66. $a_{2n+1} = 0$ (odd coefficients are zero)

68. Answers will vary.

70. $\theta = 60°, v_0 = 64, k = \dfrac{1}{16}, g = -32$

$$y = \sqrt{3}x - \frac{32x^2}{2(64)^2(1/2)^2} - \frac{(1/16)(32)x^3}{3(64)^3(1/2)^3} - \frac{(1/16)^2(32)x^4}{4(64)^4(1/2)^4} - \cdots$$

$$= \sqrt{3}x - 32\left[\frac{2^2x^2}{2(64)^2} + \frac{2^3x^3}{3(64)^3 16} + \frac{2^4x^4}{4(64)^4(16)^2} + \cdots\right]$$

$$= \sqrt{3}x - 32\sum_{n=2}^{\infty}\frac{2^n x^n}{n(64)^n(16)^{n-2}} = \sqrt{3}x - 32\sum_{n=2}^{\infty}\frac{x^n}{n(32)^n(16)^{n-2}}$$

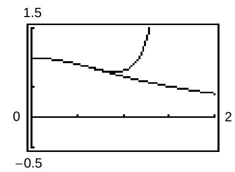
72. (a) $f(x) = \dfrac{\ln(x^2+1)}{x^2}$.

From Exercise 8, you obtain

$$P = \frac{1}{x^2}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n+1}$$

$$P_8 = 1 - \frac{x^2}{2} + \frac{x^4}{3} - \frac{x^6}{4} + \frac{x^8}{5}$$

(b)



(c) $F(x) = \displaystyle\int_0^x \frac{\ln(t^2+1)}{t^2}\,dt$

$G(x) = \displaystyle\int_0^x P_8(t)\,dt$

x	0.25	0.50	0.75	1.00	1.50	2.00
$F(x)$	0.2475	0.4810	0.6920	0.8776	1.1798	1.4096
$G(x)$	0.2475	0.4810	0.6920	0.8805	5.3064	652.21

(d) The curves are nearly identical for $0 < x < 1$. Hence, the integrals nearly agree on that interval.

74. Assume $e = p/q$ is rational. Let $N > q$ and form the following.

$$e - \left[1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{N!}\right] = \frac{1}{(N+1)!} + \frac{1}{(N+2)!} + \cdots$$

Set $a = N!\left[e - \left(1 + 1 + \ldots + \dfrac{1}{N!}\right)\right]$, a positive integer. But,

$$a = N!\left[\frac{1}{(N+1)!} + \frac{1}{(N+2)!} + \cdots\right] = \frac{1}{N+1} + \frac{1}{(N+1)(N+2)} + \cdots < \frac{1}{N+1} + \frac{1}{(N+1)^2} + \cdots$$

$$= \frac{1}{N+1}\left[1 + \frac{1}{N+1} + \frac{1}{(N+1)^2} + \ldots\right] = \frac{1}{N+1}\left[\frac{1}{1 - \left(\frac{1}{N+1}\right)}\right] = \frac{1}{N}, \text{ a contradiction.}$$

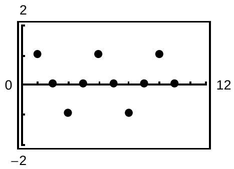
Review Exercises for Chapter 8

2. $a_n = \dfrac{n}{n^2+1}$

4. $a_n = 4 - \dfrac{n}{2}$: 3.5, 3, . . .
Matches (c)

6. $a_n = 6\left(-\dfrac{2}{3}\right)^{n-1}$: 6, −4, . . .
Matches (b)

8. $a_n = \sin \dfrac{n\pi}{2}$



The sequence seems to diverge (oscillates).

$$\sin \frac{n\pi}{2}: \ 1, 0, -1, 0, 1, 0, \ldots$$

10. $\lim_{n\to\infty} \dfrac{1}{\sqrt{n}} = 0$

Converges

12. $\lim_{n\to\infty} \dfrac{n}{\ln(n)} = \lim_{n\to\infty} \dfrac{1}{1/n} = \infty$

Diverges

14. $\lim_{n\to\infty} \left(1 + \dfrac{1}{2n}\right)^n = \lim_{k\to\infty} \left[\left(1 + \dfrac{1}{k}\right)^k\right]^{1/2} = e^{1/2}$

Converges; $k = 2n$

16. Let $\quad y = (b^n + c^n)^{1/n}$

$$\ln y = \frac{\ln(b^n + c^n)}{n}$$

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{1}{b^n + c^n}(b^n \ln b + c^n \ln c)$$

Assume $b \geq c$ and note that the terms

$$\frac{b^n \ln b + c^n \ln c}{b^n + c^n} = \frac{b^n \ln b}{b^n + c^n} + \frac{c^n \ln c}{b^n + c^n}$$

converge as $n \to \infty$. Hence a_n converges.

