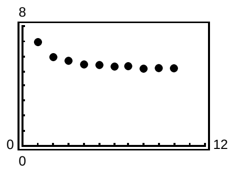
Review Exercises for Chapter 8

1. $a_n = \dfrac{1}{n!}$

3. $a_n = 4 + \dfrac{2}{n}$: 6, 5, 4.67, . . .
Matches (a)

5. $a_n = 10(0.3)^{n-1}$: 10, 3, . . .
Matches (d)

7. $a_n = \dfrac{5n + 2}{n}$



The sequence seems to converge to 5.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5n+2}{n}$$
$$= \lim_{n\to\infty} \left(5 + \frac{2}{n}\right) = 5$$

9. $\lim_{n\to\infty} \dfrac{n+1}{n^2} = 0$
Converges

11. $\lim_{n\to\infty} \dfrac{n^3}{n^2+1} = \infty$

13. $\lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right) = \lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right)\dfrac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \lim_{n\to\infty} \dfrac{1}{\sqrt{n+1} + \sqrt{n}} = 0$ Converges

15. $\lim_{n\to\infty} \dfrac{\sin(n)}{\sqrt{n}} = 0$
Converges

17. $A_n = 5000\left(1 + \dfrac{0.05}{4}\right)^n = 5000(1.0125)^n$
$n = 1, 2, 3$

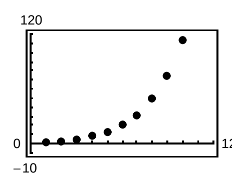
(a) $A_1 = 5062.50$ $A_5 \approx 5320.41$
$A_2 \approx 5125.78$ $A_6 \approx 5386.92$
$A_3 \approx 5189.85$ $A_7 \approx 5454.25$
$A_4 \approx 5254.73$ $A_8 \approx 5522.43$

(b) $A_{40} \approx 8218.10$

19. (a)

k	5	10	15	20	25
S_k	13.2	113.3	873.8	6448.5	50,500.3

(b)

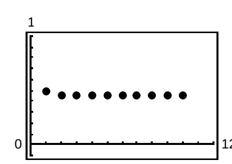


(c) The series diverges $\left(\text{geometric } r = \frac{3}{2} > 1\right)$

21. (a)

k	5	10	15	20	25
S_k	0.4597	0.4597	0.4597	0.4597	0.4597

(b)



(c) The series converges by the Alternating Series Test.

23. Converges. Geometric series, $r = 0.82$, $|r| < 1$.

25. Diverges. nth Term Test. $\lim_{n\to\infty} a_n \neq 0$.

27. $\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n$

Geometric series with $a = 1$ and $r = \frac{2}{3}$.

$$S = \frac{a}{1-r} = \frac{1}{1-(2/3)} = \frac{1}{1/3} = 3$$

29. $\sum_{n=0}^{\infty}\left(\frac{1}{2^n} - \frac{1}{3^n}\right) = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n - \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n$

$$= \frac{1}{1-(1/2)} - \frac{1}{1-(1/3)} = 2 - \frac{3}{2} = \frac{1}{2}$$

31. $0.\overline{09} = 0.09 + 0.0009 + 0.000009 + \cdots = 0.09(1 + 0.01 + 0.0001 + \cdots) = \sum_{n=0}^{\infty}(0.09)(0.01)^n = \frac{0.09}{1-0.01} = \frac{1}{11}$

33. $D_1 = 8$

$D_2 = 0.7(8) + 0.7(8) = 16(0.7)$

$\vdots$

$D = 8 + 16(0.7) + 16(0.7)^2 + \cdots + 16(0.7)^n + \cdots$

$$= -8 + \sum_{n=0}^{\infty} 16(0.7)^n = -8 + \frac{16}{1-0.7} = 45\tfrac{1}{3} \text{ meters}$$

35. See Exercise 86 in Section 8.2.

$$A = \frac{P(e^{rt}-1)}{e^{r/12}-1}$$

$$= \frac{200(e^{(0.06)(2)}-1)}{e^{0.06/12}-1}$$

$$\approx \$5087.14$$

37. $\int_1^{\infty} x^{-4}\ln(x)\,dx = \lim_{b\to\infty}\left[-\frac{\ln x}{3x^3} - \frac{1}{9x^3}\right]_1^b$

$$= 0 + \frac{1}{9} = \frac{1}{9}$$

By the Integral Test, the series converges.

