CHAPTER 8
Infinite Series

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Section 8.1 Sequences

Solutions to Even-Numbered Exercises

2. $a_n = \dfrac{2n}{n+3}$

$a_1 = \dfrac{2}{4} = \dfrac{1}{2}$

$a_2 = \dfrac{4}{5}$

$a_3 = \dfrac{6}{6} = 1$

$a_4 = \dfrac{8}{7}$

$a_5 = \dfrac{10}{8} = \dfrac{5}{4}$

4. $a_n = \left(-\dfrac{2}{3}\right)^n$

$a_1 = -\dfrac{2}{3}$

$a_2 = \dfrac{4}{9}$

$a_3 = -\dfrac{8}{27}$

$a_4 = \dfrac{16}{81}$

$a_5 = -\dfrac{32}{243}$

6. $a_n = \cos\dfrac{n\pi}{2}$

$a_1 = \cos\dfrac{\pi}{2} = 0$

$a_2 = \cos\pi = -1$

$a_3 = \cos\dfrac{3\pi}{2} = 0$

$a_4 = \cos 2\pi = 1$

$a_5 = \cos\dfrac{5\pi}{2} = 0$

8. $a_n = (-1)^{n+1}\left(\dfrac{2}{n}\right)$

$a_1 = \dfrac{2}{1} = 2$

$a_2 = -\dfrac{2}{2} = -1$

$a_3 = \dfrac{2}{3}$

$a_4 = -\dfrac{2}{4} = -\dfrac{1}{2}$

$a_5 = \dfrac{2}{5}$

10. $a_n = 10 + \dfrac{2}{n} + \dfrac{6}{n^2}$

$a_1 = 10 + 2 + 6 = 18$

$a_2 = 10 + 1 + \dfrac{3}{2} = \dfrac{25}{2}$

$a_3 = 10 + \dfrac{2}{3} + \dfrac{2}{3} = \dfrac{34}{3}$

$a_4 = 10 + \dfrac{1}{2} + \dfrac{3}{8} = \dfrac{87}{8}$

$a_5 = 10 + \dfrac{2}{5} + \dfrac{6}{25} = \dfrac{266}{25}$

12. $a_n = \dfrac{3n!}{(n-1)!} = 3n$

$a_1 = 3(1) = 3$

$a_2 = 3(2) = 6$

$a_3 = 3(3) = 9$

$a_4 = 3(4) = 12$

$a_5 = 3(5) = 15$

14. $a_1 = 4,\ a_{k+1} = \left(\dfrac{k+1}{2}\right)a_k$

$a_2 = \left(\dfrac{1+1}{2}\right)a_1 = 4$

$a_3 = \left(\dfrac{2+1}{2}\right)a_2 = 6$

$a_4 = \left(\dfrac{3+1}{2}\right)a_3 = 12$

$a_5 = \left(\dfrac{4+1}{2}\right)a_4 = 30$

16. $a_1 = 6,\ a_{k+1} = \dfrac{1}{3}a_k^{\,2}$

$a_2 = \dfrac{1}{3}a_1^{\,2} = \dfrac{1}{3}(6^2) = 12$

$a_3 = \dfrac{1}{3}a_2^{\,2} = \dfrac{1}{3}(12^2) = 48$

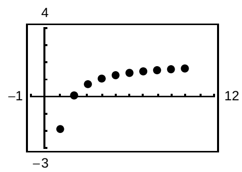
$a_4 = \dfrac{1}{3}a_3^{\,2} = \dfrac{1}{3}(48^2) = 768$

$a_5 = \dfrac{1}{3}a_4^{\,2} = \dfrac{1}{3}(768)^2 = 196{,}608$

18. Because the sequence tends to 8 as n tends to infinity, it matches (a).

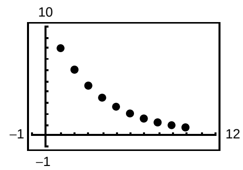
20. This sequence increases for a few terms, then decreases $a_2 = \frac{16}{2} = 8$. Matches (b).

22.



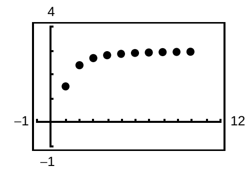
$$a_n = 2 - \frac{4}{n},\ n = 1, . . . , 10$$

24.



$$a_n = 8(0.75)^{n-1},\ n = 1, 2, . . . , 10$$

26.



$$a_n = \frac{3n^2}{n^2+1},\ n = 1, . . . , 10$$

28. $a_n = \dfrac{n+6}{2}$

$$a_5 = \frac{5+6}{2} = \frac{11}{2}$$

$$a_6 = \frac{6+6}{2} = 6$$

30. $a_{n+1} = 2a_n,\ a_1 = 5$

$$a_5 = 2(40) = 80$$

$$a_6 = 2(80) = 160$$

32. $\dfrac{25!}{23!} = \dfrac{23!(24)(25)}{23!}$

$$= (24)(25) = 600$$

34. $\dfrac{(n+2)!}{n!} = \dfrac{n!(n+1)(n+2)}{n!}$

$$= (n+1)(n+2)$$

36. $\dfrac{(2n+2)!}{(2n)!} = \dfrac{(2n)!(2n+1)(2n+2)}{(2n)!}$

$$= (2n+1)(2n+2)$$

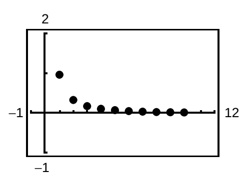
38. $\displaystyle\lim_{n\to\infty}\left(5 - \frac{1}{n^2}\right) = 5 - 0 = 5$

40. $\displaystyle\lim_{n\to\infty}\frac{5n}{\sqrt{n^2+4}} = \lim_{n\to\infty}\frac{5}{\sqrt{1+(4/n^2)}}$

$$= \frac{5}{1} = 5$$

42. $\displaystyle\lim_{n\to\infty}\cos\left(\frac{2}{n}\right) = 1$

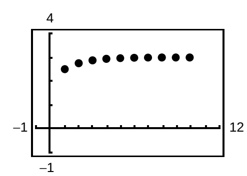
44.



The graph seems to indicate that the sequence converges to 0. Analytically,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n^{3/2}} = \lim_{x\to\infty}\frac{1}{x^{3/2}} = 0.$$

46.



The graph seems to indicate that the sequence converges to 3. Analytically,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(3 - \frac{1}{2^n}\right) = 3 - 0 = 3.$$

48. $\displaystyle\lim_{n\to\infty}\left[1 + (-1)^n\right]$

does not exist, (alternates between 0 and 2), diverges.

50. $\displaystyle\lim_{n\to\infty}\frac{\sqrt[3]{n}}{\sqrt[3]{n}+1} = 1$, converges

52. $\displaystyle\lim_{n\to\infty}\frac{1+(-1)^n}{n^2} = 0$, converges

54. $\displaystyle\lim_{n\to\infty}\frac{\ln\sqrt{n}}{n} = \lim_{n\to\infty}\frac{1/2\ln n}{n}$

$$= \lim_{n\to\infty}\frac{1}{2n} = 0, \text{ converges}$$

(L'Hôpital's Rule)

56. $\displaystyle\lim_{n\to\infty}(0.5)^n = 0$, converges

58. $\displaystyle\lim_{n\to\infty}\frac{(n-2)!}{n!} = \lim_{n\to\infty}\frac{1}{n(n-1)} = 0$, converges

60. $\lim_{n\to\infty}\left(\frac{n^2}{2n+1}-\frac{n^2}{2n-1}\right)=\lim_{n\to\infty}\frac{-2n^2}{4n^2-1}=-\frac{1}{2}$, converges

62. $a_n = n\sin\frac{1}{n}$

Let $f(x) = x\sin\frac{1}{x}$.

$$\lim_{x\to\infty} x\sin\frac{1}{x}=\lim_{x\to\infty}\frac{\sin(1/x)}{1/x}=\lim_{x\to\infty}\frac{(-1/x^2)\cos(1/x)}{-1/x^2}=\lim_{x\to\infty}\cos\frac{1}{x}=\cos 0=1 \text{ (L'Hôpital's Rule)}$$

or,

$$\lim_{x\to\infty}\frac{\sin(1/x)}{1/x}=\lim_{y\to 0^+}\frac{\sin(y)}{y}=1. \text{ Therefore } \lim_{n\to\infty} n\sin\frac{1}{n}=1.$$

64. $\lim_{n\to\infty} 2^{1/n} = 2^0 = 1$, converges

66. $\lim_{n\to\infty}\frac{\cos \pi n}{n^2}=0$, converges

68. $a_n = 4n-1$

70. $a_n = \frac{(-1)^{n-1}}{n^2}$

72. $a_n = \frac{n+2}{3n-1}$

74. $a_n = (-1)^n\frac{3^{n-2}}{2^{n-1}}$

76. $a_n = 1+\frac{2^n-1}{2^n}$

$= \frac{2^{n+1}-1}{2^n}$

78. $a_n = \frac{1}{n!}$

80. $a_n = \frac{x^{n-1}}{(n-1)!}$

82. Let $f(x) = \frac{3x}{x+2}$. Then $f'(x) = \frac{6}{(x+2)^2}$.

Thus, f is increasing which implies $\{a_n\}$ is increasing.

$|a_n| < 3$, bounded

84. $a_n = ne^{-n/2}$

$a_1 = 0.6065$

$a_2 = 0.7358$

$a_3 = 0.6694$

Not monotonic; $|a_n| \le 0.7358$, bounded

86. $a_n = \left(-\frac{2}{3}\right)^n$

$a_1 = -\frac{2}{3}$

$a_2 = \frac{4}{9}$

$a_3 = -\frac{8}{27}$

Not monotonic; $|a_n| \le \frac{2}{3}$, bounded

88. $a_n = \left(\frac{3}{2}\right)^n < \left(\frac{3}{2}\right)^{n+1} = a_{n+1}$

Monotonic; $\lim_{n\to\infty} a_n = \infty$, not bounded

90. $a_n = \frac{\cos n}{n}$

$a_1 = 0.5403$

$a_2 = -0.2081$

$a_3 = -0.3230$

$a_4 = -0.1634$

Not monotonic; $|a_n| \le 1$, bounded

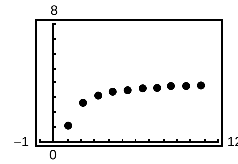
92. (a) $a_n = 4 - \dfrac{3}{n}$

$\left|4 - \dfrac{3}{n}\right| < 4 \implies$ bounded

$a_n = 4 - \dfrac{3}{n} < 3 - \dfrac{4}{n+1} = a_{n+1} \implies$ monotonic

Therefore, $\{a_n\}$ converges.

(b)



$$\lim_{n\to\infty}\left(4 - \frac{3}{n}\right) = 4$$

94. (a) $a_n = 4 + \dfrac{1}{2^n}$

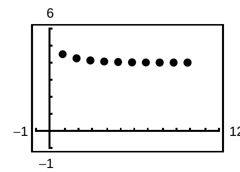
$\left|4 + \dfrac{1}{2^n}\right| \le 4.5 \implies \{a_n\}$ bounded

$a_n = 4 + \dfrac{1}{2^n} > 4 + \dfrac{1}{2^{n+1}}$

$= a_{n+1} \implies \{a_n\}$ monotonic

Therefore, $\{a_n\}$ converges.

(b)



$$\lim_{n\to\infty}\left(4 + \frac{1}{2^n}\right) = 4$$

96. $A_n = 100(101)[(1.01)^n - 1]$

(a) $A_1 = \$101.00$

$A_2 = \$203.01$

$A_3 = \$306.04$

$A_4 = \$410.10$

$A_5 = \$515.20$

$A_6 = \$621.35$

(b) $A_{60} = \$8248.64$

(c) $A_{240} = \$99{,}914.79$

98. The first sequence because every other point is below the x-axis.

100. Impossible. The sequence converges by Theorem 8.5.

102. Impossible. An unbounded sequence diverges.

104. $P_n = 16{,}000(1.045)^n$

$P_1 = \$16{,}720.00$

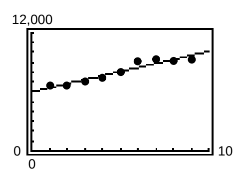
$P_2 = \$17{,}472.40$

$P_3 \approx \$18{,}258.66$

$P_4 \approx \$19{,}080.30$

$P_5 \approx \$19{,}938.91$

106. (a) $a_n = 410.9212n + 6003.8545$



(b) For 2004, $n = 14$ and $a_n = 11{,}757$, or $11,757,000,000.

108. $a_n = \left(1 + \dfrac{1}{n}\right)^n$

$a_1 = 2.0000$

$a_2 = 2.2500$

$a_3 \approx 2.3704$

$a_4 \approx 2.4414$

$a_5 \approx 2.4883$

$a_6 \approx 2.5216$

$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e$$

110. Since

$$\lim_{n\to\infty} s_n = L > 0,$$

there exists for each $\epsilon > 0$, an integer N such that $|s_n - L| < \epsilon$ for every $n > N$. Let $\epsilon = L > 0$ and we have,

$$|s_n - L| < L,\ -L < s_n - L < L, \text{ or } 0 < s_n < 2L$$

for each $n > N$.

112. If $\{a_n\}$ is bounded, monotonic and nonincreasing, then $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots$. Then

$$-a_1 \leq -a_2 \leq -a_3 \leq \cdots \leq -a_n \leq \cdots$$

is a bounded, monotonic, nondecreasing sequence which converges by the first half of the theorem. Since $\{-a_n\}$ converges, then so does $\{a_n\}$.

114. True

116. True

118. $x_0 = 1, x_n = \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}}, n = 1, 2, \ldots$

$x_1 = 1.5$ $\quad x_6 = 1.414214$

$x_2 = 1.41667$ $\quad x_7 = 1.414214$

$x_3 = 1.414216$ $\quad x_8 = 1.414114$

$x_4 = 1.414214$ $\quad x_9 = 1.414214$

$x_5 = 1.414214$ $\quad x_{10} = 1.414214$

The limit of the sequence appears to be $\sqrt{2}$. In fact, this sequence is Newton's Method applied to $f(x) = x^2 - 2$.

Section 8.2 Series and Convergence

2. $S_1 = \frac{1}{6} \approx 0.1667$

$S_2 = \frac{1}{6} + \frac{1}{6} \approx 0.3333$

$S_3 = \frac{1}{6} + \frac{1}{6} + \frac{3}{20} \approx 0.4833$

$S_4 = \frac{1}{6} + \frac{1}{6} + \frac{3}{20} + \frac{2}{15} \approx 0.6167$

$S_5 = \frac{1}{6} + \frac{1}{6} + \frac{3}{20} + \frac{2}{15} + \frac{5}{42} \approx 0.7357$

4. $S_1 = 1$

$S_2 = 1 + \frac{1}{3} \approx 1.3333$

$S_3 = 1 + \frac{1}{3} + \frac{1}{5} \approx 1.5333$

$S_4 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{9} \approx 1.6444$

$S_5 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \frac{1}{11} \approx 1.7354$

6. $S_1 = 1$

$S_2 = 1 - \frac{1}{2} = 0.5$

$S_3 = 1 - \frac{1}{2} + \frac{1}{6} \approx 0.6667$

$S_4 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} \approx 0.6250$

$S_5 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} \approx 0.6333$

8. $\sum_{n=0}^{\infty} \left(\frac{4}{3}\right)^n$ Geometric series

$r = \frac{4}{3} > 1$

Diverges by Theorem 8.6

10. $\sum_{n=0}^{\infty} 2(-1.03)^n$ Geometric series

$|r| = 1.03 > 1$

Diverges by Theorem 8.6

12. $\sum_{n=1}^{\infty} \frac{n}{2n+3}$

$\lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2} \neq 0$

Diverges by Theorem 8.9

14. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+(1/n^2)}} = 1 \neq 0$

Diverges by Theorem 8.9

16. $\sum_{n=1}^{\infty} \frac{n!}{2^n}$

$\lim_{n\to\infty} \frac{n!}{2^n} = \infty$

Diverges by Theorem 8.9

18. $\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = 1 + \frac{2}{3} + \frac{4}{9} + \cdots$

$S_0 = 1, S_1 = \frac{5}{3}, S_2 \approx 2.11, \ldots$

Matches graph (b).

Analytically, the series is geometric:

$$\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = \frac{1}{1-2/3} = \frac{1}{1/3} = 3$$

20. $\sum_{n=0}^{\infty}\frac{17}{3}\left(-\frac{8}{9}\right)^n = \frac{17}{3}\left[1 - \frac{8}{9} + \frac{64}{81} - \cdots\right]$

$S_0 = \frac{17}{3}, S_1 \approx 0.63, S_3 \approx 5.1, \ldots$

Matches (d).

Analytically, the series is geometric:

$$\sum_{n=0}^{\infty}\frac{17}{3}\left(-\frac{8}{9}\right)^n = \frac{17/3}{1-(-8/9)} = \frac{17/3}{17/9} = 3$$

22. $\sum_{n=1}^{\infty}\frac{1}{n(n+2)} = \sum_{n=1}^{\infty}\left(\frac{1}{2n} - \frac{1}{2(n+2)}\right) = \left(\frac{1}{2}-\frac{1}{6}\right) + \left(\frac{1}{4}-\frac{1}{8}\right) + \left(\frac{1}{6}-\frac{1}{10}\right) + \left(\frac{1}{8}-\frac{1}{12}\right) + \left(\frac{1}{10}-\frac{1}{14}\right) + \cdots$

$$\sum_{n=1}^{\infty}\frac{1}{n(n+2)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}\right] = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

24. $\sum_{n=0}^{\infty} 2\left(-\frac{1}{2}\right)^n$

Geometric series with $|r| = \left|-\frac{1}{2}\right| < 1$.

Converges by Theorem 8.6

26. $\sum_{n=0}^{\infty}(-0.6)^n$

Geometric series with $|r| = |-0.6| < 1$.

Converges by Theorem 8.6

28. (a) $\sum_{n=1}^{\infty}\frac{4}{n(n+4)} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+4}\right)$

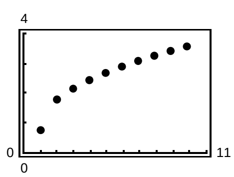
$= \left(1 - \frac{1}{5}\right) + \left(\frac{1}{2} - \frac{1}{6}\right) + \left(\frac{1}{3} - \frac{1}{7}\right) + \left(\frac{1}{4} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{9}\right) + \left(\frac{1}{6} - \frac{1}{10}\right) + \cdots$

$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} \approx 2.0833$

(b)

n	5	10	20	50	100
S_n	1.5377	1.7607	1.9051	2.0071	2.0443

(c)



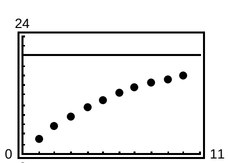
(d) The terms of the series decrease in magnitude slowly. Thus, the sequence of partial sums approaches the sum slowly.

30. (a) $\sum_{n=1}^{\infty} 3(0.85)^{n-1} = \frac{3}{1-0.85} = 20$ (Geometric series)

(b)

n	5	10	20	50	100
S_n	11.1259	16.0625	19.2248	19.9941	19.999998

(c)

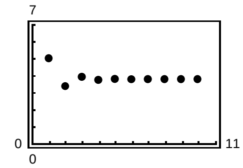


32. (a) $\sum_{n=1}^{\infty} 5\left(-\frac{1}{3}\right)^{n-1} = \frac{5}{1-(-1/3)} = \frac{15}{4} = 3.75$

(b)

n	5	10	20	50	100
S_n	3.7654	3.7499	3.7500	3.7500	3.7500

(c)



(d) The terms of the series decrease in magnitude rapidly. Thus, the sequence of partial sums approaches the sum rapidly.

