CHAPTER 9
Conics, Parametric Equations, and Polar Coordinates

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Section 9.1 Conics and Calculus

Solutions to Even-Numbered Exercises

2. $x^2 = 8y$

Vertex: $(0, 0)$

$p = 2 > 0$

Opens upward
Matches graph (a).

4. $\dfrac{(x-2)^2}{16} + \dfrac{(y+1)^2}{4} = 1$

Center: $(2, -1)$
Ellipse
Matches (b)

6. $\dfrac{x^2}{9} + \dfrac{y^2}{9} = 1$

Circle radius 3.
Matches (g)

8. $\dfrac{(x-2)^2}{9} - \dfrac{y^2}{4} = 1$

Hyperbola

Center: $(-2, 0)$

Horizontal transverse axis.
Matches (d)

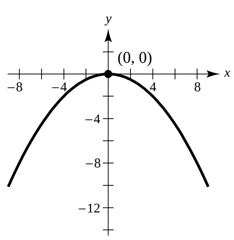
10. $x^2 + 8y = 0$

$$x^2 = 4(-2)y$$

Vertex: $(0, 0)$

Focus: $(0, -2)$

Directrix: $y = 2$



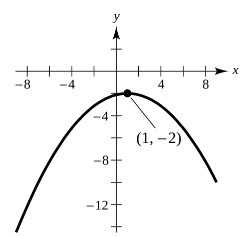
12. $(x-1)^2 + 8(y+2) = 0$

$$(x-1)^2 = 4(-2)(y+2)$$

Vertex: $(1, -2)$

Focus: $(1, -4)$

Directrix: $y = 0$



14. $y^2 + 6y + 8x + 25 = 0$

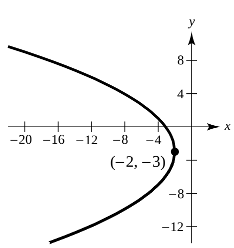
$$y^2 + 6y + 9 = -8x - 25 + 9$$

$$(y+3)^2 = 4(-2)(x+2)$$

Vertex: $(-2, -3)$

Focus: $(-4, -3)$

Directrix: $x = 0$



16. $y^2 + 4y + 8x - 12 = 0$

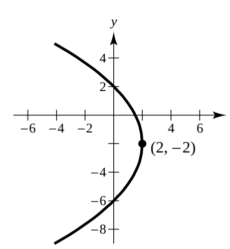
$$y^2 + 4y + 4 = -8x + 12 + 4$$

$$(y+2)^2 = 4(-2)(x-2)$$

Vertex: $(2, -2)$

Focus: $(0, -2)$

Directrix: $x = 4$



18. $y = -\frac{1}{6}(x^2 - 8x + 6) = -\frac{1}{6}(x^2 - 8x + 16 - 10)$

$-6y = (x - 4)^2 - 10$

$-6y + 10 = (x - 4)^2$

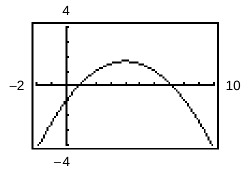
$(x - 4)^2 = -6\left(y - \frac{5}{3}\right)$

$(x - 4)^2 = 4\left(-\frac{3}{2}\right)\left(y - \frac{5}{3}\right)$

Vertex: $\left(4, \frac{5}{3}\right)$

Focus: $\left(4, \frac{1}{6}\right)$

Directrix: $y = \frac{19}{6}$



20. $x^2 - 2x + 8y + 9 = 0$

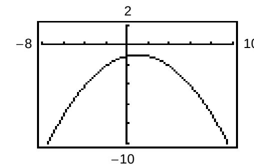
$x^2 - 2x + 1 = -8y - 9 + 1$

$(x - 1)^2 = 4(-2)(y + 1)$

Vertex: $(1, -1)$

Focus: $(1, -3)$

Directrix: $y = 1$



22. $(x + 1)^2 = 4(-2)(y - 2)$

$x^2 + 2x + 8y - 15 = 0$

24. Vertex: $(0, 2)$

$(y - 2)^2 = 4(2)(x - 0)$

$y^2 - 8x - 4y + 4 = 0$

26. $y = 4 - (x - 2)^2 = 4x - x^2$

$x^2 - 4x + y = 0$

28. From Example 2: $4p = 8$ or $p = 2$

Vertex: $(4, 0)$

$(x - 4)^2 = 8(y - 0)$

$x^2 - 8x - 8y + 16 = 0$

30. $5x^2 + 7y^2 = 70$

$$\frac{x^2}{14} + \frac{y^2}{10} = 1$$

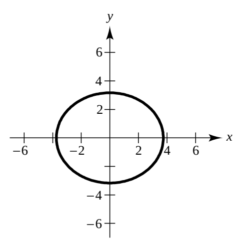
$a^2 = 14, b^2 = 10, c^2 = 4$

Center: $(0, 0)$

Foci: $(\pm 2, 0)$

Vertices: $\left(\pm\sqrt{14}, 0\right)$

$$e = \frac{2}{\sqrt{14}} = \frac{\sqrt{14}}{7}$$



32. $\dfrac{(x + 2)^2}{1} + \dfrac{(y + 4)^2}{1/4} = 1$

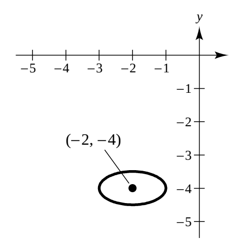
$a^2 = 1, b^2 = \dfrac{1}{4}, c^2 = \dfrac{3}{4}$

Center: $(-2, -4)$

Foci: $\left(-2 \pm \dfrac{\sqrt{3}}{2}, -4\right)$

Vertices: $(-1, -4), (-3, -4)$

$$e = \frac{\sqrt{3}}{2}$$



34. $16x^2 + 25y^2 - 64x + 150y + 279 = 0$

$16(x^2 - 4x + 4) + 25(y^2 + 6y + 0) = -279 + 64 + 225$

$= 10$

$$\frac{(x - 2)^2}{(5/8)} + \frac{(y + 3)^2}{(2/5)} = 1$$

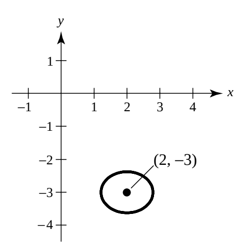
$a^2, \dfrac{5}{8}, b^2 = \dfrac{2}{5}, c^2 = a^2 - b^2 = \dfrac{9}{40}$

Center: $(2, -3)$

Foci: $\left(2 \pm \dfrac{3\sqrt{10}}{20}, -3\right)$

Vertices: $\left(2 \pm \dfrac{\sqrt{10}}{4}, -3\right)$

$$e = \frac{c}{a} = \frac{3}{5}$$



36. $36x^2 + 9y^2 + 48x - 36y + 43 = 0$

$$36\left(x^2 + \frac{4}{3}x + \frac{4}{9}\right) + 9(y^2 - 4y + 4) = -43 + 16 + 36$$

$$= 9$$

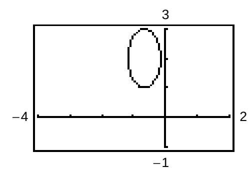
$$\frac{[x + (2/3)]^2}{1/4} + \frac{(y - 2)^2}{1} = 1$$

$a^2 = 1, b^2 = \frac{1}{4}, c^2 = \frac{3}{4}$

Center: $\left(-\frac{2}{3}, 2\right)$

Foci: $\left(-\frac{2}{3}, 2 \pm \frac{\sqrt{3}}{2}\right)$

Vertices: $\left(-\frac{2}{3}, 3\right), \left(-\frac{2}{3}, 1\right)$



Solve for y:

$$9(y^2 - 4y + 4) = -36x^2 - 48x - 43 + 36$$

$$(y - 2)^2 = \frac{-(36x^2 + 48x + 7)}{9}$$

$$y = 2 \pm \frac{1}{3}\sqrt{-(36x^2 + 48x + 7)}$$ (Graph each of these separately.)

38. $2x^2 + y^2 + 4.8x - 6.4y + 3.12 = 0$

$$50x^2 + 25y^2 + 120x - 160y + 78 = 0$$

$$50\left(x^2 + \frac{12}{5}x + \frac{36}{25}\right) + 25\left(y^2 - \frac{32}{5}y + \frac{256}{25}\right) = -78 + 72 + 256 = 250$$

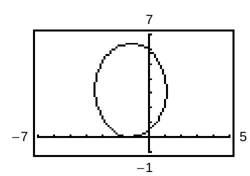
$$\frac{[x + (6/5)]^2}{5} + \frac{[y - (16/5)]^2}{10} = 1$$

$a^2 = 10, b^2 = 5, c^2 = 5$

Center: $\left(-\frac{6}{5}, \frac{16}{5}\right)$

Foci: $\left(-\frac{6}{5}, \frac{16}{5} \pm \sqrt{5}\right)$

Vertices: $\left(-\frac{6}{5}, \frac{16}{5} \pm \sqrt{10}\right)$



Solve for y: $(y^2 - 6.4y + 10.24) = -2x^2 - 4.8x - 3.12 + 10.24$

$$(y - 3.2)^2 = 7.12 - 4x - 2x^2$$

$$y = 3.2 \pm \sqrt{7.12 - 4x - 2x^2}$$ (Graph each of these separately.)

40. Vertices: $(0, 2), (4, 2)$

Eccentricity: $\frac{1}{2}$

Horizontal major axis

Center: $(2, 2)$

$a = 2, c = 1 \Rightarrow b = \sqrt{3}$

$$\frac{(x - 2)^2}{4} + \frac{(y - 2)^2}{3} = 1$$

42 Foci: $(0, \pm 5)$

Major axis length: 14

Vertical major axis

Center: $(0, 0)$

$c = 5, a = 7 \Rightarrow b = \sqrt{24}$

$$\frac{x^2}{24} + \frac{y^2}{49} = 1$$

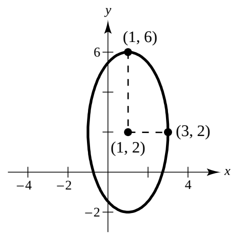
44. Center: $(1, 2)$

Vertical major axis

Points on ellipse: $(1, 6)$, $(3, 2)$

From the sketch, we can see that $h = 1, k = 2, a = 4, b = 2$

$$\frac{(x-1)^2}{4} + \frac{(y-2)^2}{16} = 1.$$



46. $\dfrac{x^2}{25} - \dfrac{y^2}{9} = 1$

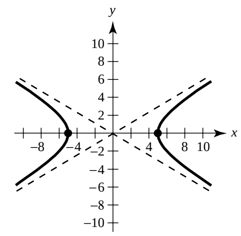
$a = 5, b = 3, c = \sqrt{a^2 + b^2} = \sqrt{34}$

Center: $(0, 0)$

Vertices: $(\pm 5, 0)$

Foci: $\left(\pm\sqrt{34}, 0\right)$

Asymptotes: $y = \pm\dfrac{3}{5}x$

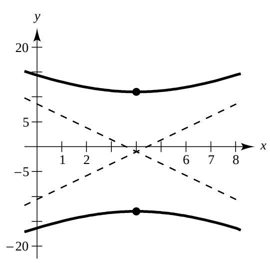


48. $\dfrac{(y+1)^2}{12^2} - \dfrac{(x-4)^2}{5^2} = 1$

$a = 12, b = 5, c = \sqrt{a^2 + b^2} = 13$

Center: $(4, -1)$
Vertices: $(4, 11)$, $(4, -13)$
Foci: $(4, -14)$, $(4, 12)$

Asymptotes: $y = -1 \pm \dfrac{12}{5}(x - 4)$



50. $y^2 - 9x^2 + 36x - 72 = 0$

$$y^2 - 9(x^2 - 4x + 4) = 72 - 36 = 36$$

$$\frac{y^2}{36} - \frac{(x-2)^2}{4} = 1$$

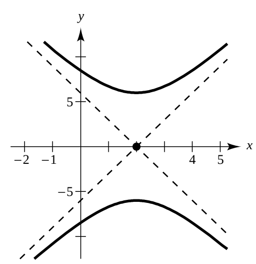
$a = 6, b = 2, c = \sqrt{a^2 + b^2} = 2\sqrt{10}$

Center: $(2, 0)$

Vertices: $(2, 6)$, $(2, -6)$

Foci: $\left(2, 2\sqrt{10}\right), \left(2, -2\sqrt{10}\right)$

Asymptotes: $y = \pm 3(x - 2)$



52. $9(x^2 + 6x + 9) - 4(y^2 - 2y + 1) = -78 + 81 - 4 = -1$

$$9(x+3)^2 - 4(y-1)^2 = -1$$

$$\frac{(y-1)^2}{1/4} - \frac{(x+3)^2}{1/9} = 1$$

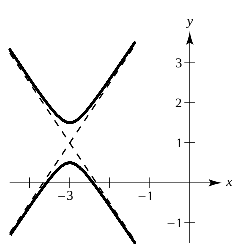
$a = \dfrac{1}{2}, b = \dfrac{1}{3}, c = \dfrac{\sqrt{13}}{6}$

Center: $(-3, 1)$

Vertices: $\left(-3, \dfrac{1}{2}\right), \left(-3, \dfrac{3}{2}\right)$

Foci: $\left(-3, 1 \pm \dfrac{1}{6}\sqrt{13}\right)$

Asymptotes: $y = 1 \pm \dfrac{3}{2}(x + 3)$



54.
$$9x^2 - y^2 + 54x + 10y + 55 = 0$$

$$9(x^2 + 6x + 9) - (y^2 - 10y + 25) = -55 + 81 - 25 = 1$$

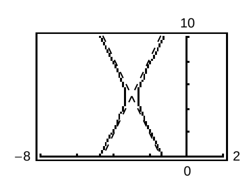
$$\frac{(x+3)^2}{1/9} - \frac{(y-5)^2}{1} = 1$$

$a = \dfrac{1}{3}, b = 1, c = \dfrac{\sqrt{10}}{3}$

Center: $(-3, 5)$

Vertices: $\left(-3 \pm \dfrac{1}{3}, 5\right)$

Foci: $\left(-3 \pm \dfrac{\sqrt{10}}{3}, 5\right)$



Solve for y:

$$y^2 - 10y + 25 = 9x^2 + 54x + 55 + 25$$

$$(y-5)^2 = 9x^2 + 54x + 80$$

$$y = 5 \pm \sqrt{9x^2 + 54x + 80}$$

(Graph each curve separately.)

56. $3y^2 - x^2 + 6x - 12y = 0$

$3(y^2 - 4y + 4) - (x^2 - 6x + 9) = 0 + 12 - 9 = 3$

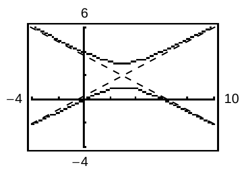
$$\frac{(y-2)^2}{1} - \frac{(x-3)^2}{3} = 1$$

$a = 1, b = \sqrt{3}, c = 2$

Center: $(3, 2)$

Vertices: $(3, 1), (3, 3)$

Foci: $(3, 0), (3, 4)$



Solve for y:

$3(y^2 - 4y + 4) = x^2 - 6x + 12$

$(y - 2)^2 = \frac{x^2 - 6x + 12}{3}$

$y = 2 \pm \sqrt{\frac{x^2 - 6x + 12}{3}}$

(Graph each curve separately.)

58. Vertices: $(0, \pm 3)$

Asymptotes: $y = \pm 3x$

Vertical transverse axis

$a = 3$

Slopes of asymptotes: $\pm\frac{a}{b} = \pm 3$

Thus, $b = 1$. Therefore,

$$\frac{y^2}{9} - \frac{x^2}{1} = 1.$$

60. Vertices: $(2, \pm 3)$

Foci: $(2, \pm 5)$

Vertical transverse axis

Center: $(2, 0)$

$a = 3, c = 5, b^2 = c^2 - a^2 = 16$

Therefore, $\frac{y^2}{9} - \frac{(x-2)^2}{16} = 1.$

62. Center: $(0, 0)$

Vertex: $(3, 0)$

Focus: $(5, 0)$

Horizontal transverse axis

$a = 3, c = 5, b^2 = c^2 - a^2 = 16$

Therefore, $\frac{x^2}{9} - \frac{y^2}{16} = 1.$

64. Focus: $(10, 0)$

Asymptotes: $y = \pm\frac{3}{4}x$

Horizontal transverse axis

Center: $(0, 0)$ since asymptotes intersect at the origin.

$c = 10$

Slopes of asymptotes: $\pm\frac{b}{a} = \pm\frac{3}{4}$ and $b = \frac{3}{4}a$

$c^2 = a^2 + b^2 = 100$

Solving these equations, we have $a^2 = 64$ and $b^2 = 36$. Therefore, the equation is

$$\frac{x^2}{64} - \frac{y^2}{36} = 1.$$

66. (a) $\frac{y^2}{4} - \frac{x^2}{2} = 1, y^2 - 2x^2 = 4, 2yy' - 4x = 0,$

$y' = \frac{4x}{2y} = \frac{2x}{y}$

At $x = 4$: $y = \pm 6, y' = \frac{\pm 2(4)}{6} = \pm\frac{4}{3}$

At $(4, 6)$: $y - 6 = -\frac{4}{3}(x - 4)$ or $4x - 3y + 2 = 0$

At $(4, -6)$: $y + 6 = -\frac{4}{3}(x - 4)$ or $4x + 3y + 2 = 0$

(b) From part (a) we know that the slopes of the normal lines must be $\mp 3/4$.

At $(4, 6)$: $y - 6 = -\frac{3}{4}(x - 4)$ or $3x + 4y - 36 = 0$

At $(4, -6)$: $y + 6 = \frac{3}{4}(x - 4)$ or $3x - 4y - 36 = 0$

68. $4x^2 - y^2 - 4x - 3 = 0$

$A = 4, C = -1$

$AC < 0$

Hyperbola

70. $25x^2 - 10x - 200y - 119 = 0$

$A = 25, C = 0$

Parabola

72. $y^2 - x - 4y - 5 = 0$

$A = 0, C = 1$

Parabola

74. $2x^2 - 2xy = 3y - y^2 - 2xy$

$2x^2 + y^2 - 3y = 0$

$A = 2, C = 1, AC > 0$

Ellipse

76. $9x^2 + 54x + 81 = 36 - 4(y^2 - 4y + 4)$

$9x^2 + 4y^2 + 54x - 16y + 61 = 0$

$A = 9, C = 4, AC > 0$

Ellipse

78. (a) An ellipse is the set of all points (x, y), the sum of whose distance from two distinct fixed points (foci) is constant.

(b) $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ or $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$

80. $e = \frac{c}{a}, c = \sqrt{a^2 - b^2} \quad 0 < e < 1$

For $e \approx 0$, the ellipse is nearly circular.

For $e \approx 1$, the ellipse is elongated.

82. Assume that the vertex is at the origin.

(a) $x^2 = 4py$

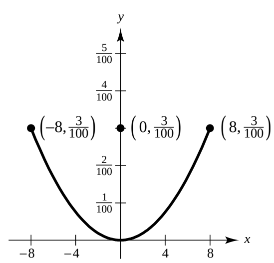
$8^2 = 4p\left(\frac{3}{100}\right)$

$\frac{1600}{3} = p$

$x^2 = 4\left(\frac{1600}{3}\right)y = \frac{6400}{3}y$

(b) The deflection is 1 cm when

$$y = \frac{2}{100} \implies x = \pm\sqrt{\frac{128}{3}} \approx \pm 6.53 \text{ meters.}$$



84. (a) Without loss of generality, place the coordinate system so that the equation of the parabola is $x^2 = 4py$ and, hence,

$$y' = \left(\frac{1}{2p}\right)x.$$

Therefore, for distinct tangent lines, the slopes are unequal and the lines intersect.

