CHAPTER 8
Infinite Series

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C H A P T E R 8
Infinite Series

Section 8.1 Sequences

Solutions to Odd-Numbered Exercises

1. $a_n = 2^n$

$a_1 = 2^1 = 2$

$a_2 = 2^2 = 4$

$a_3 = 2^3 = 8$

$a_4 = 2^4 = 16$

$a_5 = 2^5 = 32$

3. $a_n = \left(-\frac{1}{2}\right)^n$

$a_1 = \left(-\frac{1}{2}\right)^1 = -\frac{1}{2}$

$a_2 = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}$

$a_3 = \left(-\frac{1}{2}\right)^3 = -\frac{1}{8}$

$a_4 = \left(-\frac{1}{2}\right)^4 = \frac{1}{16}$

$a_5 = \left(-\frac{1}{2}\right)^5 = -\frac{1}{32}$

5. $a_n = \sin \frac{n\pi}{2}$

$a_1 = \sin \frac{\pi}{2} = 1$

$a_2 = \sin \pi = 0$

$a_3 = \sin \frac{3\pi}{2} = -1$

$a_4 = \sin 2\pi = 0$

$a_5 = \sin \frac{5\pi}{2} = 1$

7. $a_n = \frac{(-1)^{n(n+1)/2}}{n^2}$

$a_1 = \frac{(-1)^1}{1^2} = -1$

$a_2 = \frac{(-1)^3}{2^2} = -\frac{1}{4}$

$a_3 = \frac{(-1)^6}{3^2} = \frac{1}{9}$

$a_4 = \frac{(-1)^{10}}{4^2} = \frac{1}{16}$

$a_5 = \frac{(-1)^{15}}{5^2} = -\frac{1}{25}$

9. $a_n = 5 - \frac{1}{n} + \frac{1}{n^2}$

$a_1 = 5 - 1 + 1 = 5$

$a_2 = 5 - \frac{1}{2} + \frac{1}{4} = \frac{19}{4}$

$a_3 = 5 - \frac{1}{3} + \frac{1}{9} = \frac{43}{9}$

$a_4 = 5 - \frac{1}{4} + \frac{1}{16} = \frac{77}{16}$

$a_5 = 5 - \frac{1}{5} + \frac{1}{25} = \frac{121}{25}$

11. $a_n = \frac{3^n}{n!}$

$a_1 = \frac{3}{1!} = 3$

$a_2 = \frac{3^2}{2!} = \frac{9}{2}$

$a_3 = \frac{3^3}{3!} = \frac{27}{6}$

$a_4 = \frac{3^4}{4!} = \frac{81}{24}$

$a_5 = \frac{3^5}{5!} = \frac{243}{120}$

13. $a_1 = 3, a_{k+1} = 2(a_k - 1)$

$a_2 = 2(a_1 - 1)$

$= 2(3 - 1) = 4$

$a_3 = 2(a_2 - 1)$

$= 2(4 - 1) = 6$

$a_4 = 2(a_3 - 1)$

$= 2(6 - 1) = 10$

$a_5 = 2(a_4 - 1)$

$= 2(10 - 1) = 18$

15. $a_1 = 32, a_{k+1} = \frac{1}{2}a_k$

$a_2 = \frac{1}{2}a_1 = \frac{1}{2}(32) = 16$

$a_3 = \frac{1}{2}a_2 = \frac{1}{2}(16) = 8$

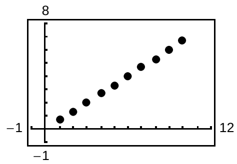
$a_4 = \frac{1}{2}a_3 = \frac{1}{2}(8) = 4$

$a_5 = \frac{1}{2}a_4 = \frac{1}{2}(4) = 2$

17. Because $a_1 = 8/(1 + 1) = 4$ and $a_2 = 8/(2 + 1) = \frac{8}{3}$, the sequence matches graph (d).

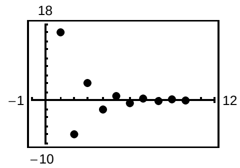
19. This sequence decreases and $a_1 = 4$, $a_2 = 4(0.5) = 2$. Matches (c).

21.



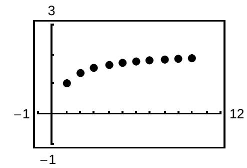
$$a_n = \frac{2}{3}n, n = 1, . . . , 10$$

23.



$$a_n = 16(-0.5)^{n-1}, n = 1, . . . , 10$$

25.



$$a_n = \frac{2n}{n + 1}, n = 1, 2, . . . , 10$$

27. $a_n = 3n - 1$

$a_5 = 3(5) - 1 = 14$

$a_6 = 3(6) - 1 = 17$

Add 3 to preceeding term.

29. $a_n = \dfrac{3}{(-2)^{n-1}}$

$$a_n = \frac{3}{(-2)^4} = \frac{3}{16}$$

$$a_6 = \frac{3}{(-2)^5} = -\frac{3}{32}$$

Multiply the preceeding term by $-\frac{1}{2}$.

31. $\dfrac{10!}{8!} = \dfrac{8!(9)(10)}{8!}$

$$= (9)(10) = 90$$

33. $\dfrac{(n + 1)!}{n!} = \dfrac{n!(n + 1)}{n!}$

$$= n + 1$$

35. $\dfrac{(2n - 1)!}{(2n + 1)!} = \dfrac{(2n - 1)!}{(2n - 1)!(2n)(2n + 1)}$

$$= \frac{1}{2n(2n + 1)}$$

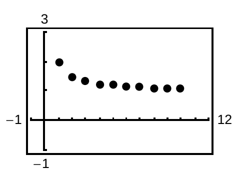
37. $\displaystyle\lim_{n\to\infty} \frac{5n^2}{n^2 + 2} = 5$

39. $\displaystyle\lim_{n\to\infty} \frac{2n}{\sqrt{n^2 + 1}} = \lim_{n\to\infty} \frac{2}{\sqrt{1 + (1/n^2)}}$

$$= \frac{2}{1} = 2$$

41. $\displaystyle\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = 0$

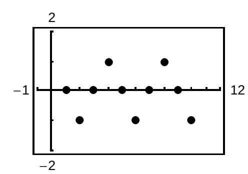
43.



The graph seems to indicate that the sequence converges to 1. Analytically,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n + 1}{n} = \lim_{x\to\infty} \frac{x + 1}{x} = \lim_{x\to\infty} 1 = 1.$$

45.



The graph seems to indicate that the sequence diverges. Analytically, the sequence is

$$\{a_n\} = \{0, -1, 0, 1, 0, -1, . . .\}.$$

Hence, $\displaystyle\lim_{n\to\infty} a_n$ does not exist.

47. $\displaystyle\lim_{n\to\infty} (-1)^n\left(\frac{n}{n + 1}\right)$

does not exist (oscillates between -1 and 1), diverges.

49. $\displaystyle\lim_{n\to\infty} \frac{3n^2 - n + 4}{2n^2 + 1} = \frac{3}{2}$, converges

51. $\displaystyle\lim_{n\to\infty} \frac{1 + (-1)^n}{n} = 0$, converges

53. $\displaystyle\lim_{n\to\infty} \frac{\ln(n^3)}{2n} = \lim_{n\to\infty} \frac{3}{2}\frac{\ln(n)}{n}$

$$= \lim_{n\to\infty} \frac{3}{2}\left(\frac{1}{n}\right) = 0, \text{ converges}$$

(L'Hôpital's Rule)

55. $\lim_{n\to\infty}\left(\frac{3}{4}\right)^n = 0$, converges

57. $\lim_{n\to\infty}\frac{(n+1)!}{n!} = \lim_{n\to\infty}(n+1) = \infty$, diverges

59. $\lim_{n\to\infty}\left(\frac{n-1}{n} - \frac{n}{n-1}\right) = \lim_{n\to\infty}\frac{(n-1)^2 - n^2}{n(n-1)}$

$= \lim_{n\to\infty}\frac{1-2n}{n^2-n} = 0$, converges

61. $\lim_{n\to\infty}\frac{n^p}{e^n} = 0$, converges

$(p > 0, n \geq 2)$

63. $a_n = \left(1 + \frac{k}{n}\right)^n$

$$\lim_{n\to\infty}\left(1 + \frac{k}{n}\right)^n = \lim_{u\to 0}\left[(1+u)^{1/u}\right]^k = e^k$$

where $u = \frac{k}{n}$, converges

65. $\lim_{n\to\infty}\frac{\sin n}{n} = \lim_{n\to\infty}(\sin n)\frac{1}{n} = 0$, converges

67. $a_n = 3n - 2$

69. $a_n = n^2 - 2$

71. $a_n = \frac{n+1}{n+2}$

73. $a_n = \frac{(-1)^{n-1}}{2^{n-2}}$

75. $a_n = 1 + \frac{1}{n} = \frac{n+1}{n}$

77. $a_n = \frac{n}{(n+1)(n+2)}$

79. $a_n = \frac{(-1)^{n-1}}{1\cdot 3\cdot 5\cdots(2n-1)} = \frac{(-1)^{n-1}2^n n!}{(2n)!}$

81. $a_n = 4 - \frac{1}{n} < 4 - \frac{1}{n+1} = a_{n+1}$,

monotonic; $|a_n| < 4$ bounded.

83. $\frac{n}{2^{n+2}} \overset{?}{\geq} \frac{n+1}{2^{(n+1)+2}}$

$2^{n+3}n \overset{?}{\geq} 2^{n+2}(n+1)$

$2n \overset{?}{\geq} n+1$

$n \geq 1$

Hence, $n \geq 1$

$2n \geq n+1$

$2^{n+3}n \geq 2^{n+2}(n+1)$

$\frac{n}{2^{n+2}} \geq \frac{n+1}{2^{(n+1)+2}}$

$a_n \geq a_{n+1}$

True; monotonic; $|a_n| \leq \frac{1}{8}$, bounded

85. $a_n = (-1)^n\left(\frac{1}{n}\right)$

$a_1 = -1$

$a_2 = \frac{1}{2}$

$a_3 = -\frac{1}{3}$

Not monotonic; $|a_n| \leq 1$, bounded

87. $a_n = \left(\frac{2}{3}\right)^n > \left(\frac{2}{3}\right)^{n+1} = a_{n+1}$

Monotonic; $|a_n| \leq \frac{2}{3}$, bounded

89. $a_n = \sin\left(\frac{n\pi}{6}\right)$

$a_1 = 0.500$

$a_2 = 0.8660$

$a_3 = 1.000$

$a_4 = 0.8660$

Not monotonic; $|a_n| \leq 1$, bounded

91. (a) $a_n = 5 + \frac{1}{n}$

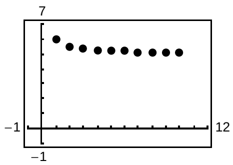
$$\left|5 + \frac{1}{n}\right| \le 6 \implies \{a_n\} \text{ bounded}$$

$$a_n = 5 + \frac{1}{n} > 5 + \frac{1}{n+1}$$

$$= a_{n+1} \implies \{a_n\} \text{ monotonic}$$

Therefore, $\{a_n\}$ converges.

(b)



$$\lim_{n\to\infty}\left(5 + \frac{1}{n}\right) = 5$$

93. (a) $a_n = \frac{1}{3}\left(1 - \frac{1}{3^n}\right)$

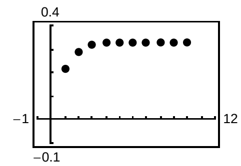
$$\left|\frac{1}{3}\left(1 - \frac{1}{3^n}\right)\right| < \frac{1}{3} \implies \{a_n\} \text{ bounded}$$

$$a_n = \frac{1}{3}\left(1 - \frac{1}{3^n}\right) < \frac{1}{3}\left(1 - \frac{1}{3^{n+1}}\right)$$

$$= a_{n+1} \implies \{a_n\} \text{ monotonic}$$

Therefore, $\{a_n\}$ converges.

(b)



$$\lim_{n\to\infty}\left[\frac{1}{3}\left(1 - \frac{1}{3^n}\right)\right] = \frac{1}{3}$$

95. $A_n = P\left[1 + \frac{r}{12}\right]^n$

(a) $\lim_{n\to\infty} A_n = \infty$, divergent. The amount will grow arbitrarily large over time.

(b) $A_n = 9000\left[1 + \frac{0.115}{12}\right]^n$

$A_1 = \$9086.25$ $A_6 = \$9530.06$

$A_2 = \$9173.33$ $A_7 = \$9621.39$

$A_3 = \$9261.24$ $A_8 = \$9713.59$

$A_4 = \$9349.99$ $A_9 = \$9806.68$

$A_5 = \$9439.60$ $A_{10} = \$9900.66$

97. (a) A sequence is a function whose domain is the set of positive integers.

(b) A sequence converges if it has a limit.

(c) A bounded monotonic sequence is a sequence that has nondecreasing or nonincreasing terms, and an upper and lower bound.

99. $a_n = 10 - \frac{1}{n}$

101. $a_n = \frac{3n}{4n + 1}$

103. (a) $A_n = (0.8)^n(2.5)$ billion

(b) $A_1 = \$2$ billion

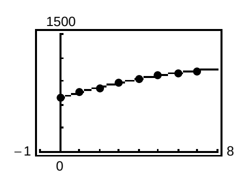
$A_2 = \$1.6$ billion

$A_3 = \$1.28$ billion

$A_4 = \$1.024$ billion

(c) $\lim_{n\to\infty} (0.8)^n(2.5) = 0$

105. (a) $a_n = -3.7262n^2 + 75.9167n + 684.25$



(b) For 2004, $n = 14$ and $a_{14} \approx 1017$, or \$1017.

107. $a_n = \dfrac{10^n}{n!}$

(a) $a_9 = a_{10} = \dfrac{10^9}{9!}$

$= \dfrac{1{,}000{,}000{,}000}{362{,}880}$

$= \dfrac{1{,}562{,}500}{567}$

(b) Decreasing

(c) Factorials increase more rapidly than exponentials.

109. $\{a_n\} = \left\{\sqrt[n]{n}\right\} = \{n^{1/n}\}$

$a_1 = 1^{1/1} = 1$

$a_2 = \sqrt{2} \approx 1.4142$

$a_3 = \sqrt[3]{3} \approx 1.4422$

$a_4 = \sqrt[4]{4} \approx 1.4142$

$a_5 = \sqrt[5]{5} \approx 1.3797$

$a_6 = \sqrt[6]{6} \approx 1.3480$

Let $y = \lim_{n\to\infty} n^{1/n}$.

$$\ln y = \lim_{n\to\infty}\left(\frac{1}{n}\ln n\right)$$

$$= \lim_{n\to\infty}\frac{\ln n}{n} = \lim_{n\to\infty}\frac{1/n}{1} = 0$$

Since $\ln y = 0$, we have $y = e^0 = 1$. Therefore, $\lim_{n\to\infty}\sqrt[n]{n} = 1$.

111. $a_{n+2} = a_n + a_{n+1}$

(a) $a_1 = 1$ $\quad a_7 = 8 + 5 = 13$

$a_2 = 1$ $\quad a_8 = 13 + 8 = 21$

$a_3 = 1 + 1 = 2$ $\quad a_9 = 21 + 13 = 34$

$a_4 = 2 + 1 = 3$ $\quad a_{10} = 34 + 21 = 55$

$a_5 = 3 + 2 = 5$ $\quad a_{11} = 55 + 34 = 89$

$a_6 = 5 + 3 = 8$ $\quad a_{12} = 89 + 55 = 144$

(b) $b_n = \dfrac{a_{n+1}}{a_n}, n \ge 1$

$b_1 = \dfrac{1}{1} = 1$ $\quad b_6 = \dfrac{13}{8}$

$b_2 = \dfrac{2}{1} = 2$ $\quad b_7 = \dfrac{21}{13}$

$b_3 = \dfrac{3}{2}$ $\quad b_8 = \dfrac{34}{21}$

$b_4 = \dfrac{5}{3}$ $\quad b_9 = \dfrac{55}{34}$

$b_5 = \dfrac{8}{5}$ $\quad b_{10} = \dfrac{89}{55}$

(c) $1 + \dfrac{1}{b_{n-1}} = 1 + \dfrac{1}{a_n/a_{n-1}}$

$= 1 + \dfrac{a_{n-1}}{a_n}$

$= \dfrac{a_n + a_{n-1}}{a_n} = \dfrac{a_{n+1}}{a_n} = b_n$

(d) If $\lim_{n\to\infty} b_n = \rho$, then $\lim_{n\to\infty}\left(1 + \dfrac{1}{b_{n-1}}\right) = \rho$.

