50. $a = 4, c = 5, b = 3, e = \dfrac{5}{4}$

$$r^2 = \frac{-9}{1 - (25/16)\cos^2\theta}$$

52. $a = 2, b = 1, c = \sqrt{3}, e = \dfrac{\sqrt{3}}{2}$

$$r^2 = \frac{1}{1 - (3/4)\cos^2\theta}$$

54. $A = 2\left[\dfrac{1}{2}\displaystyle\int_{-\pi/2}^{\pi/2}\left(\frac{2}{3 - 2\sin\theta}\right)^2 d\theta\right] = 4\displaystyle\int_{-\pi/2}^{\pi/2}\frac{1}{(3 - 2\sin\theta)^2}\,d\theta \approx 3.37$

56. (a) $r = \dfrac{ed}{1 - e\cos\theta}$

When $\theta = 0, r = c + a = ea + a = a(1 + e)$.

Therefore,

$$a(1 + e) = \frac{ed}{1 - e}$$

$$a(1 + e)(1 - e) = ed$$

$$a(1 - e^2) = ed.$$

Thus, $r = \dfrac{(1 - e^2)a}{1 - e\cos\theta}$.

(b) The perihelion distance is $a - c = a - ea = a(1 - e)$.

When $\theta = \pi,\ r = \dfrac{(1 - e^2)a}{1 + e} = a(1 - e)$.

The aphelion distance is $a + c = a + ea = a(1 + e)$.

When $\theta = 0,\ r = \dfrac{(1 - e^2)a}{1 - e} = a(1 + e)$.

58. $a = 1.427 \times 10^9$ km

$e = 0.0543$

$$r = \frac{(1 - e^2)a}{1 - e\cos\theta} = \frac{1.422792505 \times 10^9}{1 - 0.0543\cos\theta}$$

Perihelion distance: $a(1 - e) = 1.3495139 \times 10^9$ km

Aphelion distance: $a(1 + e) = 1.5044861 \times 10^9$ km

60. $a = 36.0 \times 10^6$ mi, $e = 0.206$

$$r = \frac{(1 - e^2)a}{1 - e\cos\theta} \approx \frac{34.472 \times 10^6}{1 - 0.206\cos\theta}$$

Perihelion distance: $a(1 - e) = 28.582 \times 10^6$ mi

Aphelion distance: $a(1 + e) = 43.416 \times 10^6$ mi

62.

$$r = a\sin\theta + b\cos\theta$$

$$r^2 = ar\sin\theta + br\cos\theta$$

$$x^2 + y^2 = ay + bx$$

$x^2 + y^2 - bx - ay = 0$ represents a circle.

Review Exercises for Chapter 9

2. Matches (b) - hyperbola

4. Matches (c) - hyperbola

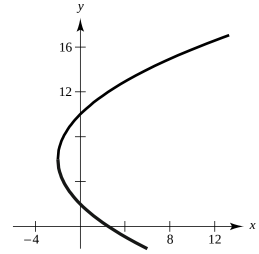
6. $y^2 - 12y - 8x + 20 = 0$

$$y^2 - 12y + 36 = 8x - 20 + 36$$

$$(y - 6)^2 = 4(2)(x + 2)$$

Parabola

Vertex: $(-2, 6)$



8. $4x^2 + y^2 - 16x + 15 = 0$

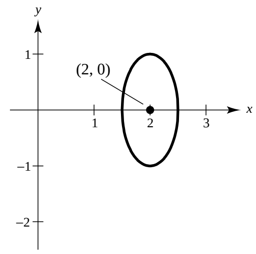
$4(x^2 - 4x + 4) + y^2 = -15 + 16$

$$\frac{(x-2)^2}{1/4} + \frac{y^2}{1} = 1$$

Ellipse

Center: $(2, 0)$

Vertices: $(2, \pm 1)$



10. $4x^2 - 4y^2 - 4x + 8y - 11 = 0$

$$4\left(x^2 - x + \frac{1}{4}\right) - 4(y^2 - 2y + 1) = 11 + 1 - 4$$

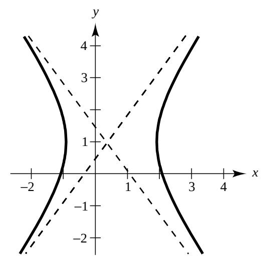
$$\frac{[x - (1/2)]^2}{2} - \frac{(y-1)^2}{2} = 1$$

Hyperbola

Center: $\left(\frac{1}{2}, 1\right)$

Vertices: $\left(\frac{1}{2} \pm \sqrt{2}, 1\right)$

Asymptotes: $y = 1 \pm \left(x - \frac{1}{2}\right)$



12. Vertex: $(4, 2)$

Focus: $(4, 0)$

Parabola opens downward

$p = -2$

$(x - 4)^2 = 4(-2)(y - 2)$

$x^2 - 8x + 8y = 0$

14. Center: $(0, 0)$

Solution points: $(1, 2), (2, 0)$

Substituting the values of the coordinates of the given points into

$$\left(\frac{x^2}{b^2}\right) + \left(\frac{y^2}{a^2}\right) = 1,$$

we obtain the system

$$\left(\frac{1}{b^2}\right) + \left(\frac{4}{a^2}\right) = 1,\ 4/b^2 = 1.$$

Solving the system, we have

$$a^2 = \frac{16}{3} \text{ and } b^2 = 4,\ \left(\frac{x^2}{4}\right) + \left(\frac{3y^2}{16}\right) = 1.$$