(b) $x^2 - 4x - 4y = 0$

$2x - 4 - 4\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{1}{2}x - 1$

At $(0, 0)$, the slope is -1: $y = -x$. At $(6, 3)$, the slope is 2: $y = 2x - 9$. Solving for x,

$-x = 2x - 9$

$-3x = -9$

$x = 3$

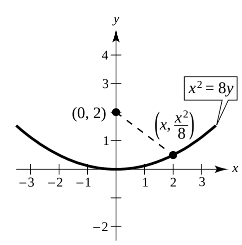
$y = -3.$

Point of intersection: $(3, -3)$

86. The focus of $x^2 = 8y = 4(2)y$ is $(0, 2)$. The distance from a point on the parabola, $(x, x^2/8)$, and the focus, $(0, 2)$, is

$$d = \sqrt{(x-0)^2 + \left(\frac{x^2}{8} - 2\right)^2}.$$

Since d is minimized when d^2 is minimized, it is sufficient to minimize the function



$$f(x) = x^2 + \left(\frac{x^2}{8} - 2\right)^2.$$

$$f'(x) = 2x + 2\left(\frac{x^2}{8} - 2\right)\left(\frac{x}{4}\right) = \frac{x^3}{16} + x.$$

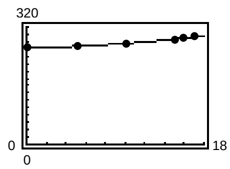
$f'(x) = 0$ implies that

$$\frac{x^3}{16} + x = x\left(\frac{x^2}{16} + 1\right) = 0 \implies x = 0.$$

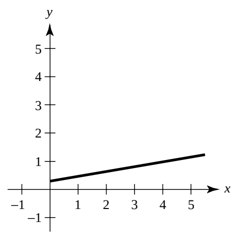
This is a minimum by the First Derivative Test. Hence, the closest point to the focus is the vertex, $(0, 0)$.

88. (a) $C = 0.0853t^2 + 0.2917t + 263.3559$

(b)



(c) $\dfrac{dC}{dt} = 0.1706t + 0.2971$



The consumption of fruits is increasing at a rate of 0.1706 pounds/year.

90.

$$x = \frac{1}{4}y^2$$

$$x' = \frac{1}{2}y$$

$$1 + (x')^2 = 1 + \frac{y^2}{4}$$

$$s = \int_0^4 \sqrt{1 + \left(\frac{y^2}{4}\right)}\,dy = \frac{1}{2}\int_0^4 \sqrt{4 + y^2}\,dy$$

$$= \frac{1}{4}\Big[y\sqrt{4 + y^2} + 4\ln\left|y + \sqrt{4 + y^2}\right|\Big]_0^4$$

$$= \frac{1}{4}\Big[4\sqrt{20} + 4\ln\left|4 + \sqrt{20}\right| - 4\ln 2\Big]$$

$$= 2\sqrt{5} + \ln\left(2 + \sqrt{5}\right) \approx 5.916$$

92. $x^2 = 20y$

$$y = \frac{x^2}{20}$$

$$y' = \frac{x}{10}$$

$$S = 2\pi\int_0^r x\sqrt{1 + \left(\frac{x}{10}\right)^2}\,dx \quad = 2\pi\int_0^r \frac{x\sqrt{100 + x^2}}{10}\,dx$$

$$= \left[\frac{\pi}{10}\cdot\frac{2}{3}(100 + x^2)^{3/2}\right]_0^r = \frac{\pi}{15}\left[(100 + r^2)^{3/2} - 1000\right]$$

94. $A = 2\displaystyle\int_0^h \sqrt{4py}\,dy$

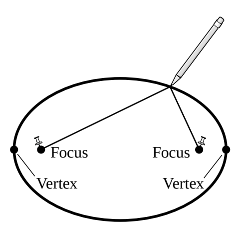
$$= 4\sqrt{p}\int_0^h y^{1/2}\,dy$$

$$= \left[4\sqrt{p}\left(\frac{2}{3}\right)y^{3/2}\right]_0^h$$

$$= \frac{8}{3}\sqrt{p}h^{3/2}$$

96. (a) At the vertices we notice that the string is horizontal and has a length of $2a$.

(b) The thumbtacks are located at the foci and the length of string is the constant sum of the distances from the foci.



98.

$$e = \frac{c}{a}$$

$$0.0167 = \frac{c}{149{,}570{,}000}$$

$$c \approx 2{,}497{,}819$$

Least distance: $a - c = 147{,}072{,}181$ km

Greatest distance: $a + c = 152{,}067{,}819$ km

100. $e = \dfrac{A - P}{A + P}$

$$= \frac{(122{,}000 + 4000) - (119 + 4000)}{(122{,}000 + 4000) + (119 + 4000)}$$

$$= \frac{121{,}881}{130{,}119} \approx 0.9367$$

102.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(b^2/a^2)} = 1$$

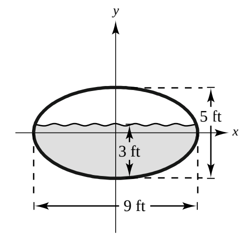
$$\frac{x^2}{a^2} + \frac{y^2}{a^2(a^2 - c^2)/a^2} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

As $e \to 0$, $1 - e^2 \to 1$ and we have

$\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2} = 1$ or the circle $x^2 + y^2 = a^2$.

104. $\dfrac{x^2}{(4.5)^2} + \dfrac{y^2}{(2.5)^2} = 1$



$$x^2 = (4.5)^2\left[1 - \frac{y^2}{(2.5)^2}\right]$$

$$x = \pm\frac{9}{5}\sqrt{(2.5)^2 - y^2}$$

$V =$ (Area of bottom)(Length) $+$ (Area of top)(Length)

$$V = \left[\frac{\pi(4.5)(2.5)}{2}\right](16) + 16\int_0^{0.5} \frac{9}{5}\sqrt{(2.5)^2 - y^2}\, dy \quad \text{(Recall: Area of ellipse is } \pi ab.)$$

$$= 90\pi + \frac{144}{5}\cdot\frac{1}{2}\left[y\sqrt{(2.5)^2 - y^2} + (2.5)^2 \arcsin\frac{y}{2.5}\right]_0^{0.5} = 90\pi + \frac{72}{5}\left[0.5\sqrt{6} + (2.5)^2 \arcsin\frac{1}{5}\right] \approx 318.5 \text{ ft}^3$$

106. $9x^2 + 4y^2 + 36x - 24y + 36 = 0$

$$18x + 8yy' + 36 - 24y' = 0$$

$$(8y - 24)y' = -(18x + 36)$$

$$y' = \frac{-(18x + 36)}{8y - 24}$$

$y' = 0$ when $x = -2$. y' undefined when $y = 3$.
At $x = -2$, $y = 0$ or 6.

Endpoints of major axis: $(-2, 0)$, $(-2, 6)$

At $y = 3$, $x = 0$ or -4.

Endpoints of minor axis: $(0, 3)$, $(-4, 3)$

Note: Equation of ellipse is $\dfrac{(x + 2)^2}{4} + \dfrac{(y - 3)^2}{9} = 1$

108. (a) $A = 4\displaystyle\int_0^4 \frac{3}{4}\sqrt{16 - x^2}\, dx = \frac{3}{2}\left[x\sqrt{16 - x^2} + 16 \arcsin\frac{x}{4}\right]_0^4 = 12\pi$

(b) **Disk:** $V = 2\pi\displaystyle\int_0^4 \frac{9}{16}(16 - x^2)\, dx = \frac{9\pi}{8}\left[\left(16x - \frac{1}{3}x^3\right)\right]_0^4 = 48\pi$

$$y = \frac{3}{4}\sqrt{16 - x^2}$$

$$y' = \frac{-3x}{4\sqrt{16 - x^2}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{9x^2}{16(16 - x^2)}}$$

$$S = 2(2\pi)\int_0^4 \frac{3}{4}\sqrt{16 - x^2}\sqrt{\frac{16(16 - x^2) + 9x^2}{16(16 - x^2)}}\, dx = 4\pi\int_0^4 \frac{3}{4}\sqrt{16 - x^2}\,\frac{\sqrt{256 - 7x^2}}{4\sqrt{16 - x^2}}\, dx = \frac{3\pi}{4}\int_0^4 \sqrt{256 - 7x^2}\, dx$$

$$= \frac{3\pi}{8\sqrt{7}}\left[\sqrt{7}x\sqrt{256 - 7x^2} + 256 \arcsin\frac{\sqrt{7}x}{16}\right]_0^4 = \frac{3\pi}{8\sqrt{7}}\left(48\sqrt{7} + 256 \arcsin\frac{\sqrt{7}}{4}\right) \approx 138.93$$

—CONTINUED—

108. —CONTINUED—

(c) **Shell:** $V = 4\pi\int_0^4 x\left[\frac{3}{4}\sqrt{16 - x^2}\right]dx = 3\pi\left[\left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)(16 - x^2)^{3/2}\right]_0^4 = 64\pi$

$$x = \frac{4}{3}\sqrt{9 - y^2}$$

$$x' = \frac{-4y}{3\sqrt{9 - y^2}}$$

$$\sqrt{1 + (x')^2} = \sqrt{1 + \frac{16y^2}{9(9 - y^2)}}$$

$$S = 2(2\pi)\int_0^3 \frac{4}{3}\sqrt{9 - y^2}\sqrt{\frac{9(9 - y^2) + 16y^2}{9(9 - y^2)}}\,dy$$

$$= 4\pi\int_0^3 \frac{4}{9}\sqrt{81 + 7y^2}\,dy$$

$$= \frac{16}{9}\left(\frac{\pi}{2\sqrt{7}}\right)\left[\sqrt{7}y\sqrt{81 + 7y^2} + 81\ln\left|\sqrt{7}y + \sqrt{81 + 7y^2}\right|\right]_0^3$$

$$= \frac{8\pi}{9\sqrt{7}}3\sqrt{7}(12) + 81\ln\left(3\sqrt{7} + 12\right) - 81\ln 9\Big] \approx 168.53$$

110. (a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$$

$$y' = -\frac{xb^2}{ya^2}$$

At P, $y' = -\frac{b^2}{a^2}\cdot\frac{x_0}{y_0} = m.$

(b) Slope of line through $(-c, 0)$ and (x_0, y_0): $m_1 = \frac{y_0}{x_0 + c}$

Slope of line through $(c, 0)$ and (x_0, y_0): $m_2 = \frac{y_0}{x_0 - c}$

(c) $\tan\alpha = \frac{m_2 - m}{1 + m_2 m} = \frac{\frac{y_0}{x_0 - c} - \left(-\frac{b^2x_0}{a^2y_0}\right)}{1 + \left(\frac{y_0}{x_0 - c}\right)\left(-\frac{b^2x_0}{a^2y_0}\right)} = \frac{a^2y_0^2 + b^2x_0(x_0 - c)}{a^2y_0(x_0 - c) - b^2x_0y_0}$

$$= \frac{a^2y_0^2 + b^2x_0^2 - b^2x_0c}{x_0y_0(a^2 - b^2) - a^2y_0c} = \frac{a^2b^2 - b^2x_0c}{x_0y_0c^2 - a^2y_0c} = \frac{b^2(a^2 - x_0c)}{y_0c(x_0c - a^2)} = -\frac{b^2}{y_0c}$$

$$\alpha = \arctan\left(-\frac{b^2}{y_0c}\right) = -\arctan\left(\frac{b^2}{y_0c}\right)$$

$$\tan\beta = \frac{m_1 - m}{1 + m_1 m} = \frac{\frac{y_0}{x_0 + c} - \left(-\frac{b^2x_0}{a^2y_0}\right)}{1 + \left(\frac{y_0}{x_0 + c}\right)\left(-\frac{b^2x_0}{a^2y_0}\right)} = \frac{a^2y_0^2 + b^2x_0(x_0 + c)}{a^2y_0(x_0 + c) - b^2x_0y_0}$$

$$= \frac{a^2y_0^2 + b^2x_0^2 + b^2x_0c}{a^2x_0y_0 + a^2cy_0 - b^2x_0y_0} = \frac{a^2b^2 + b^2x_0c}{x_0y_0(a^2 - b^2) + a^2cy_0} = \frac{b^2(a^2 + x_0c)}{y_0c(x_0c + a^2)} = \frac{b^2}{y_0c}$$

$$\beta = \arctan\left(\frac{b^2}{y_0c}\right)$$

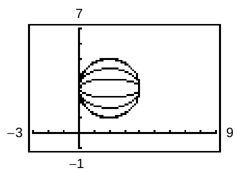
Since $|\alpha| = |\beta|$, the tangent line to an ellipse at a point P makes equal angles with the lines through P and the foci.

112. (a) $e = \dfrac{c}{a} = \dfrac{\sqrt{a^2 + b^2}}{a} \Rightarrow (ea)^2 - a^2 = b^2$. Hence,

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2(1-e^2)} = 1.$$

(b) $\dfrac{(x-2)^2}{4} + \dfrac{(y-3)^2}{4(1-e^2)} = 1$



(c) As e approaches 0, the ellipse approaches a circle.

114. The transverse axis is vertical since $(-3, 0)$ and $(-3, 3)$ are the foci.

Center: $\left(-3, \dfrac{3}{2}\right)$

$c = \dfrac{3}{2}, 2a = 2, b^2 = c^2 - a^2 = \dfrac{5}{4}$

Therefore, the equation is

$$\frac{[y - (3/2)]^2}{1} - \frac{(x+3)^2}{5/4} = 1.$$

116. Center: $(0, 0)$

Horizontal transverse axis

Foci: $(\pm c, 0)$

Vertices: $(\pm a, 0)$

The difference of the distances from any point on the hyperbola is constant. At a vertex, this constant difference is

$$(a + c) - (c - a) = 2a.$$

Now, for any point (x, y) on the hyperbola, the difference of the distances between (x, y) and the two foci must also be $2a$.

$$\sqrt{(x-c)^2 + (y-0)^2} - \sqrt{(x+c)^2 + (y-0)^2} = 2a$$

$$\sqrt{(x-c)^2 + y^2} = 2a + \sqrt{(x+c)^2 + y^2}$$

$$(x-c)^2 + y^2 = 4a^2 + 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2$$

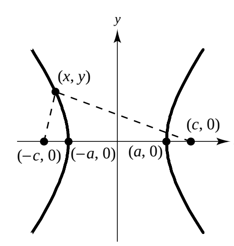
$$-4xc - 4a^2 = 4a\sqrt{(x+c)^2 + y^2}$$

$$-(xc + a^2) = a\sqrt{(x+c)^2 + y^2}$$

$$x^2c^2 + 2a^2cx + a^4 = a^2[x^2 + 2cx + c^2 + y^2]$$

$$x^2(c^2 - a^2) - a^2y^2 = a^2(c^2 - a^2)$$

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$$



Since $a^2 + b^2 = c^2$, we have $(x^2/a^2) - (y^2/b^2) = 1$.

118. $c = 150, 2a = 0.001(186{,}000), a = 93,$
$b = \sqrt{150^2 - 93^2} = \sqrt{13{,}851}$

$$\frac{x^2}{93^2} - \frac{y^2}{13{,}851} = 1$$

When $y = 75$, we have

$$x^2 = 93^2\left(1 + \frac{75^2}{13{,}851}\right)$$

$$x \approx 110.3 \text{ miles.}$$

120.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{2x}{a^2} - \frac{2yy'}{b^2} = 0 \text{ or } y' = \frac{b^2x}{a^2y}$$

$$y - y_0 = \frac{b^2x_0}{a^2y_0}(x - x_0)$$

$$a^2y_0y - a^2y_0^2 = b^2x_0x - b^2x_0^2$$

$$b^2x_0^2 - a^2y_0^2 = b^2x_0x - a^2y_0y$$

$$a^2b^2 = b^2x_0x - a^2y_0y$$

$$\frac{x_0x}{a^2} - \frac{y_0y}{b^2} = 1$$

122. $Ax^2 + Cy^2 + Dx + Ey + F = 0$ (Assume $A \neq 0$ and $C \neq 0$; see (b) below)

$$A\left(x^2 + \frac{D}{A}x\right) + C\left(y^2 + \frac{E}{C}y\right) = -F$$

$$A\left(x^2 + \frac{D}{A}x + \frac{D^2}{4A^2}\right) + C\left(y^2 + \frac{E}{C}y + \frac{E^2}{4C^2}\right) = -F + \frac{D^2}{4A} + \frac{E^2}{4C} = R$$

$$\frac{\left[x + \left(\frac{D}{2A}\right)\right]^2}{C} + \frac{\left[y + \left(\frac{E}{2C}\right)\right]^2}{A} = \frac{R}{AC}$$

(a) If $A = C$, we have

$$\left(x + \frac{D}{2A}\right)^2 + \left(y + \frac{E}{2C}\right)^2 = \frac{R}{A}$$

which is the standard equation of a circle.

(c) If $AC > 0$, we have

$$\frac{\left[x + \left(\frac{D}{2A}\right)\right]^2}{\left|\frac{R}{A}\right|} + \frac{\left[y + \left(\frac{E}{2C}\right)\right]^2}{\left|\frac{R}{C}\right|} = 1$$

which is the equation of an ellipse.

(b) If $C = 0$, we have

$$A\left(x + \frac{D}{2A}\right)^2 = -F - Ey + \frac{D^2}{4A}.$$

If $A = 0$, we have

$$C\left(y + \frac{E}{2C}\right)^2 = -F - Dx + \frac{E^2}{4C}.$$

These are the equations of parabolas.

(d) If $AC < 0$, we have

$$\frac{\left[x + \left(\frac{D}{2A}\right)\right]^2}{\left|\frac{R}{A}\right|} - \frac{\left[y + \left(\frac{E}{2C}\right)\right]^2}{\left|\frac{R}{C}\right|} = \pm 1$$

which is the equation of a hyperbola.

124. True

126. False. The y^4 term should be y^2.

128. True

Section 9.2 Plane Curves and Parametric Equations

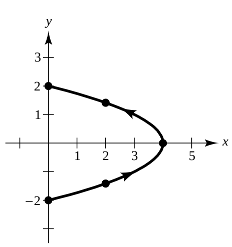
2. $x = 4\cos^2\theta$ $y = 2\sin\theta$

$0 \leq x \leq 4$ $-2 \leq y \leq 2$

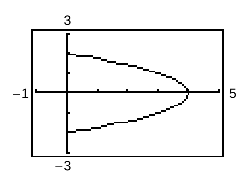
(a)

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
x	0	2	4	2	0
y	-2	$-\sqrt{2}$	0	$\sqrt{2}$	2

(b)



(c)



(d) $$\frac{x}{4} = \cos^2\theta$$

$$\frac{y^2}{4} = \sin^2\theta$$

$$\frac{x}{4} + \frac{y^2}{4} = 1$$

$$x = 4 - y^2, \ -2 \leq y \leq 2$$

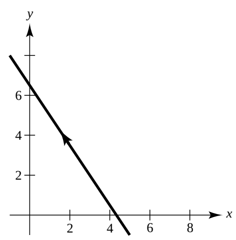
(e) The graph would be oriented in the opposite direction.

4. $x = 3 - 2t$

$y = 2 + 3t$

$y = 2 + 3\left(\dfrac{3 - x}{2}\right)$

$2y + 3x - 13 = 0$



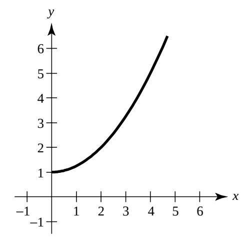
6. $x = 2t^2$

$y = t^4 + 1$

$y = \left(\dfrac{x}{2}\right)^2 + 1 = \dfrac{x^2}{4} + 1, x \geq 0$

For $t < 0$, the orientation is right to left.

For $t > 0$, the orientation is left to right.



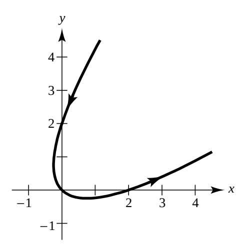
8. $x = t^2 + t,\ y = t^2 - t$

Subtracting the second equation from the first, we have

$x - y = 2t \quad \text{or} \quad t = \dfrac{x - y}{2}$

$y = \dfrac{(x - y)^2}{4} - \dfrac{x - y}{2}$

t	-2	-1	0	1	2
x	2	0	0	2	6
y	6	2	0	0	2



Since the discriminant is

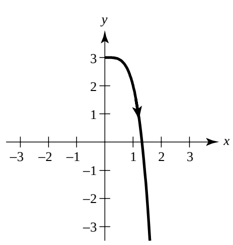
$B^2 - 4AC = (-2)^2 - 4(1)(1) = 0,$

the graph is a rotated parabola.