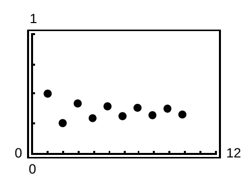
18. (a) $V_n = 120{,}000(0.70)^n$, $n = 1, 2, 3, 4, 5$

(b) $V_5 = 120{,}000(0.70)^5 = \$20{,}168.40$

20. (a)

k	5	10	15	20	25
S_k	0.3917	0.3228	0.3627	0.3344	0.3564

(b)

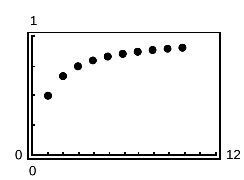


(c) The series converges by the Alternating Series Test.

22. (a)

k	5	10	15	20	25
S_k	0.8333	0.9091	0.9375	0.9524	0.9615

(b)



(c) The series converges, by the limit comparison test with $\sum \dfrac{1}{n^2}$.

24. Diverges. Geometric series, $r = 1.82 > 1$.

26. Diverges. nth Term Test, $\lim_{n\to\infty} a_n = \dfrac{2}{3}$.

28. $\displaystyle\sum_{n=0}^{\infty} \frac{2^{n+2}}{3^n} = 4\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = 4(3) = 12$

See Exercise 27.

30.
$$\sum_{n=0}^{\infty}\left[\left(\frac{2}{3}\right)^n - \frac{1}{(n+1)(n+2)}\right] = \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n - \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$
$$= \frac{1}{1-(2/3)} - \left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots\right] = 3 - 1 = 2$$

32. $0.\overline{923076} = 0.923076[1 + 0.000001 + (0.000001)^2 + \cdots]$

$$= \sum_{n=0}^{\infty} (0.923076)(0.000001)^n = \frac{0.923076}{1 - 0.000001} = \frac{923{,}076}{999{,}999} = \frac{12(76{,}923)}{13(76{,}923)} = \frac{12}{13}$$

34. $S = \sum_{n=0}^{39} 32{,}000(1.055)^n = \frac{32{,}000(1 - 1.055^{40})}{1 - 1.055}$

$\approx \$4{,}371{,}379.65$

36. See Exercise 86 in Section 8.2.

$$A = P\left(\frac{12}{r}\right)\left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right]$$

$$= 100\left(\frac{12}{0.065}\right)\left[\left(1 + \frac{0.065}{12}\right)^{120} - 1\right]$$

$\approx \$16{,}840.32$

38. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/4}}$

Divergent p-series, $p = \frac{3}{4} < 1$

40. $\sum_{n=1}^{\infty}\left(\frac{1}{n^2} - \frac{1}{2^n}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{2^n}$

The first series is a convergent p-series and the second series is a convergent geometric series. Therefore, their difference converges.

42. $\sum_{n=1}^{\infty} \frac{n + 1}{n(n + 2)}$

$$\lim_{n\to\infty} \frac{(n + 1)/n(n + 2)}{1/n} = \lim_{n\to\infty} \frac{n + 1}{n + 2} = 1$$

By a limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$, the series diverges.

44. Since $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges, $\sum_{n=1}^{\infty} \frac{1}{3^n - 5}$ converges by the Limit Comparison Test.

46. $\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n}}{n + 1}$

$$a_{n+1} = \frac{\sqrt{n + 1}}{n + 2} \leq \frac{\sqrt{n}}{n + 1} = a_n$$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n + 1} = 0$$

By the Alternating Series Test, the series converges.

48. Converges by the Alternating Series Test.

$$a_{n+1} = \frac{3\ln(n + 1)}{n + 1} < \frac{3\ln n}{n} = a_n, \lim_{n\to\infty} \frac{3\ln n}{n} = 0$$

50. $\sum_{n=1}^{\infty} \frac{n!}{e^n}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n + 1)!}{e^{n+1}} \cdot \frac{e^n}{n!}\right|$$

$$= \lim_{n\to\infty} \frac{n + 1}{e} = \infty$$

By the Ratio Test, the series diverges.