34. $\sum_{n=1}^{\infty} \frac{4}{n(n+2)} = 2\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right) = 2\left[\left(1-\frac{1}{3}\right) + \left(\frac{1}{2}-\frac{1}{4}\right) + \left(\frac{1}{3}-\frac{1}{5}\right) + \cdots\right] = 2\left(1+\frac{1}{2}\right) = 3$

36. $\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n+3)} = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{2n+1} - \frac{1}{2n+3}\right) = \frac{1}{2}\left[\left(\frac{1}{3}-\frac{1}{5}\right) + \left(\frac{1}{5}-\frac{1}{7}\right) + \left(\frac{1}{7}-\frac{1}{9}\right) + \ldots\right] = \frac{1}{2}\left(\frac{1}{3}\right) = \frac{1}{6}$

38. $\sum_{n=0}^{\infty} 6\left(\frac{4}{5}\right)^n = \frac{6}{1-(4/5)} = 30$ (Geometric)

40. $\sum_{n=0}^{\infty} 2\left(-\frac{2}{3}\right)^n = \frac{2}{1-(-2/3)} = \frac{6}{5}$

42. $\sum_{n=0}^{\infty} 8\left(\frac{3}{4}\right)^n = \frac{8}{1-(3/4)} = 32$

44. $\sum_{n=0}^{\infty} 4\left(-\frac{1}{2}\right)^n = \frac{4}{1-(-1/2)} = \frac{8}{3}$

46. $\sum_{n=1}^{\infty} [(0.7)^n + (0.9)^n] = \sum_{n=0}^{\infty}\left(\frac{7}{10}\right)^n + \sum_{n=0}^{\infty}\left(\frac{9}{10}\right)^n - 2 = \frac{1}{1-(7/10)} + \frac{1}{1-(9/10)} - 2 = \frac{10}{3} + 10 - 2 = \frac{34}{3}$

48. $0.81\overline{81} = \sum_{n=0}^{\infty} \frac{81}{100}\left(\frac{1}{100}\right)^n$

Geometric series with $a = \frac{81}{100}$ and $r = \frac{1}{100}$

$S = \frac{a}{1-r} = \frac{81/100}{1-(1/100)} = \frac{81}{99} = \frac{9}{11}$

50. $0.215\overline{15} = \frac{1}{5} + \sum_{n=0}^{\infty} \frac{3}{200}\left(\frac{1}{100}\right)^n$

Geometric series with $a = \frac{3}{200}$ and $r = \frac{1}{100}$

$S = \frac{1}{5} + \frac{a}{1-r} = \frac{1}{5} + \frac{3/200}{99/100} = \frac{71}{330}$

52. $\sum_{n=1}^{\infty} \frac{n+1}{2n-1}$

$\lim_{n\to\infty} \frac{n+1}{2n-1} = \frac{1}{2} \neq 0$

Diverges by Theorem 8.9

54. $\sum_{n=1}^{\infty} \frac{1}{n(n+3)} = \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+3}\right)$

$= \frac{1}{3}\left[\left(1-\frac{1}{4}\right) + \left(\frac{1}{2}-\frac{1}{5}\right) + \left(\frac{1}{3}-\frac{1}{6}\right) + \left(\frac{1}{4}-\frac{1}{7}\right) + \left(\frac{1}{5}-\frac{1}{8}\right) + \left(\frac{1}{6}-\frac{1}{9}\right) + \cdots\right]$

$= \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) = \frac{11}{18}$, converges

56. $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

$\lim_{n\to\infty} \frac{3^n}{n^3} = \lim_{n\to\infty} \frac{(\ln 2)3^n}{3n^2} = \lim_{n\to\infty} \frac{(\ln 2)^2 3^n}{6\text{n}} = \lim_{n\to\infty} \frac{(\ln n)^3 3^n}{6} = \infty$

(by L'Hôpital's Rule) Diverges by Theorem 8.9

58. $\sum_{n=0}^{\infty} \frac{1}{4^n}$

Geometric series with $r = \frac{1}{4}$

Converges by Theorem 8.6

60. $\sum_{n=1}^{\infty} \frac{2^n}{100}$

Geometric series with $r = 2$

Diverges by Theorem 8.6

62. $\sum_{n=1}^{\infty} \left(1 + \frac{k}{n}\right)^n$

$\lim_{n\to\infty} \left(1 + \frac{k}{n}\right)^n = e^k \neq 0$

Diverges by Theorem 8.9

64. $\lim_{n\to\infty} a_n = 5$ means that the limit of the sequence $\{a_n\}$ is 5.

$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots = 5$ means that the limit of the partial sums is 5.

66. If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

68. (a) $(-x/2)$ is the common ratio.

(b) $1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} = \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n = \frac{1}{1 - (-x/2)}$

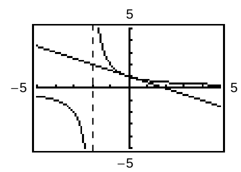
$= \frac{2}{2 + x}, \; |x| < 2$

Geometric series:

$a = 1, r = -\frac{x}{2}, \left|-\frac{x}{2}\right| < 1 \Rightarrow |x| < 2$

(c) $y_1 = \frac{2}{2 + x}$

$y_2 = 1 - \frac{x}{2}$



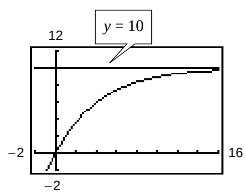
70. $f(x) = 2\left[\frac{1 - 0.8^x}{1 - 0.8}\right]$

Horizontal asymptote: $y = 10$

$\sum_{n=0}^{\infty} 2\left(\frac{4}{5}\right)^n$

$S = \frac{2}{1 - (4/5)} = 10$

The horizontal asymptote is the sum of the series. $f(n)$ is the n^{th} partial sum.



72. $\frac{1}{2^n} < 0.0001$

$10{,}000 < 2^n$

This inequality is true when $n = 14$.

$(0.01)^n < 0.0001$

$10{,}000 < 10^n$

This inequality is true when $n = 5$. This series converges at a faster rate.

74. $V(t) = 225{,}000(1 - 0.3)^n = (0.7)^n(225{,}000)$

$V(5) = (0.7)^5(225{,}000) = \$37{,}815.75$

76. $\sum_{i=0}^{n-1} 100(0.60)^i = \frac{100[1 - 0.6^n]}{1 - 0.6}$

$= 250(1 - 0.6^n)$ million dollars

Sum $= 250$ million dollars

78. The ball in Exercise 77 takes the following times for each fall.

$s_1 = -16t^2 + 16$ $s_1 = 0$ if $t = 1$

$s_2 = -16t^2 + 16(0.81)$ $s_2 = 0$ if $t = 0.9$

$s_3 = -16t^2 + 16(0.81)^2$ $s_3 = 0$ if $t = (0.9)^2$

⋮   ⋮

$s_n = -16t^2 + 16(0.81)^{n-1}$ $s_n = 0$ if $t = (0.9)^{n-1}$

Beginning with s_2, the ball takes the same amount of time to bounce up as it takes to fall. The total elapsed time before the ball comes to rest is

$$t = 1 + 2\sum_{n=1}^{\infty} (0.9)^n = -1 + 2\sum_{n=0}^{\infty} (0.9)^n$$

$$= -1 + \frac{2}{1 - 0.9} = 19 \text{ seconds.}$$

80. $P(n) = \frac{1}{3}\left(\frac{2}{3}\right)^n$

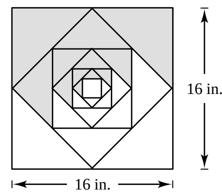
$P(2) = \frac{1}{3}\left(\frac{2}{3}\right)^2 = \frac{4}{27}$

$\sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n = \frac{1/3}{1 - (2/3)} = 1$

82. (a) $64 + 32 + 16 + 8 + 4 + 2 = 126 \text{ in.}^2$

(b) $\sum_{n=0}^{\infty} 64\left(\frac{1}{2}\right)^n = \frac{64}{1 - (1/2)} = 128 \text{ in.}^2$

Note: This is one-half of the area of the original square!



84. Surface area $= 4\pi(1)^2 + 9\left(4\pi\left(\frac{1}{3}\right)^2\right) + 9^2 \cdot 4\pi\left(\frac{1}{9}\right)^2 + \cdots = 4[\pi + \pi + \cdots] = \infty$

86. $$\sum_{n=0}^{12t-1} P\left(1 + \frac{r}{12}\right)^n = \frac{P\left[1 - \left(1 + \frac{r}{12}\right)^{12t}\right]}{1 - \left(1 + \frac{r}{12}\right)}$$

$$= P\left(-\frac{12}{r}\right)\left[\left(1 - \left(1 + \frac{r}{12}\right)^{12t}\right]\right.$$

$$= P\left(\frac{12}{r}\right)\left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right]$$

$$\sum_{n=0}^{12t-1} P(e^{r/12})^n = \frac{P(1 - (e^{r/12})^{12t})}{1 - e^{r/12}} = \frac{P(e^{rt} - 1)}{e^{r/12} - 1}$$

88. $P = 75, r = 0.05, t = 25$

(a) $A = 75\left(\frac{12}{0.05}\right)\left[\left(1 + \frac{0.05}{12}\right)^{12(25)} - 1\right] \approx \$44,663.23$

(b) $A = \frac{75(e^{0.05(25)} - 1)}{e^{0.05/12} - 1} \approx \$44,732.85$

90. $P = 20, r = 0.06, t = 50$

(a) $A = 20\left(\frac{12}{0.06}\right)\left[\left(1 + \frac{0.06}{12}\right)^{12(50)} - 1\right] \approx \$75,743.82$

(b) $A = \frac{20(e^{0.06(50)} - 1)}{e^{0.06/12} - 1} \approx \$76,151.45$

92. $T = 40,000 + 40,000(1.04) + \cdots + 40,000(1.04)^{39}$

$$= \sum_{n=0}^{39} 40,000(1.04)^n$$

$$= 40,000\left(\frac{1 - 1.04^{40}}{1 - 1.04}\right)$$

$$\approx \$3,801,020$$

94. $x = 0.a_1a_2a_3 \ldots a_k\overline{a_1a_2a_3 \ldots a_k}$

$$= 0.a_1a_2a_3 \ldots a_k\left[1 + \frac{1}{10^k} + \left(\frac{1}{10^k}\right)^2 + \left(\frac{1}{10^k}\right)^3 + \cdots\right]$$

$$= 0.a_1a_2a_3 \ldots a_k \sum_{n=0}^{\infty}\left(\frac{1}{10^k}\right)^n$$

$$= 0.a_1a_2a_3 \ldots a_k\left[\frac{1}{1 - (1/10^k)}\right] = \text{a rational number}$$

96. Let $\{S_n\}$ be the sequence of partial sums for the convergent series $\sum_{n=1}^{\infty} a_n = L$. Then

$\lim_{n\to\infty} S_n = L$ and since $R_n = \sum_{k=n+1}^{\infty} a_k = L - S_n$,

we have

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} (L - S_n) = \lim_{n\to\infty} L - \lim_{n\to\infty} S_n = L - L = 0.$$

98. If $\Sigma (a_n + b_n)$ converged, then $\Sigma (a_n + b_n) - \Sigma a_n = \Sigma b_n$ would converge, which is a contradiction.

Thus, $\Sigma (a_n + b_n)$ diverges.

100. True

102. True; $\lim_{n\to\infty} \frac{n}{1000(n+1)} = \frac{1}{1000} \neq 0$

104. $\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \cdots = \sum_{n=0}^{\infty} \frac{1}{r}\left(\frac{1}{r}\right)^n = \frac{1/r}{1-(1/r)} = \frac{1}{r-1} \quad \left(\text{since } \left|\frac{1}{r}\right| < 1\right)$

This is a geometric series which converges if $\left|\frac{1}{r}\right| < 1 \iff |r| > 1$.

Section 8.3 The Integral Test and *p*-Series

2. $\sum_{n=1}^{\infty} \frac{2}{3n+5}$

Let $f(x) = \frac{2}{3x+5}$.

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{2}{3x+5}\,dx = \left[\frac{2}{3}\ln(3x+5)\right]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

4. $\sum_{n=1}^{\infty} ne^{-n/2}$

Let $f(x) = xe^{-x/2}$.

f is positive, continuous, and decreasing for $x \geq 3$.

Since $f'(x) = \frac{2-x}{2e^{x/2}} < 0$ for $x \geq 3$.

$$\int_3^{\infty} xe^{-x/2}\,dx = \left[-2(x+2)e^{-x/2}\right]_3^{\infty} = 10e^{-3/2}$$

Converges by Theorem 8.10

6. $\sum_{n=1}^{\infty} \frac{1}{2n+1}$

Let $f(x) = \frac{1}{2x+1}$.

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{2x+1}\,dx = \left[\ln\sqrt{2x+1}\right]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

8. $\sum_{n=1}^{\infty} \frac{n}{n^2+3}$

Let $f(x) = \frac{x}{x^2+3}$.

$f(x)$ is positive, continuous, and decreasing for $x \geq 2$ since

$$f'(x) = \frac{3-x^2}{(x^2+3)} < 0 \text{ for } x \geq 2.$$

$$\int_1^{\infty} \frac{x}{x^2+3}\,dx = \left[\ln\sqrt{x^2+3}\right]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

10. $\sum_{n=1}^{\infty} n^k e^{-n}$

Let $f(x) = \frac{x^k}{e^x}$.

f is positive, continuous, and decreasing for $x > k$ since

$$f'(x) = \frac{x^{k-1}(k-x)}{e^x} < 0$$

for $x > k$. We use integration by parts.

$$\int_1^{\infty} x^k e^{-x}\,dx = \left[-x^k e^{-x}\right]_1^{\infty} + k\int_1^{\infty} x^{k-1} e^{-x}\,dx$$

$$= \frac{1}{e} + \frac{k}{e} + \frac{k(k-1)}{e} + \cdots + \frac{k!}{e}$$

Converges by Theorem 8.10

12. $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$

Let $f(x) = \frac{1}{x^{1/3}}$.

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x^{1/3}}\,dx = \left[\frac{3}{2}x^{2/3}\right]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

14. $\sum_{n=1}^{\infty} \frac{3}{n^{5/3}}$

Convergent p-series with $p = \frac{5}{3} > 1$

16. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Convergent p-series with $p = 2 > 1$

18. $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$

Divergent p-series with $p = \frac{2}{3} < 1$

20. $\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$

Convergent p-series with $p = \pi > 1$

22. $\sum_{n=1}^{\infty} \frac{2}{n} = \frac{2}{1} + \frac{2}{2} + \frac{2}{3} + \cdots$

$S_1 = 2$

$S_2 = 3$

$S_3 \approx 3.67$

Matches (d)

Diverges—harmonic series

24. $\sum_{n=1}^{\infty} \frac{2}{n^2} = 2 + \frac{2}{2^2} + \frac{2}{3^2} + \cdots$

$S_1 = 2$

$S_2 = 2.5$

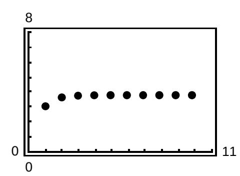
$S_3 \approx 2.722$

Matches (c)

Converges—p-series with $p = 2 > 1$.

26. (a)

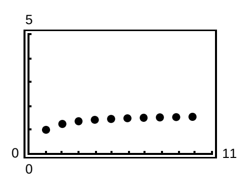
n	5	10	20	50	100
S_n	3.7488	3.75	3.75	3.75	3.75



The partial sums approach the sum 3.75 very rapidly.

(b)

n	5	10	20	50	100
S_n	1.4636	1.5498	1.5962	1.6251	1.635



The partial sums approach the sum $\frac{\pi^2}{6} \approx 1.6449$ slower than the series in part (a).

28. $\xi(x) = \sum_{n=1}^{\infty} n^{-x} = \sum_{n=1}^{\infty} \frac{1}{n^x}$

Converges for $x > 1$ by Theorem 8.11.

30. $\sum_{n=2}^{\infty} \frac{\ln n}{n^p}$

If $p = 1$, then the series diverges by the Integral Test. If $p \neq 1$,

$$\int_2^{\infty} \frac{\ln x}{x^p}\,dx = \int_2^{\infty} x^{-p} \ln x\,dx = \left[\frac{x^{-p+1}}{(-p+1)^2}\left[-1 + (-p+1)\ln x\right]\right]_2^{\infty}.$$ (Use Integration by Parts.)

Converges for $-p + 1 < 0$ or $p > 1$.

32. A series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is a p-series, $p > 0$.

The p-series converges if $p > 1$ and diverges if $0 < p \leq 1$.

34. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

36. From Exercise 35, we have:

$$0 \le S - S_N \le \int_N^{\infty} f(x)\,dx$$

$$S_N \le S \le S_N + \int_N^{\infty} f(x)\,dx$$

$$\sum_{n=1}^{N} a_n \le S \le \sum_{n=1}^{N} a_n + \int_N^{\infty} f(x)\,dx$$

38. $S_4 = 1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} \approx 1.0363$

$$R_4 \le \int_4^{\infty} \frac{1}{x^5}\,dx = \left[-\frac{1}{4x^4}\right]_4^{\infty} \approx 0.0010$$

$$1.0363 \le \sum_{n=1}^{\infty} \frac{1}{n^5} \le 1.0363 + 0.0010 = 1.0373$$

40. $S_{10} = \frac{1}{2(\ln 2)^3} + \frac{1}{3(\ln 3)^3} + \frac{1}{4(\ln 4)^3} + \cdots + \frac{1}{11(\ln 11)^3} \approx 1.9821$

$$R_{10} \le \int_{10}^{\infty} \frac{1}{(x+1)[\ln(x+1)]^3}\,dx = \left[-\frac{1}{2[\ln(x+1)]^2}\right]_{10}^{\infty} = \frac{1}{2(\ln 11)^3} \approx 0.0870$$

$$1.9821 \le \sum_{n=1}^{\infty} \frac{1}{(n+1)[\ln(n+1)]^3} \le 1.9821 + 0.0870 = 2.0691$$

42. $S_4 = \frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \frac{1}{e^4} \approx 0.5713$

$$R_4 \le \int_4^{\infty} e^{-x}\,dx = \left[-e^{-x}\right]_4^{\infty} \approx 0.0183$$

$$0.5713 \le \sum_{n=0}^{\infty} e^{-n} \le 0.5713 + 0.0183 = 0.5896$$

44. $0 \le R_N \le \int_N^{\infty} \frac{1}{x^{3/2}}\,dx = \left[-\frac{2}{x^{1/2}}\right]_N^{\infty} = \frac{2}{\sqrt{N}} < 0.001$

$$N^{-1/2} < 0.0005$$

$$\sqrt{N} > 2000$$

$$N \ge 4{,}000{,}000$$

46. $R_N \le \int_N^{\infty} e^{-x/2}\,dx = \left[-2e^{-x/2}\right]_N^{\infty} = \frac{2}{e^{N/2}} < 0.001$

$$\frac{2}{e^{N/2}} < 0.001$$

$$e^{N/2} > 2000$$

$$\frac{N}{2} > \ln 2000$$

$$N > 2\ln 2000 \approx 15.2$$

$$N \ge 16$$

48. $R_n \le \int_N^{\infty} \frac{2}{x^2+5}\,dx = 2\left[\frac{1}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right)\right]_N^{\infty}$

$$= \frac{2}{\sqrt{5}}\left(\frac{\pi}{2} - \arctan\left(\frac{N}{\sqrt{5}}\right)\right) < 0.001$$

$$\frac{\pi}{2} - \arctan\left(\frac{N}{\sqrt{5}}\right) < 0.001118$$

$$1.56968 < \arctan\left(\frac{N}{\sqrt{5}}\right)$$

$$\frac{N}{\sqrt{5}} > \tan 1.56968$$

$$N \ge 2004$$

50. (a) $\int_{10}^{\infty} \frac{1}{x^p}\,dx = \left[\frac{x^{-p+1}}{-p+1}\right]_{10}^{\infty} = \frac{1}{(p-1)10^{p-1}}, p > 1$

(b) $f(x) = \frac{1}{x^p}$

$$R_{10}(p) = \sum_{n=11}^{\infty} \frac{1}{n^p} \le \text{Area under the graph of } f \text{ over the interval } [10, \infty)$$

(c) The horizontal asymptote is $y = 0$. As n increases, the error decreases.