10. $x = \sqrt[4]{t}, t \geq 0$

$y = 3 - t$

$y = 3 - x^4, x \geq 0$

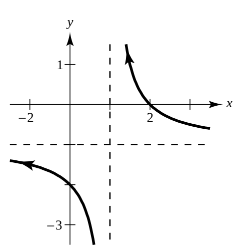


12. $x = 1 + \dfrac{1}{t}$

$y = t - 1$

$x = 1 + \dfrac{1}{t}$ implies $t = \dfrac{1}{x - 1}$

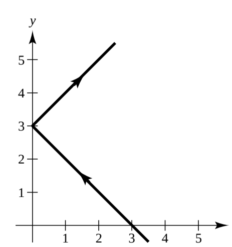
$y = \dfrac{1}{x - 1} - 1$



14. $x = |t - 1|$

$y = t + 2$

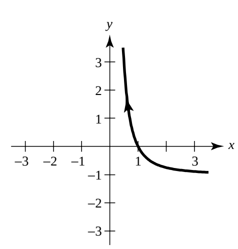
$x = |(y - 2) - 1| = |y - 3|$



16. $x = e^{-t}, x > 0$

$y = e^{2t} - 1$

$y = x^{-2} - 1 = \dfrac{1}{x^2} - 1, x > 0$



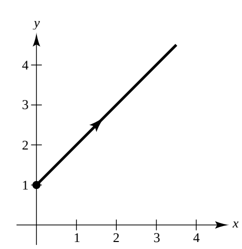
18. $x = \tan^2 \theta$

$y = \sec^2 \theta$

$\sec^2 \theta = \tan^2 \theta + 1$

$y = x + 1$

$x \geq 0$

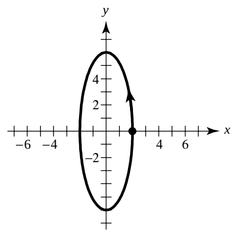


20. $x = 2\cos\theta$

$y = 6\sin\theta$

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{6}\right)^2 = \cos^2 + \sin^2\theta = 1$$

$$\frac{x^2}{4} + \frac{y^2}{36} = 1 \text{ ellipse}$$



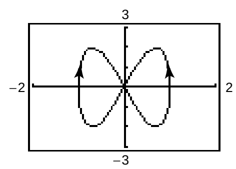
22. $x = \cos\theta$

$y = 2\sin 2\theta$

$y = 4\sin\theta\cos\theta$

$1 - x^2 = \sin^2\theta$

$y = \pm 4x\sqrt{1 - x^2}$



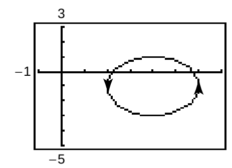
24. $x = 4 + 2\cos\theta$

$y = -1 + 2\sin\theta$

$(x - 4)^2 = 4\cos^2\theta$

$(y + 1)^2 = 4\sin^2\theta$

$(x - 4)^2 + (y + 1)^2 = 4$

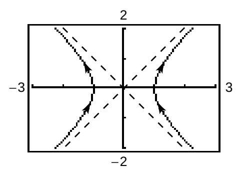


26. $x = \sec\theta$

$y = \tan\theta$

$x^2 = \sec^2\theta$

$y^2 = \tan^2\theta$

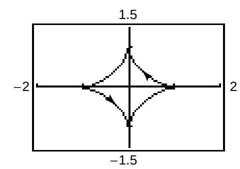


28. $x = \cos^3\theta$

$y = \sin^3\theta$

$x^{2/3} = \cos^2\theta$

$y^{2/3} = \sin^2\theta$

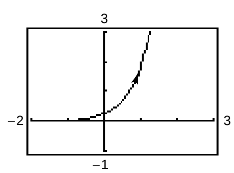


30. $x = \ln 2t$

$y = t^2$

$$t = \frac{e^x}{2}$$

$$y = \frac{e^{2x}}{r} = \frac{1}{4}e^{2x}$$



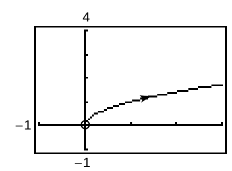
32. $x = e^{2t}$

$y = e^t$

$y^2 = x$

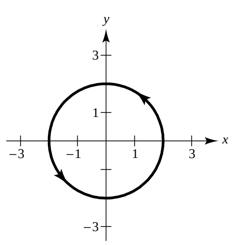
$y > 0$

$y = \sqrt{x},\ x > 0$



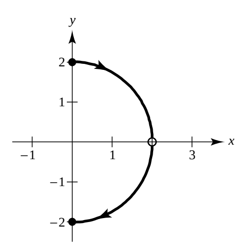
34. By eliminating the parameters in (a) – (d), we get $x^2 + y^2 = 4$. They differ from each other in orientation and in restricted domains. These curves are all smooth.

(a) $x = 2\cos\theta,\ y = 2\sin\theta$



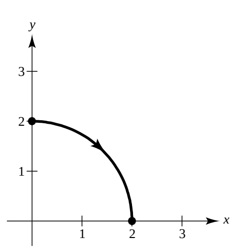
(b) $x = \dfrac{\sqrt{4t^2 - 1}}{|t|} = \sqrt{4 - \dfrac{1}{t^2}}$ $\qquad y = \dfrac{1}{t}$

$x \geq 0,\ x \neq 2$ $\qquad y \neq 0$



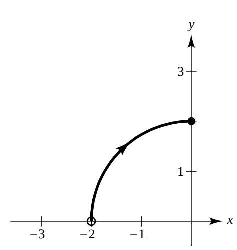
(c) $x = \sqrt{t}$ $\qquad y = \sqrt{4 - t}$

$x \geq 0$ $\qquad y \geq 0$



(d) $x = -\sqrt{4 - e^{2t}}$ $\qquad y = e^t$

$-2 < x \leq 0$ $\qquad y > 0$



36. The orientations are reversed. The graphs are the same. They are both smooth.

38. The set of points (x, y) corresponding to the rectangular equation of a set of parametric equations does not show the orientation of the curve nor any restriction on the domain of the original parametric equations.

40.

$$x = h + r\cos\theta$$
$$y = k + r\sin\theta$$
$$\cos\theta = \frac{x-h}{r}$$
$$\sin\theta = \frac{y-k}{r}$$
$$\cos^2\theta + \sin^2\theta = \frac{(x-h)^2}{r^2} + \frac{(y-k)^2}{r^2} = 1$$
$$(x-h)^2 + (y-k)^2 = r^2$$

42.

$$x = h + a\sec\theta$$
$$y = k + b\tan\theta$$
$$\frac{x-h}{a} = \sec\theta$$
$$\frac{y-k}{b} = \tan\theta$$
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

44. From Exercise 39 we have

$$x = 1 + 4t$$
$$y = 4 - 6t.$$

Solution not unique

46. From Exercise 40 we have

$$x = -3 + 3\cos\theta$$
$$y = 1 + 3\sin\theta.$$

Solution not unique

48. From Exercise 41 we have

$$a = 5, c = 3 \implies b = 4$$
$$x = 4 + 5\cos$$
$$y = 2 + 4\sin\theta.$$

Center: $(4, 2)$
Solution not unique

50. From Exercise 42 we have

$$a = 1, c = 2 \implies b = \sqrt{3}$$
$$x = \sqrt{3}\tan\theta$$
$$y = \sec\theta.$$

Center: $(0, 0)$
Solution not unique
The transverse axis is vertical, therefore, x and y are interchanged.

52. $y = \dfrac{2}{x-1}$

Example

$$x = t, y = \frac{2}{t-1}$$
$$x = -t, y = \frac{2}{-t-1}$$

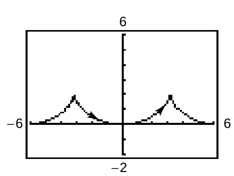
54. $y = x^2$

Example

$x = t,\quad y = t^2$
$x = t^3,\quad y = t^6$

56. $x = \theta + \sin\theta$

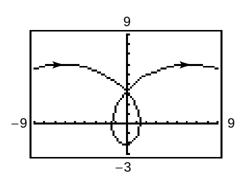
$y = 1 - \cos\theta$



Not smooth at $x = (2n-1)\pi$

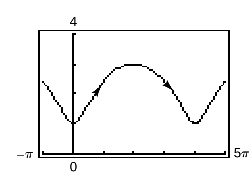
58. $x = 2\theta - 4\sin\theta$

$y = 2 - 4\cos\theta$



60. $x = 2\theta - \sin\theta$

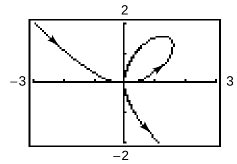
$y = 2 - \cos\theta$



Smooth everywhere

62. $x = \dfrac{3t}{1 + t^3}$

$y = \dfrac{3t^2}{1 + t^3}$



Smooth everywhere

64. Each point (x, y) in the plane is determined by the plane curve $x = f(t)$, $y = g(t)$. For each t, plot (x, y). As t increases, the curve is traced out in a specific direction called the orientation of the curve.

66. (a) Matches (ii) because $-1 \le x \le 0$ and $1 \le y \le 2$.

(b) Matches (i) because $x = (y - 2)^2 - 1$ for all y.

68. $x = \cos^3 \theta$

$y = 2 \sin^2 \theta$

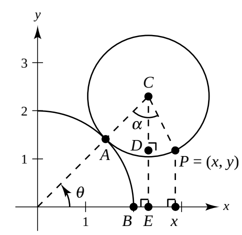
Matches (a)

70. $x = \cot \theta$

$y = 4 \sin \theta \cos \theta$

Matches (c)

72. Let the circle of radius 1 be centered at C. A is the point of tangency on the line OC. $OA = 2$, $AC = 1$, $OC = 3$. $P = (x, y)$ is the point on the curve being traced out as the angle θ changes $\widehat{AB} = \widehat{AP}$. $\widehat{AB} = 2\theta$ and $\widehat{AP} = \alpha \implies \alpha = 2\theta$. Form the right triangle $\triangle CDP$. The angle $OCE = (\pi/2) - \theta$ and



$$\angle DCP = \alpha - \left(\frac{\pi}{2} - \theta\right) = \alpha + \theta - \left(\frac{\pi}{2}\right) = 3\theta - \left(\frac{\pi}{2}\right).$$

$$x = OE + Ex = 3 \sin\left(\frac{\pi}{2} - \theta\right) + \sin\left(3\theta - \frac{\pi}{2}\right) = 3 \cos \theta - \cos 3\theta$$

$$y = EC - CD = 3 \sin \theta - \cos\left(3\theta - \frac{\pi}{2}\right) = 3 \sin \theta - \sin 3\theta$$

Hence, $x = 3 \cos \theta - \cos 3\theta,\ y = 3 \sin \theta - \sin 3\theta.$

74. False. Let $x = t^2$ and $y = t$. Then $x = y^2$ and y is not a function of x.

76. (a) $x = (v_0 \cos \theta)t$

$y = h + (v_0 \sin \theta)t - 16t^2$

$$t = \frac{x}{v_0 \cos \theta} \implies y = h + (v_0 \sin \theta)\frac{x}{v_0 \cos \theta} - 16\left(\frac{x}{v_0 \cos \theta}\right)^2$$

$$y = h + (\tan \theta)x - \frac{16 \sec^2 \theta}{v_0^2}x^2$$

(b) $y = 5 + x - 0.005x^2 = h + (\tan \theta)x - \dfrac{16 \sec^2 \theta}{v_0^2}x^2$

$h = 5$, $\tan \theta = 1 \implies \theta = \dfrac{\pi}{4}$, and

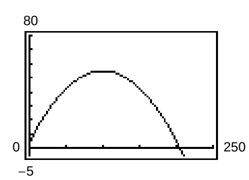
$$0.005 = \frac{16 \sec^2(\pi/4)}{v_0^2} = \frac{16}{v_0^2}(2)$$

$$v_0^2 = \frac{32}{0.005} = 6400 \implies v_0 = 80.$$

Hence, $x = (80 \cos(45°))t$

$y = 5 + (80 \sin(45°))t - 16t^2.$

(c)



(d) Maximum height: $y = 55$ (at $x = 100$)

Range: 204.88

Section 9.3 Parametric Equations and Calculus

2. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-1}{(1/3)t^{-2/3}} = -3t^{2/3}$

4. $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{(-1/2)e^{-\theta/2}}{2e^{\theta}} = -\dfrac{1}{4}e^{-3\theta/2} = \dfrac{-1}{4e^{3\theta/2}}$

6. $x = \sqrt{t},\ y = 3t - 1$

$$\frac{dy}{dx} = \frac{3}{1/(2\sqrt{t})} = 6\sqrt{t} = 6 \quad \text{when } t = 1.$$

$$\frac{d^2y}{dx^2} = \frac{3/\sqrt{t}}{1/(2\sqrt{t})} = 6 \ \text{ concave upwards}$$

8. $x = t^2 + 3t + 2, y = 2t$

$$\frac{dy}{dx} = \frac{2}{2t + 3} = \frac{2}{3} \text{ when } t = 0.$$

$$\frac{d^2y}{dx^2} = \frac{-2(2)/(2t + 3)}{2t + 3} = \frac{-4}{(2t + 3)^2} = \frac{-4}{9} \text{ when t} = 0.$$

concave downward

10. $x = \cos\theta,\ y = 3\sin\theta$

$$\frac{dy}{dx} = \frac{3\cos\theta}{-\sin\theta} = -3\cot\theta \cdot \frac{dy}{dx} \text{ is undefined when } \theta = 0.$$

$$\frac{d^2y}{dx^2} = \frac{3\csc^2\theta}{-\sin\theta} = \frac{-3}{\sin^3\theta} \cdot \frac{d^2y}{dx^2} \text{ is undefined when } \theta = 0.$$

12. $x = \sqrt{t},\ y = \sqrt{t - 1}$

$$\frac{dy}{dx} = \frac{1/(2\sqrt{t-1})}{1/(2\sqrt{t})}$$

$$= \frac{\sqrt{t}}{\sqrt{t-1}} = \sqrt{2} \text{ when } t = 2.$$

$$\frac{d^2y}{dx^2} = \frac{\left[\sqrt{t-1}/(2\sqrt{t}) - \sqrt{t}(1/2\sqrt{t-1})\right]/(t-1)}{1/(2\sqrt{t})}$$

$$= \frac{-1}{(t-1)^{3/2}} = -1 \text{ when } t = 2.$$

concave downward

14. $x = \theta - \sin\theta,\ y = 1 - \cos\theta$

$$\frac{dy}{dx} = \frac{\sin\theta}{1 - \cos\theta} = 0 \text{ when } \theta = \pi.$$

$$\frac{d^2y}{dx^2} = \frac{\dfrac{[(1 - \cos\theta)\cos\theta - \sin^2\theta]}{(1 - \cos\theta)^2}}{(1 - \cos\theta)}$$

$$= \frac{-1}{(1 - \cos\theta)^2} = -\frac{1}{4} \text{ when } \theta = \pi.$$

concave downward

16. $x = 2 - 3\cos\theta, y = 3 + 2\sin\theta$

$$\frac{dy}{dx} = \frac{2\cos\theta}{3\sin\theta} = \frac{2}{3}\cot\theta$$

At $(-1, 3),\ \theta = 0,$ and $\dfrac{dy}{dx}$ is undefined.

Tangent line: $x = -1$

At $(2, 5),\ \theta = \dfrac{\pi}{2},$ and $\dfrac{dy}{dx} = 0.$

Tangent line: $y = 5$

At $\left(\dfrac{4 + 3\sqrt{3}}{2}, 2\right),\ \theta = \dfrac{7\pi}{6},$ and $\dfrac{dy}{dx} = \dfrac{2\sqrt{3}}{3}.$

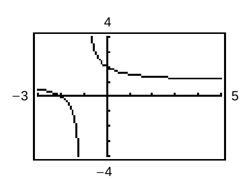
Tangent line:

$$y - 2 = \frac{2\sqrt{3}}{3}\left(x - \frac{4 + 3\sqrt{3}}{2}\right)$$

$$2\sqrt{3}x - 3y - 4\sqrt{3} - 3 = 0$$

18. $x = t - 1,\ y = \dfrac{1}{t} + 1,\ t = 1$

(a)



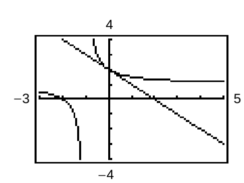
(b) At $t = 1,\ (x, y) = (0, 2),$ and

$$\frac{dx}{dt} = 1,\ \frac{dy}{dt} = -1,\ \frac{dy}{dx} = -1$$

(c) $\dfrac{dy}{dx} = -1.$ At $(0, 2),\ y - 2 = -1(x - 0)$

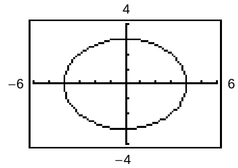
$$y = -x + 2$$

(d)



20. $x = 4\cos\theta,\ y = 3\sin\theta,\ \theta = \dfrac{3\pi}{4}$

(a)



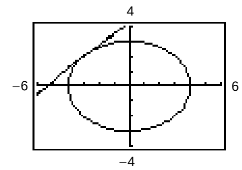
(b) At $\theta = \dfrac{3\pi}{4},\ (x, y) = \left(\dfrac{-4}{\sqrt{2}}, \dfrac{3}{\sqrt{2}}\right)$, and

$$\frac{dx}{dt} = -2\sqrt{2},\ \frac{dy}{dt} = -\frac{3\sqrt{2}}{2},\ \frac{dy}{dx} = \frac{3}{4}$$

(c) $\dfrac{dy}{dx} = \dfrac{3}{4}$. At $\left(\dfrac{-4}{\sqrt{2}}, \dfrac{3}{\sqrt{2}}\right)$, $\quad y - \dfrac{3}{\sqrt{2}} = \dfrac{3}{4}\left(x + \dfrac{4}{\sqrt{2}}\right)$

$$y = \frac{3}{4}x + 3\sqrt{2}$$

(d)



22. $x = t^2 - t, y = t^3 - 3t - 1$ crosses itself at the point $(x, y) = (2, 1)$.

At this point, $t = -1$ or $t = 2$.

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t - 1}$$

At $t = -1, \dfrac{dy}{dx} = 0$ and $y = 1$. Tangent Line

At $t = 2, \dfrac{dy}{dt} = \dfrac{9}{3} = 3$ and $y - 1 = 3(x - 2)$ or $y = 3x - 5$. Tangent Line

24. $x = 2\theta,\ y = 2(1 - \cos\theta)$

Horizontal tangents: $\dfrac{dy}{d\theta} = 2\sin\theta = 0$ when $\theta = 0,\ \pm\pi,\ \pm 2\pi, \ldots.$

Points: $(4n\pi, 0),\ (2[2n - 1]\pi, 4)$ where n is an integer.

Points shown: $(0, 0),\ (2\pi, 4),\ (4\pi, 0)$

Vertical tangents: $\dfrac{dx}{d\theta} = 2 \neq 0$; none

26. $x = t + 1,\ y = t^2 + 3t$

Horizontal tangents: $\dfrac{dy}{dt} = 2t + 3 = 0$ when $t = -\dfrac{3}{2}$.

Point: $\left(-\dfrac{1}{2}, -\dfrac{9}{4}\right)$

Vertical tangents: $\dfrac{dx}{dt} = 1 \neq 0$; none

28. $x = t^2 - t + 2,\ y = t^3 - 3t$

Horizontal tangents: $\dfrac{dy}{dt} = 3t^2 - 3 = 0$ when $t = \pm 1$.

Points: $(2, -2), (4, 2)$

Vertical tangents: $\dfrac{dx}{dt} = 2t - 1 = 0$ when $t = \dfrac{1}{2}$.

Point: $\left(\dfrac{7}{4}, -\dfrac{11}{8}\right)$

30. $x = \cos\theta, y = 2\sin 2\theta$

Horizontal tangents: $\dfrac{dy}{d\theta} = 4\cos 2\theta = 0$ when $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$.

Points: $\left(\dfrac{\sqrt{2}}{2}, 2\right), \left(-\dfrac{\sqrt{2}}{2}, -2\right), \left(-\dfrac{\sqrt{2}}{2}, 2\right), \left(\dfrac{\sqrt{2}}{2}, -2\right)$

Vertical tangents: $\dfrac{dx}{d\theta} = -\sin\theta = 0$ when $\theta = 0, \pi$.

Points: $(1, 0), (-1, 0)$

32. $x = 4\cos^2\theta,\ y = 2\sin\theta$

Horizontal tangents: $\dfrac{dy}{d\theta} = 2\cos\theta = 0$ when $\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}.$

Since $dx/d\theta = 0$ at $\pi/2$ and $3\pi/2$, exclude them.

Vertical tangents: $\dfrac{dx}{d\theta} = -8\cos\theta\sin\theta = 0$ when $\theta = 0, \pi.$

Point: $(4, 0)$

34. $x = \cos^2\theta,\ y = \cos\theta$

Horizontal tangents: $\dfrac{dy}{d\theta} = -\sin\theta = 0$ when $x = 0, \pi.$

Since $dx/d\theta = 0$ at these values, exclude them.

Vertical tangents: $\dfrac{dx}{d\theta} = -2\cos\theta\sin\theta = 0$ when $\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}.$

(Exclude $0, \pi$.)