52. $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2 \cdot 5 \cdot 8 \cdots (3n - 1)}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1 \cdot 3 \cdots (2n - 1)(2n + 1)}{2 \cdot 5 \cdots (3n - 1)(3n + 2)} \cdot \frac{2 \cdot 5 \cdots (3n - 1)}{1 \cdot 3 \cdots (2n - 1)}\right| = \lim_{n\to\infty} \frac{2n + 1}{3n + 2} = \frac{2}{3}$$

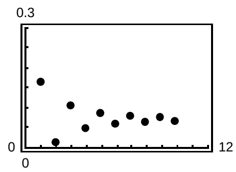
By the Ratio Test, the series converges.

54. (a) The series converges by the Alternating Series Test.

(b)

x	5	10	15	20	25
S_n	0.0871	0.0669	0.0734	0.0702	0.0721

(c)



(d) The sum is approximately 0.0714.

56. No. Let $a_n = \dfrac{3937.5}{n^2}$, then $a_{75} = 0.7$. The series $\displaystyle\sum_{n=1}^{\infty} \frac{3937.5}{n^2}$ is a convergent p-series.

58. $f(x) = \tan x \qquad f\left(-\dfrac{\pi}{4}\right) = -1$

$f'(x) = \sec^2 x \qquad f'\left(-\dfrac{\pi}{4}\right) = 2$

$f''(x) = 2\sec^2 x \tan x \qquad f''\left(-\dfrac{\pi}{4}\right) = -4$

$f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x \qquad f'''\left(-\dfrac{\pi}{4}\right) = 16$

$P_3(x) = -1 + 2\left(x + \dfrac{\pi}{4}\right) - 2\left(x + \dfrac{\pi}{4}\right)^2 + \dfrac{8}{3}\left(x + \dfrac{\pi}{4}\right)^3$

60. $\cos(0.75) \approx 1 - \dfrac{(0.75)^2}{2!} + \dfrac{(0.75)^4}{4!} - \dfrac{(0.75)^6}{6!} \approx 0.7317$

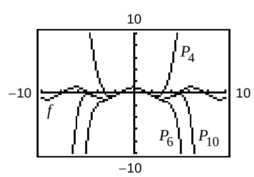
62. $e^{-0.25} \approx 1 - 0.25 + \dfrac{(0.25)^2}{2!} - \dfrac{(0.25)^3}{3!} + \dfrac{(0.25)^4}{4!} \approx 0.779$

64. $f(x) = \cos x$

$P_4(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!}$

$P_6(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!}$

$P_{10}(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dfrac{x^8}{8!} - \dfrac{x^{10}}{10!}$



66. $\displaystyle\sum_{n=0}^{\infty} (2x)^n$

Geometric series which converges only if $|2x| < 1$ or $-\frac{1}{2} < x < \frac{1}{2}$.

68. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n(x-2)^n}{n}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}(x-2)^{n+1}}{n+1}\cdot\frac{n}{3^n(x-2)^n}\right|$$
$$= 3|x-2|$$

$R = \dfrac{1}{3}$

Center: 2

Since the series converges at $\frac{5}{3}$ and diverges at $\frac{7}{3}$, the interval of convergence is $\frac{5}{3} \le x < \frac{7}{3}$.

70. $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{2^n} = \sum_{n=0}^{\infty}\left(\frac{x-2}{2}\right)^n$

Geometric series which converges only if

$\left|\dfrac{x-2}{2}\right| < 1$ or $0 < x < 4$.

72.

$$y = \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n}}{2^n n!}$$

$$y' = \sum_{n=1}^{\infty} \frac{(-3)^n (2n) x^{2n-1}}{2^n n!} = \sum_{n=0}^{\infty} \frac{(-3)^{n+1}(2n+2)x^{2n+1}}{2^{n+1}(n+1)!}$$

$$y'' = \sum_{n=0}^{\infty} \frac{(-3)^{n+1}(2n+2)(2n+1)x^{2n}}{2^{n+1}(n+1)!}$$

$$y'' + 3xy' + 3y = \sum_{n=0}^{\infty} \frac{(-3)^{n+1}(2n+2)(2n+1)x^{2n}}{2^{n+1}(n+1)!} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}3^{n+2}(2n+2)x^{2n+2}}{2^{n+1}(n+1)!} + \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}3^{n+1}(2n+2)x^{2n}}{2^n n!} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}3^{n+2}x^{2n+2}}{2^n n!} + \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1}x^{2n}}{2^n n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1}x^{2n}}{2^n n!}[-(2n+1)+1] + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}3^{n+2}x^{2n+2}}{2^n n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1}x^{2n}}{2^n n!}(-2n) + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}3^{n+2}x^{2n+2}}{2^n n!}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}3^{n+1}x^{2n}}{2^n n!}(2n) + \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n+1}x^{2n}}{2^{n-1}(n-1)!} \cdot \frac{2n}{2n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n+1}x^{2n}}{2^n n!}[-2n+2n] = 0$$