39. $\sum_{n=1}^{\infty}\left(\frac{1}{n^2} - \frac{1}{n}\right) = \sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=1}^{\infty}\frac{1}{n}$

Since the second series is a divergent p-series while the first series is a convergent p-series, the difference diverges.

41. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^3+2n}}$

$$\lim_{n\to\infty}\frac{1/\sqrt{n^3+2n}}{1/(n^{3/2})} = \lim_{n\to\infty}\frac{n^{3/2}}{\sqrt{n^3+2n}} = 1$$

By a limit comparison test with the convergent p-series

$\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$, the series converges.

43. $\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}$

$$a_n = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}$$

$$= \left(\frac{3}{2}\cdot\frac{5}{4}\cdots\frac{2n-1}{2n-2}\right)\frac{1}{2n} > \frac{1}{2n}$$

Since $\sum_{n=1}^{\infty}\frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (harmonic series), so does the original series.

45. Converges by the Alternating Series Test (Conditional convergence)

47. Diverges by the nth Term Test

49. $\sum_{n=1}^{\infty}\frac{n}{e^{n^2}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{e^{(n+1)^2}}\cdot\frac{e^{n^2}}{n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{e^{n^2}(n+1)}{e^{n^2+2n+1}n}\right|$$

$$= \lim_{n\to\infty}\left(\frac{1}{e^{2n+1}}\right)\left(\frac{n+1}{n}\right)$$

$$= (0)(1) = 0 < 1$$

By the Ratio Test, the series converges.

51. $\sum_{n=1}^{\infty}\frac{2^n}{n^3}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)^3}\cdot\frac{n^3}{2^n}\right|$$

$$= \lim_{n\to\infty}\frac{2n^3}{(n+1)^3} = 2$$

Therefore, by the Ratio Test, the series diverges.

53. (a) Ratio Test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)(3/5)^{n+1}}{n(3/5)^n}$

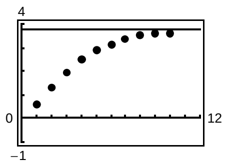
$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)\left(\frac{3}{5}\right) = \frac{3}{5} < 1$$

Converges

(b)

x	5	10	15	20	25
S_n	2.8752	3.6366	3.7377	3.7488	3.7499

(c)



(d) The sum is approximately 3.75.

55. (a) $\int_N^{\infty}\frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_N^{\infty} = \frac{1}{N}$

N	5	10	20	30	40
$\sum_{n=1}^{N}\frac{1}{n^2}$	1.4636	1.5498	1.5962	1.6122	1.6202
$\int_N^{\infty}\frac{1}{x^2}\,dx$	0.2000	0.1000	0.0500	0.0333	0.0250

(b) $\int_N^{\infty}\frac{1}{x^5}\,dx = \left[-\frac{1}{4x^4}\right]_N^{\infty} = \frac{1}{4N^4}$

N	5	10	20	30	40
$\sum_{n=1}^{N}\frac{1}{n^5}$	1.0367	1.0369	1.0369	1.0369	1.0369
$\int_N^{\infty}\frac{1}{x^5}\,dx$	0.0004	0.0000	0.0000	0.0000	0.0000

The series in part (b) converges more rapidly. The integral values represent the remainders of the partial sums.

57. $f(x) = e^{-x/2}$ $\quad f(0) = 1$

$f'(x) = -\frac{1}{2}e^{-x/2}$ $\quad f'(0) = -\frac{1}{2}$

$f''(x) = \frac{1}{4}e^{-x/2}$ $\quad f''(0) = \frac{1}{4}$

$f'''(x) = -\frac{1}{8}e^{-x/2}$ $\quad f'''(0) = -\frac{1}{8}$

$$P_3(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!}$$

$$= 1 - \frac{1}{2}x + \frac{1}{4}\frac{x^2}{2!} - \frac{1}{8}\frac{x^3}{3!}$$

$$= 1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3$$

59. $\sin(95^\circ) = \sin\left(\frac{95\pi}{180}\right) \approx \frac{95\pi}{180} - \frac{(95\pi)^3}{180^3 3!} + \frac{(95\pi)^5}{180^5 5!} - \frac{(95\pi)^7}{180^7 7!} + \frac{(95\pi)^9}{180^9 9!} \approx 0.996$

61. $\ln(1.75) \approx (0.75) - \frac{(0.75)^2}{2} + \frac{(0.75)^3}{3} - \frac{(0.75)^4}{4} + \frac{(0.75)^5}{5} - \frac{(0.75)^6}{6} + \cdots + \frac{(0.75)^{15}}{15} \approx 0.560$

63. $f(x) = \cos x,\ c = 0$

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}$$

$$\left|f^{(n+1)}(z)\right| \le 1 \Rightarrow R_n(x) \le \frac{x^{n+1}}{(n+1)!}$$

(a) $R_n(x) \le \dfrac{(0.5)^{n+1}}{(n+1)!} < 0.001$

This inequality is true for $n = 4$.

