CHAPTER 9
Conics, Parametric Equations, and Polar Coordinates

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Section 9.1 Conics and Calculus

Solutions to Odd-Numbered Exercises

1. $y^2 = 4x$

Vertex: $(0, 0)$

$p = 1 > 0$

Opens to the right
Matches graph (h).

3. $(x + 3)^2 = -2(y - 2)$

Vertex: $(-3, 2)$

$p = -\frac{1}{2} < 0$

Opens downward
Matches graph (e).

5. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

Center: $(0, 0)$
Ellipse
Matches (f)

7. $\dfrac{y^2}{16} - \dfrac{x^2}{1} = 1$

Hyperbola

Center: $(0, 0)$

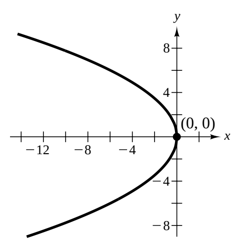
Vertical transverse axis.
Matches (c)

9. $y^2 = -6x = 4\left(-\frac{3}{2}\right)x$

Vertex: $(0, 0)$

Focus: $\left(-\frac{3}{2}, 0\right)$

Directrix: $x = \frac{3}{2}$



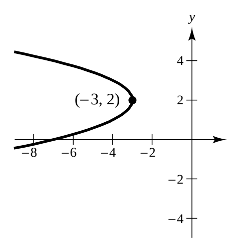
11. $(x + 3) + (y - 2)^2 = 0$

$$(y - 2)^2 = 4\left(-\tfrac{1}{4}\right)(x + 3)$$

Vertex: $(-3, 2)$

Focus: $(-3.25, 2)$

Directrix: $x = -2.75$



13. $y^2 - 4y - 4x = 0$

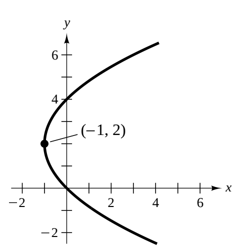
$$y^2 - 4y + 4 = 4x + 4$$

$$(y - 2)^2 = 4(1)(x + 1)$$

Vertex: $(-1, 2)$

Focus: $(0, 2)$

Directrix: $x = -2$



15. $x^2 + 4x + 4y - 4 = 0$

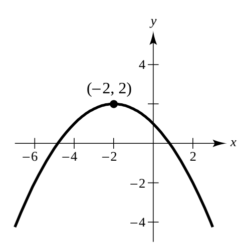
$$x^2 + 4x + 4 = -4y + 4 + 4$$

$$(x + 2)^2 = 4(-1)(y - 2)$$

Vertex: $(-2, 2)$

Focus: $(-2, 1)$

Directrix: $y = 3$



17. $y^2 + x + y = 0$

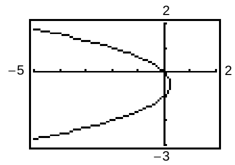
$y^2 + y + \frac{1}{4} = -x + \frac{1}{4}$

$\left(y + \frac{1}{2}\right)^2 = 4\left(-\frac{1}{4}\right)\left(x - \frac{1}{4}\right)$

Vertex: $\left(\frac{1}{4}, -\frac{1}{2}\right)$

Focus: $\left(0, -\frac{1}{2}\right)$

Directrix: $x = \frac{1}{2}$



19. $y^2 - 4x - 4 = 0$

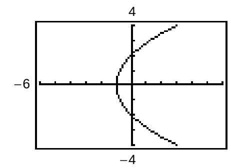
$y^2 = 4x + 4$

$= 4(1)(x + 1)$

Vertex: $(-1, 0)$

Focus: $(0, 0)$

Directrix: $x = -2$



21. $(y - 2)^2 = 4(-2)(x - 3)$

$y^2 - 4y + 8x - 20 = 0$

23. $(x - h)^2 = 4p(y - k)$

$x^2 = 4(6)(y - 4)$

$x^2 - 24y + 96 = 0$

25. $y = 4 - x^2$

$x^2 + y - 4 = 0$

27. Since the axis of the parabola is vertical, the form of the equation is $y = ax^2 + bx + c$. Now, substituting the values of the given coordinates into this equation, we obtain

$$3 = c,\ 4 = 9a + 3b + c,\ 11 = 16a + 4b + c.$$

Solving this system, we have $a = \frac{5}{3}, b = -\frac{14}{3}, c = 3$. Therefore,

$$y = \frac{5}{3}x^2 - \frac{14}{3}x + 3 \text{ or } 5x^2 - 14x - 3y + 9 = 0.$$

29. $x^2 + 4y^2 = 4$

$\frac{x^2}{4} + \frac{y^2}{1} = 1$

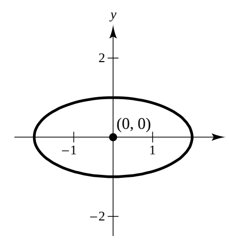
$a^2 = 4, b^2 = 1, c^2 = 3$

Center: $(0, 0)$

Foci: $\left(\pm\sqrt{3}, 0\right)$

Vertices: $(\pm 2, 0)$

$e = \frac{\sqrt{3}}{2}$



31. $\frac{(x - 1)^2}{9} + \frac{(y - 5)^2}{25} = 1$

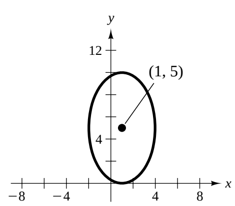
$a^2 = 25, b^2 = 9, c^2 = 16$

Center: $(1, 5)$

Foci: $(1, 9), (1, 1)$

Vertices: $(1, 10), (1, 0)$

$e = \frac{4}{5}$



33. $9x^2 + 4y^2 + 36x - 24y + 36 = 0$

$9(x^2 + 4x + 4) + 4(y^2 - 6y + 9) = -36 + 36 + 36$

$= 36$

$\frac{(x + 2)^2}{4} + \frac{(y - 3)^2}{9} = 1$

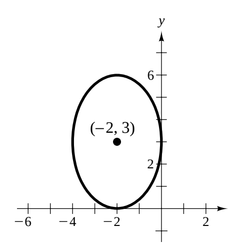
$a^2 = 9, b^2 = 4, c^2 = 5$

Center: $(-2, 3)$

Foci: $\left(-2, 3 \pm \sqrt{5}\right)$

Vertices: $(-2, 6), (-2, 0)$

$e = \frac{\sqrt{5}}{3}$



35. $12x^2 + 20y^2 - 12x + 40y - 37 = 0$

$$12\left(x^2 - x + \frac{1}{4}\right) + 20(y^2 + 2y + 1) = 37 + 3 + 20$$

$$= 60$$

$$\frac{[x - (1/2)]^2}{5} + \frac{(y + 1)^2}{3} = 1$$

$a^2 = 5, b^2 = 3, c^2 = 2$

Center: $\left(\frac{1}{2}, -1\right)$

Foci: $\left(\frac{1}{2} \pm \sqrt{2}, -1\right)$

Vertices: $\left(\frac{1}{2} \pm \sqrt{5}, -1\right)$

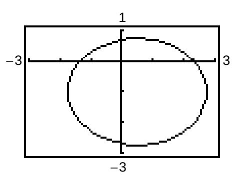
Solve for y:

$$20(y^2 + 2y + 1) = -12x^2 + 12x + 37 + 20$$

$$(y + 1)^2 = \frac{57 + 12x - 12x^2}{20}$$

$$y = -1 \pm \sqrt{\frac{57 + 12x - 12x^2}{20}}$$

(Graph each of these separately.)



37. $x^2 + 2y^2 - 3x + 4y + 0.25 = 0$

$$\left(x^2 - 3x + \frac{9}{4}\right) + 2(y^2 + 2y + 1) = -\frac{1}{4} + \frac{9}{4} + 2 = 4$$

$$\frac{[x - (3/2)]^2}{4} + \frac{(y + 1)^2}{2} = 1$$

$a^2 = 4, b^2 = 2, c^2 = 2$

Center: $\left(\frac{3}{2}, -1\right)$

Foci: $\left(\frac{3}{2} \pm \sqrt{2}, -1\right)$

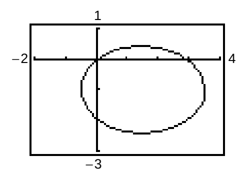
Vertices: $\left(-\frac{1}{2}, -1\right), \left(\frac{7}{2}, -1\right)$

Solve for y: $2(y^2 + 2y + 1) = -x^2 + 3x - \frac{1}{4} + 2$

$$(y + 1)^2 = \frac{1}{2}\left(\frac{7}{4} + 3x - x^2\right)$$

$$y = -1 \pm \sqrt{\frac{7 + 12x - 4x^2}{8}}$$

(Graph each of these separately.)



39. Center: $(0, 0)$
Focus: $(2, 0)$
Vertex: $(3, 0)$
Horizontal major axis

$a = 3, c = 2 \implies b = \sqrt{5}$

$$\frac{x^2}{9} + \frac{y^2}{5} = 1$$

41. Vertices: $(3, 1), (3, 9)$
Minor axis length: 6
Vertical major axis
Center: $(3, 5)$

$a = 4, b = 3$

$$\frac{(x - 3)^2}{9} + \frac{(y - 5)^2}{16} = 1$$

43. Center: $(0, 0)$
Horizontal major axis
Points on ellipse: $(3, 1), (4, 0)$

Since the major axis is horizontal,

$$\left(\frac{x^2}{a^2}\right) + \left(\frac{y^2}{b^2}\right) = 1.$$

Substituting the values of the coordinates of the given points into this equation, we have

$$\left(\frac{9}{a^2}\right) + \left(\frac{1}{b^2}\right) = 1, \text{ and } \frac{16}{a^2} = 1.$$

The solution to this system is $a^2 = 16, b^2 = 16/7$.

Therefore,

$$\frac{x^2}{16} + \frac{y^2}{16/7} = 1, \frac{x^2}{16} + \frac{7y^2}{16} = 1.$$

45. $\dfrac{y^2}{1} - \dfrac{x^2}{4} = 1$

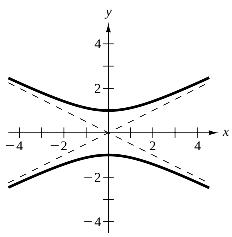
$a = 1, b = 2, c = \sqrt{5}$

Center: $(0, 0)$

Vertices: $(0, \pm 1)$

Foci: $(0, \pm\sqrt{5})$

Asymptotes: $y = \pm\frac{1}{2}x$



47. $\dfrac{(x-1)^2}{4} - \dfrac{(y+2)^2}{1} = 1$

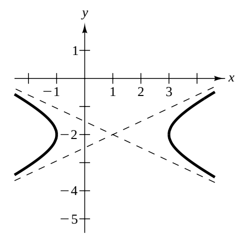
$a = 2, b = 1, c = \sqrt{5}$

Center: $(1, -2)$

Vertices: $(-1, -2), (3, -2)$

Foci: $(1 \pm \sqrt{5}, -2)$

Asymptotes: $y = -2 \pm \frac{1}{2}(x - 1)$



49. $9x^2 - y^2 - 36x - 6y + 18 = 0$

$9(x^2 - 4x + 4) - (y^2 + 6y + 9) = -18 + 36 - 9$

$$\frac{(x-2)^2}{1} - \frac{(y+3)^2}{9} = 1$$

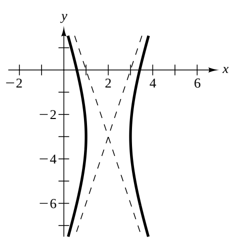
$a = 1, b = 3, c = \sqrt{10}$

Center: $(2, -3)$

Vertices: $(1, -3), (3, -3)$

Foci: $(2 \pm \sqrt{10}, -3)$

Asymptotes: $y = -3 \pm 3(x - 2)$



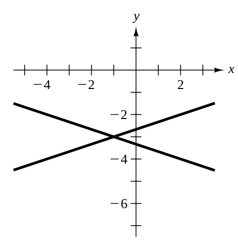
51. $x^2 - 9y^2 + 2x - 54y - 80 = 0$

$(x^2 + 2x + 1) - 9(y^2 + 6y + 9) = 80 + 1 - 81 = 0$

$(x + 1)^2 - 9(y + 3)^2 = 0$

$y + 3 = \pm\frac{1}{3}(x + 1)$

Degenerate hyperbola is two lines intersecting at $(-1, -3)$.



53. $9y^2 - x^2 + 2x + 54y + 62 = 0$

$9(y^2 + 6y + 9) - (x^2 - 2x + 1) = -62 - 1 + 81 = 18$

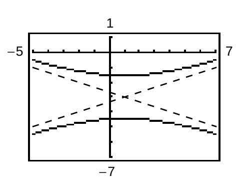
$$\frac{(y+3)^2}{2} - \frac{(x-1)^2}{18} = 1$$

$a = \sqrt{2}, b = 3\sqrt{2}, c = 2\sqrt{5}$

Center: $(1, -3)$

Vertices: $(1, -3 \pm \sqrt{2})$

Foci: $(1, -3 \pm 2\sqrt{5})$



Solve for y:

$$9(y^2 + 6y + 9) = x^2 - 2x - 62 + 81$$

$$(y + 3)^2 = \frac{x^2 - 2x + 19}{9}$$

$$y = -3 \pm \frac{1}{3}\sqrt{x^2 - 2x + 19}$$

(Graph each curve separately.)

55. $3x^2 - 2y^2 - 6x - 12y - 27 = 0$

$3(x^2 - 2x + 1) - 2(y^2 + 6y + 9) = 27 + 3 - 18 = 12$

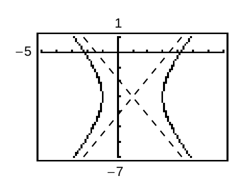
$$\frac{(x-1)^2}{4} - \frac{(y+3)^2}{6} = 1$$

$a = 2, b = \sqrt{6}, c = \sqrt{10}$

Center: $(1, -3)$

Vertices: $(-1, -3), (3, -3)$

Foci: $(1 \pm \sqrt{10}, -3)$



Solve for y:

$$2(y^2 + 6y + 9) = 3x^2 - 6x - 27 + 18$$

$$(y + 3)^2 = \frac{3x^2 - 6x - 9}{2}$$

$$y = -3 \pm \sqrt{\frac{3(x^2 - 2x - 3)}{2}}$$

(Graph each curve separately.)

57. Vertices: $(\pm 1, 0)$
Asymptotes: $y = \pm 3x$
Horizontal transverse axis
Center: $(0, 0)$

$a = 1, \pm\frac{b}{a} = \pm\frac{b}{1} = \pm 3 \implies b = 3$

Therefore, $\frac{x^2}{1} - \frac{y^2}{9} = 1$.

59. Vertices: $(2, \pm 3)$
Point on graph: $(0, 5)$
Vertical transverse axis
Center: $(2, 0)$

$a = 3$

Therefore, the equation is of the form

$$\frac{y^2}{9} - \frac{(x-2)^2}{b^2} = 1.$$

Substituting the coordinates of the point $(0, 5)$, we have

$$\frac{25}{9} - \frac{4}{b^2} = 1 \quad \text{or} \quad b^2 = \frac{9}{4}.$$

Therefore, the equation is $\frac{y^2}{9} - \frac{(x-2)^2}{9/4} = 1$.

61. Center: $(0, 0)$
Vertex: $(0, 2)$
Focus: $(0, 4)$
Vertical transverse axis

$a = 2, c = 4, b^2 = c^2 - a^2 = 12$

Therefore, $\frac{y^2}{4} - \frac{x^2}{12} = 1$.

63. Vertices: $(0, 2), (6, 2)$

Asymptotes: $y = \frac{2}{3}x, y = 4 - \frac{2}{3}x$

Horizontal transverse axis

Center: $(3, 2)$

$a = 3$

Slopes of asymptotes: $\pm\frac{b}{a} = \pm\frac{2}{3}$

Thus, $b = 2$. Therefore,

$$\frac{(x-3)^2}{9} - \frac{(y-2)^2}{4} = 1.$$

65. (a) $\frac{x^2}{9} - y^2 = 1, \frac{2x}{9} - 2yy' = 0, \frac{x}{9y} = y'$

At $x = 6$: $y = \pm\sqrt{3}, y' = \frac{\pm 6}{9\sqrt{3}} = \frac{\pm 2\sqrt{3}}{9}$

At $(6, \sqrt{3})$: $y - \sqrt{3} = \frac{2\sqrt{3}}{9}(x - 6)$

or $2x - 3\sqrt{3}y - 3 = 0$

At $(6, -\sqrt{3})$: $y + \sqrt{3} = \frac{-2\sqrt{3}}{9}(x - 6)$

or $2x + 3\sqrt{3}y - 3 = 0$

(b) From part (a) we know that the slopes of the normal lines must be $\mp 9/(2\sqrt{3})$.

At $(6, \sqrt{3})$: $y - \sqrt{3} = -\frac{9}{2\sqrt{3}}(x - 6)$

or $9x + 2\sqrt{3}y - 60 = 0$

At $(6, -\sqrt{3})$: $y + \sqrt{3} = \frac{9}{2\sqrt{3}}(x - 6)$

or $9x - 2\sqrt{3}y - 60 = 0$

67. $x^2 + 4y^2 - 6x + 16y + 21 = 0$
$A = 1, C = 4$
$AC = 4 > 0$
Ellipse

69. $y^2 - 4y - 4x = 0$
$A = 0, C = 1$
Parabola

71. $4x^2 + 4y^2 - 16y + 15 = 0$
$A = C = 4$
Circle

73. $9x^2 + 9y^2 - 36x + 6y + 34 = 0$
$A = C = 9$
Circle

75. $3x^2 - 6x + 3 = 6 + 2y^2 + 4y + 2$

$3x^2 - 2y^2 - 6x - 4y - 5 = 0$

$A = 3, C = -2, AC < 0$

Hyperbola

77. (a) A parabola is the set of all points (x, y) that are equidistant from a fixed line (directrix) and a fixed point (focus) not on the line.

(b) $(x - h)^2 = 4p(y - k)$ or $(y - k)^2 = 4p(x - h)$

(c) See Theorem 9.2.

79. (a) A hyperbola is the set of all points (x, y) for which the absolute value of the difference between the distances from two distance fixed points (foci) is constant.

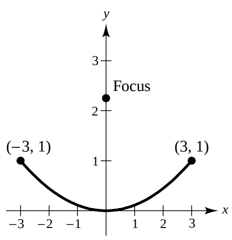
(b) $\dfrac{(x - h)^2}{a^2} - \dfrac{(y - k)^2}{b^2} = 1$ or $\dfrac{(y - k)^2}{a^2} - \dfrac{(x - h)^2}{b^2} = 1$

(c) $y = k \pm \dfrac{b}{a}(x - h)$ or $y = k \pm \dfrac{a}{b}(x - h)$

81. Assume that the vertex is at the origin.

$$x^2 = 4py$$
$$(3)^2 = 4p(1)$$
$$\frac{9}{4} = p$$

The pipe is located $\frac{9}{4}$ meters from the vertex.



83. $y = ax^2$

$y' = 2ax$

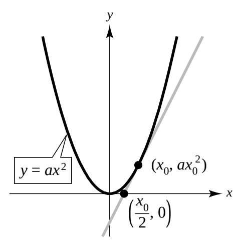
The equation of the tangent line is

$$y - ax_0^2 = 2ax_0(x - x_0) \text{ or } y = 2ax_0x - ax_0^2.$$

Let $y = 0$. Then:

$$-ax_0^2 = 2ax_0x - 2ax_0^2$$
$$ax_0^2 = 2ax_0x$$

Therefore, $\dfrac{x_0}{2} = x$ is the x-intercept.