Since $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b_{n-1}$ we have,

$1 + (1/\rho) = \rho.$

$\rho + 1 = \rho^2$

$0 = \rho^2 - \rho - 1$

$\rho = \dfrac{1 \pm \sqrt{1+4}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$

Since a_n, and thus b_n, is positive,

$\rho = \left(1 + \sqrt{5}\right)/2 \approx 1.6180.$

113. True

115. True

117. $a_1 = \sqrt{2} \approx 1.4142$

$a_2 = \sqrt{2 + \sqrt{2}} \approx 1.8478$

$a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}} \approx 1.9616$

$a_4 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}} \approx 1.9904$

$a_5 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}} \approx 1.9976$

$\{a_n\}$ is increasing and bounded by 2, and hence converges to L. Letting $\lim_{n\to\infty} a_n = L$ implies that $\sqrt{2 + L} = L \implies L = 2$. Hence, $\lim_{n\to\infty} a_n = 2$.

Section 8.2 Series and Convergence

1. $S_1 = 1$

$S_2 = 1 + \frac{1}{4} = 1.2500$

$S_3 = 1 + \frac{1}{4} + \frac{1}{9} \approx 1.3611$

$S_4 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \approx 1.4236$

$S_5 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} \approx 1.4636$

3. $S_1 = 3$

$S_2 = 3 - \frac{9}{2} = -1.5$

$S_3 = 3 - \frac{9}{2} + \frac{27}{4} = 5.25$

$S_4 = 3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} = -4.875$

$S_5 = 3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} + \frac{243}{16} = 10.3125$

5. $S_1 = 3$

$S_2 = 3 + \frac{3}{2} = 4.5$

$S_3 = 3 + \frac{3}{2} + \frac{3}{4} = 5.250$

$S_4 = 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} = 5.625$

$S_5 = 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16} = 5.8125$

7. $\sum_{n=0}^{\infty} 3\left(\frac{3}{2}\right)^n$ Geometric series

$r = \frac{3}{2} > 1$

Diverges by Theorem 8.6

9. $\sum_{n=0}^{\infty} 1000(1.055)^n$ Geometric series

$r = 1.055 > 1$

Diverges by Theorem 8.6

11. $\sum_{n=1}^{\infty} \frac{n}{n+1}$

$\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$

Diverges by Theorem 8.9

13. $\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$

$\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 \neq 0$

Diverges by Theorem 8.9

15. $\sum_{n=0}^{\infty} \frac{2^n+1}{2^{n+1}}$

$\lim_{n\to\infty} \frac{2^n+1}{2^{n+1}} = \lim_{n\to\infty} \frac{1+2^{-n}}{2} = \frac{1}{2} \neq 0$

Diverges by Theorem 8.9

17. $\sum_{n=0}^{\infty} \frac{9}{4}\left(\frac{1}{4}\right)^n = \frac{9}{4}\left[1 + \frac{1}{4} + \frac{1}{16} + \cdots\right]$

$S_0 = \frac{9}{4}, S_1 = \frac{9}{4} \cdot \frac{5}{4} = \frac{45}{16}, S_2 = \frac{9}{4} \cdot \frac{21}{16} \approx 2.95, \ldots$

Matches graph (c).

Analytically, the series is geometric:

$$\sum_{n=0}^{\infty} \left(\frac{9}{4}\right)\left(\frac{1}{4}\right)^n = \frac{9/4}{1-1/4} = \frac{9/4}{3/4} = 3$$

19. $\sum_{n=0}^{\infty} \frac{15}{4}\left(-\frac{1}{4}\right)^n = \frac{15}{4}\left[1 - \frac{1}{4} + \frac{1}{16} - \cdots\right]$

$S_0 = \frac{15}{4}, S_1 = \frac{45}{16}, S_2 \approx 3.05, \ldots$

Matches graph (a).

Analytically, the series is geometric:

$$\sum_{n=0}^{\infty} \frac{15}{4}\left(-\frac{1}{4}\right)^n = \frac{15/4}{1-(-1/4)} = \frac{15/4}{5/4} = 3$$

21. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots$

$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1$

23. $\sum_{n=0}^{\infty} 2\left(\frac{3}{4}\right)^n$

Geometric series with $r = \frac{3}{4} < 1$.

Converges by Theorem 8.6

25. $\sum_{n=0} (0.9)^n$

Geometric series with $r = 0.9 < 1$.

Converges by Theorem 8.6

27. (a) $\displaystyle\sum_{n=1}^{\infty} \frac{6}{n(n+3)} = 2\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+3}\right)$

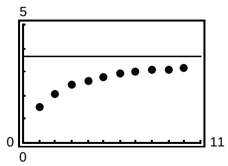
$$= 2\left[\left(1 - \frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \cdots\right]$$

$$= 2\left[1 + \frac{1}{2} + \frac{1}{3}\right] = \frac{11}{3} \approx 3.667$$

(b)

n	5	10	20	50	100
S_n	2.7976	3.1643	3.3936	3.5513	3.6078

(c)



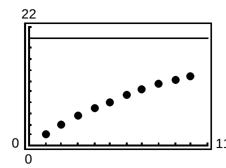
(d) The terms of the series decrease in magnitude slowly. Thus, the sequence of partial sums approaches the sum slowly.

29. (a) $\displaystyle\sum_{n=1}^{\infty} 2(0.9)^{n-1} = \sum_{n=0}^{\infty} 2(0.9)^{n} = \frac{2}{1-0.9} = 20$

(b)

n	5	10	20	50	100
S_n	8.1902	13.0264	17.5685	19.8969	19.9995

(c)



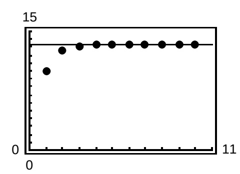
(d) The terms of the series decrease in magnitude slowly. Thus, the sequence of partial sums approaches the sum slowly.

31. (a) $\displaystyle\sum_{n=1}^{\infty} 10(0.25)^{n-1} = \frac{10}{1-0.25} = \frac{40}{3} \approx 13.3333$

(b)

n	5	10	20	50	100
S_n	13.3203	13.3333	13.3333	13.3333	13.3333

(c)



(d) The terms of the series decrease in magnitude rapidly. Thus, the sequence of partial sums approaches the sum rapidly.

33. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \sum_{n=2}^{\infty}\left(\frac{1/2}{n-1} - \frac{1/2}{n+1}\right) = \frac{1}{2}\sum_{n=2}^{\infty}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$

$$= \frac{1}{2}\left[\left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \cdots\right]$$

$$= \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4}$$

35. $\displaystyle\sum_{n=1}^{\infty} \frac{8}{(n+1)(n+2)} = 8\sum_{n=1}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) = 8\left[\left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots\right] = 8\left(\frac{1}{2}\right) = 4$

37. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-(1/2)} = 2$

39. $\displaystyle\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n = \frac{1}{1-(-1/2)} = \frac{2}{3}$

41. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n = \frac{1}{1-(1/10)} = \frac{10}{9}$

43. $\displaystyle\sum_{n=0}^{\infty} 3\left(-\frac{1}{3}\right)^n = \frac{3}{1-(-1/3)} = \frac{9}{4}$

45. $\sum_{n=0}^{\infty}\left(\frac{1}{2^n}-\frac{1}{3^n}\right)=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n-\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n$

$$=\frac{1}{1-(1/2)}-\frac{1}{1-(-1/3)}$$

$$=2-\frac{3}{2}=\frac{1}{2}$$

47. $0.\overline{4}=\sum_{n=0}^{\infty}\frac{4}{10}\left(\frac{1}{10}\right)^n$

Geometric series with $a=\frac{4}{10}$ and $r=\frac{1}{10}$

$$S=\frac{a}{1-r}=\frac{4/10}{1-(1/10)}=\frac{4}{9}$$

49. $0.075\overline{75}=\sum_{n=0}^{\infty}\frac{3}{40}\left(\frac{1}{100}\right)^n$

Geometric series with $a=\frac{3}{40}$ and $r=\frac{1}{100}$

$$S=\frac{a}{1-r}=\frac{3/40}{99/100}=\frac{5}{66}$$

51. $\sum_{n=1}^{\infty}\frac{n+10}{10n+1}$

$$\lim_{n\to\infty}\frac{n+10}{10n+1}=\frac{1}{10}\neq 0$$

Diverges by Theorem 8.9

53. $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right)=\left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\cdots=1+\frac{1}{2}=\frac{3}{2}$, converges

55. $\sum_{n=1}^{\infty}\frac{3n-1}{2n+1}$

$$\lim_{n\to\infty}\frac{3n-1}{2n+1}=\frac{3}{2}\neq 0$$

Diverges by Theorem 8.9

57. $\sum_{n=0}^{\infty}\frac{4}{2^n}=4\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$

Geometric series with $r=\frac{1}{2}$

Converges by Theorem 8.6

59. $\sum_{n=0}^{\infty}(1.075)^n$

Geometric series with $r=1.075$

Diverges by Theorem 8.6

61. $\sum_{n=2}^{\infty}\frac{n}{\ln n}$

$$\lim_{n\to\infty}\frac{n}{\ln n}=\lim_{n\to\infty}\frac{1}{1/n}=\infty$$

(by L'Hôpital's Rule) Diverges by Theorem 8.9

63. See definition, page 567.

65. The series given by

$$\sum_{n=0}^{\infty}ar^n=a+ar+ar^2+\cdots+ar^n+\cdots,\ a\neq 0$$

is a geometric series with ratio r. When $0<|r|<1$, the series converges to $\frac{a}{1-r}$. The series diverges if $|r|\geq 1$.

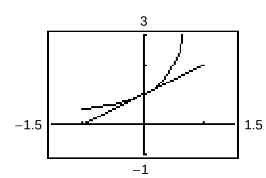
67. (a) x is the common ratio.

(b) $1+x+x^2+\cdots=\sum_{n=0}^{\infty}x^n=\frac{1}{1-x},\ |x|<1$

Geometric series: $a=1$, $r=x$, $|x|<1$

(c) $y_1=\frac{1}{1-x}$

$y_2=1+x$

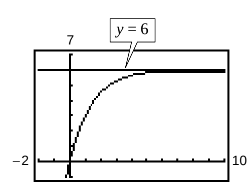


69. $f(x)=3\left[\frac{1-0.5^x}{1-0.5}\right]$

Horizontal asymptote: $y=6$

$$\sum_{n=0}^{\infty}3\left(\frac{1}{2}\right)^n$$

$$S=\frac{3}{1-(1/2)}=6$$



The horizontal asymptote is the sum of the series. $f(n)$ is the n^{th} partial sum.

71. $\dfrac{1}{n(n+1)} < 0.001$

$$10{,}000 < n^2 + n$$

$$0 < n^2 + n - 10{,}000$$

$$n = \frac{-1 \pm \sqrt{1^2 - 4(1)(-10{,}000)}}{2}$$

Choosing the positive value for n we have $n \approx 99.5012$. The first *term* that is less than 0.001 is $n = 100$.

$$\left(\frac{1}{8}\right)^n < 0.001$$

$$10{,}000 < 8^n$$

This inequality is true when $n = 5$. This series converges at a faster rate.

73. $\displaystyle\sum_{i=0}^{n-1} 8000(0.9)^i = \frac{8000\left[1 - (0.9)^{(n-1)+1}\right]}{1 - 0.9}$

$= 80{,}000(1 - 0.9^n),\; n > 0$

75. $\displaystyle\sum_{i=0}^{n-1} 100(0.75)^i = \frac{100\left[1 - 0.75^{(n-1)+1}\right]}{1 - 0.75}$

$= 400(1 - 0.75^n)$ million dollars.

Sum $= 400$ million dollars

77. $D_1 = 16$

$$D_2 = \underbrace{0.81(16)}_{\text{up}} + \underbrace{0.81(16)}_{\text{down}} = 32(0.81)$$

$$D_3 = 16(0.81)^2 + 16(0.81)^2 = 32(0.81)^2$$

$$\vdots$$

$$D = 16 + 32(0.81) + 32(0.81)^2 + \cdots = -16 + \sum_{n=0}^{\infty} 32(0.81)^n = -16 + \frac{32}{1 - 0.81} = 152.42 \text{ ft}$$

79. $P(n) = \dfrac{1}{2}\left(\dfrac{1}{2}\right)^n$

$$P(2) = \frac{1}{2}\left(\frac{1}{2}\right)^2 = \frac{1}{8}$$

$$\sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^n = \frac{1/2}{1 - (1/2)} = 1$$

81. (a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty}\frac{1}{2}\left(\frac{1}{2}\right)^n = \frac{1}{2}\,\frac{1}{(1 - (1/2))} = 1$

(b) No, the series is not geometric.

(c) $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = 2$

83. Present Value $\displaystyle = \sum_{n=1}^{19} 50{,}000\left(\frac{1}{1.06}\right)^n$

$\displaystyle = \sum_{n=0}^{18} \frac{50{,}000}{1.06}\left(\frac{1}{1.06}\right)^n \quad , \; r = \frac{1}{1.06}$

$\displaystyle = \frac{50{,}000}{1.06}\left(\frac{1 - 1.06^{-19}}{1 - 1.06^{-1}}\right)$

$\approx \$557{,}905.82$

The present value is less than $1,000,000. After accruing interest over 20 years, it attains its full value.

85. $\displaystyle w = \sum_{i=0}^{n-1} 0.01(2)^i = \frac{0.01(1 - 2^n)}{1 - 2} = 0.01(2^n - 1)$

(a) When $n = 29$: $w = \$5{,}368{,}709.11$

(b) When $n = 30$: $w = \$10{,}737{,}418.23$

(c) When $n = 31$: $w = \$21{,}474{,}836.47$

87. $P = 50, r = 0.03, t = 20$

(a) $A = 50\left(\frac{12}{0.03}\right)\left[\left(1 + \frac{0.03}{12}\right)^{12(20)} - 1\right] \approx \$16,415.10$

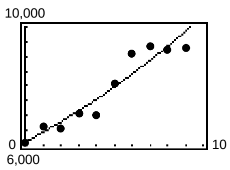
(b) $A = \frac{50 - (e^{0.03(20)} - 1)}{e^{0.03/12} - 1} \approx \$16,421.83$

89. $P = 100, r = 0.04, t = 40$

(a) $A = 100\left(\frac{12}{0.04}\right)\left[\left(1 + \frac{0.04}{12}\right)^{12(40)} - 1\right] \approx \$118,196.13$

(b) $A = \frac{100(e^{0.04(40)} - 1)}{e^{0.04/12} - 1} \approx \$118,393.43$

91. (a) $a_n = 6110.1832(1.0544)^x = 6110.1832e^{0.05297n}$



(b) 78,530 or $78,530,000,000

(c) Total $= \sum_{n=0}^{9} a_n \approx 78,449$ or $78,449,000,000

93. $x = 0.749999\ldots = 0.74 + \sum_{n=0}^{\infty} 0.009(0.1)^n$

$= 0.74 + \frac{0.009}{1 - 0.1}$

$= 0.74 + 0.01 = 0.75$

95. By letting $S_0 = 0$, we have $a_n = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k = S_n - S_{n-1}$. Thus,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (S_n - S_{n-1}) = \sum_{n=1}^{\infty} (S_n - S_{n-1} + c - c) = \sum_{n=1}^{\infty} [(c - S_{n-1}) - (c - S_n)].$$

97. Let $\sum a_n = \sum_{n=0}^{\infty} 1$ and $\sum b_n = \sum_{n=0}^{\infty} (-1)$.

Both are divergent series.

$$\sum (a_n + b_n) = \sum_{n=0}^{\infty} [1 + (-1)] = \sum_{n=0}^{\infty} [1 - 1] = 0$$

99. False. $\lim_{n\to\infty} \frac{1}{n} = 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

101. False

$$\sum_{n=1}^{\infty} ar^n = \left(\frac{a}{1 - r}\right) - a$$

The formula requires that the geometric series begins with $n = 0$.

103. Let H represent the half-life of the drug. If a patient receives n equal doses of P units each of this drug, administered at equal time interval of length t, the total amount of the drug in the patient's system at the time the last dose is administered is given by

$$T_n = P + Pe^{kt} + Pe^{2kt} + \cdots + Pe^{(n-1)kt}$$

where $k = -(\ln 2)/H$. One time interval *after* the last dose is administered is given by

$$T_{n+1} = Pe^{kt} + Pe^{2kt} + Pe^{3kt} + \cdots + Pe^{nkt}.$$

Two time intervals *after* the last dose is administered is given by

$$T_{n+2} = Pe^{2kt} + Pe^{3kt} + Pe^{4kt} + \cdots + Pe^{(n+1)kt}$$

and so on. Since $k < 0$, $T_{n+s} \to 0$ as $s \to \infty$, where s is an integer.

Section 8.3 The Integral Test and *p*-Series

1. $\sum_{n=1}^{\infty} \frac{1}{n+1}$

Let $f(x) = \frac{1}{x+1}$.

f is positive, continuous and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x+1}\,dx = \Big[\ln(x+1)\Big]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

3. $\sum_{n=1}^{\infty} e^{-n}$

Let $f(x) = e^{-x}$.