16. Foci: $(0, \pm 8)$

Asymptotes: $y = \pm 4x$

Center: $(0, 0)$

Vertical transverse axis

$c = 8$

$y = \frac{a}{b}x = 4x$ asymptote $\rightarrow a = 4b$

$b^2 = c^2 - a^2 = 64 - (4b)^2 \implies 17b^2 = 64$

$\implies b^2 = \frac{64}{17} \implies a^2 = \frac{1024}{17}$

$$\frac{y^2}{1024/17} - \frac{x^2}{64/17} = 1$$

18. $\frac{x^2}{4} + \frac{y^2}{25} = 1,\ a = 5,\ b = 2,\ c = \sqrt{21},\ e = \frac{\sqrt{21}}{5}$

By Example 5 of Section 9.1,

$$C = 20\int_0^{\pi/2} \sqrt{1 - \frac{21}{25}\sin^2\theta}\, d\theta \approx 23.01.$$

20. $y = \frac{1}{200}x^2$

(a) $x^2 = 200y$

$x^2 = 4(50)y$

Focus: $(0, 50)$

(b) $y = \frac{1}{200}x^2$

$y' = \frac{1}{100}x$

$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{10{,}000}}$

$S = 2\pi\int_0^{100} x\sqrt{1 + \frac{x^2}{10{,}000}}\,dx \approx 38{,}294.49$

22. (a) $A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx = \frac{4b}{a}\left(\frac{1}{2}\right)\left[x\sqrt{a^2 - x^2} + a^2\arcsin\left(\frac{x}{a}\right)\right]_0^a = \pi ab$

(b) **Disk:** $V = 2\pi\int_0^b \frac{a^2}{b^2}(b^2 - y^2)\,dy = \frac{2\pi a^2}{b^2}\int_0^b (b^2 - y^2)\,dy = \frac{2\pi a^2}{b^2}\left[b^2y - \frac{1}{3}y^3\right]_0^b = \frac{4}{3}\pi a^2 b$

$$S = 4\pi\int_0^b \frac{a}{b}\sqrt{b^2 - y^2}\left(\frac{\sqrt{b^4 + (a^2 - b^2)y^2}}{b\sqrt{b^2 - y^2}}\right)dy$$

$$= \frac{4\pi a}{b^2}\int_0^b \sqrt{b^4 + c^2y^2}\,dy = \frac{2\pi a}{b^2c}\left[cy\sqrt{b^4 + c^2y^2} + b^4\ln\left|cy + \sqrt{b^4 + c^2y^2}\right|\right]_0^b$$

$$= \frac{2\pi a}{b^2c}\left[b^2c\sqrt{b^2 + c^2} + b^4\ln\left|cb + b\sqrt{b^2 + c^2}\right| - b^4\ln(b^2)\right]$$

$$= 2\pi a^2 + \frac{\pi ab^2}{c}\ln\left(\frac{c + a}{e}\right)^2 = 2\pi a^2 + \left(\frac{\pi b^2}{e}\right)\ln\left(\frac{1 + e}{1 - e}\right)$$

(c) **Disk:** $V = 2\pi\int_0^a \frac{b^2}{a^2}(a^2 - x^2)\,dx = \frac{2\pi b^2}{a^2}\int_0^a (a^2 - x^2)\,dx = \frac{2\pi b^2}{a^2}\left[a^2x - \frac{1}{3}x^3\right]_0^a = \frac{4}{3}\pi ab^2$

$$S = 2(2\pi)\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\left(\frac{\sqrt{a^4 - (a^2 - b^2)x^2}}{a\sqrt{a^2 - x^2}}\right)dx$$

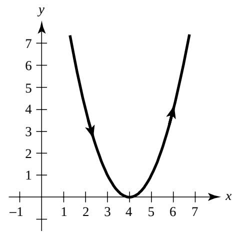
$$= \frac{4\pi b}{a^2}\int_0^a \sqrt{a^4 - c^2x^2}\,dx = \frac{2\pi b}{a^2c}\left[cx\sqrt{a^4 - c^2x^2} + a^4\arcsin\left(\frac{cx}{a^2}\right)\right]_0^a$$

$$= \frac{a\pi b}{a^2c}\left[a^2c\sqrt{a^2 - c^2} + a^4\arcsin\left(\frac{c}{a}\right)\right] = 2\pi b^2 + 2\pi\left(\frac{ab}{e}\right)\arcsin(e)$$