85. (a) Consider the parabola $x^2 = 4py$. Let m_0 be the slope of the one tangent line at (x_1, y_1) and therefore, $-1/m_0$ is the slope of the second at (x_2, y_2). From the derivative given in Exercise 32 we have:

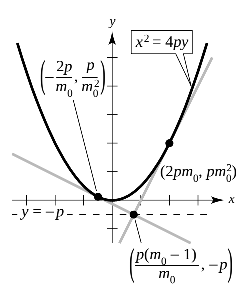
$$m_0 = \frac{1}{2p}x_1 \text{ or } x_1 = 2pm_0$$
$$\frac{-1}{m_0} = \frac{1}{2p}x_2 \text{ or } x_2 = \frac{-2p}{m_0}$$

Substituting these values of x into the equation $x^2 = 4py$, we have the coordinates of the points of tangency $(2pm_0, pm_0^2)$ and $(-2p/m_0, p/m_0^2)$ and the equations of the tangent lines are

$$(y - pm_0^2) = m_0(x - 2pm_0) \quad \text{and} \quad \left(y - \frac{p}{m_0^2}\right) = \frac{-1}{m_0}\left(x + \frac{2p}{m_0}\right).$$

The point of intersection of these lines is

$$\left(\frac{p(m_0^2 - 1)}{m_0}, -p\right) \text{ and is on the directrix, } y = -p.$$



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85. —CONTINUED—

(b) $x^2 - 4x - 4y + 8 = 0$

$$(x-2)^2 = 4(y-1).\ \text{Vertex } (2, 1)$$

$$2x - 4 - 4\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{1}{2}x - 1$$

At $(-2, 5)$, $dy/dx = -2$. At $\left(3, \frac{5}{4}\right)$, $dy/dx = \frac{1}{2}$.

Tangent line at $(-2, 5)$: $y - 5 = -2(x+2) \implies 2x + y - 1 = 0$.

Tangent line at $\left(3, \frac{5}{4}\right)$: $y - \frac{5}{4} = \frac{1}{2}(x-3) \implies 2x - 4y - 1 = 0$.

Since $m_1 m_2 = (-2)\left(\frac{1}{2}\right) = -1$, the lines are perpendicular.

Point of intersection: $-2x + 1 = \frac{1}{2}x - \frac{1}{4}$

$$-\frac{5}{2}x = -\frac{5}{4}$$

$$x = \frac{1}{2}$$

$$y = 0$$

Directrix: $y = 0$ and the point of intersection $\left(\frac{1}{2}, 0\right)$ lies on this line.

87. $y = x - x^2$

$$\frac{dy}{dx} = 1 - 2x$$

At (x_1, y_1) on the mountain, $m = 1 - 2x_1$. Also, $m = \dfrac{y_1 - 1}{x_1 + 1}$.

$$\frac{y_1 - 1}{x_1 + 1} = 1 - 2x_1$$

$$(x_1 - x_1^2) - 1 = (1 - 2x_1)(x_1 + 1)$$

$$-x_1^2 + x_1 - 1 = -2x_1^2 - x_1 + 1$$

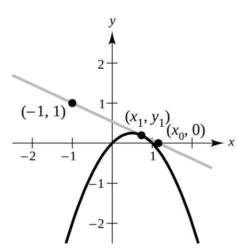
$$x_1^2 + 2x_1 - 2 = 0$$

$$x_1 = \frac{-2 \pm \sqrt{2^2 - 4(1)(-2)}}{2(1)} = \frac{-2 \pm 2\sqrt{3}}{2} = -1 \pm \sqrt{3}$$

Choosing the positive value for x_1, we have $x_1 = -1 + \sqrt{3}$.

$$m = 1 - 2\left(-1 + \sqrt{3}\right) = 3 - 2\sqrt{3}$$

$$m = \frac{0 - 1}{x_0 + 1} = -\frac{1}{x_0 + 1}$$



Thus, $-\dfrac{1}{x_0 + 1} = 3 - 2\sqrt{3}$

$$\frac{-1}{3 - 2\sqrt{3}} = x_0 + 1$$

$$\frac{3 + 2\sqrt{3}}{3} - 1 = x_0$$

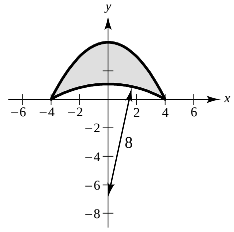
$$\frac{2\sqrt{3}}{3} = x_0.$$

The closest the receiver can be to the hill is $\left(2\sqrt{3}/3\right) - 1 \approx 0.155$.

89. Parabola

Vertex: $(0, 4)$

$$x^2 = 4p(y - 4)$$
$$4^2 = 4p(0 - 4)$$
$$p = -1$$
$$x^2 = -4(y - 4)$$
$$y = 4 - \frac{x^2}{4}$$



Circle

Center: $(0, k)$

Radius: 8

$$x^2 + (y - k)^2 = 64$$
$$4^2 + (0 - k)^2 = 64$$
$$k^2 = 48$$
$$k = -4\sqrt{3} \quad \text{(Center is on the negative } y\text{-axis.)}$$
$$x^2 + \left(y + 4\sqrt{3}\right)^2 = 64$$
$$y = -4\sqrt{3} \pm \sqrt{64 - x^2}$$

Since the y-value is positive when $x = 0$, we have $y = -4\sqrt{3} + \sqrt{64 - x^2}$.

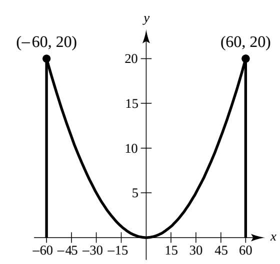
$$A = 2\int_0^4 \left[\left(4 - \frac{x^2}{4}\right) - \left(-4\sqrt{3} + \sqrt{64 - x^2}\right)\right] dx$$
$$= 2\left[4x - \frac{x^3}{12} + 4\sqrt{3}x - \frac{1}{2}\left(x\sqrt{64 - x^2} + 64 \arcsin\frac{x}{8}\right)\right]_0^4$$
$$= 2\left[16 - \frac{64}{12} + 16\sqrt{3} - 2\sqrt{48} - 32 \arcsin\frac{1}{2}\right]$$
$$= \frac{16\left(4 + 3\sqrt{3} - 2\pi\right)}{3} \approx 15.536 \text{ square feet}$$

91. (a) Assume that $y = ax^2$.

$$20 = a(60)^2 \implies a = \frac{2}{360} = \frac{1}{180} \implies y = \frac{1}{180}x^2$$

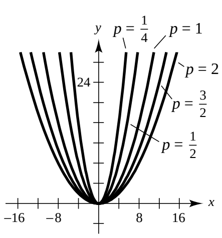
(b) $f(x) = \frac{1}{180}x^2,\ f'(x) = \frac{1}{90}x$

$$S = 2\int_0^{60} \sqrt{1 + \left(\frac{1}{90}x\right)^2}\, dx = \frac{2}{90}\int_0^{60} \sqrt{90^2 + x^2}\, dx$$
$$= \frac{2}{90}\frac{1}{2}\left[x\sqrt{90^2 + x^2} + 90^2 \ln\left|x + \sqrt{90^2 + x^2}\right|\right]_0^{60} \quad \text{(formula 26)}$$
$$= \frac{1}{90}\left[60\sqrt{11{,}700} + 90^2 \ln\left(60 + \sqrt{11{,}700}\right) - 90^2 \ln 90\right]$$
$$= \frac{1}{90}\left[1800\sqrt{13} + 90^2 \ln\left(60 + 30\sqrt{13}\right) - 90^2 \ln 90\right]$$
$$= 20\sqrt{13} + 90 \ln\left(\frac{60 + 30\sqrt{13}}{90}\right)$$
$$= 10\left[2\sqrt{13} + 9 \ln\left(\frac{2 + \sqrt{13}}{3}\right)\right] \approx 128.4 \text{ m}$$

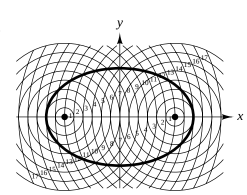


93. $x^2 = 4py,\ p = \frac{1}{4}, \frac{1}{2}, 1, \frac{3}{2}, 2$

As p increases, the graph becomes wider.



95.



97. $a = \frac{5}{2}, b = 2, c = \sqrt{\left(\frac{5}{2}\right)^2 - (2)^2} = \frac{3}{2}$

The tacks should be placed 1.5 feet from the center. The string should be $2a = 5$ feet long.

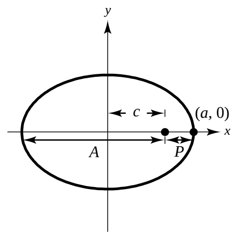
99. $e = \frac{c}{a}$

$A + P = 2a$

$a = \frac{A + P}{2}$

$c = a - P = \frac{A + P}{2} - P = \frac{A - P}{2}$

$e = \frac{c}{a} = \frac{(A - P)/2}{(A + P)/2} = \frac{A - P}{A + P}$



101. $e = \frac{A - P}{A + P} = \frac{35.34au - 0.59au}{35.34au + 0.59au} \approx 0.9672$

103. $\frac{x^2}{10^2} + \frac{y^2}{5^2} = 1$

$\frac{2x}{10^2} + \frac{2yy'}{5^2} = 0$

$y' = \frac{-5^2x}{10^2y} = \frac{-x}{4y}$

At $(-8, 3)$: $y' = \frac{8}{12} = \frac{2}{3}$

The equation of the tangent line is $y - 3 = \frac{2}{3}(x + 8)$. It will cross the y-axis when $x = 0$ and $y = \frac{2}{3}(8) + 3 = \frac{25}{3}$.

105. $16x^2 + 9y^2 + 96x + 36y + 36 = 0$

$32x + 18yy' + 96 + 36y' = 0$

$y'(18y + 36) = -(32x + 96)$

$y' = \frac{-(32x + 96)}{18y + 36}$

$y' = 0$ when $x = -3$. y' is undefined when $y = -2$.
At $x = -3, y = 2$ or -6.
Endpoints of major axis: $(-3, 2), (-3, -6)$

At $y = -2, x = 0$ or -6.
Endpoints of minor axis: $(0, -2), (-6, -2)$

Note: Equation of ellipse is $\frac{(x + 3)^2}{9} + \frac{(y + 2)^2}{16} = 1$

107. (a) $A = 4\int_0^2 \frac{1}{2}\sqrt{4 - x^2}\, dx = \left[x\sqrt{4 - x^2} + 4 \arcsin\left(\frac{x}{2}\right)\right]_0^2 = 2\pi$ [or, $A = \pi ab = \pi(2)(1) = 2\pi$]

(b) **Disk:** $V = 2\pi\int_0^2 \frac{1}{4}(4 - x^2)\, dx = \frac{1}{2}\pi\left[4x - \frac{1}{3}x^3\right]_0^2 = \frac{8\pi}{3}$

$y = \frac{1}{2}\sqrt{4 - x^2}$

$y' = \frac{-x}{2\sqrt{4 - x^2}}$

$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{16 - 4x^2}} = \sqrt{\frac{16 - 3x^2}{4y}}$

$S = 2(2\pi)\int_0^2 y\left(\frac{\sqrt{16 - 3x^2}}{4y}\right) dx = \frac{\pi}{2\sqrt{3}}\left[\sqrt{3}x\sqrt{16 - 3x^2} + 16 \arcsin\left(\frac{\sqrt{3}x}{4}\right)\right]_0^2 = \frac{2\pi}{9}\left(9 + 4\sqrt{3}\pi\right) \approx 21.48$

—CONTINUED—

107. —CONTINUED—

(c) **Shell:** $V = 2\pi\int_0^2 x\sqrt{4 - x^2}\,dx = -\pi\int_0^2 -2x(4 - x^2)^{1/2}\,dx = -\frac{2\pi}{3}\Big[(4 - x^2)^{3/2}\Big]_0^2 = \frac{16\pi}{3}$

$$x = 2\sqrt{1 - y^2}$$

$$x' = \frac{-2y}{\sqrt{1 - y^2}}$$

$$\sqrt{1 + (x')^2} = \sqrt{1 + \frac{4y^2}{1 - y^2}} = \frac{\sqrt{1 + 3y^2}}{\sqrt{1 - y^2}}$$

$$S = 2(2\pi)\int_0^1 2\sqrt{1 - y^2}\,\frac{\sqrt{1 + 3y^2}}{\sqrt{1 - y^2}}\,dy = 8\pi\int_0^1 \sqrt{1 + 3y^2}\,dy$$

$$= \frac{8\pi}{2\sqrt{3}}\Big[\sqrt{3}y\sqrt{1 + 3y^2} + \ln\left|\sqrt{3}y + \sqrt{1 + 3y^2}\right|\Big]_0^1 = \frac{4\pi}{3}\left|6 + \sqrt{3}\ln\left(2 + \sqrt{3}\right)\right| \approx 34.69$$

109. From Example 5,

$$C = 4a\int_0^{\pi/2} \sqrt{1 - e^2\sin^2\theta}\,d\theta$$

For $\frac{x^2}{25} + \frac{y^2}{49} = 1$, we have

$a = 7, b = 5, c = \sqrt{49 - 25} = 2\sqrt{6}, e = \frac{c}{a} = \frac{2\sqrt{6}}{7}$.

$$C = 4(7)\int_0^{\pi/2} \sqrt{1 - \frac{24}{49}\sin^2\theta}\,d\theta$$

$$\approx 28(1.3558) \approx 37.9614$$

111. Area circle $= \pi r^2 = 100\pi$

Area ellipse $= \pi ab = \pi a(10)$

$2(100\pi) = 10\pi a \implies a = 20$

Hence, the length of the major axis is $2a = 40$.

113. The transverse axis is horizontal since $(2, 2)$ and $(10, 2)$ are the foci (see definition of hyperbola).

Center: $(6, 2)$

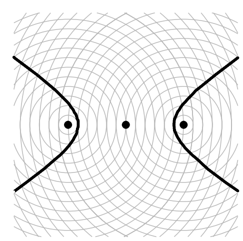
$c = 4, 2a = 6, b^2 = c^2 - a^2 = 7$

Therefore, the equation is

$$\frac{(x - 6)^2}{9} - \frac{(y - 2)^2}{7} = 1.$$

115. $2a = 10 \implies a = 5$

$c = 6 \implies b = \sqrt{11}$



117. Time for sound of bullet hitting target to reach (x, y): $\frac{2c}{v_m} + \frac{\sqrt{(x - c)^2 + y^2}}{v_s}$

Time for sound of rifle to reach (x, y): $\frac{\sqrt{(x + c)^2 + y^2}}{v_s}$

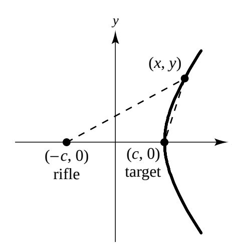
Since the times are the same, we have: $\frac{2c}{v_m} + \frac{\sqrt{(x - c)^2 + y^2}}{v_s} = \frac{\sqrt{(x + c)^2 + y^2}}{v_s}$

$$\frac{4c^2}{{v_m}^2} + \frac{4c}{v_m v_s}\sqrt{(x - c)^2 + y^2} + \frac{(x - c)^2 + y^2}{{v_s}^2} = \frac{(x + c)^2 + y^2}{{v_s}^2}$$

$$\sqrt{(x - c)^2 + y^2} = \frac{{v_m}^2 x - {v_s}^2 c}{v_s v_m}$$

$$\left(1 - \frac{{v_m}^2}{{v_s}^2}\right)x^2 + y^2 = \left(\frac{{v_s}^2}{{v_m}^2} - 1\right)c^2$$

$$\frac{x^2}{c^2{v_s}^2/{v_m}^2} - \frac{y^2}{c^2({v_m}^2 - {v_s}^2)/{v_m}^2} = 1$$



119. The point (x, y) lies on the line between $(0, 10)$ and $(10, 0)$. Thus, $y = 10 - x$. The point also lies on the hyperbola $(x^2/36) - (y^2/64) = 1$. Using substitution, we have:

$$\frac{x^2}{36} - \frac{(10-x)^2}{64} = 1$$

$$16x^2 - 9(10-x)^2 = 576$$

$$7x^2 + 180x - 1476 = 0$$

$$x = \frac{-180 \pm \sqrt{180^2 - 4(7)(-1476)}}{2(7)} = \frac{-180 \pm 192\sqrt{2}}{14} = \frac{-90 \pm 96\sqrt{2}}{7}$$

Choosing the positive value for x we have:

$$x = \frac{-90 + 96\sqrt{2}}{7} \approx 6.538 \text{ and } y = \frac{160 - 96\sqrt{2}}{7} \approx 3.462$$

121. $$\frac{x^2}{a^2} + \frac{2y^2}{b^2} = 1 \implies \frac{2y^2}{b^2} = 1 - \frac{x^2}{a^2},\ c^2 = a^2 - b^2$$

$$\frac{x^2}{a^2 - b^2} - \frac{2y^2}{b^2} = 1 \implies \frac{2y^2}{b^2} = \frac{x^2}{a^2 - b^2} - 1$$

$$1 - \frac{x^2}{a^2} = \frac{x^2}{a^2 - b^2} - 1 \implies 2 = x^2\left(\frac{1}{a^2} + \frac{1}{a^2 - b^2}\right)$$

$$x^2 = \frac{2a^2(a^2 - b^2)}{2a^2 - b^2} \implies x = \pm\frac{\sqrt{2}a\sqrt{a^2 - b^2}}{\sqrt{2a^2 - b^2}} = \pm\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}$$

$$\frac{2y^2}{b^2} = 1 - \frac{1}{a^2}\left(\frac{2a^2c^2}{2a^2 - b^2}\right) \implies \frac{2y^2}{b^2} = \frac{b^2}{2a^2 - b^2}$$

$$y^2 = \frac{b^4}{2(2a^2 - b^2)} \implies y = \pm\frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}}$$

There are four points of intersection: $\left(\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}, \pm\frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}}\right)$, $\left(-\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}, \pm\frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}}\right)$

$$\frac{x^2}{a^2} + \frac{2y^2}{b^2} = 1 \implies \frac{2x}{a^2} + \frac{4yy'}{b^2} = 0 \implies y'_e = -\frac{b^2x}{2a^2y}$$

$$\frac{x^2}{a^2 - b^2} - \frac{2y^2}{b^2} = 1 \implies \frac{2x}{c^2} - \frac{4yy'}{b^2} = 0 \implies y'_h = \frac{b^2x}{2c^2y}$$

At $\left(\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}, \frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}}\right)$, the slopes of the tangent lines are:

$$y'_e = \frac{-b^2\left(\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}\right)}{2a^2\left(\frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}}\right)} = -\frac{c}{a} \quad \text{and} \quad y'_h = \frac{b^2\left(\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}\right)}{2c^2\left(\frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}}\right)} = \frac{a}{c}$$

Since the slopes are negative reciprocals, the tangent lines are perpendicular. Similarly, the curves are perpendicular at the other three points of intersection.