74. $\dfrac{3}{2+x} = \dfrac{3/2}{1+(x/2)} = \dfrac{3/2}{1-(-x/2)} = \dfrac{a}{1-r}$

$$\sum_{n=0}^{\infty} \frac{3}{2}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n 3x^n}{2^{n+1}}$$

76. Integral: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 3x^{n+1}}{(n+1)2^{n+1}}$

78. $8 - 2(x-3) + \dfrac{1}{2}(x-3)^2 - \dfrac{1}{8}(x-3)^3 + \cdots = \displaystyle\sum_{n=0}^{\infty} 8\left[\frac{-(x-3)}{4}\right]^n = \frac{8}{1-[-(x-3)/4]}$

$$= \frac{32}{4+(x-3)} = \frac{32}{1+x}, \quad -1 < x < 7$$

80.

$$f(x) = \cos x$$

$$f'(x) = -\sin x$$

$$f''(x) = -\cos x$$

$$f'''(x) = \sin x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{f^{(n)}(-\pi/4)[x+(\pi/4)]^n}{n!} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x+\frac{\pi}{4}\right) - \frac{\sqrt{2}}{2 \cdot 2!}\left(x+\frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{2 \cdot 3!}\left(x+\frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{2 \cdot 4!}\left(x+\frac{\pi}{4}\right)^4 + \cdots$$

$$= \frac{\sqrt{2}}{2}\left[1 + \left(x+\frac{\pi}{4}\right) + \sum_{n=1}^{\infty} \frac{(-1)^{[n(n+1)]/2}[x+(\pi/4)]^{n+1}}{(n+1)!}\right]$$

82.

$$f(x) = \csc(x)$$

$$f'(x) = -\csc(x)\cot(x)$$

$$f''(x) = \csc^3(x) + \csc(x)\cot^2(x)$$

$$f'''(x) = -5\csc^3(x)\cot(x) - \csc(x)\cot^3(x)$$

$$f^{(4)}(x) = 5\csc^5(x) + 15\csc^3(x)\cot^2(x) + \csc(x)\cot^4(x)$$

$$\csc(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/2)[x-(\pi/2)]^n}{n!} = 1 + \frac{1}{2!}\left(x-\frac{\pi}{2}\right)^2 + \frac{5}{4!}\left(x-\frac{\pi}{2}\right)^4 + \cdots$$

84. $f(x) = x^{1/2}$

$f'(x) = \frac{1}{2}x^{-1/2}$

$f''(x) = -\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)x^{-3/2}$

$f'''(x) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)x^{-5/2}$

$f^{(4)}(x) = -\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)x^{-7/2}, \cdots$

$$\sqrt{x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(4)(x-4)^n}{n!}$$

$$= 2 + \frac{(x-4)}{2^2} - \frac{(x-4)^2}{2^5 2!} + \frac{1 \cdot 3(x-4)^3}{2^8 3!} - \frac{1 \cdot 3 \cdot 5(x-4)^4}{2^{11} 4!} + \cdots$$

$$= 2 + \frac{(x-4)}{2^2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots (2n-3)(x-4)^n}{2^{3n-1} n!}$$

86. $h(x) = (1+x)^{-3}$

$h'(x) = -3(1+x)^{-4}$

$h''(x) = 12(1+x)^{-5}$

$h'''(x) = -60(1+x)^{-6}$

$h^{(4)}(x) = 360(1+x)^{-7}$

$h^{(5)}(x) = -2520(1+x)^{-8}$

$$\frac{1}{(1+x)^3} = 1 - 3x + \frac{12x^2}{2!} - \frac{60x^3}{3!} + \frac{360x^4}{4!} - \frac{2520x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)! x^n}{2n!} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)(n+1)x}{2}$$

88. $\ln x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}, \quad 0 < x \le 2$

$$\ln\left(\frac{6}{5}\right) = \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{(6/5)-1}{n}\right)^n$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{5^n n} \approx 0.1823$$

90. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty$

$$e^{2/3} = \sum_{n=0}^{\infty} \frac{(2/3)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{3^n n!} \approx 1.9477$$