24. $x = t + 4, y = t^2$

$t = x - 4 \implies y = (x - 4)^2$

Parabola

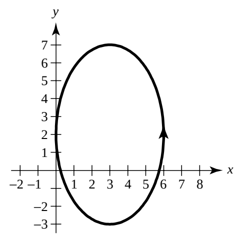


26. $x = 3 + 3\cos\theta, y = 2 + 5\sin\theta$

$\left(\frac{x - 3}{3}\right)^2 + \left(\frac{y - 2}{5}\right)^2 = 1$

$\frac{(x - 3)^2}{9} + \frac{(y - 2)^2}{25} = 1$

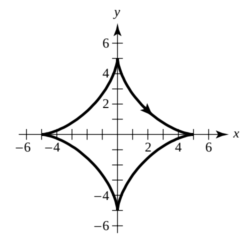
Ellipse



28. $x = 5\sin^3\theta, y = 5\cos^3\theta$

$\left(\frac{x}{5}\right)^{2/3} + \left(\frac{y}{5}\right)^{2/3} = 1$

$x^{2/3} + y^{2/3} = 5^{2/3}$



30. $(x - h)^2 + (y - k)^2 = r^2$

$(x - 5)^2 + (y - 3)^2 = 2^2 = 4$

32. $a = 4,\ c = 5,\ b^2 = c^2 - a^2 = 9,\ \dfrac{y^2}{16} - \dfrac{x^2}{9} = 1$

Let $\dfrac{y^2}{16} = \sec^2\theta$ and $\dfrac{x^2}{9} = \tan^2\theta$.

Then $x = 3\tan\theta$ and $y = 4\sec\theta$.

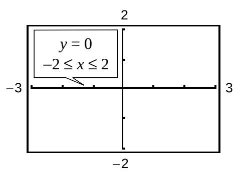
34. $x = (a - b)\cos t + b\cos\left(\dfrac{a-b}{b}t\right)$

$y = (a - b)\sin t - b\sin\left(\dfrac{a-b}{b}t\right)$

(a) $a = 2, b = 1$

$x = \cos t + \cos t = 2\cos t$

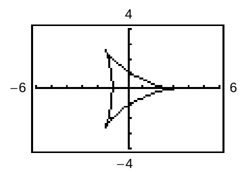
$y = \sin t - \sin t = 0$



(b) $a = 3, b = 1$

$x = 2\cos t + \cos 2t$

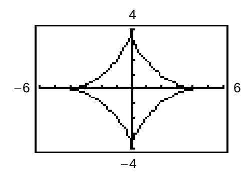
$y = 2\sin t - \sin 2t$



(c) $a = 4, b = 1$

$x = 3\cos t + \cos 3t$

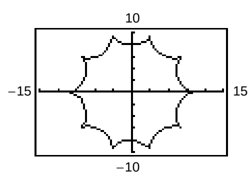
$y = 3\sin t - \sin 3t$



(d) $a = 10, b = 1$

$x = 9\cos t + \cos 9t$

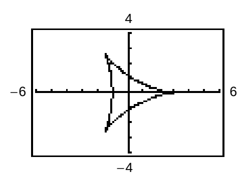
$y = 9\sin t - \sin 9t$



(e) $a = 3, b = 2$

$x = \cos t + 2\cos\dfrac{t}{2}$

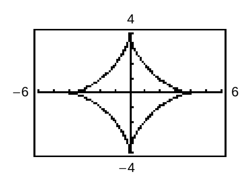
$y = \sin t - 2\sin\dfrac{t}{2}$



(f) $a = 4, b = 3$

$x = \cos t + 3\cos\dfrac{t}{3}$

$y = \sin t - 3\sin\dfrac{t}{3}$

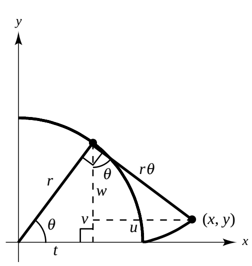


36. $x = t + u = r\cos\theta + r\theta\sin\theta$

$= r(\cos\theta + \theta\sin\theta)$

$y = v - w = r\sin\theta - r\theta\cos\theta$

$= r(\sin\theta - \theta\cos\theta)$



38. $x = t + 4$

$y = t^2$

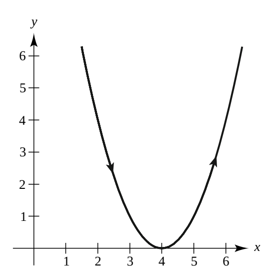
(a) $\dfrac{dy}{dx} = \dfrac{2t}{1} = 2t = 0$ when $t = 0$.

Point of horizontal tangency: $(4, 0)$

(b) $t = x - 4$

$y = (x - 4)^2$

(c)



40. $x = \dfrac{1}{t}$

$y = t^2$

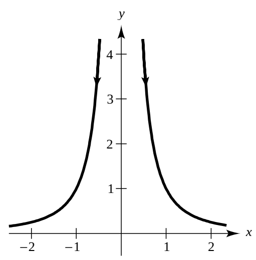
(a) $\dfrac{dy}{dx} = \dfrac{2t}{-1/t^2} = -2t^3$

No horizontal tangents $\quad (t \neq 0)$

(b) $t = \dfrac{1}{x}$

$y = \dfrac{1}{x^2}$

(c)



42. $x = 2t - 1$

$y = \dfrac{1}{t^2 - 2t}$

(a) $\dfrac{dy}{dx} = \dfrac{-(t^2 - 2t)^{-2}(2t - 2)}{2}$

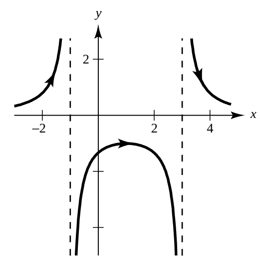
$= \dfrac{1 - t}{t^2(t - 2)^2} = 0$ when $t = 1$.