123. False. See the definition of a parabola.

125. True

127. False. $y^2 - x^2 + 2x + 2y = 0$ yields two intersecting lines.

129. True

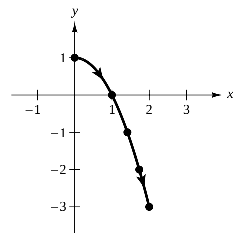
Section 9.2 Plane Curves and Parametric Equations

1. $x = \sqrt{t}$, $y = 1 - t$

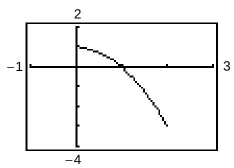
(a)

t	0	1	2	3	4
x	0	1	$\sqrt{2}$	$\sqrt{3}$	2
y	1	0	-1	-2	-3

(b)



(c)



(d) $x^2 = t$

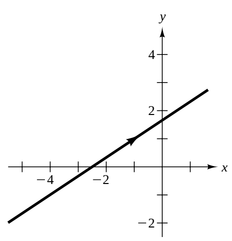
$y = 1 - x^2, x \geq 0$

3. $x = 3t - 1$

$y = 2t + 1$

$y = 2\left(\dfrac{x + 1}{3}\right) + 1$

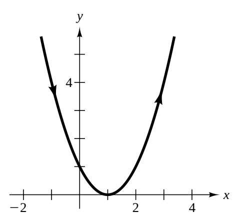
$2x - 3y + 5 = 0$



5. $x = t + 1$

$y = t^2$

$y = (x - 1)^2$

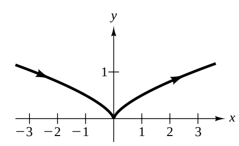


7. $x = t^3$

$y = \frac{1}{2}t^2$

$x = t^3$ implies $t = x^{1/3}$

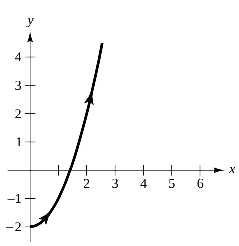
$y = \frac{1}{2}x^{2/3}$



9. $x = \sqrt{t}, t \geq 0$

$y = t - 2$

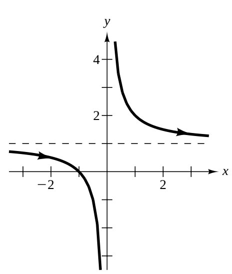
$y = x^2 - 2, x \geq 0$



11. $x = t - 1$

$y = \dfrac{t}{t - 1}$

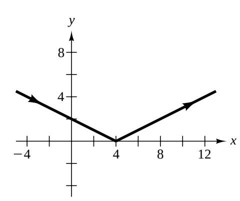
$y = \dfrac{x + 1}{x}$



13. $x = 2t$

$y = |t - 2|$

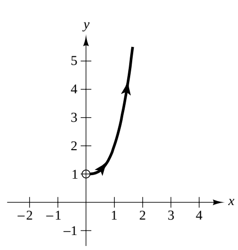
$y = \left|\dfrac{x}{2} - 2\right| = \dfrac{|x - 4|}{2}$



15. $x = e^t, x > 0$

$y = e^{3t} + 1$

$y = x^3 + 1, x > 0$



17. $x = \sec\theta$

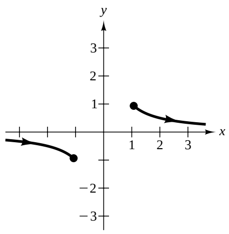
$y = \cos\theta$

$0 \le \theta < \frac{\pi}{2},\ \frac{\pi}{2} < \theta \le \pi$

$xy = 1$

$y = \frac{1}{x}$

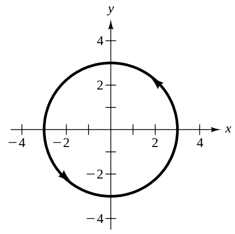
$|x| \ge 1,\ |y| \le 1$



19. $x = 3\cos\theta,\ y = 3\sin\theta$

Squaring both equations and adding, we have

$x^2 + y^2 = 9.$



21.

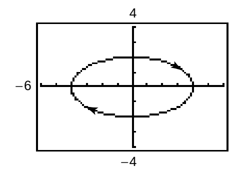
$x = 4\sin 2\theta$

$y = 2\cos 2\theta$

$\frac{x^2}{16} = \sin^2 2\theta$

$\frac{y^2}{4} = \cos^2 2\theta$

$\frac{x^2}{16} + \frac{y^2}{4} = 1$



23.

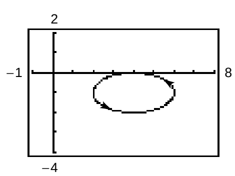
$x = 4 + 2\cos\theta$

$y = -1 + \sin\theta$

$\frac{(x-4)^2}{4} = \cos^2\theta$

$\frac{(y+1)^2}{1} = \sin^2\theta$

$\frac{(x-4)^2}{4} + \frac{(y+1)^2}{1} = 1$



25.

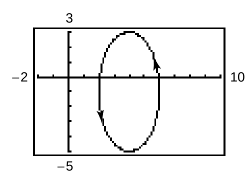
$x = 4 + 2\cos\theta$

$y = -1 + 4\sin\theta$

$\frac{(x-4)^2}{4} = \cos^2\theta$

$\frac{(y+1)^2}{16} = \sin^2\theta$

$\frac{(x-4)^2}{4} + \frac{(y+1)^2}{16} = 1$



27.

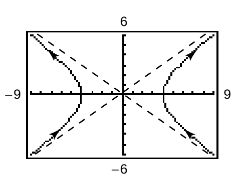
$x = 4\sec\theta$

$y = 3\tan\theta$

$\frac{x^2}{16} = \sec^2\theta$

$\frac{y^2}{9} = \tan^2\theta$

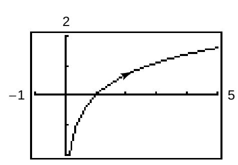
$\frac{x^2}{16} - \frac{y^2}{9} = 1$



29. $x = t^3$

$y = 3\ln t$

$y = 3\ln\sqrt[3]{x} = \ln x$



31.

$x = e^{-t}$

$y = e^{3t}$

$e^t = \frac{1}{x}$

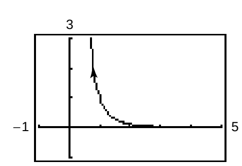
$e^t = \sqrt[3]{y}$

$\sqrt[3]{y} = \frac{1}{x}$

$y = \frac{1}{x^3}$

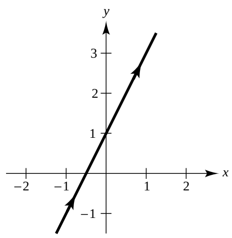
$x > 0$

$y > 0$



33. By eliminating the parameters in (a) – (d), we get $y = 2x + 1$. They differ from each other in orientation and in restricted domains. These curves are all smooth except for (b).

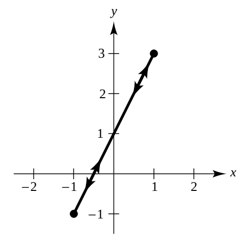
(a) $x = t, \ y = 2t + 1$



(b) $x = \cos\theta \qquad y = 2\cos\theta + 1$

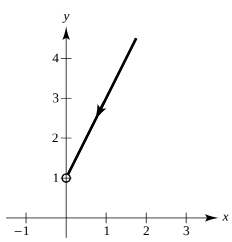
$-1 \le x \le 1 \qquad -1 \le y \le 3$

$\dfrac{dx}{d\theta} = \dfrac{dy}{d\theta} = 0$ when $\theta = 0, \pm\pi, \pm 2\pi, \ldots$



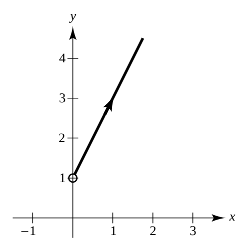
(c) $x = e^{-t} \qquad y = 2e^{-t} + 1$

$x > 0 \qquad y > 1$



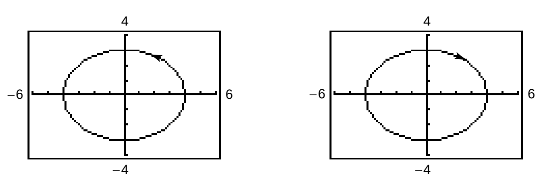
(d) $x = e^{t} \qquad y = 2e^{t} + 1$

$x > 0 \qquad y > 1$



35. The curves are identical on $0 < \theta < \pi$. They are both smooth. Represent $y = 2(1 - x^2)$

37. (a)



(b) The orientation of the second curve is reversed.

(c) The orientation will be reversed.

(d) Many answers possible. For example, $x = 1 + t$, $y = 1 + 2t$, and $x = 1 - t$, $x = 1 - 2t$.

39.

$$x = x_1 + t(x_2 - x_1)$$
$$y = y_1 + t(y_2 - y_1)$$
$$\frac{x - x_1}{x_2 - x_1} = t$$
$$y = y_1 + \left(\frac{x - x_1}{x_2 - x_1}\right)(y_2 - y_1)$$
$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$
$$y - y_1 = m(x - x_1)$$

41.

$$x = h + a\cos\theta$$
$$y = k + b\sin\theta$$
$$\frac{x - h}{a} = \cos\theta$$
$$\frac{y - k}{b} = \sin\theta$$
$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

43. From Exercise 39 we have

$x = 5t$

$y = -2t.$

Solution not unique

45. From Exercise 40 we have

$x = 2 + 4\cos\theta$

$y = 1 + 4\sin\theta.$

Solution not unique

47. From Exercise 41 we have

$a = 5, c = 4 \implies b = 3$

$x = 5\cos\theta$

$y = 3\sin\theta.$

Center: $(0, 0)$
Solution not unique

49. From Exercise 42 we have

$a = 4, c = 5 \implies b = 3$

$x = 4 \sec \theta$

$y = 3 \tan \theta.$

Center: $(0, 0)$
Solution not unique

51. $y = 3x - 2$

Example

$x = t, \quad y = 3t - 2$

$x = t - 3, \quad y = 3t - 11$

53. $y = x^3$

Example

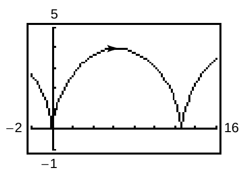
$x = t, \quad y = t^3$

$x = \sqrt[3]{t}, \quad y = t$

$x = \tan t, \quad y = \tan^3 t$

55. $x = 2(\theta - \sin \theta)$

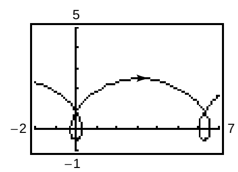
$y = 2(1 - \cos \theta)$



Not smooth at $\theta = 2n\pi$

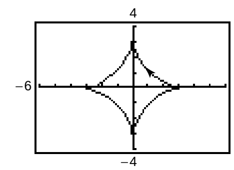
57. $x = \theta - \frac{3}{2} \sin \theta$

$y = 1 - \frac{3}{2} \cos \theta$



59. $x = 3 \cos^3 \theta$

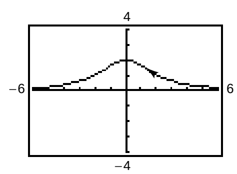
$y = 3 \sin^3 \theta$



Not smooth at $(x, y) = (\pm 3, 0)$ and $(0, \pm 3)$, or $\theta = \frac{1}{2}n\pi$.

61. $x = 2 \cot \theta$

$y = 2 \sin^2 \theta$



Smooth everywhere

63. See definition on page 665.

65. A plane curve C, represented by $x = f(t), y = g(t)$, is smooth if f' and g' are continuous and not simultaneously 0. See page 670.

67. $x = 4 \cos \theta$

$y = 2 \sin 2\theta$

Matches (d)

69. $x = \cos \theta + \theta \sin \theta$

$y = \sin \theta - \theta \cos \theta$

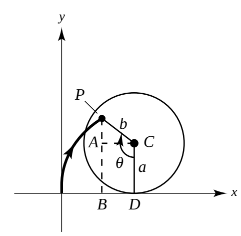
Matches (b)

71. When the circle has rolled θ radians, we know that the center is at $(a\theta, a)$.

$$\sin \theta = \sin(180° - \theta) = \frac{|AC|}{b} = \frac{|BD|}{b} \quad \text{or} \quad |BD| = b \sin \theta$$

$$\cos \theta = -\cos(180° - \theta) = \frac{|AP|}{-b} \quad \text{or} \quad |AP| = -b \cos \theta$$

Therefore, $x = a\theta - b \sin \theta$ and $y = a - b \cos \theta$.



73. False

$x = t^2 \implies x \geq 0$

$x = t^2 \implies y \geq 0$

The graph of the parametric equations is only a portion of the line $y = x$.

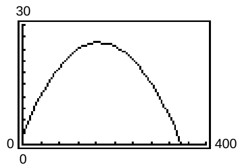
75. (a) $100 \text{ mi/hr} = \frac{(100)(5280)}{3600} = \frac{440}{3} \text{ ft/sec}$

$$x = (v_0 \cos\theta)t = \left(\frac{440}{3}\cos\theta\right)t$$

$$y = h + (v_0 \sin\theta)t - 16t^2$$

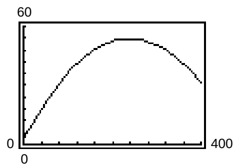
$$= 3 + \left(\frac{440}{3}\sin\theta\right)t - 16t^2$$

(b)



It is not a home run—when $x = 400$, $y \le 20$.

(c)



Yes, it's a home run when $x = 400$, $y > 10$.

(d) We need to find the angle θ (and time t) such that

$$x = \left(\frac{440}{3}\cos\theta\right)t = 400$$

$$y = 3 + \left(\frac{440}{3}\sin\theta\right)t - 16t^2 = 10.$$

From the first equation $t = 1200/440\cos\theta$. Substituting into the second equation,

$$10 = 3 + \left(\frac{440}{3}\sin\theta\right)\left(\frac{1200}{440\cos\theta}\right) - 16\left(\frac{1200}{440\cos\theta}\right)^2$$

$$7 = 400\tan\theta - 16\left(\frac{120}{44}\right)^2 \sec^2\theta$$

$$= 400\tan\theta - 16\left(\frac{120}{44}\right)^2(\tan^2\theta + 1).$$

We now solve the quadratic for $\tan\theta$:

$$16\left(\frac{120}{44}\right)^2 \tan^2\theta - 400\tan\theta + 7 + 16\left(\frac{120}{44}\right)^2 = 0$$

$\tan\theta \approx 0.35185 \implies \theta \approx 19.4°$

Section 9.3 Parametric Equations and Calculus

1. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-4}{2t} = \frac{-2}{t}$

3. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2\cos t\sin t}{2\sin t\cos t} = -1$

$\left[\text{Note: } x + y = 1 \implies y = 1 - x \text{ and } \frac{dy}{dt} = -1\right]$

5. $x = 2t,\ y = 3t - 1$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3}{2}$$

$\frac{d^2y}{dx^2} = 0$ Line

7. $x = t + 1,\ y = t^2 + 3t$

$\frac{dy}{dx} = \frac{2t + 3}{1} = 1$ when $t = -1$.

$\frac{d^2y}{dx^2} = 2$ concave upwards

9. $x = 2\cos\theta,\ y = 2\sin\theta$

$\frac{dy}{dx} = \frac{2\cos\theta}{-2\sin\theta} = -\cot\theta = -1$ when $\theta = \frac{\pi}{4}$.

$\frac{d^2y}{dx^2} = \frac{\csc^2\theta}{-2\sin\theta} = \frac{-\csc^3\theta}{2} = -\sqrt{2}$ when $\theta = \frac{\pi}{4}$.

concave downward

11. $x = 2 + \sec\theta,\ y = 1 + 2\tan\theta$

$$\frac{dy}{dx} = \frac{2\sec^2\theta}{\sec\theta\tan\theta}$$

$= \frac{2\sec\theta}{\tan\theta} = 2\csc\theta = 4$ when $\theta = \frac{\pi}{6}$.

$$\frac{d^2y}{dx^2} = \frac{-2\csc\theta\cot\theta}{\sec\theta\tan\theta}$$

$= -2\cot^3\theta = -6\sqrt{3}$ when $\theta = \frac{\pi}{6}$.

concave downward

13. $x = \cos^3 \theta, \ y = \sin^3 \theta$

$$\frac{dy}{dx} = \frac{3 \sin^2 \theta \cos \theta}{-3 \cos^2 \theta \sin \theta}$$

$$= -\tan \theta = -1 \text{ when } \theta = \frac{\pi}{4}.$$

$$\frac{d^2y}{dx^2} = \frac{-\sec^2 \theta}{-3 \cos^2 \theta \sin \theta} = \frac{1}{3 \cos^4 \theta \sin \theta}$$

$$= \frac{\sec^4 \theta \csc \theta}{3} = \frac{4\sqrt{2}}{3} \text{ when } \theta = \frac{\pi}{4}.$$

concave upward

15. $x = 2 \cot \theta, y = 2 \sin^2 \theta$

$$\frac{dy}{dx} = \frac{4 \sin \theta \cos \theta}{-2 \csc^2 \theta} = -2 \sin^3 \theta \cos \theta$$

At $\left(-\frac{2}{\sqrt{3}}, \frac{3}{2}\right)$, $\theta = \frac{2\pi}{3}$, and $\frac{dy}{dx} = \frac{3\sqrt{3}}{8}$.

Tangent line: $$y - \frac{3}{2} = \frac{3\sqrt{3}}{8}\left(x + \frac{2}{\sqrt{3}}\right)$$

$$3\sqrt{3}x - 8y + 18 = 0$$

At $(0, 2)$, $\theta = \frac{\pi}{2}$, and $\frac{dy}{dx} = 0$.

Tangent line: $y - 2 = 0$

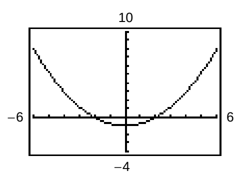
At $\left(2\sqrt{3}, \frac{1}{2}\right)$, $\theta = \frac{\pi}{6}$, and $\frac{dy}{dx} = -\frac{\sqrt{3}}{8}$.

Tangent line: $$y - \frac{1}{2} = -\frac{\sqrt{3}}{8}\left(x - 2\sqrt{3}\right)$$

$$\sqrt{3}x + 8y - 10 = 0$$

17. $x = 2t, \ y = t^2 - 1, \ t = 2$

(a)



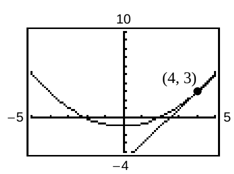
(b) At $t = 2$, $(x, y) = (4, 3)$, and

$$\frac{dx}{dt} = 2, \ \frac{dy}{dt} = 4, \ \frac{dy}{dx} = 2$$

(c) $\frac{dy}{dx} = 2$. At $(4, 3)$, $y - 3 = 2(x - 4)$

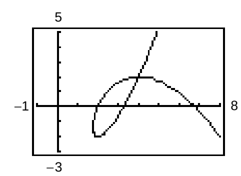
$$y = 2x - 5$$

(d)



19. $x = t^2 - t + 2, \ y = t^3 - 3t, \ t = -1$

(a)



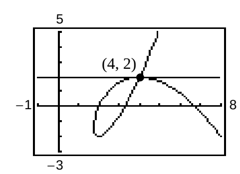
(b) At $t = -1$, $(x, y) = (4, 2)$, and

$$\frac{dx}{dt} = -3, \ \frac{dy}{dt} = 0, \ \frac{dy}{dx} = 0$$

(c) $\frac{dy}{dx} = 0$. At $(4, 2)$, $y - 2 = 0(x - 4)$

$$y = 2$$

(d)



21. $x = 2 \sin 2t, y = 3 \sin t$ crosses itself at the origin, $(x, y) = (0, 0)$.

At this point, $t = 0$ or $t = \pi$.

$$\frac{dy}{dx} = \frac{3 \cos t}{4 \cos 2t}$$

At $t = 0$: $\frac{dy}{dx} = \frac{3}{4}$ and $y = \frac{3}{4}x$. Tangent Line

At $t = \pi$, $\frac{dy}{dx} = -\frac{3}{4}$ and $y = \frac{-3}{4}x$ Tangent Line

23. $x = \cos\theta + \theta\sin\theta,\ y = \sin\theta - \theta\cos\theta$

Horizontal tangents: $\dfrac{dy}{d\theta} = \theta\sin\theta = 0$ when $\theta = 0,\ \pi,\ 2\pi,\ 3\pi, \ldots.$

Points: $(-1, [2n-1]\pi),\ (1, 2n\pi)$ where n is an integer.

