x + \frac{4}{3}\right] + C$$
$$= \frac{2}{15}(x-5)^{3/2}[3x+10] + C$$

13. $$\int x^2 \sin 2x\,dx = -\frac{1}{2}x^2\cos 2x + \int x\cos 2x\,dx$$
$$= -\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x - \frac{1}{2}\int \sin 2x\,dx$$
$$= -\frac{1}{2}x^2\cos 2x + \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x + C$$

(1) $dv = \sin 2x\,dx \implies v = -\frac{1}{2}\cos 2x$

$u = x^2 \implies du = 2x\,dx$

(2) $dv = \cos 2x\,dx \implies v = \frac{1}{2}\sin 2x$

$u = x \implies du = dx$

15. $$\int x\arcsin 2x\,dx = \frac{x^2}{2}\arcsin 2x - \int \frac{x^2}{\sqrt{1-4x^2}}\,dx$$
$$= \frac{x^2}{2}\arcsin 2x - \frac{1}{8}\int \frac{2(2x)^2}{\sqrt{1-(2x)^2}}\,dx$$
$$= \frac{x^2}{2}\arcsin 2x - \frac{1}{8}\left(\frac{1}{2}\right)\left[-(2x)\sqrt{1-4x^2} + \arcsin 2x\right] + C$$ (by Formula 43 of Integration Tables)
$$= \frac{1}{16}\left[(8x^2-1)\arcsin 2x + 2x\sqrt{1-4x^2}\right] + C$$

$dv = x\,dx \implies v = \frac{x^2}{2}$

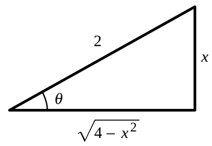
$u = \arcsin 2x \implies du = \frac{2}{\sqrt{1-4x^2}}\,dx$

17. $$\int \cos^3(\pi x - 1)\,dx = \int [1 - \sin^2(\pi x - 1)]\cos(\pi x - 1)\,dx$$
$$= \frac{1}{\pi}\left[\sin(\pi x - 1) - \frac{1}{3}\sin^3(\pi x - 1)\right] + C$$
$$= \frac{1}{3\pi}\sin(\pi x - 1)[3 - \sin^2(\pi x - 1)] + C$$
$$= \frac{1}{3\pi}\sin(\pi x - 1)[3 - (1 - \cos^2(\pi x - 1))] + C$$
$$= \frac{1}{3\pi}\sin(\pi x - 1)[2 + \cos^2(\pi x - 1)] + C$$

19. $$\int \sec^4\left(\frac{x}{2}\right)dx = \int\left[\tan^2\left(\frac{x}{2}\right) + 1\right]\sec^2\left(\frac{x}{2}\right)dx$$
$$= \int \tan^2\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)dx + \int \sec^2\left(\frac{x}{2}\right)dx$$
$$= \frac{2}{3}\tan^3\left(\frac{x}{2}\right) + 2\tan\left(\frac{x}{2}\right) + C = \frac{2}{3}\left[\tan^3\left(\frac{x}{2}\right) + 3\tan\left(\frac{x}{2}\right)\right] + C$$

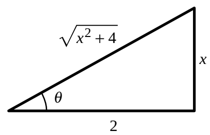
21. $$\int \frac{1}{1-\sin\theta}\,d\theta = \int \frac{1+\sin\theta}{\cos^2\theta}\,d\theta = \int(\sec^2\theta + \sec\theta\tan\theta)\,d\theta = \tan\theta + \sec\theta + C$$

23. $$\int \frac{-12}{x^2\sqrt{4-x^2}}\,dx = \int \frac{-24\cos\theta\,d\theta}{(4\sin^2\theta)(2\cos\theta)}$$
$$= -3\int \csc^2\theta\,d\theta$$
$$= 3\cot\theta + C$$
$$= \frac{3\sqrt{4-x^2}}{x} + C$$



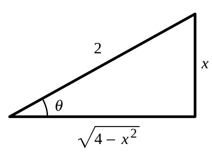
$x = 2\sin\theta,\ dx = 2\cos\theta\,d\theta,\ \sqrt{4-x^2} = 2\cos\theta$