52. $\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right) = \sum_{n=2}^{\infty} \ln\left(\frac{n^2-1}{n^2}\right) = \sum_{n=2}^{\infty} \ln\frac{(n+1)(n-1)}{n^2} = \sum_{n=2}^{\infty} [\ln(n+1) + \ln(n-1) - 2\ln n]$

$$= (\ln 3 + \ln 1 - 2\ln 2) + (\ln 4 + \ln 2 - 2\ln 3) + (\ln 5 + \ln 3 - 2\ln 4) + (\ln 6 + \ln 4 - 2\ln 5) + (\ln 7 + \ln 5 - 2\ln 6) + (\ln 8 + \ln 6 - 2\ln 7) + (\ln 9 + \ln 7 - 2\ln 8) + \cdots = -\ln 2$$

54. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

Let $f(x) = \frac{1}{x\sqrt{x^2-1}}$.

f is positive, continuous, and decreasing for $x \geq 2$.

$$\int_2^{\infty} \frac{1}{x\sqrt{x^2-1}}\,dx = \Big[\operatorname{arcsec} x\Big]_2^{\infty} = \frac{\pi}{2} - \frac{\pi}{3}$$

Converges by Theorem 8.10

56. $3\sum_{n=1}^{\infty} \frac{1}{n^{0.95}}$

p-series with $p = 0.95$

Diverges by Theorem 8.11

58. $\sum_{n=0}^{\infty} (1.075)^n$

Geometric series with $r = 1.075$

Diverges by Theorem 8.6

60. $\sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n^3}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n^3}$

Since these are both convergent p-series, the difference is convergent.

62. $\sum_{n=2}^{\infty} \ln(n)$

$\lim_{n\to\infty} \ln(n) = \infty$

Diverges by Theorem 8.9

64. $\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$

Let $f(x) = \frac{\ln x}{x^3}$.

f is positive, continuous, and decreasing for $x \geq 2$ since $f'(x) = \frac{1 - 3\ln x}{x^4} < 0$ for $x \geq 2$.

$$\int_2^{\infty} \frac{\ln x}{x^3}\,dx = \left[-\frac{\ln x}{2x^2}\right]_2^{\infty} + \frac{1}{2}\int_2^{\infty} \frac{1}{x^3}\,dx = \frac{\ln 2}{8} + \left[-\frac{1}{4x^2}\right]_2^{\infty} = \frac{\ln 2}{8} + \frac{1}{16}$$ (Use Integration by Parts.)

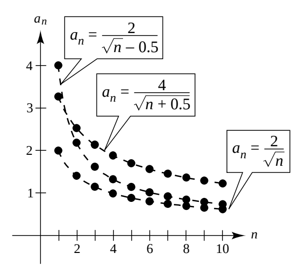
Converges by Theorem 8.10. See Exercise 14.

Section 8.4 Comparisons of Series

2. (a) $\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}} = 2 + \frac{2}{\sqrt{2}} + \cdots \quad S_1 = 2$

$\sum_{n=1}^{\infty} \frac{2}{\sqrt{n} - 0.5} = \frac{2}{0.5} + \frac{2}{\sqrt{2} - 0.5} + \cdots \quad S_1 = 4$

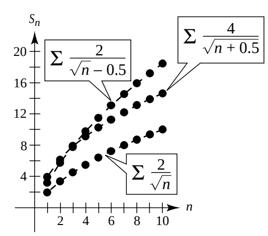
$\sum_{n=1}^{\infty} \frac{4}{\sqrt{n + 0.5}} = \frac{4}{\sqrt{1.5}} + \frac{4}{\sqrt{2.5}} + \cdots \; S_1 \approx 3.3$



(b) The first series is a p-series. It diverges $\left(p = \frac{1}{2} < 1\right)$.

(c) The magnitude of the terms of the other two series are greater than the corresponding terms of the divergent p-series. Hence, the other two series diverge.

(d) The larger the magnitude of the terms, the larger the magnitude of the terms of the sequence of partial sums.



4. $\dfrac{1}{3n^2+2} < \dfrac{1}{3n^2}$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$$

converges by comparison with the convergent p-series

$$\frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

6. $\dfrac{1}{\sqrt{n}-1} > \dfrac{1}{\sqrt{n}}$ for $n \geq 2$.

Therefore,

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$$

diverges by comparison with the divergent p-series

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}.$$

8. $\dfrac{3^n}{4^n+5} < \left(\dfrac{3}{4}\right)^n$

Therefore,

$$\sum_{n=0}^{\infty} \frac{3^n}{4^n+5}$$

converges by comparison with the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n.$$

10. $\dfrac{1}{\sqrt{n^3+1}} < \dfrac{1}{n^{3/2}}$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$$

converges by comparison with the convergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$

12. $\dfrac{1}{4\sqrt[3]{n}-1} > \dfrac{1}{4\sqrt[4]{n}}$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n}-1}$$

diverges by comparison with the divergent p-series

$$\frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}}.$$

14. $\dfrac{4^n}{3^n-1} > \dfrac{4^n}{3^n}$

Therefore,

$$\sum_{n=1}^{\infty} \frac{4^n}{3^n-1}$$

diverges by comparison with the divergent geometric series

$$\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n.$$

16. $\displaystyle\lim_{n\to\infty} \frac{2/(3^n-5)}{1/3^n} = \lim_{n\to\infty} \frac{2\cdot 3^n}{3^n-5} = 2$

Therefore,

$$\sum_{n=1}^{\infty} \frac{2}{3^n-5}$$

converges by a limit comparison with the convergent geometric series

$$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n.$$

18. $\displaystyle\lim_{n\to\infty} \frac{3/\sqrt{n^2-4}}{1/n} = \lim_{n\to\infty} \frac{3n}{\sqrt{n^2-4}} = 3$

Therefore,

$$\sum_{n=3}^{\infty} \frac{3}{\sqrt{n^2-4}}$$

diverges by a limit comparison with the divergent harmonic series

$$\sum_{n=3}^{\infty} \frac{1}{n}.$$

20. $\displaystyle\lim_{n\to\infty} \frac{\dfrac{5n-3}{n^2-2n+5}}{1/n} = \lim_{n\to\infty} \frac{5n^2-3n}{n^2-2n+5} = 5$

Therefore,

$$\sum_{n=1}^{\infty} \frac{5n-3}{n^2-2n+5}$$

diverges by a limit comparison with the divergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

22. $\displaystyle\lim_{n\to\infty} \frac{\dfrac{1}{n(n^2+1)}}{1/n^3} = \lim_{n\to\infty} \frac{n^3}{n^3+n} = 1$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n(n^2+1)}$$

converges by a limit comparison with the convergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}.$$

24. $\lim_{n\to\infty} \dfrac{n/[(n+1)2^{n-1}]}{1/(2^{n-1})} = \lim_{n\to\infty} \dfrac{n}{n+1} = 1$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)2^{n-1}}$$

converges by a limit comparison with the convergent geometric series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}.$$

26. $\lim_{n\to\infty} \dfrac{5/\left(n+\sqrt{n^2+4}\right)}{1/n} = \lim_{n\to\infty} \dfrac{5n}{n+\sqrt{n^2+4}} = \dfrac{5}{2}$

Therefore,

$$\sum_{n=1}^{\infty} \frac{5}{n+\sqrt{n^2+4}}$$

diverges by a limit comparison with the divergent harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

28. $\lim_{n\to\infty} \dfrac{\tan(1/n)}{1/n} = \lim_{n\to\infty} \dfrac{(-1/n^2)\sec^2(1/n)}{-1/n^2} = \lim_{n\to\infty} \sec^2\left(\dfrac{1}{n}\right) = 1$

Therefore,

$$\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$$

diverges by a limit comparison with the divergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

30. $\displaystyle\sum_{n=0}^{\infty} 5\left(-\frac{1}{5}\right)^n$

Converges

Geometric series with $r = -\frac{1}{5}$

32. $\displaystyle\sum_{n=4}^{\infty} \frac{1}{3n^2-2n-15}$

Converges

Limit comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

34. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \left(\frac{1}{4}-\frac{1}{5}\right) + \cdots = \frac{1}{2}$

Converges; telescoping series

36. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$

Converges; telescoping series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+3}\right)$$

38. If $j < k-1$, then $k-j > 1$. The p-series with $p = k-j$ converges and since

$$\lim_{n\to\infty} \frac{P(n)/Q(n)}{1/n^{k-j}} = L > 0,$$

the series $\displaystyle\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)}$ converges by the limit comparison test. Similarly, if $j \ge k-1$, then $k-j \le 1$ which implies that

$$\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)}$$

diverges by the limit comparison test.

40. $\dfrac{1}{3} + \dfrac{1}{8} + \dfrac{1}{15} + \dfrac{1}{24} + \dfrac{1}{35} + \cdots = \displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1},$

which converges since the degree of the numerator is two less than the degree of the denominator.

42. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$

diverges since the degree of the numerator is only one less than the degree of the denominator.

44. $\lim_{n\to\infty} \frac{n}{\ln n} = \lim_{n\to\infty} \frac{1}{1/n} = \lim_{n\to\infty} n = \infty \neq 0$

Therefore,

$$\sum_{n=2}^{\infty} \frac{1}{\ln n} \text{ diverges.}$$

46. See Theorem 8.13, page 585.
One example is

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}} \text{ diverges because } \lim_{n\to\infty} \frac{1/\sqrt{n-1}}{1/\sqrt{n}} = 1$$

and

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \text{ diverges } (p\text{-series}).$$

48. This is not correct. The beginning terms do not affect the convergence or divergence of a series.

In fact,

$$\frac{1}{1000} + \frac{1}{1001} + \cdots = \sum_{n=1000}^{\infty} \frac{1}{n} \text{ diverges (harmonic)}$$

and

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges } (p\text{-series}).$$

50. $\frac{1}{200} + \frac{1}{210} + \frac{1}{220} + \cdots = \sum_{n=0}^{\infty} \frac{1}{200 + 10n}$,

diverges

52. $\frac{1}{201} + \frac{1}{208} + \frac{1}{227} + \frac{1}{264} + \cdots = \sum_{n=1}^{\infty} \frac{1}{200 + n^3}$,

converges

54. (a) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \sum_{n=1}^{\infty} \frac{1}{4n^2 - 4n + 1}$

converges since the degree of the numerator is two less than the degree of the denominator. (See Exercise 38.)

(b)

n	5	10	20	50	100
S_n	1.1839	1.02087	1.2212	1.2287	1.2312

(c) $\sum_{n=3}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} - S_2 \approx 0.1226$

(d) $\sum_{n=10}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} - S_9 \approx 0.0277$

56. True

58. False. Let $a_n = 1/n$, $b_n = 1/n$, $c_n = 1/n^2$. Then, $a_n \leq b_n + c_n$, but $\sum_{n=1}^{\infty} c_n$ converges.

60. Since $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n a_n = \sum_{n=1}^{\infty} a_n^2$ converges by Exercise 59.

62. $\sum \frac{1}{n^2}$ converge, and hence so does $\sum \left(\frac{1}{n^2}\right)^2 = \sum \frac{1}{n^4}$.

64. (a) $\sum a_n = \sum \frac{1}{n^3}$ and $\sum b_n = \sum \frac{1}{n^2}$. Since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/n^3}{1/n^2} = \lim_{n\to\infty} \frac{1}{n} = 0 \quad \text{and} \quad \sum \frac{1}{n^2}$$

converges, so does $\sum \frac{1}{n^3}$.

(b) $\sum a_n = \sum \frac{1}{\sqrt{n}}$ and $\sum b_n = \sum \frac{1}{n}$. Since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/\sqrt{n}}{1/n} = \lim_{n\to\infty} \sqrt{n} = \infty \quad \text{and} \quad \sum \frac{1}{n}$$

diverges, so does $\sum \frac{1}{\sqrt{n}}$.

Section 8.5 Alternating Series

2. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}6}{n^2} = \frac{6}{1} - \frac{6}{4} + \frac{6}{9} - \cdots$

$S_1 = 6, S_2 = 4.5$

Matches (d)

4. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}10}{n2^n} = \frac{10}{2} - \frac{10}{8} + \cdots$

$S_1 = 5, S_2 = 3.75$

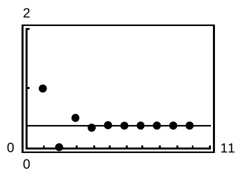
Matches (a)

6. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} = \frac{1}{e} \approx 0.3679$

(a)

n	1	2	3	4	5	6	7	8	9	10
S_n	1	0	0.5	0.3333	0.375	0.3667	0.3681	0.3679	0.3679	0.3679

(b)



(c) The points alternate sides of the horizontal line that represents the sum of the series. The distance between successive points and the line decreases.

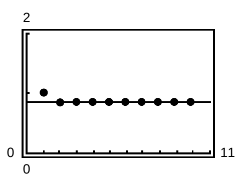
(d) The distance in part (c) is always less than the magnitude of the next series.

8. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} = \sin(1) \approx 0.8415$

(a)

n	1	2	3	4	5	6	7	8	9	10
S_n	1	0.8333	0.8417	0.8415	0.8415	0.8415	0.8415	0.8415	0.8415	0.8415

(b)



(c) The points alternate sides of the horizontal line that represents the sum of the series. The distance between successive points and the line decreases.

(d) The distance in part (c) is always less than the magnitude of the next series.

10. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{2n-1}$

$\lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2}$

Diverges by the nth Term Test.

12. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$

$a_{n+1} = \frac{1}{\ln(n+2)} < \frac{1}{\ln(n+1)} = a_n$

$\lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0$

Converges by Theorem 8.14

14. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{n^2+1}$

$a_{n+1} = \frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1} = a_n$

$\lim_{n\to\infty} \frac{n}{n^2+1} = 0$

Converges by Theorem 8.14

16. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n^2}{n^2+5}$

$\lim_{n\to\infty} \frac{n^2}{n^2+5} = 1$

Diverges by nth Term Test

18. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\ln(n+1)}{n+1}$

$a_{n+1} = \frac{\ln[(n+1)+1]}{(n+1)+1} < \frac{\ln(n+1)}{n+1}$ for $n \geq 2$

$\lim_{n\to\infty} \frac{\ln(n+1)}{n+1} = \lim_{n\to\infty} \frac{1/(n+1)}{1} = 0$

Converges by Theorem 8.14

20. $\sum_{n=1}^{\infty} \frac{1}{n}\sin\left[\frac{(2n-1)\pi}{2}\right] = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

Converges; (see Exercise 9)

22. $\sum_{n=1}^{\infty} \frac{1}{n}\cos n\pi = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

Converges; (see Exercise 9)

24. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$

$a_{n+1} = \frac{1}{(2n+3)!} < \frac{1}{(2n+1)!} = a_n$

$\lim_{n\to\infty} \frac{1}{(2n+1)!} = 0$

Converges by Theorem 8.14

26. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{\sqrt[3]{n}}$

$\lim_{n\to\infty} \frac{n^{1/2}}{n^{1/3}} = \lim_{n\to\infty} n^{1/6} = \infty$

Diverges by the nth Term Test

28. $\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{e^n + e^{-n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2e^n)}{e^{2n}+1}$

Let $f(x) = \frac{2e^x}{e^{2x}+1}$. Then

$$f'(x) = \frac{2e^{2x}(1-e^{2x})}{(e^{2x}+1)^2} < 0 \text{ for } x > 0.$$

Thus, $f(x)$ is decreasing for $x > 0$ which implies $a_{n+1} < a_n$.

$$\lim_{n\to\infty} \frac{2e^n}{e^{2n}+1} = \lim_{n\to\infty} \frac{2e^n}{2e^{2n}} = \lim_{n\to\infty} \frac{1}{e^n} = 0$$

The series converges by Theorem 8.14.

30. $S_6 = \sum_{n=1}^{6} \frac{4(-1)^{n+1}}{\ln(n+1)} \approx 2.7067$

$|R_6| = |S - S_6| \leq a_7 = \frac{4}{\ln 8} \approx 1.9236;\ 0.7831 \leq S \leq 4.6303$

32. $S_6 = \sum_{n=1}^{6} \frac{(-1)^{n+1}n}{2^n} = 0.1875$

$|R_6| = |S - S_6| \leq a_7 = \frac{7}{2^7} \approx 0.05469;\ 0.1328 \leq S \leq 0.2422$

34. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}$

(a) By Theorem 8.15,

$$|R_n| \leq a_{N+1} = \frac{1}{2^{N+1}(N+1)!} < 0.001.$$

This inequality is valid when $N = 4$.

(b) We may approximate the series by

$$\sum_{n=0}^{4} \frac{(-1)^n}{2^n n!} = 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{48} + \frac{1}{348} \approx 0.607.$$

(5 terms. Note that the sum begins with $n = 0$.)

36. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$

(a) By Theorem 8.15,

$$|R_N| \leq a_{N+1} = \frac{1}{(2N+2)!} < 0.001.$$

This inequality is valid when $N = 3$.

(b) We may approximate the series by

$$\sum_{n=0}^{3} \frac{(-1)^n}{(2n)!} = 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} \approx 0.540.$$

(4 terms. Note that the sum begins with $n = 0$.)

38. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4^n n}$

(a) By Theorem 8.15,

$$|R_N| \le a_{N+1} = \frac{1}{4^{N+1}(N+1)} < 0.001.$$

This inequality is valid when $N = 3$.

(b) We may approximate the series by

$$\sum_{n=1}^{3} \frac{(-1)^{n+1}}{4^n n} = \frac{1}{4} - \frac{1}{32} + \frac{1}{192} \approx 0.224.$$

(3 terms)

40. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$

By Theorem 8.15, $|R_N| \le a_{N+1} = \dfrac{1}{(N+1)^4} < 0.001.$

This inequality is valid when $N = 5$.

42. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$

The given series converges by the Alternating Series Test, but does not converge absolutely since the series

$$\sum_{n=1}^{\infty} \frac{1}{n+1}$$

diverges by the Integral Test. Therefore, the series converge conditionally.

44. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ which is a convergent p-series.

Therefore, the given series converges absolutely.

46. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n+3)}{n+10}$

$\displaystyle\lim_{n\to\infty} \frac{2n+3}{n+10} = 2$ Therefore, the series diverges by the

nth Term Test.

48. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{e^{n^2}}$

$$\sum_{n=0}^{\infty} \frac{1}{e^{n^2}}$$

converges by a comparison to the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n.$$

Therefore, the given series converges absolutely.

50. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{1.5}}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.5}}$ is a convergent p-series.

Therefore, the given series converge absolutely.

52. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+4}}$

The given series converges by the Alternating Series Test, but

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+4}}$$

diverges by a limit comparison to the divergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

Therefore, the given series converges conditionally.

54. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \arctan n$

$\displaystyle\lim_{n\to\infty} \arctan n = \frac{\pi}{2} \neq 0$ Therefore, the series diverges by

the nth Term Test.

56. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin[(2n-1)\pi/2]}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

The given series converges by the Alternating Series Test, but

$$\sum_{n=1}^{\infty} \left|\frac{\sin[(2n-1)\pi/2]}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

is a divergent p-series. Therefore, the series converges conditionally.

58. $|S - S_n| = |R_n| \le a_{n+1}$ (Theorem 8.15)

60. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ (Alternating Harmonic Series)

62. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$

If $p = 0$, then

$$\lim_{n\to\infty} \frac{1}{n^p} = 1$$

and the series diverges. If $p > 0$, then

$$\lim_{n\to\infty} \frac{1}{n^p} = 0 \text{ and } \frac{1}{(n+1)^p} < \frac{1}{n^p}.$$

Therefore, the series converge by the Alternating Series Test.

64. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges, but $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

66. (a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$

converges absolutely (by comparison) for

$-1 < x < 1,$

since

$$\left|\frac{x^n}{n}\right| < |x^n| \text{ and } \sum x^n$$

is a convergent geometric series for $-1 < x < 1$.

(b) When $x = -1$, we have the convergent alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

When $x = 1$, we have the divergent harmonic series

$$\frac{1}{n}.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

converges conditionally for $x = -1$.

68. True, equivalent to Theorem 8.16

70. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^2+5}$

converges by limit comparison to convergent p-series

$$\sum \frac{1}{n^2}.$$

72. Converges by limit comparison to convergent geometric series $\displaystyle\sum \frac{1}{2^n}$.

74. Diverges by nth Term Test. $\displaystyle\lim_{n\to\infty} a_n = \frac{3}{2}$

76. Converges (conditionally) by Alternating Series Test.

78. Diverges by comparison to Divergent Harmonic Series:

$$\frac{\ln n}{n} > \frac{1}{n} \text{ for } n \ge 3.$$

Section 8.6 The Ratio and Root Tests

2. $\dfrac{(2k-2)!}{(2k)!} = \dfrac{(2k-2)!}{(2k)(2k-1)(2k-2)!} = \dfrac{1}{(2k)(2k-1)}$

4. Use the Principle of Mathematical Induction. When $k = 3$, the formula is valid since $\dfrac{1}{1} = \dfrac{2^3 3!(3)(5)}{6!} = 1$. Assume that

$$\frac{1}{1 \cdot 3 \cdot 5 \cdots (2n-5)} = \frac{2^n n!(2n-3)(2n-1)}{(2n)!}$$

and show that

$$\frac{1}{1 \cdot 3 \cdot 5 \cdots (2n-5)(2n-3)} = \frac{2^{n+1}(n+1)!(2n-1)(2n+1)}{(2n+2)!}.$$

To do this, note that:

$$\begin{aligned}\frac{1}{1 \cdot 3 \cdot 5 \cdots (2n-5)(2n-3)} &= \frac{1}{1 \cdot 3 \cdot 5 \cdots (2n-5)} \cdot \frac{1}{(2n-3)}\\ &= \frac{2^n n!\cancel{(2n-3)}(2n-1)}{(2n)!} \cdot \frac{1}{\cancel{(2n-3)}}\\ &= \frac{2^n n!(2n-1)}{(2n)!} \cdot \frac{(2n+1)(2n+2)}{(2n+1)(2n+2)}\\ &= \frac{2^n(2)(n+1)n!(2n-1)(2n+1)}{(2n)!(2n+1)(2n+2)}\\ &= \frac{2^{n+1}(n+1)!(2n-1)(2n+1)}{(2n+2)!}\end{aligned}$$

The formula is valid for all $n \geq 3$.

6. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n \left(\frac{1}{n!}\right) = \frac{3}{4} + \frac{9}{16}\left(\frac{1}{2}\right) + \cdots$

$S_1 = \dfrac{3}{4}, S_2 \approx 1.03$

Matches (c)

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}4}{(2n)!} = \frac{4}{2} - \frac{4}{24} + \cdots$

$S_1 = 2$

Matches (b)

10. $\displaystyle\sum_{n=0}^{\infty} 4e^{-n} = 4 + \frac{4}{e} + \cdots$

$S_1 = 4$

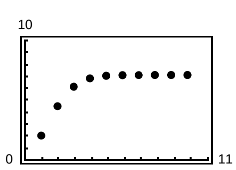
Matches (e)

12. (a) Ratio Test: $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\dfrac{(n+1)^2+1}{(n+1)!}}{\dfrac{n^2+1}{n!}} = \lim_{n\to\infty}\left(\frac{n^2+2n+2}{n^2+1}\right)\left(\frac{1}{n+1}\right) = 0 < 1$. Converges

(b)

n	5	10	15	20	25
S_n	7.0917	7.1548	7.1548	7.1548	7.1548

(c)



(d) The sum is approximately 7.15485

(e) The more rapidly the terms of the series approach 0, the more rapidly the sequence of the partial sums approaches the sum of the series.

14. $\sum_{n=0}^{\infty} \frac{3^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}}{(n+1)!}\cdot\frac{n!}{3^n}\right| = \lim_{n\to\infty}\frac{3}{n+1} = 0$$

Therefore, by the Ratio Test, the series converges.

16. $\sum_{n=1}^{\infty} n\left(\frac{3}{2}\right)^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)3^{n+1}}{2^{n+1}}\cdot\frac{2^n}{n3^n}\right| = \lim_{n\to\infty}\frac{3(n+1)}{2n} = \frac{3}{2}$$

Therefore, by the Ratio Test, the series diverges.

18. $\sum_{n=1}^{\infty} \frac{n^3}{2^n}$

$$\lim_{n\to\infty}\left|\frac{a_n+1}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^3/2^{n+1}}{n^3/2^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^3}{2n^3}\right| = \frac{1}{2}$$

Therefore, by the Ratio Test, the series converges.

20. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2)}{n(n+1)}$

$$a_{n+1} = \frac{n+3}{(n+1)(n+2)} \le \frac{n+2}{n(n+1)} = a_n$$

$$\lim_{n\to\infty}\frac{n+2}{n(n+1)} = 0$$

Therefore, by Theorem 8.14, the series converges.

Note: The Ratio Test is inconclusive since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1.$

The series converges conditionally.

22. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(3/2)^n}{n^2}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(3/2)^{n+1}}{n^2+2n+1}\cdot\frac{n^2}{(3/2)^n}\right| = \lim_{n\to\infty}\frac{3n^2}{2(n^2+2n+1)} = \frac{3}{2} > 1$$

Therefore, by the Ratio Test, the series diverges.

24. $\sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)!}{(n+1)^5}\cdot\frac{n^5}{(2n)!}\right| = \lim_{n\to\infty}\frac{(2n+2)(2n+1)n^5}{(n+1)^5} = \infty$$

Therefore, by the Ratio Test, the series diverges.

26. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n}\right| = \lim_{n\to\infty}\frac{(n+1)(n+1)^n n!}{(n+1)n!n^n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = e > 1$$

Therefore, by the Ratio Test, the series diverges.

28. $\sum_{n=0}^{\infty} \frac{(n!)^2}{(3n)!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{[(n+1)!]^2}{(3n+3)!}\cdot\frac{(3n)!}{(n!)^2}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{(3n+3)(3n+2)(3n+1)} = 0$$

Therefore, by the Ratio Test, the series converges.

30. $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{4n}}{(2n+1)!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{4n+4}}{(2n+3)!}\cdot\frac{(2n+1)!}{2^{4n}}\right| = \lim_{n\to\infty}\frac{2^4}{(2n+3)(2n+2)} = 0$$

Therefore, by the Ratio Test, the series converges.

32. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, 2 \cdot 4 \cdot 6 \cdots 2n}{2 \cdot 5 \cdot 8 \cdots (3n-1)}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2 \cdot 4 \cdots 2n(2n+2)}{2 \cdot 5 \cdots (3n-1)(3n+2)} \cdot \frac{2 \cdot 5 \cdots (3n-1)}{2 \cdot 4 \cdots 2n}\right| = \lim_{n\to\infty}\frac{2n+2}{3n+2} = \frac{2}{3}$$

Therefore, by the Ratio Test, the series converges.

Note: The first few terms of this series are $-\frac{2}{2} + \frac{2 \cdot 4}{2 \cdot 5} - \frac{2 \cdot 4 \cdot 6}{2 \cdot 5 \cdot 8} + \cdots$

34. (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1}{(n+1)^4} \cdot \frac{n^4}{1}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^4 = 1$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1}{(n+1)^p} \cdot \frac{n^p}{1}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^p = 1$$

36. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{2n}{n+1}\right)^n}$$

$$= \lim_{n\to\infty}\frac{2n}{n+1} = 2$$

Therefore, by the Root Test, the series diverges.

38. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{-3n}{2n+1}\right)^{3n}$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{-3n}{2n+1}\right)^{3n}\right|}$$

$$= \lim_{n\to\infty}\left(\frac{3n}{2n+1}\right)^3 = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$$

Therefore, by the Root Test, the series diverges.

40. $\displaystyle\sum_{n=0}^{\infty} e^{-n}$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\frac{1}{e^n}} = \frac{1}{e}$$

Therefore, by the Root Test, the series converges.

42. $\displaystyle\sum_{n=0}^{\infty} \frac{n+1}{3^n}$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\frac{n+1}{3^n}} = \lim_{n\to\infty}\frac{\sqrt[n]{n+1}}{3}$$

$$\text{Let } y = \lim_{n\to\infty}\sqrt[x]{x+1}$$

$$\ln y = \lim_{n\to\infty}\left(\ln\sqrt[x]{x+1}\right)$$

$$= \lim_{n\to\infty}\frac{1}{x}\ln(x+1)$$

$$= \lim_{n\to\infty}\frac{\ln(x+1)}{x} = \frac{1}{x+1} = 0.$$

Since $\ln y = 0$, $y = e^0 = 1$, so

$$\lim_{n\to\infty}\frac{\sqrt{n+1}}{3} = \frac{1}{3}.$$

Therefore, by the Root Test, the series converges.

44. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n} = 5\sum_{n=1}^{\infty}\frac{1}{n}$

This is the divergent harmonic series.

46. $\sum_{n=1}^{\infty} \left(\frac{\pi}{4}\right)^n$

Since $\pi/4 < 1$,this is convergent geometric series.

48. $\sum_{n=1}^{\infty} \frac{n}{2n^2+1}$

$$\lim_{n\to\infty} \frac{n/(2n^2+1)}{1/n} = \lim_{n\to\infty} \frac{n^2}{2n^2+1} = \frac{1}{2} > 0$$

This series diverges by limit comparison to the divergent harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

50. $\sum_{n=1}^{\infty} \frac{10}{3\sqrt{n^3}}$

$$\lim_{n\to\infty} \frac{10/3n^{3/2}}{1/n^{3/2}} = \frac{10}{3}$$

Therefore, the series converges by a limit comparison test with the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$

52. $\sum_{n=1}^{\infty} \frac{2^n}{4n^2-1}$

$$\lim_{n\to\infty} \frac{2^n}{4n^2-1} = \lim_{n\to\infty} \frac{(\ln 2)2^n}{8n} = \lim_{n\to\infty} \frac{(\ln 2)^2 2^n}{8} = \infty$$

Therefore, the series diverges by the nth Term Test for Divergence.

54. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln(n)}$

$$a_{n+1} = \frac{1}{(n+1)\ln(n+1)} \le \frac{1}{n\ln(n)} = a_n$$

$$\lim_{n\to\infty} \frac{1}{n\ln(n)} = 0$$

Therefore, by the Alternating Series Test, the series converges.

56. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$

$$\frac{\ln(n)}{n^2} \le \frac{1}{n^{3/2}}$$

Therefore, the series converges by comparison with the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$

58. $\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n2^n}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{3^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n2^n}{3^n}\right| = \lim_{n\to\infty} \frac{3n}{2(n+1)} = \frac{3}{2}$$

Therefore, by the Ratio Test, the series diverges.

60. $\sum_{n=1}^{\infty} \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{18^n (2n-1)n!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{3 \cdot 5 \cdot 7 \cdots (2n+1)(2n+3)}{18^{n+1}(2n+1)(2n-1)n!} \cdot \frac{18^n (2n-1)n!}{3 \cdot 5 \cdot 7 \cdots (2n+1)}\right| = \lim_{n\to\infty} \frac{(2n+3)(2n-1)}{18(2n+1)(2n-1)} = \frac{2}{18} = \frac{1}{9}$$

Therefore, by the Ratio Test, the series converge.

62. (b) and (c)

$$\sum_{n=0}^{\infty} (n+1)\left(\frac{3}{4}\right)^n = \sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^{n-1}$$

$$= 1 + 2\left(\frac{3}{4}\right) + 3\left(\frac{3}{4}\right)^2 + 4\left(\frac{3}{4}\right)^3 + \cdots$$

64. (a) and (b) are the same.

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{(n-1)2^{n-1}} = \frac{1}{2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \cdots$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n} = \frac{1}{2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \cdots$$

66. Replace n with $n+2$.

$$\sum_{n=2}^{\infty} \frac{2^n}{(n-2)!} = \sum_{n=0}^{\infty} \frac{2^{n+2}}{n!}$$

68. $$\sum_{k=0}^{\infty} \frac{(-3)^k}{1 \cdot 3 \cdot 5 \ldots (2k+1)} = \sum_{k=0}^{\infty} \frac{(-3)^k 2^k k!}{(2k)!(2k+1)} = \sum_{k=0}^{\infty} \frac{(-6)^k k!}{(2k+1)!} \approx 0.40967$$

(See Exercise 3 and use 10 terms, $k = 9$.)

70. See Theorem 8.18.

72. One example is $\displaystyle\sum_{n=1}^{\infty}\left(-100 + \frac{1}{n}\right)$.

74. Assume that

$\lim_{n\to\infty} |a_{n+1}/a_n| = L > 1$ or that $\lim_{n\to\infty} |a_{n+1}/a_n| = \infty$.

Then there exists $N > 0$ such that $|a_{n+1}/a_n| > 1$ for all $n > N$. Therefore,

$$|a_{n+1}| > |a_n|, \quad n > N \Longrightarrow \lim_{n\to\infty} a_n \neq 0 \Longrightarrow \sum a_n \text{ diverges}$$

76. The differentiation test states that if

$$\sum_{n=1}^{\infty} U_n$$

is an infinite series with real terms and $f(x)$ is a real function such that $f(1/n) = U_n$ for all positive integers n and $d^2 f/dx^2$ exists at $x = 0$, then

$$\sum_{n=1}^{\infty} U_n$$

converges absolutely if $f(0) = f'(0) = 0$ and diverges otherwise. Below are some examples.

Convergent Series	Divergent Series
$\sum \frac{1}{n^3}, f(x) = x^3$	$\sum \frac{1}{n}, f(x) = x$
$\sum\left(1 - \cos\frac{1}{n}\right), f(x) = 1 - \cos x$	$\sum \sin\frac{1}{n}, f(x) = \sin x$

Section 8.7 Taylor Polynomials and Approximations

2. $y = \frac{1}{8}x^4 - \frac{1}{2}x^2 + 1$

y-axis symmetry

Three relative extrema

Matches (c)

4. $y = e^{-1/2}\left[\frac{1}{3}(x-1)^3 - (x-1) + 1\right]$

Cubic

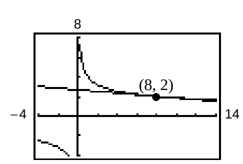
Matches (b)

6. $f(x) = \dfrac{4}{\sqrt[3]{x}} = 4x^{-1/3}$ $\qquad f(8) = 2$

$f'(x) = -\dfrac{4}{3}x^{-4/3}$ $\qquad f'(8) = -\dfrac{1}{12}$

$$P_1(x) = f(8) + f'(8)(x-8) = 2 + \left(-\frac{1}{12}\right)(x-8)$$

$$P_1(x) = -\frac{1}{12}x + \frac{8}{3}$$



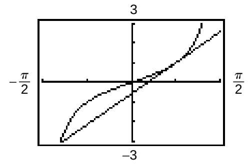
8. $f(x) = \tan x \qquad f\left(\dfrac{\pi}{4}\right) = 1$

$f'(x) = \sec^2 x \qquad f'\left(\dfrac{\pi}{4}\right) = 2$

$$P_1 = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)$$

$$= 1 + 2\left(x - \frac{\pi}{4}\right)$$

$$P_1(x) = 2x + 1 - \frac{\pi}{2}$$



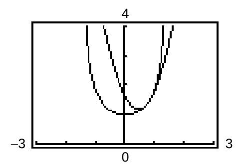
10. $f(x) = \sec x \qquad f\left(\dfrac{\pi}{4}\right) = \sqrt{2}$

$f'(x) = \sec x \tan x \qquad f'\left(\dfrac{\pi}{4}\right) = \sqrt{2}$

$f''(x) = \sec^3 x + \sec x \tan^2 x \qquad f''\left(\dfrac{\pi}{4}\right) = 3\sqrt{2}$

$$P_2(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + \frac{f''(\pi/4)}{2}\left(x - \frac{\pi}{4}\right)^2$$

$$P_2(x) = \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right) + \frac{3}{2}\sqrt{2}\left(x - \frac{\pi}{4}\right)^2$$



x	−2.15	0.585	0.685	$\pi/4$	0.885	0.985	1.785
$f(x)$	−1.8270	1.1995	1.2913	1.4142	1.5791	1.8088	−4.7043
$P_2(x)$	15.5414	1.2160	1.2936	1.4142	1.5761	1.7810	4.9475

12. $f(x) = x^2e^x,\ f(0) = 0$

(a) $f'(x) = (x^2 + 2x)e^x \qquad f'(0) = 0$

$f''(x) = (x^2 + 4x + 2)e^x \qquad f''(0) = 2$

$f'''(x) = (x^2 + 6x + 6)e^x \qquad f'''(0) = 6$

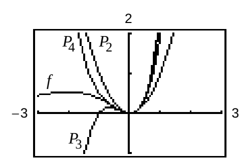
$f^{(4)}(x) = (x^2 + 8x + 12)e^x \qquad f^{(4)}(0) = 12$

$$P_2(x) = \frac{2x^2}{2!} = x^2$$

$$P_3(x) = x^2 + \frac{6x^3}{3!} = x^2 + x^3$$

$$P_4(x) = x^2 + x^3 + \frac{12x^4}{4!} = x^2 + x^3 + \frac{x^4}{2}$$

(b)



(c) $f''(0) = 2 = P_2''(0)$

$f'''(0) = 6 = P_3'''(0)$

$f^{(4)}(0) = 12 = P_4^{(4)}(0)$

(d) $f^{(n)}(0) = P_n^{(n)}(0)$

14. $f(x) = e^{-x} \qquad f(0) = 1$

$f'(x) = -e^{-x} \qquad f'(0) = -1$

$f''(x) = e^{-x} \qquad f''(0) = 1$

$f'''(x) = -e^{-x} \qquad f'''(0) = -1$

$f^{(4)}(x) = e^{-x} \qquad f^{(4)}(0) = 1$

$f^{(5)}(x) = -e^{-x} \qquad f^{(5)}(0) = -1$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120}$$