Points shown: $(1, 0),\ (-1, \pi),\ (1, -2\pi)$

Vertical tangents: $\dfrac{dx}{d\theta} = \theta\cos\theta = 0$ when $\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \ldots.$

Points: $\left(\dfrac{(-1)^{n+1}(2n-1)\pi}{2}, (-1)^{n+1}\right)$

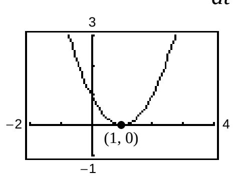
Points shown: $\left(\dfrac{\pi}{2}, 1\right),\ \left(-\dfrac{3\pi}{2}, -1\right),\ \left(\dfrac{5\pi}{2}, 1\right)$

25. $x = 1 - t,\ y = t^2$

Horizontal tangents: $\dfrac{dy}{dt} = 2t = 0$ when $t = 0.$

Point: $(1, 0)$

Vertical tangents: $\dfrac{dx}{dt} = -1 \neq 0$; none

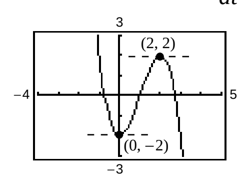


27. $x = 1 - t,\ y = t^3 - 3t$

Horizontal tangents: $\dfrac{dy}{dt} = 3t^2 - 3 = 0$ when $t = \pm 1.$

Points: $(0, -2), (2, 2)$

Vertical tangents: $\dfrac{dx}{dt} = -1 \neq 0$; none



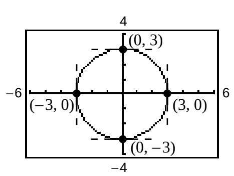
29. $x = 3\cos\theta, y = 3\sin\theta$

Horizontal tangents: $\dfrac{dy}{d\theta} = 3\cos\theta = 0$ when $\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}.$

Points: $(0, 3), (0, -3)$

Vertical tangents: $\dfrac{dx}{d\theta} = -3\sin\theta = 0$ when $\theta = 0, \pi.$

Points: $(3, 0), (-3, 0)$



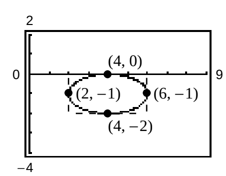
31. $x = 4 + 2\cos\theta,\ y = -1 + \sin\theta$

Horizontal tangents: $\dfrac{dy}{d\theta} = \cos\theta = 0$ when $\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}.$

Points: $(4, 0), (4, -2)$

Vertical tangents: $\dfrac{dx}{d\theta} = -2\sin\theta = 0$ when $x = 0, \pi.$

Points: $(6, -1), (2, -1)$

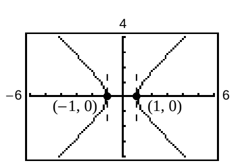


33. $x = \sec\theta,\ y = \tan\theta$

Horizontal tangents: $\dfrac{dy}{d\theta} = \sec^2\theta \neq 0$; none

Vertical tangents: $\dfrac{dx}{d\theta} = \sec\theta\tan\theta = 0$ when $x = 0, \pi.$

Points: $(1, 0), (-1, 0)$



35. $x = t^2, y = 2t, 0 \le t \le 2$

$$\frac{dx}{dt} = 2t, \frac{dy}{dt} = 2, \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4t^2 + 4 = 4(t^2 + 1)$$

$$s = 2\int_0^2 \sqrt{t^2 + 1}\, dt$$

$$= \left[t\sqrt{t^2 + 1} + \ln\left|t + \sqrt{t^2 + 1}\right|\right]_0^2$$

$$= 2\sqrt{5} + \ln\left(2 + \sqrt{5}\right) \approx 5.916$$

37. $x = e^{-t}\cos t,\ y = e^{-t}\sin t,\ 0 \le t \le \dfrac{\pi}{2}$

$$\frac{dx}{dt} = -e^{-t}(\sin t + \cos t),\ \frac{dy}{dt} = e^{-t}(\cos t - \sin t)$$

$$s = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\pi/2} \sqrt{2e^{-2t}}\, dt = -\sqrt{2}\int_0^{\pi/2} e^{-t}(-1)\, dt$$

$$= \Big[-\sqrt{2}e^{-t}\Big]_0^{\pi/2} = \sqrt{2}(1 - e^{-\pi/2}) \approx 1.12$$

39. $x = \sqrt{t},\ y = 3t - 1,\ \dfrac{dx}{dt} = \dfrac{1}{2\sqrt{t}},\ \dfrac{dy}{dt} = 3$

$$S = \int_0^1 \sqrt{\frac{1}{4t} + 9}\, dt = \frac{1}{2}\int_0^1 \frac{\sqrt{1 + 36t}}{\sqrt{t}}\, dt$$

$$= \frac{1}{6}\int_0^6 \sqrt{1 + u^2}\, du$$

$$= \frac{1}{12}\Big[\ln\left(\sqrt{1 + u^2} + u\right) + u\sqrt{1 + u^2}\Big]_0^6$$

$$= \frac{1}{12}\Big[\ln\left(\sqrt{37} + 6\right) + 6\sqrt{37}\Big] \approx 3.249$$

$$u = 6\sqrt{t},\ du = \frac{3}{\sqrt{t}}\, dt$$

41. $x = a\cos^3\theta,\ y = a\sin^3\theta,\ \dfrac{dx}{d\theta} = -3a\cos^2\theta\sin\theta,$

$$\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$S = 4\int_0^{\pi/2} \sqrt{9a^2\cos^4\theta\sin^2 + 9a^2\sin^4\theta\cos^2\theta}\, d\theta$$

$$= 12a\int_0^{\pi/2} \sin\theta\cos\theta\sqrt{\cos^2\theta + \sin^2\theta}\, d\theta$$

$$= 6a\int_0^{\pi/2} \sin 2\theta\, d\theta = \Big[-3a\cos 2\theta\Big]_0^{\pi/2} = 6a$$

43. $x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta),$

$$\frac{dx}{d\theta} = a(1 - \cos\theta),\ \frac{dy}{d\theta} = a\sin\theta$$

$$S = 2\int_0^{\pi} \sqrt{a^2(1 - \cos\theta)^2 + a^2\sin^2\theta}\, d\theta$$

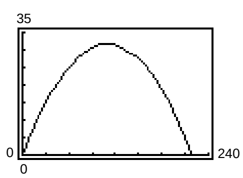
$$= 2\sqrt{2}a\int_0^{\pi} \sqrt{1 - \cos\theta}\, d\theta$$

$$= 2\sqrt{2}a\int_0^{\pi} \frac{\sin\theta}{\sqrt{1 + \cos\theta}}\, d\theta$$

$$= \Big[-4\sqrt{2}a\sqrt{1 + \cos\theta}\Big]_0^{\pi} = 8a$$

45. $x = (90\cos 30°)t,\ y = (90\sin 30°)t - 16t^2$

(a)



(b) Range: 219.2 ft

(c) $\dfrac{dx}{dt} = 90\cos 30°,\ \dfrac{dy}{dt} = 90\sin 30° - 32t.$

$y = 0$ for $t = \dfrac{45}{16}.$

$$s = \int_0^{45/16} \sqrt{(90\cos 30°)^2 + (90\sin 30° - 32t)^2}\, dt$$

$$= 230.8 \text{ ft}$$

(d) $y = 0 \implies (90\sin\theta)t = 16t^2 \implies t = \dfrac{90}{16}\sin\theta$

$$x = (90\cos\theta)t = \frac{90^2}{16}\cos\theta\sin\theta = \frac{90^2}{32}\sin 2\theta$$

$$x'(\theta) = \frac{90^2}{32}2\cos 2\theta = 0 \implies \theta = 45°$$

By the First Derivative Test, $\theta = 45°\left(\dfrac{\pi}{4}\right)$ maximizes the range.

$$\frac{dx}{dt} = 90\cos\theta,$$

$$\frac{dy}{dt} = 90\sin\theta - 32 = 90\sin\theta - 32\left(\frac{90}{16}\sin\theta\right) = -90\sin\theta$$

$$s = \int_0^{(90/16)\sin\theta} \sqrt{(90\cos\theta)^2 + (-90\sin\theta)^2}\, dt$$

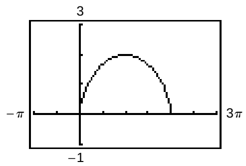
$$= \int_0^{(90/16)\sin\theta} 90\, dt = 90t\Big]_0^{(90/16)\sin\theta}$$

$$= \frac{90^2}{16}\sin\theta$$

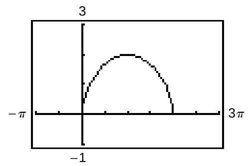
$$\frac{ds}{d\theta} = \frac{90^2}{16}\cos\theta = 0 \implies \theta = \frac{\pi}{2}$$

By the First Derivative Test, $\theta = 90°$ maximizes the arc length.

47. (a) $x = t - \sin t$, $y = 1 - \cos t$, $0 \le t \le 2\pi$



$x = 2t - \sin(2t)$, $y = 1 - \cos(2t)$, $0 \le t \le \pi$



(b) The average speed of the particle on the second path is twice the average speed of a particle on the first path.

(c) $x = \frac{1}{2}t - \sin\left(\frac{1}{2}t\right)$

$y = 1 - \cos\left(\frac{1}{2}t\right)$

The time required for the particle to traverse the same path is $t = 4\pi$.

49. $x = t,\ y = 2t,\ \dfrac{dx}{dt} = 1,\ \dfrac{dy}{dt} = 2$

(a) $S = 2\pi\displaystyle\int_0^4 2t\sqrt{1+4}\,dt = 4\sqrt{5}\pi\int_0^4 t\,dt$

$= \Big[2\sqrt{5}\pi t^2\Big]_0^4 = 32\pi\sqrt{5}$

(b) $S = 2\pi\displaystyle\int_0^4 t\sqrt{1+4}\,dt = 2\sqrt{5}\pi\int_0^4 t\,dt$

$= \Big[\sqrt{5}\pi t^2\Big]_0^4 = 16\pi\sqrt{5}$

51. $x = 4\cos\theta,\ y = 4\sin\theta,\ \dfrac{dx}{d\theta} = -4\sin\theta,\ \dfrac{dy}{d\theta} = 4\cos\theta$

$$S = 2\pi\int_0^{\pi/2} 4\cos\theta\sqrt{(-4\sin\theta)^2 + (4\cos\theta)^2}\,d\theta$$

$$= 32\pi\int_0^{\pi/2}\cos\theta\,d\theta = \Big[32\pi\sin\theta\Big]_0^{\pi/2} = 32\pi$$

53. $x = a\cos^3\theta,\ y = a\sin^3\theta,\ \dfrac{dx}{d\theta} = -3a\cos^2\theta\sin\theta,\ \dfrac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

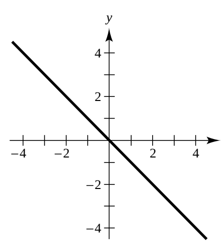
$$S = 4\pi\int_0^{\pi/2} a\sin^3\theta\sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\,d\theta = 12a^2\pi\int_0^{\pi/2}\sin^4\theta\cos\theta\,d\theta = \frac{12\pi a^2}{5}\Big[\sin^5\theta\Big]_0^{\pi/2} = \frac{12}{5}\pi a^2$$

55. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$

See Theorem 9.7, page 675.

57. One possible answer is the graph given by

$x = t,\ y = -t.$



59. $s = \displaystyle\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

See Theorem 9.8, page 678.

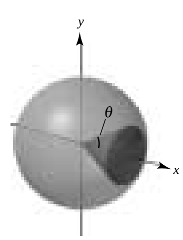
61. $x = r\cos\phi,\ y = r\sin\phi$

$S = 2\pi\displaystyle\int_0^\theta r\sin\phi\sqrt{r^2\sin^2\phi + r^2\cos^2\phi}\,d\phi$

$= 2\pi r^2\displaystyle\int_0^\theta \sin\phi\,d\phi$

$= \Big[-2\pi r^2\cos\phi\Big]_0^\theta$

$= 2\pi r^2(1 - \cos\theta)$



63. $x = \sqrt{t},\ y = 4 - t,\ 0 \le t \le 4$

$$A = \int_0^4 (4-t)\frac{1}{2\sqrt{t}}\,dt = \frac{1}{2}\int_0^4 (4t^{-1/2} - t^{1/2})\,dt = \left[\frac{1}{2}\left(8\sqrt{t} - \frac{2}{3}t\sqrt{t}\right)\right]_0^4 = \frac{16}{3}$$

$$\bar{x} = \frac{3}{16}\int_0^4 (4-t)\sqrt{t}\left(\frac{1}{2\sqrt{t}}\right)dt = \frac{3}{32}\int_0^4 (4-t)\,dt = \left[\frac{3}{32}\left(4t - \frac{t^2}{2}\right)\right]_0^4 = \frac{3}{4}$$

$$\bar{y} = \frac{3}{32}\int_0^4 (4-t)^2\frac{1}{2\sqrt{t}}\,dt = \frac{3}{64}\int_0^4 [16t^{-1/2} - 8t^{1/2} + t^{3/2}]\,dt = \frac{3}{64}\left[32\sqrt{t} - \frac{16}{3}t\sqrt{t} + \frac{2}{5}t^2\sqrt{t}\right]_0^4 = \frac{8}{5}$$

$$(\bar{x}, \bar{y}) = \left(\frac{3}{4}, \frac{8}{5}\right)$$

65. $x = 3\cos\theta,\ y = 3\sin\theta,\ \dfrac{dx}{d\theta} = -3\sin\theta$

$$\begin{aligned} V &= 2\pi\int_{\pi/2}^0 (3\sin\theta)^2(-3\sin\theta)\,d\theta \\ &= -54\pi\int_{\pi/2}^0 \sin^3\theta\,d\theta \\ &= -54\pi\int_{\pi/2}^0 (1-\cos^2\theta)\sin\theta\,d\theta \\ &= -54\pi\left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_{\pi/2}^0 = 36\pi \end{aligned}$$

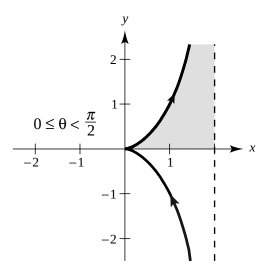
67. $x = 2\sin^2\theta$

$y = 2\sin^2\theta\tan\theta$

$\dfrac{dx}{d\theta} = 4\sin\theta\cos\theta$

$$A = \int_0^{\pi/2} 2\sin^2\theta\tan\theta(4\sin\theta\cos\theta)\,d\theta = 8\int_0^{\pi/2}\sin^4\theta\,d\theta$$

$$= 8\left[\frac{-\sin^3\theta\cos\theta}{4} - \frac{3}{8}\sin\theta\cos\theta + \frac{3}{8}\theta\right]_0^{\pi/2} = \frac{3\pi}{2}$$



69. πab is area of ellipse (d).

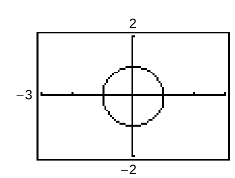
71. $6\pi a^2$ is area of cardioid (f).

73. $\frac{8}{3}ab$ is area of hourglass (a).

75. (a) $x = \dfrac{1-t^2}{1+t^2},\ y = \dfrac{2t}{1+t^2},\ -20 \le t \le 20$

The graph is the circle $x^2 + y^2 = 1$, except the point $(-1, 0)$.

Verify: $x^2 + y^2 = \left(\dfrac{1-t^2}{1+t^2}\right)^2 + \left(\dfrac{2t}{1+t^2}\right)^2 = \dfrac{1 - 2t^2 + t^4 + 4t^2}{(1+t^2)^2} = \dfrac{(1+t^2)^2}{(1+t^2)^2} = 1$



(b) As t increases from -20 to 0, the speed increases, and as t increases from 0 to 20, the speed decreases.

77. False

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left[\dfrac{g'(t)}{f'(t)}\right]}{f'(t)} = \frac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^3}$$

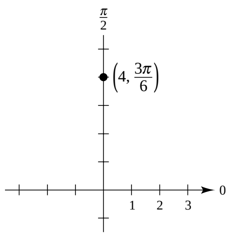
Section 9.4 Polar Coordinates and Polar Graphs

1. $\left(4, \frac{\pi}{2}\right)$

$x = 4\cos\left(\frac{\pi}{2}\right) = 0$

$y = 4\sin\left(\frac{\pi}{2}\right) = 4$

$(x, y) = (0, 4)$

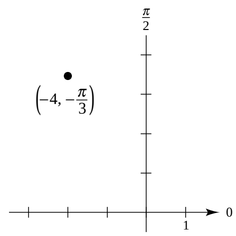


3. $\left(-4, -\frac{\pi}{3}\right)$

$x = -4\cos\left(-\frac{\pi}{3}\right) = -2$

$y = -4\sin\left(-\frac{\pi}{3}\right) = 2\sqrt{3}$

$(x, y) = \left(-2, 2\sqrt{3}\right)$

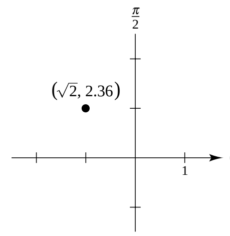


5. $\left(\sqrt{2}, 2.36\right)$

$x = \sqrt{2}\cos(2.36) \approx -1.004$

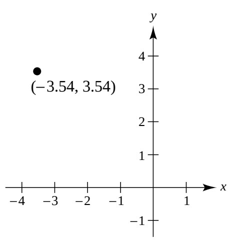
$y = \sqrt{2}\sin(2.36) \approx 0.996$

$(x, y) = (-1.004, 0.996)$



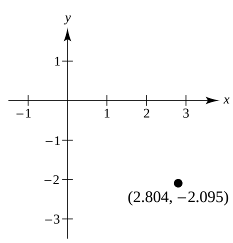
7. $(r, \theta) = \left(5, \frac{3\pi}{4}\right)$

$(x, y) = (-3.5355, 3.5355)$



9. $(r, \theta) = (-3.5, 2.5)$

$(x, y) = (2.804, -2.095)$

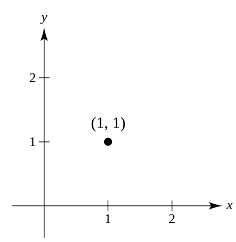


11. $(x, y) = (1, 1)$

$r = \pm\sqrt{2}$

$\tan\theta = 1$

$\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \left(\sqrt{2}, \frac{\pi}{4}\right), \left(-\sqrt{2}, \frac{5\pi}{4}\right)$

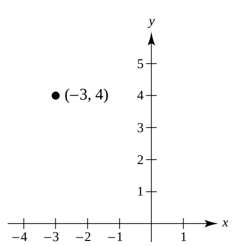


13. $(x, y) = (-3, 4)$

$r = \pm\sqrt{9 + 16} = \pm 5$

$\tan\theta = -\frac{4}{3}$

$\theta \approx 2.214, 5.356, (5, 2.214), (-5, 5.356)$



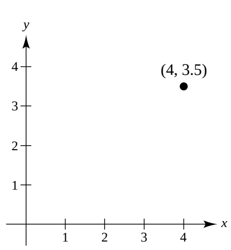
15. $(x, y) = (3, -2)$

$(r, \theta) = (3.606, -0.588)$

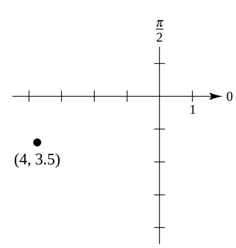
17. $(x, y) = \left(\frac{5}{2}, \frac{4}{3}\right)$