25. $$x = 2\tan\theta$$
$$dx = 2\sec^2\theta\,d\theta$$
$$4 + x^2 = 4\sec^2\theta$$



$$\int \frac{x^3}{\sqrt{4+x^2}}\,dx = \int \frac{8\tan^3\theta}{2\sec\theta}\,2\sec^2\theta\,d\theta$$
$$= 8\int \tan^3\theta\sec\theta\,d\theta$$
$$= 8\int (\sec^2\theta - 1)\tan\theta\sec\theta\,d\theta$$
$$= 8\left[\frac{\sec^3\theta}{3} - \sec\theta\right] + C$$
$$= 8\left[\frac{(x^2+4)^{3/2}}{24} - \frac{\sqrt{x^2+4}}{2}\right] + C$$
$$= \sqrt{x^2+4}\left[\frac{1}{3}(x^2+4) - 4\right] + C$$
$$= \frac{1}{3}x^2\sqrt{x^2+4} - \frac{8}{3}\sqrt{x^2+4} + C$$
$$= \frac{1}{3}(x^2+4)^{1/2}(x^2-8) + C$$

27. $$\int \sqrt{4-x^2}\,dx = \int (2\cos\theta)(2\cos\theta)\,d\theta$$
$$= 2\int (1 + \cos 2\theta)\,d\theta$$
$$= 2\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$
$$= 2(\theta + \sin\theta\cos\theta) + C$$
$$= 2\left[\arcsin\left(\frac{x}{2}\right) + \frac{x}{2}\left(\frac{\sqrt{4-x^2}}{2}\right)\right] + C$$
$$= \frac{1}{2}\left[4\arcsin\left(\frac{x}{2}\right) + x\sqrt{4-x^2}\right] + C$$



$x = 2\sin\theta,\ dx = 2\cos\theta\,d\theta,\ \sqrt{4-x^2} = 2\cos\theta$

29. (a) $\displaystyle\int \frac{x^3}{\sqrt{4+x^2}}\,dx = 8\int \frac{\sin^3\theta}{\cos^4\theta}\,d\theta$

$$= 8\int (\cos^{-4}\theta - \cos^{-2}\theta)\sin\theta\,d\theta$$

$$= \frac{8}{3}\sec\theta(\sec^2\theta - 3) + C$$

$$= \frac{\sqrt{4+x^2}}{3}(x^2 - 8) + C$$

$x = 2\tan\theta,\ dx = 2\sec^2\theta\,d\theta$

(b) $\displaystyle\int \frac{x^3}{\sqrt{4+x^2}}\,dx = \int (u^2 - 4)\,du$

$$= \frac{1}{3}u^3 - 4u + C$$

$$= \frac{u}{3}(u^2 - 12) + C$$

$$= \frac{\sqrt{4+x^2}}{3}(x^2 - 8) + C$$

$u^2 = 4 + x^2,\ 2u\,du = 2x\,dx$

(c) $\displaystyle\int \frac{x^3}{\sqrt{4+x^2}}\,dx = x^2\sqrt{4+x^2} - \int 2x\sqrt{4+x^2}\,dx$

$$= x^2\sqrt{4+x^2} - \frac{2}{3}(4+x^2)^{3/2} + C = \frac{\sqrt{4+x^2}}{3}(x^2 - 8) + C$$

$dv = \dfrac{x}{\sqrt{4+x^2}}\,dx \implies v = \sqrt{4+x^2}$

$u = x^2 \implies du = 2x\,dx$

31. $\dfrac{x - 28}{x^2 - x - 6} = \dfrac{A}{x-3} + \dfrac{B}{x+2}$

$x - 28 = A(x+2) + B(x-3)$

$x = -2 \implies -30 = B(-5) \implies B = 6$

$x = 3 \implies -25 = A(5) \implies A = -5$

$$\int \frac{x-28}{x^2-6-6}\,dx = \int\left(\frac{-5}{x-3} + \frac{6}{x+2}\right)dx = -5\ln|x-3| + 6\ln|x+2| + C$$