16. $f(x) = e^{3x}$ $f(0) = 1$

$f'(x) = 3e^{3x}$ $f'(0) = 3$

$f''(x) = 9e^{3x}$ $f''(0) = 9$

$f'''(x) = 27e^{3x}$ $f'''(0) = 27$

$f^{(4)}(x) = 81e^{3x}$ $f^{(4)}(0) = 81$

$$P_4(x) = 1 + 3x + \frac{9}{2!}x^2 + \frac{27}{3!}x^3 + \frac{81}{4!}x^4 = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4$$

18. $f(x) = \sin \pi x$ $f(0) = 0$

$f'(x) = \pi \cos \pi x$ $f'(0) = \pi$

$f''(x) = -\pi^2 \sin \pi x$ $f''(0) = 0$

$f'''(x) = -\pi^3 \cos \pi x$ $f'''(0) = -\pi^3$

$$P_3(x) = 0 + \pi x + \frac{0}{2!}x^2 + \frac{-\pi^3}{3!}x^3 = \pi x - \frac{\pi^3}{6}x^3$$

20. $f(x) = x^2e^{-x}$ $f(0) = 0$

$f'(x) = 2xe^{-x} - x^2e^{-x}$ $f'(0) = 0$

$f''(x) = 2e^{-x} - 4xe^{-x} + x^2e^{-x}$ $f''(0) = 2$

$f'''(x) = -6e^{-x} + 6xe^{-x} - x^2e^{-x}$ $f'''(0) = -6$

$f^{(4)}(x) = 12e^{-x} - 8xe^{-x} + x^2e^{-x}$ $f^{(4)}(0) = 12$

$$P_4(x) = 0 + 0x + \frac{2}{2!}x^2 + \frac{-6}{3!}x^3 + \frac{12}{4!}x^4$$

$$= x^2 - x^3 + \frac{1}{2}x^4$$

22. $f(x) = \dfrac{x}{x+1} = \dfrac{x+1-1}{x+1} = 1 - (x+1)^{-1}$ $f(0) = 0$

$f'(x) = (x+1)^{-2}$ $f'(0) = 1$

$f''(x) = -2(x+1)^{-3}$ $f''(0) = -2$

$f'''(x) = 6(x+1)^{-4}$ $f'''(0) = 6$

$f^{(4)}(x) = -24(x+1)^{-5}$ $f^{(4)}(0) = -24$

$$P_4(x) = 0 + 1(x) - \frac{2}{2}x^2 + \frac{6}{6}x^3 - \frac{24}{24}x^4 = x - x^2 + x^3 - x^4$$

24. $f(x) = \tan x$ $f(0) = 0$

$f'(x) = \sec^2 x$ $f'(0) = 1$

$f''(x) = 2\sec^2 x \tan x$ $f''(0) = 0$

$f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$ $f'''(0) = 2$

$$P_3(x) = 0 + 1(x) + 0 + \frac{2}{6}x^3 = x + \frac{1}{3}x^3$$

26. $f(x) = 2x^{-2}$ $f(2) = \dfrac{1}{2}$

$f'(x) = -4x^{-3}$ $f'(2) = -\dfrac{1}{2}$

$f''(x) = 12x^{-4}$ $f''(2) = \dfrac{3}{4}$

$f'''(x) = -48x^{-5}$ $f'''(x) = -\dfrac{3}{2}$

$f^{(4)}(x) = 240x^{-6}$ $f^{(4)}(x) = \dfrac{15}{4}$

$$P_4(x) = \frac{1}{2} - \frac{1}{2}(x-2) + \frac{3}{8}(x-2)^2 - \frac{1}{4}(x-2)^3 + \frac{5}{32}(x-2)^4$$

28. $f(x) = x^{1/3}$ $\qquad f(8) = 2$

$f'(x) = \frac{1}{3}x^{-2/3}$ $\qquad f'(8) = \frac{1}{12}$

$f''(x) = -\frac{2}{9}x^{-5/3}$ $\qquad f''(8) = -\frac{1}{144}$

$f'''(x) = \frac{10}{27}x^{-8/3}$ $\qquad f'''(8) = \frac{10}{27} \cdot \frac{1}{2^8} = \frac{5}{3456}$

$$P_3(x) = 2 + \frac{1}{12}(x - 8) - \frac{1}{288}(x - 8)^2 + \frac{5}{20{,}736}(x - 8)^3$$

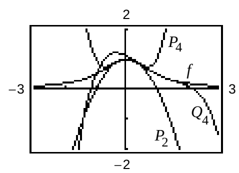
30. $f(x) = x^2 \cos x$ $\qquad f(\pi) = -\pi^2$

$f'(x) = \cos x - x^2 \sin x$ $\qquad f'(\pi) = -2\pi$

$f''(x) = 2\cos x - 4x \sin x - x^2 \cos x$ $\qquad f''(\pi) = -2 + \pi^2$

$$P_2(x) = -\pi^2 - 2\pi(x - \pi) + \frac{(\pi^2 - 2)}{2}(x - \pi)^2$$

32. $f(x) = \dfrac{1}{x^2 + 1}$



$f'(x) = \dfrac{-2x}{(x^2 + 1)^2}$

$f''(x) = \dfrac{2(3x^2 - 1)}{(x^2 + 1)^3}$

$f'''(x) = \dfrac{24x(1 - x^2)}{(x^2 + 1)^4}$

$f^{(4)}(x) = \dfrac{24(5x^4 - 10x^2 + 1)}{(x^2 + 1)^5}$

(a) $n = 2, c = 0$

$$P_2(x) = 1 + 0x + \frac{-2}{2!}x^2 = 1 - x^2$$

(b) $n = 4, c = 0$

$$P_4(x) = 1 + 0x + \frac{-2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{24}{4!}x^4 = 1 - x^2 + x^4$$

(c) $n = 4, c = 1$

$$Q_4(x) = \frac{1}{2} + \left(-\frac{1}{2}\right)(x - 1) + \frac{1/2}{2!}(x - 1)^2 + \frac{0}{3!}(x - 1)^3 + \frac{-3}{4!}(x - 1)^4 = \frac{1}{2} - \frac{1}{2}(x - 1) + \frac{1}{4}(x - 1)^2 - \frac{1}{8}(x - 1)^4$$

34. $f(x) = \ln x$

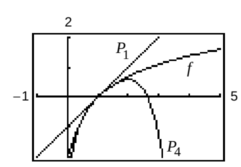
$P_1(x) = x - 1$

$P_4(x) = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \frac{1}{4}(x - 1)^4$

(a)

x	1.00	1.25	1.50	1.75	2.00
$\ln x$	0.0000	0.2231	0.4055	0.5596	0.6931
$P_1(x)$	0.0000	0.2500	0.5000	0.7500	1.0000
$P_4(x)$	0.0000	0.2230	0.4010	0.5303	0.5833

(b)



(c) As the distance increases, the accuracy decreases.

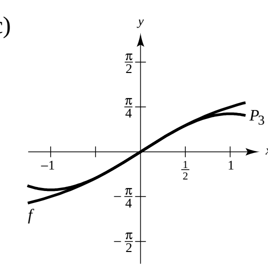
36. (a) $f(x) = \arctan x$

$$P_3(x) = x - \frac{x^3}{3}$$

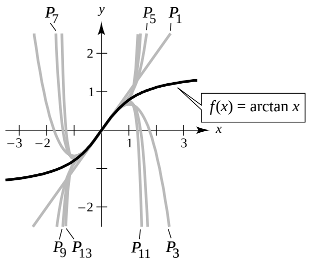
(b)

x	-0.75	-0.50	-0.25	0	0.25	0.50	0.75
$f(x)$	-0.6435	-0.4636	-0.2450	0	0.2450	0.4636	0.6435
$P_3(x)$	-0.6094	-0.4583	-0.2448	0	0.2448	0.4583	0.6094

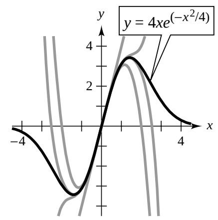
(c)



38. $f(x) = \arctan x$



40. $f(x) = 4xe^{-x^2/4}$



42. $f(x) = x^2e^{-x} \approx x^2 - x^3 + \dfrac{1}{2}x^4$

$f\left(\dfrac{1}{5}\right) \approx 0.0328$

44. $f(x) = x^2 \cos x \approx -\pi^2 - 2\pi(x - \pi) + \left(\dfrac{\pi^2 - 2}{2}\right)(x - \pi)^2$

$f\left(\dfrac{7\pi}{8}\right) \approx -6.7954$

46. $f(x) = e^x; f^{(6)}(x) = e^x \implies$ Max on $[0, 1]$ is e^1.

$R_5(x) \le \dfrac{e^1}{6!}(1)^6 \approx 0.00378 = 3.78 \times 10^{-3}$

48. $f(x) = \arctan x; f^{(4)}(x) = \dfrac{24x(x^2 + 1)}{(1 - x^2)^4}$

$\implies$Max on $[0, 0.4]$ is $f^{(4)}(0.4) \approx 22.3672$.

$R_3(x) \le \dfrac{22.3672}{4!}(0.4)^4 \approx 0.0239$

50. $f(x) = e^x$

$f^{(n+1)}(x) = e^x$

Max on $[0, 0.6]$ is $e^{0.6} \approx 1.8221$.

$R_n \le \dfrac{1.8221}{(n + 1)!}(0.6)^{n+1} < 0.001$

By trial and error, $n = 5$.

52. $f(x) = \cos(\pi x^2)$

$$g(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$f(x) = g(\pi x^2)$$

$$= 1 - \frac{(\pi x^2)^2}{2!} + \frac{(\pi x^2)^4}{4!} - \frac{(\pi x^2)^6}{6!} + \cdots$$

$$= 1 - \frac{\pi^2 x^4}{2!} + \frac{\pi^4 x^8}{4!} - \frac{\pi^6 x^{12}}{6!} + \cdots$$

$$f(0.6) = 1 - \frac{\pi^2}{2!}(0.6)^4 + \frac{\pi^4}{4!}(0.6)^8 - \frac{\pi^6}{6!}(0.6)^{12} + \cdots$$

Since this is an alternating series,

$$R_n \le a_{n+1} = \frac{\pi^{2n}}{(2n)!}(0.6)^{4n} < 0.0001.$$

By trial and error, $n = 4$. Using 4 terms $f(0.6) \approx 0.4257$.

54. $f(x) = \sin x \approx x - \dfrac{x^3}{3!}$

$$|R_3(x)| = \left|\frac{\sin z}{4!}x^4\right| \le \frac{|x^4|}{4!} < 0.001$$

$$x^4 < 0.024$$

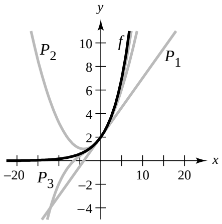
$$|x| < 0.3936$$

$$-0.3936 < x < 0.3936$$

56. $f(c) = P_2(c)$, $f'(c) = P_2{}'(c)$, and $f''(c) = P_2{}''(c)$

58. See Theorem 8.19, page 611.

60.



62. (a) $P_5(x) = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}$ for $f(x) = \sin x$

$$P_5{}'(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

This is the Maclaurin polynomial of degree 4 for $g(x) = \cos x$.

(b) $Q_6(x) = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!}$ for $\cos x$

$$Q_6{}'(x) = -x + \frac{x^3}{3!} - \frac{x^5}{5!} = -P_5(x)$$

(c) $R(x) = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!}$

$$R'(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

The first four terms are the same!

64. Let f be an odd function and P_n be the n^{th} Maclaurin polynomial for f. Since f is odd, f' is even:

$$f'(-x) = \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h\to 0} \frac{-f(x-h) + f(x)}{h} = \lim_{h\to 0} \frac{f(x+(-h)) - f(x)}{-h} = f'(x).$$

Similarly, f'' is odd, f''' is even, etc. Therefore, $f, f'', f^{(4)}$, etc. are all odd functions, which implies that $f(0) = f''(0) = \cdots = 0$. Hence, in the formula

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \cdots$$ all the coefficients of the even power of x are zero.

66. Let $P_n(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n$ where $a_i = \dfrac{f^{(i)}(c)}{i!}$.

$$P_n(c) = a_0 = f(c)$$

For $1 \le k \le n$, $P_n{}^{(k)}(c) = a_n k! = \left(\dfrac{f^{(k)}(c)}{k!}\right)k! = f^{(k)}(c)$.

Section 8.8 Power Series

2. Centered at 0

4. Centered at π

6. $\displaystyle\sum_{n=0}^{\infty} (2x)^n$

$$L = \lim_{n\to\infty} \left|\frac{u_n + 1}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(2x)^{n+1}}{(2x)^n}\right| = 2|x|$$

$$2|x| < 1 \implies R = \frac{1}{2}$$

8. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^n}$

$$L = \lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(-1)^n x^n}\right|$$

$$= \frac{1}{2}|x|$$

$$\frac{1}{2}|x| < 1 \implies R = 2$$

10. $\sum_{n=0}^{\infty} \frac{(2n)!x^{2n}}{n!}$

$$L = \lim_{n\to\infty}\left|\frac{u_n + 1}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)!x^{2n+2}/(n+1)!}{(2n)!x^{2n}/n!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)x^2}{(n+1)}\right| = \infty$$

The series only converges at $x = 0$. $R = 0$.

12. $\sum_{n=0}^{\infty}\left(\frac{x}{k}\right)^n$

Since the series is geometric, it converges only if $|x/k| < 1$ or $-k < x < k$.

14. $\sum_{n=0}^{\infty} (-1)^{n+1}(n+1)x^n$

$$\lim_{n\to\infty}\left|\frac{u_n + 1}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(n+2)x^{n+1}}{(-1)^n(n+1)x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+2)x}{n+1}\right| = |x|$$

Interval: $-1 < x < 1$

When $x = 1$, the series $\sum_{n=0}^{\infty} (-1)^{n+1}(n+1)$ diverges.

When $x = -1$, the series $\sum_{n=0}^{\infty} -(n+1)$ diverges.

Therefore, the interval of convergence is $-1 < x < 1$.

16. $\sum_{n=0}^{\infty} \frac{(3x)^n}{(2n)!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(3x)^{n+1}}{(2n+1)!}\cdot\frac{(2n)!}{(3x)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{3x}{(2n+2)(2n+1)}\right| = 0$$

Therefore, the interval of convergence is $-\infty < x < \infty$.

18. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n+1)(n+2)}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{(n+2)(n+3)}\cdot\frac{(n+1)(n+2)}{(-1)^n x^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)x}{n+3}\right| = |x|$$

Interval: $-1 < x < 1$

When $x = 1$, the alternating series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)(n+2)}$ converges.

When $x = -1$, the series $\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)}$ converges by limit comparison to $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Therefore, the interval of convergence is $-1 \le x \le 1$.

20. $\sum_{n=0}^{\infty} \frac{(-1)^n n!(x-4)^n}{3^n}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)!(x-4)^{n+1}}{3^{n+1}}\cdot\frac{3^n}{(-1)^n n!(x-4)^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x-4)}{3}\right| = \infty$$

$R = 0$

Center: $x = 4$

Therefore, the series converges only for $x = 4$.

22. $\sum_{n=0}^{\infty} \frac{(x-2)^{n+1}}{(n+1)4^{n+1}}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+2}}{(n+2)4^{n+2}} \cdot \frac{(n+1)4^{n+1}}{(x-2)^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(x-2)(n+1)}{4(n+2)}\right| = \frac{1}{4}|x-2|$$

$R = 4$

Center: $x = 2$

Interval: $-4 < x - 2 < 4$ or $-2 < x < 6$

When $x = -2$, the alternating series $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)}$ converges.

When $x = 6$, the series $\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges.

Therefore, the interval of convergence is $-2 \le x < 6$.

24. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-c)^n}{nc^n}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-c)^{n+1}}{(n+1)c^{n+1}} \cdot \frac{nc^n}{(-1)^{n+1}(x-c)^n}\right| = \lim_{n\to\infty}\left|\frac{n(x-c)}{c(n+1)}\right| = \frac{1}{c}|x-c|$$

$R = c$

Center: $x = c$

Interval: $-c < x - c < c$ or $0 < x < 2c$

When $x = 0$, the p-series $\sum_{n=1}^{\infty} \frac{-1}{n}$ diverges.

When $x = 2c$, the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges. Therefore, the interval of convergence is $0 < x \le 2c$.

26. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+3}}{(2n+3)} \cdot \frac{(2n+1)}{(-1)^n x^{2n+1}}\right| = \lim_{n\to\infty}\left|\frac{(2n+1)}{(2n+3)}x^2\right| = |x^2|$$

$R = 1$

Interval: $-1 < x < 1$

When $x = 1$, $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges.

When $x = -1$, $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$ converges.

Therefore, the interval of convergence is $-1 \le x \le 1$.

28. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+2}}{(n+1)!} \cdot \frac{n!}{(-1)^n x^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^2}{n+1}\right| = 0$$

Therefore, the interval of convergence is $-\infty < x < \infty$.

30. $\sum_{n=1}^{\infty} \frac{n!x^n}{(2n)!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{n!x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)x}{(2n+2)(2n+1)}\right| = 0$$

Therefore, the interval of convergence is $-\infty < x < \infty$.

32. $\displaystyle\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n+1)} (x^{2n+1})$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{2 \cdot 4 \cdots (2n)(2n+2)x^{2n+3}}{3 \cdot 5 \cdot 7 \cdots (2n+1)(2n+3)} \cdot \frac{3 \cdot 5 \cdots (2n+1)}{2 \cdot 4 \cdots (2n)x^{2n+1}}\right| = \lim_{n\to\infty} \left|\frac{(2n+2)x^2}{(2n+3)}\right| = |x^2|$$

$R = 1$

When $x = \pm 1$, the series diverges by comparing it to

$$\sum_{n=1}^{\infty} \frac{1}{2n+1}$$

which diverges. Therefore, the interval of convergence is $-1 < x < 1$.

34. $\displaystyle\sum_{n=1}^{\infty} \frac{n!(x-c)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!(x-c)^{n+1}}{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!(x-c)}\right| = \lim_{n\to\infty} \left|\frac{(n+1)(x-c)}{2n+1}\right| = \frac{1}{2}|x-c|$$

$R = 2$

Interval: $-2 < x - c < 2$ or $c - 2 < x < c + 2$

The series diverges at the endpoints. Therefore, the interval of convergence is $c - 2 < x < c + 2$.

$$\left[\frac{n!(c+2-c)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \frac{n!\,2^2}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)} > 1\right]$$

36. (a) $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n}$, $0 < x \le 10$

(b) $f'(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^{n-1}}{5^n}$, $0 < x < 10$

(c) $f''(x) = \displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}(n-1)(x-5)^{n-2}}{5^n}$, $0 < x < 10$

(d) $\displaystyle\int f(x)\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^{n+1}}{n(n+1)5^n}$, $0 \le x \le 10$

38. (a) $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n}$, $1 < x \le 3$

(b) $f'(x) = \displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}(x-2)^{n-1}$, $1 < x < 3$

(c) $f''(x) = \displaystyle\sum_{n=2}^{\infty} (-1)^{n+1}(n-1)(x-2)^{n-2}$, $1 < x < 3$

(d) $\displaystyle\int f(x)\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^{n+1}}{n(n+1)}$, $1 \le x \le 3$

40. $g(2) = \displaystyle\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 1 + \frac{2}{3} + \frac{4}{9} + \cdots$

$S_1 = 1$, $S_2 = 1.67$. Matches (a)

42. $g(-2) = \displaystyle\sum_{n=0}^{\infty} \left(-\frac{2}{3}\right)^n$ alternating. Matches (d)

44. The set of all values of x for which the power series converges is the interval of convergence. If the power series converges for all x, then the radius of convergence is $R = \infty$. If the power series converges at only c, then $R = 0$. Otherwise, according to Theorem 8.20, there exists a real number $R > 0$ (radius of convergence) such that the series converges absolutely for $|x - c| < R$ and diverges for $|x - c| > R$.