$(r, \theta) = (2.833, 0.490)$

19. (a) $(x, y) = (4, 3.5)$

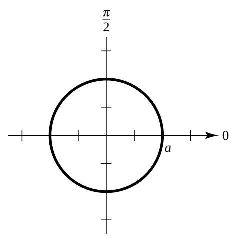


(b) $(r, \theta) = (4, 3.5)$



21. $x^2 + y^2 = a^2$

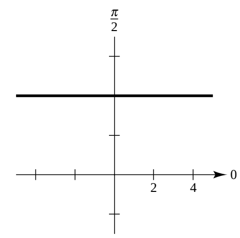
$r = a$



23. $y = 4$

$r \sin \theta = 4$

$r = 4 \csc \theta$

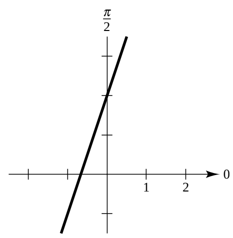


25. $3x - y + 2 = 0$

$3r \cos \theta - r \sin \theta + 2 = 0$

$r(3 \cos \theta - \sin \theta) = -2$

$$r = \frac{-2}{3 \cos \theta - \sin \theta}$$

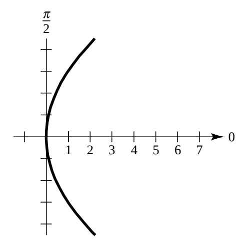


27. $y^2 = 9x$

$r^2 \sin^2 \theta = 9r \cos \theta$

$$r = \frac{9 \cos \theta}{\sin^2 \theta}$$

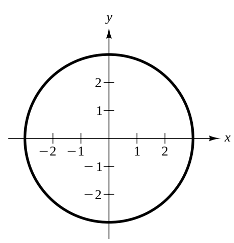
$r = 9 \csc^2 \theta \cos \theta$



29. $r = 3$

$r^2 = 9$

$x^2 + y^2 = 9$



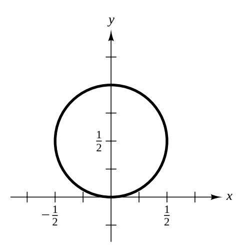
31. $r = \sin \theta$

$r^2 = r \sin \theta$

$x^2 + y^2 = y$

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$$

$x^2 + y^2 - y = 0$

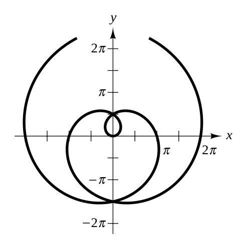


33. $r = \theta$

$\tan r = \tan \theta$

$$\tan \sqrt{x^2 + y^2} = \frac{y}{x}$$

$$\sqrt{x^2 + y^2} = \arctan \frac{y}{x}$$

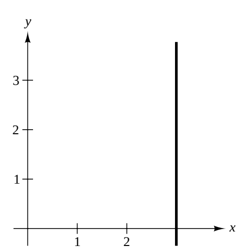


35. $r = 3 \sec \theta$

$r \cos \theta = 3$

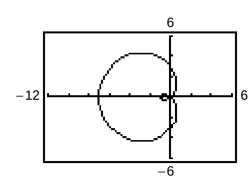
$x = 3$

$x - 3 = 0$



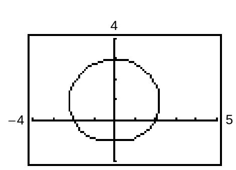
37. $r = 3 - 4 \cos \theta$

$0 \le \theta < 2\pi$



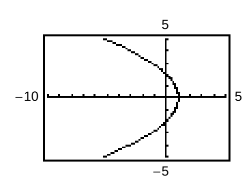
39. $r = 2 + \sin \theta$

$0 \le \theta < 2\pi$



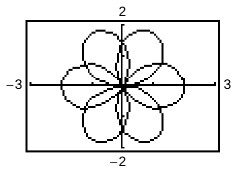
41. $r = \dfrac{2}{1 + \cos \theta}$

Traced out once on $-\pi < \theta < \pi$



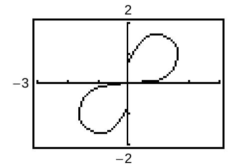
43. $r = 2\cos\left(\frac{3\theta}{2}\right)$

$0 \le \theta < 4\pi$



45. $r^2 = 4\sin 2\theta$

$0 \le \theta < \frac{\pi}{2}$



47.

$$\begin{aligned} r &= 2(h\cos\theta + k\sin\theta) \\ r^2 &= 2r(h\cos\theta + k\sin\theta) \\ r^2 &= 2[h(r\cos\theta) + k(r\sin\theta)] \\ x^2 + y^2 &= 2(hx + ky) \\ x^2 + y^2 - 2hx - 2ky &= 0 \\ (x^2 - 2hx + h^2) + (y^2 - 2ky + k^2) &= 0 + h^2 + k^2 \\ (x - h)^2 + (y - k)^2 &= h^2 + k^2 \end{aligned}$$

Radius: $\sqrt{h^2 + k^2}$

Center: (h, k)

49. $\left(4, \frac{2\pi}{3}\right), \left(2, \frac{\pi}{6}\right)$

$$d = \sqrt{4^2 + 2^2 - 2(4)(2)\cos\left(\frac{2\pi}{3} - \frac{\pi}{6}\right)}$$

$$= \sqrt{20 - 16\cos\frac{\pi}{2}} = 2\sqrt{5} \approx 4.5$$

51. $(2, 0.5),\ (7, 1.2)$

$$d = \sqrt{2^2 + 7^2 - 2(2)(7)\cos(0.5 - 1.2)}$$

$$= \sqrt{53 - 28\cos(-0.7)} \approx 5.6$$

53. $r = 2 + 3\sin\theta$

$$\frac{dy}{dx} = \frac{3\cos\theta\sin\theta + \cos\theta(2 + 3\sin\theta)}{3\cos\theta\cos\theta - \sin\theta(2 + 3\sin\theta)}$$

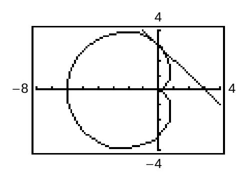
$$= \frac{2\cos\theta(3\sin\theta + 1)}{3\cos 2\theta - 2\sin\theta} = \frac{2\cos\theta(3\sin\theta + 1)}{6\cos^2\theta - 2\sin\theta - 3}$$

At $\left(5, \frac{\pi}{2}\right)$, $\frac{dy}{dx} = 0$.

At $(2, \pi)$, $\frac{dy}{dx} = -\frac{2}{3}$.

At $\left(-1, \frac{3\pi}{2}\right)$, $\frac{dy}{dx} = 0$.

55. (a), (b) $r = 3(1 - \cos\theta)$



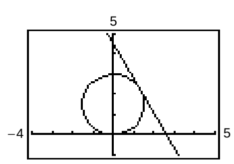
$(r, \theta) = \left(3, \frac{\pi}{2}\right) \implies (x, y) = (0, 3)$

Tangent line: $y - 3 = -1(x - 0)$

$y = -x + 3$

(c) At $\theta = \frac{\pi}{2}, \frac{dy}{dx} = -1.0$.

57. (a), (b) $r = 3\sin\theta$



$(r, \theta) = \left(\frac{3\sqrt{3}}{2}, \frac{\pi}{3}\right) \implies (x, y) = \left(\frac{3\sqrt{3}}{4}, \frac{9}{4}\right)$

Tangent line: $y - \frac{9}{4} = -\sqrt{3}\left(x - \frac{3\sqrt{3}}{4}\right)$

$$y = -\sqrt{3}x + \frac{9}{2}$$

(c) At $\theta = \frac{\pi}{3}, \frac{dy}{dx} = -\sqrt{3} \approx -1.732$.

59. $r = 1 - \sin\theta$

$$\frac{dy}{d\theta} = (1 - \sin\theta)\cos\theta - \cos\theta\sin\theta$$

$$= \cos\theta(1 - 2\sin\theta) = 0$$

$\cos\theta = 0, \sin\theta = \frac{1}{2} \implies \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$

Horizontal tangents: $\left(2, \frac{3\pi}{2}\right), \left(\frac{1}{2}, \frac{\pi}{6}\right), \left(\frac{1}{2}, \frac{5\pi}{6}\right)$

$$\frac{dx}{d\theta} = (-1 + \sin\theta)\sin\theta - \cos\theta\cos\theta$$

$$= -\sin\theta + \sin^2\theta + \sin^2\theta - 1$$

$$= 2\sin^2\theta - \sin\theta - 1$$

$$= (2\sin\theta + 1)(\sin\theta - 1) = 0$$

$\sin\theta = 1, \sin\theta = -\frac{1}{2} \implies \theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

Vertical tangents: $\left(\frac{3}{2}, \frac{7\pi}{6}\right), \left(\frac{3}{2}, \frac{11\pi}{6}\right)$

61. $r = 2\csc\theta + 3$

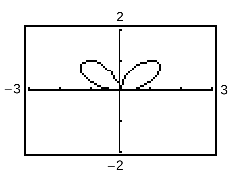
$$\frac{dy}{d\theta} = (2\csc\theta + 3)\cos\theta + (-2\csc\theta\cot\theta)\sin\theta$$

$$= 3\cos\theta = 0$$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

Horizontal: $\left(5, \frac{\pi}{2}\right), \left(1, \frac{3\pi}{2}\right)$

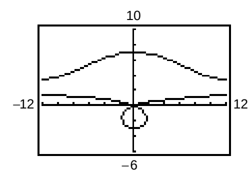
63. $r = 4\sin\theta\cos^2\theta$



Horizontal tangents:

(0, 0), (1.4142, 0.7854), (1.4142, 2.3562)

65. $r = 2\csc\theta + 5$



Horizontal tangents: $\left(7, \frac{\pi}{2}\right), \left(3, \frac{3\pi}{2}\right)$

67.

$$r = 3\sin\theta$$

$$r^2 = 3r\sin\theta$$

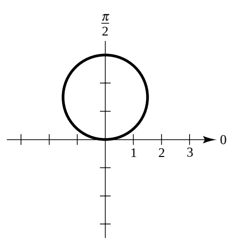
$$x^2 + y^2 = 3y$$

$$x^2 + \left(y - \frac{3}{2}\right)^2 = \frac{9}{4}$$

Circle $r = \frac{3}{2}$

Center: $\left(0, \frac{3}{2}\right)$

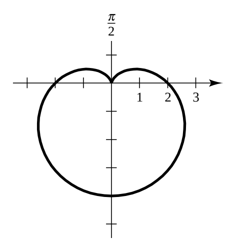
Tangent at the pole: $\theta = 0$



69. $r = 2(1 - \sin\theta)$

Cardioid

Symmetric to y-axis, $\theta = \frac{\pi}{2}$



71. $r = 2\cos(3\theta)$

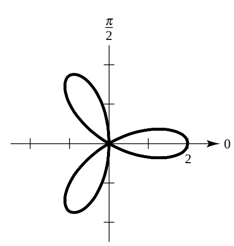
Rose curve with three petals

Symmetric to the polar axis

Relative extrema: (2, 0), $\left(-2, \frac{\pi}{3}\right), \left(2, \frac{2\pi}{3}\right)$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	2	0	$-\sqrt{2}$	-2	0	2	0	-2

Tangents at the pole: $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$



73. $r = 3 \sin 2\theta$

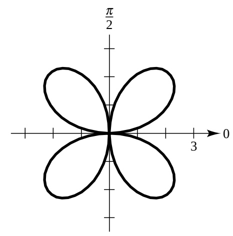
Rose curve with four petals

Symmetric to the polar axis, $\theta = \dfrac{\pi}{2}$, and pole

Relative extrema: $\left(\pm 3, \dfrac{\pi}{4}\right), \left(\pm 3, \dfrac{5\pi}{4}\right)$

Tangents at the pole: $\theta = 0, \dfrac{\pi}{2}$

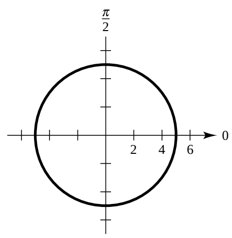
($\theta = \pi,\ 3\pi/2$ give the same tangents.)



75. $r = 5$

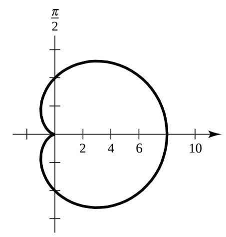
Circle radius: 5

$x^2 + y^2 = 25$



77. $r = 4(1 + \cos \theta)$

Cardioid

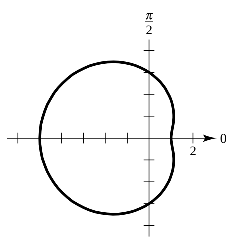


79. $r = 3 - 2 \cos \theta$

Limaçon

Symmetric to polar axis

θ	0	$\dfrac{\pi}{3}$	$\dfrac{\pi}{2}$	$\dfrac{2\pi}{3}$	π
r	1	2	3	4	5

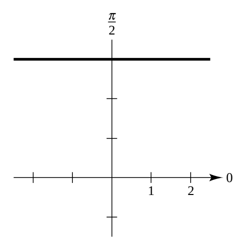


81. $r = 3 \csc \theta$

$r \sin \theta = 3$

$y = 3$

Horizontal line



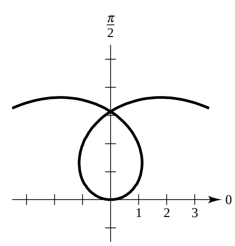
83. $r = 2\theta$

Spiral of Archimedes

Symmetric to $\theta = \dfrac{\pi}{2}$

θ	0	$\dfrac{\pi}{4}$	$\dfrac{\pi}{2}$	$\dfrac{3\pi}{4}$	π	$\dfrac{5\pi}{4}$	$\dfrac{3\pi}{2}$
r	0	$\dfrac{\pi}{2}$	π	$\dfrac{3\pi}{2}$	2π	$\dfrac{5\pi}{2}$	3π

Tangent at the pole: $\theta = 0$



85. $r^2 = 4 \cos(2\theta)$

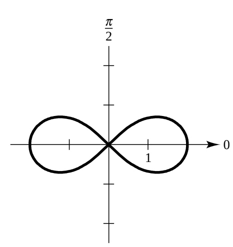
Lemniscate

Symmetric to the polar axis, $\theta = \dfrac{\pi}{2}$, and pole

Relative extrema: $(\pm 2, 0)$

θ	0	$\dfrac{\pi}{6}$	$\dfrac{\pi}{4}$
r	± 2	$\pm\sqrt{2}$	0

Tangents at the pole: $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$

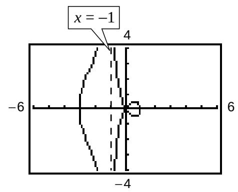


87. Since

$$r = 2 - \sec\theta = 2 - \frac{1}{\cos\theta},$$

the graph has polar axis symmetry and the lengths at the pole are

$$\theta = \frac{\pi}{3}, \frac{-\pi}{3}.$$



Furthermore,

$$r \Longrightarrow -\infty \text{ as } \theta \Longrightarrow \frac{\pi}{2^-}$$

$$r \Longrightarrow \infty \text{ as } \theta \Longrightarrow -\frac{\pi}{2^+}.$$

Also, $r = 2 - \dfrac{1}{\cos\theta} = 2 - \dfrac{r}{r\cos\theta} = 2 - \dfrac{r}{x}$

$$rx = 2x - r$$

$$r = \frac{2x}{1 + x}.$$

Thus, $r \Longrightarrow \pm\infty$ as $x \Longrightarrow -1$.

89. $r = \dfrac{2}{\theta}$

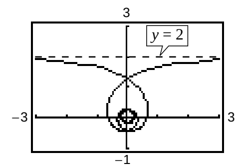
Hyperbolic spiral

$r \Longrightarrow \infty$ as $\theta \Longrightarrow 0$

$$r = \frac{2}{\theta} \Longrightarrow \theta = \frac{2}{r} = \frac{2\sin\theta}{r\sin\theta} = \frac{2\sin\theta}{y}$$

$$y = \frac{2\sin\theta}{\theta}$$

$$\lim_{\theta\to 0} \frac{2\sin\theta}{\theta} = \lim_{\theta\to 0} \frac{2\cos\theta}{1} = 2$$



91. The rectangular coordinate system consists of all points of the form (x, y) where x is the directed distance from the y-axis to the point, and y is the directed distance from the x-axis to the point. Every point has a unique representation.

The polar coordinate system uses (r, θ) to designate the location of a point.

r is the directed distance to the origin and θ is the angle the point makes with the positive x-axis, measured clockwise.

Point do not have a unique polar representation.

93. $r = a$ circle

$\theta = b$ line

95. $r = 2\sin\theta$ circle

Matches (c)

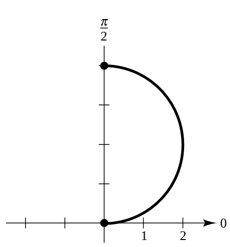
97. $r = 3(1 + \cos\theta)$

Cardioid

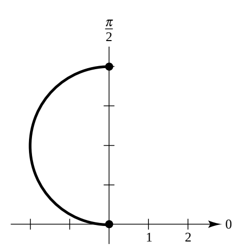
Matches (a)

99. $r = 4\sin\theta$

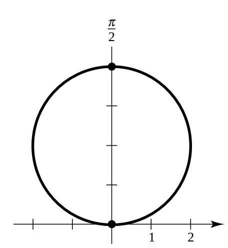
(a) $0 \le \theta \le \dfrac{\pi}{2}$



(b) $\dfrac{\pi}{2} \le \theta \le \pi$



(c) $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

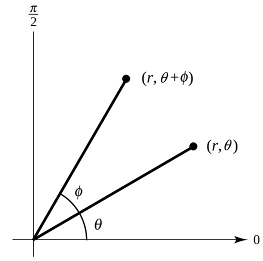


101. Let the curve $r = f(\theta)$ be rotated by ϕ to form the curve $r = g(\theta)$. If (r_1, θ_1) is a point on $r = f(\theta)$, then $(r_1, \theta_1 + \phi)$ is on $r = g(\theta)$. That is,

$$g(\theta_1 + \phi) = r_1 = f(\theta_1).$$

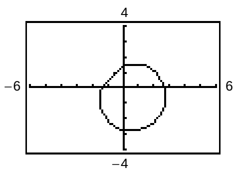
Letting $\theta = \theta_1 + \phi$, or $\theta_1 = \theta - \phi$, we see that

$$g(\theta) = g(\theta_1 + \phi) = f(\theta_1) = f(\theta - \phi).$$

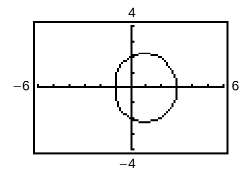


103. $r = 2 - \sin\theta$

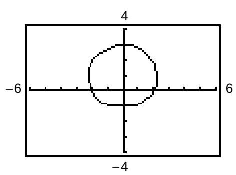
(a) $r = 2 - \sin\left(\theta - \dfrac{\pi}{4}\right) = 2 - \dfrac{\sqrt{2}}{2}(\sin\theta - \cos\theta)$



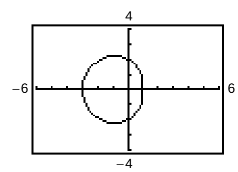
(b) $r = 2 - (-\cos\theta) = 2 + \cos\theta$



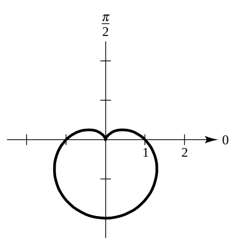
(c) $r = 2 - (-\sin\theta) = 2 + \sin\theta$



(d) $r = 2 - \cos\theta$



105. (a) $r = 1 - \sin\theta$

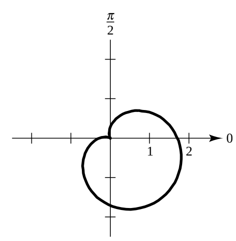


(b) $r = 1 - \sin\left(\theta - \dfrac{\pi}{4}\right)$

Rotate the graph of

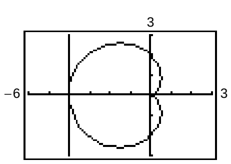
$r = 1 - \sin\theta$

through the angle $\pi/4$.



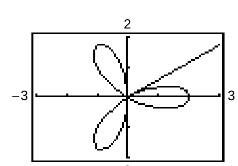
107. $\tan\psi = \dfrac{r}{dr/d\theta} = \dfrac{2(1 - \cos\theta)}{2\sin\theta}$

At $\theta = \pi$, $\tan\psi$ is undefined $\implies \psi = \dfrac{\pi}{2}$.

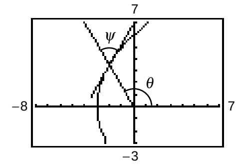


109. $\tan\psi = \dfrac{r}{dr/d\theta} = \dfrac{2\cos 3\theta}{-6\sin 3\theta}$

At $\theta = \dfrac{\pi}{6}$, $\tan\psi = 0 \implies \psi = 0$.



111. $r = \dfrac{6}{1 - \cos\theta} = 6(1 - \cos\theta)^{-1} \Rightarrow \dfrac{dr}{d\theta} = \dfrac{6\sin\theta}{(1 - \cos\theta)^2}$



$$\tan\psi = \frac{r}{\dfrac{dr}{d\theta}} = \frac{\dfrac{6}{(1 - \cos\theta)}}{\dfrac{6\sin\theta}{(1 - \cos\theta)^2}} = \frac{1 - \cos\theta}{\sin\theta}$$

At $\theta = \dfrac{2\pi}{3}$, $\tan\psi = \dfrac{1 - \left(-\dfrac{1}{2}\right)}{\dfrac{\sqrt{3}}{2}} = \sqrt{3}$.