Point of horizontal tangency: $(1, -1)$

(b) $t = \dfrac{x + 1}{2}$

$y = \dfrac{1}{[(x + 1)/2]^2 - 2[(x + 1)/2]} = \dfrac{4}{(x - 3)(x + 1)}$

(c)



44. $x = 6 \cos \theta$

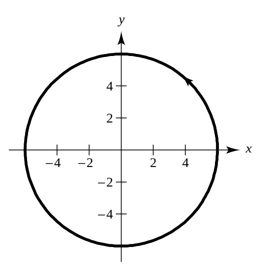
$y = 6 \sin \theta$

(a) $\dfrac{dy}{dx} = \dfrac{6 \cos \theta}{-6 \sin \theta} = -\cot \theta = 0$ when $\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

Points of horizontal tangency: $(0, 6), (0, -6)$

(b) $\left(\dfrac{x}{6}\right)^2 + \left(\dfrac{y}{6}\right)^2 = 1$

(c)



46. $x = e^t$

$y = e^{-t}$

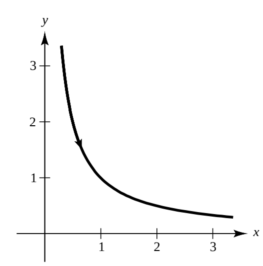
(a) $\dfrac{dy}{dx} = \dfrac{-e^{-t}}{e^t} = -\dfrac{1}{e^{2t}} = -\dfrac{1}{x^2}$

No horizontal tangents

(b) $t = \ln x$

$y = e^{-\ln x} = e^{\ln(1/x)} = \dfrac{1}{x}, x > 0$

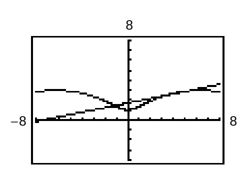
(c)



48. $x = 2\theta - \sin \theta$

$y = 2 - \cos \theta$

(a), (c)



(b) At $\theta = \dfrac{\pi}{6}, \dfrac{dx}{d\theta} \approx 1.134, \left(2 - \dfrac{\sqrt{3}}{2}\right),$

$\dfrac{dy}{dt} = 0.5,$ and $\dfrac{dy}{dx} \approx 0.441$

50. $x = 6 \cos \theta$

$y = 6 \sin \theta$

$\dfrac{dx}{d\theta} = -6 \sin \theta$

$\dfrac{dy}{d\theta} = 6 \cos \theta$

$$s = \int_0^{\pi} \sqrt{36 \sin^2 \theta + 36 \cos^2 \theta}\, d\theta = \Big[6\theta\Big]_0^{\pi} = 6\pi$$

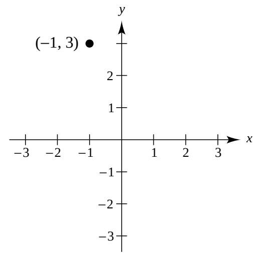
(one-half circumference of circle)

52. $(x, y) = (-1, 3)$

$$r = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$$

$$\theta = \arctan(-3) \approx 1.89\ (108.43°)$$

$$(r, \theta) = \left(\sqrt{10}, 1.89\right), \left(-\sqrt{10}, 5.03\right)$$



54.

$$r = 10$$
$$r^2 = 100$$
$$x^2 + y^2 = 100$$

56.

$$r = \frac{1}{2 - \cos\theta}$$
$$2r - r\cos\theta = 1$$
$$2\left(\pm\sqrt{x^2 + y^2}\right) - x = 1$$
$$4(x^2 + y^2) = (x + 1)^2$$
$$3x^2 + 4y^2 - 2x - 1 = 0$$

58.

$$r = 4\sec\left(\theta - \frac{\pi}{3}\right) = \frac{4}{\cos[\theta - (\pi/3)]}$$
$$= \frac{4}{(1/2)\cos\theta + \left(\sqrt{3}/2\right)\sin\theta}$$
$$r\left(\cos\theta + \sqrt{3}\sin\theta\right) = 8$$
$$x + \sqrt{3}\,y = 8$$

60.

$$\theta = \frac{3\pi}{4}$$
$$\tan\theta = -1$$
$$\frac{y}{x} = -1$$
$$y = -x$$

62.

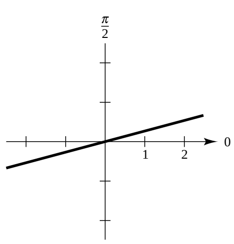
$$x^2 + y^2 - 4x = 0$$
$$r^2 - 4r\cos\theta = 0$$
$$r = 4\cos\theta$$

64.

$$(x^2 + y^2)\left(\arctan\frac{y}{x}\right)^2 = a^2$$
$$r^2\theta^2 = a^2$$

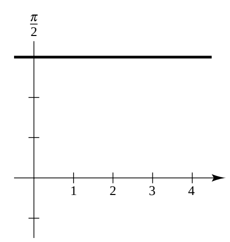
66. $\theta = \dfrac{\pi}{12}$

Line



68. $r = 3\csc\theta$, $r\sin\theta = 3$, $y = 3$

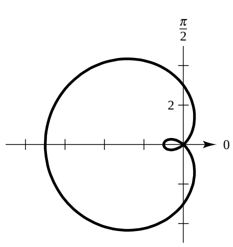
Horizontal line



70. $r = 3 - 4\cos\theta$

Limaçon

Symmetric to polar axis

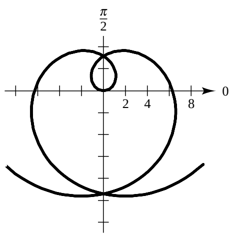


θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	-1	1	3	5	7

72. $r = 2\theta$

Spiral

Symmetric to $\theta = \pi/2$



θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
r	0	$\frac{\pi}{5}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π