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} e^{-x}\,dx = \Big[-e^{-x}\Big]_1^{\infty} = \frac{1}{e}$$

Converges by Theorem 8.10

5. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Let $f(x) = \frac{1}{x^2+1}$.

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \Big[\arctan x\Big]_1^{\infty} = \frac{\pi}{4}$$

Converges by Theorem 8.10

7. $\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n+1}$

Let $f(x) = \frac{\ln(x+1)}{x+1}$

f is positive, continuous, and decreasing for $x \geq 2$ since

$$f'(x) = \frac{1-\ln(x+1)}{(x+1)^2} < 0 \text{ for } x \geq 2.$$

$$\int_1^{\infty} \frac{\ln(x+1)}{x+1}\,dx = \left[\frac{\ln^2(x+1)}{2}\right]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

9. $\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k+c}$

Let $f(x) = \frac{x^{k-1}}{x^k+c}$.

f is positive, continuous, and decreasing for $x > \sqrt[k]{c(k-1)}$ since

$$f'(x) = \frac{x^{k-2}[c(k-1)-x^k]}{(x^k+c)^2} < 0$$

for $x > \sqrt[k]{c(k-1)}$.

$$\int_1^{\infty} \frac{x^{k-1}}{x^k+c}\,dx = \left[\frac{1}{k}\ln(x^k+c)\right]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

11. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

Let $f(x) = \frac{1}{x^3}$.

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x^3}\,dx = \left[-\frac{1}{2x^2}\right]_1^{\infty} = \frac{1}{2}$$

Converges by Theorem 8.10

13. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/5}}$

Divergent p-series with $p = \frac{1}{5} < 1$

15. $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

Divergent p-series with $p = \frac{1}{2} < 1$

17. $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

Convergent p-series with $p = \frac{3}{2} > 1$

19. $\sum_{n=1}^{\infty} \frac{1}{n^{1.04}}$

Convergent p-series with $p = 1.04 > 1$

21. $\sum_{n=1}^{\infty} \frac{2}{\sqrt[4]{n^3}} = \frac{2}{1} + \frac{2}{2^{3/4}} + \frac{2}{3^{3/4}} + \cdots$

$S_1 = 2$

$S_2 \approx 3.189$

$S_3 \approx 4.067$

Matches (a)

Diverges—p-series with $p = \frac{3}{4} < 1$

23. $\sum_{n=1}^{\infty} \frac{2}{n\sqrt{n}} = 2 + 2/2^{3/2} + 2/3^{3/2} + \cdots$

$S_1 = 2$

$S_2 \approx 2.707$

$S_3 \approx 3.092$

Matches (b)

Converges—p-series with $p = 3/2 > 1$

25. No. Theorem 8.9 says that if the series converges, then the terms a_n tend to zero. Some of the series in Exercises 21-24 converge because the terms tend to 0 very rapidly.

27. $\sum_{n=1}^{N} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{N} > M$

(a)

M	2	4	6	8
N	4	31	227	1674

(b) No. Since the terms are decreasing (approaching zero), more and more terms are required to increase the partial sum by 2.

29. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$

If $p = 1$, then the series diverges by the Integral Test. If $p \neq 1$,

$$\int_2^{\infty} \frac{1}{x(\ln x)^p}\,dx = \int_2^{\infty} (\ln x)^{-p}\frac{1}{x}\,dx = \left[\frac{(\ln x)^{-p+1}}{-p+1}\right]_2^{\infty}.$$

Converges for $-p + 1 < 0$ or $p > 1$.

31. Let f be positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$. Then,

$$\sum_{n=1}^{\infty} a_n \text{ and } \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge (Theorem 8.10). See Example 1, page 578.

33. Your friend is not correct. The series

$$\sum_{n=10,000}^{\infty} \frac{1}{n} = \frac{1}{10,000} + \frac{1}{10,001} + \cdots$$

is the harmonic series, starting with the 10,000th term, and hence diverges.

35. Since f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, we have,

$$R_N = S - S_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n = \sum_{n=N+1}^{\infty} a_n > 0.$$

Also, $R_N = S - S_N = \sum_{n=N+1}^{\infty} a_n \leq a_{N+1} + \int_{N+1}^{\infty} f(x)\,dx \leq \int_N^{\infty} f(x)\,dx$. Thus,

$$0 \leq R_N \leq \int_N^{\infty} f(x)\,dx.$$

37. $S_6 = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} \approx 1.0811$

$$R_6 \leq \int_6^{\infty} \frac{1}{x^4}\,dx = \left[-\frac{1}{3x^3}\right]_6^{\infty} \approx 0.0015$$

$$1.0811 \leq \sum_{n=1}^{\infty} \frac{1}{n^4} \leq 1.0811 + 0.0015 = 1.0826$$

39. $S_{10} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26} + \frac{1}{37} + \frac{1}{50} + \frac{1}{65} + \frac{1}{82} + \frac{1}{101} \approx 0.9818$

$$R_{10} = \leq \int_{10}^{\infty} \frac{1}{x^2+1}\,dx = \Big[\arctan x\Big]_{10}^{\infty} = \frac{\pi}{2} - \arctan 10 \approx 0.0997$$

$$0.9818 \leq \sum_{n=1}^{\infty} \frac{1}{n^5} \leq 0.9818 + 0.0997 = 1.0815$$

41. $S_4 = \frac{1}{e} + \frac{2}{e^4} + \frac{3}{e^9} + \frac{4}{e^{16}} \approx 0.4049$

$$R_4 \leq \int_4^{\infty} xe^{-x^2}\,dx = \left[-\frac{1}{2}e^{-x^2}\right]_4^{\infty} = 5.6 \times 10^{-8}$$

$$0.4049 \leq \sum_{n=1}^{\infty} ne^{-n^2} \leq 0.4049 + 5.6 \times 10^{-8}$$

43. $0 \leq R_N \leq \int_N^{\infty} \frac{1}{x^4}\,dx = \left[-\frac{1}{3x^3}\right]_N^{\infty} = \frac{1}{3N^3} < 0.001$

$$\frac{1}{N^3} < 0.003$$

$$N^3 > 333.33$$

$$N > 6.93$$

$$N \geq 7$$

45. $R_N \leq \int_N^{\infty} e^{-5x}\,dx = \left[-\frac{1}{5}e^{-5x}\right]_N^{\infty} = \frac{e^{-5N}}{5} < 0.001$

$$\frac{1}{e^{5N}} < 0.005$$

$$e^{5N} > 200$$

$$5N > \ln 200$$

$$N > \frac{\ln 200}{5}$$

$$N > 1.0597$$

$$N \geq 2$$

47. $R_N \leq \int_N^{\infty} \frac{1}{x^2+1}\,dx = \Big[\arctan x\Big]_N^{\infty}$

$$= \frac{\pi}{2} - \arctan N < 0.001$$

$$-\arctan N < -1.5698$$

$$\arctan N > 1.5698$$

$$N > \tan 1.5698$$

$$N \geq 1004$$

49. (a) $\sum_{n=2}^{\infty} \frac{1}{n^{1.1}}$. This is a convergent p-series with $p = 1.1 > 1$.

$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ is a divergent series. Use the Integral Test.

$$\int_2^{\infty} \frac{1}{x \ln x}\,dx = \Big[\ln|\ln x|\Big]_2^{\infty} = \infty$$

(b) $\sum_{n=2}^{6} \frac{1}{n^{1.1}} = \frac{1}{2^{1.1}} + \frac{1}{3^{1.1}} + \frac{1}{4^{1.1}} + \frac{1}{5^{1.1}} + \frac{1}{6^{1.1}} \approx 0.4665 + 0.2987 + 0.2176 + 0.1703 + 0.1393$

$$\sum_{n=2}^{6} \frac{1}{n \ln n} = \frac{1}{2\ln 2} + \frac{1}{3\ln 3} + \frac{1}{4\ln 4} + \frac{1}{5\ln 5} + \frac{1}{6\ln 6} \approx 0.7213 + 0.3034 + 0.1803 + 0.1243 + 0.0930$$

The terms of the convergent series **seem** to be larger than those of the divergent series!

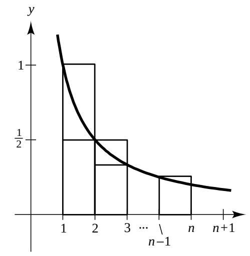
(c) $\frac{1}{n^{1.1}} < \frac{1}{n \ln n}$

$$n \ln n < n^{1.1}$$

$$\ln n < n^{0.1}$$

This inequality holds when $n \geq 3.5 \times 10^{15}$. Or, $n > e^{40}$. Then $\ln e^{40} = 40 < (e^{40})^{0.1} = e^4 \approx 55$.

51. (a) Let $f(x) = 1/x$. f is positive, continuous, and decreasing on $[1, \infty)$.



$$S_n - 1 \le \int_1^n \frac{1}{x}\,dx$$

$$S_n - 1 \le \ln n$$

Hence, $S_n \le 1 + \ln n$. Similarly,

$$S_n \ge \int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1).$$

Thus, $\ln(n+1) \le S_n \le 1 + \ln n$.

(b) Since $\ln(n+1) \le S_n \le 1 + \ln n$, we have $\ln(n+1) - \ln n \le S_n - \ln n \le 1$. Also, since $\ln x$ is an increasing function, $\ln(n+1) - \ln n > 0$ for $n \ge 1$. Thus, $0 \le S_n - \ln n \le 1$ and the sequence $\{a_n\}$ is bounded.

(c) $a_n - a_{n+1} = [S_n - \ln n] - [S_{n+1} - \ln(n+1)] = \displaystyle\int_n^{n+1} \frac{1}{x}\,dx - \frac{1}{n+1} \ge 0$

Thus, $a_n \ge a_{n+1}$ and the sequence is decreasing.

(d) Since the sequence is bounded and monotonic, it converges to a limit, γ.

(e) $a_{100} = S_{100} - \ln 100 \approx 0.5822$ (Actually $\gamma \approx 0.577216$.)

53. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$

Let $f(x) = \dfrac{1}{2x-1}$.

f is positive, continuous, and decreasing for $x \ge 1$.

$$\int_1^{\infty} \frac{1}{2x-1}\,dx = \Big[\ln\sqrt{2x-1}\Big]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

55. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt[4]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$

p-series with $p = \frac{5}{4}$

Converges by Theorem 8.11

57. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$

Geometric series with $r = \frac{2}{3}$

Converges by Theorem 8.6

59. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+(1/n^2)}} = 1 \ne 0$$

Diverges by Theorem 8.9

61. $\displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e \ne 0$$

Fails nth Term Test

Diverges by Theorem 8.9

63. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

Let $f(x) = \dfrac{1}{x(\ln x)^3}$.

f is positive, continuous and decreasing for $x \ge 2$.

$$\int_2^{\infty} \frac{1}{x(\ln x)^3}\,dx = \int_2^{\infty} (\ln x)^{-3}\frac{1}{x}\,dx = \left[\frac{(\ln x)^{-2}}{-2}\right]_2^{\infty} = \left[-\frac{1}{2(\ln x)^2}\right]_2^{\infty} = \frac{1}{2(\ln 2)^2}$$

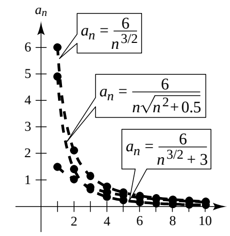
Converges by Theorem 8.10. See Exercise 13.

Section 8.4 Comparisons of Series

1. (a) $\sum_{n=1}^{\infty} \frac{6}{n^{3/2}} = \frac{6}{1} + \frac{6}{2^{3/2}} + \cdots \quad S_1 = 6$

$\sum_{n=1}^{\infty} \frac{6}{n^{3/2} + 3} = \frac{6}{4} + \frac{6}{2^{3/2} + 3} + \cdots \quad S_1 = \frac{3}{2}$

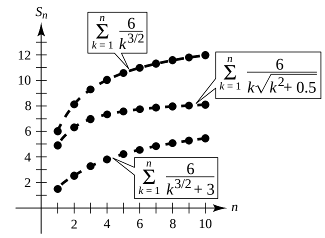
$\sum_{n=1}^{\infty} \frac{6}{n\sqrt{n^2 + 0.5}} = \frac{6}{1\sqrt{1.5}} + \frac{6}{2\sqrt{4.5}} + \cdots \quad S_1 = \frac{6}{\sqrt{1.5}} \approx 4.9$



(b) The first series is a p-series. It converges $(p = 3/2 > 1)$.

(c) The magnitude of the terms of the other two series are less than the corresponding terms at the convergent p-series. Hence, the other two series converge.

(d) The smaller the magnitude of the terms, the smaller the magnitude of the terms of the sequence of partial sums.



3. $\frac{1}{n^2 + 1} < \frac{1}{n^2}$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$$

converges by comparison with the convergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

5. $\frac{1}{n - 1} > \frac{1}{n}$ for $n \geq 2$

Therefore,

$$\sum_{n=2}^{\infty} \frac{1}{n - 1}$$

diverges by comparison with the divergent p-series

$$\sum_{n=2}^{\infty} \frac{1}{n}.$$

7. $\frac{1}{3^n + 1} < \frac{1}{3^n}$

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{3^n + 1}$$

converges by comparison with the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n.$$

9. For $n \geq 3$, $\frac{\ln n}{n + 1} > \frac{1}{n + 1}$.

Therefore,

$$\sum_{n=1}^{\infty} \frac{\ln n}{n + 1}$$

diverges by comparison with the divergent series

$$\sum_{n=1}^{\infty} \frac{1}{n + 1}.$$

Note: $\sum_{n=1}^{\infty} \frac{1}{n + 1}$ diverges by the integral test.

11. For $n > 3$, $\frac{1}{n^2} > \frac{1}{n!}$.

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{n!}$$

converges by comparison with the convergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

13. $\frac{1}{e^{n^2}} \leq \frac{1}{e^n}$

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{e^{n^2}}$$

converges by comparison with the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n.$$

15. $\lim_{n\to\infty} \frac{n/(n^2+1)}{1/n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

diverges by a limit comparison with the divergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

17. $\lim_{n\to\infty} \frac{1/\sqrt{n^2+1}}{1/n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = 1$

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}}$$

diverges by a limit comparison with the divergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

19. $\lim_{n\to\infty} \frac{\frac{2n^2-1}{3n^5+2n+1}}{1/n^3} = \lim_{n\to\infty} \frac{2n^5-n^3}{3n^5+2n+1} = \frac{2}{3}$

Therefore,

$$\sum_{n=1}^{\infty} \frac{2n^2-1}{3n^5+2n+1}$$

converges by a limit comparison with the convergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}.$$

21. $\lim_{n\to\infty} \frac{\frac{n+3}{n(n+2)}}{1/n} = \lim_{n\to\infty} \frac{n^2+3n}{n^2+2n} = 1$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n+3}{n(n+2)}$$

diverges by a limit comparison with the divergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

23. $\lim_{n\to\infty} \frac{1/\left(n\sqrt{n^2+1}\right)}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{n\sqrt{n^2+1}} = 1$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$$

converges by a limit comparison with the convergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

25. $\lim_{n\to\infty} \frac{(n^{k-1})/(n^k+1)}{1/n} = \lim_{n\to\infty} \frac{n^k}{n^k+1} = 1$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k+1}$$

diverges by a limit comparison with the divergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

27. $\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{n\to\infty} \frac{(-1/n^2)\cos(1/n)}{-1/n^2}$

$$= \lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

diverges by a limit comparison with the divergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

29. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Diverges

p-series with $p = \frac{1}{2}$

31. $\sum_{n=1}^{\infty} \frac{1}{3^n+2}$

Converges

Direct comparison with $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$

33. $\sum_{n=1}^{\infty} \frac{n}{2n+3}$

Diverges; nth Term Test

$\lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2} \neq 0$

35. $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$

Converges; integral test

37. $\lim_{n\to\infty} \frac{a_n}{1/n} = \lim_{n\to\infty} na_n$ by given conditions $\lim_{n\to\infty} na_n$ is finite and nonzero.

Therefore,

$$\sum_{n=1}^{\infty} a_n$$

diverges by a limit comparison with the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

39. $\frac{1}{2} + \frac{2}{5} + \frac{3}{10} + \frac{4}{17} + \frac{5}{26} + \cdots = \sum_{n=1}^{\infty} \frac{n}{n^2+1}$,

which diverges since the degree of the numerator is only one less than the degree of the denominator.

41. $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$

converges since the degree of the numerator is three less than the degree of the denominator.