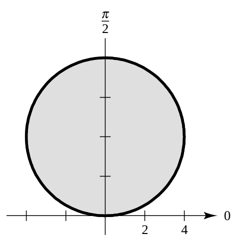
$\psi = \dfrac{\pi}{3}$, $(60°)$

113. True

115. True

Section 9.5 Area and Arc Length in Polar Coordinates

1. (a) $r = 8\sin\theta$



$A = \pi(4)^2 = 16\pi$

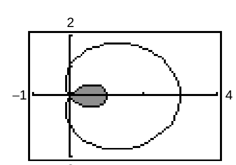
(b) $$A = 2\left(\frac{1}{2}\right)\int_0^{\pi/2}\left[8\sin\theta\right]^2 d\theta$$
$$= 64\int_0^{\pi/2}\sin^2\theta\, d\theta$$
$$= 32\int_0^{\pi/2}(1 - \cos 2\theta)\, d\theta$$
$$= 32\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = 16\pi$$

3. $A = 2\left[\dfrac{1}{2}\displaystyle\int_0^{\pi/6}(2\cos 3\theta)^2\, d\theta\right] = 2\left[\theta + \dfrac{1}{6}\sin 6\theta\right]_0^{\pi/6} = \dfrac{\pi}{3}$

5. $$A = 2\left[\frac{1}{2}\int_0^{\pi/4}(\cos 2\theta)^2\, d\theta\right]$$
$$= \frac{1}{2}\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\pi/4} = \frac{\pi}{8}$$

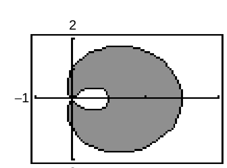
7. $$A = 2\left[\frac{1}{2}\int_{-\pi/2}^{\pi/2}(1 - \sin\theta)^2\, d\theta\right]$$
$$= \left[\frac{3}{2}\theta + 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = \frac{3\pi}{2}$$

9. $$A = 2\left[\frac{1}{2}\int_{2\pi/3}^{\pi}(1 + 2\cos\theta)^2\, d\theta\right]$$
$$= \left[3\theta + 4\sin\theta + \sin 2\theta\right]_{2\pi/3}^{\pi} = \frac{2\pi - 3\sqrt{3}}{2}$$



11. The area inside the outer loop is

$$2\left[\frac{1}{2}\int_0^{2\pi/3}(1 + 2\cos\theta)^2\, d\theta\right] = \left[3\theta + 4\sin\theta + \sin 2\theta\right]_0^{2\pi/3} = \frac{4\pi + 3\sqrt{3}}{2}.$$



From the result of Exercise 9, the area between the loops is

$$A = \left(\frac{4\pi + 3\sqrt{3}}{2}\right) - \left(\frac{2\pi - 3\sqrt{3}}{2}\right) = \pi + 3\sqrt{3}.$$

13. $r = 1 + \cos\theta$

$r = 1 - \cos\theta$

Solving simultaneously,

$$1 + \cos\theta = 1 - \cos\theta$$
$$2\cos\theta = 0$$
$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}.$$

Replacing r by $-r$ and θ by $\theta + \pi$ in the first equation and solving, $-1 + \cos\theta = 1 - \cos\theta$, $\cos\theta = 1$, $\theta = 0$. Both curves pass through the pole, $(0, \pi)$, and $(0, 0)$, respectively.

Points of intersection: $\left(1, \frac{\pi}{2}\right), \left(1, \frac{3\pi}{2}\right), (0, 0)$

15. $r = 1 + \cos\theta$

$r = 1 - \sin\theta$

Solving simultaneously,

$$1 + \cos\theta = 1 - \sin\theta$$
$$\cos\theta = -\sin\theta$$
$$\tan\theta = -1$$
$$\theta = \frac{3\pi}{4}, \frac{7\pi}{4}.$$

Replacing r by $-r$ and θ by $\theta + \pi$ in the first equation and solving, $-1 + \cos\theta = 1 - \sin\theta$, $\sin\theta + \cos\theta = 2$, which has no solution. Both curves pass through the pole, $(0, \pi)$, and $(0, \pi/2)$, respectively.

Points of intersection: $\left(\frac{2 - \sqrt{2}}{2}, \frac{3\pi}{4}\right), \left(\frac{2 + \sqrt{2}}{2}, \frac{7\pi}{4}\right), (0, 0)$

17. $r = 4 - 5\sin\theta$

$r = 3\sin\theta$

Solving simultaneously,

$$4 - 5\sin\theta = 3\sin\theta$$
$$\sin\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}.$$

Both curves pass through the pole, $(0, \arcsin 4/5)$, and $(0, 0)$, respectively.

Points of intersection: $\left(\frac{3}{2}, \frac{\pi}{6}\right), \left(\frac{3}{2}, \frac{5\pi}{6}\right), (0, 0)$

19. $r = \dfrac{\theta}{2}$

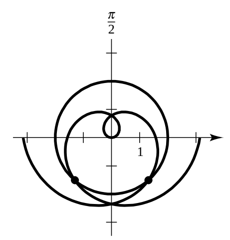
$r = 2$

Solving simultaneously, we have

$\theta/2 = 2$, $\theta = 4$.

Points of intersection:

$(2, 4), (-2, -4)$

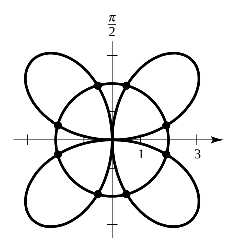


21. $r = 4\sin 2\theta$

$r = 2$

$r = 4\sin 2\theta$ is the equation of a rose curve with four petals and is symmetric to the polar axis, $\theta = \pi/2$, and the pole. Also, $r = 2$ is the equation of a circle of radius 2 centered at the pole. Solving simultaneously,

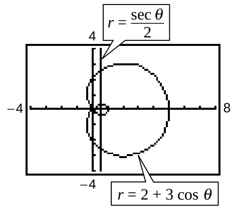
$$4\sin 2\theta = 2$$
$$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$
$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}.$$



Therefore, the points of intersection for one petal are $(2, \pi/12)$ and $(2, 5\pi/12)$. By symmetry, the other points of intersection are $(2, 7\pi/12)$, $(2, 11\pi/12)$, $(2, 13\pi/12)$, $(2, 17\pi/12)$, $(2, 19\pi/12)$, and $(2, 23\pi/12)$.

23. $r = 2 + 3\cos\theta$

$r = \dfrac{\sec\theta}{2}$



The graph of $r = 2 + 3\cos\theta$ is a limaçon with an inner loop $(b > a)$ and is symmetric to the polar axis. The graph of $r = (\sec\theta)/2$ is the vertical line $x = 1/2$. Therefore, there are four points of intersection. Solving simultaneously,

$$2 + 3\cos\theta = \frac{\sec\theta}{2}$$

$$6\cos^2\theta + 4\cos\theta - 1 = 0$$

$$\cos\theta = \frac{-2 \pm \sqrt{10}}{6}$$

$$\theta = \arccos\left(\frac{-2 + \sqrt{10}}{6}\right) \approx 1.376$$

$$\theta = \arccos\left(\frac{-2 - \sqrt{10}}{6}\right) \approx 2.6068.$$

Points of intersection: $(-0.581, \pm 2.607)$, $(2.581, \pm 1.376)$

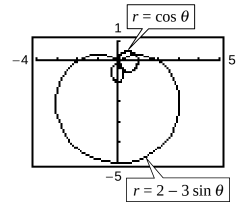
25. $r = \cos\theta$

$r = 2 - 3\sin\theta$

Points of intersection:

$(0, 0)$, $(0.935, 0.363)$, $(0.535, -1.006)$

The graphs reach the pole at different times (θ values).

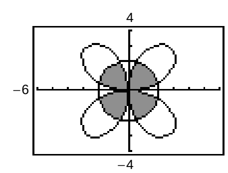


27. From Exercise 21, the points of intersection for one petal are $(2, \pi/12)$ and $(2, 5\pi/12)$. The area within one petal is

$$A = \frac{1}{2}\int_0^{\pi/12} (4\sin 2\theta)^2\, d\theta + \frac{1}{2}\int_{\pi/12}^{5\pi/12} (2)^2\, d\theta + \frac{1}{2}\int_{5\pi/12}^{\pi/2} (4\sin 2\theta)^2\, d\theta$$

$$= 16\int_0^{\pi/12} \sin^2(2\theta)\, d\theta + 2\int_{\pi/12}^{5\pi/12} d\theta \text{ (by symmetry of the petal)}$$

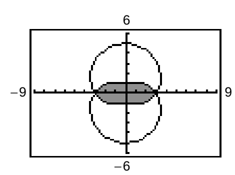
$$= 8\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/12} + \left[2\theta\right]_{\pi/12}^{5\pi/12} = \frac{4\pi}{3} - \sqrt{3}.$$



$$\text{Total area} = 4\left(\frac{4\pi}{3} - \sqrt{3}\right) = \frac{16\pi}{3} - 4\sqrt{3} = \frac{4}{3}\left(4\pi - 3\sqrt{3}\right)$$

29. $A = 4\left[\dfrac{1}{2}\displaystyle\int_0^{\pi/2} (3 - 2\sin\theta)^2\, d\theta\right]$

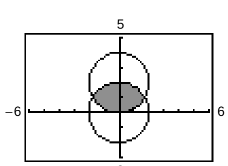
$$= 2\left[11\theta + 12\cos\theta - \sin(2\theta)\right]_0^{\pi/2} = 11\pi - 24$$



31. $A = 2\left[\dfrac{1}{2}\displaystyle\int_0^{\pi/6} (4\sin\theta)^2\, d\theta + \dfrac{1}{2}\displaystyle\int_{\pi/6}^{\pi/2} (2)^2\, d\theta\right]$

$$= 16\left[\frac{1}{2}\theta - \frac{1}{4}\sin(2\theta)\right]_0^{\pi/6} + \left[4\theta\right]_{\pi/6}^{\pi/2}$$

$$= \frac{8\pi}{3} - 2\sqrt{3} = \frac{2}{3}\left(4\pi - 3\sqrt{3}\right)$$



33. $A = 2\left[\dfrac{1}{2}\displaystyle\int_0^{\pi} [a(1 + \cos\theta)]^2\, d\theta\right] - \dfrac{a^2\pi}{4}$

$$= a^2\left[\frac{3}{2}\theta + 2\sin\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi} - \frac{a^2\pi}{4}$$

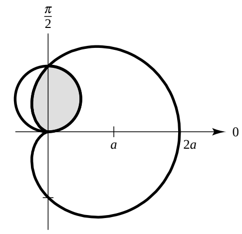
$$= \frac{3a^2\pi}{2} - \frac{a^2\pi}{4} = \frac{5a^2\pi}{4}$$

35. $A = \dfrac{\pi a^2}{8} + \dfrac{1}{2}\displaystyle\int_{\pi/2}^{\pi} [a(1 + \cos\theta)]^2\, d\theta$

$= \dfrac{\pi a^2}{8} + \dfrac{a^2}{2}\displaystyle\int_{\pi/2}^{\pi} \left(\dfrac{3}{2} + 2\cos\theta + \dfrac{\cos 2\theta}{2}\right) d\theta$

$= \dfrac{\pi a^2}{8} + \dfrac{a^2}{2}\left[\dfrac{3}{2}\theta + 2\sin\theta + \dfrac{\sin 2\theta}{4}\right]_{\pi/2}^{\pi}$

$= \dfrac{\pi a^2}{8} + \dfrac{a^2}{2}\left[\dfrac{3\pi}{2} - \dfrac{3\pi}{4} - 2\right] = \dfrac{a^2}{2}[\pi - 2]$

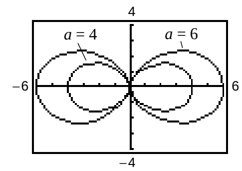


37. (a) $r = a\cos^2\theta$

$r^3 = ar^2\cos^2\theta$

$(x^2 + y^2)^{3/2} = ax^2$

(b)



(c) $A = 4\left(\dfrac{1}{2}\right)\displaystyle\int_0^{\pi/2} [(6\cos^2\theta)^2 - (4\cos^2\theta)^2]\, d\theta = 40\displaystyle\int_0^{\pi/2} \cos^4\theta\, d\theta = 10\displaystyle\int_0^{\pi/2} (1 + \cos 2\theta)^2\, d\theta$

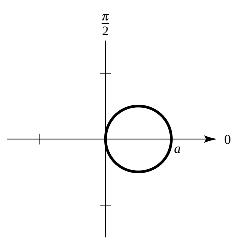
$= 10\displaystyle\int_0^{\pi/2} \left(1 + 2\cos 2\theta + \dfrac{1 - \cos 4\theta}{2}\right) d\theta = 10\left[\dfrac{3}{2}\theta + \sin 2\theta + \dfrac{1}{8}\sin 4\theta\right]_0^{\pi/2} = \dfrac{15\pi}{2}$

39. $r = a\cos(n\theta)$

For $n = 1$:

$r = a\cos\theta$

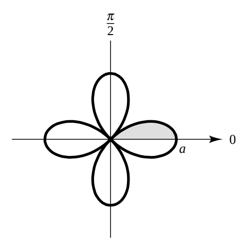
$A = \pi\left(\dfrac{a}{2}\right)^2 = \dfrac{\pi a^2}{4}$



For $n = 2$:

$r = a\cos 2\theta$

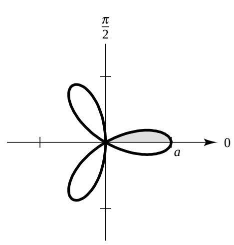
$A = 8\left(\dfrac{1}{2}\right)\displaystyle\int_0^{\pi/4} (a\cos 2\theta)^2\, d\theta = \dfrac{\pi a^2}{2}$



For $n = 3$:

$r = a\cos 3\theta$

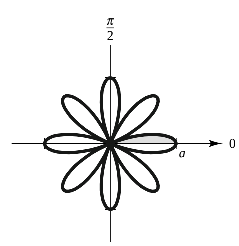
$A = 6\left(\dfrac{1}{2}\right)\displaystyle\int_0^{\pi/6} (a\cos 3\theta)^2\, d\theta = \dfrac{\pi a^2}{4}$



For $n = 4$:

$r = a\cos 4\theta$

$A = 16\left(\dfrac{1}{2}\right)\displaystyle\int_0^{\pi/8} (a\cos 4\theta)^2\, d\theta = \dfrac{\pi a^2}{2}$



In general, the area of the region enclosed by $r = a\cos(n\theta)$ for $n = 1, 2, 3, \ldots$ is $(\pi a^2)/4$ if n is odd and is $(\pi a^2)/2$ if n is even.

41. $r = a$

$r' = 0$

$$s = \int_0^{2\pi} \sqrt{a^2 + 0^2}\, d\theta = \Big[a\theta\Big]_0^{2\pi} = 2\pi a$$

(circumference of circle of radius a)

43. $r = 1 + \sin\theta$

$r' = \cos\theta$

$$s = 2\int_{\pi/2}^{3\pi/2} \sqrt{(1 + \sin\theta)^2 + (\cos\theta)^2}\, d\theta$$

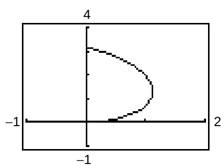
$$= 2\sqrt{2}\int_{\pi/2}^{3\pi/2} \sqrt{1 + \sin\theta}\, d\theta$$

$$= 2\sqrt{2}\int_{\pi/2}^{3\pi/2} \frac{-\cos\theta}{\sqrt{1 - \sin\theta}}\, d\theta$$

$$= \Big[4\sqrt{2}\sqrt{1 - \sin\theta}\Big]_{\pi/2}^{3\pi/2}$$

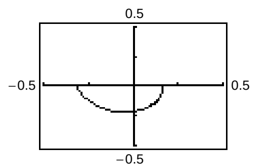
$$= 4\sqrt{2}\left(\sqrt{2} - 0\right) = 8$$

45. $r = 2\theta,\ 0 \le \theta \le \dfrac{\pi}{2}$



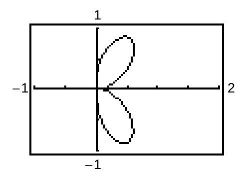
Length ≈ 4.16

47. $r = \dfrac{1}{\theta},\ \pi \le \theta \le 2\pi$



Length ≈ 0.71

49. $r = \sin(3\cos\theta),\ 0 \le \theta \le \pi$



Length ≈ 4.39

51. $r = 6\cos\theta$

$r' = -6\sin\theta$

$$S = 2\pi\int_0^{\pi/2} 6\cos\theta\sin\theta\sqrt{36\cos^2\theta + 36\sin^2\theta}\, d\theta$$

$$= 72\pi\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta$$

$$= \Big[36\pi\sin^2\theta\Big]_0^{\pi/2}$$

$$= 36\pi$$

53. $r = e^{a\theta}$

$r' = ae^{a\theta}$

$$S = 2\pi\int_0^{\pi/2} e^{a\theta}\cos\theta\sqrt{(e^{a\theta})^2 + (ae^{a\theta})^2}\, d\theta$$

$$= 2\pi\sqrt{1 + a^2}\int_0^{\pi/2} e^{2a\theta}\cos\theta\, d\theta$$

$$= 2\pi\sqrt{1 + a^2}\left[\frac{e^{2a\theta}}{4a^2 + 1}(2a\cos\theta + \sin\theta)\right]_0^{\pi/2}$$

$$= \frac{2\pi\sqrt{1 + a^2}}{4a^2 + 1}(e^{\pi a} - 2a)$$

55. $r = 4\cos 2\theta$

$r' = -8\sin 2\theta$

$$S = 2\pi\int_0^{\pi/4} 4\cos 2\theta\sin\theta\sqrt{16\cos^2 2\theta + 64\sin^2 2\theta}\, d\theta$$

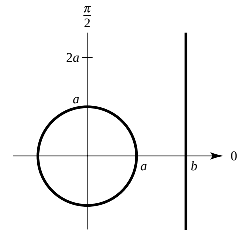
$$= 32\pi\int_0^{\pi/4} \cos 2\theta\sin\theta\sqrt{\cos^2 2\theta + 4\sin^2 2\theta}\, d\theta \approx 21.87$$

57. $$\text{Area} = \frac{1}{2}\int_\alpha^\beta [f(\theta)]^2 d\theta = \frac{1}{2}\int_\alpha^\beta r^2\, d\theta$$

$$\text{Arc length} = \int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

59. (a) is correct: $s \approx 33.124$.

61. Revolve $r = a$ about the line $r = b \sec \theta$ where $b > a > 0$.



$$f(\theta) = a$$

$$f'(\theta) = 0$$

$$S = 2\pi \int_0^{2\pi} [b - a \cos \theta] \sqrt{a^2 + 0^2}\, d\theta$$

$$= 2\pi a \Big[b\theta - a \sin \theta \Big]_0^{2\pi}$$

$$= 2\pi a(2\pi b) = 4\pi^2 ab$$

63. False. $f(\theta) = 1$ and $g(\theta) = -1$ have the same graphs.

65. In parametric form,

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Using θ instead of t, we have $x = r \cos \theta = f(\theta) \cos \theta$ and $y = r \sin \theta = f(\theta) \sin \theta$. Thus,

$$\frac{dx}{d\theta} = f'(\theta) \cos \theta - f(\theta) \sin \theta \text{ and } \frac{dy}{d\theta} = f'(\theta) \sin \theta + f(\theta) \cos \theta.$$

It follows that

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = [f(\theta)]^2 + [f'(\theta)]^2.$$

Therefore, $s = \displaystyle\int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$

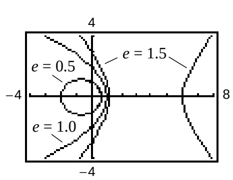
Section 9.6 Polar Equations of Conics and Kepler's Laws

1. $r = \dfrac{2e}{1 + e \cos \theta}$

(a) $e = 1$, $r = \dfrac{2}{1 + \cos \theta}$, parabola

(b) $e = 0.5$, $r = \dfrac{1}{1 + 0.5 \cos \theta} = \dfrac{2}{2 + \cos \theta}$, ellipse

(c) $e = 1.5$, $r = \dfrac{3}{1 + 1.5 \cos \theta} = \dfrac{6}{2 + 3 \cos \theta}$, hyperbola

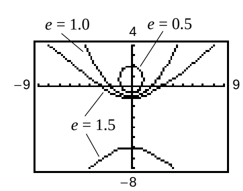


3. $r = \dfrac{2e}{1 - e \sin \theta}$

(a) $e = 1$, $r = \dfrac{2}{1 - \sin \theta}$, parabola

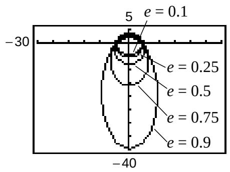
(b) $e = 0.5$, $r = \dfrac{1}{1 - 0.5 \sin \theta} = \dfrac{2}{2 - \sin \theta}$, ellipse

(c) $e = 1.5$, $r = \dfrac{3}{1 - 1.5 \sin \theta} = \dfrac{6}{2 - 3 \sin \theta}$, hyperbola



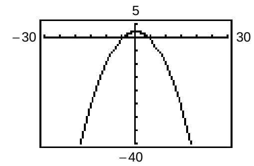
5. $r = \dfrac{4}{1 + e \sin \theta}$

(a)



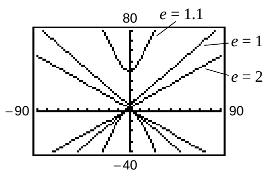
The conic is an ellipse. As $e \to 1^-$, the ellipse becomes more elliptical, and as $e \to 0^+$, it becomes more circular.