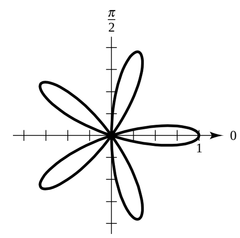
74. $r = \cos(5\theta)$

Rose curve with five petals

Symmetric to polar axis

Relative extrema: $(1, 0), \left(-1, \frac{\pi}{5}\right), \left(1, \frac{2\pi}{5}\right), \left(-1, \frac{3\pi}{5}\right), \left(1, \frac{4\pi}{5}\right)$

Tangents at the pole: $\theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{9\pi}{10}$



76. $r^2 = \cos(2\theta)$

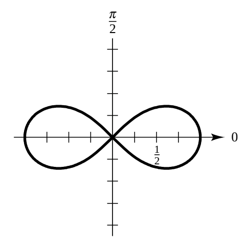
Lemniscate

Symmetric to the polar axis

Relative extrema: $(\pm 1, 0)$

Tangents at the pole: $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

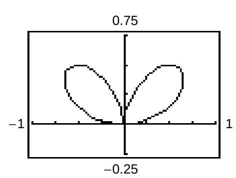
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$
r	± 1	$\pm\frac{\sqrt{2}}{2}$	0



78. $r = 2 \sin\theta \cos^2\theta$

Bifolium

Symmetric to $\theta = \pi/2$



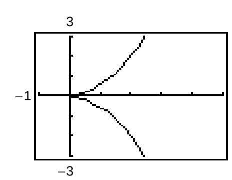
80. $r = 4(\sec\theta - \cos\theta)$

Semicubical parabola

Symmetric to the polar axis

$r \Rightarrow \infty$ as $\theta \Rightarrow \frac{\pi^-}{2}$

$r \Rightarrow \infty$ as $\theta \Rightarrow \frac{-\pi^+}{2}$



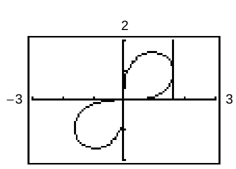
82. $r^2 = 4\sin(2\theta)$

(a) $2r\left(\frac{dr}{d\theta}\right) = 8\cos(2\theta)$

$$\frac{dr}{d\theta} = \frac{4\cos(2\theta)}{r}$$

Tangents at the pole: $\theta = 0, \frac{\pi}{2}$

(c)



(b) $$\frac{dy}{dx} = \frac{r\cos\theta + [(4\cos 2\theta \sin\theta)/r]}{-r\sin\theta + [(4\cos 2\theta\cos\theta)/r]}$$

$$= \frac{\cos(2\theta)\sin\theta + \sin(2\theta)\cos\theta}{\cos(2\theta)\cos\theta - \sin(2\theta)\sin\theta}$$

Horizontal tangents:

$\frac{dy}{dx} = 0$ when $\cos(2\theta)\sin\theta + \sin(2\theta)\cos\theta = 0$,

$\tan\theta = -\tan(2\theta),\ \theta = 0, \frac{\pi}{3}, (0, 0), \left(\pm\sqrt{2\sqrt{3}}, \frac{\pi}{3}\right)$

Vertical tangents when $\cos 2\theta\cos\theta - \sin 2\theta \sin\theta = 0$:

$\tan 2\theta \tan\theta = 1,\ \theta = 0, \frac{\pi}{6}, (0, 0), \left(\pm\sqrt{2\sqrt{3}}, \frac{\pi}{6}\right)$

84. False. There are an infinite number of polar coordinate representations of a point. For example, the point $(x, y) = (1, 0)$ has polar representations $(r, \theta) = (1, 0), (1, 2\pi), (-1, \pi)$, etc.

86. $r = a\sin\theta, r = a\cos\theta$

The points of intersection are $(a/\sqrt{2}, \pi/4)$ and $(0, 0)$. For $r = a\sin\theta$,

$$m_1 = \frac{dy}{dx} = \frac{a\cos\theta\sin\theta + a\sin\theta\cos\theta}{a\cos^2\theta - a\sin^2\theta} = \frac{2\sin\theta\cos\theta}{\cos 2\theta}.$$

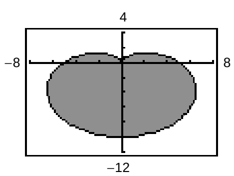
At $(a/\sqrt{2}, \pi/4)$, m_1 is undefined and at $(0, 0)$, $m_1 = 0$. For $r = a\cos\theta$,

$$m_2 = \frac{dy}{dx} = \frac{-a\sin^2\theta + a\cos^2\theta}{-a\sin\theta\cos\theta - a\cos\theta\sin\theta} = \frac{\cos 2\theta}{-2\sin\theta\cos\theta}.$$

At $(a/\sqrt{2}, \pi/4)$, $m_2 = 0$ and at $(0, 0)$, m_2 is undefined. Therefore, the graphs are orthogonal at $(a/\sqrt{2}, \pi/4)$ and $(0, 0)$.