(b) $R_n(x) \le \dfrac{(1)^{n+1}}{(n+1)!} < 0.001$

This inequality is true for $n = 6$.

(c) $R_n(x) \le \dfrac{(0.5)^{n+1}}{(n+1)!} < 0.0001$

This inequality is true for $n = 5$.

(d) $R_n(x) \le \dfrac{2^{n+1}}{(n+1)!} < 0.0001$

This inequality is true for $n = 10$.

65. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{x}{10}\right)^n$

Geometric series which converges only if $|x/10| < 1$ or $-10 < x < 10$.

67. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(x-2)^n}{(n+1)^2}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-2)^{n+1}}{(n+2)^2}\cdot\frac{(n+1)^2}{(-1)^n(x-2)^n}\right| = |x-2|$$

$R = 1$
Center: 2

Since the series converges when $x = 1$ and when $x = 3$, the interval of convergence is $1 \le x \le 3$.

69. $\displaystyle\sum_{n=0}^{\infty} n!(x-2)^n$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!(x-2)^{n+1}}{n!(x-2)^n}\right| = \infty$$

which implies that the series converges only at the center $x = 2$.

71.

$$y = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{4^n(n!)^2}$$

$$y' = \sum_{n=1}^{\infty}\frac{(-1)^n(2n)x^{2n-1}}{4^n(n!)^2} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n+2)x^{2n+1}}{4^{n+1}[(n+1)!]^2}$$

$$y'' = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n+2)(2n+1)x^{2n}}{4^{n+1}[(n+1)!]^2}$$

$$\begin{aligned}
x^2y'' + xy' + x^2y &= \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n+2)(2n+1)x^{2n+2}}{4^{n+1}[(n+1)!]^2} + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n+2)x^{2n+2}}{4^{n+1}[(n+1)!]^2} + \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{4^n(n!)^2}\\
&= \sum_{n=0}^{\infty}\left[(-1)^{n+1}\frac{(2n+2)(2n+1)}{4^{n+1}[(n+1)!]^2} + \frac{(-1)^{n+1}(2n+2)}{4^{n+1}[(n+1)!]^2} + \frac{(-1)^n}{4^n(n!)^2}\right]x^{2n+2}\\
&= \sum_{n=0}^{\infty}\left[\frac{(-1)^{n+1}(2n+2)(2n+1+1)}{4^{n+1}[(n+1)!]^2} + (-1)^n\frac{1}{4^n(n!)^2}\right]x^{2n+2}\\
&= \sum_{n=0}^{\infty}\left[\frac{(-1)^{n+1}4(n+1)^2}{4^{n+1}[(n+1)!]^2} + (-1)^n\frac{1}{4^n(n!)^2}\right]x^{2n+2}\\
&= \sum_{n=0}^{\infty}\left[\frac{(-1)^{n+1}1}{4^n(n!)^2} + (-1)^n\frac{1}{4^n(n!)^2}\right]x^{2n+2} = 0
\end{aligned}$$

73. $\dfrac{2}{3-x} = \dfrac{2/3}{1-(x/3)} = \dfrac{a}{1-r}$

$$\sum_{n=0}^{\infty}\frac{2}{3}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}\frac{2x^n}{3^{n+1}}$$

75. Derivative: $\displaystyle\sum_{n=1}^{\infty}\frac{2nx^{n-1}}{3^{n+1}}$

77. $1 + \frac{2}{3}x + \frac{4}{9}x^2 + \frac{8}{27}x^3 + \cdots = \sum_{n=0}^{\infty}\left(\frac{2x}{3}\right)^n = \frac{1}{1-(2x/3)} = \frac{3}{3-2x}, \quad -\frac{3}{2} < x < \frac{3}{2}$