46. You differentiate and integrate the power series term by term. The radius of convergence remains the same. However, the interval of convergence might change.

48. (a) $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$, $-\infty < x < \infty$ (See Exercise 11)

(b) $f'(x) = \displaystyle\sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$

(c) $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$f(0) = 1$

(d) $f(x) = e^x$

50.
$$y = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)}$$

$$y' = \sum_{n=1}^{\infty} \frac{(-1)^n 4nx^{4n-1}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)}$$

$$y'' = \sum_{n=1}^{\infty} \frac{(-1)^n 4n(4n-1)x^{4n-2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} = -x^2 + \sum_{n=2}^{\infty} \frac{(-1)^n 4nx^{4n-2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-5)}$$

$$y'' + x^2 y = -x^2 + \sum_{n=2}^{\infty} \frac{(-1)^n 4nx^{4n-2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-5)} + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n+2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} + x^2$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4(n+1)x^{4n+2}}{2^{2n+2}(n+1)! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{4n+2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} \frac{2^2(n+1)}{2^2(n+1)} = 0$$

52. $J_1(x) = x \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k+1}k!(k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1}k!(k+1)!}$

(a) $\lim_{k\to\infty} \left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to\infty} \left|\frac{(-1)^{k+1} x^{2k+3}}{2^{2k+3}(k+1)!(k+2)!} \cdot \frac{2^{2k+1}k!(k+1)!}{(-1)^k x^{2k+1}}\right| = \lim_{k\to\infty} \left|\frac{(-1)x^2}{2^2 (k+2)(k+1)}\right| = 0$

Therefore, the interval of convergence is $-\infty < x < \infty$.

(b) $J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1}k!(k+1!)}$

$$J_1'(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)x^{2k}}{2^{2k+1}k!(k+1)!}$$

$$J_1''(x) = \sum_{k=1}^{\infty} \frac{(-1)^k (2k+1)(2k)x^{2k-1}}{2^{2k+1}k!(k+1)!}$$

$$x^2 J_1'' + xJ_1' + (x^2 - 1)J_1 = \sum_{k=1}^{\infty} \frac{(-1)^k (2k+1)(2k)x^{2k+1}}{2^{2k+1}k!(k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k(2k+1)x^{2k+1}}{2^{2k+1}k!(k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1}k!(k+1)!} - \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1}k!(k+1)!}$$

$$= \left[\sum_{k=1}^{\infty} \frac{(-1)^k(2k+1)(2k)x^{2k+1}}{2^{2k+1}k!(k+1)!} + \frac{x}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k(2k+1)x^{2k+1}}{2^{2k+1}k!(k+1)!} - \frac{x}{2} - \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1}k!(k+1)!}\right] + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1}k!(k+1)!}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1}[(2k+1)(2k) + (2k+1) - 1]}{2^{2k+1}k!(k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1}k!(k+1)!}$$

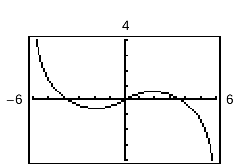
$$= \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1} 4k(k+1)}{2^{2k+1}k!(k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1}k!(k+1)!}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k-1}(k-1)!k!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1}k!(k+1)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+3}}{2^{2k+1}k!(k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1}k!(k+1)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}[(-1) + 1]}{2^{2k+1}k!(k+1)!} = 0$$

(c) $P_7(x) = \frac{x}{2} - \frac{1}{16}x^3 + \frac{1}{384}x^5 - \frac{1}{18{,}432}x^7$



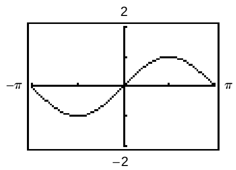
(d) $J_0'(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} 2(k+1)x^{2k+1}}{2^{2k+2}(k+1)!(k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+1}}{2^{2k+1}k!(k+1)!}$

$-J_1(x) = -\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1}k!(k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+1}}{2^{2k+1}k!(k+1)!}$

Note: $J_0'(x) = -J_1(x)$

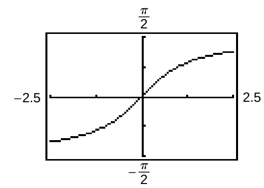
54. $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sin x$

(See Exercise 47.)



56. $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \arctan x, \ -1 \le x \le 1$

(See Exercise 38 in Section 8.7.)



58. $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}$

Replace n with $n - 1$.

60. True; if

$$\sum_{n=0}^{\infty} a_n x^n$$

converges for $x = 2$, then we know that it must converge on $(-2, 2]$.

62. True

$$\int_0^1 f(x)\,dx = \int_0^1 \left(\sum_{n=0}^{\infty} a_n x^n\right) dx = \left[\sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}\right]_0^1 = \sum_{n=0}^{\infty} \frac{a_n}{n+1}$$

Section 8.9 Representation of Functions by Power Series

2. (a) $f(x) = \frac{4}{5-x} = \frac{4/5}{1-x/5} = \frac{a}{1-r}$

$$= \sum_{n=0}^{\infty} \frac{4}{5}\left(\frac{x}{5}\right)^n = \sum_{n=0}^{\infty} \frac{4x^n}{5^{n+1}}$$

This series converges on $(-5, 5)$.

(b)
$$\begin{array}{r|l} & \frac{4}{5} + \frac{4}{25}x + \frac{4}{125}x^2 + \frac{4x^3}{625} + \cdots \\ \hline 5 - x & 4 \\ & \underline{4 - \frac{4}{5}x} \\ & \frac{4}{5}x \\ & \underline{\frac{4}{5}x - \frac{4}{25}x^2} \\ & \frac{4}{25}x^2 \\ & \underline{\frac{4}{25}x^2 - \frac{4x^3}{125}} \\ & \frac{4x^3}{125} \\ & \underline{\frac{4x^3}{125} - \frac{4x^4}{625}} \\ & \vdots \end{array}$$

4. (a) $\frac{1}{1+x} = \frac{1}{1-(-x)} = \frac{a}{1-r}$

$$= \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

This series converges on $(-1, 1)$.

(b)
$$\begin{array}{r|l} & 1 - x + x^2 - x^3 + \cdots \\ \hline 1 + x & 1 \\ & \underline{1 + x} \\ & -x \\ & \underline{-x - x^2} \\ & x^2 \\ & \underline{x^2 + x^3} \\ & -x^3 \\ & \underline{-x^3 - x^4} \\ & \vdots \end{array}$$

6. Writing $f(x)$ in the form $\dfrac{a}{1-r}$, we have

$$\frac{4}{5-x} = \frac{4}{7-(x+2)} = \frac{4/7}{1-1/7(x+2)} = \frac{a}{1-r}.$$

Therefore, the power series for $f(x)$ is given by

$$\frac{4}{5-x} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{4}{7}\left(\frac{1}{7}(x+2)\right)^n$$
$$= \sum_{n=0}^{\infty} \frac{4(x+2)^n}{7^{n+1}}.$$

$|x+2| < 7$ or $-5 < x < 9$

8. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{3}{2x-1} = \frac{3}{3+2(x-2)} = \frac{1}{1+(2/3)(x-2)} = \frac{a}{1-r}$$

which implies that $a = 1$ and $r = (-2/3)(x-2)$. Therefore, the power series for $f(x)$ is given by

$$\frac{3}{2x-1} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty}\left[-\frac{2}{3}(x-2)\right]^n,$$
$$= \sum_{n=0}^{\infty} \frac{(-2)^n(x-2)^n}{3^n},$$

$|x-2| < \frac{3}{2}$ or $\frac{1}{2} < x < \frac{7}{2}$.

10. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{1}{2x-5} = \frac{1}{-5+2x} = \frac{-1/5}{1-(2/5)x} = \frac{a}{1-r}$$

which implies that $a = -1/5$ and $r = (2/5)x$. Therefore, the power series for $f(x)$ is given by

$$\frac{1}{2x-5} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty}\left(-\frac{1}{5}\right)\left(\frac{2}{5}x\right)^n = -\sum_{n=0}^{\infty}\frac{2^n x^n}{5^{n+1}},$$

$|x| < \frac{5}{2}$ or $-\frac{5}{2} < x < \frac{5}{2}$.

12. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{4}{3x+2} = \frac{4}{8+3(x-2)} = \frac{1/2}{1+(3/8)(x-2)} = \frac{a}{1-r}$$

which implies that $a = 1/2$ and $r = (-3/8)(x-2)$. Therefore, the power series for $f(x)$ is given by

$$\frac{4}{3x+2} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty}\frac{1}{2}\left[-\frac{3}{8}(x-2)\right]^n$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-3)^n(x-2)^n}{8^n},$$

$|x-2| < \frac{8}{3}$ or $-\frac{2}{3} < x < \frac{14}{3}$.

14. $$\frac{4x-7}{2x^2+3x-2} = \frac{3}{x+2} - \frac{2}{2x-1} = \frac{3}{2+x} - \frac{2}{-1+2x} = \frac{3/2}{1+(1/2)x} + \frac{2}{1-2x}$$

Writing $f(x)$ as a sum of two geometric series, we have

$$\frac{4x-7}{2x^2+3x-2} = \sum_{n=0}^{\infty}\left(\frac{3}{2}\right)\left(-\frac{1}{2}x\right)^n + \sum_{n=0}^{\infty} 2(2x)^n = \sum_{n=0}^{\infty}\left[\frac{3(-1)^n}{2^{n+1}} + 2^{n+1}\right]x^n,\ |x| < \frac{1}{2} \text{ or } -\frac{1}{2} < x < \frac{1}{2}.$$

16. First finding the power series for $4/(4+x)$, we have

$$\frac{1}{1+(1/4)x} = \sum_{n=0}^{\infty}\left(-\frac{1}{4}x\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{4^n}$$

Now replace x with x^2.

$$\frac{4}{4+x^2} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{4^n}.$$

The interval of convergence is $|x^2| < 4$ or $-2 < x < 2$ since

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+2}}{4^{n+1}}\cdot\frac{4^n}{(-1)^n x^{2n}}\right| = \left|-\frac{x^2}{4}\right| = \frac{|x^2|}{4}.$$

18. $$h(x) = \frac{x}{x^2-1} = \frac{1}{2(1+x)} - \frac{1}{2(1-x)} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n x^n - \frac{1}{2}\sum_{n=0}^{\infty}x^n$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}[(-1)^n - 1]x^n = \frac{1}{2}[0 - 2x + 0x^2 - 2x^3 + 0x^4 - 2x^5 + \cdots]$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}(-2)x^{2n+1} = -\sum_{n=0}^{\infty}x^{2n+1},\ 1 < x < 1$$

20. By taking the second derivative, we have $\dfrac{d^2}{dx^2}\left[\dfrac{1}{x+1}\right] = \dfrac{2}{(x+1)^3}$. Therefore,

$$\begin{aligned}\frac{2}{(x+1)^3} &= \frac{d^2}{dx^2}\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]\\ &= \frac{d}{dx}\left[\sum_{n=1}^{\infty}(-1)^n n x^{n-1}\right] = \sum_{n=2}^{\infty}(-1)^n n(n-1)x^{n-2} = \sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n,\ -1 < x < 1.\end{aligned}$$

22. By integrating, we have

$$\int \frac{1}{1+x}\,dx = \ln(1+x) + C_1 \text{ and } \int \frac{1}{1-x}\,dx = -\ln(1-x) + C_2.$$

$f(x) = \ln(1-x^2) = \ln(1+x) - [-\ln(1-x)]$. Therefore,

$$\begin{aligned}\ln(1-x^2) &= \int \frac{1}{1+x}\,dx - \int \frac{1}{1-x}\,dx\\ &= \int\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]dx - \int\left[\sum_{n=0}^{\infty}x^n\right]dx = \left[C_1 + \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1}\right] - \left[C_2 + \sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}\right]\\ &= C + \sum_{n=0}^{\infty}\frac{[(-1)^n - 1]x^{n+1}}{n+1} = C + \sum_{n=0}^{\infty}\frac{-2x^{2n+2}}{2n+2} = C + \sum_{n=0}^{\infty}\frac{(-1)x^{2n+2}}{n+1}\end{aligned}$$

To solve for C, let $x = 0$ and conclude that $C = 0$. Therefore,

$$\ln(1-x^2) = -\sum_{n=0}^{\infty}\frac{x^{2n+2}}{n+1},\ -1 < x < 1$$

24. $$\begin{aligned}\frac{2x}{x^2+1} &= 2x\sum_{n=0}^{\infty}(-1)^n x^{2n} \text{ (See Exercise 23.)}\\ &= \sum_{n=0}^{\infty}(-1)^n 2x^{2n+1}\end{aligned}$$

Since $\dfrac{d}{dx}(\ln(x^2+1)) = \dfrac{2x}{x^2+1}$, we have

$$\ln(x^2+1) = \int\left[\sum_{n=0}^{\infty}(-1)^n 2x^{2n+1}\right]dx = C + \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{n+1},\ -1 \le x \le 1.$$

To solve for C, let $x = 0$ and conclude that $C = 0$. Therefore,

$$\ln(x^2+1) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{n+1},\ -1 \le x \le 1.$$

26. Since $\displaystyle\int \frac{1}{4x^2+1}\,dx = \frac{1}{2}\arctan(2x)$, we can use the result of Exercise 25 to obtain

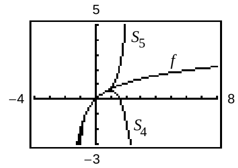
$$\arctan(2x) = 2\int \frac{1}{4x^2+1}\,dx = 2\int\left[\sum_{n=0}^{\infty}(-1)^n 4^n x^{2n}\right]dx = C + 2\sum_{n=0}^{\infty}\frac{(-1)^n 4^n x^{2n+1}}{2n+1},\ -\frac{1}{2} < x \le \frac{1}{2}.$$

To solve for C, let $x = 0$ and conclude that $C = 0$. Therefore,

$$\arctan(2x) = 2\sum_{n=0}^{\infty}\frac{(-1)^n 4^n x^{2n+1}}{2n+1},\ -\frac{1}{2} < x \le \frac{1}{2}.$$

28. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \le \ln(x+1)$

$$\le x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$$



x	0.0	0.2	0.4	0.6	0.8	1.0
$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$	0.0	0.18227	0.33493	0.45960	0.54827	0.58333
$\ln(x+1)$	0.0	0.18232	0.33647	0.47000	0.58779	0.69315
$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$	0.0	0.18233	0.33698	0.47515	0.61380	0.78333

In Exercise 35–38, $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

30. $g(x) = x - \frac{x^3}{3}$, cubic with 3 zeros.

Matches (d)

32. $g(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$,

Matches (b)

34. The approximations of degree 3, 7, 11, . . . $(4n - 1, n = 1, 2, \ldots)$ have relative extrema.

In Exercises 36 and 38, $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

36.

$$\arctan x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}$$

$$\int \arctan x^2 \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)} + C, C = 0$$

$$\int_0^{3/4} \arctan x^2 \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{(3/4)^{4n+3}}{(4n+3)(2n+1)}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{3^{4n+3}}{(4n+3)(2n+1)4^{4n+3}}$$

$$= \frac{27}{192} - \frac{2187}{344{,}064} + \frac{177{,}147}{230{,}686{,}720}$$

Since $177{,}147/230{,}686{,}720 < 0.001$, we can approximate the series by its first two terms: 0.13427

38.

$$x^2 \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{2n+1}$$

$$\int x^2 \arctan x \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+4}}{(2n+4)(2n+1)}$$

$$\int_0^{1/2} x^2 \arctan x \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+4)(2n+1)2^{2n+4}} = \frac{1}{64} - \frac{1}{1152} + \cdots$$

Since $\frac{1}{1152} < 0.001$, we can approximate the series by its first term: $\int_0^{1/2} x^2 \arctan x \, dx \approx 0.015625$.

In Exercises 40 and 42, $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n,\ |x| 1.$

40. Replace n with $n+1$.

$$\sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n$$

42. (a) $\dfrac{1}{3}\displaystyle\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n = \frac{2}{9}\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^{n-1} = \frac{2}{9}\,\frac{1}{[1-(2/3)]^2} = 2$

(b) $\dfrac{1}{10}\displaystyle\sum_{n=1}^{\infty} n\left(\frac{9}{10}\right)^n = \frac{9}{100}\sum_{n=1}^{\infty} n\left(\frac{9}{10}\right)^{n-1}$

$$= \frac{9}{100}\cdot\frac{1}{[1-(9/10)]^2} = 9$$

44. Replace x with x^2.

46. Integrate the series and multiply by (-1).

48. (a) From Exercise 47, we have

$$\arctan\frac{120}{119} - \arctan\frac{1}{239} = \arctan\frac{120}{119} + \arctan\left(-\frac{1}{239}\right)$$

$$= \arctan\left[\frac{(120/119) + (-1/239)}{1-(120/119)(-1/239)}\right] = \arctan\left(\frac{28{,}561}{28{,}561}\right) = \arctan 1 = \frac{\pi}{4}$$

(b) $2\arctan\dfrac{1}{5} = \arctan\dfrac{1}{5} + \arctan\dfrac{1}{5} = \arctan\left[\dfrac{2(1/5)}{1-(1/5)^2}\right] = \arctan\dfrac{10}{24} = \arctan\dfrac{5}{12}$

$$4\arctan\frac{1}{5} = 2\arctan\frac{1}{5} + 2\arctan\frac{1}{5} = \arctan\frac{5}{12} + \arctan\frac{5}{12} = \arctan\left[\frac{2(5/12)}{1-(5/12)^2}\right] = \arctan\frac{120}{119}$$

$$4\arctan\frac{1}{5} - \arctan\frac{1}{239} = \arctan\frac{120}{119} - \arctan\frac{1}{239} = \frac{\pi}{4} \text{ (see part (a).)}$$

50. (a) $\arctan\dfrac{1}{2} + \arctan\dfrac{1}{3} = \arctan\left[\dfrac{(1/2)+(1/3)}{1-(1/2)(1/3)}\right] = \arctan\left(\dfrac{5/6}{5/6}\right) = \dfrac{\pi}{4}$

(b) $\pi = 4\left[\arctan\dfrac{1}{2} + \arctan\dfrac{1}{3}\right]$

$$= 4\left[\frac{1}{2} - \frac{(1/2)^3}{3} + \frac{(1/2)^5}{5} - \frac{(1/2)^7}{7}\right] + 4\left[\frac{1}{3} - \frac{(1/3)^3}{3} + \frac{(1/3)^5}{5} - \frac{(1/3)^7}{7}\right]$$

$$\approx 4(0.4635) + 4(0.3217) \approx 3.14$$

52. From Exercise 51, we have

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{3^n n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(1/3)^n}{n}$$

$$= \ln\left(\frac{1}{3}+1\right) = \ln\frac{4}{3} \approx 0.2877.$$

54. From Example 5, we have $\arctan x = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$.

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1} = \sum_{n=0}^{\infty}(-1)^n\frac{(1)^{2n+1}}{2n+1}$$

$$= \arctan 1 = \frac{\pi}{4} \approx 0.7854$$

56. From Exercise 54, we have

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{3^{2n-1}(2n-1)} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{3^{2n+1}(2n+1)}$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{(1/3)^{2n+1}}{2n+1}$$

$$= \arctan\frac{1}{3} \approx 0.3218.$$

58. From Example 5, we have $\arctan x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$.

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n(2n+1)} &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(\sqrt{3})^{2n}(2n+1)}\cdot\frac{\sqrt{3}}{\sqrt{3}}\\ &= \sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n(1/\sqrt{3})^{2n+1}}{2n+1}\\ &= \sqrt{3}\arctan\frac{1}{\sqrt{3}}\\ &= \sqrt{3}\left(\frac{\pi}{6}\right) = \frac{\pi}{2\sqrt{3}}\end{aligned}$$