Point: $(0, 0)$

36. $x = t^2 + 1, y = 4t^3 + 3, -1 \le t \le 0$

$$\frac{dx}{dt} = 2t, \frac{dy}{dt} = 12t^2, \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4t^2 + 144t^4$$

$$s = \int_{-1}^{0} \sqrt{4t^2 + 144t^4}\,dt = \int_{-1}^{0} -2t\sqrt{1 + 36t^2}\,dt$$

$$= \left[\frac{-(1 + 36t^2)^{3/2}}{54}\right]_{-1}^{0} = \frac{-1}{54}(1 - 37^{3/2}) \approx 4.149$$

38. $x = \arcsin t, y = \ln\sqrt{1 - t^2}, 0 \le t \le \dfrac{1}{2}$

$$\frac{dx}{dt} = \frac{1}{\sqrt{1 - t^2}}, \frac{dy}{dt} = \frac{1}{2}\left(\frac{-2t}{1 - t^2}\right) = \frac{t}{1 - t^2}$$

$$s = \int_0^{1/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^{1/2} \sqrt{\frac{1}{(1 - t^2)^2}}\,dt = \int_0^{1/2} \frac{1}{1 - t^2}\,dt$$

$$= \left[-\frac{1}{2}\ln\left|\frac{t - 1}{t + 1}\right|\right]_0^{1/2}$$

$$= -\frac{1}{2}\ln\left(\frac{1}{3}\right) = \frac{1}{2}\ln(3) \approx 0.549$$

40. $x = t,\ y = \dfrac{t^5}{10} + \dfrac{1}{6t^3},\ \dfrac{dx}{dt} = 1,\ \dfrac{dy}{dt} = \dfrac{t^4}{2} - \dfrac{1}{2t^4}$

$$S = \int_1^2 \sqrt{1 + \left(\frac{t^4}{2} - \frac{1}{2t^4}\right)^2}\,dt =$$

$$= \int_1^2 \sqrt{\left(\frac{t^4}{2} + \frac{1}{2t^4}\right)^2}\,dt$$

$$= \int_1^2 \left(\frac{t^4}{2} + \frac{1}{2t^4}\right)dt$$

$$= \left[\frac{t^5}{10} - \frac{1}{6t^3}\right]_1^2 = \frac{779}{240}$$

42. $x = a\cos\theta,\ y = a\sin\theta,\ \dfrac{dx}{d\theta} = -a\sin\theta,\ \dfrac{dy}{d\theta} = a\cos\theta$

$$S = 4\int_0^{\pi/2} \sqrt{a^2\sin^2\theta + a^2\cos^2\theta}\,d\theta$$

$$= 4a\int_0^{\pi/2} d\theta = \left[4a\theta\right]_0^{\pi/2} = 2\pi a$$

44. $x = \cos\theta + \theta\sin\theta,\ y = \sin\theta - \theta\cos\theta,\ \dfrac{dx}{d\theta} = \theta\cos\theta$

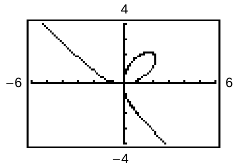
$\dfrac{dy}{d\theta} = \theta\sin\theta$

$$S = \int_0^{2\pi} \sqrt{\theta^2\cos^2\theta + \theta^2\sin^2\theta}\,d\theta$$

$$= \int_0^{2\pi} \theta\,d\theta = \left[\frac{\theta^2}{2}\right]_0^{2\pi} = 2\pi^2$$

46. $x = \dfrac{4t}{1 + t^3},\ y = \dfrac{4t^2}{1 + t^3}$

(a) $x^3 + y^3 = 4xy$



(b) $\dfrac{dy}{dt} = \dfrac{(1 + t^3)(8t) - 4t^2(3t^2)}{(1 + t^3)^2}$

$= \dfrac{4t(2 - t^3)}{(1 + t^3)^2} = 0$ when $t = 0$ or $t = \sqrt[3]{2}$.

Points: $(0, 0), \left(\dfrac{4\sqrt[3]{2}}{3}, \dfrac{4\sqrt[3]{4}}{3}\right) \approx (1.6799, 2.1165)$

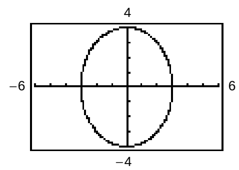
(c) $s = 2\displaystyle\int_0^1 \sqrt{\left[\frac{4(1 - 2t^3)}{(1 + t^3)^2}\right]^2 + \left[\frac{4t(2 - t^3)}{(1 + t^3)^2}\right]^2}\, dt = 2\int_0^1 \sqrt{\frac{16}{(1 + t^3)^4}\left[t^8 + 4t^6 - 4t^5 - 4t^3 + 4t^2 + 1\right]}\, dt$

$= 8\displaystyle\int_0^1 \frac{\sqrt{t^8 + 4t^6 - 4t^5 - 4t^3 + 4t^2 + 1}}{(1 + t^3)^2}\, dt \approx 6.557$

48. $x = 3\cos\theta,\ y = 4\sin\theta$

$\dfrac{dx}{d\theta} = -3\sin\theta,\ \dfrac{dy}{d\theta} = 4\cos\theta$

$s = \displaystyle\int_0^{2\pi} \sqrt{9\sin^2\theta + 16\cos^2\theta}\, d\theta \approx 22.1$



50. $x = t,\ y = 4 - 2t,\ \dfrac{dx}{dt} = 1,\ \dfrac{dy}{dt} = -2$

(a) $S = 2\pi\displaystyle\int_0^2 (4 - 2t)\sqrt{1 + 4}\, dt$

$= \left[2\sqrt{5}\pi(4t - t^2)\right]_0^2 = 8\pi\sqrt{5}$

(b) $S = 2\pi\displaystyle\int_0^2 t\sqrt{1 + 4}\, dt = \left[\sqrt{5}\pi t^2\right]_0^2 = 4\pi\sqrt{5}$

52. $x = \dfrac{1}{3}t^3, y = t + 1, 1 \le t \le 2, y$-axis

$\dfrac{dx}{dt} = t^2, \dfrac{dy}{dt} = 1$

$S = 2\pi\displaystyle\int_1^2 \frac{1}{3}t^3\sqrt{t^4 + 1}\, dt = \frac{\pi}{9}\left[(x^4 + 1)^{3/2}\right]_1^2$

$= \dfrac{\pi}{9}(17^{3/2} - 2^{3/2}) \approx 23.48$

54. $x = a\cos\theta,\ y = b\sin\theta,\ \dfrac{dx}{d\theta} = -a\sin\theta,\ \dfrac{dy}{d\theta} = b\cos\theta$

(a) $S = 4\pi\displaystyle\int_0^{\pi/2} b\sin\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta$

$= 4\pi\displaystyle\int_0^{\pi/2} ab\sin\theta\sqrt{1 - \left(\frac{a^2 - b^2}{a^2}\right)\cos^2\theta}\, d\theta = \frac{-4ab\pi}{e}\int_0^{\pi/2} (-e\sin\theta)\sqrt{1 - e^2\cos^2\theta}\, d\theta$

$= \dfrac{-2ab\pi}{e}\left[e\cos\theta\sqrt{1 - e^2\cos^2\theta} + \arcsin(e\cos\theta)\right]_0^{\pi/2} = \dfrac{-ab\pi}{e}\left[e\sqrt{1 - e^2} + \arcsin(e)\right]$

$= 2\pi b^2 + \left(\dfrac{2\pi a^2 b}{\sqrt{a^2 - b^2}}\right)\arcsin\left(\dfrac{\sqrt{a^2 - b^2}}{a}\right) = 2\pi b^2 + 2\pi\left(\dfrac{ab}{e}\right)\arcsin(e)$

$\left(e = \dfrac{\sqrt{a^2 - b^2}}{a} = \dfrac{c}{a}\text{: eccentricity}\right)$

—CONTINUED—

54. —CONTINUED—

(b) $S = 4\pi\int_0^{\pi/2} a\cos\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\,d\theta$

$$= 4\pi\int_0^{\pi/2} a\cos\theta\sqrt{b^2 + c^2\sin^2\theta}\,d\theta = \frac{4a\pi}{c}\int_0^{\pi/2} c\cos\theta\sqrt{b^2 + c^2\sin^2\theta}\,d\theta$$

$$= \frac{2a\pi}{c}\left[c\sin\theta\sqrt{b^2 + c^2\sin^2\theta} + b^2\ln\left|c\sin\theta + \sqrt{b^2 + c^2\sin^2\theta}\right|\right]_0^{\pi/2}$$

$$= \frac{2a\pi}{c}\left[c\sqrt{b^2 + c^2} + b^2\ln\left|c + \sqrt{b^2 + c^2}\right| - b^2\ln b\right]$$

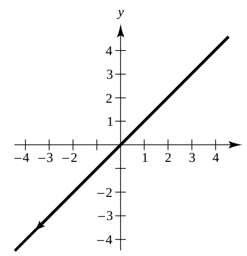
$$= 2\pi a^2 + \frac{2\pi ab^2}{\sqrt{a^2 - b^2}}\ln\left|\frac{a + \sqrt{a^2 - b^2}}{b}\right| = 2\pi a^2 + \left(\frac{\pi b^2}{e}\right)\ln\left|\frac{1 + e}{1 - e}\right|$$

56. (a) 0

(b) 4

58. One possible answer is the graph given by

$x = -t, y = -t.$



60. (a) $S = 2\pi\int_a^b g(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

(b) $S = 2\pi\int_a^b f(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

62. Let y be a continuous function of x on $a \le x \le b$. Suppose that $x = f(t)$, $y = g(t)$, and $f(t_1) = a$, $f(t_2) = b$. Then using integration by substitution, $dx = f'(t)\,dt$ and

$$\int_a^b y\,dx = \int_{t_1}^{t_2} g(t)f'(t)\,dt.$$

64. $x = \sqrt{4 - t},\ y = \sqrt{t},\ \frac{dx}{dt} = -\frac{1}{2\sqrt{4 - t}},\ 0 \le t \le 4$

$$A = \int_4^0 \sqrt{t}\left(-\frac{1}{2\sqrt{4 - t}}\right)dt = \int_0^2 \sqrt{4 - u^2}\,du = \frac{1}{2}\left[u\sqrt{4 - u^2} + 4\arcsin\frac{u}{2}\right]_0^2 = \pi$$

Let $u = \sqrt{4 - t}$, then $du = -1/\left(2\sqrt{4 - t}\right)dt$ and $\sqrt{t} = \sqrt{4 - u^2}$.

$$\bar{x} = \frac{1}{\pi}\int_4^0 \sqrt{4 - t}\sqrt{t}\left(-\frac{1}{2\sqrt{4 - t}}\right)dt = -\frac{1}{2\pi}\int_4^0 \sqrt{t}\,dt = \left[-\frac{1}{2\pi}\frac{2}{3}t^{3/2}\right]_4^0 = \frac{8}{3\pi}$$

$$\bar{y} = \frac{1}{2\pi}\int_4^0 \left(\sqrt{t}\right)^2\left(-\frac{1}{2\sqrt{4 - t}}\right)dt = -\frac{1}{4\pi}\int_4^0 \frac{t}{\sqrt{4 - t}}\,dt = -\frac{1}{4\pi}\left[\frac{-2(8 + t)}{3}\sqrt{4 - t}\right]_4^0 = \frac{8}{3\pi}$$

$$(\bar{x}, \bar{y}) = \left(\frac{8}{3\pi}, \frac{8}{3\pi}\right)$$

66. $x = \cos\theta,\ y = 3\sin\theta,\ \frac{dx}{d\theta} = -\sin\theta$

$$V = 2\pi\int_{\pi/2}^0 (3\sin\theta)^2(-\sin\theta)\,d\theta$$

$$= -18\pi\int_{\pi/2}^0 \sin^3\theta\,d\theta = -18\pi\left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_{\pi/2}^0 = 12\pi$$

68. $x = 2\cot\theta,\ y = 2\sin^2\theta,\ \dfrac{dx}{d\theta} = -2\csc^2\theta$

$$A = 2\int_{\pi/2}^{0}(2\sin^2\theta)(-2\csc^2\theta)\,d\theta = -8\int_{\pi/2}^{0}d\theta = \Big[-8\theta\Big]_{\pi/2}^{0} = 4\pi$$

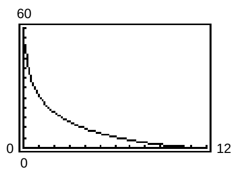
70. $\frac{3}{8}\pi a^2$ is area of asteroid (b).

72. $2\pi a^2$ is area of deltoid (c).

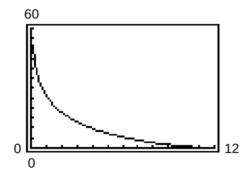
74. $2\pi ab$ is area of teardrop (e).

76. (a) $y = -12\ln\left(\dfrac{12 - \sqrt{144 - x^2}}{x}\right) - \sqrt{144 - x^2}$

$0 < x \leq 12$



(b) $x = 12\operatorname{sech}\dfrac{t}{12},\ y = t - 12\tanh\dfrac{t}{12},\ 0 \leq t$



Same as the graph in (a), but has the advantage of showing the position of the object and any given time t.

(c) $\dfrac{dy}{dx} = \dfrac{1 - \operatorname{sech}^2(t/12)}{-\operatorname{sech}(t/12)\tanh(t/12)} = -\sinh\dfrac{t}{12}$

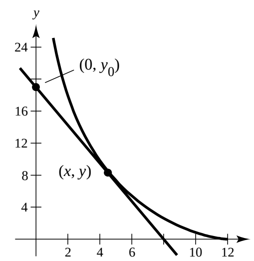
Tangent line: $y - \left(t_0 - 12\tanh\dfrac{t_0}{12}\right) = -\sinh\dfrac{t_0}{12}\left(x - 12\operatorname{sech}\dfrac{t_0}{12}\right)$

$$y = t_0 - \left(\sinh\frac{t_0}{12}\right)x$$

y-intercept: $(0, t_0)$

Distance between $(0, t_0)$ and (x, y): $d = \sqrt{\left(12\operatorname{sech}\dfrac{t_0}{12}\right)^2 + \left(-12\tanh\dfrac{t_0}{12}\right)^2} = 12$

$$d = 12 \text{ for any } t \geq 0.$$



78. False. Both dx/dt and dy/dt are zero when $t = 0$. By eliminating the parameter, we have $y = x^{2/3}$ which does not have a horizontal tangent at the origin.

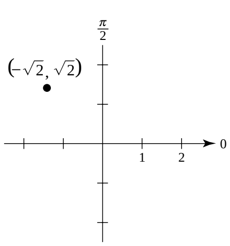
Section 9.4 Polar Coordinates and Polar Graphs

2. $\left(-2, \dfrac{7\pi}{4}\right)$

$x = -2\cos\left(\dfrac{7\pi}{4}\right) = -\sqrt{2}$

$y = -2\sin\left(\dfrac{7\pi}{4}\right) = \sqrt{2}$

$(x, y) = \left(-\sqrt{2}, \sqrt{2}\right)$

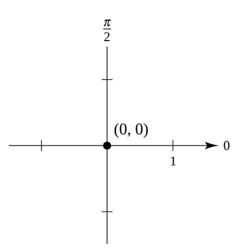


4. $\left(0, -\dfrac{7\pi}{6}\right)$

$x = 0\cos\left(-\dfrac{7\pi}{6}\right) = 0$

$y = 0\sin\left(-\dfrac{7\pi}{6}\right) = 0$

$(x, y) = (0, 0)$

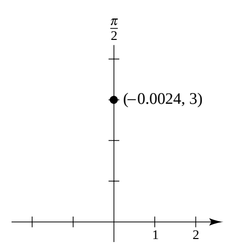


6. $(-3, -1.57)$

$x = -3\cos(-1.57) \approx -0.0024$

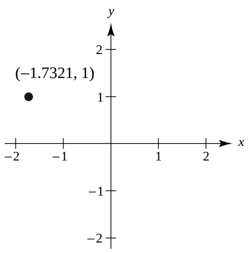
$y = -3\sin(-1.57) \approx 3$

$(x, y) = (-0.0024, 3)$



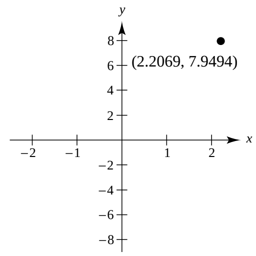
8. $(r, \theta) = \left(-2, \dfrac{11\pi}{6}\right)$

$(x, y) = (-1.7321, 1)$



10. $(r, \theta) = (8.25, 1.3)$

$(x, y) = (2.2069, 7.9494)$

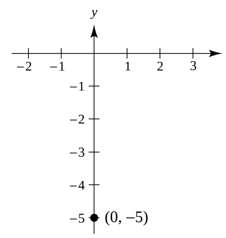


12. $(x, y) = (0, -5)$

$r = \pm 5$

$\tan\theta$ undefined

$\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \left(5, \dfrac{3\pi}{2}\right), \left(-5, \dfrac{\pi}{2}\right)$



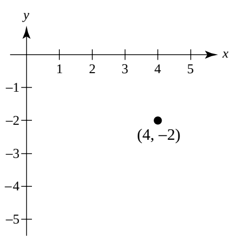
14. $(x, y) = (4, -2)$

$r = \pm\sqrt{16 + 4} = \pm 2\sqrt{5}$

$\tan\theta = -\dfrac{2}{4} = -\dfrac{1}{2}$

$\theta \approx -0.464$

$\left(2\sqrt{5}, -0.464\right), \left(-2\sqrt{5}, 2.678\right)$



16. $(x, y) = \left(3\sqrt{2}, 3\sqrt{2}\right)$

$(r, \theta) = (6, 0.785)$

18. $(x, y) = (0, -5)$

$(r, \theta) = (5, -1.571)$

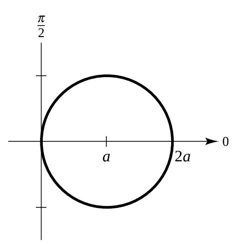
20. (a) Moving horizontally, the x-coordinate changes. Moving vertically, the y-coordinate changes.

(b) Both r and θ values change.

(c) In polar mode, horizontal (or vertical) changes result in changes in both r and θ.

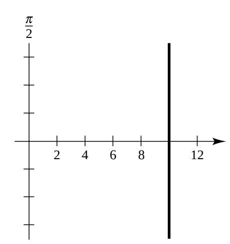
22.

$$x^2 + y^2 - 2ax = 0$$
$$r^2 - 2ar\cos\theta = 0$$
$$r(r - 2a\cos\theta) = 0$$
$$r = 2a\cos\theta$$



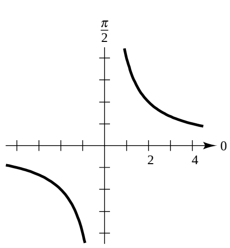
24.

$$x = 10$$
$$r\cos\theta = 10$$
$$r = 10\sec\theta$$



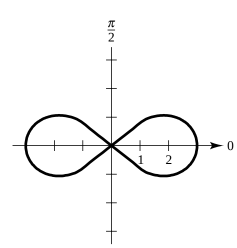
26.

$$xy = 4$$
$$(r\cos\theta)(r\sin\theta) = 4$$
$$r^2 = 4\sec\theta\csc\theta$$
$$= 8\csc 2\theta$$



28.