79.
$$f(x) = \sin(x)$$
$$f'(x) = \cos(x)$$
$$f''(x) = -\sin(x)$$
$$f'''(x) = -\cos(x), \cdots$$
$$\sin(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)[x-(3\pi/4)]^n}{n!}$$
$$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x - \frac{3\pi}{4}\right) - \frac{\sqrt{2}}{2\cdot 2!}\left(x - \frac{3\pi}{4}\right)^2 + \cdots = \frac{\sqrt{2}}{2}\sum_{n=0}^{\infty}\frac{(-1)^{n(n+1)/2}[x-(3\pi/4)]^n}{n!}$$

81. $3^x = (e^{\ln(3)})^x = e^{x\ln(3)}$ and since $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$, we have
$$3^x = \sum_{n=0}^{\infty}\frac{(x\ln 3)^n}{n!}$$
$$= 1 + x\ln 3 + \frac{x^2\ln^2 3}{2!} + \frac{x^3\ln^3 3}{3!} + \frac{x^4\ln^4 3}{4!} + \cdots.$$

83.
$$f(x) = \frac{1}{x}$$
$$f'(x) = -\frac{1}{x^2}$$
$$f''(x) = \frac{2}{x^3}$$
$$f'''(x) = -\frac{6}{x^4}, \cdots$$
$$\frac{1}{x} = \sum_{n=0}^{\infty}\frac{f^{(n)}(-1)(x+1)^n}{n!}$$
$$= \sum_{n=0}^{\infty}\frac{-n!(x+1)^n}{n!} = -\sum_{n=0}^{\infty}(x+1)^n$$

85.
$$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \cdots$$
$$(1+x)^{1/5} = 1 + \frac{x}{5} + \frac{(1/5)(-4/5)x^2}{2!} + \frac{1/5(-4/5)(-9/5)x^3}{3!} + \cdots$$
$$= 1 + \frac{1}{5}x - \frac{1\cdot 4x^2}{5^2 2!} + \frac{1\cdot 4\cdot 9x^3}{5^3 3!} - \cdots$$
$$= 1 + \frac{x}{5} + \sum_{n=2}^{\infty}\frac{(-1)^{n+1}4\cdot 9\cdot 14\cdots(5n-6)x^n}{5^n n!}$$
$$= 1 + \frac{x}{5} - \frac{2}{25}x^2 + \frac{6}{125}x^3 - \cdots$$

87.
$$\ln x = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}, \qquad 0 < x \le 2$$
$$\ln\left(\frac{5}{4}\right) = \sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{(5/4)-1}{n}\right)^n$$
$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{4^n n} \approx 0.2231$$

89.
$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}, \quad -\infty < x < \infty$$
$$e^{1/2} = \sum_{n=0}^{\infty}\frac{1}{2^n n!} \approx 1.6487$$

91. $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad -\infty < x < \infty$

$$\cos\left(\frac{2}{3}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}}{3^{2n}(2n)!} \approx 0.7859$$

93. The series for Exercise 41 converges very slowly because the terms approach 0 at a slow rate.

95. (a)

$$f(x) = e^{2x} \qquad f(0) = 1$$
$$f'(x) = 2e^{2x} \qquad f'(0) = 2$$
$$f''(x) = 4e^{2x} \qquad f''(0) = 4$$
$$f'''(x) = 8e^{2x} \qquad f'''(0) = 8$$
$$e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \cdots$$
$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \cdots$$

(b)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \cdots$$
$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \cdots$$

(c)

$$e^{2x} = e^x \cdot e^x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)$$
$$= 1 + (x + x) + \left(x^2 + \frac{x^2}{2} + \frac{x^2}{2}\right) + \left(\frac{x^3}{6} + \frac{x^3}{6} + \frac{x^3}{2} + \frac{x^3}{2}\right) + \cdots = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \cdots$$

97.

$$\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}$$
$$\frac{\sin t}{t} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!}$$
$$\int_0^x \frac{\sin t}{t}\,dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)(2n+1)!}\right]_0^x$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}$$

99.

$$\frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n$$
$$\ln(1+t) = \int \frac{1}{1+t}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{n+1}$$
$$\frac{\ln(t+1)}{t} = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n+1}$$
$$\int_0^x \frac{\ln(t+1)}{t}\,dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{(n+1)^2}\right]_0^x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)^2}$$

101.