(b)



The conic is a parabola.

(c)



The conic is a hyperbola. As $e \to 1^+$, the hyperbolas opens more slowly, and as $e \to \infty$, they open more rapidly.

7. Parabola; Matches (c)

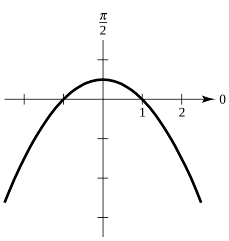
9. Hyperbola; Matches (a)

11. Ellipse; Matches (b)

13. $r = \dfrac{-1}{1 - \sin \theta}$

Parabola since $e = 1$

Vertex: $\left(-\dfrac{1}{2}, \dfrac{3\pi}{2}\right)$

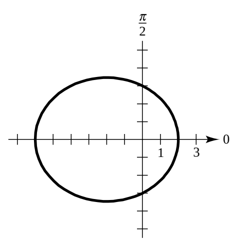


15. $r = \dfrac{6}{2 + \cos \theta}$

$= \dfrac{3}{1 + (1/2) \cos \theta}$

Ellipse since $e = \dfrac{1}{2} < 1$

Vertices: $(2, 0)$, $(6, \pi)$



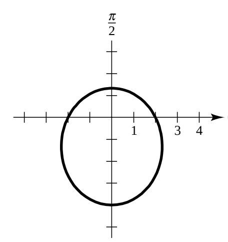
17. $r(2 + \sin \theta) = 4$

$r = \dfrac{4}{2 + \sin \theta}$

$= \dfrac{2}{1 + (1/2) \sin \theta}$

Ellipse since $e = \dfrac{1}{2} < 1$

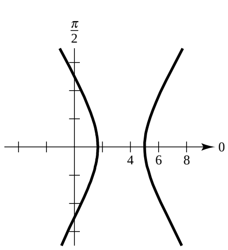
Vertices: $\left(\dfrac{4}{3}, \dfrac{\pi}{2}\right)$, $\left(4, \dfrac{3\pi}{2}\right)$



19. $r = \dfrac{5}{-1 + 2 \cos \theta} = \dfrac{-5}{1 - 2 \cos \theta}$

Hyperbola since $e = 2 > 1$

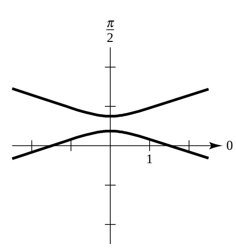
Vertices: $(5, 0)$, $\left(-\dfrac{5}{3}, \pi\right)$

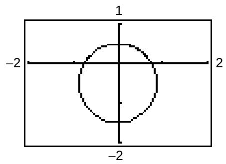


21. $r = \dfrac{3}{2 + 6 \sin \theta} = \dfrac{3/2}{1 + 3 \sin \theta}$

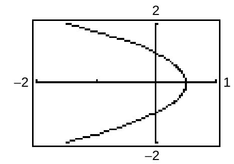
Hyperbola since $e = 3 > 1$

Vertices: $\left(\dfrac{3}{8}, \dfrac{\pi}{2}\right)$, $\left(-\dfrac{3}{4}, \dfrac{3\pi}{2}\right)$



23. 

Ellipse

25. 

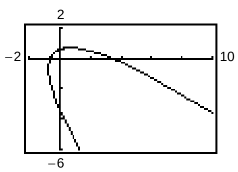
Parabola

27. $r = \dfrac{-1}{1 - \sin\left(\theta - \dfrac{\pi}{4}\right)}$

Rotate the graph of

$$r = \frac{-1}{1 - \sin\theta}$$

counterclockwise through the angle $\dfrac{\pi}{4}$.

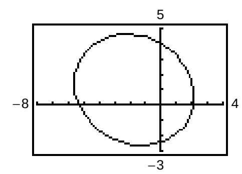


29. $r = \dfrac{6}{2 + \cos\left(\theta + \dfrac{\pi}{6}\right)}$

Rotate the graph of

$$r = \frac{6}{2 + \cos\theta}$$

clockwise through the angle $\dfrac{\pi}{6}$.



31. Change θ to $\theta + \dfrac{\pi}{4}$: $r = \dfrac{5}{5 + 3\cos\left(\theta + \dfrac{\pi}{4}\right)}$.

33. Parabola

$e = 1, x = -1, d = 1$

$$r = \frac{ed}{1 - e\cos\theta} = \frac{1}{1 - \cos\theta}$$

35. Ellipse

$e = \dfrac{1}{2}, y = 1, d = 1$

$$r = \frac{ed}{1 + e\sin\theta} = \frac{1/2}{1 + (1/2)\sin\theta} = \frac{1}{2 + \sin\theta}$$

37. Hyperbola

$e = 2, x = 1, d = 1$

$$r = \frac{ed}{1 + e\cos\theta} = \frac{2}{1 + 2\cos\theta}$$

39. Parabola

Vertex: $\left(1, -\dfrac{\pi}{2}\right)$

$e = 1, d = 2, r = \dfrac{2}{1 - \sin\theta}$

41. Ellipse

Vertices: $(2, 0), (8, \pi)$

$e = \dfrac{3}{5}, d = \dfrac{16}{3}$

$$r = \frac{ed}{1 + e\cos\theta} = \frac{16/5}{1 + (3/5)\cos\theta} = \frac{16}{5 + 3\cos\theta}$$

43. Hyperbola

Vertices: $\left(1, \dfrac{3\pi}{2}\right), \left(9, \dfrac{3\pi}{2}\right)$

$e = \dfrac{5}{4}, d = \dfrac{9}{5}$

$$r = \frac{ed}{1 - e\sin\theta} = \frac{9/4}{1 - (5/4)\sin\theta} = \frac{9}{4 - 5\sin\theta}$$

45. Ellipse if $0 < e < 1$, parabola if $e = 1$, hyperbola if $e > 1$.

47. (a) Hyperbola $(e = 2 > 1)$

(b) Ellipse $\left(e = \dfrac{1}{2} < 1\right)$

(c) Parabola $(e = 1)$

(d) Rotated hyperbola $(e = 3)$

49. $a = 5, c = 4, e = \dfrac{4}{5}, b = 3$

$$r^2 = \frac{9}{1 - (16/25)\cos^2\theta}$$

51. $a = 3, b = 4, c = 5, e = \dfrac{5}{3}$

$$r^2 = \frac{-16}{1 - (25/9)\cos^2\theta}$$

53. $A = 2\left[\dfrac{1}{2}\displaystyle\int_0^{\pi}\left(\frac{3}{2 - \cos\theta}\right)^2 d\theta\right]$

$$= 9\int_0^{\pi}\frac{1}{(2 - \cos\theta)^2}\,d\theta \approx 10.88$$

55. Vertices: $(126{,}000, 0), (4119, \pi)$

$$a = \frac{126{,}000 + 4119}{2} = 65{,}059.5,\ c = 65{,}059.5 - 4119 = 60{,}940.5,\ e = \frac{c}{a} = \frac{40{,}627}{43{,}373},\ d = 4119\left(\frac{84{,}000}{40{,}627}\right)$$

$$r = \frac{ed}{1 - e\cos\theta} = \frac{4119(84{,}000/43{,}373)}{1 - (40{,}627/43{,}373)\cos\theta} = \frac{345{,}996{,}000}{43{,}373 - 40{,}627\cos\theta}$$

When $\theta = 60^\circ$, $r = \dfrac{345{,}996{,}000}{23{,}059.5} \approx 15{,}004.49$.

Distance between the surface of the earth and the satellite is $r - 4000 = 11{,}004.49$ miles.

57. $a = 92.957 \times 10^6$ mi, $e = 0.0167$

$$r = \frac{(1 - e^2)a}{1 - e\cos\theta} = \frac{92{,}931{,}075.2223}{1 - 0.0167\cos\theta}$$

Perihelion distance: $a(1 - e) \approx 91{,}404{,}618$ mi

Aphelion distance: $a(1 + e) \approx 94{,}509{,}382$ mi

59. $a = 5.900 \times 10^9$ km, $e = 0.2481$

$$r = \frac{(1 - e^2)a}{1 - e\cos\theta} \approx \frac{5.537 \times 10^9}{1 - 0.2481\cos\theta}$$

Perihelion distance: $a(1 - e) = 4.436 \times 10^9$ km

Aphelion distance: $a(1 + e) = 7.364 \times 10^9$ km

61. $r = \dfrac{5.537 \times 10^9}{1 - 0.2481\cos\theta}$

(a) $A = \dfrac{1}{2}\displaystyle\int_0^{\pi/9}\left[\frac{5.537 \times 10^9}{1 - 0.2481\cos\theta}\right]^2 d\theta \approx 9.341 \times 10^{18}\ \text{km}^2$

$$248\left[\frac{\dfrac{1}{2}\displaystyle\int_0^{\pi/9}\left[\frac{5.537 \times 10^9}{1 - 0.2481\cos\theta}\right]^2 d\theta}{\dfrac{1}{2}\displaystyle\int_0^{2\pi}\left[\frac{5.537 \times 10^9}{1 - 0.2481\cos\theta}\right]^2 d\theta}\right] \approx 21.867\ \text{yr}$$

(b) $\dfrac{1}{2}\displaystyle\int_{\pi}^{\alpha - \pi}\left[\frac{5.537 \times 10^9}{1 - 0.2481\cos\theta}\right]^2 d\theta = 9.341 \times 10^{18}$

$\alpha \approx \pi + 0.8995$ rad

In part (a) the ray swept through a smaller angle to generate the same area since the length of the ray is longer than in part (b).

(c) $r' = \dfrac{(-5.537 \times 10^9)(0.2481\sin\theta)}{(1 - 0.2481\cos\theta)^2}$

$$s = \int_0^{\pi/9}\sqrt{\left(\frac{5.537 \times 10^9}{1 - 0.2481\cos\theta}\right)^2 + \left[\frac{-1.3737297 \times 10^9\sin\theta}{(1 - 0.2481\cos\theta)^2}\right]^2}\,d\theta \approx 2.559 \times 10^9\ \text{km}$$

$$\frac{2.559 \times 10^9\ \text{km}}{21.867\ \text{yr}} \approx 1.17 \times 10^8\ \text{km/yr}$$

$$s = \int_{\pi}^{\pi + 0.899}\sqrt{\left(\frac{5.537 \times 10^9}{1 - 0.2481\cos\theta}\right)^2 + \left[\frac{-1.3737297 \times 10^9\sin\theta}{(1 - 0.2481\cos\theta)^2}\right]^2}\,d\theta \approx 4.119 \times 10^9\ \text{km}$$

$$\frac{4.119 \times 10^9\ \text{km}}{21.867\ \text{yr}} \approx 1.88 \times 10^8\ \text{km/yr}$$

63. $r_1 = \dfrac{ed}{1 + \sin\theta}$ and $r_2 = \dfrac{ed}{1 - \sin\theta}$

Points of intersection: $(ed, 0)$, (ed, π)

$$r_1\!: \frac{dy}{dx} = \frac{\left(\dfrac{ed}{1+\sin\theta}\right)(\cos\theta) + \left(\dfrac{-ed\cos\theta}{(1+\sin\theta)^2}\right)(\sin\theta)}{\left(\dfrac{-ed}{1+\sin\theta}\right)(\sin\theta) + \left(\dfrac{-ed\cos\theta}{(1+\sin\theta)^2}\right)(\cos\theta)}$$

At $(ed, 0)$, $\dfrac{dy}{dx} = -1$. At (ed, π), $\dfrac{dy}{dx} = 1$.

$$r_2\!: \frac{dy}{dx} = \frac{\left(\dfrac{ed}{1-\sin\theta}\right)(\cos\theta) + \left(\dfrac{ed\cos\theta}{(1-\sin\theta)^2}\right)(\sin\theta)}{\left(\dfrac{-ed}{1-\sin\theta}\right)(\sin\theta) + \left(\dfrac{ed\cos\theta}{(1-\sin\theta)^2}\right)(\cos\theta)}$$

At $(ed, 0)$, $\dfrac{dy}{dx} = 1$. At (ed, π), $\dfrac{dy}{dx} = -1$.

Therefore, at $(ed, 0)$ we have $m_1m_2 = (-1)(1) = -1$, and at (ed, π) we have $m_1m_2 = 1(-1) = -1$. The curves intersect at right angles.

Review Exercises for Chapter 9

1. Matches (d) - ellipse

3. Matches (a) - parabola

5. $16x^2 + 16y^2 - 16x + 24y - 3 = 0$

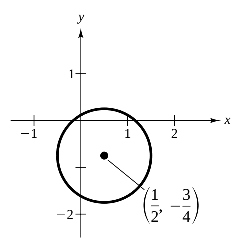
$$\left(x^2 - x + \frac{1}{4}\right) + \left(y^2 + \frac{3}{2}y + \frac{9}{16}\right) = \frac{3}{16} + \frac{1}{4} + \frac{9}{16}$$

$$\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{3}{4}\right)^2 = 1$$

Circle

Center: $\left(\frac{1}{2}, -\frac{3}{4}\right)$

Radius: 1



7. $3x^2 - 2y^2 + 24x + 12y + 24 = 0$

$$3(x^2 + 8x + 16) - 2(y^2 - 6y + 9) = -24 + 48 - 18$$

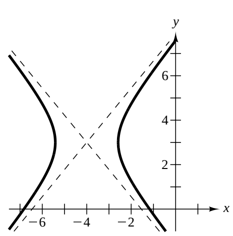
$$\frac{(x+4)^2}{2} - \frac{(y-3)^2}{3} = 1$$

Hyperbola

Center: $(-4, 3)$

Vertices: $\left(-4 \pm \sqrt{2}, 3\right)$

Asymptotes: $y = 3 \pm \sqrt{\dfrac{3}{2}}(x + 4)$



9. $3x^2 + 2y^2 - 12x + 12y + 29 = 0$

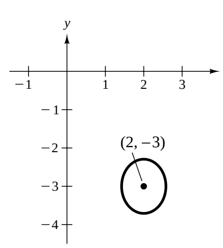
$$3(x^2 - 4x + 4) + 2(y^2 + 6y + 9) = -29 + 12 + 18$$

$$\frac{(x-2)^2}{1/3} + \frac{(y+3)^2}{1/2} = 1$$

Ellipse

Center: $(2, -3)$

Vertices: $\left(2, -3 \pm \dfrac{\sqrt{2}}{2}\right)$



11. Vertex: $(0, 2)$

Directrix: $x = -3$

Parabola opens to the right

$p = 3$

$(y - 2)^2 = 4(3)(x - 0)$

$y^2 - 4y - 12x + 4 = 0$

13. Vertices: $(-3, 0), (7, 0)$

Foci: $(0, 0), (4, 0)$

Horizontal major axis

Center: $(2, 0)$

$a = 5, c = 2, b = \sqrt{21}$

$\dfrac{(x-2)^2}{25} + \dfrac{y^2}{21} = 1$

15. Vertices: $(\pm 4, 0)$

Foci: $(\pm 6, 0)$

Center: $(0, 0)$

Horizontal transverse axis

$a = 4, c = 6, b = \sqrt{36 - 16} = 2\sqrt{5}$

$\dfrac{x^2}{16} - \dfrac{y^2}{20} = 1$

17. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1, a = 3, b = 2, c = \sqrt{5}, e = \dfrac{\sqrt{5}}{3}$

By Example 5 of Section 9.1,

$$C = 12\int_0^{\pi/2} \sqrt{1 - \left(\frac{5}{9}\right)\sin^2\theta}\, d\theta \approx 15.87.$$

19. $y = x - 2$ has a slope of 1. The perpendicular slope is -1.

$$y = x^2 - 2x + 2$$

$$\frac{dy}{dx} = 2x - 2 = -1 \text{ when } x = \frac{1}{2} \text{ and } y = \frac{5}{4}.$$

Perpendicular line: $y - \dfrac{5}{4} = -1\left(x - \dfrac{1}{2}\right)$

$$4x + 4y - 7 = 0$$

21. (a) $V = (\pi ab)(\text{Length}) = 12\pi(16) = 192\pi \text{ ft}^3$

(b) $$F = 2(62.4)\int_{-3}^{3}(3 - y)\frac{4}{3}\sqrt{9 - y^2}\, dy = \frac{8}{3}(62.4)\left[3\int_{-3}^{3}\sqrt{9 - y^2}\, dy - \int_{-3}^{3} y\sqrt{9 - y^2}\, dy\right]$$

$$= \frac{8}{3}(62.4)\left[\frac{3}{2}\left(y\sqrt{9 - y^2} + 9\arcsin\frac{y}{3}\right) + \frac{1}{3}(9 - y^2)^{3/2}\right]_{-3}^{3}$$

$$= \frac{8}{3}(62.4)\left[\frac{3}{2}\left(\frac{9\pi}{2}\right) - \frac{3}{2}\left(-\frac{9\pi}{2}\right)\right] = \frac{8}{3}(62.4)\left(\frac{27\pi}{2}\right) \approx 7057.274$$

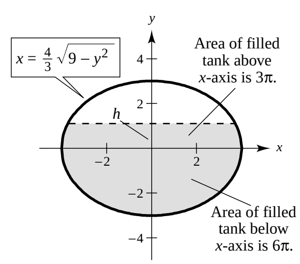
(c) You want $\frac{3}{4}$ of the total area of 12π covered. Find h so that

$$2\int_0^h \frac{4}{3}\sqrt{9 - y^2}\, dy = 3\pi$$

$$\int_0^h \sqrt{9 - y^2}\, dy = \frac{9\pi}{8}$$

$$\frac{1}{2}\left[y\sqrt{9 - y^2} + 9\arcsin\left(\frac{y}{3}\right)\right]_0^h = \frac{9\pi}{8}$$

$$h\sqrt{9 - h^2} + 9\arcsin\left(\frac{h}{3}\right) = \frac{9\pi}{4}.$$



By Newton's Method, $h \approx 1.212$. Therefore, the total height of the water is $1.212 + 3 = 4.212$ ft.