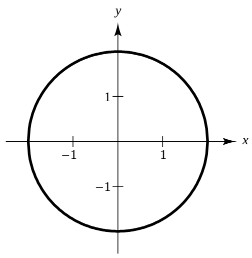
$$(x^2 + y^2)^2 - 9(x^2 - y^2) = 0$$
$$(r^2)^2 - 9(r^2\cos^2\theta - r^2\sin^2\theta) = 0$$
$$r^2[r^2 - 9(\cos 2\theta)] = 0$$
$$r^2 = 9\cos 2\theta$$



30. $r = -2$

$r^2 = 4$

$x^2 + y^2 = 4$



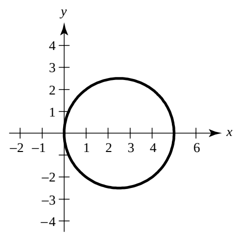
32. $r = 5\cos\theta$

$r^2 = 5r\cos\theta$

$x^2 + y^2 = 5x$

$x^2 - 5x + \frac{25}{4} + y^2 = \frac{25}{4}$

$\left(x - \frac{5}{2}\right)^2 + y^2 = \left(\frac{5}{2}\right)^2$

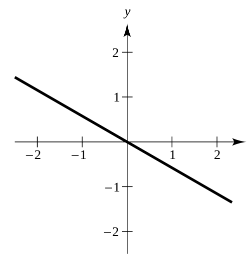


34. $\theta = \frac{5\pi}{6}$

$\tan\theta = \tan\frac{5\pi}{6}$

$\frac{y}{x} = -\frac{\sqrt{3}}{3}$

$y = -\frac{\sqrt{3}}{3}x$

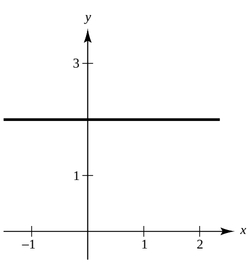


36. $r = 2\csc\theta$

$r\sin\theta = 2$

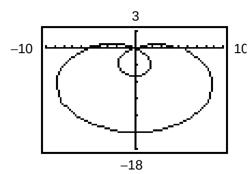
$y = 2$

$y - 2 = 0$



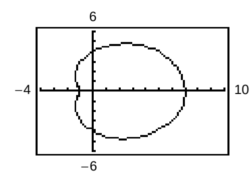
38. $r = 5(1 - 2\sin\theta)$

$0 \le \theta < 2\pi$



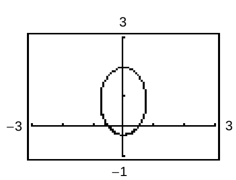
40. $r = 4 + 3\cos\theta$

$0 \le \theta < 2\pi$



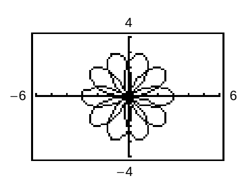
42. $r = \frac{2}{4 - 3\sin\theta}$

Traced out once on $0 \le \theta \le 2\pi$



44. $r = 3\sin\left(\frac{5\theta}{2}\right)$

$0 \le \theta < 4\pi$

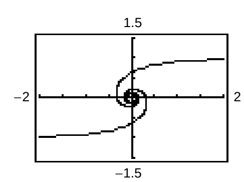


46. $r^2 = \frac{1}{\theta}$.

Graph as

$$r_1 = \frac{1}{\sqrt{\theta}},\ r_2 = -\frac{1}{\sqrt{\theta}}.$$

It is traced out once on $[0, \infty)$.



48. (a) The rectangular coordinates of (r_1, θ_1) are $(r_1 \cos \theta_1, r_1 \sin \theta_1)$. The rectangular coordinates of (r_2, θ_2) are $(r_2 \cos \theta_2, r_2 \sin \theta_2)$.

$$\begin{aligned} d^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &= (r_2 \cos \theta_2 - r_1 \cos \theta_1)^2 + (r_2 \sin \theta_2 - r_1 \sin \theta_1)^2 \\ &= r_2^2 \cos^2 \theta_2 - 2r_1r_2 \cos \theta_1 \cos \theta_2 + r_1^2 \cos^2 \theta_1 + r_2^2 \sin^2 \theta_2^2 - 2r_1r_2 \sin \theta_1 \sin \theta_2 + r_1^2 \sin^2 \theta_1 \\ &= r_2^2 (\cos^2 \theta_2 + \sin^2 \theta_2) + r_1^2 (\cos^2 \theta_1 + \sin^2 \theta_1) - 2\, r_1r_2(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) \\ &= r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2) \\ d &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2)} \end{aligned}$$

(b) If $\theta_1 = \theta_2$, the points lie on the same line passing through the origin. In this case,

$$\begin{aligned} d &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(0)} \\ &= \sqrt{(r_1 - r_2)^2} = |r_1 - r_2| \end{aligned}$$

(c) If $\theta_1 - \theta_2 = 90°$, then $\cos(\theta_1 - \theta_2) = 0$ and $d = \sqrt{r_1^2 + r_2^2}$, the Pythagorean Theorem!

(d) Many answers are possible. For example, consider the two points $(r_1, \theta_1) = (1, 0)$ and $(r_2, \theta_2) = (2, \pi/2)$.

$$d = \sqrt{1 + 2^2 - 2(1)(2) \cos\left(0 - \frac{\pi}{2}\right)} = \sqrt{5}$$

Using $(r_1, \theta_1) = (-1, \pi)$ and $(r_2, \theta_2) = [2, (5\pi/2)]$, $d = \sqrt{(-1)^2 + (2)^2 - 2(-1)(2) \cos\left(\pi - \frac{5\pi}{2}\right)} = \sqrt{5}$.

You always obtain the same distance.

50. $\left(10, \frac{7\pi}{6}\right)$, $(3, \pi)$

$$\begin{aligned} d &= \sqrt{10^2 + 3^2 - 2(10)(3) \cos\left(\frac{7\pi}{6} - \pi\right)} \\ &= \sqrt{109 - 60 \cos \frac{\pi}{6}} = \sqrt{109 - 30\sqrt{3}} \approx 7.6 \end{aligned}$$

52. $(4, 2.5)$, $(12, 1)$

$$\begin{aligned} d &= \sqrt{4^2 + 12^2 - 2(4)(12) \cos(2.5 - 1)} \\ &= \sqrt{160 - 96 \cos 1.5} \approx 12.3 \end{aligned}$$

54. $r = 2(1 - \sin \theta)$

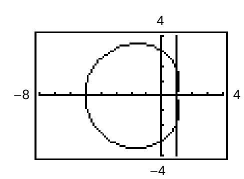
$$\frac{dy}{dx} = \frac{-2 \cos \theta \sin \theta + 2 \cos \theta(1 - \sin \theta)}{-2 \cos \theta \cos \theta - 2 \sin \theta(1 - \sin \theta)}$$

At $(2, 0)$, $\frac{dy}{dx} = -1$.

At $\left(3, \frac{7\pi}{6}\right)$, $\frac{dy}{dx}$ is undefined.

At $\left(4, \frac{3\pi}{2}\right)$, $\frac{dy}{dx} = 0$.

56. (a), (b) $r = 3 - 2 \cos \theta$

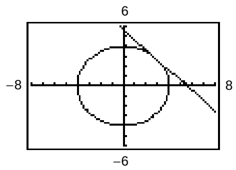


$(r, \theta) = (1, 0) \implies (x, y) = (1, 0)$

Tangent line: $x = 1$

(c) At $\theta = 0$, $\frac{dy}{dx}$ does not exist (vertical tangent).

58. (a), (b) $r = 4$



at $(r, \theta) = \left(4, \frac{\pi}{4}\right) \Rightarrow (x, y) = \left(2\sqrt{2}, 2\sqrt{2}\right)$

Tangent line: $y - 2\sqrt{2} = -1\left(x - 2\sqrt{2}\right)$

$$y = -x + 4\sqrt{2}$$

(c) At $\theta = \frac{\pi}{4}, \frac{dy}{dx} = -1.$

60. $r = a \sin \theta$

$$\frac{dy}{d\theta} = a \sin \theta \cos \theta + a \cos \theta \sin \theta$$

$$= 2a \sin \theta \cos \theta = 0$$

$$\theta = 0, \ \frac{\pi}{2}, \ \pi, \ \frac{3\pi}{2}$$

$$\frac{dx}{d\theta} = -a \sin^2 \theta + a \cos^2 \theta = a(1 - 2 \sin^2 \theta) = 0$$

$$\sin \theta = \pm\frac{1}{\sqrt{2}}, \ \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

Horizontal: $(0, 0), \left(a, \frac{\pi}{2}\right)$

Vertical: $\left(\frac{a\sqrt{2}}{2}, \frac{\pi}{4}\right), \left(\frac{a\sqrt{2}}{2}, \frac{3\pi}{4}\right)$

62. $r = a \sin \theta \cos^2 \theta$

$$\frac{dy}{d\theta} = a \sin \theta \cos^3 \theta + [-2a \sin^2 \theta \cos \theta + a \cos^3 \theta] \sin \theta$$

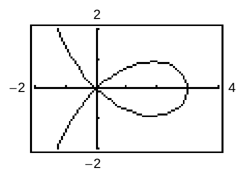
$$= 2a[\sin \theta \cos^3 \theta - \sin^3 \theta \cos \theta]$$

$$= 2a \sin \theta \cos \theta(\cos^2 \theta - \sin^2 \theta) = 0$$

$$\theta = 0, \ \tan^2 \theta = 1, \ \theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

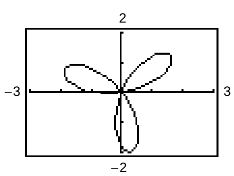
Horizontal: $\left(\frac{\sqrt{2}a}{4}, \frac{\pi}{4}\right), \left(\frac{\sqrt{2}a}{4}, \frac{3\pi}{4}\right), (0, 0)$

64. $r = 3 \cos 2\theta \sec \theta$



Horizontal tangents: $(2.133, \pm 0.4352)$

66. $r = 2 \cos(3\theta - 2)$



Horizontal tangents:

$(1.894, 0.776), (1.755, 2.594), (1.998, -1.442)$

68.

$$r = 3 \cos \theta$$

$$r^2 = 3r \cos \theta$$

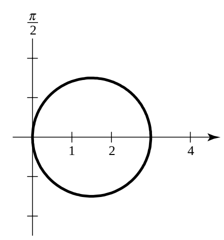
$$x^2 + y^2 = 3x$$

$$\left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}$$

Circle: $r = \frac{3}{2}$

Center: $\left(\frac{3}{2}, 0\right)$

Tangent at pole: $\theta = \frac{\pi}{2}$

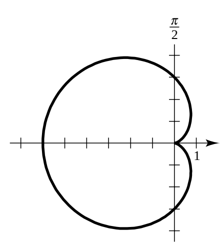


70. $r = 3(1 - \cos \theta)$

Cardioid

Symmetric to polar axis since r is a function of $\cos \theta$.

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	0	$\frac{3}{2}$	3	$\frac{9}{2}$	6



72. $r = -\sin(5\theta)$

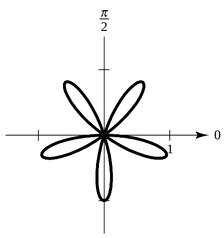
Rose curve with five petals

Symmetric to $\theta = \dfrac{\pi}{2}$

Relative extrema occur when

$$\frac{dr}{d\theta} = -5\cos(5\theta) = 0 \text{ at } \theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}.$$

Tangents at the pole: $\theta = 0, \dfrac{\pi}{5}, \dfrac{2\pi}{5}, \dfrac{3\pi}{5}, \dfrac{4\pi}{5}$



74. $r = 3\cos 2\theta$

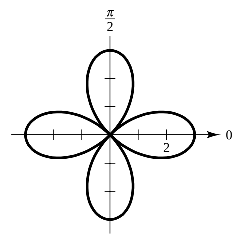
Rose curve with four petals

Symmetric to the polar axis, $\theta = \dfrac{\pi}{2}$, and pole

Relative extrema: $(3, 0), \left(-3, \dfrac{\pi}{2}\right), (3, \pi), \left(-3, \dfrac{3\pi}{2}\right)$

Tangents at the pole: $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$

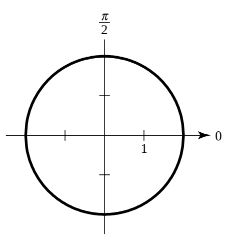
$\theta = \dfrac{5\pi}{4}$ and $\dfrac{7\pi}{4}$ given the same tangents.



76. $r = 2$

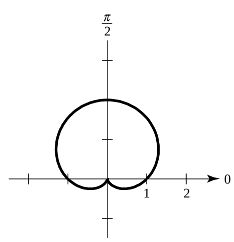
Circle radius: 2

$x^2 + y^2 = 4$



78. $r = 1 + \sin\theta$

Cardioid

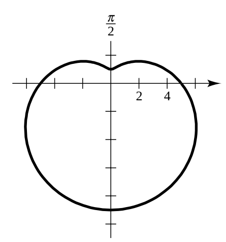


80. $r = 5 - 4\sin\theta$

Limaçon

Symmetric to $\theta = \dfrac{\pi}{2}$

θ	$-\dfrac{\pi}{2}$	$-\dfrac{\pi}{6}$	0	$\dfrac{\pi}{6}$	$\dfrac{\pi}{2}$
r	9	7	5	3	1

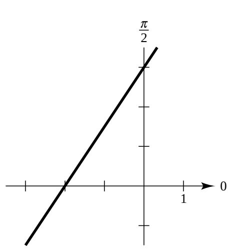


82. $$r = \frac{6}{2\sin\theta - 3\cos\theta}$$

$2r\sin\theta - 3r\cos\theta = 6$

$2y - 3x = 6$

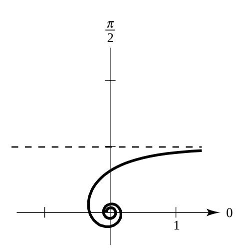
Line



84. $r = \dfrac{1}{\theta}$

Hyperbolic spiral

θ	$\dfrac{\pi}{4}$	$\dfrac{\pi}{2}$	$\dfrac{3\pi}{4}$	π	$\dfrac{5\pi}{4}$	$\dfrac{3\pi}{2}$
r	$\dfrac{4}{\pi}$	$\dfrac{2}{\pi}$	$\dfrac{4}{3\pi}$	$\dfrac{1}{\pi}$	$\dfrac{4}{5\pi}$	$\dfrac{2}{3\pi}$



86. $r^2 = 4 \sin \theta$

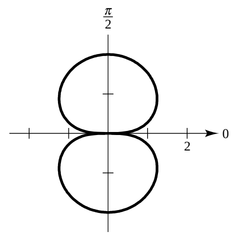
Lemniscate

Symmetric to the polar axis, $\theta = \dfrac{\pi}{2}$, and pole

Relative extrema: $\left(\pm 2, \dfrac{\pi}{2}\right)$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	π
r	0	$\pm\sqrt{2}$	± 2	$\pm\sqrt{2}$	0

Tangent at the pole: $\theta = 0$



88. Since

$$r = 2 + \csc \theta = 2 + \frac{1}{\sin \theta},$$

the graphs has symmetry with respect to $\theta = \pi/2$. Furthermore,

$$r \Rightarrow \infty \text{ as } \theta \Rightarrow 0^+$$

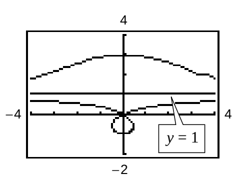
$$r \Rightarrow \infty \text{ as } \theta \Rightarrow \pi^-.$$

Also, $r = 2 + \dfrac{1}{\sin \theta} = 2 + \dfrac{r}{\sin \theta} = 2 + \dfrac{r}{y}$

$$ry = 2y + r$$

$$r = \frac{2y}{y - 1}.$$

Thus, $r \Rightarrow \pm\infty$ as $y \Rightarrow 1$.



90. $r = 2 \cos 2\theta \sec \theta$

Strophoid

$r \Rightarrow -\infty$ as $\theta \Rightarrow \dfrac{\pi^-}{2}$

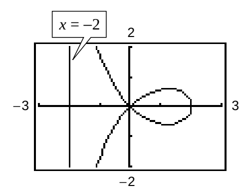
$r \Rightarrow \infty$ as $\theta \Rightarrow \dfrac{-\pi^+}{2}$

$$r = 2 \cos 2\theta \sec \theta = 2(2 \cos^2 \theta - 1) \sec \theta$$

$$r \cos \theta = 4 \cos^2 \theta - 2$$

$$x = 4 \cos^2 \theta - 2$$

$$\lim_{\theta \to \pm\pi/2} (4 \cos^2 \theta - 2) = -2$$



92. $x = r \cos \theta, y = r \sin \theta$

$x^2 + y^2 = r^2, \tan \theta = \dfrac{y}{x}$

94. Slope of tangent line to graph of $r = f(\theta)$ at (r, θ) is

$$\frac{dy}{dx} = \frac{f(\theta)\cos \theta + f'(\theta)\sin \theta}{-f(\theta)\sin \theta + f'(\theta)\cos \theta}.$$

If $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, then $\theta = \alpha$ is tangent at the pole.

96. $r = 4 \cos 2\theta$

Rose curve

Matches (b)

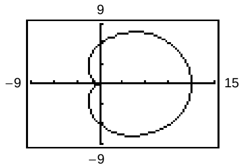
98. $r = 2 \sec \theta$

Line

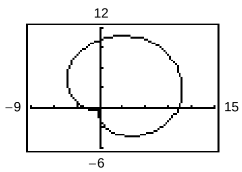
Matches (d)

100. $r = 6[1 + \cos(\theta - \phi)]$

(a) $\phi = 0, r = 6[1 + \cos\theta]$



(b) $\theta = \frac{\pi}{4}, r = 6\left[1 + \cos\left(\theta - \frac{\pi}{4}\right)\right]$



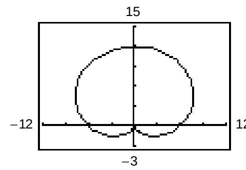
The graph of $r = 6[1 + \cos\theta]$ is rotated through the angle $\pi/4$.

(c) $\theta = \frac{\pi}{2}$

$$r = 6\left[1 + \cos\left(\theta - \frac{\pi}{2}\right)\right]$$

$$= 6\left[1 + \cos\theta\cos\frac{\pi}{2} + \sin\theta\sin\frac{\pi}{2}\right]$$

$$= 6[1 + \sin\theta]$$



The graph of $r = 6[1 + \cos\theta]$ is rotated through the angle $\pi/2$.

102. (a) $\sin\left(\theta - \frac{\pi}{2}\right) = \sin\theta\cos\left(\frac{\pi}{2}\right) - \cos\theta\sin\left(\frac{\pi}{2}\right)$

$$= -\cos\theta$$

$$r = f\left[\sin\left(\theta - \frac{\pi}{2}\right)\right]$$

$$= f(-\cos\theta)$$

(b) $\sin(\theta - \pi) = \sin\theta\cos\pi - \cos\theta\sin\pi$

$$= -\sin\theta$$

$$r = f[\sin(\theta - \pi)]$$

$$= f(-\sin\theta)$$

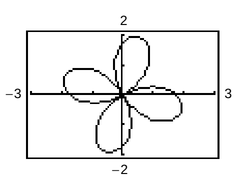
(c) $\sin\left(\theta - \frac{3\pi}{2}\right) = \sin\theta\cos\left(\frac{3\pi}{2}\right) - \cos\theta\sin\left(\frac{3\pi}{2}\right)$

$$= \cos\theta$$

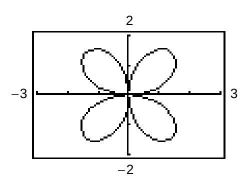
$$r = f\left[\sin\left(\theta - \frac{3\pi}{2}\right)\right] = f(\cos\theta)$$

104. $r = 2\sin 2\theta = 4\sin\theta\cos\theta$

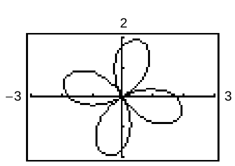
(a) $r = 4\sin\left(\theta - \frac{\pi}{6}\right)\cos\left(\theta - \frac{\pi}{6}\right)$



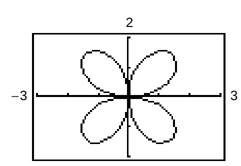
(b) $r = 4\sin\left(\theta - \frac{\pi}{2}\right)\cos\left(\theta - \frac{\pi}{2}\right) = -4\sin\theta\cos\theta$



(c) $r = 4\sin\left(\theta - \frac{2\pi}{3}\right)\cos\left(\theta - \frac{2\pi}{3}\right)$



(d) $r = 4\sin(\theta - \pi)\cos(\theta - \pi) = 4\sin\theta\cos\theta$

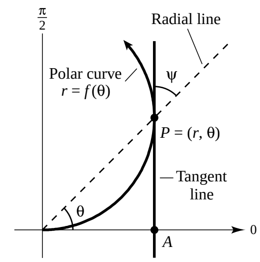


106. By Theorem 9.11, the slope of the tangent line through A and P is

$$\frac{f\cos\theta + f'\sin\theta}{-f\sin\theta + f'\cos\theta}$$

This is equal to

$$\tan(\theta + \psi) = \frac{\tan\theta + \tan\psi}{1 - \tan\theta\tan\psi} = \frac{\sin\theta + \cos\theta\tan\psi}{\cos\theta - \sin\theta\tan\psi}.$$



Equating the expressions and cross-multiplying, you obtain

$$(f\cos\theta + f'\sin\theta)(\cos\theta - \sin\theta\tan\psi) = (\sin\theta + \cos\theta\tan\psi)(-f\sin\theta + f'\cos\theta)$$

$$f\cos^2\theta - f\cos\theta\sin\theta\tan\psi + f'\sin\theta\cos\theta - f'\sin^2\theta\tan\psi = -f\sin^2\theta - f\sin\theta\cos\theta\tan\psi + f'\sin\theta\cos\theta + f'\cos^2\theta\tan\psi$$

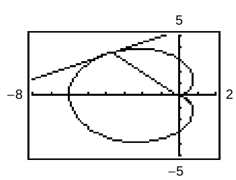
$$f(\cos^2\theta + \sin^2\theta) = f'\tan\psi(\cos^2\theta + \sin^2\theta)$$

$$\tan\psi = \frac{f}{f'} = \frac{r}{dr/d\theta}.$$

108. $\tan\psi = \dfrac{r}{dr/d\theta} = \dfrac{3(1 - \cos\theta)}{3\sin\theta}$

At $\theta = \dfrac{3\pi}{4}$, $\tan\psi = \dfrac{1 + \left(\sqrt{2}/2\right)}{\sqrt{2}} = \dfrac{2 + \sqrt{2}}{\sqrt{2}}$.