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$
$$\frac{\arctan x}{\sqrt{x}} = \sqrt{x} - \frac{x^{5/2}}{3} + \frac{x^{9/2}}{5} - \frac{x^{13/2}}{7} + \frac{x^{17/2}}{9} - \cdots$$
$$\lim_{x\to 0} \frac{\arctan x}{\sqrt{x}} = 0$$

By L'Hôpital's Rule, $\lim_{x\to 0} \frac{\arctan x}{\sqrt{x}} = \lim_{x\to 0} \frac{\left(\frac{1}{1+x^2}\right)}{\left(\frac{1}{2\sqrt{x}}\right)} = \lim_{x\to 0} \frac{2\sqrt{x}}{1+x^2} = 0.$

Problem Solving for Chapter 8

1. (a) $1\left(\frac{1}{3}\right) + 2\left(\frac{1}{9}\right) + 4\left(\frac{1}{27}\right) + \cdots = \sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n = \frac{1/3}{1 - (2/3)} = 1$

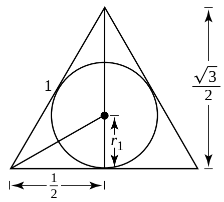
(b) $0, \frac{1}{3}, \frac{2}{3}, 1$, etc.

(c) $\lim_{n\to\infty} C_n = 1 - \sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n = 1 - 1 = 0$

3. If there are n rows, then $a_n = \dfrac{n(n+1)}{2}$.

For one circle,

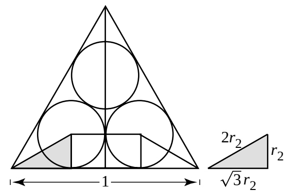
$$a_1 = 1 \text{ and } r_1 = \frac{1}{3}\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{6} = \frac{1}{2\sqrt{3}}$$



For three circles,

$$a_2 = 3 \text{ and } 1 = 2\sqrt{3}r_2 + 2r_2$$

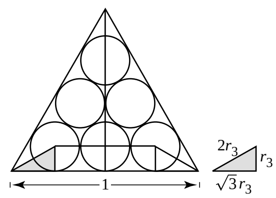
$$r_2 = \frac{1}{2 + 2\sqrt{3}}$$



For six circles,

$$a_3 = 6 \text{ and } 1 = 2\sqrt{3}r_3 + 4r_3$$

$$r_3 = \frac{1}{2\sqrt{3} + 4}$$



Continuing this pattern, $r_n = \dfrac{1}{2\sqrt{3} + 2(n-1)}$.

$$\text{Total Area} = (\pi r_n^{\,2})a_n = \pi\left(\frac{1}{2\sqrt{3} + 2(n-1)}\right)^2 \frac{n(n+1)}{2}$$

$$A_n = \frac{\pi}{2}\,\frac{n(n+1)}{\left[2\sqrt{3} + 2(n+1)^2\right]}$$

$$\lim_{n\to\infty} A_n = \frac{\pi}{2}\cdot\frac{1}{4} = \frac{\pi}{8}$$

5. (a) $\sum a_n x^n = 1 + 2x + 3x^2 + x^3 + 2x^4 + 3x^5 + \cdots$

$$= (1 + x^3 + x^6 + \cdots) + 2(x + x^4 + x^7 + \cdots) + 3(x^2 + x^5 + x^8 + \cdots)$$

$$= (1 + x^3 + x^6 + \cdots)[1 + 2x + 3x^2]$$

$$= (1 + 2x + 3x^2)\frac{1}{1 - x^3}$$

$R = 1$ because each series in the second line has $R = 1$.

(b) $\sum a_n x^n = (a_0 + a_1x + \cdots + a_{p-1}x^{p-1}) + (a_0x^p + a_1x^{p+1} + \cdots) + \cdots$

$$= a_0(1 + x^p + \cdots) + a_1x(1 + x^p + \cdots) + \cdots + a_{p-1}x^{p-1}(1 + x^p + \cdots)$$

$$= (a_0 + a_1x + \cdots + a_{p-1}x^{p-1})(1 + x^p + \cdots)$$

$$= (a_0 + a_1x + \cdots + a_{p-1}x^{p-1})\frac{1}{1 - x^p}.$$

$R = 1$

7. $$e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

$$xe^x = x + x^2 + \frac{x^3}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$$

$$\int xe^x\,dx = xe^x - e^x + C = \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)n!}$$

Letting $x = 0$, $C = 1$. Letting $x = 1$,

$$1 = \sum_{n=0}^{\infty} \frac{1}{(n+2)n!} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{(n+2)n!}.$$

Thus, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+2)n!} = \frac{1}{2}$.