(d) Area of ends $= 2(12\pi) = 24\pi$

Area of sides $= (\text{Perimeter})(\text{Length})$

$$= 16\int_0^{\pi/2}\left(\sqrt{1 - \left(\frac{7}{16}\right)\sin^2\theta}\right) d\theta(16) \quad [\text{from Example 5 of Section 9.1}]$$

$$\approx 256\left(\frac{\pi/2}{12}\right)\left[\sqrt{1 - \left(\frac{7}{16}\right)\sin^2(0)} + 4\sqrt{1 - \left(\frac{7}{16}\right)\sin^2\left(\frac{\pi}{8}\right)} + 2\sqrt{1 - \left(\frac{7}{16}\right)\sin^2\left(\frac{\pi}{4}\right)}\right.$$

$$\left. + 4\sqrt{1 - \left(\frac{7}{16}\right)\sin^2\left(\frac{3\pi}{8}\right)} + \sqrt{1 - \left(\frac{7}{16}\right)\sin^2\left(\frac{\pi}{2}\right)}\right] \approx 353.65$$

Total area $= 24\pi + 353.65 \approx 429.05$

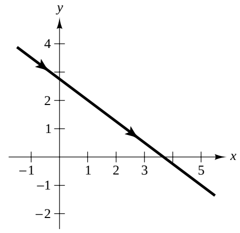
23. $x = 1 + 4t, y = 2 - 3t$

$$t = \frac{x-1}{4} \Rightarrow y = 2 - 3\left(\frac{x-1}{4}\right)$$

$$y = -\frac{3}{4}x + \frac{11}{4}$$

$$4y + 3x - 11 = 0$$

Line

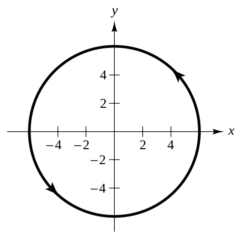


25. $x = 6\cos\theta, y = 6\sin\theta$

$$\left(\frac{x}{6}\right)^2 + \left(\frac{y}{6}\right)^2 = 1$$

$$x^2 + y^2 = 36$$

Circle

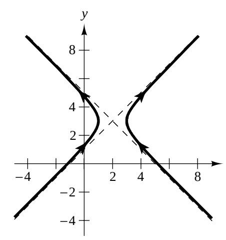


27. $x = 2 + \sec\theta, y = 3 + \tan\theta$

$$(x-2)^2 = \sec^2\theta = 1 + \tan^2\theta = 1 + (y-3)^2$$

$$(x-2)^2 - (y-3)^2 = 1$$

Hyperbola



29. $x = 3 + (3 - (-2))t = 3 + 5t$

$y = 2 + (2 - 6)t = 2 - 4t$

(other answers possible)

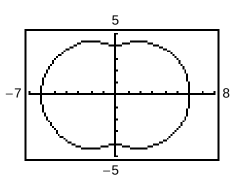
31. $\dfrac{(x+3)^2}{16} + \dfrac{(y-4)^2}{9} = 1$

Let $\dfrac{(x+3)^2}{16} = \cos^2\theta$ and $\dfrac{(y-4)^2}{9} = \sin^2\theta.$

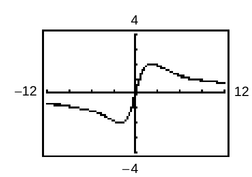
Then $x = -3 + 4\cos\theta$ and $y = 4 + 3\sin\theta.$

33. $x = \cos 3\theta + 5\cos\theta$

$y = \sin 3\theta + 5\sin\theta$



35. (a) $x = 2\cot\theta, y = 4\sin\theta\cos\theta, 0 < \theta < \pi$



(b) $$\begin{aligned}(4 + x^2)y &= (4 + 4\cot^2\theta)4\sin\theta\cos\theta \\ &= 16\csc^2\theta \cdot \sin\theta \cdot \cos\theta \\ &= 16\frac{\cos\theta}{\sin\theta} \\ &= 8(2\cot\theta) \\ &= 8x\end{aligned}$$

37. $x = 1 + 4t$

$y = 2 - 3t$

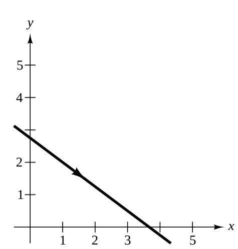
(a) $\dfrac{dy}{dx} = -\dfrac{3}{4}$

No horizontal tangents

(b) $t = \dfrac{x-1}{4}$

$$y = 2 - \frac{3}{4}(x-1) = \frac{-3x+11}{4}$$

(c)



39. $x = \dfrac{1}{t}$

$y = 2t + 3$

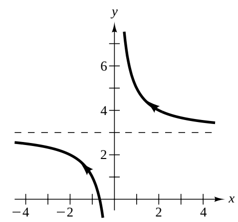
(a) $\dfrac{dy}{dx} = \dfrac{2}{-1/t^2} = -2t^2$

No horizontal tangents
$(t \neq 0)$

(b) $t = \dfrac{1}{x}$

$y = \dfrac{2}{x} + 3$

(c)



41. $x = \dfrac{1}{2t + 1}$

$y = \dfrac{1}{t^2 - 2t}$

(a) $\dfrac{dy}{dx} = \dfrac{\dfrac{-(2t-2)}{(t^2-2t)^2}}{\dfrac{-2}{(2t+1)^2}} = \dfrac{(t-1)(2t+1)^2}{t^2(t-2)^2} = 0$ when $t = 1$.

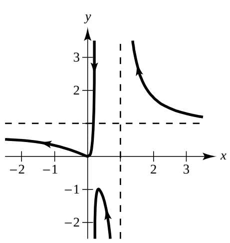
Point of horizontal tangency: $\left(\frac{1}{3}, -1\right)$

(b) $2t + 1 = \dfrac{1}{x} \implies t = \dfrac{1}{2}\left(\dfrac{1}{x} - 1\right)$

$$y = \frac{1}{\frac{1}{2}\left(\frac{1-x}{x}\right)\left[\frac{1}{2}\left(\frac{1-x}{x}\right) - 2\right]}$$

$$= \frac{4x^2}{(1-x)^2 - 4x(1-x)} = \frac{4x^2}{(5x-1)(x-1)}$$

(c)



43. $x = 3 + 2\cos\theta$

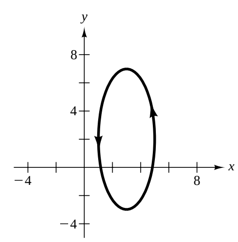
$y = 2 + 5\sin\theta$

(a) $\dfrac{dy}{dx} = \dfrac{5\cos\theta}{-2\sin\theta} = -2.5\cot\theta = 0$ when $\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

Points of horizontal tangency: $(3, 7), (3, -3)$

(b) $\dfrac{(x-3)^2}{4} + \dfrac{(y-2)^2}{25} = 1$

(c)



45. $x = \cos^3\theta$

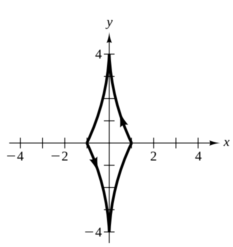
$y = 4\sin^3\theta$

(a) $\dfrac{dy}{dx} = \dfrac{12\sin^2\theta\cos\theta}{3\cos^2\theta(-\sin\theta)} = \dfrac{-4\sin\theta}{\cos\theta} = -4\tan\theta = 0$ when $\theta = 0, \pi$.

But, $\dfrac{dy}{dt} = \dfrac{dx}{dt} = 0$ at $\theta = 0, \pi$. Hence no points of horizontal tangency.

(b) $x^{2/3} + \left(\dfrac{y}{4}\right)^{2/3} = 1$

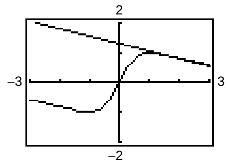
(c)



47. $x = \cot\theta$

$y = \sin 2\theta = 2\sin\theta\cos\theta$

(a), (c)



(b) At $\theta = \dfrac{\pi}{6}, \dfrac{dx}{d\theta} = -4, \dfrac{dy}{d\theta} = 1$, and $\dfrac{dy}{dx} = -\dfrac{1}{4}$

49. $x = r(\cos\theta + \theta\sin\theta)$

$y = r(\sin\theta - \theta\cos\theta)$

$$\frac{dx}{d\theta} = r\theta\cos\theta$$

$$\frac{dy}{d\theta} = r\theta\sin\theta$$

$$s = r\int_0^{\pi} \sqrt{\theta^2\cos^2\theta + \theta^2\sin^2\theta}\,d\theta$$

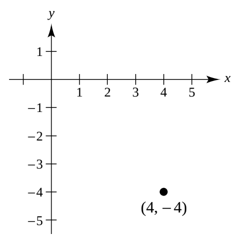
$$= r\int_0^{\pi} \theta\,d\theta = \frac{r}{2}\Big[\theta^2\Big]_0^{\pi} = \frac{1}{2}\pi^2 r$$

51. $(x, y) = (4, -4)$

$r = \sqrt{4^2 + (-4)^2} = 4\sqrt{2}$

$\theta = 7\dfrac{\pi}{4}$

$(r, \theta) = \left(4\sqrt{2}, \dfrac{7\pi}{4}\right), \left(-4\sqrt{2}, \dfrac{3\pi}{4}\right)$



53.

$$r = 3\cos\theta$$
$$r^2 = 3r\cos\theta$$
$$x^2 + y^2 = 3x$$
$$x^2 + y^2 - 3x = 0$$

55.

$$r = -2(1 + \cos\theta)$$
$$r^2 = -2r(1 + \cos\theta)$$
$$x^2 + y^2 = -2\left(\pm\sqrt{x^2 + y^2}\right) - 2x$$
$$(x^2 + y^2 + 2x)^2 = 4(x^2 + y^2)$$

57.

$$r^2 = \cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$r^4 = r^2\cos^2\theta - r^2\sin^2\theta$$
$$(x^2 + y^2)^2 = x^2 - y^2$$

59.

$$r = 4\cos 2\theta\sec\theta$$
$$= 4(2\cos^2\theta - 1)\left(\frac{1}{\cos\theta}\right)$$
$$r\cos\theta = 8\cos^2\theta - 4$$
$$x = 8\left(\frac{x^2}{x^2 + y^2}\right) - 4$$
$$x^3 + xy^2 = 4x^2 - 4y^2$$
$$y^2 = x^2\left(\frac{4 - x}{4 + x}\right)$$

61. $(x^2 + y^2)^2 = ax^2y$

$$r^4 = a(r^2\cos^2\theta)(r\sin\theta)$$
$$r = a\cos^2\theta\sin\theta$$

63. $x^2 + y^2 = a^2\left(\arctan\dfrac{y}{x}\right)^2$

$$r^2 = a^2\theta^2$$

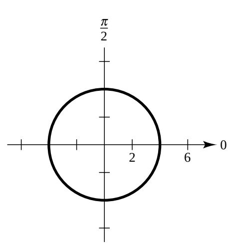
65. $r = 4$

Circle of radius 4

Centered at the pole

Symmetric to polar axis,

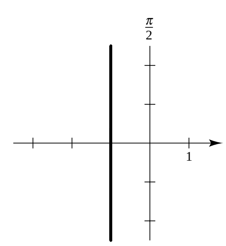
$\theta = \pi/2$, and pole



67. $r = -\sec\theta = \dfrac{-1}{\cos\theta}$

$r\cos\theta = -1, x = -1$

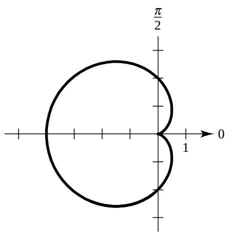
Vertical line



69. $r = -2(1 + \cos\theta)$

Cardioid

Symmetric to polar axis

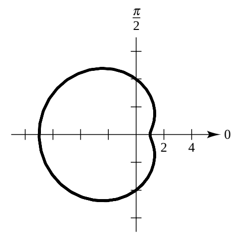


θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	-4	-3	-2	-1	0

71. $r = 4 - 3\cos\theta$

Limaçon

Symmetric to polar axis



θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	1	$\frac{5}{2}$	4	$\frac{11}{2}$	7

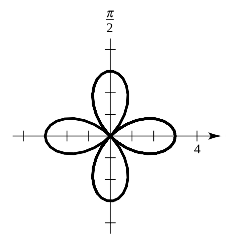
73. $r = -3\cos(2\theta)$

Rose curve with four petals

Symmetric to polar axis, $\theta = \dfrac{\pi}{2}$, and pole

Relative extrema: $(-3, 0), \left(3, \dfrac{\pi}{2}\right), (-3, \pi), \left(3, \dfrac{3\pi}{2}\right)$

Tangents at the pole: $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$



75. $r^2 = 4\sin^2(2\theta)$

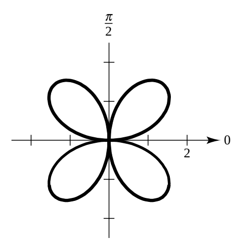
$r = \pm 2\sin(2\theta)$

Rose curve with four petals

Symmetric to the polar axis, $\theta = \dfrac{\pi}{2}$, and pole

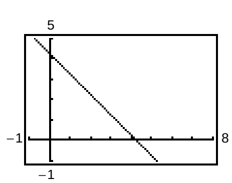
Relative extrema: $\left(\pm 2, \dfrac{\pi}{4}\right), \left(\pm 2, \dfrac{3\pi}{4}\right)$

Tangents at the pole: $\theta = 0, \dfrac{\pi}{2}$



77. $r = \dfrac{3}{\cos[\theta - (\pi/4)]}$

Graph of $r = 3\sec\theta$ rotated through an angle of $\pi/4$



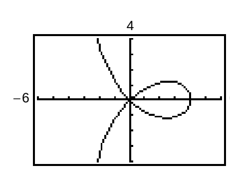
79. $r = 4\cos 2\theta \sec\theta$

Strophoid

Symmetric to the polar axis

$r \Rightarrow -\infty$ as $\theta \Rightarrow \dfrac{\pi^-}{2}$

$r \Rightarrow -\infty$ as $\theta \Rightarrow \dfrac{-\pi^+}{2}$



81. $r = 1 - 2\cos\theta$

(a) The graph has polar symmetry and the tangents at the pole are

$$\theta = \frac{\pi}{3}, -\frac{\pi}{3}.$$

(b) $\dfrac{dy}{dx} = \dfrac{2\sin^2\theta + (1 - 2\cos\theta)\cos\theta}{2\sin\theta\cos\theta - (1 - 2\cos\theta)\sin\theta}$

Horizontal tangents: $-4\cos^2\theta + \cos\theta + 2 = 0$, $\cos\theta = \dfrac{-1 \pm \sqrt{1 + 32}}{-8} = \dfrac{1 \pm \sqrt{33}}{8}$

When $\cos\theta = \dfrac{1 \pm \sqrt{33}}{8}$, $r = 1 - 2\left(\dfrac{1 + \sqrt{33}}{8}\right) = \dfrac{3 \mp \sqrt{33}}{4}$,

$$\left[\frac{3 - \sqrt{33}}{4}, \arccos\left(\frac{1 + \sqrt{33}}{8}\right)\right] \approx (-0.686, 0.568)$$

$$\left[\frac{3 - \sqrt{33}}{4}, -\arccos\left(\frac{1 + \sqrt{33}}{8}\right)\right] \approx (-0.686, -0.568)$$

$$\left[\frac{3 + \sqrt{33}}{4}, \arccos\left(\frac{1 - \sqrt{33}}{8}\right)\right] \approx (2.186, 2.206)$$

$$\left[\frac{3 + \sqrt{33}}{4}, -\arccos\left(\frac{1 - \sqrt{33}}{8}\right)\right] \approx (2.186, -2.206).$$

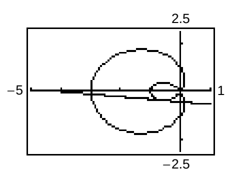
Vertical tangents:

$$\sin\theta(4\cos\theta - 1) = 0, \sin\theta = 0, \cos\theta = \frac{1}{4},$$

$$\theta = 0, \pi, \theta = \pm\arccos\left(\frac{1}{4}\right), (-1, 0), (3, \pi)$$

$$\left(\frac{1}{2}, \pm\arccos\frac{1}{4}\right) \approx (0.5, \pm 1.318)$$

(c)



83. Circle: $r = 3\sin\theta$

$$\frac{dy}{dx} = \frac{3\cos\theta\sin\theta + 3\sin\theta\cos\theta}{3\cos\theta\cos\theta - 3\sin\theta\sin\theta} = \frac{\sin 2\theta}{\cos^2\theta - \sin^2\theta} = \tan 2\theta \text{ at } \theta = \frac{\pi}{6}, \frac{dy}{dx} = \sqrt{3}$$

Limaçon: $r = 4 - 5\sin\theta$

$$\frac{dy}{dx} = \frac{-5\cos\theta\sin\theta + (4 - 5\sin\theta)\cos\theta}{-5\cos\theta\cos\theta - (4 - 5\sin\theta)\sin\theta} \text{ at } \theta = \frac{\pi}{6}, \frac{dy}{dx} = \frac{\sqrt{3}}{9}$$

Let α be the angle between the curves:

$$\tan\alpha = \frac{\sqrt{3} - \left(\sqrt{3}/9\right)}{1 + (1/3)} = \frac{2\sqrt{3}}{3}.$$

Therefore, $\alpha = \arctan\left(\dfrac{2\sqrt{3}}{3}\right) \approx 49.1°$.

85. $r = 1 + \cos\theta, r = 1 - \cos\theta$

The points $(1, \pi/2)$ and $(1, 3\pi/2)$ are the two points of intersection (other than the pole). The slope of the graph of $r = 1 + \cos\theta$ is

$$m_1 = \frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{-\sin^2\theta + \cos\theta(1 + \cos\theta)}{-\sin\theta\cos\theta - \sin\theta(1 + \cos\theta)}.$$

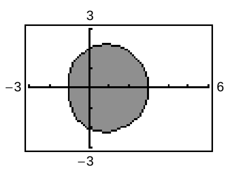
At $(1, \pi/2)$, $m_1 = -1/-1 = 1$ and at $(1, 3\pi/2)$, $m_1 = -1/1 = -1$. The slope of the graph of $r = 1 - \cos\theta$ is

$$m_2 = \frac{dy}{dx} = \frac{\sin^2\theta + \cos\theta(1 - \cos\theta)}{\sin\theta\cos\theta - \sin\theta(1 - \cos\theta)}.$$

At $(1, \pi/2)$, $m_2 = 1/-1 = -1$ and at $(1, 3\pi/2)$, $m_2 = 1/1 = 1$. In both cases, $m_1 = -1/m_2$ and we conclude that the graphs are orthogonal at $(1, \pi/2)$ and $(1, 3\pi/2)$.

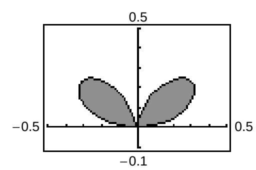
87. $r = 2 + \cos\theta$

$$A = 2\left[\frac{1}{2}\int_0^{\pi} (2 + \cos\theta)^2\, d\theta\right] \approx 14.14 \quad \left(\frac{9\pi}{2}\right)$$



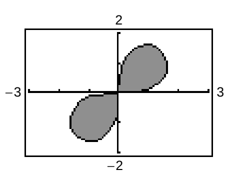
89. $r = \sin\theta \cdot \cos^2\theta$

$$A = 2\left[\frac{1}{2}\int_0^{\pi/2} (\sin\theta\cos^2\theta)^2\, d\theta\right] \approx 0.10 \left(\frac{\pi}{32}\right)$$



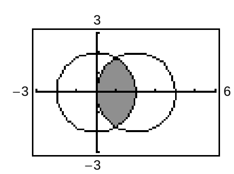
91. $r^2 = 4\sin 2\theta$

$$A = 2\left[\frac{1}{2}\int_0^{\pi/2} 4\sin 2\theta\, d\theta\right] = 4$$



93. $r = 4\cos\theta, r = 2$

$$A = 2\left[\frac{1}{2}\int_0^{\pi/3} 4\, d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2} (4\cos\theta)^2\, d\theta\right] \approx 4.91$$

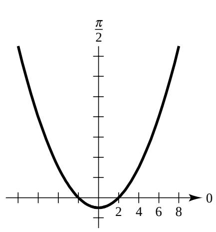


95. $s = 2\int_0^{\pi} \sqrt{a^2(1 - \cos\theta)^2 + a^2\sin^2\theta}\, d\theta$

$$= 2\sqrt{2}\, a \int_0^{\pi} \sqrt{1 - \cos\theta}\, d\theta = 2\sqrt{2}\, a \int_0^{\pi} \frac{\sin\theta}{\sqrt{1 + \cos\theta}}\, d\theta = \left[-4\sqrt{2}\, a(1 + \cos\theta)^{1/2}\right]_0^{\pi} = 8a$$

97. $r = \dfrac{2}{1 - \sin\theta}, e = 1$

Parabola



99. $r = \dfrac{6}{3 + 2\cos\theta} = \dfrac{2}{1 + (2/3)\cos\theta}, e = \dfrac{2}{3}$

Ellipse