33. $\dfrac{x^2 + 2x}{(x-1)(x^2+1)} = \dfrac{A}{x-1} + \dfrac{Bx + C}{x^2+1}$

$x^2 + 2x = A(x^2+1) + (Bx+C)(x-1)$

Let $x = 1$: $3 = 2A \implies A = \dfrac{3}{2}$

Let $x = 0$: $0 = A - C \implies C = \dfrac{3}{2}$

Let $x = 2$: $8 = 5A + 2B + C \implies B = -\dfrac{1}{2}$

$$\int \frac{x^2+2x}{x^3 - x^2 + x - 1}\,dx = \frac{3}{2}\int\frac{1}{x-1}\,dx - \frac{1}{2}\int\frac{x-3}{x^2+1}\,dx$$

$$= \frac{3}{2}\int\frac{1}{x-1}\,dx - \frac{1}{4}\int\frac{2x}{x^2+1}\,dx + \frac{3}{2}\int\frac{1}{x^2+1}\,dx$$

$$= \frac{3}{2}\ln|x-1| - \frac{1}{4}\ln|x^2+1| + \frac{3}{2}\arctan x + C$$

$$= \frac{1}{4}[6\ln|x-1| - \ln(x^2+1) + 6\arctan x] + C$$

35. $\dfrac{x^2}{x^2+2x-15} = 1 + \dfrac{15-2x}{x^2+2x-15}$

$$\frac{15-2x}{(x-3)(x+5)} = \frac{A}{x-3} + \frac{B}{x+5}$$

$$15 - 2x = A(x+5) + B(x-3)$$

Let $x = 3$: $\quad 9 = 8A \implies A = \dfrac{9}{8}$

Let $x = -5$: $25 = -8B \implies B = -\dfrac{25}{8}$

$$\int \frac{x^2}{x^2+2x-15}\,dx = \int dx + \frac{9}{8}\int \frac{1}{x-3}\,dx - \frac{25}{8}\int \frac{1}{x+5}\,dx$$

$$= x + \frac{9}{8}\ln|x-3| - \frac{25}{8}\ln|x+5| + C$$

37. $\displaystyle\int \frac{x}{(2+3x)^2}\,dx = \frac{1}{9}\left[\frac{2}{2+3x} + \ln|2+3x|\right] + C$

(Formula 4)

39. $\displaystyle\int \frac{x}{1+\sin x^2}\,dx = \frac{1}{2}\int \frac{1}{1+\sin u}\,du \qquad (u = x^2)$

$$= \frac{1}{2}[\tan u - \sec u] + C \qquad \text{(Formula 56)}$$

$$= \frac{1}{2}[\tan x^2 - \sec x^2] + C$$

41. $\displaystyle\int \frac{x}{x^2+4x+8}\,dx = \frac{1}{2}\left[\ln|x^2+4x+8| - 4\int \frac{1}{x^2+4x+8}\,dx\right] \qquad \text{(Formula 15)}$

$$= \frac{1}{2}[\ln|x^2+4x+8|] - 2\left[\frac{2}{\sqrt{32-16}}\arctan\left(\frac{2x+4}{\sqrt{32-16}}\right)\right] + C \quad \text{(Formula 14)}$$

$$= \frac{1}{2}\ln|x^2+4x+8| - \arctan\left(1 + \frac{x}{2}\right) + C$$

43. $\displaystyle\int \frac{1}{\sin \pi x \cos \pi x}\,dx = \frac{1}{\pi}\int \frac{1}{\sin \pi x \cos \pi x}(\pi)\,dx \quad (u = \pi x)$

$$= \frac{1}{\pi}\ln|\tan \pi x| + C \qquad \text{(Formula 58)}$$

45. $dv = dx \implies v = x$

$u = (\ln x)^n \implies du = n(\ln x)^{n-1}\dfrac{1}{x}dx$

$$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$$

47. $\displaystyle\int \theta \sin\theta \cos\theta\,d\theta = \frac{1}{2}\int \theta \sin 2\theta\,d\theta$

$$= -\frac{1}{4}\theta\cos 2\theta + \frac{1}{4}\int \cos 2\theta\,d\theta = -\frac{1}{4}\theta\cos 2\theta + \frac{1}{8}\sin 2\theta + C = \frac{1}{8}(\sin 2\theta - 2\theta\cos 2\theta) + C$$