9. Let $a_1 = \displaystyle\int_0^{\pi} \frac{\sin x}{x}\,dx$, $a_2 = -\displaystyle\int_{\pi}^{2\pi} \frac{\sin x}{x}\,dx$, $a_3 = \displaystyle\int_{2\pi}^{3\pi} \frac{\sin x}{x}\,dx$, etc.

Then,

$$\int_0^{\infty} \frac{\sin x}{x}\,dx = a_1 - a_2 + a_3 - a_4 + \cdots.$$

Since $\lim_{n\to\infty} a_n = 0$ and $a_{n+1} < a_n$, this series converges.

11. (a) $a_1 = 3.0$

$a_2 \approx 1.73205$

$a_3 \approx 2.17533$

$a_4 \approx 2.27493$

$a_5 \approx 2.29672$

$a_6 \approx 2.30146$

$\lim_{n\to\infty} a_n = \dfrac{1 + \sqrt{13}}{2}$ [See part (b) for proof.]

(b) Use mathematical induction to show the sequence is increasing. Clearly, $a_2 = \sqrt{a + a_1} = \sqrt{a\sqrt{a}} > \sqrt{a} = a_1$.

Now assume $a_n > a_{n-1}$. Then

$$a_n + a > a_{n-1} + a$$
$$\sqrt{a_n + a} > \sqrt{a_{n-1} + a}$$
$$a_{n+1} > a_n.$$

Use mathematical induction to show that the sequence is bounded above by a. Clearly, $a_1 = \sqrt{a} < a$.

Now assume $a_n < a$. Then $a > a_n$ and $a - 1 > 1$ implies

$$a(a-1) > a_n(1)$$
$$a^2 - a > a_n$$
$$a^2 > a_n + a$$
$$a > \sqrt{a_n + a} = a_{n+1}.$$

Hence, the sequence converges to some number L. To find L, assume $a_{n+1} \approx a_n \approx L$:

$$L = \sqrt{a + L} \implies L^2 = a + L \implies L^2 - L - a = 0$$

$$L = \frac{1 \pm \sqrt{1 + 4a}}{2}.$$

Hence, $L = \dfrac{1 + \sqrt{1 + 4a}}{2}$.

13. (a) $\sum_{n=1}^{\infty} \frac{1}{2^{n+(-1)^n}} = \frac{1}{2^{1-1}} + \frac{1}{2^{2+1}} + \frac{1}{2^{3-1}} + \frac{1}{2^{4+1}} + \frac{1}{2^{5-1}} + \cdots$

$$S_1 = \frac{1}{2^0} = 1$$

$$S_1 = 1 + \frac{1}{8} = \frac{9}{8}$$

$$S_3 = \frac{9}{8} + \frac{1}{4} = \frac{11}{8}$$

$$S_4 = \frac{11}{8} + \frac{1}{32} = \frac{45}{32}$$

$$S_5 = \frac{45}{32} + \frac{1}{16} = \frac{47}{32}$$

(b) $\frac{a_{n+1}}{a_n} = \frac{2^{n+(-1)^n}}{2^{(n+1)+(-1)^{n+1}}} = \frac{2^{(-1)^n}}{2^{1+(-1)^{n+1}}}$

This sequence is $\frac{1}{8}, 2, \frac{1}{8}, 2, \ldots$ which diverges.

(c) $\sqrt[n]{\frac{1}{2^{n+(-1)^n}}} = \left(\frac{1}{2^n \cdot 2^{(-1)^n}}\right)^{1/n}$

$$= \frac{1}{2 \cdot \sqrt[n]{2^{(-1)^n}}} \rightarrow \frac{1}{2} < 1 \text{ converges because } \{2^{(-1)^n}\} = \tfrac{1}{2}, 2, \tfrac{1}{2}, 2, \ldots \text{ and } \sqrt[n]{1/2} \rightarrow 1 \text{ and } \sqrt[n]{2} \rightarrow 1.$$

15. $S_6 = 130 + 70 + 40 = 240$

$S_7 = 240 + 130 + 70 = 440$

$S_8 = 440 + 240 + 130 = 810$

$S_9 = 810 + 440 + 240 = 1490$

$S_{10} = 1490 + 810 + 440 = 2740$