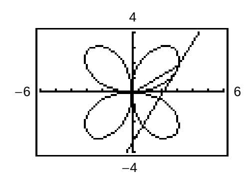
$\psi = \arctan\left(\dfrac{2 + \sqrt{2}}{\sqrt{2}}\right) \approx 1.041 (\approx 59.64°)$



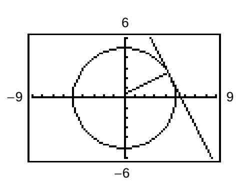
110. $\tan\psi = \dfrac{r}{dr/d\theta} = \dfrac{4\sin 2\theta}{8\cos 2\theta}$

At $\theta = \dfrac{\pi}{6}$, $\tan\psi = \dfrac{\sin(\pi/3)}{2\cos(\pi/3)} = \dfrac{\sqrt{3}}{2}$.

$\psi = \arctan\left(\dfrac{\sqrt{3}}{2}\right) \approx 0.7137 (\approx 40.89°)$



112. $\tan\psi = \dfrac{r}{dr/d\theta} = \dfrac{5}{0}$ undefined $\Rightarrow \psi = \dfrac{\pi}{2}$.

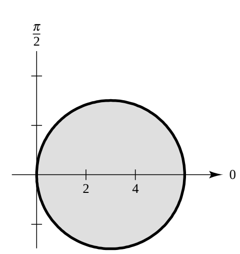


114. True

116. True

Section 9.5 Area and Arc Length in Polar Coordinates

2. (a) $r = 3\cos\theta$



$A = \pi\left(\dfrac{3}{2}\right)^2 = \dfrac{9\pi}{4}$

(b)
$$A = 2\left(\frac{1}{2}\right)\int_0^{\pi/2}\left[3\cos\theta\right]^2 d\theta$$
$$= 9\int_0^{\pi/2}\cos^2\theta\, d\theta$$
$$= \frac{9}{2}\int_0^{\pi/2}(1 + \cos 2\theta)\, d\theta$$
$$= \frac{9}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \frac{9\pi}{4}$$

4. $A = 2\left[\frac{1}{2}\int_0^{\pi/4} (6 \sin 2\theta)^2\, d\theta\right] = 36\int_0^{\pi/4} \sin^2 2\theta\, d\theta$

$= 36\int_0^{\pi/4} \frac{1 - \cos 4\theta}{2}\, d\theta$

$= 18\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/4}$

$= 18\left[\frac{\pi}{4}\right] = \frac{9\pi}{2}$

6. $A = 2\left[\frac{1}{2}\int_0^{\pi/10} (\cos 5\theta)^2\, d\theta\right]$

$= \frac{1}{2}\left[\theta + \frac{1}{10}\sin(10\,\theta)\right]_0^{\pi/10} = \frac{\pi}{20}$

8. $A = 2\left[\frac{1}{2}\int_0^{\pi/2} (1 - \sin\theta)^2\, d\theta\right]$

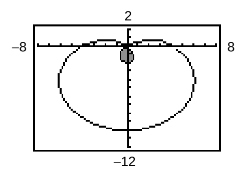
$= \left[\frac{3}{2}\theta + 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} = \frac{3\pi - 8}{4}$

10. $A = 2\left[\frac{1}{2}\int_{\arcsin(2/3)}^{\pi/2} (4 - 6\sin\theta)^2\, d\theta\right.$

$= \int_{\arcsin(2/3)}^{\pi/2} \left[16 - 48\sin\theta + 36\sin^2\theta\right] d\theta$

$= \int_{\arcsin(2/3)}^{\pi/2} \left[16 - 48\sin\theta + 36\left(\frac{1 - \cos 2\theta}{2}\right)\right] d\theta$

$= \left[34\theta + 48\cos\theta - 9\sin 2\theta\right]_{\arcsin(2/3)}^{\pi/2} \approx 1.7635$

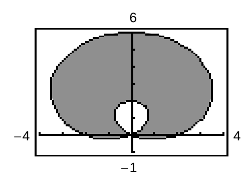


12. Four times the area in Exercise 11, $A = 4\left(\pi + 3\sqrt{3}\right)$. More specifically, we see that the area inside the outer loop is

$$2\left[\frac{1}{2}\int_{-\pi/6}^{\pi/2} (2(1 + 2\sin\theta))^2\, d\theta\right] = \int_{-\pi/6}^{\pi/2} (4 + 16\sin\theta + 16\sin^2\theta)\, d\theta = 8\pi + 6\sqrt{3}.$$

The area inside the inner loop is

$$2\,\frac{1}{2}\left[\int_{7\pi/6}^{3\pi/2} (2(1 + 2\sin\theta))^2\, d\theta\right] = 4\pi - 6\sqrt{3}.$$



Thus, the area between the loops is $\left(8\pi + 6\sqrt{3}\right) - \left(4\pi - 6\sqrt{3}\right) = 4\pi + 12\sqrt{3}$.

14. $r = 3(1 + \sin\theta)$

$r = 3(1 - \sin\theta)$

Solving simultaneously,

$3(1 + \sin\theta) = 3(1 - \sin\theta)$

$2\sin\theta = 0$

$\theta = 0, \pi.$

Replacing r by $-r$ and θ by $\theta + \pi$ in the first equation and solving, $-3(1 - \sin\theta) = 3(1 - \sin\theta)$, $\sin\theta = 1$, $\theta = \pi/2$. Both curves pass through the pole, $(0, 3\pi/2)$, and $(0, \pi/2)$, respectively.

Points of intersection: $(3, 0)$, $(3, \pi)$, $(0, 0)$

16. $r = 2 - 3\cos\theta$

$r = \cos\theta$

Solving simultaneously,

$2 - 3\cos\theta = \cos\theta$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}.$

Both curves pass through the pole, $(0, \arccos 2/3)$, and $(0, \pi/2)$, respectively.

Points of intersection: $\left(\frac{1}{2}, \frac{\pi}{3}\right), \left(\frac{1}{2}, \frac{5\pi}{3}\right), (0, 0)$

18. $r = 1 + \cos\theta$

$r = 3\cos\theta$

Solving simultaneously,

$$1 + \cos\theta = 3\cos\theta$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}.$$

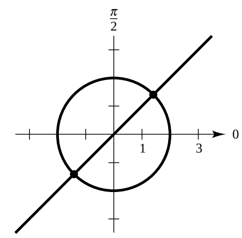
Both curves pass through the pole, $(0, \pi)$, and $(0, \pi/2)$, respectively.

Points of intersection: $\left(\frac{3}{2}, \frac{\pi}{3}\right), \left(\frac{3}{2}, \frac{5\pi}{3}\right), (0, 0)$

20. $\theta = \dfrac{\pi}{4}$

$r = 2$

Line of slope 1 passing through the pole and a circle of radius 2 centered at the pole.

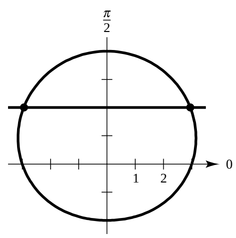


Points of intersection:

$$\left(2, \frac{\pi}{4}\right), \left(-2, \frac{\pi}{4}\right)$$

22. $r = 3 + \sin\theta$

$r = 2\csc\theta$



The graph of $r = 3 + \sin\theta$ is a limaçon symmetric to $\theta = \pi/2$, and the graph of $r = 2\csc\theta$ is the horizontal line $y = 2$. Therefore, there are two points of intersection. Solving simultaneously,

$$3 + \sin\theta = 2\csc\theta$$

$$\sin^2\theta + 3\sin\theta - 2 = 0$$

$$\sin\theta = \frac{-3 \pm \sqrt{17}}{2}$$

$$\theta = \arcsin\left(\frac{\sqrt{17} - 3}{2}\right) \approx 0.596.$$

Points of intersection:

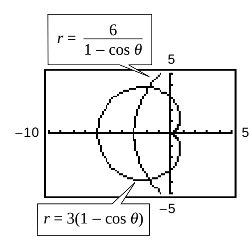
$$\left(\frac{\sqrt{17} + 3}{2}, \arcsin\left(\frac{\sqrt{17} - 3}{2}\right)\right),$$

$$\left(\frac{\sqrt{17} + 3}{2}, \pi - \arcsin\left(\frac{\sqrt{17} - 3}{2}\right)\right),$$

$(3.56, 0.596), (3.56, 2.545)$

24. $r = 3(1 - \cos\theta)$

$r = \dfrac{6}{1 - \cos\theta}$



The graph of $r = 3(1 - \cos\theta)$ is a cardioid with polar axis symmetry. The graph of

$$r = 6/(1 - \cos\theta)$$

is a parabola with focus at the pole, vertex$(3, \pi)$, and polar axis symmetry. Therefore, there are two points of intersection. Solving simultaneously,

$$3(1 - \cos\theta) = \frac{6}{1 - \cos\theta}$$

$$(1 - \cos\theta)^2 = 2$$

$$\cos\theta = 1 \pm \sqrt{2}$$

$$\theta = \arccos\left(1 - \sqrt{2}\right).$$

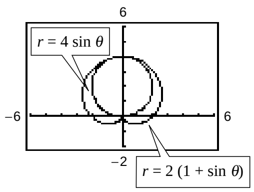
Points of intersection: $\left(3\sqrt{2}, \arccos\left(1 - \sqrt{2}\right)\right) \approx (4.243, 1.998), \left(3\sqrt{2}, 2\pi - \arccos\left(1 - \sqrt{2}\right)\right) \approx (4.243, 4.285)$

26. $r = 4 \sin \theta$

$r = 2(1 + \sin \theta)$

Points of intersection: $(0, 0)$, $\left(4, \dfrac{\pi}{2}\right)$

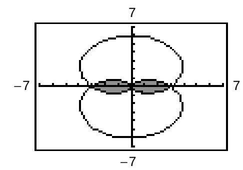
The graphs reach the pole at different times (θ values).



28. $A = 4\left[\dfrac{1}{2}\displaystyle\int_0^{\pi/2} 9(1 - \sin \theta)^2 \, d\theta\right]$

$= 18 \displaystyle\int_0^{\pi/2} (1 - \sin \theta)^2 \, d\theta = \dfrac{9}{2}(3\pi - 8)$

(from Exercise 14)



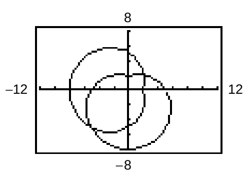
30. $r = 5 - 3 \sin \theta$ and $r = 5 - 3 \cos \theta$ intersect at $\theta = \pi/4$ and $\pi = 5\pi/4$.

$A = 2\left[\dfrac{1}{2}\displaystyle\int_{\pi/4}^{5\pi/4} (5 - 3 \sin \theta)^2 \, d\theta\right]$

$= \left[\dfrac{59}{2}\theta + 30 \cos \theta - \dfrac{9}{4} \sin 2\theta\right]_{\pi/4}^{5\pi/4}$

$= \left(\dfrac{59}{2}\left(\dfrac{5\pi}{4}\right) - 30\dfrac{\sqrt{2}}{2} - \dfrac{9}{4}\right) - \left(\dfrac{59}{2}\left(\dfrac{\pi}{4}\right) + 30\dfrac{\sqrt{2}}{2} - \dfrac{9}{4}\right)$

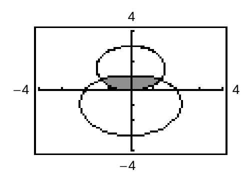
$= \dfrac{59\pi}{2} - 30\sqrt{2} \approx 50.251$



32. $A = 2\left[\dfrac{1}{2}\displaystyle\int_{\pi/6}^{\pi/2} (3 \sin \theta)^2 \, d\theta - \dfrac{1}{2}\displaystyle\int_{\pi/6}^{\pi/2} (2 - \sin \theta)^2 \, d\theta\right]$

$= \displaystyle\int_{\pi/6}^{\pi/2} (-4 \cos 2\theta + 4 \sin \theta) \, d\theta$

$= \Big[-2 \sin(2\theta) - 4 \cos \theta\Big]_{\pi/6}^{\pi/2} = 3\sqrt{3}$



34. Area = Area of $r = 2a \cos \theta$ − Area of sector − twice area between $r = 2a \cos \theta$ and the lines

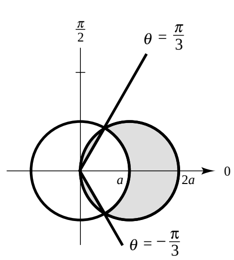
$\theta = \dfrac{\pi}{3}, \theta = \dfrac{\pi}{2}.$

$A = \pi a^2 - \left(\dfrac{\pi}{3}\right)a^2 - 2\left[\dfrac{1}{2}\displaystyle\int_{\pi/3}^{\pi/2} (2a \cos \theta)^2 \, d\theta\right]$

$= \dfrac{2\pi a^2}{3} - 2a^2 \displaystyle\int_{\pi/3}^{\pi/2} (1 + \cos 2\theta) \, d\theta$

$= \dfrac{2\pi a^2}{3} - 2a^2\left[\theta + \dfrac{\sin 2\theta}{2}\right]_{\pi/3}^{\pi/2}$

$= \dfrac{2\pi a^2}{3} - 2a^2\left[\dfrac{\pi}{2} - \dfrac{\pi}{3} - \dfrac{\sqrt{3}}{4}\right] = \dfrac{2\pi a^2 + 3\sqrt{3}a^2}{6}$



36. $r = a \cos \theta, r = a \sin \theta$

$\tan \theta = 1, \theta = \pi/4$

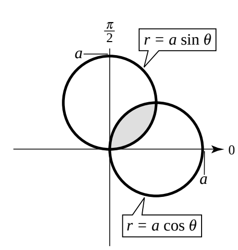
$A = 2\left[\dfrac{1}{2}\displaystyle\int_0^{\pi/2} (a \cos \theta)^2 \, d\theta\right]$

$= a^2 \displaystyle\int_0^{\pi/4} \dfrac{1 + \cos 2\theta}{2} \, d\theta$

$= \dfrac{1}{2}a^2\left[\theta + \dfrac{\sin 2\theta}{2}\right]_0^{\pi/4}$

$= \dfrac{1}{2}a^2\left[\dfrac{\pi}{4} + \dfrac{1}{2}\right]$

$= \dfrac{1}{4}a^2 + \dfrac{1}{8}a^2\pi$



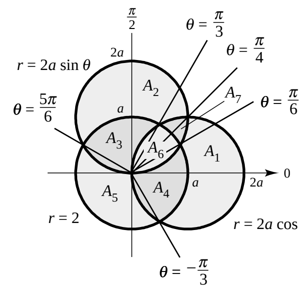
38. By symmetry, $A_1 = A_2$ and $A_3 = A_4$.

$$A_1 = A_2 = \frac{1}{2}\int_{-\pi/3}^{\pi/6}\left[(2a\cos\theta)^2 - (a)^2\right]d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/4}\left[(2a\cos\theta)^2 - (2a\sin\theta)^2\right]d\theta$$

$$= \frac{a^2}{2}\int_{-\pi/3}^{\pi/6}(4\cos^2\theta - 1)\,d\theta + 2a^2\int_{\pi/6}^{\pi/4}\cos 2\theta\,d\theta$$

$$= \frac{a^2}{2}\Big[\theta + \sin 2\theta\Big]_{-\pi/3}^{\pi/6} + a^2\Big[\sin 2\theta\Big]_{\pi/6}^{\pi/4} = \frac{a^2}{2}\left(\frac{\pi}{2} + \sqrt{3}\right) + a^2\left(1 - \frac{\sqrt{3}}{2}\right) = a^2\left(\frac{\pi}{4} + 1\right)$$

$$A_3 = A_4 = \frac{1}{2}\left(\frac{\pi}{2}\right)a^2 = \frac{\pi a^2}{4}$$



$$A_5 = \frac{1}{2}\left(\frac{5\pi}{6}\right)a^2 - 2\left(\frac{1}{2}\right)\int_{5\pi/6}^{\pi}(2a\sin\theta)^2\,d\theta$$

$$= \frac{5\pi a^2}{12} - 2a^2\int_{5\pi/6}^{\pi}(1 - \cos 2\theta)\,d\theta$$

$$= \frac{5\pi a^2}{12} - a^2\Big[2\theta - \sin 2\theta\Big]_{5\pi/6}^{\pi} = \frac{5\pi a^2}{12} - a^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) = a^2\left(\frac{\pi}{12} + \frac{\sqrt{3}}{2}\right)$$

$$A_6 = 2\left(\frac{1}{2}\right)\int_0^{\pi/6}(2a\sin\theta)^2\,d\theta + 2\left(\frac{1}{2}\right)\int_{\pi/6}^{\pi/4}a^2\,d\theta$$

$$= 2a^2\int_0^{\pi/6}(1 - \cos 2\theta)\,d\theta + \Big[a^2\theta\Big]_{\pi/6}^{\pi/4}$$

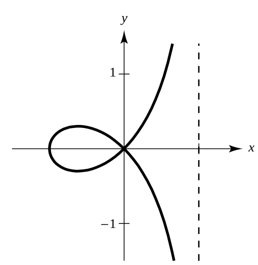
$$= a^2\Big[2\theta - \sin 2\theta\Big]_0^{\pi/6} + \frac{\pi a^2}{12} = a^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) + \frac{\pi a^2}{12} = a^2\left(\frac{5\pi}{12} - \frac{\sqrt{3}}{2}\right)$$

$$A_7 = 2\left(\frac{1}{2}\right)\int_{\pi/6}^{\pi/4}\left[(2a\sin\theta)^2 - (a)^2\right]d\theta$$

$$= a^2\int_{\pi/6}^{\pi/4}(4\sin^2\theta - 1)\,d\theta = a^2\Big[\theta - \sin 2\theta\Big]_{\pi/6}^{\pi/4} = a^2\left(\frac{\pi}{12} - 1 + \frac{\sqrt{3}}{2}\right)$$

[**Note:** $A_1 + A_6 + A_7 + A_4 = \pi a^2 =$ area of circle of radius a]

40. $r = \sec\theta - 2\cos\theta,\ -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$



$$r\cos\theta = 1 - 2\cos^2\theta$$

$$x = 1 - 2\frac{r^2\cos^2\theta}{r^2} = 1 - 2\left(\frac{x^2}{x^2 + y^2}\right)$$

$$(x^2 + y^2)x = x^2 + y^2 - 2x^2$$

$$y^2(x - 1) = -x^2 - x^3$$

$$y^2 = \frac{x^2(1 + x)}{1 - x}$$

$$A = 2\left(\frac{1}{2}\right)\int_0^{\pi/4}(\sec\theta - 2\cos\theta)^2\,d\theta$$

$$= \int_0^{\pi/4}(\sec^2\theta - 4 + 4\cos^2\theta)\,d\theta = \int_0^{\pi/4}(\sec^2\theta - 4 + 2(1 + \cos 2\theta))\,d\theta = \Big[\tan\theta - 2\theta + \sin 2\theta\Big]_0^{\pi/4} = 2 - \frac{\pi}{2}$$

42. $r = 2a\cos\theta$

$r' = -2a\sin\theta$

$$s = \int_{-\pi/2}^{\pi/2} \sqrt{(2a\cos\theta)^2 + (-2a\sin\theta)^2}\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 2a\,d\theta = \Big[2\theta\Big]_{-\pi/2}^{\pi/2} = 2\pi a$$

44. $r = 8(1 + \cos\theta),\ 0 \le \theta \le 2\pi$

$r' = -8\sin\theta$

$$s = 2\int_0^{\pi} \sqrt{[8(1+\cos\theta)]^2 + (-8\sin\theta]^2}\,d\theta$$

$$= 16\int_0^{\pi} \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}\,d\theta$$

$$= 16\sqrt{2}\int_0^{\pi} \sqrt{1+\cos\theta}\,d\theta$$

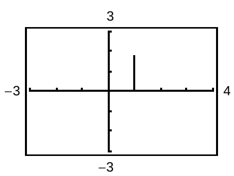
$$= 16\sqrt{2}\int_0^{\pi} \sqrt{1+\cos\theta}\cdot\left(\frac{\sqrt{1-\cos\theta}}{\sqrt{1-\cos\theta}}\right)d\theta$$

$$= 16\sqrt{2}\int_0^{\pi} \frac{\sin\theta}{\sqrt{1-\cos\theta}}\,d\theta$$

$$= \Big[32\sqrt{2}\sqrt{1-\cos\theta}\Big]_0^{\pi}$$

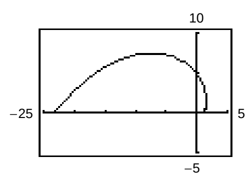
$$= 64$$

46. $r = \sec\theta,\ 0 \le \theta \le \dfrac{\pi}{3}$



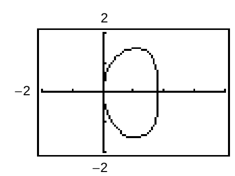
Length ≈ 1.73 (exact $\sqrt{3}$)

48. $r = e^{\theta},\ 0 \le \theta \le \pi$



Length ≈ 31.31

50. $r = 2\sin(2\cos\theta),\ 0 \le \theta \le \pi$



Length ≈ 7.78

52. $r = a\cos\theta$

$r' = -a\sin\theta$

$$S = 2\pi\int_0^{\pi/2} a\cos\theta\,(\cos\theta)\sqrt{a^2\cos\theta + a^2\sin^2\theta}\,d\theta$$

$$= 2\pi a^2\int_0^{\pi/2}\cos^2\theta\,d\theta = \pi a^2\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta$$

$$= \left[\pi a^2\left(\theta + \frac{\sin 2\theta}{2}\right)\right]_0^{\pi/2} = \frac{\pi^2 a^2}{2}$$

54. $r = a(1+\cos\theta)$

$r' = -a\sin\theta$

$$S = 2\pi\int_0^{\pi} a(1+\cos\theta)\sin\theta\sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\,d\theta = 2\pi a^2\int_0^{\pi}\sin\theta(1+\cos\theta)\sqrt{2+2\cos\theta}\,d\theta$$

$$= -2\sqrt{2}\pi a^2\int_0^{\pi}(1+\cos\theta)^{3/2}(-\sin\theta)\,d\theta = -\frac{4\sqrt{2}\pi a^2}{5}\Big[(1+\cos\theta)^{5/2}\Big]_0^{\pi} = \frac{32\pi a^2}{5}$$

56. $r = \theta$

$r' = 1$

$$S = 2\pi\int_0^{\pi}\theta\sin\theta\sqrt{\theta^2+1}\,d\theta \approx 42.32$$

58. The curves might intersect for different values of θ:

See page 696.