Section 8.10 Taylor and Maclaurin Series

2. For $c = 0$, we have

$$\begin{aligned}f(x) &= e^{3x}\\ f^{(n)}(x) &= 3^n e^{3x} \implies f^{(n)}(0) = 3^n\\ e^{3x} &= 1 + 3x + \frac{9x^2}{2!} + \frac{27x^3}{3!} + \cdots = \sum_{n=0}^{\infty}\frac{(3x)^n}{n!}\end{aligned}$$

4. For $c = \pi/4$, we have:

$$\begin{aligned}f(x) &= \sin x & f\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2}\\ f'(x) &= \cos x & f'\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2}\\ f''(x) &= -\sin x & f''\left(\frac{\pi}{4}\right) &= -\frac{\sqrt{2}}{2}\\ f'''(x) &= -\cos x & f'''\left(\frac{\pi}{4}\right) &= -\frac{\sqrt{2}}{2}\\ f^{(4)}(x) &= \sin x & f^{(4)}\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2}\end{aligned}$$

and so on. Therefore we have:

$$\begin{aligned}\sin x &= \sum_{n=0}^{\infty}\frac{f^{(n)}(\pi/4)[x-(\pi/4)]^n}{n!}\\ &= \frac{\sqrt{2}}{2}\left[1 + \left(x - \frac{\pi}{4}\right) - \frac{[x-(\pi/4)]^2}{2!} - \frac{[x-(\pi/4)]^3}{3!} + \frac{[x-(\pi/4)]^4}{4!} + \cdots\right]\\ &= \frac{\sqrt{2}}{2}\left\{\sum_{n=0}^{\infty}\frac{(-1)^{n(n+1)/2}[x-(\pi/4)]^{n+1}}{(n+1)!} + 1\right\}\end{aligned}$$

6. For $c = 1$, we have:

$$\begin{aligned}f(x) &= e^x\\ f^{(n)}(x) &= e^x \implies f^{(n)}(1) = e\\ e^x &= \sum_{n=0}^{\infty}\frac{f^{(n)}(1)(x-1)^n}{n!} = e\left[1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \frac{(x-1)^4}{4!} + \cdots\right] = e\sum_{n=0}^{\infty}\frac{(x-1)^n}{n!}\end{aligned}$$

8. For $c = 0$, we have:

$$\begin{aligned} f(x) &= \ln(x^2+1) & f(0) &= 0 \\ f'(x) &= \frac{2x}{x^2+1} & f'(0) &= 0 \\ f''(x) &= \frac{2-2x^2}{(x^2+1)^2} & f''(0) &= 2 \\ f'''(x) &= \frac{4x(x^2-3)}{(x^2+1)^3} & f'''(0) &= 0 \\ f^{(4)}(x) &= \frac{12(-x^4+6x^2-1)}{(x^2+1)^4} & f^{(4)}(0) &= -12 \\ f^{(5)}(x) &= \frac{48x(x^4-10x^2+5)}{(x^2+1)^5} & f^{(5)}(0) &= 0 \\ f^{(6)}(x) &= \frac{-240(5x^6-15x^4+15x^2-1)}{(x^2+1)^6} & f^{(6)}(0) &= 240 \end{aligned}$$

and so on. Therefore, we have:

$$\begin{aligned} \ln(x^2+1) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 0 + 0x + \frac{2x^2}{2!} + \frac{0x^3}{3!} - \frac{12x^4}{4!} + \frac{0x^5}{5!} + \frac{240x^6}{6!} + \cdots \\ &= x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1} \end{aligned}$$

10. For $c = 0$, we have;

$$\begin{aligned} f(x) &= \tan(x) & f(0) &= 0 \\ f'(x) &= \sec^2(x) & f'(0) &= 1 \\ f''(x) &= 2\sec^2(x)\tan(x) & f''(0) &= 0 \\ f'''(x) &= 2[\sec^4(x) + 2\sec^2(x)\tan^2(x)] & f'''(0) &= 2 \\ f^{(4)}(x) &= 8[\sec^4(x)\tan(x) + \sec^2(x)\tan^3(x)] & f^{(4)}(0) &= 0 \\ f^{(5)}(x) &= 8[2\sec^6(x) + 11\sec^4(x)\tan^2(x) + 2\sec^2(x)\tan^4(x)] & f^{(5)}(0) &= 16 \end{aligned}$$

$$\tan(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = x + \frac{2x^3}{3!} + \frac{16x^5}{5!} + \cdots = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots$$

12. The Maclaurin Series for $f(x) = e^{-2x}$ is $\displaystyle\sum_{n=0}^{\infty} \frac{(-2x)^n}{n!}$.

$f^{(n+1)}(x) = (-2)^{n+1}e^{-2x}$. Hence, by Taylor's Theorem,

$$0 \le |Rn(x)| = \left|\frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}\right| = \left|\frac{(-2)^{n+1}e^{-2z}}{(n+1)!}x^{n+1}\right|.$$

Since $\displaystyle\lim_{n\to\infty} \left|\frac{(-2)^{n+1}x^{n+1}}{(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{(2x)^{n+1}}{(n+1)!}\right| = 0$, it follows that $Rn(x) \to 0$ as $n \to \infty$.

Hence, the Maclaurin Series for e^{-2x} converges to e^{-2x} for all x.

14. Since $(1+x)^{-k} = 1 - kx + \frac{k(k+1)x^2}{2!} - \frac{k(k+1)(k+2)x^3}{3!} + \cdots$, we have

$$\left[1+(-x)\right]^{-1/2} = 1 + \left(\frac{1}{2}\right)x + \frac{(1/2)(3/2)x^2}{2!} + \frac{(1/2)(3/2)(5/2)x^3}{3!} + \cdots$$

$$= 1 + \frac{x}{2} + \frac{(1)(3)x^2}{2^2 2!} + \frac{(1)(3)(5)x^3}{2^3 3!} + \cdots$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)x^n}{2^n n!}$$

16. Since $(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \cdots$, we have

$$(1+x)^{1/3} = 1 + \left(\frac{1}{3}\right)x + \frac{(1/3)(-2/3)x^2}{2!} + \frac{(1/3)(-2/3)(-5/3)x^3}{3!} + \cdots$$

$$= 1 + \frac{x}{3} - \frac{2x^2}{3^2 2!} + \frac{2 \cdot 5x^3}{3^3 3!} - \frac{2 \cdot 5 \cdot 8x^4}{3^4 4!} + \cdots$$

$$= 1 + \frac{x}{3} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 2 \cdot 5 \cdot 8 \cdots (3n-4)}{3^n n!}.$$

18. Since $(1+x)^{1/2} = 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}\, 1 \cdot 3 \cdot 5 \cdots (2n-3)x^n}{2^n n!}$ (Exercise 14)

we have $(1+x^3)^{1/2} = 1 + \frac{x^3}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}\, 1 \cdot 3 \cdot 5 \cdots (2n-3)x^{3n}}{2^n n!}$.

20.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^{-3x} = \sum_{n=0}^{\infty} \frac{(-3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n x^n}{n!} = 1 - 3x + \frac{9x^2}{2!} - \frac{27x^3}{3!} + \frac{81x^4}{4!} - \frac{243x^5}{5!} + \cdots$$

22.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\cos 4x = \sum_{n=0}^{\infty} \frac{(-1)^n (4x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n} x^{2n}}{(2n)!}$$

$$= 1 - \frac{16x^2}{2!} + \frac{256x^4}{4!} - \cdots$$

24.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$2 \sin x^3 = 2\sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{(2n+1)!}$$

$$= 2\left[x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots\right]$$

$$= 2x^3 - \frac{2x^9}{3!} + \frac{2x^{15}}{5!} - \cdots$$

26.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$e^x + e^{-x} = 2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \cdots$$

$$2 \cos h(x) = e^x + e^{-x} = \sum_{n=0}^{\infty} 2\frac{x^{2n}}{(2n)!}$$

28. The formula for the binomial series gives $(1+x)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^n}{2^n n!}$, which implies that

$$(1+x^2)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^{2n}}{2^n n!}$$

$$\ln\left(x+\sqrt{x^2+1}\right) = \int \frac{1}{\sqrt{x^2+1}}\,dx$$

$$= x + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \ldots (2n-1)^{2n+1}}{2^n(2n+1)n!}$$

$$= x - \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} - \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots.$$

30. $x \cos x = x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$

$$= x - \frac{x^3}{2!} + \frac{x^5}{4!} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n)!}$$

32. $\frac{\arcsin x}{x} = \sum_{n=0}^{\infty} \frac{(2n)!x^{2n+1}}{(2^n n!)^2(2n+1)} \cdot \frac{1}{x}$

$$= \sum_{n=0}^{\infty} \frac{(2n)!x^{2n}}{(2^n n!)^2(2n+1)}, \; x \neq 0$$

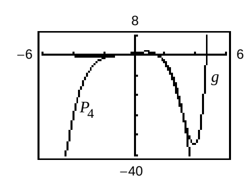
34. $e^{ix} + e^{-ix} = 2 - \frac{2x^2}{2!} + \frac{2x^4}{4!} - \frac{2x^6}{6!} + \cdots$ (See Exercise 33.)

$$\frac{e^{ix} + e^{-ix}}{2} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \cos(x)$$

36. $g(x) = e^x \cos x$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^4}{6} + \frac{x^4}{24} + \cdots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right)$$

$$= 1 + x + \left(\frac{x^2}{2} - \frac{x^2}{2}\right) + \left(\frac{x^3}{6} - \frac{x^3}{2}\right) + \left(\frac{x^4}{24} - \frac{x^4}{4} + \frac{x^4}{24}\right) + \cdots = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$$

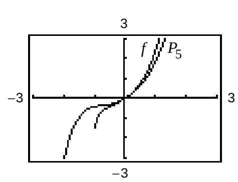


38. $f(x) = e^x \ln(1+x)$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right)$$

$$= x + \left(x^2 - \frac{x^2}{2}\right) + \left(\frac{x^3}{3} - \frac{x^3}{2} + \frac{x^3}{2}\right) + \left(-\frac{x^4}{4} + \frac{x^4}{3} - \frac{x^4}{4} + \frac{x^4}{6}\right) + \left(\frac{x^5}{5} - \frac{x^5}{4} + \frac{x^5}{6} - \frac{x^5}{12} + \frac{x^5}{24}\right) + \cdots$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{3x^5}{40} + \cdots$$



40. $f(x) = \dfrac{e^x}{1+x}$. Divide the series for e^x by $(1 + x)$.

$$\begin{array}{r} 1 + \dfrac{x^2}{2} - \dfrac{x^3}{3} + \dfrac{3x^4}{8} + \cdots \\ 1 + x \overline{\Big)\, 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24} + \dfrac{x^5}{120} + \cdots} \end{array}$$

$$\underline{1 + x}$$

$$0 + \frac{x^2}{2} + \frac{x^3}{6}$$

$$\underline{\frac{x^2}{2} + \frac{x^3}{2}}$$

$$-\frac{x^3}{3} + \frac{x^4}{24}$$

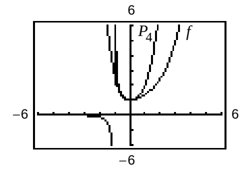
$$\underline{-\frac{x^3}{3} - \frac{x^4}{3}}$$

$$\frac{3x^4}{8} + \frac{x^5}{120}$$

$$\underline{\frac{3x^4}{8} + \frac{3x^5}{8}}$$

$$\vdots$$

$$f(x) = 1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{3x^4}{8} - \cdots$$



42. $y = x - \dfrac{x^3}{2!} + \dfrac{x^5}{4!} = x\left(1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!}\right) \approx x \cos x.$

Matches (b)

44. $y = x^2 - x^3 + x^4 = x^2(1 - x + x^2) \approx x^2\left(\dfrac{1}{1+x}\right).$

Matches (d)

46. $$\int_0^x \sqrt{1+t^3}\, dt = \int_0^x \left[1 + \frac{t^3}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3) t^{3n}}{2^n n!}\right] dt$$

$$= \left[t + \frac{t^4}{8} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3) t^{3n+1}}{(3n+1)2^n n!}\right]_0^x$$

$$= x + \frac{x^4}{8} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3) x^{3n+1}}{(3n+1)2^n n!}$$

48. Since $\sin(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$, we have

$\sin(1) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots \approx 0.8415.$ (4 terms)

50. Since $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$, we have $e^{-1} = 1 - 1 + \dfrac{1}{2!} - \dfrac{1}{3!} + \dfrac{1}{4!} - \dfrac{1}{5!} + \cdots$

and $\dfrac{e-1}{e} = 1 - e^{-1} = 1 - \dfrac{1}{2!} + \dfrac{1}{3!} - \dfrac{1}{4!} + \dfrac{1}{5!} - \dfrac{1}{7!} + \cdots = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \approx 0.6321.$ (6 terms)

52. Since

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$

we have $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x \to 0} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1.$

54. $\displaystyle\int_0^{1/2} \frac{\arctan x}{x}\,dx = \int_0^{1/2}\left(1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \cdots\right)dx = \left[x - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \frac{x^7}{7^2} + \cdots\right]_0^{1/2}$

Since $1/(9^2 2^9) < 0.0001$, we have

$$\int_0^{1/2} \frac{\arctan x}{x}\,dx \approx \left(\frac{1}{2} - \frac{1}{3^2 2^3} + \frac{1}{5^2 2^5} - \frac{1}{7^2 2^7} + \frac{1}{9^2 2^9}\right) \approx 0.4872.$$

Note: We are using $\displaystyle\lim_{x\to 0^+} \frac{\arctan x}{x} = 1$.

56. $\displaystyle\int_{0.5}^1 \cos\sqrt{x}\,dx = \int_{0.5}^1\left(1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} - \cdots\right)dx = \left[x - \frac{x^2}{2(2!)} + \frac{x^3}{3(4!)} - \frac{x^4}{4(6!)} + \frac{x^5}{5(8!)} - \cdots\right]_{0.5}^1$

Since $\frac{1}{210,600}(1 - 0.5^5) < 0.0001$, we have

$$\int_{0.5}^1 \cos\sqrt{x}\,dx \approx \left[(1 - 0.5) - \frac{1}{4}(1 - 0.5^2) + \frac{1}{72}(1 - 0.5^3) - \frac{1}{2880}(1 - 0.5^4) + \frac{1}{201,600}(1 - 0.5)^5\right] \approx 0.3243.$$

58. $\displaystyle\int_0^{1/4} x\ln(x+1)\,dx = \int_0^{1/4}\left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \cdots\right)dx$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4\cdot 2} + \frac{x^5}{5\cdot 3} - \frac{x^6}{6\cdot 4} + \cdots\right]_0^{1/4}$$

Since $\dfrac{(1/4)^5}{15} < 0.0001$,

$$\int_0^{1/4} x\ln(x+1)\,dx \approx \frac{(1/4)^3}{3} - \frac{(1/4)^4}{8} \approx 0.00472.$$

60. From Exercise 19, we have

$$\frac{1}{\sqrt{2\pi}}\int_1^2 e^{-x^2/2}dx = \frac{1}{\sqrt{2\pi}}\int_1^2 \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^n n!}\,dx = \frac{1}{\sqrt{2\pi}}\left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2^n n!(2n+1)}\right]_1^2$$

$$= \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n(2^{n+1}-1)}{2^n n!(2n+1)}$$

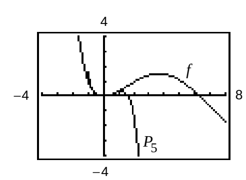
$$\approx \frac{1}{\sqrt{2\pi}}\left[1 - \frac{7}{2\cdot 1\cdot 3} + \frac{31}{2^2\cdot 2!\cdot 5} - \frac{127}{2^3\cdot 3!\cdot 7} + \frac{511}{2^4\cdot 4!\cdot 9} - \frac{2047}{2^5\cdot 5!\cdot 11}\right.$$

$$\left. + \frac{8191}{2^6\cdot 6!\cdot 13} - \frac{32,767}{2^7\cdot 7!\cdot 15} + \frac{131,071}{2^8\cdot 8!\cdot 17} - \frac{524,287}{2^9\cdot 9!\cdot 19}\right] \approx 0.1359.$$

62. $f(x) = \sin\dfrac{x}{2}\ln(1+x)$

$$P_5(x) = \frac{x^2}{2} - \frac{x^3}{4} + \frac{7x^4}{48} - \frac{11x^5}{96}$$

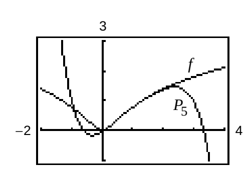
The polynomial is a reasonable approximation on the interval $(-0.60, 0.73)$.



64. $f(x) = \sqrt[3]{x}\cdot\arctan x,\ c = 1$

$$P_5(x) \approx 0.7854 + 0.7618(x-1) - 0.3412\left[\frac{(x-1)^2}{2!}\right] - 0.0424\left[\frac{(x-1)^3}{3!}\right]$$

$$+ 1.3025\left[\frac{(x-1)^4}{4!}\right] - 5.5913\left[\frac{(x-1)^5}{5!}\right]$$



The polynomial is a reasonable approximation on the interval $(0.48, 1.75)$.

66. $a_{2n+1} = 0$ (odd coefficients are zero)

68. Answers will vary.

70. $\theta = 60°, v_0 = 64, k = \dfrac{1}{16}, g = -32$

$$y = \sqrt{3}x - \frac{32x^2}{2(64)^2(1/2)^2} - \frac{(1/16)(32)x^3}{3(64)^3(1/2)^3} - \frac{(1/16)^2(32)x^4}{4(64)^4(1/2)^4} - \cdots$$

$$= \sqrt{3}x - 32\left[\frac{2^2x^2}{2(64)^2} + \frac{2^3x^3}{3(64)^3 16} + \frac{2^4x^4}{4(64)^4(16)^2} + \cdots\right]$$

$$= \sqrt{3}x - 32\sum_{n=2}^{\infty}\frac{2^n x^n}{n(64)^n(16)^{n-2}} = \sqrt{3}x - 32\sum_{n=2}^{\infty}\frac{x^n}{n(32)^n(16)^{n-2}}$$

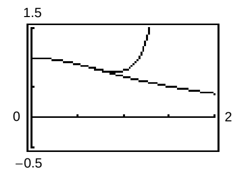
72. (a) $f(x) = \dfrac{\ln(x^2+1)}{x^2}$.

From Exercise 8, you obtain

$$P = \frac{1}{x^2}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n+1}$$

$$P_8 = 1 - \frac{x^2}{2} + \frac{x^4}{3} - \frac{x^6}{4} + \frac{x^8}{5}$$

(b)



(c) $F(x) = \displaystyle\int_0^x \frac{\ln(t^2+1)}{t^2}\,dt$

$G(x) = \displaystyle\int_0^x P_8(t)\,dt$

x	0.25	0.50	0.75	1.00	1.50	2.00
$F(x)$	0.2475	0.4810	0.6920	0.8776	1.1798	1.4096
$G(x)$	0.2475	0.4810	0.6920	0.8805	5.3064	652.21

(d) The curves are nearly identical for $0 < x < 1$. Hence, the integrals nearly agree on that interval.