43. $\lim_{n\to\infty} n\left(\frac{n^3}{5n^4+3}\right) = \lim_{n\to\infty} \frac{n^4}{5n^4+3} = \frac{1}{5} \neq 0$

Therefore,

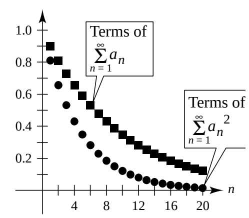
$$\sum_{n=1}^{\infty} \frac{n^3}{5n^4+3} \text{ diverges.}$$

45. See Theorem 8.12, page 583. One example is $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges because

$$\frac{1}{n^2+1} < \frac{1}{n^2} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges (p-series).

47.



For $0 < a_n < 1$, $0 < a_n^2 < a_n < 1$. Hence, the lower terms are those of $\sum a_n^2$.

49. $\frac{1}{200} + \frac{1}{400} + \frac{1}{600} + \cdots = \sum_{n=1}^{\infty} \frac{1}{200n}$, diverges

51. $\frac{1}{201} + \frac{1}{204} + \frac{1}{209} + \frac{1}{216} = \sum_{n=1}^{\infty} \frac{1}{200+n^2}$, converges

53. Some series diverge or converge very slowly. You cannot decide convergence or divergence of a series by comparing the first few terms.

55. False. Let $a_n = 1/n^3$ and $b_n = 1/n^2$. $0 < a_n \leq b_n$ and both

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converge.

57. True

59. Since $\sum_{n=1}^{\infty} b_n$ converges, $\lim_{n\to\infty} b_n = 0$. There exists N such that $b_n < 1$ for $n > N$. Thus,

$$a_n b_n < a_n \text{ for } n > N \text{ and } \sum_{n=1}^{\infty} a_n b_n$$

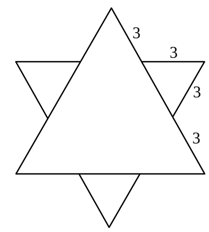
converges by comparison to the convergent series $\sum_{i=1}^{\infty} a_n$.

61. $\sum \frac{1}{n^2}$ and $\sum \frac{1}{n^3}$ both converge, and hence so does $\sum \left(\frac{1}{n^2}\right)\left(\frac{1}{n^3}\right) = \sum \frac{1}{n^5}$.

63. (a) Suppose Σb_n converges and Σa_n diverges. Then there exists N such that $0 < b_n < a_n$ for $n \geq N$. This means that $1 < a_n/b_n$ for $n \geq N$. Therefore, $\lim_{n\to\infty} a_n/b_n \neq 0$. Thus, Σa_n must also converge.

(b) Suppose Σb_n diverges and Σa_n converges. Then there exists N such that $0 < a_n < b_n$ for $n \geq N$. This means that $0 < a_n/b_n < 1$ for $n \geq N$. Therefore, $\lim_{n\to\infty} a_n/b_n \neq \infty$. Thus, Σa_n must also diverge.

65. Start with one triangle whose sides have length 9. At the nth step, each side is replaced by four smaller line segments each having $\frac{1}{3}$ the length of the original side.



#Sides	Length of sides
3	9
$3\cdot 4$	$9\left(\frac{1}{3}\right)$
$3\cdot 4^2$	$9\left(\frac{1}{3}\right)^2$
$\vdots$	
$3\cdot 4^n$	$9\left(\frac{1}{3}\right)^n$

At the nth step there are $3\cdot 4^n$ sides, each of length $9\left(\frac{1}{3}\right)^n$. At the next step, there are $3\cdot 4^n$ new triangles of side $9\left(\frac{1}{3}\right)^{n+1}$. The area of an equilateral triangle of side x is $\frac{1}{4}\sqrt{3}\,x^2$. Thus, the new triangles each have area

$$9\frac{\sqrt{3}}{4}\left(\frac{1}{3^{n+1}}\right)^2 = \frac{\sqrt{3}}{4}\frac{1}{3^{2n}}.$$

The area of the $3\cdot 4^n$ new triangles is

$$(3\cdot 4^n)\left(\frac{\sqrt{3}}{4}\frac{1}{3^{2n}}\right) = \frac{3\sqrt{3}}{4}\left(\frac{4}{9}\right)^n.$$

The total area is the infinite sum

$$\frac{9\sqrt{3}}{4} + \sum_{n=0}^{\infty}\frac{3\sqrt{3}}{4}\left(\frac{4}{9}\right)^n = \frac{9\sqrt{3}}{4} + \frac{3\sqrt{3}}{4}\left(\frac{1}{1-4/9}\right) = \frac{9\sqrt{3}}{4} + \frac{3\sqrt{3}}{4}\left(\frac{9}{5}\right) = \frac{18\sqrt{3}}{5}.$$

The perimeter is infinite, since at step n there are $3\cdot 4^n$ sides of length $9\left(\frac{1}{3}\right)^n$. Thus, the perimeter at step n is $27\left(\frac{4}{3}\right)^n \to \infty$.

Section 8.5 Alternating Series

1. $\displaystyle\sum_{n=1}^{\infty}\frac{6}{n^2} = \frac{6}{1} + \frac{6}{4} + \frac{6}{9} + \cdots$

$S_1 = 6, S_2 = 7.5$

Matches (b)

3. $\displaystyle\sum_{n=1}^{\infty}\frac{10}{n2^n} = \frac{10}{2} + \frac{10}{8} + \cdots$

$S_1 = 5, S_2 = 6.25$

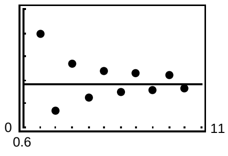
Matches (c)

5. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4} \approx 0.7854$

(a)

n	1	2	3	4	5	6	7	8	9	10
S_n	1	0.6667	0.8667	0.7238	0.8349	0.7440	0.8209	0.7543	0.8131	0.7605

(b)



(c) The points alternate sides of the horizontal line that represents the sum of the series. The distance between successive points and the line decreases.

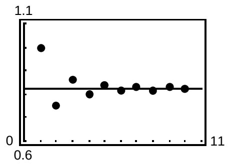
(d) The distance in part (c) is always less than the magnitude of the next term of the series.

7. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12} \approx 0.8225$

(a)

n	1	2	3	4	5	6	7	8	9	10
S_n	1	0.75	0.8611	0.7986	0.8386	0.8108	0.8312	0.8156	0.8280	0.8180

(b)



(c) The points alternate sides of the horizontal line that represents the sum of the series. The distance between successive points and the line decreases.

(d) The distance in part (c) is always less than the magnitude of the next term in the series.

9. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

$a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$

$\lim_{n\to\infty} \frac{1}{n} = 0$

Converges by Theorem 8.14.

11. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$

$a_{n+1} = \frac{1}{2(n+1)-1} < \frac{1}{2n-1} = a_n$

$\lim_{n\to\infty} \frac{1}{2n-1} = 0$

Converges by Theorem 8.14

13. $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+1}$

$\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$

Diverges by the nth Term Test

15. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

$a_{n+1} = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = a_n$

$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

Converges by Theorem 8.14

17. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{\ln(n+1)}$

$\lim_{n\to\infty} \frac{n+1}{\ln(n+1)} = \lim_{n\to\infty} \frac{1}{1/(n+1)} = \lim_{n\to\infty} (n+1) = \infty$

Diverges by the nth Term Test

19. $\sum_{n=1}^{\infty} \sin\left[\frac{(2n-1)\pi}{2}\right] = \sum_{n=1}^{\infty} (-1)^{n+1}$

Diverges by the nth Term Test

21. $\sum_{n=1}^{\infty} \cos n\pi = \sum_{n=1}^{\infty} (-1)^n$

Diverges by the nth Term Test

23. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

$a_{n+1} = \frac{1}{(n+1)!} < \frac{1}{n!} = a_n$

$\lim_{n\to\infty} \frac{1}{n!} = 0$

Converges by Theorem 8.14

25. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n+2}$

$$a_{n+1} = \frac{\sqrt{n+1}}{(n+1)+2} < \frac{\sqrt{n}}{n+2} \text{ for } n \geq 2$$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n+2} = 0$$

Converges by Theorem 8.14

27. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2)}{e^n - e^{-n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2e^n)}{e^{2n} - 1}$

Let $f(x) = \dfrac{2e^x}{e^{2x} - 1}$. Then

$$f'(x) = \frac{-2e^x(e^{2x}+1)}{(e^{2x}-1)^2} < 0.$$

Thus, $f(x)$ is decreasing. Therefore, $a_{n+1} < a_n$, and

$$\lim_{n\to\infty} \frac{2e^n}{e^{2n}-1} = \lim_{n\to\infty} \frac{2e^n}{2e^{2n}} = \lim_{n\to\infty} \frac{1}{e^n} = 0.$$

The series converges by Theorem 8.14.

29. $\displaystyle S_6 = \sum_{n=1}^{6} \frac{3(-1)^{n+1}}{n^2} = 2.4325$

$$|R_6| = |S - S_6| \leq a_7 = \frac{3}{49} \approx 0.0612;\ 2.3713 \leq S \leq 2.4937$$

31. $\displaystyle S_6 = \sum_{n=0}^{5} \frac{2(-1)^n}{n!} \approx 0.7333$

$$|R_6| = |S - S_6| \leq a_7 = \frac{2}{6!} = 0.002778;\ 0.7305 \leq S \leq 0.7361$$

33. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

(a) By Theorem 8.15,

$$|R_N| \leq a_{N+1} = \frac{1}{(N+1)!} < 0.001.$$

This inequality is valid when $N = 6$.

(b) We may approximate the series by

$$\sum_{n=0}^{6} \frac{(-1)^n}{n!} = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} \approx 0.368.$$

(7 terms. Note that the sum begins with $n = 0$.)

35. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$

(a) By Theorem 8.15,

$$|R_N| \leq a_{N+1} = \frac{1}{[2(N+1)+1]!} < 0.001.$$

This inequality is valid when $N = 2$.

(b) We may approximate the series by

$$\sum_{n=0}^{2} \frac{(-1)^n}{(2n+1)!} = 1 - \frac{1}{6} + \frac{1}{120} \approx 0.842.$$

(3 terms. Note that the sum begins with $n = 0$.)

37. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

(a) By Theorem 8.15,

$$|R_N| \leq a_{N+1} = \frac{1}{N+1} < 0.001.$$

This inequality is valid when $N = 1000$.

(b) We may approximate the series by

$$\sum_{n=1}^{1000} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1000} \approx 0.693.$$

(1000 terms)

39. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^3 - 1}$

By Theorem 8.15,

$$|R_N| \leq a_{N+1} = \frac{1}{2(N+1)^3 - 1} < 0.001.$$

This inequality is valid when $N = 7$.

41. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)^2}$

$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ converges by comparison to the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Therefore, the given series converge absolutely.

43. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

The given series converges by the Alternating Series Test, but does not converge absolutely since

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

is a divergent p-series. Therefore, the series converges conditionally.

45. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{(n+1)^2}$

$\lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$

Therefore, the series diverges by the nth Term Test.

47. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$

The given series converges by the Alternating Series Test, but does not converge absolutely since the series

$$\sum_{n=2}^{\infty} \frac{1}{\ln n}$$

diverges by comparison to the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

Therefore, the series converges conditionally.

49. $\sum_{n=2}^{\infty} \frac{(-1)^n n}{n^3-1}$

$\sum_{n=2}^{\infty} \frac{n}{n^3-1}$

converges by a limit comparison to the convergent p-series

$$\sum_{n=2}^{\infty} \frac{1}{n^2}.$$

Therefore, the given series converges absolutely.

51. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$

$\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}$

is convergent by comparison to the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

since

$$\frac{1}{(2n+1)!} < \frac{1}{2^n} \text{ for } n > 0.$$

Therefore, the given series converges absolutely.

53. $\sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$

The given series converges by the Alternating Series Test, but

$$\sum_{n=0}^{\infty} \frac{|\cos n\pi|}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1}$$

diverges by a limit comparison to the divergent harmonic series,

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$\lim_{n\to\infty} \frac{|\cos n\pi|/(n+1)}{1/n} = 1$, therefore the series converges conditionally.

55. $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series. Therefore, the given series converges absolutely.

57. An alternating series is a series whose terms alternate in sign. See Theorem 8.14.

59. $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges.

$\sum a_n$ is conditionally convergent if $\sum |a_n|$ diverges, but $\sum a_n$ converges.

61. (b). The partial sums alternate above and below the horizontal line representing the sum.

63. Since $\sum_{n=1}^{\infty} |a_n|$ converges we have

$$\lim_{n\to\infty} |a_n| = 0.$$

Thus, there must exist an $N > 0$ such that $|a_N| < 1$ for all $n > N$ and it follows that $a_n^2 \le |a_n|$ for all $n > N$. Hence, by the Comparison Test,

$$\sum_{n=1}^{\infty} a_n^2$$

converges. Let $a_n = 1/n$ to see that the converse is false.

65. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, hence so does $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

67. False

Let $a_n = \frac{(-1)^n}{n}$.

69. $\sum_{n=1}^{\infty} \frac{10}{n^{3/2}} = 10 \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ convergent p-series

71. Diverges by nth Term Test. $\lim_{n\to\infty} a_n = \infty$

73. Convergent Geometric Series $\left(r = \frac{7}{8} < 1\right)$

75. Convergent Geometric Series $\left(r = \frac{1}{\sqrt{e}}\right)$ or Integral Test

77. Converges (absolutely) by Alternating Series Test

79. The first term of the series is zero, not one. You cannot regroup series terms arbitrarily.

Section 8.6 The Ratio and Root Tests

1. $$\frac{(n+1)!}{(n-2)!} = \frac{(n+1)(n)(n-1)(n-2)!}{(n-2)!}$$
$$= (n+1)(n)(n-1)$$

3. Use the Principle of Mathematical Induction. When $k = 1$, the formula is valid since $1 = \frac{(2(1))!}{2^1 \cdot 1!}$. Assume that

$$1 \cdot 3 \cdot 5 \cdot \cdot \cdot (2n-1) = \frac{(2n)!}{2^n n!}$$

and show that

$$1 \cdot 3 \cdot 5 \cdot \cdot \cdot (2n-1)(2n+1) = \frac{(2n+2)!}{2^{n+1}(n+1)!}.$$

—CONTINUED—

3. —CONTINUED—

To do this, note that:

$$1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1) = [1 \cdot 3 \cdot 5 \cdots (2n-1)](2n+1)$$
$$= \frac{(2n)!}{2^n n!} \cdot (2n+1)$$
$$= \frac{(2n)!(2n+1)}{2^n n!} \cdot \frac{(2n+2)}{2(n+1)}$$
$$= \frac{(2n)!(2n+1)(2n+2)}{2^{n+1} n!(n+1)}$$
$$= \frac{(2n+2)!}{2^{n+1}(n+1)}$$

The formula is valid for all $n \geq 1$.

5. $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^n = 1\left(\frac{3}{4}\right) + 2\left(\frac{9}{16}\right) + \cdots$

$S_1 = \frac{3}{4}, S_2 \approx 1.875$

Matches (d)

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{n!} = 9 - \frac{3^3}{2} + \cdots$

$S_1 = 9$

Matches (f)

9. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4n}{5n-3}\right)^n = \frac{4}{2} + \left(\frac{8}{7}\right)^2 + \cdots$

$S_1 = 2$

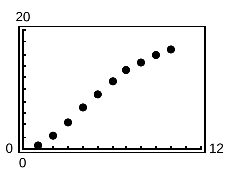
Matches (a)

11. (a) Ratio Test: $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)^2(5/8)^{n+1}}{n^2(5/8)^n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2\frac{5}{8} = \frac{5}{8} < 1.$ Converges

(b)

n	5	10	15	20	25
S_n	9.2104	16.7598	18.8016	19.1878	19.2491

(c)



(d) The sum is approximately 19.26.

(e) The more rapidly the terms of the series approach 0, the more rapidly the sequence of the partial sums approaches the sum of the series.

13. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{3^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{3^{n+1}} \cdot \frac{3^n}{n!}\right|$$
$$= \lim_{n\to\infty}\frac{n+1}{3} = \infty$$

Therefore, by the Ratio Test, the series diverges.

15. $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^n$

$$\lim_{n\to\infty}\left|\frac{a_n+1}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(3/4)^{n+1}}{n(3/4)^n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{3(n+1)}{4n}\right| = \frac{3}{4}$$

Therefore, by the Ratio Test, the series converges.

17. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n}\right|$$
$$= \lim_{n\to\infty}\frac{n+1}{2n} = \frac{1}{2}$$

Therefore, by the Ratio Test, the series converges.

19. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n}\right|$$
$$= \lim_{n\to\infty}\frac{2n^2}{(n+1)^2} = 2$$

Therefore, by the Ratio Test, the series diverges.

21. $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}\right| = \lim_{n\to\infty} \frac{2}{n+1} = 0$$

Therefore, by the Ratio Test, the series converges.