60. (a) $S = 2\pi \int_{\alpha}^{\beta} f(\theta) \sin\theta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$

(b) $S = 2\pi \int_{\alpha}^{\beta} f(\theta) \cos\theta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$

62. $r = 8\cos\theta,\ 0 \le \theta \le \pi$

(a) $A = \frac{1}{2}\int_0^{\pi} r^2\, d\theta = \frac{1}{2}\int_0^{\pi} 64\cos^2\theta\, d\theta = 32\int_0^{\pi} \frac{1 + \cos 2\theta}{2}\, d\theta = 16\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi} = 16\pi$

(Area circle $= \pi r^2 = \pi 4^2 = 16\pi$)

(b)

θ	0.2	0.4	0.6	0.8	1.0	1.2	1.4
A	6.32	12.14	17.06	20.80	23.27	24.60	25.08

(c), (d) For $\frac{1}{4}$ of area $(4\pi \approx 12.57)$: 0.42

For $\frac{1}{2}$ of area $(8\pi \approx 25.13)$: 1.57 $(\pi/2)$

For $\frac{3}{4}$ of area $(12\pi \approx 37.70)$: 2.73

(e) No, it does not depend on the radius.

64. False. $f(\theta) = 0$ and $g(\theta) = \sin 2\theta$ have only one point of intersection.

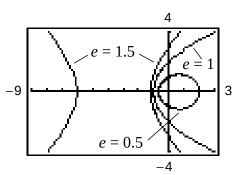
Section 9.6 Polar Equations of Conics and Kepler's Laws

2. $r = \dfrac{2e}{1 - e\cos\theta}$

(a) $e = 1,\ r = \dfrac{2}{1 - \cos\theta}$, parabola

(b) $e = 0.5,\ r = \dfrac{1}{1 - 0.5\cos\theta} = \dfrac{2}{2 - \cos\theta}$, ellipse

(c) $e = 1.5,\ r = \dfrac{3}{1 - 1.5\cos\theta} = \dfrac{6}{2 - 3\cos\theta}$, hyperbola

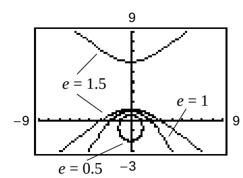


4. $r = \dfrac{2e}{1 + e\sin\theta}$

(a) $e = 1,\ r = \dfrac{2}{1 + \sin\theta}$, parabola

(b) $e = 0.5,\ r = \dfrac{1}{1 + 0.5\sin\theta} = \dfrac{2}{2 + \sin\theta}$, ellipse

(c) $e = 1.5,\ r = \dfrac{3}{1 + 1.5\sin\theta} = \dfrac{6}{2 + 3\sin\theta}$, hyperbola



6. $r = \dfrac{4}{1 - 0.4\cos\theta}$

(a) Because $e = 0.4 < 1$, the conic is an ellipse with vertical directrix to the left of the pole.

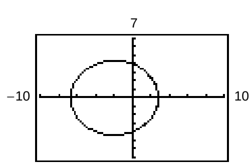
(b) $r = \dfrac{4}{1 + 0.4\cos\theta}$

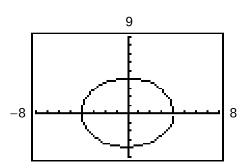
The ellipse is shifted to the left. The vertical directrix is to the right of the pole

$$r = \frac{4}{1 - 0.4\sin\theta}.$$

The ellipse has a horizontal directrix below the pole.

(c)





8. Ellipse; Matches (f)

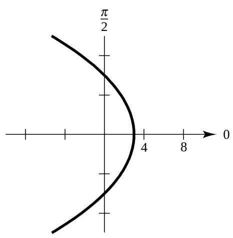
10. Parabola; Matches (e)

12. Hyperbola; Matches (d)

14. $r = \dfrac{6}{1 + \cos\theta}$

Parabola since $e = 1$

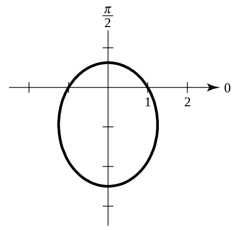
Vertex: $(3, 0)$



16. $r = \dfrac{5}{5 + 3\sin\theta} = \dfrac{1}{1 + (3/5)\sin\theta}$

Ellipse since $e = \dfrac{3}{5} < 1$

Vertices: $\left(\dfrac{5}{8}, \dfrac{\pi}{2}\right), \left(\dfrac{5}{2}, \dfrac{3\pi}{2}\right)$



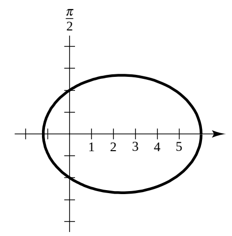
18. $r(3 - 2\cos\theta) = 6$

$r = \dfrac{6}{3 - 2\cos\theta}$

$= \dfrac{2}{1 - (2/3)\cos\theta}$

Ellipse since $e = \dfrac{2}{3} < 1$

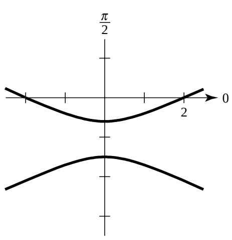
Vertices: $(6, 0), \left(\dfrac{6}{5}, \pi\right)$



20. $r = \dfrac{-6}{3 + 7\sin\theta} = \dfrac{-2}{1 + (7/3)\sin\theta}$

Hyperbola since $e = \dfrac{7}{3} > 1$.

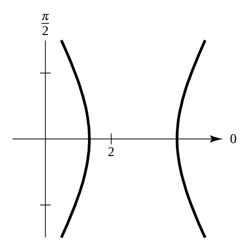
Vertices: $\left(-\dfrac{3}{5}, \dfrac{\pi}{2}\right), \left(\dfrac{3}{2}, \dfrac{3\pi}{2}\right)$



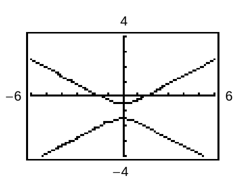
22. $r = \dfrac{4}{1 + 2\cos\theta}$

Hyperbola since $e = 2 > 1$

Vertices: $\left(\dfrac{4}{3}, 0\right), (-4, \pi)$

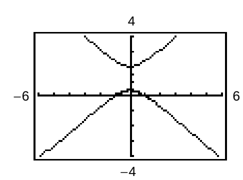


24.



Hyperbola

26.

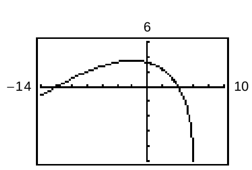


Hyperbola

28. $r = \dfrac{6}{1 + \cos\left(\theta - \dfrac{\pi}{3}\right)}$

Rotate the graph of $r = \dfrac{6}{1 + \cos\theta}$

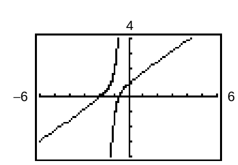
counterclockwise through the angle $\dfrac{\pi}{3}$.



30. $r = \dfrac{-6}{3 + 7\sin(\theta + (2\pi/3))}$

Rotate graph of $r = \dfrac{-6}{3 + 7\sin\theta}$.

Clockwise through angle of $2\pi/3$.



32. Change θ to $\theta - \frac{\pi}{6}$: $r = \dfrac{2}{1 + \sin\left(\theta - \frac{\pi}{6}\right)}$

34. Parabola

$e = 1, y = 1, d = 1$

$$r = \frac{ed}{1 + e\sin\theta} = \frac{1}{1 + \sin\theta}$$

36. Ellipse

$e = \frac{3}{4}, y = -2, d = 2$

$$r = \frac{ed}{1 - e\sin\theta} = \frac{2(3/4)}{1 - (3/4)\sin\theta} = \frac{6}{4 - 3\sin\theta}$$

38. Hyperbola

$e = \frac{3}{2}, x = -1, d = 1$

$$r = \frac{ed}{1 - e\cos\theta} = \frac{3/2}{1 - (3/2)\cos\theta} = \frac{3}{2 - 3\cos\theta}$$

40. Parabola

Vertex: $(5, \pi)$

$e = 1, d = 10$

$$r = \frac{ed}{1 - e\cos\theta} = \frac{10}{1 - \cos\theta}$$

42. Ellipse

Vertices: $\left(2, \frac{\pi}{2}\right), \left(4, \frac{3\pi}{2}\right)$

$e = \frac{1}{3}, d = 8$

$$r = \frac{ed}{1 + e\sin\theta} = \frac{8/3}{1 + (1/3)\sin\theta} = \frac{8}{3 + \sin\theta}$$

44. Hyperbola

Vertices: $(2, 0), (10, 0)$

$e = \frac{3}{2}, d = \frac{10}{3}$

$$r = \frac{ed}{1 + e\cos\theta} = \frac{5}{1 + (3/2)\cos\theta} = \frac{10}{2 + 3\cos\theta}$$

46. $r = \dfrac{4}{1 + \sin\theta}$ is a parabola with horizontal directrix above the pole.

(a) Parabola with vertical directrix to left pole.

(b) Parabola with horizontal directrix below pole.

(c) Parabola with vertical directrix to right of pole.

(d) Parabola (b) rotated counterclockwise $\pi/4$.

48. (a)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$x^2b^2 + y^2a^2 = a^2b^2$$

$$b^2r^2\cos^2\theta + a^2r^2\sin^2\theta = a^2b^2$$

$$r^2[b^2\cos^2\theta + a^2(1 - \cos^2\theta)] = a^2b^2$$

$$r^2[a^2 + \cos^2\theta(b^2 - a^2)] = a^2b^2$$

$$r^2 = \frac{a^2b^2}{a^2 + (b^2 - a^2)\cos^2\theta} = \frac{a^2b^2}{a^2 - c^2\cos^2\theta}$$

$$= \frac{b^2}{1 - (c/a)^2\cos^2\theta} = \frac{b^2}{1 - e^2\cos^2\theta}$$

(b)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$x^2b^2 - y^2a^2 = a^2b^2$$

$$b^2r^2\cos^2\theta - a^2r^2\sin^2\theta = a^2b^2$$

$$r^2[b^2\cos^2\theta - a^2(1 - \cos^2\theta)] = a^2b^2$$

$$r^2[-a^2 + \cos^2\theta(a^2 + b^2)] = a^2b^2$$

$$r^2 = \frac{a^2b^2}{-a^2 + c^2\cos^2\theta} = \frac{b^2}{-1 + (c^2/a^2)\cos^2\theta}$$

$$= \frac{-b^2}{1 - e^2\cos^2\theta}$$

50. $a = 4, c = 5, b = 3, e = \frac{5}{4}$

$$r^2 = \frac{-9}{1 - (25/16)\cos^2\theta}$$

52. $a = 2, b = 1, c = \sqrt{3}, e = \frac{\sqrt{3}}{2}$

$$r^2 = \frac{1}{1 - (3/4)\cos^2\theta}$$

54. $A = 2\left[\frac{1}{2}\int_{-\pi/2}^{\pi/2}\left(\frac{2}{3 - 2\sin\theta}\right)^2 d\theta\right] = 4\int_{-\pi/2}^{\pi/2}\frac{1}{(3 - 2\sin\theta)^2}\,d\theta \approx 3.37$

56. (a) $r = \frac{ed}{1 - e\cos\theta}$

When $\theta = 0, r = c + a = ea + a = a(1 + e)$.

Therefore,

$$a(1 + e) = \frac{ed}{1 - e}$$

$$a(1 + e)(1 - e) = ed$$

$$a(1 - e^2) = ed.$$

Thus, $r = \frac{(1 - e^2)a}{1 - e\cos\theta}$.

(b) The perihelion distance is $a - c = a - ea = a(1 - e)$.

When $\theta = \pi,\ r = \frac{(1 - e^2)a}{1 + e} = a(1 - e)$.

The aphelion distance is $a + c = a + ea = a(1 + e)$.

When $\theta = 0,\ r = \frac{(1 - e^2)a}{1 - e} = a(1 + e)$.

58. $a = 1.427 \times 10^9$ km

$e = 0.0543$

$$r = \frac{(1 - e^2)a}{1 - e\cos\theta} = \frac{1.422792505 \times 10^9}{1 - 0.0543\cos\theta}$$

Perihelion distance: $a(1 - e) = 1.3495139 \times 10^9$ km

Aphelion distance: $a(1 + e) = 1.5044861 \times 10^9$ km

60. $a = 36.0 \times 10^6$ mi, $e = 0.206$

$$r = \frac{(1 - e^2)a}{1 - e\cos\theta} \approx \frac{34.472 \times 10^6}{1 - 0.206\cos\theta}$$

Perihelion distance: $a(1 - e) = 28.582 \times 10^6$ mi

Aphelion distance: $a(1 + e) = 43.416 \times 10^6$ mi

62.

$$r = a\sin\theta + b\cos\theta$$

$$r^2 = ar\sin\theta + br\cos\theta$$

$$x^2 + y^2 = ay + bx$$

$x^2 + y^2 - bx - ay = 0$ represents a circle.

Review Exercises for Chapter 9

2. Matches (b) - hyperbola

4. Matches (c) - hyperbola

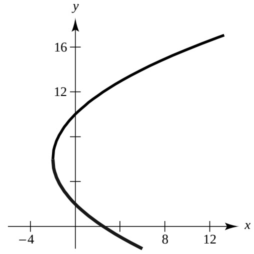
6. $y^2 - 12y - 8x + 20 = 0$

$$y^2 - 12y + 36 = 8x - 20 + 36$$

$$(y - 6)^2 = 4(2)(x + 2)$$

Parabola

Vertex: $(-2, 6)$



8. $4x^2 + y^2 - 16x + 15 = 0$

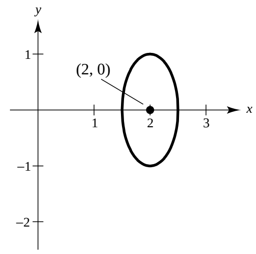
$4(x^2 - 4x + 4) + y^2 = -15 + 16$

$$\frac{(x-2)^2}{1/4} + \frac{y^2}{1} = 1$$

Ellipse

Center: $(2, 0)$

Vertices: $(2, \pm 1)$



10. $4x^2 - 4y^2 - 4x + 8y - 11 = 0$

$4\left(x^2 - x + \frac{1}{4}\right) - 4(y^2 - 2y + 1) = 11 + 1 - 4$

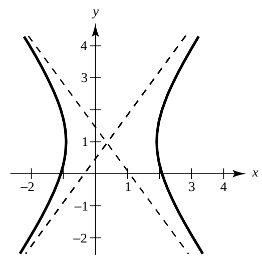
$$\frac{[x - (1/2)]^2}{2} - \frac{(y-1)^2}{2} = 1$$

Hyperbola

Center: $\left(\frac{1}{2}, 1\right)$

Vertices: $\left(\frac{1}{2} \pm \sqrt{2}, 1\right)$

Asymptotes: $y = 1 \pm \left(x - \frac{1}{2}\right)$



12. Vertex: $(4, 2)$

Focus: $(4, 0)$

Parabola opens downward

$p = -2$

$(x-4)^2 = 4(-2)(y-2)$

$x^2 - 8x + 8y = 0$

14. Center: $(0, 0)$

Solution points: $(1, 2), (2, 0)$

Substituting the values of the coordinates of the given points into

$$\left(\frac{x^2}{b^2}\right) + \left(\frac{y^2}{a^2}\right) = 1,$$

we obtain the system

$$\left(\frac{1}{b^2}\right) + \left(\frac{4}{a^2}\right) = 1,\ 4/b^2 = 1.$$

Solving the system, we have

$$a^2 = \frac{16}{3} \text{ and } b^2 = 4,\ \left(\frac{x^2}{4}\right) + \left(\frac{3y^2}{16}\right) = 1.$$

16. Foci: $(0, \pm 8)$

Asymptotes: $y = \pm 4x$

Center: $(0, 0)$

Vertical transverse axis

$c = 8$

$y = \frac{a}{b}x = 4x$ asymptote $\rightarrow a = 4b$

$b^2 = c^2 - a^2 = 64 - (4b)^2 \implies 17b^2 = 64$

$\implies b^2 = \frac{64}{17} \implies a^2 = \frac{1024}{17}$

$$\frac{y^2}{1024/17} - \frac{x^2}{64/17} = 1$$

18. $\frac{x^2}{4} + \frac{y^2}{25} = 1,\ a = 5,\ b = 2,\ c = \sqrt{21},\ e = \frac{\sqrt{21}}{5}$

By Example 5 of Section 9.1,

$$C = 20\int_0^{\pi/2} \sqrt{1 - \frac{21}{25}\sin^2\theta}\, d\theta \approx 23.01.$$

20. $y = \frac{1}{200}x^2$

(a) $x^2 = 200y$

$x^2 = 4(50)y$

Focus: $(0, 50)$

(b) $y = \frac{1}{200}x^2$

$y' = \frac{1}{100}x$

$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{10{,}000}}$

$S = 2\pi\int_0^{100} x\sqrt{1 + \frac{x^2}{10{,}000}}\,dx \approx 38{,}294.49$

22. (a) $A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx = \frac{4b}{a}\left(\frac{1}{2}\right)\left[x\sqrt{a^2 - x^2} + a^2\arcsin\left(\frac{x}{a}\right)\right]_0^a = \pi ab$