92. $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty$

$$\sin\left(\frac{1}{3}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{2n+1}(2n+1)!} \approx 0.3272$$

94. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots$

$$xe^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = x + x^2 + \frac{x^3}{2!} + \cdots$$

$$\int_0^1 xe^x\,dx = \Big[xe^x - e^x\Big]_0^1 = (e - e) - (0 - 1) = 1$$

$$\int_0^1 \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}\,dx = \sum_{n=0}^{\infty} \left[\frac{x^{n+2}}{(n+2)n!}\right]_0^1 = \sum_{n=0}^{\infty} \frac{1}{(n+2)n!} = 1$$

96. (a)

$$\begin{aligned} f(x) &= \sin 2x & f(0) &= 0\\ f'(x) &= 2\cos 2x & f'(0) &= 2\\ f''(x) &= -4\sin 2x & f''(0) &= 0\\ f'''(x) &= -8\cos 2x & f'''(0) &= -8\\ f^{(4)}(x) &= 16\sin 2x & f^{(4)}(0) &= 0\\ f^{(5)}(x) &= 32\cos 2x & f^{(5)}(0) &= 32\\ f^{(6)}(x) &= -64\sin 2x & f^{(6)}(0) &= 0\\ f^{(7)}(x) &= -128\cos 2x & f^{(7)}(0) &= -128 \end{aligned}$$

$$\sin 2x = 0 + 2x + \frac{0x^2}{2!} - \frac{8x^3}{3!} + \frac{0x^4}{4!} + \frac{32x^5}{5!} + \frac{0x^6}{6!} - \frac{128x^7}{7!} + \cdots = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots$$

(b) $\sin x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$\sin 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots$$

$$= 2x - \frac{8x^3}{6} + \frac{32x^5}{120} - \frac{128x^7}{5040} + \cdots = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots$$

(c) $\sin 2x = 2\sin x\cos x$

$$= 2\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right)$$

$$= 2\left[x + \left(-\frac{x^3}{2} - \frac{x^3}{6}\right) + \left(\frac{x^5}{24} + \frac{x^5}{12} + \frac{x^5}{120}\right) + \left(-\frac{x^7}{720} - \frac{x^7}{144} - \frac{x^7}{240} - \frac{x^7}{5040}\right) + \cdots\right]$$

$$= 2\left[x - \frac{2x^3}{3} + \frac{2x^5}{15} - \frac{4x^7}{315} + \cdots\right] = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots$$

98.

$$\cos t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!}$$

$$\cos\frac{\sqrt{t}}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{2^{2n}(2n)!}$$

$$\int_0^x \cos\frac{\sqrt{t}}{2}\,dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{2^{2n}(2n)!(n+1)}\right]_0^x$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{2^{2n}(2n)!(n+1)}$$

100.

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$$

$$e^t - 1 = \sum_{n=1}^{\infty} \frac{t^n}{n!}$$

$$\frac{e^t - 1}{t} = \sum_{n=1}^{\infty} \frac{t^{n-1}}{n!}$$

$$\int_0^x \frac{e^t - 1}{t}\,dt = \left[\sum_{n=1}^{\infty} \frac{t^n}{n \cdot n!}\right]_0^x = \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$$

102.

$$\arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots$$

$$\frac{\arcsin x}{x} = 1 + \frac{x^2}{2 \cdot 3} + \frac{1 \cdot 3x^4}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^6}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots$$

$$\lim_{x\to 0} \frac{\arcsin x}{x} = 1$$

By L'Hôpital's Rule, $\displaystyle\lim_{x\to 0} \frac{\arcsin x}{x} = \lim_{x\to 0} \frac{\left(\frac{1}{\sqrt{1-x^2}}\right)}{1} = 1.$

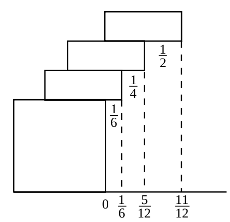
Problem Solving for Chapter 8

2. Let $S = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

Then
$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$
$$= S + \frac{1}{2^2} + \frac{1}{4^2} + \cdots$$
$$= S + \frac{1}{2^2}\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$
$$= S + \frac{1}{2^2}\left(\frac{\pi^2}{6}\right).$$

Thus, $S = \frac{\pi^2}{6} - \frac{1}{4}\frac{\pi^2}{6} = \frac{\pi^2}{6}\left(\frac{3}{4}\right) = \frac{\pi^2}{8}.$

4. (a) Position the three blocks as indicated in the figure. The bottom block extends $1/6$ over the edge of the table, the middle block extends $1/4$ over the edge of the bottom block, and the top block extends $1/2$ over the edge of the middle block.