23. $\sum_{n=1}^{\infty} \frac{n!}{n3^n}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!}{(n+1)3^{n+1}} \cdot \frac{n3^n}{n!}\right| = \lim_{n\to\infty} \frac{n}{3} = \infty$$

Therefore, by the Ratio Test, the series diverges.

25. $\sum_{n=0}^{\infty} \frac{4^n}{n!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{4^{n+1}}{(n+1)!} \cdot \frac{n!}{4^n}\right| = \lim_{n\to\infty} \frac{4}{n+1} = 0$$

Therefore, by the Ratio Test, the series converges.

27. $\sum_{n=0}^{\infty} \frac{3^n}{(n+1)^n}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{3^{n+1}}{(n+2)^{n+1}} \cdot \frac{(n+1)^n}{3^n}\right| = \lim_{n\to\infty} \frac{3(n+1)^n}{(n+2)^{n+1}} = \lim_{n\to\infty} \frac{3}{n+2}\left(\frac{n+1}{n+2}\right)^n = (0)\left(\frac{1}{e}\right) = 0$$

To find $\lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^n$, let $y = \lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^n$. Then,

$$\ln y = \lim_{n\to\infty} n \ln\left(\frac{n+1}{n+2}\right) = \lim_{n\to\infty} \frac{\ln[(n+1)/(n+2)]}{1/n} = \frac{0}{0}$$

$$\ln y = \lim_{n\to\infty} \frac{[(1)/(n+1)] - [(1)/(n+2)]}{-(1/n^2)} = -1 \text{ by L'Hôpital's Rule}$$

$$y = e^{-1} = \frac{1}{e}.$$

Therefore, by the Ratio Test, the series converges.

29. $\sum_{n=0}^{\infty} \frac{4^n}{3^n + 1}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{4^{n+1}}{3^{n+1}+1} \cdot \frac{3^n+1}{4^n}\right| = \lim_{n\to\infty} \frac{4(3^n+1)}{3^{n+1}+1} = \lim_{n\to\infty} \frac{4(1+1/3^n)}{3+1/3^n} = \frac{4}{3}$$

Therefore, by the Ratio Test, the series diverges.

31. $\sum_{n=0}^{\infty} \frac{(-1)^{n+1} n!}{1 \cdot 3 \cdot 5 \cdots (2n+1)}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!}{1 \cdot 3 \cdot 5 \cdots (2n+1)(2n+3)} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{n!}\right| = \lim_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2}$$

Therefore, by the Ratio Test, the series converges.

Note: The first few terms of this series are $-1 + \frac{1}{1 \cdot 3} - \frac{2!}{1 \cdot 3 \cdot 5} + \frac{3!}{1 \cdot 3 \cdot 5 \cdot 7} - \cdots$

33. (a) $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1}{(n+1)^{3/2}} \cdot \frac{n^{3/2}}{1}\right| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{3/2} = 1$$

(b) $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1}{(n+1)^{1/2}} \cdot \frac{n^{1/2}}{1}\right| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{1/2} = 1$$

35. $\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{2n+1}\right)^n}$$

$$= \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$$

Therefore, by the Root Test, the series converges.

37. $\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^n}{(\ln n)^n}\right|}$$

$$= \lim_{n\to\infty} \frac{1}{|\ln n|} = 0$$

Therefore, by the Root Test, the series converges.

39. $\sum_{n=1}^{\infty} \left(2\sqrt[n]{n} + 1\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(2\sqrt[n]{n} + 1\right)^n} = \lim_{n\to\infty} \left(2\sqrt[n]{n} + 1\right)$$

To find $\lim_{n\to\infty} \sqrt[n]{n}$, let $y = \lim_{n\to\infty} \sqrt[x]{x}$. Then

$$\ln y = \lim_{n\to\infty} \left(\ln \sqrt[x]{x}\right) = \lim_{n\to\infty} \frac{1}{x} \ln x = \lim_{n\to\infty} \frac{\ln x}{x} = \lim_{n\to\infty} \frac{1/x}{1} = 0.$$

Thus, $\ln y = 0$, so $y = e^0 = 1$ and $\lim_{n\to\infty} \left(2\sqrt[n]{n} + 1\right) = 2(1) + 1 = 3$. Therefore, by the Root Test, the series diverges.

41. $\sum_{n=3}^{\infty} \frac{1}{(\ln n)^n}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{(\ln n)^n}} = \lim_{n\to\infty} \frac{1}{\ln n} = 0$$

Therefore, by the Root Test, the series converges.

43. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 5}{n}$

$$a_{n+1} = \frac{5}{n+1} < \frac{5}{n} = a_n$$

$$\lim_{n\to\infty} \frac{5}{n} = 0$$

Therefore, by the Alternating Series Test, the series converges (conditional convergence).

45. $\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}} = 3\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

This is convergent p-series.

47. $\sum_{n=1}^{\infty} \frac{2n}{n+1}$

$$\lim_{n\to\infty} \frac{2n}{n+1} = 2 \neq 0$$

This diverges by the nth Term Test for Divergence.

49. $\sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-2}}{2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 3^n 3^{-2}}{2^n} = \sum_{n=1}^{\infty} \frac{1}{9}\left(-\frac{3}{2}\right)^n$

Since $|r| = \frac{3}{2} > 1$, this is a divergent geometric series.

51. $\sum_{n=1}^{\infty} \frac{10n+3}{n2^n}$

$$\lim_{n\to\infty} \frac{(10n+3)/n2^n}{1/2^n} = \lim_{n\to\infty} \frac{10n+3}{n} = 10$$

Therefore, the series converges by a limit comparison test with the geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n.$$

53. $\sum_{n=1}^{\infty} \frac{\cos(n)}{2^n}$

$$\left|\frac{\cos(n)}{2^n}\right| \le \frac{1}{2^n}$$

Therefore, the series

$$\sum_{n=1}^{\infty} \left|\frac{\cos(n)}{2^n}\right|$$

converges by comparison with the geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n.$$

55. $\sum_{n=1}^{\infty} \frac{n7^n}{n!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)7^{n+1}}{(n+1)!} \cdot \frac{n!}{n7^n}\right| = \lim_{n\to\infty} \frac{7}{n} = 0$$

Therefore, by the Ratio Test, the series converges.

57. $\sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-1}}{n!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{3^n}{(n+1)!} \cdot \frac{n!}{3^{n-1}}\right| = \lim_{n\to\infty} \frac{3}{n+1} = 0$$

Therefore, by the Ratio Test, the series converges.

59. $\sum_{n=1}^{\infty} \frac{(-3)^n}{3 \cdot 5 \cdot 7 \cdots (2n+1)}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-3)^{n+1}}{3 \cdot 5 \cdot 7 \cdots (2n+1)(2n+3)} \cdot \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{(-3)^n}\right| = \lim_{n\to\infty} \frac{3}{2n+3} = 0$$

Therefore, by the Ratio Test, the series converges.

61. (a) and (c)

$$\sum_{n=1}^{\infty} \frac{n5^n}{n!} = \sum_{n=0}^{\infty} \frac{(n+1)5^{n+1}}{(n+1)!}$$

$$= 5 + \frac{(2)(5)^2}{2!} + \frac{(3)(5)^3}{3!} + \frac{(4)(5)^4}{4!} + \cdots$$

63. (a) and (b) are the same.

65. Replace n with $n + 1$.

$$\sum_{n=1}^{\infty} \frac{n}{4^n} = \sum_{n=0}^{\infty} \frac{n+1}{4^{n+1}}$$

67. Since

$$\frac{3^{10}}{2^{10}\, 10!} = 1.59 \times 10^{-5},$$

use 9 terms.

$$\sum_{k=1}^{9} \frac{(-3)^k}{2^k\, k!} \approx -0.7769$$

69. See Theorem 8.17, page 597.

71. No. Let $a_n = \frac{1}{n + 10{,}000}$.

The series $\sum_{n=1}^{\infty} \frac{1}{n + 10{,}000}$ diverges.

73. The series converges absolutely. See Theorem 8.17.

75. First, let

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = r < 1$$

and choose R such that $0 \le r < R < 1$. There must exist some $N > 0$ such that $\sqrt[n]{|a_n|} < R$ for all $n > N$. Thus, for $n > N$, we $|a_n| < R^n$ and since the geometric series

$$\sum_{n=0}^{\infty} R^n$$

converges, we can apply the Comparison Test to conclude that

$$\sum_{n=1}^{\infty} |a_n|$$

converges which in turn implies that $\sum_{n=1}^{\infty} a_n$ converges.

Second, let

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = r > R > 1.$$

Then there must exist some $M > 0$ such that $\sqrt[n]{|a_n|} > R$ for all $n > M$. Thus, for $n > M$, we have $|a_n| > R_n > 1$ which implies that $\lim_{n\to\infty} a_n \neq 0$ which in turn implies that

$$\sum_{n=1}^{\infty} a_n \text{ diverges.}$$

Section 8.7 Taylor Polynomials and Approximations

1. $y = -\frac{1}{2}x^2 + 1$

Parabola

Matches (d)

3. $y = e^{-1/2}[(x + 1) + 1]$

Linear

Matches (a)

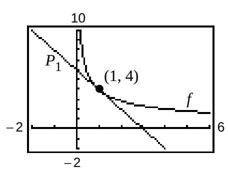
5. $f(x) = \dfrac{4}{\sqrt{x}} = 4x^{-1/2}$ $\quad f(1) = 4$

$f'(x) = -2x^{-3/2}$ $\quad f'(1) = -2$

$P_1(x) = f(1) + f'(1)(x - 1)$

$= 4 + (-2)(x - 1)$

$P_1(x) = -2x + 6$

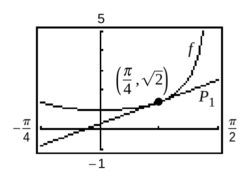


7. $f(x) = \sec x$ $\quad f\left(\dfrac{\pi}{4}\right) = \sqrt{2}$

$f'(x) = \sec x \tan x$ $\quad f'\left(\dfrac{\pi}{4}\right) = \sqrt{2}$

$P_1(x) = f\left(\dfrac{\pi}{4}\right) + f'\left(\dfrac{\pi}{4}\right)\left(x - \dfrac{\pi}{4}\right)$

$P_1(x) = \sqrt{2} + \sqrt{2}\left(x - \dfrac{\pi}{4}\right)$



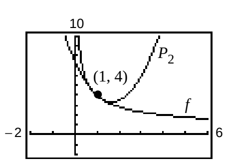
9. $f(x) = \dfrac{4}{\sqrt{x}} = 4x^{-1/2}$ $\quad f(1) = 4$

$f'(x) = -2x^{-3/2}$ $\quad f'(1) = -2$

$f''(x) = 3x^{-5/2}$ $\quad f''(1) = 3$

$P_2 = f(1) + f'(1)(x - 1) + \dfrac{f''(1)}{2}(x - 1)^2$

$= 4 - 2(x - 1) + \dfrac{3}{2}(x - 1)^2$



x	0	0.8	0.9	1.0	1.1	1.2	2
$f(x)$	Error	4.4721	4.2164	4.0	3.8139	3.6515	2.8284
$P_2(x)$	7.5	4.46	4.215	4.0	3.815	3.66	3.5

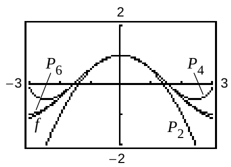
11. $f(x) = \cos x$

$P_2(x) = 1 - \frac{1}{2}x^2$

$P_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$

$P_6(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$

(a)



(b) $f'(x) = -\sin x \qquad P_2'(x) = -x$

$f''(x) = -\cos x \qquad P_2''(x) = -1$

$f''(0) = P_2''(0) = -1$

$f'''(x) = \sin x \qquad P_4'''(x) = x$

$f^{(4)}(x) = \cos x \qquad P_4^{(4)}(x) = 1$

$f^{(4)}(0) = 1 = P_4^{(4)}(0)$

$f^{(5)}(x) = -\sin x \qquad P_6^{(5)}(x) = -x$

$f^{(6)}(x) = -\cos x \qquad P^{(6)}(x) = -1$

$f^{(6)}(0) = -1 = P_6^{(6)}(0)$

(c) In general, $f^{(n)}(0) = P_n^{(n)}(0)$ for all n.

13. $f(x) = e^{-x} \qquad f(0) = 1$

$f'(x) = -e^{-x} \qquad f'(0) = -1$

$f''(x) = e^{-x} \qquad f''(0) = 1$

$f'''(x) = -e^{-x} \qquad f'''(0) = -1$

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$= 1 - x + \frac{x^2}{2} - \frac{x^3}{6}$$

15. $f(x) = e^{2x} \qquad f(0) = 1$

$f'(x) = 2e^{2x} \qquad f'(0) = 2$

$f''(x) = 4e^{2x} \qquad f''(0) = 4$

$f'''(x) = 8e^{2x} \qquad f'''(0) = 8$

$f^{(4)}(x) = 16^{2x} \qquad f^{(4)}(0) = 16$

$$P_4(x) = 1 + 2x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 + \frac{16}{4!}x^4$$

$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4$$

17. $f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) = 0$

$f^{(5)}(x) = \cos x \qquad f^{(5)}(0) = 1$

$$P_5(x) = 0 + (1)x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5$$

$$= x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

19. $f(x) = xe^x \qquad f(0) = 0$

$f'(x) = xe^x + e^x \qquad f'(0) = 1$

$f''(x) = xe^x + 2e^x \qquad f''(0) = 2$

$f'''(x) = xe^x + 3e^x \qquad f'''(0) = 3$

$f^{(4)}(x) = xe^x + 4e^x \qquad f^{(4)}(0) = 4$

$$P_4(x) = 0 + x + \frac{2}{2!}x^2 + \frac{3}{3!}x^3 + \frac{4}{4!}x^4$$

$$= x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4$$

21. $f(x) = \dfrac{1}{x+1} \qquad f(0) = 1$

$f'(x) = -\dfrac{1}{(x+1)^2} \qquad f'(0) = -1$

$f''(x) = \dfrac{2}{(x+1)^2} \qquad f''(0) = 2$

$f'''(x) = \dfrac{-6}{(x+1)^4} \qquad f'''(0) = -6$

$f^{(4)}(x) = \dfrac{24}{(x+1)^5} \qquad f^{(4)}(0) = 24$

$$P_4(x) = 1 - x + \frac{2}{2!}x^2 + \frac{-6}{3!}x^3 + \frac{24}{4!}x^4$$

$$= 1 - x + x^2 - x^3 + x^4$$

23. $f(x) = \sec x \qquad f(0) = 1$

$f'(x) = \sec x \tan x \qquad f'(0) = 0$

$f''(x) = \sec^3 x + \sec x \tan^2 x \qquad f''(0) = 1$

$$P_2(x) = 1 + 0x + \frac{1}{2!}x^2 = 1 + \frac{1}{2}x^2$$

25. $f(x) = \frac{1}{x}$ $\quad f(1) = 1$

$f'(x) = -\frac{1}{x^2}$ $\quad f'(1) = -1$

$f''(x) = \frac{2}{x^3}$ $\quad f''(1) = 2$

$f'''(x) = -\frac{6}{x^4}$ $\quad f'''(1) = -6$

$f^{(4)}(x) = \frac{24}{x^5}$ $\quad f^{(4)}(1) = 24$

$$P_4(x) = 1 - (x-1) + \frac{2}{2!}(x-1)^2 + \frac{-6}{3!}(x-1)^3 + \frac{24}{4!}(x-1)^4$$