(b) **Disk:** $V = 2\pi\int_0^b \frac{a^2}{b^2}(b^2 - y^2)\,dy = \frac{2\pi a^2}{b^2}\int_0^b (b^2 - y^2)\,dy = \frac{2\pi a^2}{b^2}\left[b^2y - \frac{1}{3}y^3\right]_0^b = \frac{4}{3}\pi a^2 b$

$$S = 4\pi\int_0^b \frac{a}{b}\sqrt{b^2 - y^2}\left(\frac{\sqrt{b^4 + (a^2 - b^2)y^2}}{b\sqrt{b^2 - y^2}}\right)dy$$

$$= \frac{4\pi a}{b^2}\int_0^b \sqrt{b^4 + c^2y^2}\,dy = \frac{2\pi a}{b^2c}\left[cy\sqrt{b^4 + c^2y^2} + b^4\ln\left|cy + \sqrt{b^4 + c^2y^2}\right|\right]_0^b$$

$$= \frac{2\pi a}{b^2c}\left[b^2c\sqrt{b^2 + c^2} + b^4\ln\left|cb + b\sqrt{b^2 + c^2}\right| - b^4\ln(b^2)\right]$$

$$= 2\pi a^2 + \frac{\pi ab^2}{c}\ln\left(\frac{c + a}{e}\right)^2 = 2\pi a^2 + \left(\frac{\pi b^2}{e}\right)\ln\left(\frac{1 + e}{1 - e}\right)$$

(c) **Disk:** $V = 2\pi\int_0^a \frac{b^2}{a^2}(a^2 - x^2)\,dx = \frac{2\pi b^2}{a^2}\int_0^a (a^2 - x^2)\,dx = \frac{2\pi b^2}{a^2}\left[a^2x - \frac{1}{3}x^3\right]_0^a = \frac{4}{3}\pi ab^2$

$$S = 2(2\pi)\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\left(\frac{\sqrt{a^4 - (a^2 - b^2)x^2}}{a\sqrt{a^2 - x^2}}\right)dx$$

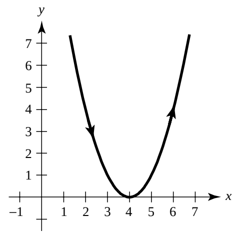
$$= \frac{4\pi b}{a^2}\int_0^a \sqrt{a^4 - c^2x^2}\,dx = \frac{2\pi b}{a^2c}\left[cx\sqrt{a^4 - c^2x^2} + a^4\arcsin\left(\frac{cx}{a^2}\right)\right]_0^a$$

$$= \frac{a\pi b}{a^2c}\left[a^2c\sqrt{a^2 - c^2} + a^4\arcsin\left(\frac{c}{a}\right)\right] = 2\pi b^2 + 2\pi\left(\frac{ab}{e}\right)\arcsin(e)$$

24. $x = t + 4, y = t^2$

$t = x - 4 \implies y = (x - 4)^2$

Parabola

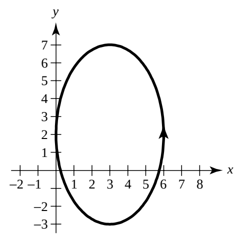


26. $x = 3 + 3\cos\theta, y = 2 + 5\sin\theta$

$$\left(\frac{x - 3}{3}\right)^2 + \left(\frac{y - 2}{5}\right)^2 = 1$$

$$\frac{(x - 3)^2}{9} + \frac{(y - 2)^2}{25} = 1$$

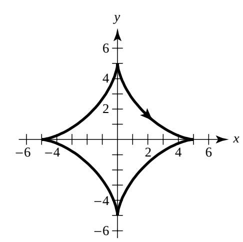
Ellipse



28. $x = 5\sin^3\theta, y = 5\cos^3\theta$

$$\left(\frac{x}{5}\right)^{2/3} + \left(\frac{y}{5}\right)^{2/3} = 1$$

$$x^{2/3} + y^{2/3} = 5^{2/3}$$



30. $(x - h)^2 + (y - k)^2 = r^2$

$(x - 5)^2 + (y - 3)^2 = 2^2 = 4$

32. $a = 4,\ c = 5,\ b^2 = c^2 - a^2 = 9,\ \dfrac{y^2}{16} - \dfrac{x^2}{9} = 1$

Let $\dfrac{y^2}{16} = \sec^2\theta$ and $\dfrac{x^2}{9} = \tan^2\theta$.

Then $x = 3\tan\theta$ and $y = 4\sec\theta$.

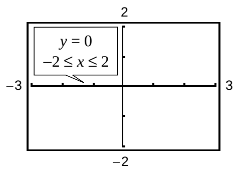
34. $x = (a - b)\cos t + b\cos\left(\dfrac{a-b}{b}t\right)$

$y = (a - b)\sin t - b\sin\left(\dfrac{a-b}{b}t\right)$

(a) $a = 2, b = 1$

$x = \cos t + \cos t = 2\cos t$

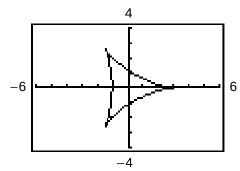
$y = \sin t - \sin t = 0$



(b) $a = 3, b = 1$

$x = 2\cos t + \cos 2t$

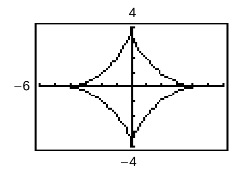
$y = 2\sin t - \sin 2t$



(c) $a = 4, b = 1$

$x = 3\cos t + \cos 3t$

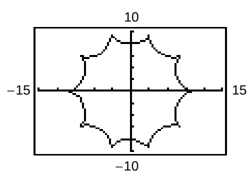
$y = 3\sin t - \sin 3t$



(d) $a = 10, b = 1$

$x = 9\cos t + \cos 9t$

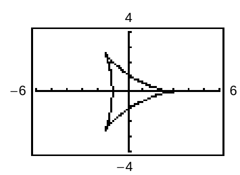
$y = 9\sin t - \sin 9t$



(e) $a = 3, b = 2$

$x = \cos t + 2\cos\dfrac{t}{2}$

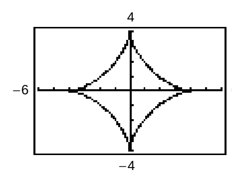
$y = \sin t - 2\sin\dfrac{t}{2}$



(f) $a = 4, b = 3$

$x = \cos t + 3\cos\dfrac{t}{3}$

$y = \sin t - 3\sin\dfrac{t}{3}$

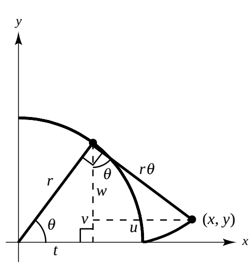


36. $x = t + u = r\cos\theta + r\theta\sin\theta$

$= r(\cos\theta + \theta\sin\theta)$

$y = v - w = r\sin\theta - r\theta\cos\theta$

$= r(\sin\theta - \theta\cos\theta)$



38. $x = t + 4$

$y = t^2$

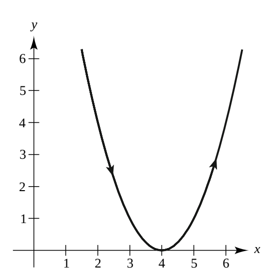
(a) $\dfrac{dy}{dx} = \dfrac{2t}{1} = 2t = 0$ when $t = 0$.

Point of horizontal tangency: $(4, 0)$

(b) $t = x - 4$

$y = (x - 4)^2$

(c)



40. $x = \frac{1}{t}$

$y = t^2$

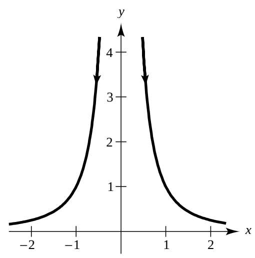
(a) $\frac{dy}{dx} = \frac{2t}{-1/t^2} = -2t^3$

No horizontal tangents $(t \neq 0)$

(b) $t = \frac{1}{x}$

$y = \frac{1}{x^2}$

(c)



42. $x = 2t - 1$

$y = \frac{1}{t^2 - 2t}$

(a) $\frac{dy}{dx} = \frac{-(t^2 - 2t)^{-2}(2t - 2)}{2}$

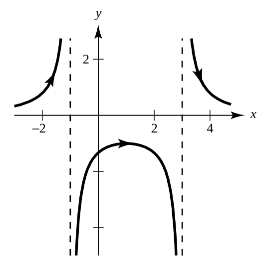
$= \frac{1 - t}{t^2(t - 2)^2} = 0$ when $t = 1$.

Point of horizontal tangency: $(1, -1)$

(b) $t = \frac{x + 1}{2}$

$y = \frac{1}{[(x + 1)/2]^2 - 2[(x + 1)/2]} = \frac{4}{(x - 3)(x + 1)}$

(c)



44. $x = 6 \cos \theta$

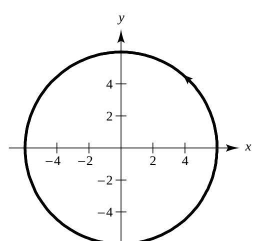
$y = 6 \sin \theta$

(a) $\frac{dy}{dx} = \frac{6 \cos \theta}{-6 \sin \theta} = -\cot \theta = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

Points of horizontal tangency: $(0, 6), (0, -6)$

(b) $\left(\frac{x}{6}\right)^2 + \left(\frac{y}{6}\right)^2 = 1$

(c)



46. $x = e^t$

$y = e^{-t}$

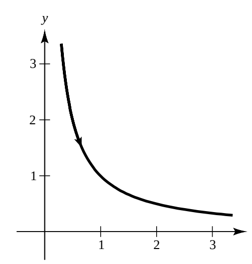
(a) $\frac{dy}{dx} = \frac{-e^{-t}}{e^t} = -\frac{1}{e^{2t}} = -\frac{1}{x^2}$

No horizontal tangents

(b) $t = \ln x$

$y = e^{-\ln x} = e^{\ln(1/x)} = \frac{1}{x}, x > 0$

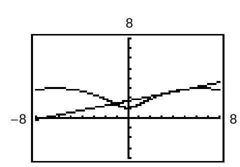
(c)



48. $x = 2\theta - \sin \theta$

$y = 2 - \cos \theta$

(a), (c)



(b) At $\theta = \frac{\pi}{6}, \frac{dx}{d\theta} \approx 1.134, \left(2 - \frac{\sqrt{3}}{2}\right),$

$\frac{dy}{dt} = 0.5,$ and $\frac{dy}{dx} \approx 0.441$

50. $x = 6 \cos \theta$

$y = 6 \sin \theta$

$\frac{dx}{d\theta} = -6 \sin \theta$

$\frac{dy}{d\theta} = 6 \cos \theta$

$$s = \int_0^{\pi} \sqrt{36 \sin^2 \theta + 36 \cos^2 \theta}\, d\theta = \Big[6\theta\Big]_0^{\pi} = 6\pi$$

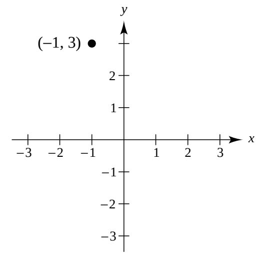
(one-half circumference of circle)

52. $(x, y) = (-1, 3)$

$$r = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$$

$$\theta = \arctan(-3) \approx 1.89\ (108.43^\circ)$$

$$(r, \theta) = \left(\sqrt{10}, 1.89\right), \left(-\sqrt{10}, 5.03\right)$$



54.
$$r = 10$$
$$r^2 = 100$$
$$x^2 + y^2 = 100$$

56.
$$r = \frac{1}{2 - \cos\theta}$$
$$2r - r\cos\theta = 1$$
$$2\left(\pm\sqrt{x^2 + y^2}\right) - x = 1$$
$$4(x^2 + y^2) = (x + 1)^2$$
$$3x^2 + 4y^2 - 2x - 1 = 0$$

58.
$$r = 4\sec\left(\theta - \frac{\pi}{3}\right) = \frac{4}{\cos[\theta - (\pi/3)]}$$
$$= \frac{4}{(1/2)\cos\theta + \left(\sqrt{3}/2\right)\sin\theta}$$
$$r\left(\cos\theta + \sqrt{3}\sin\theta\right) = 8$$
$$x + \sqrt{3}\,y = 8$$

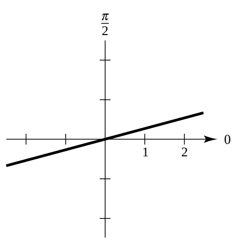
60.
$$\theta = \frac{3\pi}{4}$$
$$\tan\theta = -1$$
$$\frac{y}{x} = -1$$
$$y = -x$$

62.
$$x^2 + y^2 - 4x = 0$$
$$r^2 - 4r\cos\theta = 0$$
$$r = 4\cos\theta$$

64.
$$(x^2 + y^2)\left(\arctan\frac{y}{x}\right)^2 = a^2$$
$$r^2\theta^2 = a^2$$

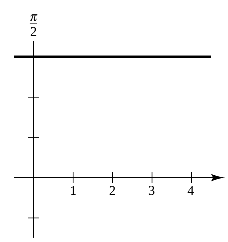
66. $\theta = \dfrac{\pi}{12}$

Line



68. $r = 3\csc\theta$, $r\sin\theta = 3$, $y = 3$

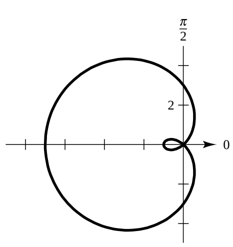
Horizontal line



70. $r = 3 - 4\cos\theta$

Limaçon

Symmetric to polar axis

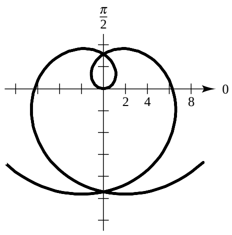


θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	-1	1	3	5	7

72. $r = 2\theta$

Spiral

Symmetric to $\theta = \pi/2$



θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
r	0	$\frac{\pi}{5}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π

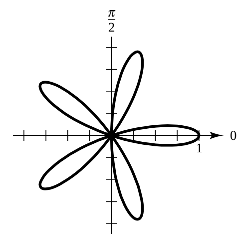
74. $r = \cos(5\theta)$

Rose curve with five petals

Symmetric to polar axis

Relative extrema: $(1, 0), \left(-1, \frac{\pi}{5}\right), \left(1, \frac{2\pi}{5}\right), \left(-1, \frac{3\pi}{5}\right), \left(1, \frac{4\pi}{5}\right)$

Tangents at the pole: $\theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{9\pi}{10}$



76. $r^2 = \cos(2\theta)$

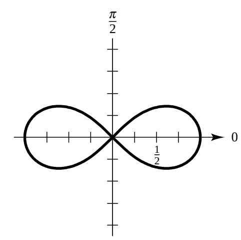
Lemniscate

Symmetric to the polar axis

Relative extrema: $(\pm 1, 0)$

Tangents at the pole: $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

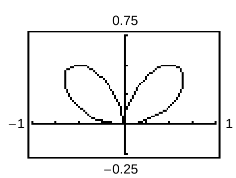
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$
r	± 1	$\pm\frac{\sqrt{2}}{2}$	0



78. $r = 2\sin\theta\cos^2\theta$

Bifolium

Symmetric to $\theta = \pi/2$



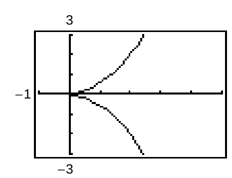
80. $r = 4(\sec\theta - \cos\theta)$

Semicubical parabola

Symmetric to the polar axis

$r \implies \infty$ as $\theta \implies \frac{\pi^-}{2}$

$r \implies \infty$ as $\theta \implies \frac{-\pi^+}{2}$



82. $r^2 = 4\sin(2\theta)$

(a) $2r\left(\frac{dr}{d\theta}\right) = 8\cos(2\theta)$

$$\frac{dr}{d\theta} = \frac{4\cos(2\theta)}{r}$$

Tangents at the pole: $\theta = 0, \frac{\pi}{2}$

(b) $$\frac{dy}{dx} = \frac{r\cos\theta + [(4\cos 2\theta\sin\theta)/r]}{-r\sin\theta + [(4\cos 2\theta\cos\theta)/r]}$$

$$= \frac{\cos(2\theta)\sin\theta + \sin(2\theta)\cos\theta}{\cos(2\theta)\cos\theta - \sin(2\theta)\sin\theta}$$

Horizontal tangents:

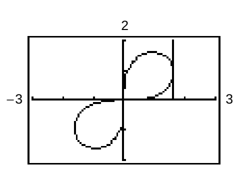
$\frac{dy}{dx} = 0$ when $\cos(2\theta)\sin\theta + \sin(2\theta)\cos\theta = 0$,

$\tan\theta = -\tan(2\theta)$, $\theta = 0, \frac{\pi}{3}, (0, 0), \left(\pm\sqrt{2\sqrt{3}}, \frac{\pi}{3}\right)$

Vertical tangents when $\cos 2\theta\cos\theta - \sin 2\theta\sin\theta = 0$:

$\tan 2\theta\tan\theta = 1$, $\theta = 0, \frac{\pi}{6}, (0, 0), \left(\pm\sqrt{2\sqrt{3}}, \frac{\pi}{6}\right)$

(c)



84. False. There are an infinite number of polar coordinate representations of a point. For example, the point $(x, y) = (1, 0)$ has polar representations $(r, \theta) = (1, 0), (1, 2\pi), (-1, \pi)$, etc.

86. $r = a\sin\theta, r = a\cos\theta$

The points of intersection are $\left(a/\sqrt{2}, \pi/4\right)$ and $(0, 0)$. For $r = a\sin\theta$,

$$m_1 = \frac{dy}{dx} = \frac{a\cos\theta\sin\theta + a\sin\theta\cos\theta}{a\cos^2\theta - a\sin^2\theta} = \frac{2\sin\theta\cos\theta}{\cos 2\theta}.$$

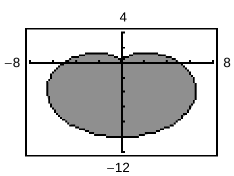
At $\left(a/\sqrt{2}, \pi/4\right)$, m_1 is undefined and at $(0, 0)$, $m_1 = 0$. For $r = a\cos\theta$,

$$m_2 = \frac{dy}{dx} = \frac{-a\sin^2\theta + a\cos^2\theta}{-a\sin\theta\cos\theta - a\cos\theta\sin\theta} = \frac{\cos 2\theta}{-2\sin\theta\cos\theta}.$$

At $\left(a/\sqrt{2}, \pi/4\right)$, $m_2 = 0$ and at $(0, 0)$, m_2 is undefined. Therefore, the graphs are orthogonal at $\left(a/\sqrt{2}, \pi/4\right)$ and $(0, 0)$.

88. $r = 5(1 - \sin\theta)$

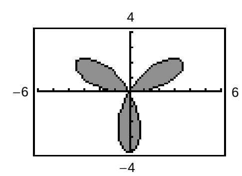
$$A = 2\left[\frac{1}{2}\int_{\pi/2}^{3\pi/2}[5(1 - \sin\theta)]^2\right]d\theta \approx 117.81 \quad \left(75\,\frac{\pi}{2}\right)$$



90. $r = 4\sin 3\theta$

$$A = 3\left[\frac{1}{2}\int_0^{\pi/3}(4\sin 3\theta)^2\,d\theta\right.$$

$$\approx 12.57 \quad (4\pi)$$

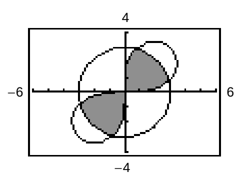


92. $r = 3, r^2 = 18\sin 2\theta$

$9 = r^2 = 18\sin 2\theta$

$\sin 2\theta = \dfrac{1}{2}$

$\theta = \dfrac{\pi}{12}$

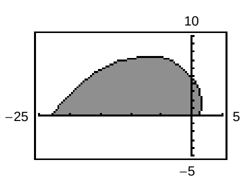


$$A = 2\left[\frac{1}{2}\int_0^{\pi/12} 18\sin 2\theta\,d\theta + \frac{1}{2}\int_{\pi/12}^{5\pi/12} 9\,d\theta + \frac{1}{2}\int_{5\pi/12}^{\pi/2} 18\sin 2\theta\,d\theta\right]$$

$$\approx 1.2058 + 9.4248 + 1.2058 \approx 11.84$$

94. $r = e^\theta, 0 \le \theta \le \pi$

$$A = \frac{1}{2}\int_0^{\pi}(e^\theta)^2\,d\theta \approx 133.62$$



96. $r = a\cos 2\theta, \dfrac{dr}{d\theta} = -2a\sin 2\theta$

$$s = 8\int_0^{\pi/4}\sqrt{a^2\cos^2 2\theta + 4a^2\sin^2 2\theta}\,d\theta$$

$$= 8a\int_0^{\pi/4}\sqrt{1 + 3\sin^2 2\theta}\,d\theta \quad \text{(Simpson's Rule: } n = 4)$$

$$\approx \frac{\pi a}{6}[1 + 4(1.1997) + 2(1.5811) + 4(1.8870) + 2]$$

$$\approx 9.69a$$