74. Assume $e = p/q$ is rational. Let $N > q$ and form the following.

$$e - \left[1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{N!}\right] = \frac{1}{(N+1)!} + \frac{1}{(N+2)!} + \cdots$$

Set $a = N!\left[e - \left(1 + 1 + \ldots + \dfrac{1}{N!}\right)\right]$, a positive integer. But,

$$a = N!\left[\frac{1}{(N+1)!} + \frac{1}{(N+2)!} + \cdots\right] = \frac{1}{N+1} + \frac{1}{(N+1)(N+2)} + \cdots < \frac{1}{N+1} + \frac{1}{(N+1)^2} + \cdots$$

$$= \frac{1}{N+1}\left[1 + \frac{1}{N+1} + \frac{1}{(N+1)^2} + \ldots\right] = \frac{1}{N+1}\left[\frac{1}{1 - \left(\frac{1}{N+1}\right)}\right] = \frac{1}{N}, \text{ a contradiction.}$$

Review Exercises for Chapter 8

2. $a_n = \dfrac{n}{n^2+1}$

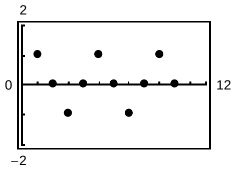
4. $a_n = 4 - \dfrac{n}{2}$: 3.5, 3, . . .

Matches (c)

6. $a_n = 6\left(-\dfrac{2}{3}\right)^{n-1}$: 6, −4, . . .

Matches (b)

8. $a_n = \sin\dfrac{n\pi}{2}$



The sequence seems to diverge (oscillates).

$$\sin\frac{n\pi}{2}:\ 1, 0, -1, 0, 1, 0, \ldots$$

10. $\lim_{n\to\infty}\dfrac{1}{\sqrt{n}} = 0$

Converges

12. $\lim_{n\to\infty}\dfrac{n}{\ln(n)} = \lim_{n\to\infty}\dfrac{1}{1/n} = \infty$

Diverges

14. $\lim_{n\to\infty}\left(1+\dfrac{1}{2n}\right)^n = \lim_{k\to\infty}\left[\left(1+\dfrac{1}{k}\right)^k\right]^{1/2} = e^{1/2}$

Converges; $k = 2n$

16. Let $\quad y = (b^n + c^n)^{1/n}$

$$\ln y = \frac{\ln(b^n + c^n)}{n}$$

$$\lim_{n\to\infty}\ln y = \lim_{n\to\infty}\frac{1}{b^n + c^n}(b^n\ln b + c^n\ln c)$$

Assume $b \geq c$ and note that the terms

$$\frac{b^n\ln b + c^n\ln c}{b^n + c^n} = \frac{b^n\ln b}{b^n + c^n} + \frac{c^n\ln c}{b^n + c^n}$$

converge as $n\to\infty$. Hence a_n converges.

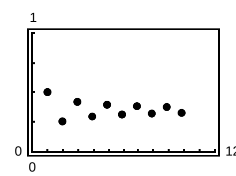
18. (a) $V_n = 120{,}000(0.70)^n$, $n = 1, 2, 3, 4, 5$

(b) $V_5 = 120{,}000(0.70)^5 = \$20{,}168.40$

20. (a)

k	5	10	15	20	25
S_k	0.3917	0.3228	0.3627	0.3344	0.3564

(b)

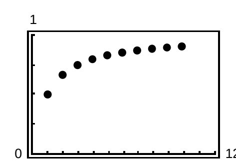


(c) The series converges by the Alternating Series Test.

22. (a)

k	5	10	15	20	25
S_k	0.8333	0.9091	0.9375	0.9524	0.9615

(b)



(c) The series converges, by the limit comparison test with $\sum \dfrac{1}{n^2}$.

24. Diverges. Geometric series, $r = 1.82 > 1$.

26. Diverges. nth Term Test, $\lim_{n\to\infty} a_n = \dfrac{2}{3}$.

28. $\displaystyle\sum_{n=0}^{\infty}\frac{2^{n+2}}{3^n} = 4\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = 4(3) = 12$

See Exercise 27.

30.
$$\sum_{n=0}^{\infty}\left[\left(\frac{2}{3}\right)^n - \frac{1}{(n+1)(n+2)}\right] = \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n - \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$
$$= \frac{1}{1-(2/3)} - \left[\left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots\right] = 3 - 1 = 2$$

32. $0.\overline{923076} = 0.923076[1 + 0.000001 + (0.000001)^2 + \cdots]$

$$= \sum_{n=0}^{\infty} (0.923076)(0.000001)^n = \frac{0.923076}{1 - 0.000001} = \frac{923{,}076}{999{,}999} = \frac{12(76{,}923)}{13(76{,}923)} = \frac{12}{13}$$

34. $S = \sum_{n=0}^{39} 32{,}000(1.055)^n = \frac{32{,}000(1 - 1.055^{40})}{1 - 1.055}$

$\approx \$4{,}371{,}379.65$

36. See Exercise 86 in Section 8.2.

$$A = P\left(\frac{12}{r}\right)\left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right]$$

$$= 100\left(\frac{12}{0.065}\right)\left[\left(1 + \frac{0.065}{12}\right)^{120} - 1\right]$$

$$\approx \$16{,}840.32$$

38. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/4}}$

Divergent p-series, $p = \frac{3}{4} < 1$

40. $\sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{2^n}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{2^n}$

The first series is a convergent p-series and the second series is a convergent geometric series. Therefore, their difference converges.

42. $\sum_{n=1}^{\infty} \frac{n + 1}{n(n + 2)}$

$$\lim_{n\to\infty} \frac{(n + 1)/n(n + 2)}{1/n} = \lim_{n\to\infty} \frac{n + 1}{n + 2} = 1$$

By a limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$, the series diverges.

44. Since $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges, $\sum_{n=1}^{\infty} \frac{1}{3^n - 5}$ converges by the Limit Comparison Test.

46. $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n + 1}$

$$a_{n+1} = \frac{\sqrt{n + 1}}{n + 2} \le \frac{\sqrt{n}}{n + 1} = a_n$$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n + 1} = 0$$

By the Alternating Series Test, the series converges.

48. Converges by the Alternating Series Test.

$$a_{n+1} = \frac{3\ln(n + 1)}{n + 1} < \frac{3\ln n}{n} = a_n, \ \lim_{n\to\infty} \frac{3\ln n}{n} = 0$$

50. $\sum_{n=1}^{\infty} \frac{n!}{e^n}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n + 1)!}{e^{n+1}} \cdot \frac{e^n}{n!}\right|$$

$$= \lim_{n\to\infty} \frac{n + 1}{e} = \infty$$

By the Ratio Test, the series diverges.

52. $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2 \cdot 5 \cdot 8 \cdots (3n - 1)}$

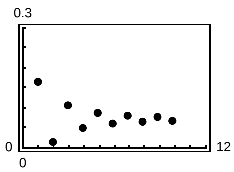
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1 \cdot 3 \cdots (2n - 1)(2n + 1)}{2 \cdot 5 \cdots (3n - 1)(3n + 2)} \cdot \frac{2 \cdot 5 \cdots (3n - 1)}{1 \cdot 3 \cdots (2n - 1)}\right| = \lim_{n\to\infty} \frac{2n + 1}{3n + 2} = \frac{2}{3}$$

By the Ratio Test, the series converges.

54. (a) The series converges by the Alternating Series Test.

(b)

x	5	10	15	20	25
S_n	0.0871	0.0669	0.0734	0.0702	0.0721

(c) 

(d) The sum is approximately 0.0714.

56. No. Let $a_n = \dfrac{3937.5}{n^2}$, then $a_{75} = 0.7$. The series $\sum_{n=1}^{\infty} \dfrac{3937.5}{n^2}$ is a convergent p-series.

58.

$f(x) = \tan x \qquad f\left(-\dfrac{\pi}{4}\right) = -1$

$f'(x) = \sec^2 x \qquad f'\left(-\dfrac{\pi}{4}\right) = 2$

$f''(x) = 2\sec^2 x \tan x \qquad f''\left(-\dfrac{\pi}{4}\right) = -4$

$f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x \qquad f'''\left(-\dfrac{\pi}{4}\right) = 16$

$$P_3(x) = -1 + 2\left(x + \frac{\pi}{4}\right) - 2\left(x + \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x + \frac{\pi}{4}\right)^3$$

60. $\cos(0.75) \approx 1 - \dfrac{(0.75)^2}{2!} + \dfrac{(0.75)^4}{4!} - \dfrac{(0.75)^6}{6!} \approx 0.7317$

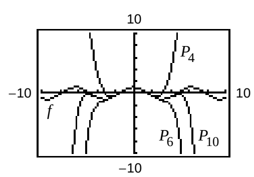
62. $e^{-0.25} \approx 1 - 0.25 + \dfrac{(0.25)^2}{2!} - \dfrac{(0.25)^3}{3!} + \dfrac{(0.25)^4}{4!} \approx 0.779$

64. $f(x) = \cos x$

$P_4(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!}$

$P_6(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!}$

$P_{10}(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dfrac{x^8}{8!} - \dfrac{x^{10}}{10!}$



66. $\sum_{n=0}^{\infty} (2x)^n$

Geometric series which converges only if $|2x| < 1$ or $-\frac{1}{2} < x < \frac{1}{2}$.

68. $\sum_{n=1}^{\infty} \dfrac{3^n (x-2)^n}{n}$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{3^{n+1}(x-2)^{n+1}}{n+1} \cdot \frac{n}{3^n (x-2)^n}\right|$$
$$= 3|x-2|$$

$R = \dfrac{1}{3}$

Center: 2

Since the series converges at $\frac{5}{3}$ and diverges at $\frac{7}{3}$, the interval of convergence is $\frac{5}{3} \le x < \frac{7}{3}$.

70. $\sum_{n=0}^{\infty} \dfrac{(x-2)^n}{2^n} = \sum_{n=0}^{\infty} \left(\dfrac{x-2}{2}\right)^n$

Geometric series which converges only if

$$\left|\frac{x-2}{2}\right| < 1 \quad \text{or} \quad 0 < x < 4.$$

72.

$$y = \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n}}{2^n n!}$$

$$y' = \sum_{n=1}^{\infty} \frac{(-3)^n (2n) x^{2n-1}}{2^n n!} = \sum_{n=0}^{\infty} \frac{(-3)^{n+1}(2n+2)x^{2n+1}}{2^{n+1}(n+1)!}$$

$$y'' = \sum_{n=0}^{\infty} \frac{(-3)^{n+1}(2n+2)(2n+1)x^{2n}}{2^{n+1}(n+1)!}$$

$$y'' + 3xy' + 3y = \sum_{n=0}^{\infty} \frac{(-3)^{n+1}(2n+2)(2n+1)x^{2n}}{2^{n+1}(n+1)!} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}3^{n+2}(2n+2)x^{2n+2}}{2^{n+1}(n+1)!} + \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}3^{n+1}(2n+2)x^{2n}}{2^n n!} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}3^{n+2}x^{2n+2}}{2^n n!} + \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1}x^{2n}}{2^n n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!}[-(2n+1)+1] + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}3^{n+2}x^{2n+2}}{2^n n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!}(-2n) + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}3^{n+2}x^{2n+2}}{2^n n!}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3^{n+1} x^{2n}}{2^n n!}(2n) + \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n+1}x^{2n}}{2^{n-1}(n-1)!} \cdot \frac{2n}{2n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!}[-2n + 2n] = 0$$

74. $\dfrac{3}{2+x} = \dfrac{3/2}{1+(x/2)} = \dfrac{3/2}{1-(-x/2)} = \dfrac{a}{1-r}$

$$\sum_{n=0}^{\infty} \frac{3}{2}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n 3x^n}{2^{n+1}}$$

76. Integral: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 3x^{n+1}}{(n+1)2^{n+1}}$

78. $$8 - 2(x-3) + \frac{1}{2}(x-3)^2 - \frac{1}{8}(x-3)^3 + \cdots = \sum_{n=0}^{\infty} 8\left[\frac{-(x-3)}{4}\right]^n = \frac{8}{1-[-(x-3)/4]}$$

$$= \frac{32}{4+(x-3)} = \frac{32}{1+x}, \quad -1 < x < 7$$

80.

$$f(x) = \cos x$$
$$f'(x) = -\sin x$$
$$f''(x) = -\cos x$$
$$f'''(x) = \sin x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{f^{(n)}(-\pi/4)[x+(\pi/4)]^n}{n!} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x+\frac{\pi}{4}\right) - \frac{\sqrt{2}}{2 \cdot 2!}\left(x+\frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{2 \cdot 3!}\left(x+\frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{2 \cdot 4!}\left(x+\frac{\pi}{4}\right)^4 + \cdots$$

$$= \frac{\sqrt{2}}{2}\left[1 + \left(x+\frac{\pi}{4}\right) + \sum_{n=1}^{\infty} \frac{(-1)^{[n(n+1)]/2}[x+(\pi/4)]^{n+1}}{(n+1)!}\right]$$

82.

$$f(x) = \csc(x)$$
$$f'(x) = -\csc(x)\cot(x)$$
$$f''(x) = \csc^3(x) + \csc(x)\cot^2(x)$$
$$f'''(x) = -5\csc^3(x)\cot(x) - \csc(x)\cot^3(x)$$
$$f^{(4)}(x) = 5\csc^5(x) + 15\csc^3(x)\cot^2(x) + \csc(x)\cot^4(x)$$

$$\csc(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/2)[x-(\pi/2)]^n}{n!} = 1 + \frac{1}{2!}\left(x-\frac{\pi}{2}\right)^2 + \frac{5}{4!}\left(x-\frac{\pi}{2}\right)^4 + \cdots$$

84. $f(x) = x^{1/2}$

$$f'(x) = \frac{1}{2}x^{-1/2}$$

$$f''(x) = -\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)x^{-3/2}$$

$$f'''(x) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)x^{-5/2}$$

$$f^{(4)}(x) = -\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)x^{-7/2}, \cdots$$

$$\sqrt{x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(4)(x-4)^n}{n!}$$

$$= 2 + \frac{(x-4)}{2^2} - \frac{(x-4)^2}{2^5 2!} + \frac{1 \cdot 3(x-4)^3}{2^8 3!} - \frac{1 \cdot 3 \cdot 5(x-4)^4}{2^{11} 4!} + \cdots$$

$$= 2 + \frac{(x-4)}{2^2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots (2n-3)(x-4)^n}{2^{3n-1} n!}$$

86. $h(x) = (1+x)^{-3}$

$$h'(x) = -3(1+x)^{-4}$$

$$h''(x) = 12(1+x)^{-5}$$

$$h'''(x) = -60(1+x)^{-6}$$

$$h^{(4)}(x) = 360(1+x)^{-7}$$

$$h^{(5)}(x) = -2520(1+x)^{-8}$$

$$\frac{1}{(1+x)^3} = 1 - 3x + \frac{12x^2}{2!} - \frac{60x^3}{3!} + \frac{360x^4}{4!} - \frac{2520x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)! x^n}{2n!} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)(n+1)x}{2}$$

88. $\ln x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}, \quad 0 < x \le 2$

$$\ln\left(\frac{6}{5}\right) = \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{(6/5)-1}{n}\right)^n$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{5^n n} \approx 0.1823$$

90. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty$

$$e^{2/3} = \sum_{n=0}^{\infty} \frac{(2/3)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{3^n n!} \approx 1.9477$$

92. $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty$

$$\sin\left(\frac{1}{3}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{2n+1}(2n+1)!} \approx 0.3272$$

94. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots$

$$xe^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = x + x^2 + \frac{x^3}{2!} + \cdots$$

$$\int_0^1 xe^x\,dx = \Big[xe^x - e^x\Big]_0^1 = (e-e) - (0-1) = 1$$

$$\int_0^1 \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}\,dx = \sum_{n=0}^{\infty} \left[\frac{x^{n+2}}{(n+2)n!}\right]_0^1 = \sum_{n=0}^{\infty} \frac{1}{(n+2)n!} = 1$$

96. (a)

$$\begin{aligned} f(x) &= \sin 2x & f(0) &= 0 \\ f'(x) &= 2\cos 2x & f'(0) &= 2 \\ f''(x) &= -4\sin 2x & f''(0) &= 0 \\ f'''(x) &= -8\cos 2x & f'''(0) &= -8 \\ f^{(4)}(x) &= 16\sin 2x & f^{(4)}(0) &= 0 \\ f^{(5)}(x) &= 32\cos 2x & f^{(5)}(0) &= 32 \\ f^{(6)}(x) &= -64\sin 2x & f^{(6)}(0) &= 0 \\ f^{(7)}(x) &= -128\cos 2x & f^{(7)}(0) &= -128 \end{aligned}$$

$$\sin 2x = 0 + 2x + \frac{0x^2}{2!} - \frac{8x^3}{3!} + \frac{0x^4}{4!} + \frac{32x^5}{5!} + \frac{0x^6}{6!} - \frac{128x^7}{7!} + \cdots = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots$$

(b) $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$\sin 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots$$

$$= 2x - \frac{8x^3}{6} + \frac{32x^5}{120} - \frac{128x^7}{5040} + \cdots = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots$$

(c) $\sin 2x = 2\sin x\cos x$

$$= 2\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right)$$

$$= 2\left[x + \left(-\frac{x^3}{2} - \frac{x^3}{6}\right) + \left(\frac{x^5}{24} + \frac{x^5}{12} + \frac{x^5}{120}\right) + \left(-\frac{x^7}{720} - \frac{x^7}{144} - \frac{x^7}{240} - \frac{x^7}{5040}\right) + \cdots\right]$$

$$= 2\left[x - \frac{2x^3}{3} + \frac{2x^5}{15} - \frac{4x^7}{315} + \cdots\right] = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots$$

98.

$$\cos t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!}$$

$$\cos\frac{\sqrt{t}}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{2^{2n}(2n)!}$$

$$\int_0^x \cos\frac{\sqrt{t}}{2}\,dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{2^{2n}(2n)!(n+1)}\right]_0^x$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{2^{2n}(2n)!(n+1)}$$

100.

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$$

$$e^t - 1 = \sum_{n=1}^{\infty} \frac{t^n}{n!}$$

$$\frac{e^t - 1}{t} = \sum_{n=1}^{\infty} \frac{t^{n-1}}{n!}$$

$$\int_0^x \frac{e^t - 1}{t}\,dt = \left[\sum_{n=1}^{\infty} \frac{t^n}{n\cdot n!}\right]_0^x = \sum_{n=1}^{\infty} \frac{x^n}{n\cdot n!}$$

102.

$$\arcsin x = x + \frac{x^3}{2\cdot 3} + \frac{1\cdot 3x^5}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7} + \cdots$$

$$\frac{\arcsin x}{x} = 1 + \frac{x^2}{2\cdot 3} + \frac{1\cdot 3x^4}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5x^6}{2\cdot 4\cdot 6\cdot 7} + \cdots$$

$$\lim_{x\to 0} \frac{\arcsin x}{x} = 1$$

By L'Hôpital's Rule, $\lim_{x\to 0} \frac{\arcsin x}{x} = \lim_{x\to 0} \frac{\left(\frac{1}{\sqrt{1-x^2}}\right)}{1} = 1.$