$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4$$

27. $f(x) = \sqrt{x}$ $\quad f(1) = 1$

$f'(x) = \frac{1}{2\sqrt{x}}$ $\quad f'(1) = \frac{1}{2}$

$f''(x) = -\frac{1}{4x\sqrt{x}}$ $\quad f''(1) = -\frac{1}{4}$

$f'''(x) = \frac{3}{8x^2\sqrt{x}}$ $\quad f'''(1) = \frac{3}{8}$

$f^{(4)}(x) = -\frac{15}{16x^3\sqrt{x}}$ $\quad f^{(4)}(1) = -\frac{15}{16}$

$$P_4(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4$$

29. $f(x) = \ln x$ $\quad f(1) = 0$

$f'(x) = \frac{1}{x}$ $\quad f'(1) = 1$

$f''(x) = -\frac{1}{x^2}$ $\quad f''(1) = -1$

$f'''(x) = \frac{2}{x^3}$ $\quad f'''(1) = 2$

$f^{(4)}(x) = -\frac{6}{x^4}$ $\quad f^{(4)}(1) = -6$

$$P_4(x) = 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$$

31. $f(x) = \tan x$

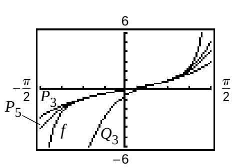
$f'(x) = \sec^2 x$

$f''(x) = 2\sec^2 x \tan x$

$f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$

$f^{(4)}(x) = 8\sec^2 x \tan^3 x + 16\sec^4 x \tan x$

$f^{(5)}(x) = 16\sec^2 x \tan^4 x + 88\sec^4 x \tan^2 x + 16\sec^6 x$



(a) $n = 3, c = 0$

$$P_3(x) = 0 + x + \frac{0}{2!}x^2 + \frac{2}{3!}x^3 = x + \frac{1}{3}x^3$$

(b) $n = 5, c = 0$

$$P_5(x) = 0 + x + \frac{0}{2!}x^2 + \frac{2}{3!}x^3 + \frac{0}{4!}x^4 + \frac{16}{5!}x^5$$

$$= x + \frac{1}{3}x^3 + \frac{2}{15}x^5$$

(c) $n = 3, c = \frac{\pi}{4}$

$$Q_3(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + \frac{4}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{16}{3!}\left(x - \frac{\pi}{4}\right)^3$$

$$= 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$$

33. $f(x) = \sin x$

$P_1(x) = x$

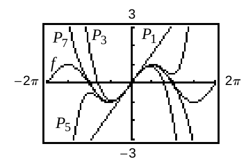
$P_3(x) = x - \frac{1}{6}x^3$

$P_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$

$P_7(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7$

(a)

x	0.00	0.25	0.50	0.75	1.00
$\sin x$	0.0000	0.2474	0.4794	0.6816	0.8415
$P_1(x)$	0.0000	0.2500	0.5000	0.7500	1.0000
$P_3(x)$	0.0000	0.2474	0.4792	0.6797	0.8333
$P_5(x)$	0.0000	0.2474	0.4794	0.6817	0.8417
$P_7(x)$	0.0000	0.2474	0.4794	0.6816	0.8415

(b) 

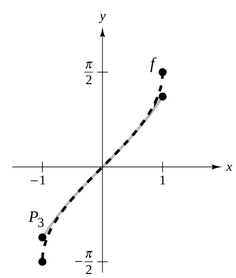
(c) As the distance increases, the accuracy decreases

35. $f(x) = \arcsin x$

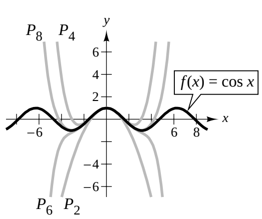
(a) $P_3(x) = x + \dfrac{x^3}{6}$

(b)

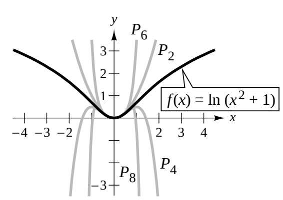
x	−0.75	−0.50	−0.25	0	0.25	0.50	0.75
$f(x)$	−0.848	−0.524	−0.253	0	0.253	0.524	0.848
$P_3(x)$	−0.820	−0.521	−0.253	0	0.253	0.521	0.820

(c) 

37. $f(x) = \cos x$



39. $f(x) = \ln(x^2 + 1)$



41. $f(x) = e^{-x} \approx 1 - x + \dfrac{x^2}{2} - \dfrac{x^3}{6}$

$f\left(\dfrac{1}{2}\right) \approx 0.6042$

43. $f(x) = \ln x \approx (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \frac{1}{4}(x - 1)^4$

$f(1.2) \approx 0.1823$

45. $f(x) = \cos x;\ f^{(5)}(x) = -\sin x \implies$ Max on $[0, 0.3]$ is 1.

$R_4(x) \leq \dfrac{1}{5!}(0.3)^5 = 2.025 \times 10^{-5}$

47. $f(x) = \arcsin x;\ f^{(4)}(x) = \dfrac{x(6x^2 + 9)}{(1 - x^2)^{7/2}} \Longrightarrow$ Max on $[0, 0.4]$ is $f^{(4)}(0.4) \approx 7.3340$.

$$R_3(x) \le \frac{7.3340}{4!}(0.4)^4 \approx 0.00782 = 7.82 \times 10^{-3}$$

49. $g(x) = \sin x$

$g^{(n+1)}(x) \le 1$ for all x

$$R_n(x) \le \frac{1}{(n+1)!}(0.3)^{n+1} < 0.001$$

By trial and error, $n = 3$.

51. $f(x) = \ln(x + 1)$

$f^{(n+1)}(x) = \dfrac{(-1)^{n+1}n!}{(x+1)^{n+1}} \Longrightarrow$ Max on $[0, 0.5]$ is $n!$.

$$R_n \le \frac{n!}{(n+1)!}(0.5)^{n+1} = \frac{(0.5)^{n+1}}{n+1} < 0.0001$$

By trial and error, $n = 9$. (See Example 9.) Using 9 terms, $\ln(1.5) \approx 0.4055$.

53.
$$f(x) = e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6},\ x < 0$$
$$R_3(x) = \frac{e^z}{4!}x^4 < 0.001$$
$$e^z x^4 < 0.024$$
$$xe^{z/4} < 0.3936$$
$$x < \frac{0.3936}{e^{z/4}} < 0.3936,\ z < 0$$
$$-0.3936 < x < 0$$

55. The graph of the approximating polynomial P and the elementary function f both pass through the point $(c, f(c))$ and the slopes of P and f agree at $(c, f(c))$. Depending on the degree of P, the nth derivatives of P and f agree at $(c, f(c))$.

57. See definition on page 607.

59. The accuracy increases as the degree increases (for values within the interval of convergence).

61. (a) $f(x) = e^x$

$$P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$$

$$g(x) = xe^x$$

$$Q_5(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5$$

$$Q_5(x) = x\,P_4(x)$$

(b) $f(x) = \sin x$

$$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$g(x) = x\sin x$$

$$Q_6(x) = x\,P_5(x) = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!}$$

(c) $g(x) = \dfrac{\sin x}{x} = \dfrac{1}{x}P_5(x) = 1 - \dfrac{x^2}{3!} + \dfrac{x^4}{5!}$

63. (a) $Q_2(x) = -1 + \dfrac{\pi^2(x+2)^2}{32}$

(b) $R_2(x) = -1 + \dfrac{\pi^2(x-6)^2}{32}$

(c) No. The polynomial will be linear.
Translations are possible at $x = -2 + 8n$.

65. Let f be an even function and P_n be the nth Maclaurin polynomial for f. Since f is even, f' is odd, f'' is even, f''' is odd, etc. (see Exercise 45). All of the odd derivatives of f are odd and thus, all of the odd powers of x will have coefficients of zero. P_n will only have terms with even powers of x.

67. As you move away from $x = c$, the Taylor Polynomial becomes less and less accurate.

Section 8.8 Power Series

1. Centered at 0

3. Centered at 2

5. $\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}$

$$L = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{n+2}\cdot\frac{n+1}{(-1)^n x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n+1}{n+2}\right||x| = |x|$$

$|x| < 1 \Longrightarrow R = 1$

7. $\sum_{n=1}^{\infty} \frac{(2x)^n}{n^2}$

$$L = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(2x)^{n+1}}{(n+1)^2}\cdot\frac{n^2}{(2x)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2n^2x}{(n+1)^2}\right| = 2|x|$$

$2|x| < 1 \Longrightarrow R = \frac{1}{2}$

9. $\sum_{n=0}^{\infty} \frac{(2x)^{2n}}{(2n)!}$

$$L = \lim_{n\to\infty}\left|\frac{u_n+1}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(2x)^{2n+2}/(2n+2)!}{(2x)^{2n}/(2n)!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(2x)^2}{(2n+2)(2n+1)}\right| = 0$$

Thus, the series converges for all x. $R = \infty$.

11. $\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$

Since the series is geometric, it converges only if $|x/2| < 1$ or $-2 < x < 2$.

13. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{n+1}\cdot\frac{n}{(-1)^n x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{nx}{n+1}\right| = |x|$$

Interval: $-1 < x < 1$

When $x = 1$, the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

When $x = -1$, the p-series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Therefore, the interval of convergence is $-1 < x \le 1$.

15. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0$$

The series converges for all x. Therefore, the interval of convergence is $-\infty < x < \infty$.

17. $\sum_{n=0}^{\infty} (2n)!\left(\frac{x}{2}\right)^n$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)!x^{n+1}}{2^{n+1}}\cdot\frac{2^n}{(2n)!x^n}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)x}{2}\right| = \infty$$

Therefore, the series converges only for $x = 0$.

19. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{4^n}$

Since the series is geometric, it converges only if $|x/4| < 1$ or $-4 < x < 4$.

21. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-5)^{n+1}}{(n+1)5^{n+1}} \cdot \frac{n5^n}{(-1)^{n+1}(x-5)^n}\right| = \lim_{n\to\infty}\left|\frac{n(x-5)}{5(n+1)}\right| = \frac{1}{5}|x-5|$$

$R = 5$

Center: $x = 5$

Interval: $-5 < x - 5 < 5$ or $0 < x < 10$

When $x = 0$, the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{-1}{n}$ diverges.

When $x = 10$, the alternating series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

Therefore, the interval of convergence is $0 < x \le 10$.

23. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-1)^{n+2}}{n+2} \cdot \frac{n+1}{(-1)^{n+1}(x-1)^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x-1)}{n+2}\right| = |x-1|$$

$R = 1$

Center: $x = 1$

Interval: $-1 < x - 1 < 1$ or $0 < x < 2$

When $x = 0$, the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges by the integral test.

When $x = 2$, the alternating series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}$ converges.

Therefore, the interval of convergence is $0 < x \le 2$.

25. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-c)^{n-1}}{c^{n-1}}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x-c)^n}{c^n} \cdot \frac{c^{n-1}}{(x-c)^{n-1}}\right| = \frac{1}{c}|x-c|$$

$R = c$

Center: $x = c$

Interval: $-c < x - c < c$ or $0 < x < 2c$

When $x = 0$, the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}$ diverges.

When $x = 2c$, the series $\displaystyle\sum_{n=1}^{\infty} 1$ diverges.

Therefore, the interval of convergence is $0 < x < 2c$.

27. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}(-2x)^{n-1}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(-2x)^n}{n+2} \cdot \frac{n+1}{n(-2x)^{n-1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-2x)(n+1)^2}{n(n+2)}\right| = 2|x|$$

$R = \frac{1}{2}$

Interval: $-\frac{1}{2} < x < \frac{1}{2}$

When $x = -\frac{1}{2}$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges by the nth Term Test.

When $x = \frac{1}{2}$, the alternating series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{n+1}$ diverges.

Therefore, the interval of convergence is $-\frac{1}{2} < x < \frac{1}{2}$.

29. $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{x^{2n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^2}{(2n+2)(2n+3)}\right| = 0$$

Therefore, the interval of convergence is $-\infty < x < \infty$.

31. $\displaystyle\sum_{n=1}^{\infty} \frac{k(k+1)\cdots(k+n-1)x^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{k(k+1)\cdots(k+n-1)(k+n)x^{n+1}}{(n+1)!}\cdot\frac{n!}{k(k+1)\cdots(k+n-1)x^n}\right| = \lim_{n\to\infty}\left|\frac{(k+n)x}{n+1}\right| = |x|$$

$R = 1$

When $x = \pm 1$, the series diverges and the interval of convergence is $-1 < x < 1$.

$$\left[\frac{k(k+1)\cdots(k+n-1)}{1\cdot 2\cdots n} \geq 1\right]$$

33. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}3\cdot 7\cdot 11\cdots(4n-1)(x-3)^n}{4^n}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}\cdot 3\cdot 7\cdot 11\cdots(4n-1)(4n+3)(x-3)^{n+1}}{4^{n+1}}\cdot\frac{4^n}{(-1)^{n+1}\cdot 3\cdot 7\cdot 11\cdots(4n-1)(x-3)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(4n+3)(x-3)}{4}\right| = \infty$$

$R = 0$

Center: $x = 3$

Therefore, the series converges only for $x = 3$.

35. (a) $f(x) = \displaystyle\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n, -2 < x < 2$ (Geometric)

(b) $f'(x) = \displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{2}\right)\left(\frac{x}{2}\right)^{n-1}, -2 < x < 2$

(c) $f''(x) = \displaystyle\sum_{n=2}^{\infty}\left(\frac{n}{2}\right)\left(\frac{n-1}{2}\right)\left(\frac{x}{2}\right)^{n-2}, -2 < x < 2$

(d) $\displaystyle\int f(x)\,dx = \sum_{n=0}^{\infty}\frac{2}{n+1}\left(\frac{x}{2}\right)^{n+1}, -2 \leq x < 2$

37. (a) $f(x) = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}, 0 < x \leq 2$

(b) $f'(x) = \displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}(x-1)^n,\ 0 < x < 2$

(c) $f''(x) = \displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}n(x-1)^{n-1}, 0 < x < 2$

(d) $\displaystyle\int f(x)\,dx = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-1)^{n+2}}{(n+1)(n+2)}, 0 \leq x \leq 2$

39. $g(1) = \displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n = 1 + \frac{1}{3} + \frac{1}{9} + \cdots$

$S_1 = 1, S_2 = 1.33$. Matches (c)

41. $g(3.1) = \displaystyle\sum_{n=0}^{\infty}\left(\frac{3.1}{3}\right)^n$ diverges. Matches (b)

43. A series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n$$

is called a power series centered at c.

45. A single point, a_n interval, or the entire real line.

47. (a) $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, -\infty < x < \infty$ (See Exercise 29.)

$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, -\infty < x < \infty$

(b) $f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = g(x)$

(c) $g'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!} = -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = -f(x)$

(d) $f(x) = \sin x$ and $g(x) = \cos x$

49.

$$y = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}$$

$$y' = \sum_{n=1}^{\infty} \frac{2nx^{2n-1}}{2^n n!}$$

$$y'' = \sum_{n=1}^{\infty} \frac{2n(2n-1)x^{2n-2}}{2^n n!}$$

$$y'' - xy' - y = \sum_{n=1}^{\infty} \frac{2n(2n-1)x^{2n-2}}{2^n n!} - \sum_{n=1}^{\infty} \frac{2nx^{2n}}{2^n n!} - \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}$$

$$= \sum_{n=1}^{\infty} \frac{2n(2n-1)x^{2n-2}}{2^n n!} - \sum_{n=0}^{\infty} \frac{(2n+1)x^{2n}}{2^n n!}$$

$$= \sum_{n=0}^{\infty} \left[\frac{(2n+2)(2n+1)x^{2n}}{2^{n+1}(n+1)!} - \frac{(2n+1)x^{2n}}{2^n n!} \cdot \frac{2(n+1)}{2(n+1)} \right]$$

$$= \sum_{n=0}^{\infty} \frac{2(n+1)x^{2n}[(2n+1) - (2n+1)]}{2^{n+1}(n+1)!} = 0$$

51. $J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}$

(a) $\lim_{k\to\infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k\to\infty} \left| \frac{(-1)^{k+1} x^{2k+2}}{2^{2k+2}[(k+1)!]^2} \cdot \frac{2^{2k}(k!)^2}{(-1)^k x^{2k}} \right| = \lim_{k\to\infty} \left| \frac{(-1)x^2}{2^2(k+1)^2} \right| = 0$

Therefore, the interval of convergence is $-\infty < x < \infty$.

(b)

$$J_0 = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{4^k(k!)^2}$$

$$J_0' = \sum_{k=1}^{\infty} (-1)^k \frac{2kx^{2k-1}}{4^k(k!)^2} = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{(2k+2)x^{2k+1}}{4^{k+1}[(k+1)!]^2}$$

$$J_0'' = \sum_{k=1}^{\infty} (-1)^k \frac{2k(2k-1)x^{2k-2}}{4^k(k!)^2} = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{(2k+2)(2k+1)x^{2k}}{4^{k+1}[(k+1)!]^2}$$

$$x^2J_0'' + xJ_0' + x^2J_0 = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{2(2k+1)x^{2k+2}}{4^{k+1}(k+1)!k!} + \sum_{k=0}^{\infty} (-1)^{k+1} \frac{2x^{2k+2}}{4^{k+1}(k+1)!k!} + \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+2}}{4^k(k!)^2}$$

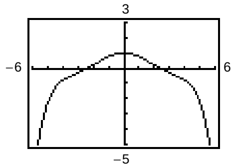
$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{4^k(k!)^2} \left[(-1)\frac{2(2k+1)}{4(k+1)} + (-1)\frac{2}{4(k+1)} + 1 \right]$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{4^k(k!)^2} \left[\frac{-4k-2}{4k+4} - \frac{2}{4k+4} + \frac{4k+4}{4k+4} \right] = 0$$

—CONTINUED—

51. —CONTINUED—

(c) $P_6(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304}$



(d) $\int_0^1 J_0 dx = \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{4^k (k!)^2} dx$

$$= \left[\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{4^k (k!)^2 (2k+1)} \right]_0^1$$

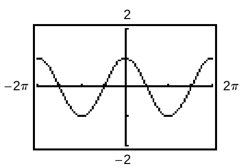
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k (k!)^2 (2k+1)}$$

$$= 1 - \frac{1}{12} + \frac{1}{320} \approx 0.92$$

(exact integral is 0.9197304101)

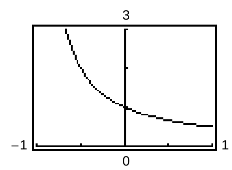
53. $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos x$

(See Exercise 47.)