$dv = \sin 2\theta\,d\theta \implies v = -\dfrac{1}{2}\cos 2\theta$

$u = \theta \implies du = d\theta$

49. $\int \frac{x^{1/4}}{1 + x^{1/2}}\,dx = 4\int \frac{u(u^3)}{1 + u^2}\,du$

$= 4\int \left(u^2 - 1 + \frac{1}{u^2 + 1}\right) du$

$= 4\left(\frac{1}{3}u^3 - u + \arctan u\right) + C$

$= \frac{4}{3}[x^{3/4} - 3x^{1/4} + 3\arctan(x^{1/4})] + C$

$y = \sqrt[4]{x},\ x = u^4,\ dx = 4u^3\,du$

51. $\int \sqrt{1 + \cos x}\,dx = \int \frac{\sin x}{\sqrt{1 - \cos x}}\,dx$

$= \int (1 - \cos x)^{-1/2}(\sin x)\,dx$

$= 2\sqrt{1 - \cos x} + C$

$u = 1 - \cos x,\ du = \sin x\,dx$

53. $\int \cos x \ln(\sin x)\,dx = \sin x \ln(\sin x) - \int \cos x\,dx$

$= \sin x \ln(\sin x) - \sin x + C$

$dv = \cos x\,dx \implies v = \sin x$

$u = \ln(\sin x) \implies du = \frac{\cos x}{\sin x}\,dx$

55. $y = \int \frac{9}{x^2 - 9}\,dx = \frac{3}{2}\ln\left|\frac{x - 3}{x + 3}\right| + C$

(by Formula 24 of Integration Tables)

57. $y = \int \ln(x^2 + x)dx = x\ln|x^2 + x| - \int \frac{2x^2 + x}{x^2 + x}\,dx$

$= x\ln|x^2 + x| - \int \frac{2x + 1}{x + 1}\,dx$

$= x\ln|x^2 + x| - \int 2dx + \int \frac{1}{x + 1}\,dx$

$= x\ln|x^2 + x| - 2x + \ln|x + 1| + C$

$dv = dx \implies v = x$

$u = \ln(x^2 + x) \implies du = \frac{2x + 1}{x^2 + x}\,dx$

59. $\int_2^{\sqrt{5}} x(x^2 - 4)^{3/2}\,dx = \left[\frac{1}{5}(x^2 - 4)^{5/2}\right]_2^{\sqrt{5}} = \frac{1}{5}$

61. $\int_1^4 \frac{\ln x}{x}\,dx = \left[\frac{1}{2}(\ln x)^2\right]_1^4 = \frac{1}{2}(\ln 4)^2 = 2(\ln 2)^2 \approx 0.961$

63. $\int_0^{\pi} x\sin x\,dx = \left[-x\cos x + \sin x\right]_0^{\pi} = \pi$

65. $A = \int_0^4 x\sqrt{4 - x}\,dx = \int_2^0 (4 - u^2)u(-2u)\,du$

$= \int_2^0 2(u^4 - 4u^2)\,du$

$= \left[2\left(\frac{u^5}{5} - \frac{4u^3}{3}\right)\right]_2^0 = \frac{128}{15}$

$u = \sqrt{4 - x},\ x = 4 - u^2,\ dx = -2u\,du$

67. By symmetry, $\bar{x} = 0,\ A = \frac{1}{2}\pi$.

$\bar{y} = \frac{2}{\pi}\left(\frac{1}{2}\right)\int_{-1}^{1}\left(\sqrt{1 - x^2}\right)^2 dx = \frac{1}{\pi}\left[x - \frac{1}{3}x^3\right]_{-1}^{1} = \frac{4}{3\pi}$

$(\bar{x}, \bar{y}) = \left(0, \frac{4}{3\pi}\right)$

69. $s = \int_0^{\pi} \sqrt{1 + \cos^2 x}\,dx \approx 3.82$

71. $\lim_{x \to 1}\left[\frac{(\ln x)^2}{x - 1}\right] = \lim_{x \to 1}\left[\frac{2(1/x)\ln x}{1}\right] = 0$

73. $\lim_{x \to \infty}\frac{e^{2x}}{x^2} = \lim_{x \to \infty}\frac{2e^{2x}}{2x} = \lim_{x \to \infty}\frac{4e^{2x}}{2} = \infty$

75. $y = \lim_{x \to \infty}(\ln x)^{2/x}$

$\ln y = \lim_{x \to \infty}\frac{2\ln(\ln x)}{x} = \lim_{x \to \infty}\left[\frac{2/(x\ln x)}{1}\right] = 0$

Since $\ln y = 0$, $y = 1$.