The centers of gravity are located at

bottom block: $\frac{1}{6} - \frac{1}{2} = -\frac{1}{3}$

middle block: $\frac{1}{6} + \frac{1}{4} - \frac{1}{2} = -\frac{1}{12}$

top block: $\frac{1}{6} + \frac{1}{4} + \frac{1}{2} - \frac{1}{2} = \frac{5}{12}.$

The center of gravity of the top 2 blocks is

$$\left(-\frac{1}{12} + \frac{5}{12}\right)/2 = \frac{1}{6},$$

which lies over the bottom block. The center of gravity of the 3 blocks is

$$\left(-\frac{1}{3} - \frac{1}{12} + \frac{5}{12}\right)/3 = 0$$

which lies over the table. Hence, the far edge of the top block lies

$$\frac{1}{6} + \frac{1}{4} + \frac{1}{2} = \frac{11}{12}$$

beyond the edge of the table.

(b) Yes. If there are n blocks, then the edge of the top block lies $\sum_{c=1}^{n} \frac{1}{2i}$ from the edge of the table. Using 4 blocks,

$$\sum_{c=1}^{4} \frac{1}{2i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} = \frac{25}{24}$$

which shows that the top block extends beyond the table.

(c) The blocks can extend any distance beyond the table because the series diverges:

$$\sum_{c=1}^{\infty} \frac{1}{2i} = \frac{1}{2}\sum_{c=1}^{\infty} \frac{1}{i} = \infty.$$

6. $a - \frac{b}{2} + \frac{a}{3} - \frac{b}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(a+b) + (a-b)}{2n}$

If $a = b$, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2a)}{2n} = a\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges conditionally.

If $a \neq b$, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(a+b)}{2n} + \sum_{n=1}^{\infty} \frac{a-b}{2n}$ diverges.

No values of a and b give absolute convergence. $a = b$ implies conditional convergence.

8.
$$e^x = 1 + x + \frac{x^2}{2!} + \cdots$$
$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \cdots + \frac{x^{12}}{6!} + \cdots$$
$$\frac{f^{(12)}(0)}{12!} = \frac{1}{6!} \implies f^{(12)}(0) = \frac{12!}{6!} = 665{,}280$$

10. (a) If $p = 1$, $\int_2^{\infty} \frac{1}{x \ln x}\,dx = \ln \ln x \Big]_2^{\infty}$ diverges.

If $p > 1$, $\int_2^{\infty} \frac{1}{x(\ln x)^p}\,dx = \lim_{b\to\infty} \left[\frac{(\ln b)^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p}\right]$ converges.

If $p < 1$, diverges.

(b) $\sum_{n=4}^{\infty} \frac{1}{n \ln(n^2)} = \frac{1}{2}\sum_{n=4}^{\infty} \frac{1}{n \ln n}$ diverges by part (a).

12. Let $b_n = a_n r^n$.

$(bn)^{1/n} = (a_n r^n)^{1/n} = a_n^{1/n} \cdot r \to Lr$ as $n \to \infty$.

$Lr < \frac{1}{r} r = 1.$

By the Root Test, $\sum b_n$ converges $\implies$ $\sum a_n r^n$ converges.

14. (a) $\frac{1}{0.99} = \frac{1}{1-0.01} = \sum_{n=0}^{\infty} (0.01)^n$
$$= 1 + 0.01 + (0.01)^2 + \cdots$$
$$= 1.010101\ldots$$

(b) $\frac{1}{0.98} = \frac{1}{1-0.02} = \sum_{n=0}^{\infty} (0.02)^n$
$$= 1 + 0.02 + (0.02)^2 + \cdots$$
$$= 1 + 0.02 + 0.0004 + \cdots$$
$$= 1.0204081632\ldots$$

16. (a) Height $= 2\left[1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots\right]$
$$= 2\sum_{n=1}^{\infty} \frac{1}{n^{1/2}} = \infty \quad \left(p\text{-series, } p = \frac{1}{2} < 1\right)$$

(b) $S = 4\pi\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots\right] 4\pi\sum_{n=1}^{\infty} \frac{1}{n} = \infty$

(c) $W = \frac{4}{3}\pi\left[1 + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \cdots\right]$
$$= \frac{4}{3}\pi\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \text{ converges.}$$