55. $f(x) = \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-x)^n$

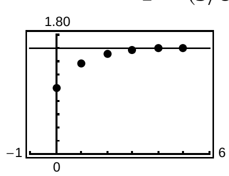
$$= \frac{1}{1-(-x)} = \frac{1}{1+x} \text{ for } -1 < x < 1$$



57. $\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$

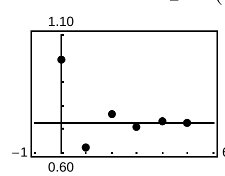
(a) $\sum_{n=0}^{\infty} \left(\frac{3/4}{2}\right)^n = \sum_{n=0}^{\infty} \left(\frac{3}{8}\right)^n$

$$= \frac{1}{1-(3/8)} = \frac{8}{5} = 1.6$$



(b) $\sum_{n=0}^{\infty} \left(\frac{-3/4}{2}\right)^n = \sum_{n=0}^{\infty} \left(-\frac{3}{8}\right)^n$

$$= \frac{1}{1-(-3/8)} = \frac{8}{11} \approx 0.7272$$



(c) The alternating series converges more rapidly. The partial sums of the series of positive terms approach the sum from below. The partial sums of the alternating series alternate sides of the horizontal line representing the sum.

(d) $\sum_{n=0}^{N} \left(\frac{3}{2}\right)^n > M$

M	10	100	1000	10,000
N	4	9	15	21

59. False;

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n 2^n}$$

converges for $x = 2$ but diverges for $x = -2$.

61. True; the radius of convergence is $R = 1$ for both series.

Section 8.9 Representation of Functions by Power Series

1. (a) $\dfrac{1}{2-x} = \dfrac{1/2}{1-(x/2)} = \dfrac{a}{1-r}$

$$= \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}$$

This series converges on $(-2, 2)$.

(b)

$$\begin{array}{r l}
 & \frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \frac{x^3}{16} + \cdots \\
2 - x \,\big) & 1 \\
 & 1 - \frac{x}{2} \\ \hline
 & \frac{x}{2} \\
 & \frac{x}{2} - \frac{x^2}{4} \\ \hline
 & \frac{x^2}{4} \\
 & \frac{x^2}{4} - \frac{x^3}{8} \\ \hline
 & \frac{x^3}{8} \\
 & \frac{x^3}{8} - \frac{x^4}{16} \\ \hline
 & \vdots
\end{array}$$

5. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{1}{2-x} = \frac{1}{-3-(x-5)} = \frac{-1/3}{1+(1/3)(x-5)}$$

which implies that $a = -1/3$ and $r = (-1/3)(x-5)$.

Therefore, the power series for $f(x)$ is given by

$$\frac{1}{2-x} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} -\frac{1}{3}\left[-\frac{1}{3}(x-5)\right]^n$$

$$= \sum_{n=0}^{\infty} \frac{(x-5)^n}{(-3)^{n+1}},\ |x-5| < 3 \text{ or } 2 < x < 8.$$

9. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{1}{2x-5} = \frac{-1}{11-2(x+3)}$$

$$= \frac{-1/11}{1-(2/11)(x+3)} = \frac{a}{1-r}$$

which implies that $a = -1/11$ and $r = (2/11)(x+3)$. Therefore, the power series for $f(x)$ is given by

$$\frac{1}{2x-5} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty}\left(-\frac{1}{11}\right)\left[\frac{2}{11}(x+3)\right]^n$$

$$= -\sum_{n=0}^{\infty} \frac{2^n(x+3)^n}{11^{n+1}},$$

$$|x+3| < \frac{11}{2} \text{ or } -\frac{17}{2} < x < \frac{5}{2}.$$

3. (a) $\dfrac{1}{2+x} = \dfrac{1/2}{1-(-x/2)} = \dfrac{a}{1-r}$

$$= \sum_{n=0}^{\infty} \frac{1}{2}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1}}$$

This series converges on $(-2, 2)$.

(b)

$$\begin{array}{r l}
 & \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \cdots \\
2 + x \,\big) & 1 \\
 & 1 + \frac{x}{2} \\ \hline
 & -\frac{x}{2} \\
 & -\frac{x}{2} - \frac{x^2}{4} \\ \hline
 & \frac{x^2}{4} \\
 & \frac{x^2}{4} + \frac{x^3}{8} \\ \hline
 & -\frac{x^3}{8} \\
 & -\frac{x^3}{8} - \frac{x^4}{16} \\ \hline
 & \vdots
\end{array}$$

7. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{3}{2x-1} = \frac{-3}{1-2x} = \frac{a}{1-r}$$

which implies that $a = -3$ and $r = 2x$.

Therefore, the power series for $f(x)$ is given by

$$\frac{3}{2x-1} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} (-3)(2x)^n$$

$$= -3\sum_{n=0}^{\infty} (2x)^n,\ |2x| < 1 \text{ or } -\frac{1}{2} < x < \frac{1}{2}.$$

11. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{3}{x+2} = \frac{3}{2+x} = \frac{3/2}{1+(1/2)x} = \frac{a}{1-r}$$

which implies that $a = 3/2$ and $r = (-1/2)x$. Therefore, the power series for $f(x)$ is given by

$$\frac{3}{x+2} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{3}{2}\left(-\frac{1}{2}x\right)^n$$

$$= 3\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1}} = \frac{3}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n,$$

$|x| < 2$ or $-2 < x < 2$.

13. $\dfrac{3x}{x^2+x-2} = \dfrac{2}{x+2} + \dfrac{1}{x-1} = \dfrac{2}{2+x} + \dfrac{1}{-1+x} = \dfrac{1}{1+(1/2)x} + \dfrac{-1}{1-x}$

Writing $f(x)$ as a sum of two geometric series, we have

$$\frac{3x}{x^2+x-2} = \sum_{n=0}^{\infty}\left(-\frac{1}{2}x\right)^n + \sum_{n=0}^{\infty}(-1)(x)^n = \sum_{n=0}^{\infty}\left[\frac{1}{(-2)^n} - 1\right]x^n.$$

The interval of convergence is $-1 < x < 1$ since

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(1-(-2)^{n+1})x^{n+1}}{(-2)^{n+1}} \cdot \frac{(-2)^n}{(1-(-2)^n)x^n}\right| = \lim_{n\to\infty}\left|\frac{(1-(-2)^{n+1})x}{-2-(-2)^{n+1}}\right| = |x|.$$

15. $\dfrac{2}{1-x^2} = \dfrac{1}{1-x} + \dfrac{1}{1+x}$

Writing $f(x)$ as a sum of two geometric series, we have

$$\frac{2}{1-x^2} = \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(1+(-1)^n)x^n = \sum_{n=0}^{\infty} 2x^{2n}.$$

The interval of convergence is $|x^2| < 1$ or $-1 < x < 1$ since $\lim\limits_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{2x^{2n+2}}{2x^2}\right| = |x^2|.$

17. $\dfrac{1}{1+x} = \sum\limits_{n=0}^{\infty}(-1)^n x^n$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty}(-1)^n(-x)^n = \sum_{n=0}^{\infty}(-1)^{2n}x^n = \sum_{n=0}^{\infty} x^n$$

$$h(x) = \frac{-2}{x^2-1} = \frac{1}{1+x} + \frac{1}{1-x} = \sum_{n=0}^{\infty}(-1)^n x^n + \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty}[(-1)^n + 1]x^n$$

$$= 2 + 0x + 2x^2 + 0x^3 + 2x^4 + 0x^5 + 2x^6 + \cdots = \sum_{n=0}^{\infty} 2x^{2n},\ -1 < x < 1 \text{ (See Exercise 15.)}$$

19. By taking the first derivative, we have $\dfrac{d}{dx}\left[\dfrac{1}{x+1}\right] = \dfrac{-1}{(x+1)^2}$. Therefore,

$$\frac{-1}{(x+1)^2} = \frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n x^n\right] = \sum_{n=1}^{\infty}(-1)^n n x^{n-1}$$

$$= \sum_{n=0}^{\infty}(-1)^{n+1}(n+1)x^n,\ -1 < x < 1.$$

21. By integrating, we have $\displaystyle\int \frac{1}{x+1}\,dx = \ln(x+1)$. Therefore,

$$\ln(x+1) = \int\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]dx = C + \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1},\ -1 < x \le 1.$$

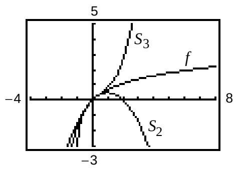
To solve for C, let $x = 0$ and conclude that $C = 0$. Therefore,

$$\ln(x+1) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1},\ -1 < x \le 1.$$

23. $\dfrac{1}{x^2+1} = \sum\limits_{n=0}^{\infty}(-1)^n(x^2)^n = \sum\limits_{n=0}^{\infty}(-1)^n x^{2n},\ -1 < x < 1$

25. Since, $\dfrac{1}{x+1} = \sum\limits_{n=0}^{\infty}(-1)^n x^n$, we have $\dfrac{1}{4x^2+1} = \sum\limits_{n=0}^{\infty}(-1)^n(4x^2)^n = \sum\limits_{n=0}^{\infty}(-1)^n 4^n x^{2n} = \sum\limits_{n=0}^{\infty}(-1)^n(2x)^{2n},\ -\dfrac{1}{2} < x < \dfrac{1}{2}.$

27. $x - \frac{x^2}{2} \le \ln(x+1) \le x - \frac{x^2}{2} + \frac{x^3}{3}$



x	0.0	0.2	0.4	0.6	0.8	1.0
$x - \frac{x^2}{2}$	0.000	0.180	0.320	0.420	0.480	0.500
$\ln(x+1)$	0.000	0.180	0.336	0.470	0.588	0.693
$x - \frac{x^2}{2} + \frac{x^3}{3}$	0.000	0.183	0.341	0.492	0.651	0.833

29. $g(x) = x$, line, Matches (c)

31. $g(x) = x - \frac{x^3}{3} + \frac{x^5}{5}$, Matches (a)

33. $f(x) = \arctan x$ is an odd function (symmetric to the origin)

In Exercises 35 and 37, $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

35. $\arctan \frac{1}{4} = \sum_{n=0}^{\infty} (-1)^n \frac{(1/4)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)4^{2n+1}} = \frac{1}{4} - \frac{1}{192} + \frac{1}{5120} + \cdots$

Since $\frac{1}{5120} < 0.001$, we can approximate the series by its first two terms: $\arctan \frac{1}{4} \approx \frac{1}{4} - \frac{1}{192} \approx 0.245$.

37.
$$\frac{\arctan x^2}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{2n+1}$$
$$\int \frac{\arctan x^2}{x}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(4n+2)(2n+1)}$$
$$\int_0^{1/2} \frac{\arctan x^2}{x}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(4n+2)(2n+1)2^{4n+2}} = \frac{1}{8} - \frac{1}{1152} + \cdots$$

Since $\frac{1}{1152} < 0.001$, we can approximate the series by its first term: $\int_0^{1/2} \frac{\arctan x^2}{x}\,dx \approx 0.125$

In Exercises 39 and 41, use $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x| < 1$.

39. (a) $\frac{1}{(1-x)^2} = \frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{d}{dx}\left[\sum_{n=0}^{\infty} x^n\right] = \sum_{n=1}^{\infty} nx^{n-1}$, $|x| < 1$

(b) $\frac{x}{(1-x)^2} = x\sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^n$, $|x| < 1$

(c) $\frac{1+x}{(1-x)^2} = \frac{1}{(1-x)^2} + \frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n(x^{n-1} + x^n)$, $|x| < 1$

$= \sum_{n=0}^{\infty} (2n+1)x^n$, $|x| < 1$

(d) $\frac{x(1+x)}{(1-x)^2} = x\sum_{n=0}^{\infty} (2n+1)x^n = \sum_{n=0}^{\infty} (2n+1)x^{n+1}$, $|x| < 1$

41. $P(n) = \left(\frac{1}{2}\right)^n$

$$E(n) = \sum_{n=1}^{\infty} nP(n) = \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \frac{1}{2}\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^{n-1}$$
$$= \frac{1}{2}\frac{1}{[1-(1/2)]^2} = 2$$

Since the probability of obtaining a head on a single toss is $\frac{1}{2}$, it is expected that, on average, a head will be obtained in two tosses.

43. Replace x with $(-x)$.

45. Replace x with $(-x)$ and multiply the series by 5.

47. Let $\arctan x + \arctan y = \theta$. Then,

$$\tan(\arctan x + \arctan y) = \tan\theta$$

$$\frac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x)\tan(\arctan y)} = \tan\theta$$

$$\frac{x+y}{1-xy} = \tan\theta$$

$$\arctan\left(\frac{x+y}{1-xy}\right) = \theta. \text{ Therefore, } \arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right) \text{ for } xy \neq 1.$$

49. (a) $2\arctan\frac{1}{2} = \arctan\frac{1}{2} + \arctan\frac{1}{2} = \arctan\left[\frac{2(1/2)}{1-(1/2)^2}\right] = \arctan\frac{4}{3}$

$$2\arctan\frac{1}{2} - \arctan\frac{1}{7} = \arctan\frac{4}{3} + \arctan\left(-\frac{1}{7}\right) = \arctan\left[\frac{(4/3)-(1/7)}{1+(4/3)(1/7)}\right] = \arctan\frac{25}{25} = \arctan 1 = \frac{\pi}{4}$$

(b) $\pi = 8\arctan\frac{1}{2} - 4\arctan\frac{1}{7} \approx 8\left[\frac{1}{2} - \frac{(0.5)^3}{3} + \frac{(0.5)^5}{5} - \frac{(0.5)^7}{7}\right] - 4\left[\frac{1}{7} - \frac{(1/7)^3}{3} + \frac{(1/7)^5}{5} - \frac{(1/7)^7}{7}\right] \approx 3.14$

51. From Exercise 21, we have

$$\ln(x+1) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n}.$$

Thus, $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2^n n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(1/2)^n}{n}$

$$= \ln\left(\frac{1}{2}+1\right) = \ln\frac{3}{2} \approx 0.4055$$

53. From Exercise 51, we have

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{5^n n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2/5)^n}{n}$$

$$= \ln\left(\frac{2}{5}+1\right) = \ln\frac{7}{5} \approx 0.3365.$$

55. From Exercise 54, we have

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^{2n+1}(2n+1)} = \sum_{n=0}^{\infty}(-1)^n\frac{(1/2)^{2n+1}}{2n+1} = \arctan\frac{1}{2} \approx 0.4636.$$

57. The series in Exercise 54 converges to its sum at a slower rate because its terms approach 0 at a much slower rate.

59. $f(x) = \displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n},\ 0 < x \le 2$

$$f(0.5) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-0.5)^n}{n} = \sum_{n=1}^{\infty} -\frac{(1/2)^n}{n}$$

$$\sum_{n=1}^{\infty} -\frac{(1/2)^n}{n} = -0.6931$$

Section 8.10 Taylor and Maclaurin Series

1. For $c = 0$, we have:

$$f(x) = e^{2x}$$

$$f^{(n)}(x) = 2^n e^{2x} \Rightarrow f^{(n)}(0) = 2^n$$

$$e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \frac{16x^4}{4!} + \cdots = \sum_{n=0}^{\infty}\frac{(2x)^n}{n!}$$

3. For $c = \pi/4$, we have:

$$f(x) = \cos(x) \qquad f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f'(x) = -\sin(x) \qquad f'\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$f''(x) = -\cos(x) \qquad f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$f'''(x) = \sin(x) \qquad f'''\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f^{(4)}(x) = \cos(x) \qquad f^{(4)}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

and so on. Therefore, we have:

$$\begin{aligned}\cos x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/4)[x - (\pi/4)]^n}{n!} \\ &= \frac{\sqrt{2}}{2}\left[1 - \left(x - \frac{\pi}{4}\right) - \frac{[x - (\pi/4)]^2}{2!} + \frac{[x - (\pi/4)]^3}{3!} + \frac{[x - (\pi/4)]^4}{4!} - \cdots\right] \\ &= \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n(n+1)/2}[x - (\pi/4)]^n}{n!}.\end{aligned}$$

[**Note:** $(-1)^{n(n+1)/2} = 1, -1, -1, 1, 1, -1, -1, 1, \ldots$]