88. $r = 5(1 - \sin\theta)$

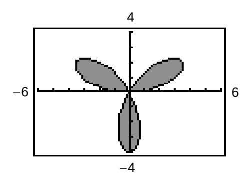
$$A = 2\left[\frac{1}{2}\int_{\pi/2}^{3\pi/2} [5(1 - \sin\theta)]^2\right] d\theta \approx 117.81 \quad \left(75\frac{\pi}{2}\right)$$



90. $r = 4\sin 3\theta$

$$A = 3\left[\frac{1}{2}\int_0^{\pi/3} (4\sin 3\theta)^2\, d\theta\right.$$

$$\approx 12.57 \quad (4\pi)$$

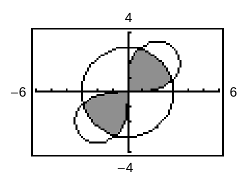


92. $r = 3, r^2 = 18\sin 2\theta$

$$9 = r^2 = 18\sin 2\theta$$

$$\sin 2\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{12}$$

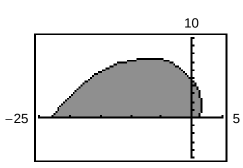


$$A = 2\left[\frac{1}{2}\int_0^{\pi/12} 18\sin 2\theta\, d\theta + \frac{1}{2}\int_{\pi/12}^{5\pi/12} 9\, d\theta + \frac{1}{2}\int_{5\pi/12}^{\pi/2} 18\sin 2\theta\, d\theta\right]$$

$$\approx 1.2058 + 9.4248 + 1.2058 \approx 11.84$$

94. $r = e^\theta, 0 \le \theta \le \pi$

$$A = \frac{1}{2}\int_0^{\pi} (e^\theta)^2\, d\theta \approx 133.62$$



96. $r = a\cos 2\theta, \dfrac{dr}{d\theta} = -2a\sin 2\theta$

$$s = 8\int_0^{\pi/4} \sqrt{a^2\cos^2 2\theta + 4a^2\sin^2 2\theta}\, d\theta$$

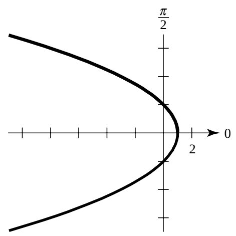
$$= 8a\int_0^{\pi/4} \sqrt{1 + 3\sin^2 2\theta}\, d\theta \quad \text{(Simpson's Rule: } n = 4\text{)}$$

$$\approx \frac{\pi a}{6}[1 + 4(1.1997) + 2(1.5811) + 4(1.8870) + 2]$$

$$\approx 9.69a$$

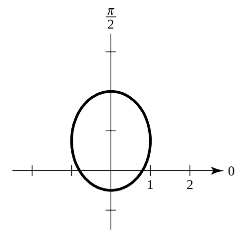
98. $r = \dfrac{2}{1 + \cos\theta}, e = 1$

Parabola



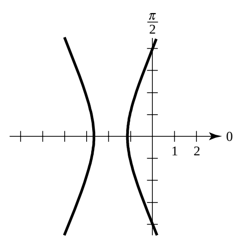
100. $r = \dfrac{4}{5 - 3\sin\theta} = \dfrac{4/5}{1 - (3/5)\sin\theta}, e = \dfrac{3}{5}$

Ellipse



102. $r = \dfrac{8}{2 - 5\cos\theta} = \dfrac{4}{1 - (5/2)\cos\theta}, e = \dfrac{5}{2}$

Hyperbola



104. Line

Slope: $\sqrt{3}$

Solution point: $(0, 0)$

$y = \sqrt{3}\,x, r\sin\theta = \sqrt{3}\,r\cos\theta,$

$\tan\theta = \sqrt{3}, \theta = \dfrac{\pi}{3}$

106. Parabola

Vertex: $\left(2, \dfrac{\pi}{2}\right)$

Focus: $(0, 0)$

$e = 1, d = 4$

$r = \dfrac{4}{1 + \sin\theta}$

108. Hyperbola

Vertices: $(1, 0), (7, 0)$

Focus: $(0, 0)$

$a = 3, c = 4, e = \dfrac{4}{3}, d = \dfrac{7}{4}$

$$r = \frac{\left(\frac{4}{3}\right)\left(\frac{7}{4}\right)}{1 + \left(\frac{4}{3}\right)\cos\theta} = \frac{7}{3 + 4\cos\theta}$$

Problem Solving for Chapter 9

2. Assume $p > 0$.

Let $y = mx + p$ be the equation of the focal chord.

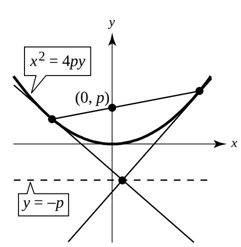
First find x-coordinates of focal chord endpoints:

$$x^2 = 4py = 4p(mx + p)$$

$$x^2 - 4pmx - 4p^2 = 0$$

$$x = \frac{4pm \pm \sqrt{16p^2m^2 + 16p^2}}{2} = 2pm \pm 2p\sqrt{m^2 + 1}$$

$$x^2 = 4py, 2x = 4py' \implies y' = \frac{x}{2p}.$$



(a) The slopes of the tangent lines at the endpoints are perpendicular because

$$\frac{1}{2p}\left[2pm + 2p\sqrt{m^2 + 1}\right]\frac{1}{2p}\left[2pm - 2p\sqrt{m^2 + 1}\right] = \frac{1}{4p^2}[4p^2m^2 - 4p^2(m^2 + 1)] = \frac{1}{4p^2}[-4p^2] = -1$$

—CONTINUED—

2. —CONTINUED—

(b) Finally, we show that the tangent lines intersect at a point on the directrix $y = -p$.