5. For $c = 1$, we have,

$$f(x) = \ln x \qquad f(1) = 0$$

$$f'(x) = \frac{1}{x} \qquad f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2} \qquad f''(1) = -1$$

$$f'''(x) = \frac{2}{x^3} \qquad f'''(1) = 2$$

$$f^{(4)}(x) = -\frac{6}{x^4} \qquad f^{(4)}(1) = -6$$

$$f^{(5)}(x) = \frac{24}{x^5} \qquad f^{(5)}(1) = 24$$

and so on. Therefore, we have:

$$\begin{aligned}\ln x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x - 1)^n}{n!} \\ &= 0 + (x - 1) - \frac{(x - 1)^2}{2!} + \frac{2(x - 1)^3}{3!} - \frac{6(x - 1)^4}{4!} + \frac{24(x - 1)^5}{5!} - \cdots \\ &= (x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \frac{(x - 1)^4}{4} + \frac{(x - 1)^5}{5} - \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(x - 1)^{n+1}}{n + 1}\end{aligned}$$

7. For $c = 0$, we have:

$$\begin{aligned} f(x) &= \sin 2x & f(0) &= 0 \\ f'(x) &= 2\cos 2x & f'(0) &= 2 \\ f''(x) &= -4\sin 2x & f''(0) &= 0 \\ f'''(x) &= -8\cos 2x & f'''(0) &= -8 \\ f^{(4)}(x) &= 16\sin 2x & f^{(4)}(0) &= 0 \\ f^{(5)}(x) &= 32\cos 2x & f^{(5)}(0) &= 32 \\ f^{(6)}(x) &= -64\sin 2x & f^{(6)}(0) &= 0 \\ f^{(7)}(x) &= -128\cos 2x & f^{(7)}(0) &= -128 \end{aligned}$$

and so on. Therefore, we have:

$$\begin{aligned} \sin 2x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 0 + 2x + \frac{0x^2}{2!} - \frac{8x^3}{3!} + \frac{0x^4}{4!} + \frac{32x^5}{5!} + \frac{0x^6}{6!} - \frac{128x^7}{7!} + \cdots \\ &= 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n(2x)^{2n+1}}{(2n+1)!} \end{aligned}$$

9. For $c = 0$, we have:

$$\begin{aligned} f(x) &= \sec(x) & f(0) &= 1 \\ f'(x) &= \sec(x)\tan(x) & f'(0) &= 0 \\ f''(x) &= \sec^3(x) + \sec(x)\tan^2(x) & f''(0) &= 1 \\ f'''(x) &= 5\sec^3(x)\tan(x) + \sec(x)\tan^3(x) & f'''(0) &= 0 \\ f^{(4)}(x) &= 5\sec^5(x) + 18\sec^3(x)\tan^2(x) + \sec(x)\tan^4(x) & f^{(4)}(0) &= 5 \end{aligned}$$

$$\sec(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \cdots$$

11. The Maclaurin series for $f(x) = \cos x$ is $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)x^{2n}}{(2n)!}$.

Because $f^{(n+1)}(x) = \pm\sin x$ or $\pm\cos x$, we have $|f^{(n+1)}(z)| \le 1$ for all z. Hence by Taylor's Theorem,

$$0 \le |Rn(x)| = \left|\frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}\right| \le \frac{|x|^{n+1}}{(n+1)!}.$$

Since $\displaystyle\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$, it follows that $Rn(x) \to 0$ as $n \to \infty$. Hence, the Maclaurin series for $\cos x$ converges to $\cos x$ for all x.

13. Since $(1 + x)^{-k} = 1 - kx + \dfrac{k(k+1)x^2}{2!} - \dfrac{k(k+1)(k+2)x^3}{3!} + \cdots$, we have

$$\begin{aligned} (1 + x)^{-2} &= 1 - 2x + \frac{2(3)x^2}{2!} - \frac{2(3)(4)x^3}{3!} + \frac{2(3)(4)(5)x^4}{5!} - \cdots = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n(n+1)x^n. \end{aligned}$$

15. $\dfrac{1}{\sqrt{4+x^2}} = \left(\dfrac{1}{2}\right)\left[1 + \left(\dfrac{x}{2}\right)^2\right]^{-1/2}$ and since $(1+x)^{-1/2} = 1 + \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, 1 \cdot 3 \cdot 5 \cdots (2n-1)x^n}{2^n n!}$, we have

$$\frac{1}{\sqrt{4+x^2}} = \frac{1}{2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n\, 1 \cdot 3 \cdot 5 \cdots (2n-1)(x/2)^{2n}}{2^n\, n!}\right] = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n\, 1 \cdot 3 \cdot 5 \cdots (2n-1)x^{2n}}{2^{3n+1} n!}.$$

17. Since $(1+x)^{1/2} = 1 + \dfrac{x}{2} + \displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}\, 1 \cdot 3 \cdot 5 \cdots (2n-3)x^n}{2^n\, n!}$ (Exercise 14)

we have $(1+x^2)^{1/2} = 1 + \dfrac{x^2}{2} + \displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}\, 1 \cdot 3 \cdot 5 \cdots (2n-3)x^{2n}}{2^n\, n!}.$

19.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^{x^2/2} = \sum_{n=0}^{\infty} \frac{(x^2/2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = 1 + \frac{x^2}{2} + \frac{x^4}{2^2 2!} + \frac{x^6}{2^3 3!} + \frac{x^8}{2^4 4!} + \cdots$$

21.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sin 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{2n+1} x^{2n+1}}{(2n+1)!} = 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \cdots$$

23.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\cos x^{3/2} = \sum_{n=0}^{\infty} \frac{(-1)^n (x^{3/2})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{(2n)!} = 1 - \frac{x^3}{2!} + \frac{x^6}{4!} - \cdots$$

25.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots$$

$$e^x - e^{-x} = 2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \frac{2x^7}{7!} + \cdots$$

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

27.

$$\cos^2(x) = \frac{1}{2}[1 + \cos(2x)]$$

$$= \frac{1}{2}\left[1 + 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} - \cdots\right]$$

$$= \frac{1}{2}\left[1 + \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}\right]$$

29.

$$x \sin x = x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

$$= x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)!}$$

31.

$$\frac{\sin x}{x} = \frac{x - (x^3/3!) + (x^5/5!) - \cdots}{x}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!},\ x \neq 0$$

33. $$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \cdots$$

$$e^{-ix} = 1 - ix + \frac{(-ix)^2}{2!} + \frac{(-ix)^3}{3!} + \frac{(-ix)^4}{4!} + \cdots = 1 - ix - \frac{x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{4!} - \frac{ix^5}{5!} - \frac{x^6}{6!} + \cdots$$

$$e^{ix} - e^{-ix} = 2ix - \frac{2ix^3}{3!} + \frac{2ix^5}{5!} - \frac{2ix^7}{7!} + \cdots$$

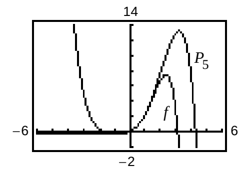
$$\frac{e^{ix} - e^{-ix}}{2i} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin(x)$$

35. $f(x) = e^x \sin x$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right)$$

$$= x + x^2 + \left(\frac{x^3}{2} - \frac{x^3}{6}\right) + \left(\frac{x^4}{6} - \frac{x^4}{6}\right) + \left(\frac{x^5}{120} - \frac{x^5}{12} + \frac{x^5}{24}\right) + \cdots$$

$$= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \cdots$$

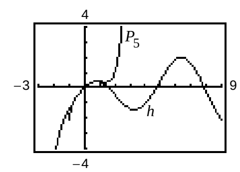


37. $h(x) = \cos x \ln(1 + x)$

$$= \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right)$$

$$= x - \frac{x^2}{2} + \left(\frac{x^3}{3} - \frac{x^3}{2}\right) + \left(\frac{x^4}{4} - \frac{x^4}{4}\right) + \left(\frac{x^5}{5} - \frac{x^5}{6} + \frac{x^5}{24}\right) + \cdots$$

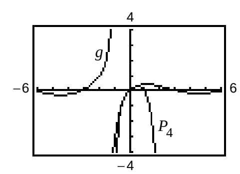
$$= x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{3x^5}{40} + \cdots$$



39. $g(x) = \dfrac{\sin x}{1 + x}$. Divide the series for $\sin x$ by $(1 + x)$.

$$\begin{array}{r} x - x^2 + \frac{5x^2}{6} - \frac{5x^4}{6} + \\ 1 + x \,\big)\, x + 0x^2 - \frac{x^3}{6} + 0x^4 + \frac{x^5}{120} + \cdots \\ \underline{x + x^2} \\ -x^2 - \frac{x^3}{6} \\ \underline{-x^2 - x^3} \\ \frac{5x^3}{6} + 0x^4 \\ \underline{\frac{5x^3}{6} + \frac{5x^4}{6}} \\ -\frac{5x^4}{6} + \frac{x^5}{120} \\ \underline{-\frac{5x^4}{6} - \frac{5x^5}{6}} \\ \vdots \end{array}$$

$$g(x) = x - x^2 + \frac{5x^3}{6} - \frac{5x^4}{6} + \cdots$$



41. $y = x^2 - \dfrac{x^4}{3!} = x\left(x - \dfrac{x^3}{3!}\right) \approx x \sin x.$

Matches (a)

43. $y = x + x^2 + \dfrac{x^3}{2!} = x\left(1 + x + \dfrac{x^2}{2!}\right) \approx xe^x.$

Matches (c)

45. $\displaystyle\int_0^x (e^{-t^2} - 1)\,dt = \int_0^x \left[\left(\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}\right) - 1\right] dt$

$$= \int_0^x \left[\sum_{n=0}^{\infty} \frac{(-1)^{n+1} t^{2n+2}}{(n+1)!}\right] dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^{n+1} t^{2n+3}}{(2n+3)(n+1)!}\right]_0^x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)(n+1)!}$$

47. Since $\displaystyle \ln x = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$

we have $\displaystyle \ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \approx 0.6931.$ (10,001 terms)

49. Since $\displaystyle e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$

we have $\displaystyle e^2 = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{2^n}{n!} \approx 7.3891.$ (12 terms)

51. Since

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+2)!}$$

$$\frac{1 - \cos}{x} = \frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} - \frac{x^7}{8!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+2)!}$$

we have $\displaystyle \lim_{x\to 0} \frac{1 - \cos x}{x} = \lim_{x\to 0} \sum_{n=0}^{\infty} \frac{(-1) x^{2n+1}}{(2n+2)!} = 0.$

53. $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx = \int_0^1 \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}\right] dx = \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}\right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!}$

Since $1/(7 \cdot 7!) < 0.0001$, we have

$$\int_0^1 \frac{\sin x}{x}\,dx = 1 - \frac{1}{3 \cdot 3!} + \frac{1}{5 \cdot 5!} - \cdots \approx 0.9461.$$

Note: We are using $\displaystyle \lim_{x\to 0^+} \frac{\sin x}{x} = 1.$

55. $\displaystyle\int_0^{\pi/2} \sqrt{x} \cos x\,dx = \int_0^{\pi/2} \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{(4n+1)/2}}{(2n)!}\right] dx = \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{(4n+3)/2}}{\left(\frac{4n+3}{2}\right)(2n)!}\right]_0^{\pi/2} = \left[\sum_{n=0}^{\infty} \frac{(-1)^n 2x^{(4n+3)/2}}{(4n+3)(2n)!}\right]_0^{\pi/2}$

Since $(\pi/2)^{19/2}/766{,}080 < 0.0001$, we have

$$\int_0^1 \sqrt{x} \cos x\,dx = 2\left[\frac{(\pi/2)^{3/2}}{3} - \frac{(\pi/2)^{7/2}}{14} + \frac{(\pi/2)^{11/2}}{264} - \frac{(\pi/2)^{15/2}}{10{,}800} + \frac{(\pi/2)^{19/2}}{766{,}080}\right] \approx 0.7040.$$

57. $\displaystyle\int_{0.1}^{0.3} \sqrt{1 + x^3}\,dx = \int_{0.1}^{0.3} \left(1 + \frac{x^3}{2} - \frac{x^6}{8} + \frac{x^9}{16} - \frac{5x^{12}}{128} + \cdots\right) dx = \left[x + \frac{x^4}{8} - \frac{x^7}{56} + \frac{x^{10}}{160} - \frac{5x^{13}}{1664} + \cdots\right]_{0.1}^{0.3}$

Since $\frac{1}{56}(0.3^7 - 0.1^7) < 0.0001$, we have

$$\int_{0.1}^{0.3} \sqrt{1 + x^3}\,dx = \left[(0.3 - 0.1) + \frac{1}{8}(0.3^4 - 0.1^4) - \frac{1}{56}(0.3^7 - 0.1^7)\right] \approx 0.2010.$$

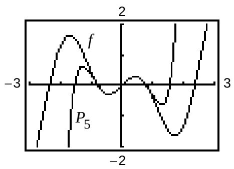
59. From Exercise 19, we have

$$\frac{1}{\sqrt{2\pi}}\int_0^1 e^{-x^2/2}\,dx = \frac{1}{\sqrt{2\pi}}\int_0^1 \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^n n!}\,dx = \frac{1}{\sqrt{2\pi}}\left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2^n n!(2n+1)}\right]_0^1 = \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!(2n+1)}$$

$$\approx \frac{1}{\sqrt{2\pi}}\left[1 - \frac{1}{2\cdot 1\cdot 3} + \frac{1}{2^2\cdot 2!\cdot 5} - \frac{1}{2^3\cdot 3!\cdot 7}\right] \approx 0.3414.$$

61. $f(x) = x\cos 2x = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n 4^n x^{2n+1}}{(2n)!}$

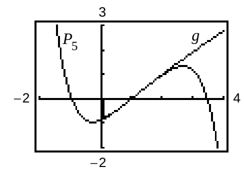
$P_5(x) = x - 2x^3 + \dfrac{2x^5}{3}$



The polynomial is a reasonable approximation on the interval $\left[-\frac{3}{4}, \frac{3}{4}\right]$.

63. $f(x) = \sqrt{x}\ln x,\ c = 1$

$P_5(x) = (x-1) - \dfrac{(x-1)^3}{24} + \dfrac{(x-1)^4}{24} - \dfrac{71(x-1)^5}{1920}$



The polynomial is a reasonable approximation on the interval $\left[\frac{1}{4}, 2\right]$.

65. See Guidelines, page 636.

67. (a) Replace x with $(-x)$.

(b) Replace x with $3x$.

(c) Multiply series by x.

(d) Replace x with $2x$, then replace x with $-2x$, and add the two together.

69. $y = \left(\tan\theta - \dfrac{g}{kv_0\cos\theta}\right)x - \dfrac{g}{k^2}\ln\left(1 - \dfrac{kx}{v_0\cos\theta}\right)$

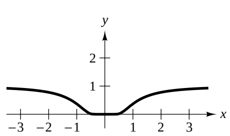
$$= (\tan\theta)x - \frac{gx}{kv_0\cos\theta} - \frac{g}{k^2}\left[-\frac{kx}{v_0\cos\theta} - \frac{1}{2}\left(\frac{kx}{v_0\cos\theta}\right)^2 - \frac{1}{3}\left(\frac{kx}{v_0\cos\theta}\right)^3 - \frac{1}{4}\left(\frac{kx}{v_o\cos\theta}\right)^4 - \cdots\right]$$

$$= (\tan\theta)x - \frac{gx}{kv_0\cos\theta} + \frac{gx}{kv_0\cos\theta} + \frac{gx^2}{2v_0^2\cos^2\theta} + \frac{gkx^3}{3v_0^3\cos^3\theta} + \frac{gk^2x^4}{4v_0^4\cos^4\theta} + \cdots\Bigg]$$

$$= (\tan\theta)x + \frac{gx^2}{2v_0^2\cos^2\theta} + \frac{kgx^3}{3v_0^3\cos^3\theta} + \frac{k^2gx^4}{4v_0^4\cos^4\theta} + \cdots$$

71. $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(a)



(b) $f'(0) = \displaystyle\lim_{x\to 0}\frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0}\frac{e^{-1/x^2} - 0}{x}$

Let $y = \displaystyle\lim_{x\to 0}\frac{e^{-1/x^2}}{x}$. Then

$$\ln y = \lim_{x\to 0}\ln\left(\frac{e^{-1/x^2}}{x}\right) = \lim_{x\to 0^+}\left[-\frac{1}{x^2} - \ln x\right] = \lim_{x\to 0^+}\left[\frac{-1 - x^2\ln x}{x^2}\right] = -\infty.$$

Thus, $y = e^{-\infty} = 0$ and we have $f'(0) = 0$.

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \cdots = 0 \neq f(x)$

This series converges to f at $x = 0$ only.

73. By the Ratio Test: $\displaystyle\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = \lim_{n\to\infty}\frac{|x|}{n+1} = 0$ which shows that $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$ converges for all x.