Let $b = 2pm + 2p\sqrt{m^2 + 1}$ and $c = 2pm - 2p\sqrt{m^2 + 1}$.

$$b^2 = 8p^2m^2 + 4p^2 + 8p^2m\sqrt{m^2 + 1}$$

$$c^2 = 8p^2m^2 + 4p^2 - 8p^2m\sqrt{m^2 + 1}$$

$$\frac{b^2}{4p} = 2pm^2 + p + 2pm\sqrt{m^2 + 1}$$

$$\frac{c^2}{4p} = 2pm^2 + p - 2pm\sqrt{m^2 + 1}$$

Tangent line at $x = b$: $y - \dfrac{b^2}{4p} = \dfrac{b}{2p}(x - b) \implies y = \dfrac{bx}{2p} - \dfrac{b^2}{4p}$

Tangent line at $x = c$: $y - \dfrac{c^2}{4p} = \dfrac{c}{2p}(x - c) \implies y = \dfrac{cx}{2p} - \dfrac{c^2}{4p}$

Intersection of tangent lines:

$$\frac{bx}{2p} - \frac{b^2}{4p} = \frac{cx}{2p} - \frac{c^2}{4p}$$

$$2bx - b^2 = 2cx - c^2$$

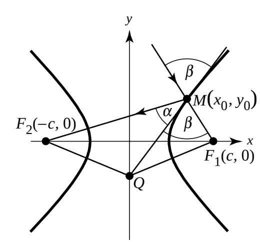
$$2x(b - c) = b^2 - c^2$$

$$2x\left(4p\sqrt{m^2 + 1}\right) = 16p^2m\sqrt{m^2 + 1}$$

$$x = 2pm$$

Finally, the corresponding y-value is $y - p$, which shows that the intersection point lies on the directrix.

4. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, $a^2 + b^2 = c^2$, $MF_2 - Mf_1 = 2a$



$$y' = \frac{b^2x}{a^2y}$$

Tangent line at $M(x_0, y_0)$: $\quad y = y_0 = \dfrac{b^2x_0}{a^2y_0}(x - x_0)$

$$\frac{yy_0 - y_0^{\,2}}{b^2} = \frac{x_0x - x_0^{\,2}}{a^2}$$

$$\frac{x_0x}{a^2} - \frac{y_0y}{b^2} = \frac{x_0^{\,2}}{a^2} + \frac{y_0^{\,2}}{b^2}$$

$$\frac{x_0x}{a^2} - \frac{y_0y}{b^2} = 1$$

At $x = 0, y = -\dfrac{b^2}{y_0} \implies Q = \left(0, -\dfrac{b^2}{y_0}\right)$.

$$QF_2 = QF_1 = \sqrt{c^2 + \frac{b^4}{y_0^{\,2}}} = d$$

$$MQ = \sqrt{x_0^{\,2} + \left(y_0 + \frac{b^2}{y_0}\right)^2} = f$$

By the Law of Cosines,

$$(F_2Q)^2 = (MF_2)^2 + (MQ)^2 - 2(MF_2)(MQ)\cos\alpha$$

$$d^2 = (MF_2)^2 + f^2 - 2f(MF_2)\cos\alpha$$

$$(F_1Q)^2 = (MF_1)^2 + f^2 - 2f(MF_1)\cos\beta$$

$$d^2 = (MF_1)^2 + f^2 - 2f(MF_1)\cos\beta.$$

$$\cos\alpha = \frac{(MF_2)^2 f^2 - d^2}{2f(MF_2)}, \cos\beta = \frac{(MF_1)^2 + f^2 - d^2}{2f(MF_1)}$$

$MF_2 = MF_1 + 2a$. Let $z = MF_1$.

Slopes: MF_1: $\dfrac{y_0}{x_0 - c}$; QF_1: $\dfrac{-b^2}{y_0c}$; QF_2: $\dfrac{b^2}{y_0c}$

—CONTINUED—

4. —CONTINUED—

To show $\alpha = \beta$, consider

$$[(MF_2)^2 + f^2 - d^2][2f(MF_1)] = [(MF_1)^2 + f^2 - d^2][2f(MF_2)]$$

$$\Leftrightarrow \quad [(z + 2a)^2 + f^2 - d^2][z] = [z^2 + f^2 - d^2][z + 2a]$$

$$\Leftrightarrow \quad z^2 + 2az = f^2 - d^2$$

$$\Leftrightarrow \quad (x_0 - c)^2 + y_0^2 + 2az = \left(x_0^2 + \left(y_0 + \frac{b^2}{y_0}\right)^2\right) - \left(c^2 + \frac{b^4}{y_0^2}\right)$$

$$\Leftrightarrow \quad az - x_0c + a^2 = 0$$

$$\Leftrightarrow \quad a\sqrt{(x_0 - c)^2 + y_0^2} = x_0c - a^2$$

$$\Leftrightarrow \quad x_0^2b^2 - a^2y_0^2 = a^2b^2$$

$$\Leftrightarrow \quad \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1.$$

Thus, $\alpha = \beta$ and the reflective property is verified.

6. (a) $y^2 = \dfrac{t^2(1 - t^2)^2}{(1 + t^2)^2}, x^2 = \dfrac{(1 - t^2)^2}{(1 + t^2)^2}$

$$\frac{1 - x}{1 + x} = \frac{1 - \left(\dfrac{1 - t^2}{1 + t^2}\right)}{1 + \left(\dfrac{1 - t^2}{1 + t^2}\right)} = \frac{2t^2}{2} = t^2$$

Thus, $y^2 = x^2\left(\dfrac{1 - x}{1 + x}\right)$.

(b)

$$r^2 \sin^2\theta = r^2\cos^2\theta\left(\frac{1 - r\cos\theta}{1 + r\cos\theta}\right)$$

$$\sin^2\theta(1 + r\cos\theta) = \cos^2\theta(1 - r\cos\theta)$$

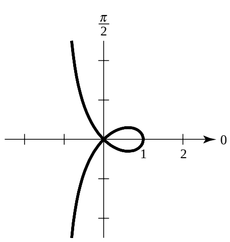
$$r\cos\theta\sin^2\theta + \sin^2\theta = \cos^2\theta - r\cos^3\theta$$

$$r\cos\theta(\sin^2\theta + \cos^2\theta) = \cos^2\theta - \sin^2\theta$$

$$r\cos\theta = \cos 2\theta$$

$$r = \cos 2\theta \cdot \sec\theta$$

(c)



(d) $r(\theta) = 0$ for $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$.

Thus, $y = x$ and $y = -x$ are tangent lines to curve at the origin.

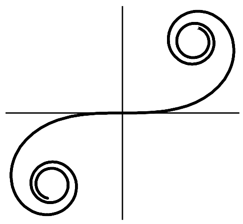
(e) $y'(t) = \dfrac{(1 + t^2)(1 - 3t^2) - (t - t^3)(2t)}{(1 + t^2)^2} = \dfrac{1 - 4t^2 - t^4}{(1 + t^2)^2} = 0$

$$t^4 + 4t^2 - 1 = 0 \implies t^2 = -2 \pm \sqrt{5} \implies x = \frac{1 - \left(-2 \pm \sqrt{5}\right)}{1 + \left(-2 \pm \sqrt{5}\right)} = \frac{3 \mp \sqrt{5}}{-1 \pm \sqrt{5}}$$

$$= \frac{3 - \sqrt{5}}{-1 + \sqrt{5}} = \frac{\sqrt{5} - 1}{2}$$

$$\left(\frac{\sqrt{5} - 1}{2}, \pm\frac{\sqrt{5} - 1}{2}\sqrt{-2 + \sqrt{5}}\right)$$

8. (a)



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(b) $(-x, -y) = \left(-\int_0^t \cos\frac{\pi u^2}{2}\,du, -\int_0^t \sin\frac{\pi u^2}{2}\,du\right)$ is on the curve whenever (x, y) is on the curve.

(c) $x'(t) = \cos\frac{\pi t^2}{2}, y'(t) = \sin\frac{\pi t^2}{2}, x'(t)^2 + y'(t)^2 = 1$

Thus, $s = \int_0^a dt = a$.

On $[-\pi, \pi]$, $s = 2\pi$.

10. $r = \dfrac{ab}{a\sin\theta + b\cos\theta}, 0 \le \theta \le \dfrac{\pi}{2}$

$$r(a\sin\theta + b\cos\theta) = ab$$

$$ay + bx = ab$$

$$\frac{y}{b} + \frac{x}{a} = 1$$

Line segment

Area $= \dfrac{1}{2}ab$

12. Let (r, θ) be on the graph.

$$\sqrt{r^2 + 1 + 2r\cos\theta}\sqrt{r^2 + 1 - 2r\cos\theta} = 1$$

$$(r^2 + 1)^2 - 4r^2\cos^2\theta = 1$$

$$r^4 + 2r^2 + 1 - 4r^2\cos^2\theta = 1$$

$$r^2(r^2 - 4\cos^2\theta + 2) = 0$$

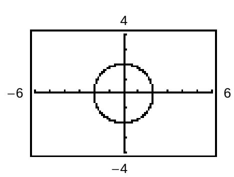
$$r^2 = 4\cos^2\theta - 2$$

$$r^2 = 2(2\cos^2\theta - 1)$$

$$r^2 = 2\cos 2\theta$$

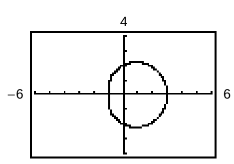
14. (a) $r = 2$

Circle radius 2



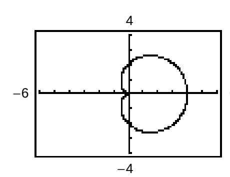
(b) $r = 2 + \cos\theta$

Convex limaçon



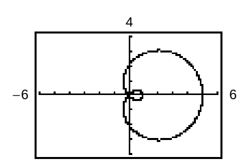
(c) $r = 2 + 2\cos\theta$

Cardioid



(d) $r = 2 + 3\cos\theta$

Limaçon with inner loop



16. The curve is produced over the interval

$0 \le \theta \le 9\pi$.